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Asymptotic behavior of a degenerate forest kinematic model with a perturbation

Lu LI*, Guillaume Cantin†

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Abstract

In this paper, we study the asymptotic behavior of the global solution to a degenerate forest kinematic model, under the action of a perturbation modelling the impact of climate change. When the main nonlinearity of the model is assumed to be monotone, we prove that the global solution converges to a stationary solution, by showing that a Lyapunov function deduced from the system satisfies a Lojasiewicz-Simon gradient inequality. Under suitable assumptions on the parameters, we prove the continuity of the flow and of the stationary solutions with respect to the perturbation parameter. Although, due to a lack of compactness, the system does not admit the global attractor, we succeed in proving the robustness of the weak attractors, by establishing the existence of a family of positively invariant regions. We also present numerical simulations of the model and experiment the behavior of the solution under the effect of several types of perturbations. Finally, we show that the forest kinematic model can lead to the emergence of chaotic patterns.

Key words. Forest kinematic model, perturbation, asymptotic behavior, Lojasiewicz-Simon gradient inequality, robustness.

AMS Subject Classification: 35K57, 35K65, 35B40, 35B41.

1 Introduction

Global Forest Resources Assessment 2020 (FRA 2020) declares that there are 4.06 billion hectares forests around the world, cover nearly 1/3 of land globally. Forests, which are found around the globe, are the largest terrestrial ecosystem of Earth by area, and are one of the most valuable ecosystems on Earth. There exist tropical moist or dry forests around the Equator, and temperate forests at the middle latitudes, boreal forests in subarctic climates (see [41]). Most importantly, forests provide a diversity of ecosystem services including biodiversity, carbon stored, purifying water, aiding in regulating climate. Meanwhile, anthropic activities and forest ecosystems interact with each other. Anthropogenic factors that can affect forests include illegal or unsustainable logging, urban sprawl, human-caused forest fires, acid rain, invasive species, etc. There are also many natural factors that can cause changes in forests over time, including forest fires, pollution, insect pests, diseases, competition between species (see [35]), as well as the impacts of climate change (see [8, 29]). However, the impact of these accidental factors on forest ecosystems can not be fully described by non-random forest dynamics. In the past decades, scientists have continued to study global warming and its impact on Earth, and it is a challenge for the scientific community to better understand the dynamics of forest ecosystems. In this paper, our aim is to study, through a mathematical modelling approach, the dynamics of forest ecosystems associated with simplified perturbations, which are caused by global warming and anthropic activities.

In [27], a forest kinematic model determined by a system of parabolic-ordinary differential equations, and describing the dynamics of a simplified forest ecosystem, was firstly investigated. Here, we consider the

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following forest kinematic model with a perturbation:

$$\frac{\partial u}{\partial t} = \alpha w - q(u) - \mu p(u), \quad \frac{\partial w}{\partial t} = \delta \Delta w - \beta w + \alpha u, \quad (1)$$

where the unknown functions u , w denote the density of trees and the density of seeds, respectively. As detailed below, u satisfies a nonlinear ordinary differential equation involving a perturbation $p(u)$ of a given nonlinearity $q(u)$, and w satisfies a linear diffusion equation in a domain $\Omega \subset \mathbb{R}^2$ modelling a geographical region occupied by a forest. This forest kinematic model belongs to a large class of degenerate reaction-diffusion systems arising in life science and admitting remarkable properties. Such degenerate reaction-diffusion systems have been widely studied, for instance in [32] to model the spread of early tumors along linear or tubular structures; in [2, 36] to model the competing species with sedentary species. Therefore, it is of long-term practical significance to study the properties of the forest kinematic model. Here, the degeneracy corresponds to the absence of diffusion in the first equation. In other words, one species of the system is sedentary (namely trees, which are typically motionless). Furthermore, the non-diffusive equation admits a non trivial nonlinearity, which is often related to a *hysteresis* process that acts in the system (see for instance [1, 25, 26, 46]). Hence, it is observed that these degenerate reaction-diffusion systems admit discontinuous patterns. Although the properties of such patterns has been widely studied, for instance in [4, 6, 11, 26, 28, 33, 34], their characteristic mechanisms are far from being completely understood. Indeed, their local stability has been studied (for instance in [7, 11, 26]), but to the best of our knowledge, their *structural* stability, that is, their behavior under the effect of a perturbation of the system, has not been analyzed. In parallel, very recent results have been established on the asymptotic behavior of the trajectories determined by these degenerate forest kinematic systems. Notably, a result of non-existence of the global attractor for a degenerate reaction-diffusion system with hysteresis has been proved in [5]. Meanwhile, the weak convergence of the solutions to a forest model towards heterogeneous stationary solutions has been established in [23] by applying the Lojasiewicz-Simon gradient inequality, and the weak convergence of the solutions to a simplified model towards discontinuous patterns has been proved in [6] by using a macroscopic mass effect under symmetry assumptions. However, it is worth pointing out that no attempt has been made to analyze the robustness of the asymptotic behavior of these dynamical systems with respect to a perturbation for now. It is precisely our aim in this paper to bring a novel contribution on this point. Therefore, we study the asymptotic behavior of the perturbed problem (1). We establish a new result on the convergence of the solutions towards stationary solutions and we prove non trivial statements on the robustness of the flow and of the weak attractors, under the action of the perturbation parameter, in a functional context which is characterized by a lack of compactness. Although it is very basic, the perturbation succeeds in faithfully reproducing ecological properties of great interest, as highlighted by our numerical results. We emphasize that our main results in this paper are based on the assumption that the perturbation is monotone. For the non-monotone case, limited results have been obtained in [5] and [13].

Besides, since the 1970's, many numerical studies show that reaction-diffusion systems can produce steady state finite amplitude spatial patterns, for instance, fronts, spirals, targets, hexagons, stripes, dissipative solitons, etc. Therefore, reaction-diffusion systems have attracted much interest as a prototype model for pattern formation (see [24, 39]). However, patterns in nature are often chaotic, rarely exactly repeating, and often involve fractals. Because extremely small differences in starting conditions can lead to widely differing outcomes. The numerical simulations we perform in Section 5.3 justify this assertion.

Our paper is organized as follows. In Section 2, we firstly present the forest kinetic model and its well-posedness results, and the Lyapunov function deduced by the dynamical system. We further characterize the L^2 - ω -limit set of the global solution and prove its non emptiness. In Section 3, under the assumptions of parameters, we prove the asymptotic convergence result (Theorem 4) by applying the Lojasiewicz-Simon gradient inequality (Proposition 11). It is remarkable that the conclusion only holds when the potential of the perturbation is convex. In Section 4, we study the long time behavior of this model when the perturbation parameter μ tends to 0. We prove the continuity of the flow (Theorem 6), and in the monotone case, we can further prove the continuity of the stationary solutions (Theorem 7), which is nontrivial, and yields the robustness of the weak attractors (Theorem 8). We also analyze the case of a strong perturbation and show how it drives the system to converge to the trivial equilibrium (Theorem 9). Finally, in Section 5, we present several numerical simulations, which help better understand how the ecotone can be shifted, and how intermediate ecosystems can emerge under different climatic perturbations. We also show that randomly generated initial conditions can lead to chaotic patterns.

2 Setting of the problem

In this section, we present the degenerate forest kinematic model under study, the assumptions on the perturbation introduced in the model, and basic results on the existence of local and global solutions.

2.1 Notations and preliminary results

Throughout this paper, C will stand for positive constants, which may depend on Ω and some other parameters, but are independent of the choice of t , and may change from line to line. $\Omega \subset \mathbb{R}^n$ is a bounded domain, and its boundary Γ is sufficiently smooth. Let $\mathcal{C}(I, X)$ (respectively $\mathcal{C}^1(I, X)$) denote the space of continuous (respectively continuously differentiable) functions defined on an interval $I \subset \mathbb{R}$ with values in a Banach space X . Let $L^p(\Omega)$ and $W^{k,p}(\Omega)$, $p \in [1, \infty]$, $k \in \mathbb{N}$ be the general Lebesgue and Sobolev spaces, equipped with the norms $\|\cdot\|_{L^p}$ and $\|\cdot\|_{W^{k,p}}$, respectively. In particular, for $p = 2$, we simply note $W^{k,2}(\Omega) = H^k(\Omega)$. If V is a Hilbert space with dual V' , its inner product is denoted (u, v) , $u, v \in V$, and the duality product in $V \times V'$ is denoted $\langle u, v \rangle$, $u \in V$, $v \in V'$. To simplify the notations, we denote $\mathbb{L}^2(\Omega) = L^2(\Omega) \times L^2(\Omega)$.

So as to guaranty the self-sufficiency of the paper, we now present an important result on semi-linear equations. Let A be a sectorial operator in X of angle $\xi < \frac{\pi}{2}$. Let η be a real coefficient such that $0 < \eta < 1$, and let F be a non-linear operator defined in $\mathcal{D}(A^\eta)$ with values in X . We consider the Cauchy problem

$$\frac{dU}{dt} + AU = F(U), \quad t > 0, \quad U(0) = U_0, \quad (2)$$

with $U_0 \in X$. We assume that F enjoys the property:

$$\|F(U) - F(V)\|_X \leq C_F(1 + \|A^\eta U\|_X^2 + \|A^\eta V\|_X^2)\|A^\eta(U - V)\|_X, \quad (3)$$

for all $U, V \in \mathcal{D}(A^\eta)$, with a positive constant C_F and a well-chosen $\eta \in (0, 1)$. The following theorem is proved in Ref. yagi2009abstract Theorem 4.4.

Theorem 1. *For all $U_0 \in X$, there exists $T_{U_0} > 0$ such that problem (2) admits a unique solution $U = U(t, U_0)$ in function space*

$$U \in \mathcal{C}((0, T_{U_0}]; \mathcal{D}(A)) \cap \mathcal{C}([0, T_{U_0}]; X) \cap \mathcal{C}^1((0, T_{U_0}]; X),$$

where T_{U_0} depends only on $\|U_0\|_X$.

2.2 The degenerate forest kinematic model

In this paper, we consider the following initial boundary value problem:

$$\begin{cases} \frac{\partial u}{\partial t} = \alpha w - q_\mu(u) & \text{in } (0, +\infty) \times \Omega, \\ \frac{\partial w}{\partial t} = \delta \Delta w - \beta w + \alpha u & \text{in } (0, +\infty) \times \Omega, \\ \frac{\partial w}{\partial \nu} = 0 & \text{on } (0, +\infty) \times \Gamma, \\ u(0, x) = u_0(x), \quad w(0, x) = w_0(x) & \text{in } \Omega, \end{cases} \quad (4)$$

in a bounded and regular domain $\Omega \subset \mathbb{R}^2$ with boundary Γ . The domain Ω models a geographical area occupied by a forest. The unknown functions $u = u(t, x)$ and $w = w(t, x)$ respectively correspond to the densities of the trees and the air-borne seeds. The biological coefficients α, β are the seed production and seed deposition rates; δ is a diffusion rate in the air; $q_\mu(u)$ denotes the mortality of the trees, which is a smooth function associated with a perturbation term, given as

$$q_\mu(u) = q(u) + \mu p(u), \quad q(u) = u[a(u - b)^2 + c], \quad \mu \geq 0, \quad u \in \mathbb{R}, \quad (5)$$

where a, b, c are positive coefficients. We also assume that the coefficients α, β, δ and μ are positive, and the perturbation $p(u)$ is continuously differentiable in \mathbb{R} and satisfies:

$$|p(s)| + |p'(s)| \leq M_1, \quad \exists M_1 > 0, \forall s \in \mathbb{R}. \quad (6)$$

A simple example of a perturbation $p(u)$ has been studied in Ref. cantin2023onthe, given by

$$p(u) = \frac{20u}{[1 + 10(u - b)^2]^2}, \quad u \in \mathbb{R}^+. \quad (7)$$

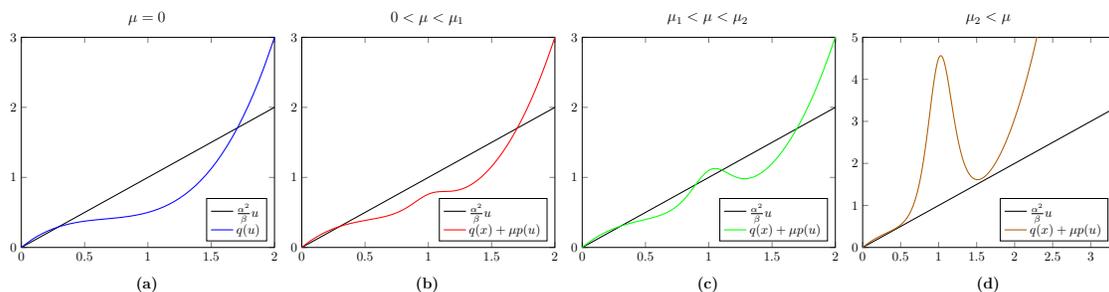


Figure 1: Effect of the perturbation parameter μ on the function $q_\mu(u)$ defined by (5) and (7).

Remark 1. If the parameters α , β and c satisfy the condition

$$\sqrt{\beta c} \leq \alpha \leq \beta, \quad (8)$$

then there exist positive constants $\mu_1 < \mu_2$ such that $q_\mu(u)$ is monotone for all $0 \leq \mu \leq \mu_1$, non monotone for $\mu > \mu_1$. For $\mu_1 < \mu < \mu_2$, $q_\mu(u)$ presents at least three monotone branches and admits at least 3 intersection points with the line $f(u) = \frac{\alpha^2}{\beta}u$ for all $0 \leq \mu < \mu_2$. For $\mu > \mu_2$, $q_\mu(u)$ admits a unique intersection point with the line $f(u) = \frac{\alpha^2}{\beta}u$. In the rest of the paper, we will always assume that the parameter condition (8) is fulfilled.

Besides, the behavior of a type of time period perturbation is also attractive to study, for instance, in the case of

$$p(u) = \sin(20u), \quad u \in \mathbb{R}^+, \quad (9)$$

it illustrates from Figure 2(b) that there exists a perturbation parameter $\mu_1 > 0$ so that the perturbation is monotone in $[0, \mu_1]$, and admits 5 intersection points with the line $f(u) = \frac{\alpha^2}{\beta}u$. For $\mu_1 < \mu < \mu_2$, $q_\mu(u)$ admits more than 5 intersection points. In Section 5, we will also perform some numerical simulations of the degenerate kinematic forest model with different perturbations and study the effect of climatic perturbations effect on the model. In particular, we will observe the emergence of ecotones in the forest ecosystem, see Section 5.2 for more information.

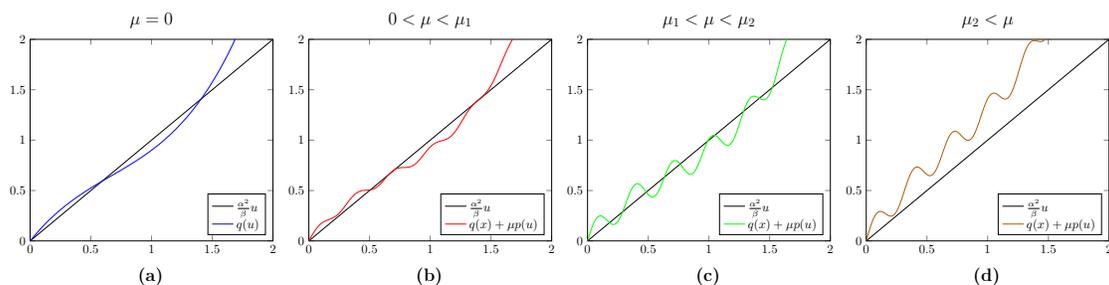


Figure 2: Effect of the perturbation parameter μ on the function q_μ defined by (5) and (9).

Note that the function $q_\mu(u)$ derives from a potential $Q_\mu(u)$ written as

$$Q_\mu(u) = \int_0^u (q(\xi) + \mu p(\xi)) d\xi, \quad u \in \mathbb{R}. \quad (10)$$

In this paper, we prove a Lojasiewicz-Simon gradient inequality under the assumption:

$$c - \frac{1}{3}ab^2 + \mu p'(u) \geq 0, \quad (11)$$

which implies that the potential $Q_\mu(u)$ given by (10) is convex. More precisely, since

$$Q_\mu''(u) = \frac{1}{3}a(3u - 2b)^2 + c - \frac{1}{3}ab^2 + \mu p'(u),$$

note that $q_0 = \min_{u \in \mathbb{R}} Q_\mu''(u) \geq 0$ is required.

Now, our aim is to prove that the degenerate forest kinetic system (4) admits local solutions. To this end, we rewrite it as a semi-linear equation in a proper Banach space.

2.3 Abstract formulation and local solutions

Following [49], we handle the degenerate forest kinetic system (4) in the Banach space X defined by

$$X = L^\infty(\Omega) \times L^2(\Omega),$$

equipped with the product norm $\|U\|_X = \|u\|_{L^\infty} + \|w\|_{L^2}$, $U = (u, w)^\top \in X$. The space of initial values is given by

$$\mathcal{K} = \{U = (u, w)^\top \in X; u, w \geq 0\}.$$

We consider the differential operator Λ defined as the realization of $-\delta\Delta + \beta$ in $L^2(\Omega)$ with the Neumann boundary condition on Γ . It is known that Λ is a positive definite self-adjoint and sectorial operator, of angle strictly less than $\frac{\pi}{2}$, with domain

$$\mathcal{D}(\Lambda) = H_N^2(\Omega) = \left\{ w \in H^2(\Omega); \frac{\partial w}{\partial \nu} = 0 \text{ on } \Gamma \right\}.$$

Hence, the diagonal operator $A = \text{diag}\{1, \Lambda\}$ is also a sectorial operator in X , with angle strictly less than $\frac{\pi}{2}$, and with domain $\mathcal{D}(A) = L^\infty(\Omega) \times \mathcal{D}(\Lambda)$. Here, we emphasize that the domain of A is *not* compactly embedded in X , although the domain of Λ is compactly embedded in $L^2(\Omega)$. This is due to the absence of diffusion in the first equation, and partly determines an original asymptotic behavior, as will be shown in Section 4.

Next, we consider an exponent $\eta \in (\frac{3}{4}, 1)$. The sectorial operator Λ admits a fractional power Λ^η whose domain is given by

$$\mathcal{D}(\Lambda^\eta) = H_N^{2\eta}(\Omega) = \left\{ w \in H^{2\eta}(\Omega); \frac{\partial w}{\partial \nu} = 0 \text{ on } \Gamma \right\},$$

where $H^{2\eta}(\Omega)$ is the interpolation space $W^{2\eta, 2}(\Omega)$. We have the continuous embeddings

$$H^{2\eta}(\Omega) \subset \mathcal{C}(\bar{\Omega}) \subset L^\infty(\Omega) \subset L^2(\Omega), \quad (12)$$

note that $\|u\|_{\mathcal{D}(\Lambda^\eta)}$ and $\|\Lambda^\eta u\|_{L^2(\Omega)}$ are equivalent. The diagonal operator A also admits a fractional power A^η and its domain is given by $\mathcal{D}(A^\eta) = L^\infty(\Omega) \times H_N^{2\eta}(\Omega)$.

In this way, the degenerate forest kinematic system (4) can be written in an abstract form

$$\begin{cases} \frac{dU}{dt} + AU = F_\mu(U), & t > 0, \\ U(0) = U_0, \end{cases} \quad (13)$$

where $F_\mu(U)$ is the nonlinear operator defined by

$$F_\mu(U) = \begin{pmatrix} \alpha w - q_\mu(u) + u \\ \alpha u \end{pmatrix}, \quad U = (u, w)^\top \in \mathcal{D}(A^\eta). \quad (14)$$

Note that the domain of the nonlinear operator F_μ is uniform with respect to the perturbation parameter μ . The following proposition establishes an estimate of F_μ which is crucial for proving the existence of local solutions to the problem (13). This estimate will also be useful for studying the asymptotic behavior of the solutions.

Proposition 1. *There exist constants $C_1, C_2 > 0$ such that the nonlinear operator F_μ defined in $\mathcal{D}(A^\eta)$ by (14) satisfies*

$$\begin{aligned} \|F_\mu(U) - F_\mu(\tilde{U})\|_X &\leq C_1(1 + \|U\|_X^2 + \|\tilde{U}\|_X^2)\|A^\eta(U - \tilde{U})\|_X \\ &\quad + \mu C_2\|A^\eta(U - \tilde{U})\|_X, \end{aligned} \quad (15)$$

for all $\mu \in \mathbb{R}^+$ and $U, \tilde{U} \in \mathcal{D}(A^\eta)$.

Proof. Let $U = (u, w)^\top, \tilde{U} = (\tilde{u}, \tilde{w})^\top \in \mathcal{D}(A^\eta)$ with $u, \tilde{u} \in L^\infty(\Omega), w, \tilde{w} \in H_N^{2\eta}(\Omega)$. We compute

$$\begin{aligned} \|F_\mu(U) - F_\mu(\tilde{U})\|_X &= \|(\alpha w - q_\mu(u) + u) - (\alpha \tilde{w} - q_\mu(\tilde{u}) + \tilde{u})\|_{L^\infty} + \|\alpha(u - \tilde{u})\|_{L^2} \\ &\leq \|\alpha(w - \tilde{w})\|_{L^\infty} + \|q_\mu(u) - q_\mu(\tilde{u})\|_{L^\infty} + \|u - \tilde{u}\|_{L^\infty} \\ &\quad + \|\alpha(u - \tilde{u})\|_{L^2}. \end{aligned}$$

By virtue of the continuous embeddings (12), we have

$$\|\alpha(w - \tilde{w})\|_{L^\infty} \leq C\|w - \tilde{w}\|_{H^{2\eta}} \leq C\|A^\eta(U - \tilde{U})\|_X, \quad (16)$$

since $\|A^\eta(U - \tilde{U})\|_X = \|u - \tilde{u}\|_{L^\infty} + \|A^\eta(w - \tilde{w})\|_{L^2}$. Similarly, we have

$$\|u - \tilde{u}\|_{L^\infty} \leq C\|A^\eta(U - \tilde{U})\|_X, \quad \|\alpha(u - \tilde{u})\|_{L^2} \leq C\|A^\eta(U - \tilde{U})\|_X. \quad (17)$$

Next, we have

$$\|q_\mu(u) - q_\mu(\tilde{u})\|_{L^\infty} \leq \|q(u) - q(\tilde{u})\|_{L^\infty} + \mu\|p(u)p(\tilde{u})\|_{L^\infty}.$$

We observe that $q(s) - q(\tilde{s}) = (s - \tilde{s})[a(s^2 + s\tilde{s} + \tilde{s}^2) - 2ab(s + \tilde{s}) + ab^2 + c]$, for all $s, \tilde{s} \in \mathbb{R}$, from which we deduce

$$\begin{aligned} \|q(u) - q(\tilde{u})\|_{L^\infty} &\leq C\|u - \tilde{u}\|_{L^\infty} (\|u\|_{L^\infty}^2 + \|u\|_{L^\infty}\|\tilde{u}\|_{L^\infty} \\ &\quad + \|\tilde{u}\|_{L^\infty}^2 + \|u\|_{L^\infty} + \|\tilde{u}\|_{L^\infty} + 1) \\ &\leq C\|A^\eta(U - \tilde{U})\|_X (1 + \|U\|_X^2 + \|\tilde{U}\|_X^2). \end{aligned}$$

In parallel, by the properties of $p(u)$, we have

$$\|p(u) - p(\tilde{u})\|_{L^\infty} \leq M_1\|u - \tilde{u}\|_{L^\infty},$$

which leads to

$$\begin{aligned} \|q_\mu(u) - q_\mu(\tilde{u})\|_{L^\infty} &\leq C_1\|A^\eta(U - \tilde{U})\|_X (1 + \|U\|_X^2 + \|\tilde{U}\|_X^2) \\ &\quad + \mu C_2\|A^\eta(U - \tilde{U})\|_X, \end{aligned} \quad (18)$$

where C, C_1 and C_2 depend on a, b and c . Finally, gathering estimates (16), (17) and (18) leads to (15). We finish the proof. \square

Therefore, we can apply Theorem 1 and address the existence of a local-in-time solution to the abstract problem (13).

Theorem 2 (Local solution). *Let $\mu \geq 0$. For all $U_0 \in \mathcal{K}$, the Cauchy problem (13) admits a unique local solution $U_\mu = (u_\mu, w_\mu)^\top$ defined on $[0, T_{U_0}]$ in the function space*

$$\begin{aligned} u_\mu &\in \mathcal{C}([0, T_{U_0}], L^\infty(\Omega)) \cap \mathcal{C}^1((0, T_{U_0}], L^\infty(\Omega)), \\ w_\mu &\in \mathcal{C}((0, T_{U_0}], H_N^2(\Omega)) \cap \mathcal{C}([0, T_{U_0}], L^2(\Omega)) \cap \mathcal{C}^1((0, T_{U_0}], L^2(\Omega)), \end{aligned}$$

where $T_{U_0} > 0$ is only determined by the norm $\|U_0\|_X$. Moreover, the local solution satisfies the estimate

$$t\|AU(t)\|_X + \|U(t)\|_X \leq C_{U_0}, \quad 0 < t \leq T_{U_0}, \quad (19)$$

where $C_{U_0} > 0$ depends only on $\|U_0\|_X$.

2.4 Global solutions and continuous dynamical system

We further prove that the local-in-time solutions to the degenerate forest kinematic system (4) are actually global and determine a continuous dynamical system. The following proposition establishes an *a priori* estimate for any local solution.

Proposition 2 (Dissipative estimate). *Let $\mu \geq 0$. There exist an exponent $\sigma > 0$ and a constant $C_3 > 0$ such that, for all $U_0 \in \mathcal{K}$, the local-in-time solution U_μ of the Cauchy problem defined by (13) and $U(0) = U_0$, defined on $[0, T_{U_0}]$ with $T_{U_0} > 0$, satisfies*

$$\|U_\mu(t)\|_X \leq C_3(e^{-\sigma t}\|U_0\|_X + 1), \quad (20)$$

for all $t \in [0, T_{U_0}]$.

Proof. Let $U_0 \in X$. We denote by $U_\mu(t)$ the local-in-time solution to the degenerate forest kinematic system (4) stemming from U_0 , defined on $[0, T_{U_0}]$ with $T_{U_0} > 0$. We introduce the energy function L defined for $t \geq 0$ by

$$L(t) = \frac{1}{2} \int_{\Omega} (u_\mu^2(t) + w_\mu^2(t)) dx.$$

The function L is continuously differentiable on \mathbb{R}^+ and we have

$$\begin{aligned} \dot{L}(t) &= \int_{\Omega} \frac{\partial u_\mu}{\partial t}(t) u_\mu(t) dx + \int_{\Omega} \frac{\partial w_\mu}{\partial t}(t) w_\mu(t) dx \\ &= 2\alpha \int_{\Omega} u_\mu(t) w_\mu(t) dx - \int_{\Omega} u_\mu(t) q_\mu(u_\mu(t)) dx \\ &\quad - \delta \int_{\Omega} |\nabla w_\mu(t)|^2 dx - \beta \int_{\Omega} w_\mu^2(t) dx, \end{aligned}$$

by applying the Green's formula with the Neumann boundary condition.

Note that

$$2\alpha \int_{\Omega} u_\mu(t) w_\mu(t) dx \leq \frac{2\alpha^2}{\beta} \int_{\Omega} u_\mu^2(t) dx + \frac{\beta}{2} \int_{\Omega} w_\mu^2(t) dx.$$

by employing the generalized Young's inequality $ab \leq \frac{1}{2\varepsilon} a^2 + \frac{\varepsilon}{2} b^2$, $\forall a, b \in \mathbb{R}$ (here, we take $\varepsilon = \frac{\beta}{2\alpha} > 0$).

Next, we claim that the following lower estimate holds for all $s \in \mathbb{R}$:

$$sq_\mu(s) \geq g_1 s^2 - g_2, \quad g_1, g_2 > 0.$$

Indeed, it follows from the properties of $p(s)$ that

$$\begin{aligned} sq_\mu(s) - (g_1 s^2 - g_2) &= s^2[a(s-b)^2 + c] + \mu sp(s) - g_1 s^2 + g_2 \\ &\geq s^2[a(s-b)^2 + c - g_1] + g_2 \\ &\geq q_2 + g_2, \end{aligned}$$

where $q_2 = \inf_{s \in \mathbb{R}} \{s^2[a(s-b)^2 + c - g_1]\}$. Since the first coefficient of the quartic polynomial is positive, there exists such a infimum q_2 . When g_1 is chosen arbitrarily large, it suffices to choose a proper g_2 such that $q_2 + g_2 \geq 0$.

Then, we deduce that

$$\dot{L}(t) \leq - \left(g_1 - \frac{2\alpha^2}{\beta} \right) \int_{\Omega} u_\mu^2(t) dx - \frac{\beta}{2} \int_{\Omega} w_\mu^2(t) dx + g_2 |\Omega|,$$

where g_1 is chosen to ensure that $g_1 - \frac{2\alpha^2}{\beta} > 0$, and we denote

$$\sigma = \min \left(g_1 - \frac{2\alpha^2}{\beta}, \frac{\beta}{2} \right)$$

to obtain

$$\dot{L}(t) \leq -2\sigma L(t) + g_2|\Omega|.$$

We thus have

$$L(t) \leq L(0)e^{-2\sigma t} + \frac{g_2|\Omega|}{2\sigma}(1 - e^{-2\sigma t}),$$

by applying Gronwall lemma, which reduces to

$$\left(\int_{\Omega} (u_{\mu}^2(t) + w_{\mu}^2(t)) dx \right)^{1/2} \leq \left(\int_{\Omega} (u_0^2 + w_0^2) dx \right)^{1/2} e^{-\sigma t} + C.$$

We obtain

$$\|U_{\mu}(t)\|_{\mathbb{L}^2} \leq C(e^{-\sigma t}\|U_0\|_{\mathbb{L}^2} + 1), \quad (21)$$

for all $t \geq 0$.

Next, it remains to show how the dissipative estimate (21), which holds in $\mathbb{L}^2(\Omega) = L^2(\Omega) \times L^2(\Omega)$, implies the stronger estimate (20) in X . Indeed, it suffices to apply similar arguments as in the proof of Proposition 11.1 in Ref. yagi2009abstract. Hence, replacing σ by new one, we can show that w and u satisfy respectively

$$\|w_{\mu}(t)\|_{L^{\infty}} \leq C[(1 + t^{-\eta})e^{-\sigma t}\|U_0\|_{L^2} + 1], \quad (22)$$

$$\|u_{\mu}(t)\|_{L^{\infty}} \leq C(e^{-\sigma t}\|U_0\|_X + 1), \quad (23)$$

for all $t > 0$.

Finally, we combine the dissipative estimates (21) and (23) to obtain (20), which completes the proof. \square

With the dissipative estimate (20), we can directly state the existence of global solutions, that determine a continuous dynamical system.

Theorem 3 (Global solutions and continuous dynamical system). *Let $\mu \geq 0$. For all $U_0 \in \mathcal{K}$, the Cauchy problem defined by (13) and $U(0) = U_0$ admits a unique global solution $U_{\mu}(t, U_0) = (u_{\mu}, w_{\mu})^{\top}$ defined on $[0, +\infty)$ in the function space*

$$\begin{aligned} u_{\mu} &\in \mathcal{C}([0, +\infty), L^{\infty}(\Omega)) \cap \mathcal{C}^1((0, +\infty), L^{\infty}(\Omega)), \\ w_{\mu} &\in \mathcal{C}((0, +\infty), H_N^2(\Omega)) \cap \mathcal{C}([0, +\infty), L^2(\Omega)) \cap \mathcal{C}^1((0, +\infty), L^2(\Omega)). \end{aligned}$$

Furthermore, the degenerate forest kinematic system (4) determines a continuous dynamical system $S_{\mu}(t)$ defined in X by

$$S_{\mu}(t)U_0 = U_{\mu}(t, U_0), \quad t \geq 0. \quad (24)$$

Remark 2 (Lack of compactness). *Using the dissipative estimate (20), it can be shown that the continuous dynamical system $S_{\mu}(t)$ admits an absorbing set $\mathcal{B}_{\mu} \subset X$ which is bounded in $\mathcal{D}(A)$ (see[49] Chapter 11, Section 4). However, we can not show that the absorbing set \mathcal{B}_{μ} is compact, hence it turns out that the study of the asymptotic behavior of the dynamical system $S_{\mu}(t)$ can not be described by means of the global attractor.[5] In Section 3, we will bypass this lack of compactness by showing the existence of a family of positively invariant regions.*

For $U_0 \in \mathcal{K}$, let $U(t) = U(t, U_0)$ be the global solution of (4). It immediately follows from the dissipative estimate in Proposition 2 that

$$\|U_{\mu}(t, U_0)\|_X \leq C(e^{-\sigma t}\|U_0\|_X + 1), \quad 0 \leq t < \infty, \quad U_0 \in \mathcal{K}.$$

In particular, we have

$$\|U_{\mu}(t, U_0)\|_X \leq C(\|U_0\|_X + 1), \quad 0 \leq t < \infty, \quad U_0 \in \mathcal{K}.$$

This, together with (19), yields that

$$\|AU_{\mu}(t, U_0)\|_X \leq (1 + t^{-1})p_0(\|U_0\|), \quad 0 \leq t < \infty, \quad U_0 \in \mathcal{K}, \quad (25)$$

where $p_0(\cdot)$ is a suitable continuous increasing function. Hence, we obtain

$$\|w_\mu(t)\|_{H^2} \leq (1+t^{-1})p_0(\|U_0\|), \quad 0 \leq t < \infty, \quad U_0 \in \mathcal{K}.$$

The latter inequality, together with (22), leads to

$$\|w_\mu(t)\|_{L^\infty} \leq (1+t^{-\eta})p_0(\|U_0\|), \quad 0 \leq t < \infty, \quad U_0 \in \mathcal{K}. \quad (26)$$

Therefore, we can further obtain the following estimates.

Proposition 3. For $0 < t < \infty$ and $U_0 \in \mathcal{K}$, the time derivative $\frac{dU_\mu}{dt}(t)$ satisfies

$$\left\| \frac{\partial u_\mu}{\partial t} \right\|_{L^\infty} \leq (1+t^{-\eta})p_1(\|U_0\|_X), \quad (27)$$

$$\left\| \frac{\partial w_\mu}{\partial t} \right\|_{L^2} \leq (1+t^{-1})p_1(\|U_0\|_X), \quad (28)$$

where $p_1(\cdot)$ is an appropriate continuous increasing function.

Proof. The first estimate (27) is derived directly from (26) and the first equation of u in model (4), and the second estimate (28) is an immediate consequence of (25). \square

Proposition 4. For $0 < t < \infty$ and $U_0 \in \mathcal{K}$, the second order time derivative $\frac{d^2U_\mu}{dt^2}(t)$ satisfies

$$\left\| \frac{\partial^2 u_\mu}{\partial t^2} \right\|_{L^\infty} \leq (1+t^{-1-\eta})p_2(\|U_0\|_X), \quad (29)$$

$$\left\| \frac{\partial^2 w_\mu}{\partial t^2} \right\|_{L^2} \leq (1+t^{-2})p_2(\|U_0\|_X), \quad (30)$$

where $p_2(\cdot)$ is an appropriate continuous increasing function.

Proof. Take the time derivative of the second equation of w in model (4) to deduce the following estimate

$$\left\| \frac{\partial^2 w_\mu}{\partial t^2} \right\|_{L^2} + \left\| \frac{\partial w_\mu}{\partial t} \right\|_{H^2} \leq C[1+(t-\tau)^{-1}] \left\| \frac{\partial w_\mu}{\partial \tau} \right\|_{L^2}, \quad 0 < \tau < t < \infty.$$

By taking $\tau = \frac{t}{2}$, we verify (30) and the estimate

$$\left\| \frac{\partial w_\mu}{\partial t} \right\|_{H^2} \leq (1+t^{-2})p_2(\|U_0\|_X), \quad 0 < t < \infty.$$

It follows that

$$\left\| \frac{\partial w_\mu}{\partial t} \right\|_{L^\infty} \leq C \left\| \frac{\partial w_\mu}{\partial t} \right\|_{H^{2\eta}} \leq (1+t^{-1-\eta})p_2(\|U_0\|_X), \quad 0 < t < \infty.$$

Then, (29) is observed by taking time derivative of the first equation of u in model (4). \square

2.5 Lyapunov function

It is easily seen that the potential $Q_\mu(u)$ defined by (10) satisfies

$$Q_\mu(u) - \alpha u w \geq -C, \quad \forall u, w \geq 0,$$

with $C > 0$. Therefore, we can prove that the dynamical system $(S_\mu(t), \mathcal{K}, X)$ determined by (24) admits a Lyapunov function given by

$$\mathcal{L}_\mu(u, w) = \int_\Omega \left[\frac{\delta}{2} |\nabla w|^2 + \frac{\beta}{2} w^2 dx - \alpha u w + Q_\mu(u) \right] dx. \quad (31)$$

In [23], such a Lyapunov function has been considered to prove the weak convergence of the orbits of a forest kinematic model towards stationary solutions. Besides, the existence of a Lyapunov function highlights the dissipative nature of the degenerate forest kinematic system (4).

Remark 3. Note that the behavior of the degenerate forest kinetic system (4) is partly governed by the potential $H_\mu(u, w)$ defined by

$$H_\mu(u, w) = \frac{\beta}{2}w^2 - \alpha uw + Q_\mu(u), \quad u, w \in \mathbb{R}. \quad (32)$$

Recall that the potential $Q_\mu(u)$ is convex under the assumption (11). However, the convexity of $Q_\mu(u)$ does not imply that $H_\mu(u, w)$ is convex. Besides, in the case where $q(u)$ is an unperturbed and non-invertible cubic function, the properties of potential $H(u, w)$ have been studied in [5].

Then, we establish the following proposition.

Proposition 5. For any trajectory $S_\mu(t)U_0 = U_\mu(t)$ of the forest kinematic model (4), $U_0 \in \mathcal{K}$, the time derivative of the global solution is bounded in $L^2((1, \infty), \mathbb{L}^2)$, i.e.,

$$\int_1^\infty \left\| \frac{dU_\mu}{dt}(t) \right\|_{\mathbb{L}^2}^2 < \infty.$$

Proof. Similar as the proof of Proposition 11.4 in Ref. yagi2009abstract, we have

$$-\int_\Omega \left(\left| \frac{\partial u}{\partial t} \right|^2 + \left| \frac{\partial w}{\partial t} \right|^2 \right) dx = \frac{d}{dt} \int_\Omega \left[\frac{\delta}{2} |\nabla w|^2 dx + \frac{\beta}{2} w^2 dx - \alpha uw + Q_\mu(u) \right] dx \leq 0. \quad (33)$$

Integrating (33) on a time interval $[1, T]$, with $T > 0$, leads to

$$\begin{aligned} \int_1^T \int_\Omega \left(\left| \frac{\partial u_\mu}{\partial t} \right|^2 + \left| \frac{\partial w_\mu}{\partial t} \right|^2 \right) dx dt &\leq \int_\Omega \left[\frac{\delta}{2} |\nabla w_\mu(1)|^2 dx + \frac{\beta}{2} w_\mu(1)^2 dx \right. \\ &\quad \left. + \alpha u_\mu(T)w_\mu(T) + Q_\mu(u_\mu(1)) \right] dx. \end{aligned}$$

Next, it follows from (20) that

$$\int_1^\infty \int_\Omega \left(\left| \frac{\partial u_\mu}{\partial t} \right|^2 + \left| \frac{\partial w_\mu}{\partial t} \right|^2 \right) dx dt < \infty,$$

which ends the proof. \square

Following [49] (Proposition 11.5), we now state the following second proposition.

Proposition 6. For any trajectory $S_\mu(t)U_0 = U_\mu(t)$ of the forest kinematic model (4), $U_0 \in \mathcal{K}$, the time derivative of the global solution $\frac{dU_\mu}{dt}(t)$ converges to 0 as $t \rightarrow \infty$ in the \mathbb{L}^2 norm.

Proof. Propositions 3 and 4 yield that $\left| \frac{d}{dt} \left\| \frac{dU_\mu}{dt}(t) \right\|_{\mathbb{L}^2}^2 \right| \leq C_{U_0}$ for $t \geq 1$, where the constant $C_{U_0} > 0$ depends only on $\|U_0\|_X$. This, together with the integrability of $\left\| \frac{dU_\mu}{dt}(t) \right\|_{\mathbb{L}^2}^2$ in $(1, \infty)$, implies the desired convergence result. \square

Remark 4. The convergence of global solutions can be obtained as long as we can prove that $\frac{dU_\mu}{dt}(t) \in L^1((1, \infty), \mathbb{L}^2)$. However, it cannot be directly deduced from the above two propositions that $\frac{dU_\mu}{dt}(t) \in L^1((1, \infty), \mathbb{L}^2)$. To overcome this problem, we prove additional properties of the Lyapunov function and of the global solutions, and we prove the Lojasiewicz-Simon gradient inequality. We then obtain further estimation of the time derivative of the global solution, and thus prove the asymptotic convergence result in Section 3.

2.6 Stationary solutions

The forest kinematic model (4) admits homogeneous and heterogeneous stationary solutions. The homogeneous stationary solutions satisfy

$$\begin{cases} \alpha w - q_\mu(u) = 0, \\ -\beta w + \alpha u = 0. \end{cases} \quad (34)$$

The solutions of the latter system correspond to the intersection points between the function $q_\mu(u)$ and the line $\frac{\alpha^2}{\beta}u$. For $\mu = 0$, system (34) admits three homogeneous stationary solutions, which are denoted by $O = (0, 0)^\top$, $U^- = (u^-, w^-)^\top$ and $U^+ = (u^+, w^+)^\top$ (with $0 < u^- < u^+$).

It follows from the approach in [28] that the homogeneous stationary solutions $O = (0, 0)^\top$, $U^+ = (u^+, w^+)^\top$ are stable, and the homogeneous stationary solution $U^- = (u^-, w^-)^\top$ is unstable. For $0 < \mu < \mu_2$ (where μ_2 is given in Remark 1), the number of homogeneous stationary solutions can be greater than 3. In the sequel, we denote by

$$U_\mu^+ = (u_\mu^+, w_\mu^+)^\top \quad (35)$$

the greatest homogeneous stationary solution of the perturbed system (34). For $\mu > \mu_2$, the trivial solution O is the only homogeneous stationary solution.

Next, the heterogeneous stationary solutions are determined by the elliptic problem

$$\begin{cases} \alpha \bar{w} - q_\mu(\bar{u}) = 0 & \text{in } \Omega, \\ \delta \Delta \bar{w} - \beta \bar{w} + \alpha \bar{u} = 0 & \text{in } \Omega, \\ \frac{\partial \bar{w}}{\partial \nu} = 0 & \text{on } \Gamma. \end{cases} \quad (36)$$

We will prove that the heterogeneous stationary solutions describe the asymptotic behavior of the forest kinematic model (4). Indeed, based on the theories in Chapter 11, Section 4.2 in [49], for each $U_0 \in \mathcal{K}$, we can introduce a modified L^2 - ω -limit set of the global solution $U_\mu(t)$ by setting

$$L^2\text{-}\omega_\mu(U_0) = \{\bar{U}_\mu \in X; \exists t_n \nearrow \infty \text{ such that } \|U_\mu(t_n) - \bar{U}_\mu\|_{L^2} \rightarrow 0\}. \quad (37)$$

Furthermore, adapting the proof of Theorem 11.5 in [49], the following proposition can be addressed.

Proposition 7. *Let (11) hold, and let $U_0 \in \mathcal{K}$. Then, the L^2 - ω -limit set $L^2\text{-}\omega_\mu(U_0)$ is nonempty. Furthermore, the set consists of equilibria of (4), i.e., $\bar{U}_\mu = (\bar{u}, \bar{w})^\top$ satisfies (36).*

Our aim in Section 3 is to prove that the global solutions $U_\mu(t) = S_\mu(t)U_0$ converge towards the heterogeneous stationary solution \bar{U}_μ .

3 Convergence towards equilibrium

In this section, we establish our main convergence result. First, we apply the same approach as in [18, 23] to prove the Łojasiewicz-Simon gradient inequality. In order to simplify the notations, we use the notation $U_\mu(t) = (u, w)^\top$ instead of $U_\mu(t) = (u_\mu, w_\mu)^\top$ to denote the global solution to (4) with the initial value $U_0 \in \mathcal{K}$, and we fix $(\bar{u}, \bar{w})^\top = \bar{U}_\mu \in L^2\text{-}\omega(U_0)$. It follows from Theorem 3 that the global solution $U_\mu(t)$ satisfies

$$\|u(t)\|_{L^\infty} + \|w(t)\|_{H^2} \leq R, \quad 1 \leq t < \infty, \quad (38)$$

where $R > 0$ depends only on $\|U_0\|_X$.

Then, we present the following propositions.

Proposition 8. *Let (11) be satisfied, and $\bar{U} = (\bar{u}, \bar{w})^\top \in L^2\text{-}\omega(U_0)$. Then, $\|\bar{u}\|_{L^\infty} \leq R$ holds. Moreover, $\bar{w} \in H_N^2(\Omega)$ and $\|\bar{w}\|_{H^2} \leq R$ hold.*

Proof. The first assertion immediately follows as a result of the weak*-compactness of the closed unit ball in $L^\infty(\Omega)$.

Let us show the second assertion. By definition of the L^2 - ω -limit set, there exists a sequence $t_n \nearrow \infty$ such that $w(t_n) \rightarrow \bar{w}$ in $L^2(\Omega)$. Since $H_N^2(\Omega)$ is sequentially weakly compact and the sequence $w(t_n)$ is a bounded sequence in $H_N^2(\Omega)$, there exists a subsequence $w(t'_n)$ such that $w(t'_n)$ has a weak limit \tilde{w} in $H_N^2(\Omega)$. Of course $w(t'_n) \rightarrow \tilde{w}$ in $L^2(\Omega)$, so $\tilde{w} = \bar{w} \in H_N^2(\Omega)$. Furthermore, we know $\|\bar{w}\|_{H^2} \leq \liminf_{t'_n \rightarrow \infty} \|w(t'_n)\|_{H^2} \leq R$. \square

Proposition 9. *Let (11) be satisfied, and $\bar{U}_\mu = (\bar{u}, \bar{w})^\top \in L^2\text{-}\omega(U_0)$. Then, there exists a time sequence $t_n \nearrow \infty$ such that $U_\mu(t_n) \rightarrow \bar{U}_\mu$ in $L^2(\Omega) \times H_N^1(\Omega)$.*

Proof. By definition of the $L^2\text{-}\omega$ -limit set, there exists a sequence $t_n \nearrow \infty$ such that $U_\mu(t_n) \rightarrow \bar{U}_\mu$ in \mathbb{L}^2 . Then, since $\|w(t) + \bar{w}\|_{H^2} \leq 2R$ for all $t \geq 1$ from (38), we know that $\|w(t_n) - \bar{w}\|_{H^1} \leq \|w(t_n) - \bar{w}\|_{H^2}^{\frac{1}{2}} \|w(t_n) - \bar{w}\|_{L^2}^{\frac{1}{2}} \leq \sqrt{2R} \|w(t_n) - \bar{w}\|_{L^2}^{\frac{1}{2}} \rightarrow 0$ as $t_n \rightarrow \infty$. \square

We then set

$$V = L^2(\Omega) \times H_N^1(\Omega) \quad (39)$$

in the following sections.

3.1 Modification of the Lyapunov function

In order to consider the Lyapunov function given by (31) on the space V , we need to modify the potential $Q_\mu(u)$ such that $Q_\mu(u) = \mathcal{O}(u^2)$ when $|u| \gg 1$, where $\mathcal{O}(\cdot)$ is big **O** notation, without changing the value \mathcal{L}_μ on the global solution $U_\mu(t)$ and \bar{U}_μ .

Therefore, the constructed function $\tilde{Q}_\mu : \mathbb{R} \rightarrow \mathbb{R}$ has following properties:

- (i) $\tilde{Q}_\mu(u) = Q_\mu(u)$ for $u \in (-1, R+1)$;
- (ii) $\tilde{Q}_\mu(u) = \mathcal{O}(u^2)$ for $u \in (-\infty, -1] \cup [R+1, \infty)$;
- (iii) $\tilde{Q}_\mu \in \mathcal{C}^2(\mathbb{R})$ and the second derivative satisfies

$$0 < q_0 \leq \tilde{Q}_\mu''(u) \leq q_1 \quad \text{for any } u \in \mathbb{R}, \quad (40)$$

with some constant $q_1 > 0$ (recall that $q_0 = c - \frac{1}{3}ab^2 + \mu p'(u)$, as defined in (11)).

Based on the modified \tilde{Q}_μ , we thus obtain the modified Lyapunov function $\tilde{\mathcal{L}}_\mu : V \rightarrow \mathbb{R}$ written as

$$\tilde{\mathcal{L}}_\mu(u, w) = \int_\Omega \left[\frac{\delta}{2} |\nabla w|^2 + \frac{\beta}{2} w^2 - \alpha w + \tilde{Q}_\mu(u) \right] dx \quad \text{for } U_\mu = (u, w)^\top \in V. \quad (41)$$

Since $\tilde{Q}_\mu(u) = Q_\mu(u)$ for $0 \leq u \leq R$, it is known that $\tilde{\mathcal{L}}_\mu(U_\mu(t)) = \mathcal{L}_\mu(U_\mu(t))$ along the global solution $U(t)$, i.e., $\tilde{Q}_\mu(\cdot)$ still plays the role of a Lyapunov function for the global solution. Furthermore, the estimation (38) ensures that $\tilde{\mathcal{L}}_\mu(\bar{U}_\mu) = \mathcal{L}_\mu(\bar{U}_\mu)$.

For such a modified Lyapunov function $\tilde{\mathcal{L}}_\mu : V \rightarrow \mathbb{R}$, we calculate the first Fréchet derivative $\tilde{\mathcal{L}}_\mu' : V \rightarrow V'$. Note that $V' = L^2(\Omega) \times H^{-1}(\Omega)$ by identifying $L^2(\Omega)'$ as $L^2(\Omega)$. Furthermore, we regard $\Lambda = -\delta\Delta + \beta$ as an isomorphism from $H_N^1(\Omega)$ onto $H^{-1}(\Omega)$.

Proposition 10. *The function $\tilde{\mathcal{L}}_\mu : V \rightarrow \mathbb{R}$ is Fréchet differentiable with its derivative*

$$\tilde{\mathcal{L}}_\mu'(U_\mu) = \begin{pmatrix} -\alpha w + q_\mu(u) \\ \Lambda w - \alpha u \end{pmatrix} \in V' \quad \text{for } U_\mu = (u, w)^\top \in V. \quad (42)$$

In particular, we have $\tilde{\mathcal{L}}_\mu'(\bar{U}_\mu) = 0$ for $\bar{U}_\mu \in L^2\text{-}\omega(U_0)$.

Proof. For $U_\mu = (u, w)^\top, \tilde{U}_\mu = (\tilde{u}, \tilde{w})^\top \in V$, we have

$$\begin{aligned} & \tilde{\mathcal{L}}_\mu(U_\mu + \tilde{U}_\mu) - \tilde{\mathcal{L}}_\mu(U_\mu) - \left\langle \begin{pmatrix} -\alpha w + q_\mu(u) \\ \Lambda w - \alpha u \end{pmatrix}, \begin{pmatrix} \tilde{u} \\ \tilde{w} \end{pmatrix} \right\rangle_{V' \times V} \\ &= \int_\Omega [-\alpha \tilde{u} \tilde{w} + \tilde{Q}_\mu(u + \tilde{u}) - \tilde{Q}_\mu(u) - \tilde{Q}'_\mu(u) \tilde{u}] dx + \frac{1}{2} \langle \Lambda \tilde{w}, \tilde{w} \rangle_{H^{-1} \times H^1}. \end{aligned}$$

Due to (40), we have

$$\left| \int_\Omega [\tilde{Q}_\mu(u + \tilde{u}) - \tilde{Q}_\mu(u) - \tilde{Q}'_\mu(u) \tilde{u}] dx \right| \leq \frac{q_1}{2} \|\tilde{u}\|_{L^2}^2,$$

thus,

$$\left| \tilde{\mathcal{L}}_\mu(U_\mu + \tilde{U}_\mu) - \tilde{\mathcal{L}}_\mu(U_\mu) - \langle \tilde{\mathcal{L}}_\mu'(U_\mu), \tilde{U}_\mu \rangle_{V' \times V} \right| \leq C_{\mathcal{L}_\mu} \|\tilde{U}_\mu\|_V^2, \quad (43)$$

where $C_{\mathcal{L}_\mu}$ is positive. The Fréchet derivative (42) is obtained.

Recall that $\tilde{U}_\mu \in L^2\text{-}\omega(U_0)$ satisfies (36). Moreover, since $0 \leq \bar{u} \leq R$ for a.e. $x \in \Omega$, we know that $\tilde{Q}'_\mu(\bar{u}(x)) = Q'_\mu(\bar{u}(x)) = q_\mu(\bar{u})$. Therefore, we have $\tilde{\mathcal{L}}'_\mu(\tilde{U}_\mu) = 0$. \square

We note that $\tilde{\mathcal{L}}'_\mu : V \rightarrow V'$ is not Fréchet differentiable. Furthermore, due to (40), it is easily derived that $\tilde{\mathcal{L}}'_\mu$ is Lipschitz continuous, i.e., there exists a constant $L_0 > 0$ such that

$$\|\tilde{\mathcal{L}}'_\mu(U_\mu) - \tilde{\mathcal{L}}'_\mu(\tilde{U}_\mu)\|_{V'} \leq L_0 \|U_\mu - \tilde{U}_\mu\|_V \quad \text{for } U_\mu, \tilde{U}_\mu \in V. \quad (44)$$

3.2 Łojasiewicz-Simon gradient inequality

We then prove that the Lyapunov function $\tilde{\mathcal{L}}_\mu(U_\mu)$ give by (41) satisfies the Łojasiewicz-Simon gradient inequality with the following form.

Proposition 11. *Set $\theta \in (0, \frac{1}{2})$. There exists $r > 0$ and $\epsilon > 0$ such that*

$$|\tilde{\mathcal{L}}_\mu(U_\mu) - \tilde{\mathcal{L}}_\mu(\tilde{U}_\mu)| \leq \epsilon \|\tilde{\mathcal{L}}'_\mu(U_\mu)\|_{V'}^{\frac{1}{1-\theta}} \quad \text{if } \|U_\mu - \tilde{U}_\mu\|_V < r, \quad (45)$$

where $\tilde{\mathcal{L}}_\mu(U_\mu)$ is given by (41).

Proof. We divide the proof of the above Łojasiewicz-Simon gradient inequality into three steps.

Step 1. Note that $\Lambda^{-1} : L^2(\Omega) \rightarrow L^2(\Omega)$ is a compact self-adjoint operator. Here, we consider the eigenvalue problem

$$\Lambda^{-1}e_n = \eta_n e_n \quad \text{in } L^2(\Omega).$$

As a result of the theory of compact self-adjoint operators, there exists a Hilbert basis $\{e_n\}_{n \in \mathbb{N}} (\subset H_N^2(\Omega))$ of $L^2(\Omega)$ and positive eigenvalues $\{\eta_n\}_{n \in \mathbb{N}}$ such that $\eta_n \searrow 0$ as $n \rightarrow +\infty$. For each $N \in \mathbb{N}$, considering orthogonal projection P_N from $L^2(\Omega)$ onto $\text{span}\{e_1, \dots, e_N\}$, then we have the following estimation

$$\|w\|_{L^2(\Omega)}^2 \leq \|P_N w\|_{L^2(\Omega)}^2 + \eta_{N+1} \langle \Lambda w, w \rangle_{H^{-1} \times H^1} \quad \text{for } w \in H_N^1(\Omega).$$

Therefore, for $U_\mu = (u, w)^\top \in V$, the mapping $\mathcal{F} : V \rightarrow V'$ is written as

$$\mathcal{F}(U_\mu) = \mathcal{L}'_\mu(U_\mu) + \begin{pmatrix} 0 \\ \lambda P_N w \end{pmatrix} = \begin{pmatrix} -\alpha w + \tilde{Q}'_\mu(u) \\ \Lambda w - \alpha u + \lambda P_N w \end{pmatrix}, \quad (46)$$

which is a coercive monotone operator if $N \in \mathbb{N}$ and $\lambda > 0$ are sufficiently large. In other words, the following proposition is confirmed.

Proposition 12. *For sufficiently large $N \in \mathbb{N}$ and $\lambda > 0$, there exists a constant $L_1 > 0$ such that*

$$\frac{1}{L_1} \|U_\mu - \tilde{U}_\mu\|_V^2 \leq \langle \mathcal{F}(U_\mu) - \mathcal{F}(\tilde{U}_\mu), U_\mu - \tilde{U}_\mu \rangle_{V' \times V} \quad \text{for } U_\mu, \tilde{U}_\mu \in V. \quad (47)$$

Proof. We firstly calculate that

$$\begin{aligned} & \langle \tilde{\mathcal{L}}'_\mu(U_\mu) - \tilde{\mathcal{L}}'_\mu(\tilde{U}_\mu), U_\mu - \tilde{U}_\mu \rangle_{V' \times V} \\ &= -2\alpha \int_\Omega (u - \tilde{u})(w - \tilde{w}) dx + \int_\Omega [\tilde{Q}'_\mu(u) - \tilde{Q}'_\mu(\tilde{u})](u - \tilde{u}) dx + \langle \Lambda(w - \tilde{w}), w - \tilde{w} \rangle_{H^{-1} \times H^1}. \end{aligned}$$

Due to $\tilde{Q}'_\mu(\xi) - \tilde{Q}'_\mu(\tilde{\xi}) = \int_0^1 \tilde{Q}''_\mu(\theta\xi + (1-\theta)\tilde{\xi}) d\theta \times (\xi - \tilde{\xi})$ for $\xi, \tilde{\xi} \in \mathbb{R}$, applying (40) again, we have

$$\int_\Omega [\tilde{Q}'_\mu(u) - \tilde{Q}'_\mu(\tilde{u})](u - \tilde{u}) dx \geq q_0 \|u - \tilde{u}\|_{L^2}^2,$$

and

$$-2\alpha \int_\Omega (u - \tilde{u})(w - \tilde{w}) dx \geq -\frac{q_0}{2} \|u - \tilde{u}\|_{L^2}^2 - \frac{2\alpha^2}{q_0} \|w - \tilde{w}\|_{L^2}^2$$

by employing the Young's inequality. Thus, it follows that

$$\langle \tilde{\mathcal{L}}'_\mu(U_\mu) - \tilde{\mathcal{L}}'_\mu(\tilde{U}_\mu), U_\mu - \tilde{U}_\mu \rangle_{V' \times V} \geq \frac{q_0}{2} \|u - \tilde{u}\|_{L^2}^2 - \frac{2\alpha^2}{q_0} \|w - \tilde{w}\|_{L^2}^2 + \langle \Lambda(w - \tilde{w}), w - \tilde{w} \rangle_{H^{-1} \times H},$$

so we have

$$\langle \mathcal{F}(U_\mu) - \mathcal{F}(\tilde{U}_\mu), U_\mu - \tilde{U}_\mu \rangle_{V' \times V} \geq \frac{q_0}{2} \|u - \tilde{u}\|_{L^2}^2 + (\lambda - \frac{2\alpha^2}{q_0}) \|w - \tilde{w}\|_{L^2}^2 + (1 - \lambda\eta_{N+1}) \langle \Lambda(w - \tilde{w}), w - \tilde{w} \rangle_{H^{-1} \times H}.$$

Then, we finish the proof by choosing $\lambda > \frac{2\alpha^2}{q_0}$ and taking $N \in \mathbb{N}$ large enough (so that η_{N+1} is sufficiently small). \square

It follows from Proposition 12 that $\mathcal{F} : V \rightarrow V'$ is injective. Furthermore, it is derived by the Browder-Minty theorem (see [44], Theorem 10.49) that \mathcal{F} is surjective. As for its inverse $\mathcal{F}^{-1} : V' \rightarrow V$, which is deduced by (47) that

$$\|\mathcal{F}^{-1}(U_\mu^*) - \mathcal{F}^{-1}(\tilde{U}_\mu^*)\|_V \leq L_1 \|U_\mu^* - \tilde{U}_\mu^*\|_{V'} \quad \text{for } U_\mu^*, \tilde{U}_\mu^* \in V'. \quad (48)$$

Step 2. As a restriction of \mathcal{F} , consider the mapping $\tilde{\mathcal{F}} : \mathcal{D} \rightarrow X$ given by

$$\tilde{\mathcal{F}}(U_\mu) = \begin{pmatrix} -\alpha w + \tilde{Q}'_\mu(u) \\ \Lambda w - \alpha u + \lambda P_N w \end{pmatrix} \in X, \quad \text{for } U_\mu = (u, w)^\top \in \mathcal{D} \quad (49)$$

Then, we obtain the following proposition. Note that the orthogonal projection P_N on $L^2(\Omega)$ is regarded as a bounded linear operator from $H_N^2(\Omega)$ to $L^2(\Omega)$.

Proposition 13. $\tilde{\mathcal{F}} : \mathcal{U}(\bar{U}_\mu) \subset \mathcal{D} \rightarrow X$ is an analytic function, where $\mathcal{U}(\bar{U}_\mu)$ is a neighborhood of \bar{U}_μ in \mathcal{D} .

Proof. It is easy to prove that mapping $\tilde{Q}_\mu : L^\infty(\Omega) \rightarrow L^\infty(\Omega)$ is analytic at $\bar{u} \in L^\infty(\Omega)$. In fact, if \tilde{Q}'_μ is analytic at $\bar{u} \in L^\infty(\Omega)$, a neighborhood of \bar{u} exists such that \tilde{Q}'_μ is analytic on its neighborhood. We omit the proof here. \square

In particular, for $U_\mu = (u, w)^\top \in \mathcal{U}(\bar{U}_\mu)$, its first derivative $\tilde{\mathcal{F}}' : \mathcal{U}(\bar{U}_\mu) \rightarrow \mathcal{L}(\mathcal{D}, X)$ is given by

$$\tilde{\mathcal{F}}'(U_\mu) = \begin{pmatrix} \tilde{Q}''_\mu(u) & -\alpha \\ -\alpha & \Lambda + \lambda P_N \end{pmatrix} \in \mathcal{L}(\mathcal{D}, X),$$

where the inverse mapping theorem theorem (see [50], Corollary 4.37) is applied. We further have the following proposition.

Proposition 14. $\tilde{\mathcal{F}}'(\bar{U}_\mu) : \mathcal{D} \rightarrow X$ is bijective.

Proof. Note that $T \in \mathcal{L}(V, V')$ given by

$$T = \begin{pmatrix} \tilde{Q}''_\mu(u) & -\alpha \\ -\alpha & \Lambda + \lambda P_N \end{pmatrix} \in \mathcal{L}(V, V').$$

By using the same approach in the proof of Proposition 12, $\frac{1}{L_1} \|U_\mu\|_V^2 \leq \langle TU_\mu, U_\mu \rangle_{V' \times V}$ for $U_\mu \in V$ is obtained. Therefore,

$$T \text{ is a linear isomorphism from } V \text{ onto } V'. \quad (50)$$

Let $U_\mu^* = (u^*, w^*)^\top \in X$. Since also $(u^*, w^*)^\top \in V'$, it follows from (50) that there exists a unique $U_\mu = (u, w)^\top \in V$ such that

$$T \begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} u\tilde{Q}''_\mu(u) - \alpha w \\ -\alpha u + \Lambda w + \lambda P_N w \end{pmatrix} = \begin{pmatrix} u^* \\ w^* \end{pmatrix}$$

It is obviously that $\Lambda w = w^* + \alpha u - \lambda P_N w \in L^2(\Omega)$, and $u = \frac{1}{\tilde{Q}''_\mu(u)}(u^* + \alpha w) \in L^\infty(\Omega)$ by employing (40). Therefore, $\tilde{\mathcal{F}}'(\bar{U}_\mu)$ is bijective from \mathcal{D} to X . \square

Due to Proposition 13, Proposition 14 and [50] Corollary 4.37, there exists a neighborhood $\mathcal{V}(\tilde{\mathcal{F}}(\bar{U}_\mu)) \subset X$ such that

$$\tilde{\mathcal{F}} : \mathcal{U}(\bar{U}_\mu) \rightarrow \mathcal{V}(\tilde{\mathcal{F}}(\bar{U}_\mu)) \text{ is an analytic diffeomorphism} \quad (51)$$

by choosing sufficiently small $\mathcal{U}(\bar{U}_\mu)$.

Step 3. Considering the finite-dimensional linear space

$$E_N = 0 \times \text{span}\{e_1, \dots, e_N\} \subset \mathcal{D}$$

equipped with the norm $\|\cdot\|_{E_N}$, we then have the norm equivalence

$$\|(0, w_N)^\top\|_{E_N} = \|w_N\|_{H^1} \text{ for } (0, w_N)^\top \in E_N. \quad (52)$$

Therefore, we present the following proposition.

Proposition 15. *There exist constants $\epsilon_0 > 0$ and $r_0 > 0$ such that*

$$|\tilde{\mathcal{L}}_\mu \circ \tilde{\mathcal{F}}^{-1}((0, \lambda P_N w)^\top) - \tilde{\mathcal{L}}_\mu(\bar{U}_\mu)|^{1-\theta} \leq \epsilon_0 \|(\tilde{\mathcal{L}}_\mu \circ \tilde{\mathcal{F}}^{-1})'((0, \lambda P_N w)^\top)\|_{E'_N} \text{ for } \|w - \bar{w}\|_{H^1} < r_0. \quad (53)$$

Proof. It is obviously that $\tilde{\mathcal{L}}_\mu$ is analytic as a function from $\mathcal{U}(\bar{U}_\mu) \subset \mathcal{D}$ to \mathbb{R} . Combined with (51), we have the function composition:

$$\tilde{\mathcal{L}}_\mu \circ \tilde{\mathcal{F}}^{-1} : \mathcal{V}(\tilde{\mathcal{F}}(\bar{U}_\mu)) \cap E_N \rightarrow \mathbb{R} \text{ is an analytic function from } E_N \text{ to } \mathbb{R}. \quad (54)$$

Then, employing the classical Lojasiewicz Theorem (see [31]) to assume that $\theta \in (0, \frac{1}{2}]$ and $\epsilon_0, r_1 > 0$ exist, such that

$$|\tilde{\mathcal{L}}_\mu \circ \tilde{\mathcal{F}}^{-1}((0, w_N)^\top) - \tilde{\mathcal{L}}_\mu \circ \tilde{\mathcal{F}}^{-1}(\tilde{\mathcal{F}}(\bar{U}))|^{1-\theta} \leq \epsilon_0 \|(\tilde{\mathcal{L}}_\mu \circ \tilde{\mathcal{F}}^{-1})'((0, w_N)^\top) - (\tilde{\mathcal{L}}_\mu \circ \tilde{\mathcal{F}}^{-1})'(\tilde{\mathcal{F}}(\bar{U}))\|_{E'_N},$$

if $(0, w_N)^\top \in E_N$ satisfies $\|(0, w_N - \lambda P_N \bar{w})^\top\|_{E_N} < r_1$. Note that

$$\begin{aligned} (\tilde{\mathcal{L}}_\mu \circ \tilde{\mathcal{F}}^{-1})'(\tilde{\mathcal{F}}(\bar{U})) &= \tilde{\mathcal{L}}'_\mu(\tilde{\mathcal{F}}^{-1}(\tilde{\mathcal{F}}(\bar{U}))) \circ (\tilde{\mathcal{F}}^{-1})'(\tilde{\mathcal{F}}(\bar{U})) \\ &= \tilde{\mathcal{L}}'_\mu(\bar{U}) \circ (\tilde{\mathcal{F}}^{-1})'(\tilde{\mathcal{F}}(\bar{U})) \\ &= 0. \end{aligned}$$

Therefore, we have

$$|\tilde{\mathcal{L}}_\mu \circ \tilde{\mathcal{F}}^{-1}((0, w_N)^\top) - \tilde{\mathcal{L}}_\mu(\bar{U})|^{1-\theta} \leq \epsilon_0 \|(\tilde{\mathcal{L}}_\mu \circ \tilde{\mathcal{F}}^{-1})'((0, w_N)^\top)\|_{E'_N} \text{ if } \|(0, w_N - \lambda P_N \bar{w})^\top\|_{E_N} < r_1. \quad (55)$$

Due to (52), there exists $r_0 > 0$ small enough so that $\|w - \bar{w}\|_{H^1} < r_0$ and $\|(0, w_N - \lambda P_N \bar{w})^\top\|_{E_N} < r_1$. Thus, (53) can be deduced by (55), we finish the proof of Proposition 15. \square

Furthermore, we estimate the right side of (53) to have

$$\begin{aligned} \|(\tilde{\mathcal{L}}_\mu \circ \tilde{\mathcal{F}}^{-1})'((0, \lambda P_N w)^\top)\|_{E'_N} &= \|(\tilde{\mathcal{L}}'_\mu(\tilde{\mathcal{F}}^{-1}((0, \lambda P_N w)^\top))) \circ (\tilde{\mathcal{F}}^{-1})'((0, \lambda P_N w)^\top)\|_{E'_N} \\ &\leq C_r \|(\tilde{\mathcal{L}}'_\mu(\tilde{\mathcal{F}}^{-1}((0, \lambda P_N w)^\top)))\|_{V'}, \end{aligned}$$

where the constant C_r is positive and depends only on r_0 , we note that it follows from (51) that

$$\|(\tilde{\mathcal{F}}^{-1})'((0, \lambda P_N w)^\top)\|_{\mathcal{L}(E_N, V)} \leq C_r \text{ if } \|w - \bar{w}\|_{H^1} < r_0.$$

For arbitrarily $u \in L^2(\Omega)$, we have

$$\begin{aligned} \|(\tilde{\mathcal{L}}'_\mu(\tilde{\mathcal{F}}^{-1}((0, \lambda P_N w)^\top)))\|_{V'} &\leq \|(\tilde{\mathcal{L}}'_\mu(\tilde{\mathcal{F}}^{-1}((0, \lambda P_N w)^\top)) - \tilde{\mathcal{L}}'_\mu((u, w)^\top))\|_{V'} + \|(\tilde{\mathcal{L}}'_\mu((u, w)^\top))\|_{V'} \\ &\leq L_0 \|(\tilde{\mathcal{F}}^{-1}((0, \lambda P_N w)^\top) - \tilde{\mathcal{F}}^{-1}(\tilde{\mathcal{F}}((u, w)^\top)))\|_V + \|(\tilde{\mathcal{L}}'_\mu(U_\mu))\|_{V'} \\ &\leq (L_0 L_1 + 1) \|(\tilde{\mathcal{L}}'_\mu(U_\mu))\|_{V'}, \end{aligned}$$

by employing (44), (48) and (46). Then, for $U_\mu = (u, w)^\top \in V$, $\|w - \bar{w}\|_{H^1} < r_0$, it follows that

$$\left| \tilde{\mathcal{L}}_\mu \circ \tilde{\mathcal{F}}^{-1}((0, \lambda P_N w)^\top) - \tilde{\mathcal{L}}_\mu(\bar{U}_\mu) \right|^{1-\theta} \leq \epsilon_0 C_r (L_0 L_1 + 1) \|\tilde{\mathcal{L}}'_\mu(U_\mu)\|_{V'}. \quad (56)$$

Besides, note that $\tilde{U}_\mu = \tilde{\mathcal{F}}^{-1}((0, \lambda P_N w)^\top) - U_\mu$, it follows from (43) that

$$\begin{aligned} & \left| \tilde{\mathcal{L}}_\mu(U_\mu) - \tilde{\mathcal{L}}_\mu \circ \tilde{\mathcal{F}}^{-1}((0, \lambda P_N w)^\top) \right| \\ &= \left| \tilde{\mathcal{L}}_\mu(U_\mu) - \tilde{\mathcal{L}}_\mu(U_\mu + \tilde{U}_\mu) \right| \\ &\leq \|\tilde{\mathcal{L}}'_\mu(U_\mu)\|_{V'} \|\tilde{\mathcal{F}}^{-1}((0, \lambda P_N w)^\top) - U_\mu\|_V + C_{\mathcal{L}_\mu} \|\tilde{\mathcal{F}}^{-1}((0, \lambda P_N w)^\top) - U_\mu\|_V^2 \\ &\leq (L_1 + C_{\mathcal{L}_\mu} L_1^2) \|\tilde{\mathcal{L}}'_\mu(U_\mu)\|_{V'}^2, \end{aligned} \quad (57)$$

where, (48) and (46) are applied.

Therefore, it is derived from (56) and (57) that

$$\begin{aligned} \left| \tilde{\mathcal{L}}_\mu(U_\mu) - \tilde{\mathcal{L}}_\mu(\bar{U}_\mu) \right| &\leq \epsilon_0 C_r (L_0 L_1 + 1) \|\tilde{\mathcal{L}}'_\mu(U_\mu)\|_{V'}^{\frac{1}{1-\theta}} + (L_1 + C_{\mathcal{L}_\mu} L_1^2) \|\tilde{\mathcal{L}}'_\mu(U_\mu)\|_{V'}^2 \\ &\leq C_0 \|\tilde{\mathcal{L}}'_\mu(U_\mu)\|_{V'}^{\frac{1}{1-\theta}} \quad \text{if } \|U_\mu - \bar{U}_\mu\|_V < r, \end{aligned}$$

where the constant C_0 is positive and depends on $\epsilon_0, C_{\mathcal{L}_\mu}, C_r, L_0, L_1$ and θ , by taking $r \in (0, r_0)$ small enough to make sure that $\|\tilde{\mathcal{L}}'_\mu(U_\mu)\|_{V'}^2 \leq \|\tilde{\mathcal{L}}'_\mu(U_\mu)\|_{V'}^{\frac{1}{1-\theta}}$. We eventually finish the proof of Proposition 11. \square

Remark 5. Note that the condition (11) is a restrictive one, otherwise (especially, when the perturbation μ is large), the approach we apply in this section can not ensure the Lojasiewicz-Simon gradient inequality still holds. Therefore, we can not further prove the following asymptotic convergence result.

3.3 Asymptotic convergence result

To prove the asymptotic convergence of global solutions, we firstly present the following lemma (see Appendix in [23] and [15] Lemma 7.1), which gives a sufficient condition in order that a function in $L^2(0, \infty)$ is also in $L^1(0, \infty)$.

Lemma 1. Let $F : (0, \infty) \rightarrow [0, \infty)$ be a nonnegative continuous function satisfying $F \in L^2(0, \infty)$. Assume that an interval $I = (T, T') \subset (0, \infty)$, an exponent $\alpha \in (1, 2)$, and a constant $\kappa > 0$ exist such that

$$\left(\int_t^\infty F(\tau)^2 d\tau \right)^\alpha \leq \kappa F(t)^2, \quad \forall t \in I. \quad (58)$$

Then, the following inequality holds:

$$\int_I F(\tau) d\tau \leq \kappa' \|F\|_{L^2(T, \infty)}^{\alpha'}, \quad (59)$$

where $\alpha' > 0$ depends only on α , and $\kappa' > 0$ depends only on α and κ .

Then, we have the following proposition.

Proposition 16. Sufficiently large time $T > 0$ and a constant $\kappa > 0$ exist such that the following assertion holds. If $t > T$ satisfies $\|U_\mu(t) - \bar{U}_\mu\|_V < r$, then

$$\left(\int_t^\infty \left\| \frac{dU_\mu}{d\tau}(\tau) \right\|_{\mathbb{L}^2}^2 d\tau \right)^{2(1-\theta)} \leq \kappa \left\| \frac{dU_\mu}{dt}(t) \right\|_{\mathbb{L}^2}^2. \quad (60)$$

Proof. Since we have $\left\| \frac{dU_\mu}{dt}(t) \right\|_{\mathbb{L}^2} \rightarrow 0$ as $t \rightarrow \infty$ due to Proposition 6, take $T > 1$ large enough such that

$$\left\| \frac{\partial u}{\partial t}(t) \right\|_{L^2} < 1, \quad \forall t > T,$$

and we fix such time $T > 1$. Set $t \in (T, \infty)$ such that $\|U_\mu(t) - \bar{U}_\mu\|_V < r$. By the definition of $L^2\text{-}\omega(U_0)$, such a time t actually exists.

By integrating equation (33) respect to time on (t, ∞) , we have

$$\int_t^\infty \int_\Omega \left(\left| \frac{\partial u}{\partial \tau}(\tau) \right|^2 + \left| \frac{\partial w}{\partial \tau}(\tau) \right|^2 \right) dx d\tau = \tilde{\mathcal{L}}_\mu(U_\mu(t)) - \tilde{\mathcal{L}}_\mu(\bar{U}_\mu),$$

it follows from (45) that

$$\int_t^\infty \left(\left\| \frac{\partial u}{\partial \tau}(\tau) \right\|_{L^2}^2 + \left\| \frac{\partial w}{\partial \tau}(\tau) \right\|_{L^2}^2 \right) d\tau \leq \epsilon \|\tilde{\mathcal{L}}'_\mu(U_\mu(t))\|_{V'}^{\frac{1}{1-\theta}}. \quad (61)$$

It follows from Proposition 10 that $\tilde{\mathcal{L}}'_\mu(U_\mu(t)) = (-\frac{\partial u}{\partial t}, -\frac{\partial w}{\partial t})^\top \in V'$, which yields

$$\|\tilde{\mathcal{L}}'_\mu(U_\mu(t))\|_{V'}^2 \leq C \left(\left\| \frac{\partial u}{\partial t}(t) \right\|_{L^2}^2 + \left\| \frac{\partial w}{\partial t}(t) \right\|_{H^{-1}}^2 \right).$$

Combined with (61), we have

$$\int_t^\infty \left\| \frac{dU_\mu}{d\tau}(\tau) \right\|_{\mathbb{L}^2}^2 d\tau \leq C' \left\| \frac{dU_\mu}{dt}(t) \right\|_{\mathbb{L}^2}^{\frac{1}{1-\theta}},$$

where, C' depends on positive constants ϵ and $\theta \in (0, \frac{1}{2})$. We thus conclude the proof. \square

Furthermore, we prove the following lemma.

Lemma 2. *There exists a sufficiently large $t_N > 0$ such that $\|U_\mu(t) - \bar{U}_\mu\|_V < r$ for all $t \geq t_N$.*

Proof. Due to Proposition 9, there exists a time sequence $t_n \nearrow \infty$ such that $U_\mu(t_n) \rightarrow \bar{U}_\mu$ in V . Thus, a sufficiently large N_0 exists to ensure that

$$\|U_\mu(t_n) - \bar{U}_\mu\|_V \leq \frac{r}{3}, \quad \forall n \geq N_0.$$

Moreover, for $n \geq N_0$, set $t'_n = \inf\{t \in (t_n, \infty); \|U_\mu(t) - \bar{U}_\mu\|_V = r\}$; in particular, $t'_n = \infty$ when $\|U_\mu(t) - \bar{U}_\mu\|_V < r$ for all $t \geq t_n$. Recalling Lemma 1, we take $F(t) = \left\| \frac{dU_\mu}{dt}(t) \right\|_{\mathbb{L}^2}$, $\alpha = 2(1-\theta)$, $T = t_n$, and $T' = t'_n$. Then, due to Proposition 16, it follows

$$\int_{t_n}^{t'_n} \left\| \frac{dU_\mu}{d\tau}(\tau) \right\|_{\mathbb{L}^2} d\tau < \kappa' \left(\int_{t_n}^\infty \left\| \frac{dU_\mu}{dt}(t) \right\|_{\mathbb{L}^2}^2 dt \right)^{\frac{\theta'}{2}},$$

where θ' depends only on θ . Therefore,

$$\|U_\mu(t'_n) - U_\mu(t_n)\|_V \leq C \int_{t_n}^{t'_n} \left\| \frac{dU_\mu}{d\tau}(\tau) \right\|_{\mathbb{L}^2} d\tau \leq C\kappa' \left(\int_{t_n}^\infty \left\| \frac{dU_\mu}{dt}(t) \right\|_{\mathbb{L}^2}^2 dt \right)^{\frac{\theta'}{2}}.$$

Then, $\|U_\mu(t'_N) - U_\mu(t_N)\|_V \leq \frac{r}{3}$ can be derived by taking $N \geq N_0$ large enough.

Besides, we know that $t'_N = \infty$, for such $N \geq N_0$. Indeed, suppose that $t'_N < \infty$. Then, $\|U_\mu(t'_N) - \bar{U}_\mu\|_V = r$. On the other hand, we have

$$\|U_\mu(t'_N) - \bar{U}_\mu\|_V \leq \|U_\mu(t'_N) - U_\mu(t_N)\|_V + \|U_\mu(t_N) - \bar{U}_\mu\|_V \leq \frac{2r}{3},$$

which is a contradiction. \square

Finally, we conclude the asymptotic convergence result.

Theorem 4. *Note that the assumption (11) holds, $U_\mu(t) = S_\mu(t)U_0$ is the global solution to the forest kinematic model (4) with initial value $U_0 \in \mathcal{K}$, and let $\bar{U}_\mu \in L^2\text{-}\omega(U_0)$ is a stationary solution to (4). Then, we have $U_\mu(t) \rightarrow \bar{U}_\mu$ in $\mathbb{L}^2(\Omega)$ as $t \rightarrow \infty$.*

Proof. Due to Lemma 1, 2 and Proposition 16, there exists a sufficiently large $t_N > 0$ such that

$$\int_t^\infty \left\| \frac{dU_\mu}{d\tau}(\tau) \right\|_{\mathbb{L}^2} d\tau \leq \kappa' \left(\int_t^\infty \left\| \frac{dU_\mu}{d\tau}(\tau) \right\|_{\mathbb{L}^2}^2 d\tau \right)^{\frac{\theta'}{2}}, \quad \forall t \geq t_N.$$

Furthermore, since the definition of $L^2\text{-}\omega(U_0)$, there exists a time sequence $t_n \nearrow \infty$ such that $U_\mu(t_n) \rightarrow \bar{U}_\mu$ in $\mathbb{L}^2(\Omega)$. Then, we have

$$\begin{aligned} \|U_\mu(t) - \bar{U}_\mu\|_{\mathbb{L}^2} &\leq \|U_\mu(t) - U_\mu(t_n)\|_{\mathbb{L}^2} + \|U_\mu(t_n) - \bar{U}_\mu\|_{\mathbb{L}^2} \\ &\leq \int_t^{t_n} \left\| \frac{dU_\mu}{d\tau}(\tau) \right\|_{\mathbb{L}^2} d\tau + \|U_\mu(t) - U_\mu(t_n)\|_{\mathbb{L}^2} \end{aligned}$$

and when t_n goes to infinity, we obtain

$$\|U_\mu(t) - \bar{U}_\mu\|_{\mathbb{L}^2} \leq \kappa' \left(\int_t^\infty \left\| \frac{dU_\mu}{d\tau}(\tau) \right\|_{\mathbb{L}^2}^2 d\tau \right)^{\frac{\theta'}{2}}.$$

Therefore, we conclude the assertion. \square

4 Robustness of the weak attractors

In this section, we analyze the robustness of the dynamical system (24) determined by the degenerate forest kinetic system (4). As explained previously in Remark 2, we have to face a lack of compactness, which is due to the absence of diffusion in the first equation of system (4). Therefore, as shown in [5], the long time behavior of the dynamical system (24) cannot be described by means of the global attractor. However, following [49], we can still analyze the weak convergence of its orbits, in the topology of the Banach space Y given by

$$Y = \mathbb{L}^2(\Omega) = L^2(\Omega) \times L^2(\Omega). \quad (62)$$

Hence, for $U_0 \in \mathcal{K}$ and $B \subset X$, we consider the weak $L^2\text{-}\omega$ -limit sets of the dynamical system $S_\mu(t)$ given by

$$\omega_Y^\mu(U_0) = \bigcap_{t \geq 0} \overline{\{S_\mu(s)U_0; t \leq s < \infty\}}^Y, \quad (63)$$

where the closures are in Y . Note that $\omega_Y^\mu(U_0)$ actually coincides with the set $L^2\text{-}\omega_\mu(U_0)$ defined in (37).

We bypass the lack of compactness by proving the existence of a family $\{\mathcal{R}_\mu\}$ of positively invariant regions, from which we deduce the continuity of the flow induced by the dynamical system (24). Combined with the asymptotic convergence result obtained in Section 3.3, we can further prove the continuity of the stationary solutions. Therefore, the robustness of the weak ω -limit sets under the effect of the perturbation parameter μ can be derived. In the end of this section, we address that the case of a small perturbation $\mu \in (0, \mu_2)$ proves to be the most interesting, since the case of a strong perturbation $\mu > \mu_2$ leads to trivial dynamics.

4.1 Positively invariant regions

The following theorem establishes the existence of a family of positively invariant regions for the dynamical system $\{S_\mu(t)\}$ defined by (24).

Theorem 5 (Positively invariant region). *Let $\mu \geq 0$. Then the region $\mathcal{R}_\mu \subset X$ defined by*

$$\mathcal{R}_\mu = \left\{ (u, w) \in X \mid (u(x), w(x)) \in R_\mu, \forall x \in \mathbb{R} \right\} \quad (64)$$

with $R_\mu = [0, u_\mu^+] \times [0, w_\mu^+]$ (where (u_μ^+, w_μ^+) is given by (35)) is positively invariant by the flow induced by the degenerate forest kinetic system (4), that is, if $U_0 \in \mathcal{R}_\mu$, then $S_\mu(t)U_0 \in \mathcal{R}_\mu$ for all $t > 0$.

Proof. Let $U_0 \in \mathcal{R}_\mu$, for $t \geq 0$, we denote by $U_\mu(t) = S_\mu(t)U_0$ the global solution of the degenerate reaction-diffusion system (4) stemming from U_0 .

We consider a cut-off function χ defined on \mathbb{R} by

$$\chi(s) = \begin{cases} \frac{1}{2}s^2 & \text{if } s < 0, \\ 0 & \text{if } s \geq 0. \end{cases}$$

We observe that χ is continuously differentiable on \mathbb{R} , and elementary computations show that the following properties hold for all $r, s \in \mathbb{R}$:

$$\chi(s) \geq 0, \quad \chi'(s) \leq 0, \tag{65}$$

$$0 \leq s\chi'(s) \leq 2\chi(s), \tag{66}$$

$$r\chi'(s) + s\chi'(r) \leq r\chi'(r) + s\chi'(s). \tag{67}$$

Now we introduce the functions ξ and ζ defined for all $t \geq 0$ by

$$\rho_1(t) = \int_{\Omega} \chi(u_\mu^+ - u_\mu(t))dx, \quad \rho_2(t) = \int_{\Omega} \chi(w_\mu^+ - w_\mu(t))dx.$$

Since $U_0(x) \in R_\mu = [0, u_\mu^+] \times [0, w_\mu^+]$ for all $x \in \Omega$, we have $\rho_1(0) = \rho_2(0) = 0$. Furthermore, since $\chi(s) \geq 0$ for all $s \in \mathbb{R}$, we have $\rho_1(t) \geq 0$ and $\rho_2(t) \geq 0$ for all $t > 0$. Since ρ_1 is continuously differentiable on \mathbb{R} , we obtain its time derivative

$$\begin{aligned} \dot{\rho}_1(t) &= \int_{\Omega} \frac{\partial(u_\mu^+ - u_\mu(t))}{\partial t} \chi'(u_\mu^+ - u_\mu(t))dx \\ &= \alpha \int_{\Omega} (w_\mu^+ - w_\mu(t)) \chi'(u_\mu^+ - u_\mu(t))dx - \int_{\Omega} [q_\mu(u_\mu^+) - q_\mu(u_\mu(t))] \chi'(u_\mu^+ - u_\mu(t))dx \\ &= \alpha \int_{\Omega} (w_\mu^+ - w_\mu(t)) \chi'(u_\mu^+ - u_\mu(t))dx - \int_{\Omega} J_\mu(u_\mu^+, u_\mu(t)) (u_\mu^+ - u_\mu(t)) \chi'(u_\mu^+ - u_\mu(t))dx \\ &\leq \alpha \int_{\Omega} (w_\mu^+ - w_\mu(t)) \chi'(u_\mu^+ - u_\mu(t))dx + 2J_\mu^* \rho_1(t) \end{aligned}$$

by applying (66) and denoting that

$$J_\mu(u_\mu^+, u_\mu(t)) = \frac{q_\mu(u_\mu^+) - q_\mu(u_\mu(t))}{u_\mu^+ - u_\mu(t)}, \quad \forall t \geq 0.$$

It is obvious that $J_\mu(u_\mu^+, u_\mu(t))$ gives the coefficient of the line that joins the points of coordinates $(u_\mu^+, q_\mu(u_\mu^+))$ and $(u_\mu(t), q_\mu(u_\mu(t)))$ in the plane \mathbb{R}^2 . Hence, there exists $J_\mu^* \geq 0$ such that

$$J_\mu(u_\mu^+, u_\mu(t)) \geq -J_\mu^*, \quad \forall t \geq 0.$$

In particular, if assumption (11) is satisfied, which ensures that $J_\mu(u_\mu^+, u_\mu(t))$ is positive, then we can easily erase the corresponding term in the above estimation.

In parallel, the time derivative of ρ_2 is given by

$$\begin{aligned} \dot{\rho}_2(t) &= \int_{\Omega} \frac{\partial(w_\mu^+ - w_\mu(t))}{\partial t} \chi'(w_\mu^+ - w_\mu(t))dx \\ &= \delta \int_{\Omega} \Delta(w_\mu^+ - w_\mu(t)) \chi'(w_\mu^+ - w_\mu(t))dx + \alpha \int_{\Omega} (u_\mu^+ - u_\mu(t)) \chi'(w_\mu^+ - w_\mu(t))dx \\ &\quad - \beta \int_{\Omega} (w_\mu^+ - w_\mu(t)) \chi'(w_\mu^+ - w_\mu(t))dx \\ &\leq \alpha \int_{\Omega} (u_\mu^+ - u_\mu(t)) \chi'(w_\mu^+ - w_\mu(t))dx \end{aligned}$$

by applying property (66) and the Green formula with the Neumann boundary condition, i.e., for all $t \geq 0$,

$$\int_{\Omega} \Delta(w_{\mu}^{+} - w_{\mu}(t)) \chi'(w_{\mu}^{+} - w_{\mu}(t)) dx = - \int_{\Omega} |\nabla(w_{\mu}^{+} - w_{\mu}(t))|^2 dx \leq 0.$$

Eventually, for all $t \geq 0$, we have

$$\begin{aligned} \rho_1(t) + \rho_2(t) &\leq \alpha \int_{\Omega} (u_{\mu}^{+} u_{\mu}(t)) \chi'(w_{\mu}^{+} w_{\mu}(t)) dx + \alpha \int_{\Omega} (w_{\mu}^{+} - w_{\mu}(t)) \chi'(u_{\mu}^{+} - u_{\mu}(t)) dx + 2J_{\mu}^{*} \rho_1(t) \\ &\leq \alpha \int_{\Omega} (w_{\mu}^{+} w_{\mu}(t)) \chi'(w_{\mu}^{+} w_{\mu}(t)) dx + \alpha \int_{\Omega} (u_{\mu}^{+} - u_{\mu}(t)) \chi'(u_{\mu}^{+} - u_{\mu}(t)) dx + 2J_{\mu}^{*} \rho_1(t) \\ &\leq 2\alpha[\rho_1(t) + \rho_2(t)] + 2J_{\mu}^{*} \rho_1(t) \\ &\leq 2(\alpha + J_{\mu}^{*})[\rho_1(t) + \rho_2(t)] \end{aligned}$$

by employing property (67). We then apply the Gronwall lemma to deduce that

$$\rho_1(t) + \rho_2(t) \leq [\rho_1(0) + \rho_2(0)] e^{2(\alpha + J_{\mu}^{*})t} \leq 0, \quad \forall t \geq 0.$$

Owning to $\rho_1(0) = \rho_2(0) = 0$, it deduce that $\rho_1(t) + \rho_2(t) = 0$, we have moreover $\rho_1(t), \rho_2(t) \geq 0$, which yields that $\rho_1 = \rho_2 \equiv 0$, which completes the proof. \square

Although the invariant region \mathcal{R}_{μ} defined by (64) depends on the perturbation parameter μ , we can prove the existence of a L^{∞} -bound for the dynamical system $S_{\mu}(t)$, which is uniform with respect to μ . Indeed, it suffices to remark that the homogeneous stationary solution $(u_{\mu}^{+}, w_{\mu}^{+})$ given by (35) that delimits \mathcal{R}_{μ} varies continuously with μ . Note that such a uniform bound could not be established for the absorbing set \mathcal{B}_{μ} derived from Theorem 3 (see Remark 2).

Corollary 1 (Uniform bound). *There exists a positive constant $M_{\mathcal{R}}$ such that, for all $\mu \in [0, \mu_2]$ and for all $U_0 \in \mathcal{R}_{\mu}$:*

$$\|S_{\mu}(t)U_0\|_{L^{\infty}(\Omega)^2} \leq M_{\mathcal{R}}. \quad (68)$$

Proof. Let $\mu \in [0, \mu_3]$ and let $U_0 \in \mathcal{R}_{\mu}$. For $t \geq 0$, we denote $(u_{\mu}(t), w_{\mu}(t)) = S_{\mu}(t)U_0$. By virtue of Theorem 5, we have

$$|u_{\mu}(t)| \leq u_{\mu}^{+}, \quad |w_{\mu}(t)| \leq w_{\mu}^{+}, \quad \forall t \geq 0.$$

Next, it is easily seen that $(u_{\mu}^{+}, w_{\mu}^{+})$ depends continuously on μ . Hence we can consider

$$u_{\max}^{+} = \max_{0 \leq \mu \leq \mu_2} u_{\mu}^{+}, \quad w_{\max}^{+} = \max_{0 \leq \mu \leq \mu_2} w_{\mu}^{+},$$

from which we deduce $|u_{\mu}(t)| \leq u_{\max}^{+}$ and $|w_{\mu}(t)| \leq w_{\max}^{+}$ for all $t \geq 0$ and for all $\mu \in [0, \mu_2]$. Finally, we introduce $M_{\mathcal{R}} = \max(u_{\max}^{+}, w_{\max}^{+})$, which achieves the proof. \square

4.2 Continuity of the stationary solutions

Note that the continuous dynamical system $S_{\mu}(t)$ of problem (4) is defined as (24), and $S_0(t)$ is generated by the unperturbed degenerate forest kinematic model, i.e. $\mu = 0$ in (4). Now we present the following theorem, which yields the continuity of the flow, and plays a significant role to prove the continuity of the stationary solutions.

Theorem 6 (Continuity of the flow). *Suppose that assumption (11) holds. Suppose moreover that $\mathcal{R}_{\mu} \subset \mathcal{R}_0$ for all $\mu \in (0, \mu_2)$. Let $U_{\mu}(t) = (u_{\mu}, w_{\mu})^{\top}$ and $U(t) = (u, w)^{\top}$ denote, respectively, the global solutions to the degenerate forest kinematic model (4) when $\mu > 0$ and $\mu = 0$. Then, there exists a positive constant $\varrho = \min\{\frac{\alpha_0}{2}, \beta\}$, such that, for all $U_0 \in \mathcal{R}_{\mu}$ and all $t \geq 0$, the following estimate is fulfilled:*

$$\|S_{\mu}(t)U_0 - S_0(t)U_0\|_Y^2 \leq \frac{\mu^2 M_1^2 |\Omega|}{q_0(\alpha - \varrho)} (e^{2(\alpha - \varrho)t} - 1). \quad (69)$$

Moreover,

(i) if $\alpha - \varrho < 0$, then $U_{\mu}(t) \xrightarrow{\mu \rightarrow 0^+} U(t)$ in Y uniformly for $t \in [0, +\infty)$;

(ii) if $\alpha - \varrho > 0$, then $U_{\mu}(t) \xrightarrow{\mu \rightarrow 0^+} U(t)$ in Y uniformly in every compact interval $[0, T]$ with $T > 0$.

Proof. Let $\varphi = u_\mu - u$, $\psi = w_\mu - w$ and $\Psi(t) = (\varphi, \psi)^\top = U_\mu(t) - U(t)$, then we can rewrite the degenerate forest kinetic system as following

$$\begin{cases} \frac{\partial \varphi}{\partial t} = \alpha \psi - q_\mu(u_\mu) + q(u) & \text{in } (0, +\infty) \times \Omega, \\ \frac{\partial \psi}{\partial t} = \delta \Delta \psi - \beta \psi + \alpha \varphi & \text{in } (0, +\infty) \times \Omega, \\ \frac{\partial \psi}{\partial \nu} = 0 & \text{on } (0, +\infty) \times \Gamma, \\ \varphi(0, t) = \varphi_0, \psi(0, t) = \psi_0 & \text{in } \Omega. \end{cases} \quad (70)$$

Multiply the first equation of (70) by φ , and integrate the product in Ω to have

$$\frac{1}{2} \frac{d}{dt} \|\varphi\|_{L^2}^2 = \alpha \int_{\Omega} \varphi \psi dx - \int_{\Omega} (q_\mu(u_\mu) - q(u)) \varphi dx,$$

we next multiply the second equation of (70) by ψ , and integrate the product in Ω to have

$$\frac{1}{2} \frac{d}{dt} \|\psi\|_{L^2}^2 = \delta \int_{\Omega} \Delta \psi \cdot \psi dx - \beta \|\psi\|_{L^2}^2 + \alpha \int_{\Omega} \varphi \psi dx.$$

Combine the above two equations to have

$$\frac{1}{2} \frac{d}{dt} \|\Psi\|_Y^2 = 2\alpha \int_{\Omega} \varphi \psi dx + \delta \int_{\Omega} \Delta \psi \cdot \psi dx - \beta \|\psi\|_{L^2}^2 - \int_{\Omega} (q_\mu(u_\mu) - q(u)) \varphi dx.$$

Note that

$$2\alpha \int_{\Omega} \varphi \psi dx \leq \alpha \|\varphi\|_{L^2}^2 + \alpha \|\psi\|_{L^2}^2$$

by employing the Young's inequality, and

$$\delta \int_{\Omega} \Delta \psi \cdot \psi dx = -\delta \|\nabla \psi\|_{L^2}^2 \leq 0$$

by employing the Green formula with Neumann boundary conditions. Besides, owing to the properties of the potential $Q_\mu(u)$, $p(u)$, the L^p interpolation inequality, and the ϵ -Young's inequality we have

$$\begin{aligned} - \int_{\Omega} (q_\mu(u_\mu) - q(u)) \varphi dx &= - \int_{\Omega} (q_\mu(u_\mu) - q_\mu(u)) \varphi dx - \int_{\Omega} \mu p(u) \varphi dx \\ &\leq -q_0 \|\varphi\|_{L^2}^2 + \mu M_1 \int_{\Omega} \varphi dx \\ &\leq -q_0 \|\varphi\|_{L^2}^2 + \epsilon \|\varphi\|_{L^2}^2 + \frac{\mu^2 M_1^2 |\Omega|}{4\epsilon} \\ &\leq -\frac{q_0}{2} \|\varphi\|_{L^2}^2 + \frac{\mu^2 M_1^2 |\Omega|}{2q_0}, \end{aligned}$$

by taking $\epsilon = \frac{q_0}{2}$. Then, it is derived from the above that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\Psi\|_Y^2 &\leq \alpha \|\varphi\|_{L^2}^2 + \alpha \|\psi\|_{L^2}^2 - \beta \|\psi\|_{L^2}^2 - \frac{q_0}{2} \|\varphi\|_{L^2}^2 + \frac{\mu^2 M_1^2 |\Omega|}{2q_0} \\ &\leq (\alpha - \varrho) \|\Psi\|_Y^2 + \frac{\mu^2 M_1^2 |\Omega|}{2q_0}, \end{aligned}$$

where $\varrho = \min\{\frac{q_0}{2}, \beta\}$, then it follows from the Gronwall lemma that

$$\begin{aligned} \|\Psi\|_Y^2 &\leq e^{2(\alpha - \varrho)t} \|\Psi_0\|_Y^2 + \frac{\mu^2 M_1^2 |\Omega|}{q_0(\alpha - \varrho)} (e^{2(\alpha - \varrho)t} - 1) \\ &\leq \frac{\mu^2 M_1^2 |\Omega|}{q_0(\alpha - \varrho)} (e^{2(\alpha - \varrho)t} - 1). \end{aligned}$$

Furthermore, if $\alpha - \varrho < 0$, it is obvious that there exists a positive constant ε small enough so that

$$\|U_\mu(t) - U(t)\|_Y \leq \frac{\varepsilon}{3} = \sqrt{\frac{2\mu^2 M_1^2 |\Omega|}{q_0(\alpha - \varrho)}}, \quad \mu \rightarrow 0^+.$$

Similarly, if $\alpha - \varrho > 0$, then there exists a positive constant $C_{\mu,t}$ depending on the perturbation parameter μ and t so that

$$\|U_\mu(t) - U(t)\|_Y \leq C_{\mu,t} = \sqrt{\frac{\mu^2 M_1^2 |\Omega|}{q_0(\alpha - \varrho)}} (e^{2(\alpha - \varrho)t} - 1), \quad \mu \rightarrow 0^+, \quad t \in [0, T], \quad T > 0.$$

We eventually complete the proof. \square

By virtue of case (i) in Theorem 6, the following main result can be deduced.

Theorem 7 (Continuity of the stationary solutions). *Suppose that the assumptions in Theorem 6 hold. Let \bar{U}_μ, \bar{U} be the stationary solutions to the forest kinematic model (4) when $\mu > 0$ and $\mu = 0$ respectively. Then, if $\alpha - \varrho < 0$, we have $\bar{U}_\mu \xrightarrow{\mu \rightarrow 0} \bar{U}$ in Y .*

Proof. It follows from Theorem 4 that $U_\mu(t) \rightarrow \bar{U}_\mu$ in Y as $t \rightarrow \infty$, i.e., for all $\varepsilon > 0$, there exists $\mu \in [0, \mu_1]$ so that

$$\|U_\mu(t) - \bar{U}_\mu\|_Y \leq \frac{\varepsilon}{3}.$$

In parallel, when $\mu = 0$, Theorem 4 still holds as long as the assumption $c - \frac{1}{3}ab^2 \geq 0$ is satisfied, i.e., $U(t) \rightarrow \bar{U}$ in Y as $t \rightarrow \infty$. Thus, we have

$$\|U(t) - \bar{U}\|_Y \leq \frac{\varepsilon}{3}.$$

Combine with the conclusion in case (i) of Theorem 6, we eventually deduce that

$$\|\bar{U}_\mu - \bar{U}\|_Y \leq \|\bar{U}_\mu - U_\mu(t)\|_Y + \|U_\mu(t) - U(t)\|_Y + \|U(t) - \bar{U}\|_Y \leq \varepsilon, \quad \mu \rightarrow 0^+, \quad t \rightarrow \infty,$$

which completes the proof. \square

Remark 6. *Note that we can not conclude that $\bar{U}_\mu \xrightarrow{\mu \rightarrow 0^+} \bar{U}$ in Y when $\alpha - \varrho > 0$. Due to the assertion in case (ii) of Theorem 6, the dissipative estimation (69) only holds on a compact time interval $[0, T]$. In particular, the above convergence results and continuity estimations are based on the assumption that $Q_\mu(u)$ is convex. Otherwise, it is still an open problem, and the difficulty comes with the perturbations.*

4.3 A robust family of weak attractors

Numerous results on the effect of a perturbation on the asymptotic behavior of a given dynamical system have been established. Very often, these results express a type of robustness of the global attractor [9, 12, 17, 19, 21, 22, 30, 37, 48], whereas results on the robustness of ω -limit sets are rare [10]. More specifically, robustness is often described by proving the upper or lower semi-continuity of the global attractor with respect to the perturbation parameter. As mentioned previously, compactness is a necessary requirement for proving such statements [38]. Here, the compactness requirement is not fulfilled. However, we can still prove that the set \mathcal{A}_μ defined by

$$\mathcal{A}_\mu = \overline{\bigcup_{U_0 \in \mathcal{R}_\mu} \omega_Y^\mu(U_0)}^Y, \quad \mu \geq 0, \quad (71)$$

where the set $\omega_Y^\mu(U_0)$ is given by (63), attracts the trajectories of the forest kinematic model (4) and varies smoothly with μ . Recall that the distance in Y between a element $u \in Y$ and a bounded set $B \subset Y$ is defined as $d_Y(u, B) = \inf_{v \in B} d_Y(u, v)$, and the semi-distance in Y between two bounded sets $B_1, B_2 \subset Y$ is defined by $d_Y(B_1, B_2) = \sup_{u \in B_1} d_Y(u, B_2)$.

Theorem 8 (A robust family of weak attractors). *Suppose that assumption (11) holds. Suppose moreover that $\mathcal{R}_\mu \subset \mathcal{R}_0$ for all $\mu \in [0, \mu_1]$. Then the set \mathcal{A}_μ defined by (71) satisfies the following properties:*

(i) \mathcal{A}_μ is invariant, closed and uniformly bounded in Y , for all $\mu \in [0, \mu_1]$;

(ii) \mathcal{A}_μ attracts in Y the trajectories of (4) starting in \mathcal{R}_μ , for all $\mu \in [0, \mu_1]$, that is:

$$\lim_{t \rightarrow +\infty} d_Y(S_\mu(t)U_0, \mathcal{A}_\mu) = 0, \quad (72)$$

for all $U_0 \in \mathcal{R}_\mu$ and for all $\mu \in [0, \mu_1]$;

(iii) if $\alpha - \varrho < 0$, then the family $\{\mathcal{A}_\mu\}_{0 \leq \mu \leq \mu_1}$ satisfies

$$d_Y(\mathcal{A}_\mu, \mathcal{A}_0) \leq C\mu, \quad (73)$$

where C is a positive constant and $\varrho = \min(\frac{\alpha_0}{2}, \beta)$.

Proof. (i) Let $\mu \in [0, \mu_1]$. It follows from the definition (71) that \mathcal{A}_μ is invariant and closed in Y . Next, let $U_\mu \in \mathcal{A}_\mu$. Then there exists a sequence $(U_{\mu,n})$ in $\bigcup_{U_0 \in \mathcal{R}_\mu} \omega_Y^\mu(U_0)$ such that $(U_{\mu,n})$ converges to U_μ in Y as n

tends to $+\infty$. By virtue of Theorem 4, we have $\omega_Y^\mu(U_0) = \{\ell^\mu(U_0)\}$, for each $U_0 \in \mathcal{R}_\mu$, where we shortly denote by $\ell^\mu(U_0)$ the limit in Y of the trajectory $S_\mu(t)U_0$. Hence, we have $U_{\mu,n} = \ell^\mu(U_{\mu,n,0})$ for each $n \geq 0$, with $U_{\mu,n,0} \in \mathcal{R}_\mu$. Now, Corollary 1 guarantees that $\|S_\mu(t)U_{\mu,n,0}\|_Y \leq \widetilde{M}_\mathcal{R}$ for all $t \geq 0$, where $\widetilde{M}_\mathcal{R}$ is a positive constant that does not depend on μ . We can deduce that $\|U_{\mu,n}\|_Y \leq \widetilde{M}_\mathcal{R}$ for all $n \geq 0$, and finally that $\|U_\mu\|_Y \leq \widetilde{M}_\mathcal{R}$, for all $U_\mu \in \mathcal{A}_\mu$ and for all μ , which proves that \mathcal{A}_μ is uniformly bounded in Y .

(ii) Let $U_0 \in \mathcal{R}_\mu$. Theorem 4 guarantees that $S_\mu(t)U_0$ converges in Y to $\ell^\mu(U_0)$ as t tends to $+\infty$, with $\omega_Y^\mu(U_0) = \{\ell^\mu(U_0)\}$, which can be written

$$\lim_{t \rightarrow +\infty} d_Y(S_\mu(t)U_0, \ell^\mu(U_0)) = 0.$$

But we have

$$d_Y(S_\mu(t)U_0, \mathcal{A}_\mu) \leq d_Y(S_\mu(t)U_0, \ell^\mu(U_0)),$$

since $\ell^\mu(U_0) \in \mathcal{A}_\mu$, which proves (72).

(iii) Let $U_\mu \in \mathcal{A}_\mu$. We consider again a sequence $(U_{\mu,n})$ in $\bigcup_{U_0 \in \mathcal{R}_\mu} \omega_Y^\mu(U_0)$ such that $(U_{\mu,n})$ converges to U_μ in Y as n tends to $+\infty$, and we write again $U_{\mu,n} = \ell^\mu(U_{\mu,n,0})$ with $U_{\mu,n,0} \in \mathcal{R}_\mu$. Now, we have

$$d_Y(U_{\mu,n}, \mathcal{A}_0) = d_Y(\ell^\mu(U_{\mu,n,0}), \mathcal{A}_0) \leq d_Y(\ell^\mu(U_{\mu,n,0}), \ell^0(U_{\mu,n,0})),$$

where $\ell^0(U_{\mu,n,0})$ denotes the limit in Y of the unperturbed trajectory $S_0(t)U_{\mu,n,0}$ (which is well defined, since $\mathcal{R}_\mu \subset \mathcal{R}_0$). Since $\alpha < \min(\frac{\alpha_0}{2}, \beta)$, Theorem 6 ensures that

$$d_Y(\ell^\mu(U_{\mu,n,0}), \ell^0(U_{\mu,n,0})) \leq C\mu,$$

for all $\mu \in [0, \mu_1]$ and for all $n \geq 0$, with $C > 0$. We can deduce that

$$d_Y(U_{\mu,n}, \mathcal{A}_0) \leq C\mu,$$

for all $n \geq 0$ and consequently that

$$d_Y(U_\mu, \mathcal{A}_0) \leq C\mu,$$

for all $U_\mu \in \mathcal{A}_\mu$. We obtain

$$d_Y(\mathcal{A}_\mu, \mathcal{A}_0) \leq C\mu,$$

which proves (73). The proof is complete. \square

Remark 7. *Theorem 8 guarantees that the set \mathcal{A}_μ defined by (71) attracts in Y the trajectories of the kinematic forest model (4). However, due to the lack of compactness, it is not known if the set \mathcal{A}_μ attracts the bounded sets of \mathcal{R}_μ .*

4.4 Case of a strong perturbation

We end this section with the case of a strong perturbation, obtained for $\mu > \mu_2$. The following theorem proves that in this case, the orbits of the dynamical system (24) converge to the trivial equilibrium.

Theorem 9. *Let $\mu > \mu_3$, then there exists $\rho^* > 0$ such that for each $U_0 \in \mathcal{K}$, the solution $U_\mu(t)$ of the degenerate forest kinetic system (4) stemming from U_0 satisfies*

$$\|U_\mu(t)\|_Y \leq \|U_0\|_Y e^{-\rho^* t}, \quad t \geq 0.$$

Proof. Since $\mu > \mu_3$, there exists $\rho > 0$ such that $q_\mu(u) \geq \left(\frac{\alpha^2}{\beta} + \rho\right)u$ if $u \geq 0$ and $q_\mu(u) \leq \left(\frac{\alpha^2}{\beta} + \rho\right)u$ if $u \leq 0$. Hence we have

$$uq_\mu(u) \geq \left(\frac{\alpha^2}{\beta} + \rho\right)u^2,$$

for all $u \in \mathbb{R}$. Now we introduce the energy function L defined for $t \geq 0$ by

$$L(t) = \frac{1}{2} \int_{\Omega} [u_\mu^2(t) + w_\mu^2(t)] dx,$$

which is continuously differentiable and we have

$$\begin{aligned} \dot{L}(t) &= \int_{\Omega} u_\mu(t) \frac{\partial u_\mu}{\partial t}(t) dx + \int_{\Omega} w_\mu(t) \frac{\partial w_\mu}{\partial t}(t) dx \\ &= 2\alpha \int_{\Omega} u_\mu(t) w_\mu(t) dx - \int_{\Omega} u_\mu(t) q_\mu(u_\mu(t)) dx + \delta \int_{\Omega} w_\mu(t) \Delta w_\mu(t) dx - \beta \int_{\Omega} w_\mu^2(t) dx \\ &\leq 2\alpha \int_{\Omega} u_\mu(t) w_\mu(t) dx - \left(\frac{\alpha^2}{\beta} + \rho\right) \int_{\Omega} u_\mu^2(t) dx - \beta \int_{\Omega} w_\mu^2(t) dx \end{aligned}$$

by applying the Green formula with the Neumann boundary and the assumption above.

Now we apply the generalized Young's inequality to write

$$\int_{\Omega} u_\mu(t) w_\mu(t) dx \leq \frac{1}{2\varepsilon} \int_{\Omega} u_\mu^2(t) dx + \frac{\varepsilon}{2} \int_{\Omega} w_\mu^2(t) dx,$$

with any $\varepsilon > 0$, then we have

$$2\alpha \int_{\Omega} u_\mu(t) w_\mu(t) dx \leq \frac{\alpha}{\varepsilon} \int_{\Omega} u_\mu^2(t) dx + \alpha\varepsilon \int_{\Omega} w_\mu^2(t) dx,$$

in which we can choose a proper ε to ensure $\frac{\alpha}{\varepsilon} = \frac{\alpha^2}{\beta} + \frac{\rho}{2}$, i.e., $\varepsilon = \frac{\alpha}{\frac{\alpha^2}{\beta} + \frac{\rho}{2}}$.

Therefore, we obtain

$$\dot{L}(t) \leq -\frac{\rho}{2} \int_{\Omega} u_\mu^2(t) dx - (\beta - \alpha\varepsilon) \int_{\Omega} w_\mu^2(t) dx,$$

where $\beta - \alpha\varepsilon > 0$ is naturally guaranteed by our choice of ε . Note that $\rho^* = \min(\rho, \beta - \alpha\varepsilon)$, we thus obtain that

$$\dot{L}(t) \leq -\rho^* L(t).$$

Employing again the Gronwall lemma, we eventually complete the proof. \square

5 Numerical simulations

In this section, we aim to illustrate with relevant numerical simulations the theoretical results established above, and to show how the perturbed forest kinetic model (4) fulfills various ecological properties of interest. First, we experiment the effect of several perturbations on the position of the *ecotone*, which corresponds to

the frontier between the forest and another ecosystem (as for instance savanna in tropical regions, or tundra in boreal regions). These perturbations are expected to model the impact of global warming on the biological dynamics of the ecosystem. However, as the complex mechanisms of global warming are not yet precisely understood, we experiment several functions for the perturbation $p(u)$ involved in (5), mainly determined by polynomial or sinusoidal expressions. Then, we present numerical results that prove that the forest kinetic model (4) can reproduce the formation of chaotic patterns.

Our numerical computations were performed using a Strang type splitting scheme [47], with a finite elements discretization in space, and a Runge-Kutta method in time. The computation code was executed in a Debian/GNU-LINUX environment, with the free software FreeFem++ [20]. For all simulations, we have considered an elliptic domain Ω of width $L = 500$ and height $\ell = 300$; we have fixed the parameters of the forest kinetic model (4) as follows:

$$\alpha = \beta = 1, \quad \delta = 10, \quad a = 0.6, \quad b = 1, \quad c = 0.9. \quad (74)$$

With these parameters, the unperturbed problem admits three homogeneous equilibrium states given by

$$O = (0, 0), \quad U^- = \left(1 - \frac{1}{\sqrt{6}}, 1 - \frac{1}{\sqrt{6}}\right), \quad U^+ = \left(1 + \frac{1}{\sqrt{6}}, 1 + \frac{1}{\sqrt{6}}\right). \quad (75)$$

5.1 Shift of the ecotone or modification of the persistence equilibrium

One of the most fascinating properties of forest ecosystems is their ability to migrate in space, while they are obviously populations of sedentary individuals. Recently, it has been proved in [5] that this indirect diffusion, when combined with a hysteresis process, can explain the separation of trajectories, that reproduces the formation of an ecotone (see [40]). As forest ecosystems are highly destabilized by climatic and anthropic perturbations, it is relevant to investigate the impact of a perturbation on the position of the ecotone. In the ecological science literature, it is well described that the ecotone can be simply shifted to the north or to the south, depending on the nature of the perturbation and on the geographical region, or can be highly modified and exhibit the formation of chaotic or fractal patterns [51]. For instance, in [14], a northward shift is proved to occur in the boreal forest-tundra ecotone; in [42], the tropical forest-savanna ecotone is studied and it is shown that climate change can modify the trees density near the ecotone; in [43], island tropical montane cloud forests are studied and it is observed that climate change might push them towards higher elevations.

Here, to reproduce these complex mechanisms, we consider two shift perturbations s_1, s_2 defined by

$$s_1(u) = auu^-(u - u^+), \quad s_2(u) = -auu^-(u - u^+). \quad (76)$$

The effect of these shift perturbations on the cubic function $q(u)$ is depicted in Figure 3: when μ increases, the persistence equilibrium U^+ is not modified. However, the saddle equilibrium U_μ^- is shifted to the left under the action of $s_1(u)$, or to the right under the action of $s_2(u)$. The shift can also be visualized on the energy levels of the potential $H_\mu(u, w)$ defined by (32), as illustrated in Figure 4.

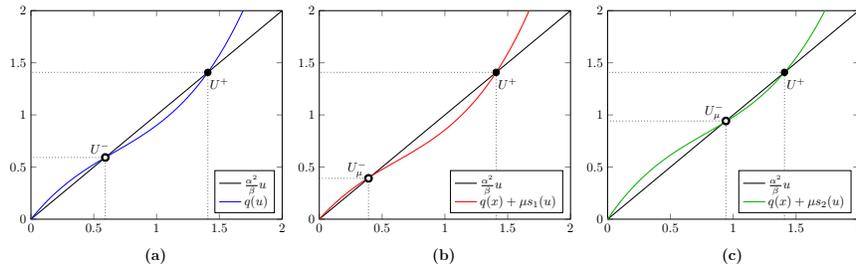


Figure 3: Effect of the shift perturbations $s_1(u), s_2(u)$ defined by (76) on the cubic function $q(u)$. **(a)** Shape of the unperturbed cubic function $q(u)$. **(b)** When μ increases, the persistence equilibrium U^+ is not modified; the saddle equilibrium U_μ^- is shifted to the left under the action of $s_1(u)$. **(c)** The saddle equilibrium U_μ^- is shifted to the right under the action of $s_2(u)$.

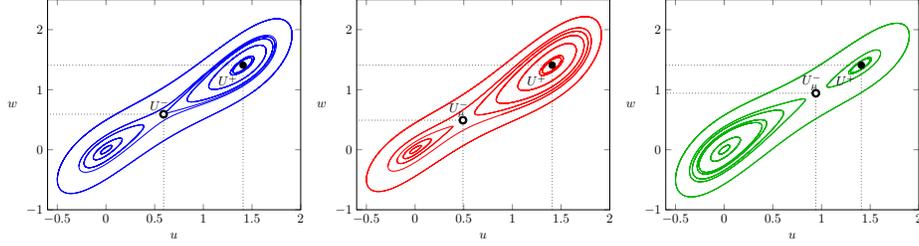


Figure 4: Effect of the shift perturbations $s_1(u)$, $s_2(u)$ defined by (76) on the energy levels of the potential $H_\mu(u, w)$ defined by (32). (a) Energy levels of the unperturbed potential. (b) Under the action of $s_1(u)$, the saddle U_μ^- is shifted to the left, whereas the persistence equilibrium U^+ is not modified. (c) Under the action of $s_2(u)$, the saddle U_μ^- is shifted to the right.

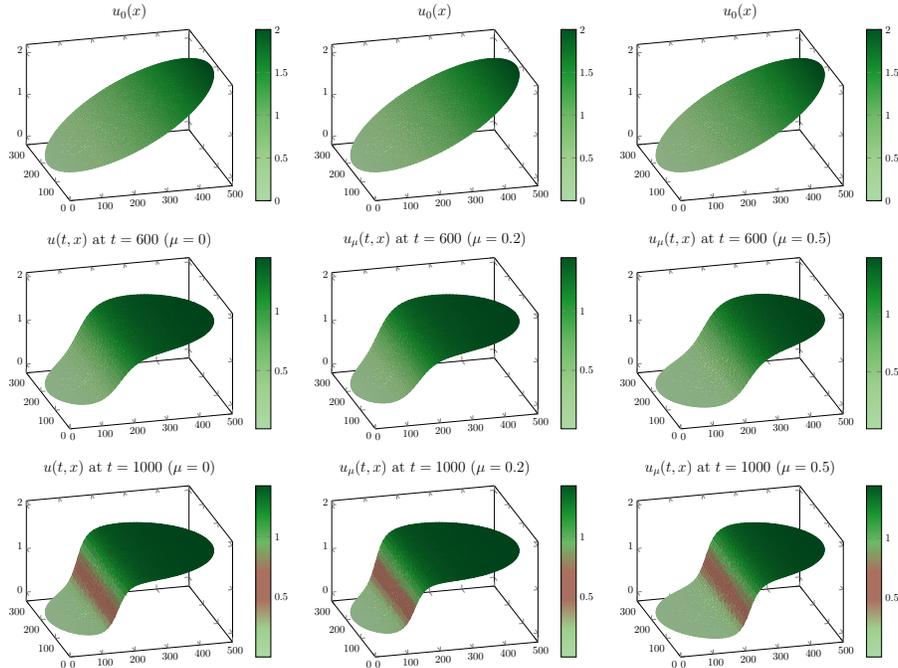


Figure 5: Numerical simulation showing the effect of the shift perturbations $s_1(u)$, $s_2(u)$ on the position of the ecotone. In the last row, the position of the ecotone is highlighted by a brown band. Under the action of $s_1(u)$, the ecotone is shifted to the left (second column). Under the action of $s_2(u)$, it is shifted to the right (third column).

Next, we have considered an initial condition $(u_0(x), w_0(x))$ which is equally distributed within the basins of attraction of the extinction equilibrium O and of the persistence equilibrium U^+ . As proved in [6], the trajectory of the forest kinetic model (4) starting from such an initial condition is expected to converge to a heterogeneous stationary solution that reproduces the ecotone. Hence, we have computed the trajectories $(u(t, x), w(t, x))$, $(u_\mu(t, x), w_\mu(t, x))$, of the unperturbed problem and of the problem perturbed by $s_1(u)$ or $s_2(u)$, respectively. The results are depicted in Figure 5, where we focus on the density of trees. In this figure, the color semantics is chosen according to intuition: deep green illustrates a high density of trees, while pale green represents a low density of trees. We have chosen a 3D view in order to visualize the formation of the ecotone. In the first column, we show the time evolution of the density of trees $u(t, x)$ of the unperturbed trajectory, from $t = 0$ (at the top), until $t = 1000$ (at the bottom). In the second and third columns, we show the time evolution of the density of trees $u_\mu(t, x)$ of the trajectories perturbed by $s_1(u)$ and $s_2(u)$, respectively. In the asymptotic phase ($t = 1000$), the ecotone is highlighted by a brown band, located at the position of the domain Ω where the density of trees decreases very rapidly. As expected, the ecotone is shifted to the left under the action of $s_1(u)$, which implies that the area occupied by the high density of trees spreads

in the domain Ω . If the perturbation parameter μ is increased, the area occupied by a high density of trees can even invade the whole domain Ω . At the opposite, the ecotone is shifted to the right under the action of $s_2(u)$, which implies that the area occupied by a high density of trees shrinks. Hence, these numerical simulations show that the perturbed forest model (4) can faithfully reproduce ecological observations of great interest.

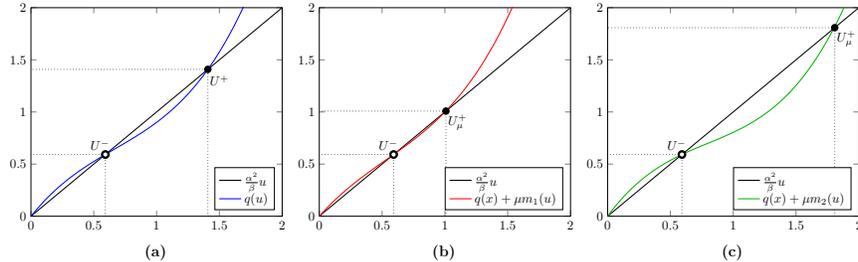


Figure 6: Effect of the perturbations $m_1(u)$, $m_2(u)$ defined by (77) on the cubic function $q(u)$. (a) Shape of the unperturbed cubic function $q(u)$. (b) The persistence equilibrium U_μ^+ decreases under the action of $m_1(u)$. (c) It increases under the action of $m_2(u)$. In parallel, the saddle point U^- is not modified.

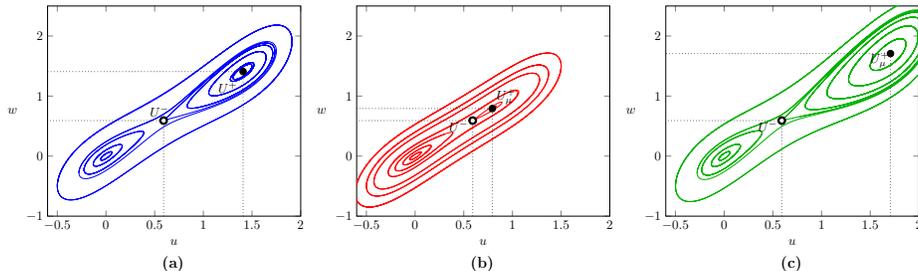


Figure 7: Effect of the perturbations $m_1(u)$, $m_2(u)$ defined by (77) on the energy levels of the potential $H_\mu(u, w)$. (a) Energy levels of the unperturbed potential. (b) The persistence equilibrium U_μ^+ decreases under the action of $m_1(u)$. (c) It increases under the action of $m_2(u)$.

Afterwards, we have also experimented two perturbations m_1 , m_2 that are expected to modify the persistence equilibrium U^+ , without impacting the position of the ecotone. The perturbations m_1 , m_2 are defined by the polynomial expressions

$$m_1(u) = auu^+(u - u^-), \quad m_2(u) = -auu^+(u - u^-). \quad (77)$$

The effects of the perturbations $m_1(u)$, $m_2(u)$ on the cubic function $q(u)$ and on the energy levels of the potential $H(u, w)$ are depicted in Figures 6 and 7, respectively. Under the action of the perturbation $m_1(u)$, the persistence equilibrium U_μ^+ is decreased, whereas it is increased under the action of the perturbation $m_2(u)$. In parallel, the saddle point U^- is not modified.

With the perturbations $m_1(u)$, $m_2(u)$ defined by (77), we have again considered an initial condition $(u_0(x), w_0(x))$ which is equally distributed within the basins of attraction of the extinction equilibrium O and of the persistence equilibrium U^+ . The perturbed trajectories starting from that initial condition are depicted in Figure 8. We observe that the trajectory behaves as expected, with a modification of the level of the persistence equilibrium U_μ^+ and no modification of the position of the ecotone. It is worth noting that this behavior reproduces a non trivial ecological transition which is well observed and described: indeed, under the action of climate change, forest ecosystems can exhibit a deep modification of their dynamics, such as, for instance, the savannization of the Amazon forest (see e.g. [45]), that leads to a sharp fall of the density of trees in the ecosystem.

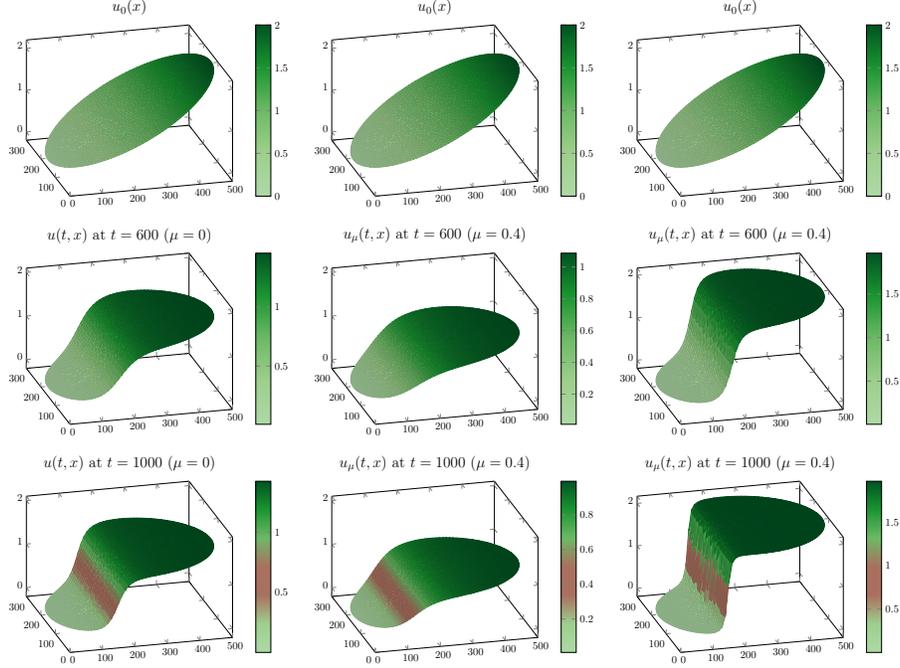


Figure 8: Numerical simulation showing the effect of the perturbations $m_1(u)$, $m_2(u)$ on the level of the persistence equilibrium U^+ . In the last row, the position of the ecotone is again highlighted by a brown band. Under the action of $m_1(u)$, the level of persistence is decreased (second column). Under the action of $m_2(u)$, it is increased (third column). In parallel, the position of the ecotone is not varied.

5.2 Emergence of intermediate ecosystems

We continue with numerical simulations of the forest kinematic model (4) perturbed by a periodic process. Hence, we consider the perturbations $p_1(u)$, $p_2(u)$ and $p_3(u)$ defined by

$$p_1(u) = \sin(20u), \quad p_2(u) = \sin(9u), \quad p_3(u) = \sin(11u). \quad (78)$$

The effects of the perturbations $p_1(u)$, $p_2(u)$ and $p_3(u)$ are depicted in Figures 9 and 10, with $\mu = 0.035$, $\mu = 0.09$ and $\mu = 0.07$, respectively. Note that these values of the perturbation parameter μ guarantee that the perturbed cubic function $q_\mu(u)$ is still monotone; however, it admits more than three intersection points with the line $w = \frac{\alpha^2}{\beta}u$. These supplementary intersection points are expected to perturb the form and the position of the ecotone. In parallel, the positions of the intersection points can modify the values of the trees density at equilibrium.

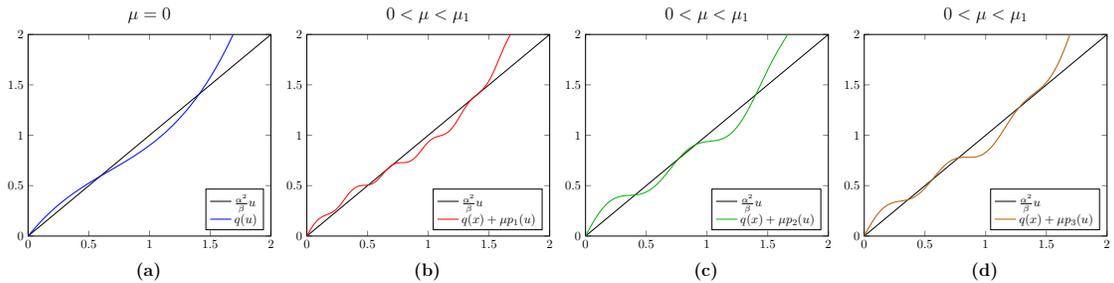


Figure 9: Effect of the perturbations $p_1(u)$, $p_2(u)$, $p_3(u)$ defined by (78) on the cubic function $q(u)$. (a) Shape of the unperturbed cubic function $q(u)$. (b)-(c)-(d) The perturbed function $q_\mu(u)$ admits more than three intersection points with the line $w = \frac{\alpha^2}{\beta}u$.

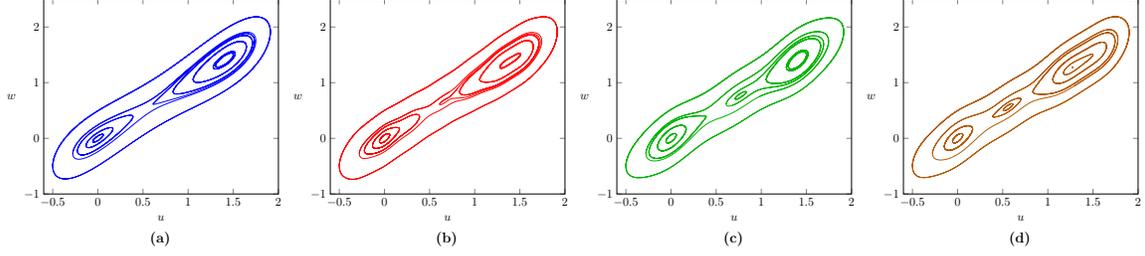


Figure 10: Effect of the perturbations $p_1(u)$, $p_2(u)$, $p_3(u)$ defined by (78) on the energy levels of the potential $H(u, w)$. (a) Energy levels of the unperturbed potential $H(u, w)$. (b)-(c)-(d) The perturbed potential $H_\mu(u, w)$ admits a supplementary saddle point and a supplementary sink.

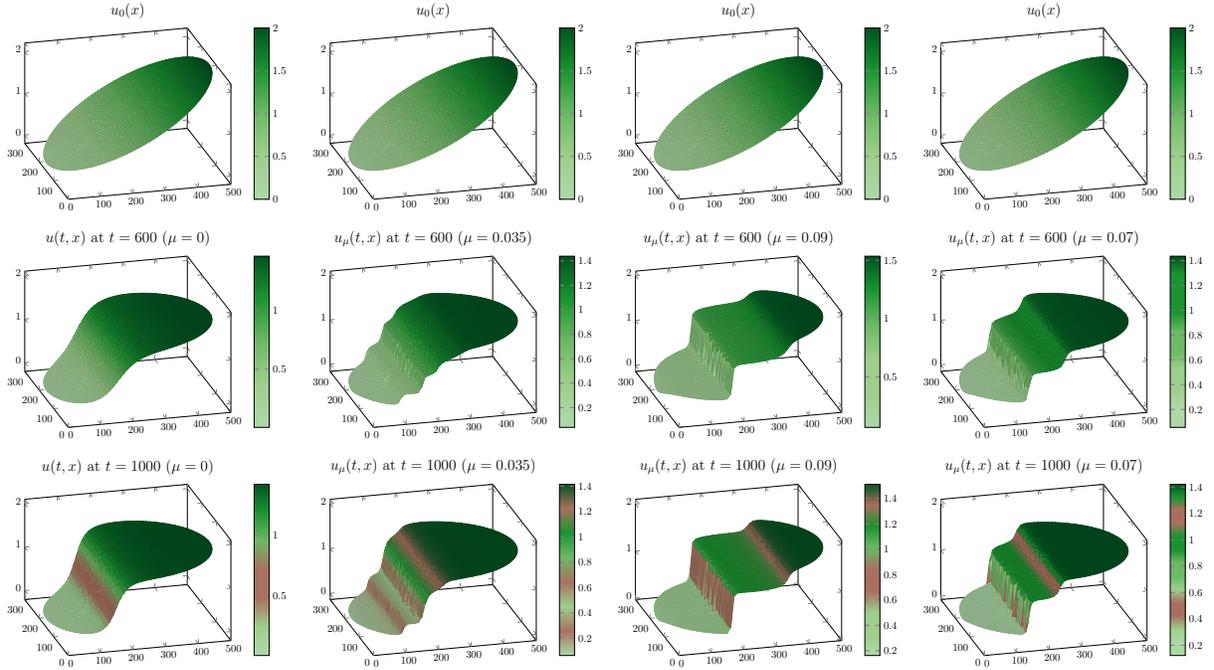


Figure 11: Numerical simulation showing the effect of the periodic perturbations $p_1(u)$, $p_2(u)$, $p_3(u)$ on the dynamics of the forest model (4). In the last row, the positions of the ecotones are highlighted by brown bands. Under the action of the perturbations, the form of the ecotone is modified and intermediate ecosystems emerge. The size and trees densities of these intermediate ecosystems are very sensitive to the nature of the perturbation.

With the perturbations $p_1(u)$, $p_2(u)$, $p_3(u)$ defined by (78), we have again considered an initial condition $(u_0(x), w_0(x))$ which is equally distributed within the basins of attraction of the extinction equilibrium O and of the persistence equilibrium U^+ . The perturbed trajectories starting from that initial condition are depicted in Figure 11. In this figure, we show the time evolution of the density of trees $u_\mu(t, x)$ for each perturbation. We observe that the position and the form of the ecotone are modified and intermediate ecosystems emerge. The size and trees densities of these intermediate ecosystems are very sensitive to the intensity and the nature of the perturbation. In each case, the perturbation leads to a decrease of the total living biomass, which is in concordance with ecological observations (see for instance [16]).

5.3 Randomly generated initial conditions lead to chaotic patterns

We end this section with numerical simulations of the forest kinetic model (4), starting from randomly generated initial conditions. We have chosen an initial condition $(u_0(x), w_0(x))$, $x \in \Omega$, using the random number generator `randreal()` of the FreeFem++ software, and we have computed two trajectories of the

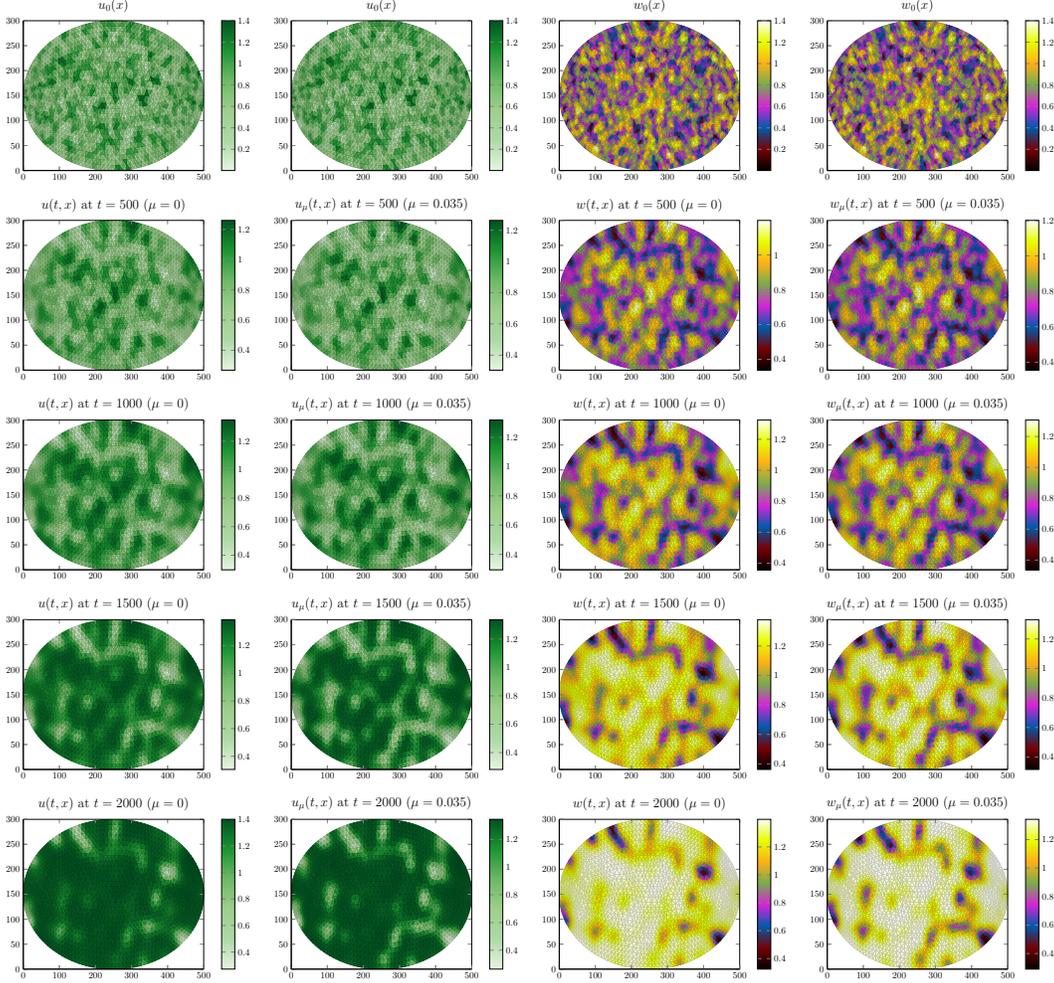


Figure 12: Numerical simulation showing the time evolution of the densities of trees $u(t, x)$, $u_\mu(t, x)$ (first and second columns) and of the densities of seeds $w(t, x)$, $w_\mu(t, x)$ (third and fourth columns), corresponding to an unperturbed trajectory and a perturbed trajectory of the forest kinetic model (4), respectively. Starting from a randomly generated initial condition, the trajectories converge to a stationary solution that exhibits spots patterns with a low density of trees and seeds.

forest kinetic model (4). The first trajectory $(u(t, x), w(t, x))$ is the solution of the unperturbed problem (that is, with $\mu = 0$), while the second trajectory $(u_\mu(t, x), w_\mu(t, x))$ is the solution of the perturbed problem, with $p(u) = \sin(20u)$ and $\mu = 0.035$.

Our results are presented in Figure 12. In this figure, the left column shows the time evolution of the density of trees $u(t, x)$ (of the unperturbed trajectory), from $t = 0$ (at the top), until $t = 2000$ (at the bottom). The color semantics is the same as in Figures 5 and 8 (deep green models a high density of trees, and pale green corresponds to a low density of trees). However, we have chosen a map view rather than a 3D view, so as to better visualize the formation of patterns. The second column shows the time evolution of the density of trees $u_\mu(t, x)$ (of the perturbed trajectory), with the same color semantics. The third and fourth columns show the time evolution of the densities of seeds $w(t, x)$ and $w_\mu(t, x)$ (unperturbed and perturbed trajectories, respectively); white and yellow correspond to a high density of seeds, while blue and brown model a low density of seeds.

We observe that the trajectories converge to a heterogeneous stationary solution that exhibits spots patterns with a low density of trees and seeds. For instance, a white spot of low density of trees is located at the position $(400, 200)$ of the domain Ω (first and second columns, at the bottom), in correspondence with a

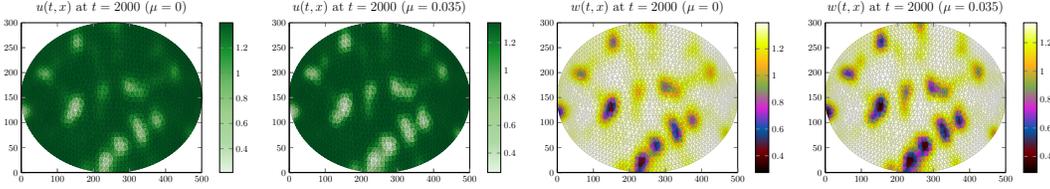


Figure 13: Numerical simulation showing the limits of two trajectories $(u(t, x), w(t, x))$, $(u_\mu(t, x), w_\mu(t, x))$ starting from another randomly generated initial condition $(u_0(x), w_0(x))$. The trajectories converge to a stationary solution that again exhibits spots patterns with a low density of trees and seeds. However, the number and location of spots are completely different than in Figure 12.

brown spot of low density of seeds, located at the same position (third and fourth columns, at the bottom). According to Theorem 8, the stationary solution is robust with respect to a variation of the perturbation parameter μ , that is, the spots are smoothly modified by an increase of μ . Hence, modifications of the spots can be identified after a careful look, but these modifications are small. However, these spots are very sensitive a change of the initial condition $(u_0(x), w_0(x))$. Indeed, we show in Figure 13 the limits of two other trajectories $(u(t, x), w(t, x))$, $(u_\mu(t, x), w_\mu(t, x))$, starting from another randomly generated initial condition $(u_0(x), w_0(x))$. As in Figure 12, we observe the formation of spots, modeling a low density of trees and seeds. While these spots are again smoothly modified by the perturbation parameter μ , their number and location are completely different in Figures 12 and 13. Therefore, these patterns admit a chaotic behavior. It is worth emphasizing that such a chaotic behavior has been intensely studied in non degenerate reaction-diffusion systems admitting a diffusion driven instability (see for instance [3], [39] and the references therein); however, chaotic patterns emerging in degenerate reaction-diffusion systems as the forest model (4) have not been as much analyzed. In particular, given an initial condition $(u_0(x), w_0(x))$, it seems very difficult to predict the position of spots of the corresponding trajectory.

6 Conclusion

In this paper, we studied the dynamics of a perturbed forest kinematic model (4). We firstly presented the normal well-posedness results. Then, we established the asymptotic convergence result by proving the Lojasiewicz-Simon gradient inequality, which is guaranteed by the monotonicity of the nonlinear perturbation $q_\mu(u)$, i.e. when assumption (11) holds. Moreover, we established the existence of a family of positively invariant regions for the dynamical system $\{S_\mu(t)\}$, and proved the continuity of the flow, which yields the continuity of the stationary solutions. We further addressed the robustness of the weak attractors, which is highly nontrivial. However, when the monotonicity assumption is violated, the set of equilibria in the weak topology contains numbers of essentially discontinuous solutions, [13] which is challenging and will be studied in our forthcoming research. We also presented the case of a strong perturbation, which leads to trivial dynamics. At last but not the least, we performed numerical simulations to illustrate our theoretical results. We showed the modification of the persistence equilibrium, and the emergence of intermediate ecosystems associated with different perturbations. We also introduced the chaotic patterns caused by randomly generated initial conditions.

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