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# Contraction Theory and Differential Passivity in the port-Hamiltonian formalism

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**Abstract:** In this work, we recall the concept of contractive dynamics and its natural extension to the notion of differential dissipativity/passivity via the use of the so-called prolonged (or extended) system dynamics, obtained by lifting the system to the tangent bundle of the underlying manifold. The new concept of dissipative differential Hamiltonian dynamics is proposed providing a weaker notion of contractive dynamical system. Furthermore, the differential Hamiltonian notion is extended to the definition of differentially passive port-Hamiltonian system. We describe explicit conditions to exploit the ‘natural’ differential Hamiltonian function as differential storage function for the port-Hamiltonian system dynamics.

*Keywords:* Contractive systems, differential passivity, port-Hamiltonian systems, dissipative Hamiltonian systems, dissipative differential Hamiltonian system

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## 1. INTRODUCTION

Since Dissipativity was introduced by J.C. Willems in Willems (1972) it has played a relevant role in systems theory, in particular in the analysis of open and interconnected systems, as it generalizes the concept of Lyapunov function to system interfacing the surrounding environment, as described in Willems (2007). It also gives the foundations to dissipativity-based nonlinear control designs, see e.g. van der Schaft (2000), Sepulchre et al. (2012), and Brogliato et al. (2007). A particular case of dissipativity is the concept of passivity. On the latter, a special class of nonlinear physical systems has been built, i.e., port-Hamiltonian systems.

The port-Hamiltonian approach to modeling and control of complex physical systems is a well-established framework, and it started/pioneered with the work by Maschke and van der Schaft (1993), van der Schaft and Maschke (1995), see also Duindam et al. (2009); van der Schaft (2000); van der Schaft et al. (2014) for a general overview of the topic including also control techniques. A particular feature of port-Hamiltonian Systems is that of describing all the main physical properties of the system under consideration, such as energy dissipation, passivity, and power conservation laws. Moreover, the formalism is particularly well suited for the interconnection of physical systems, preserving the passivity, stability, and structure in a larger Port-Hamiltonian system, Cervera et al. (2007).

Concurrently to the introduction of the dissipativity concept, the first results of contraction theory can be found in Demidovich (1967) and Willems (1970), and they are based on a constant Euclidean metric, see also Pavlov

et al. (2005). The contraction approach was born as a tool to study and describe the stability of nonlinear systems as a counterpart of Lyapunov approach, see Forni and Sepulchre (2013a). Only later on, in Lohmiller and Slotine (1998) the theory has been extended to nonlinear state dependent metric, while further developments can be found in Angeli (2002); Forni and Sepulchre (2013a); Andrieu et al. (2016).

A strictly related concept to contraction theory is that of differential dissipativity, introduced for the first time in Forni and Sepulchre (2013b). As the contraction theory gives tools for studying the neighborhood of autonomous systems trajectories, analogously the notion of differential dissipativity can be employed as a tool to study the variational input-output-state behavior. The concept has then been specialized to differential passivity in Forni et al. (2013) and described in a geometric and coordinate-free framework in van der Schaft (2013). See Reyes-Báez et al. (2018) for an application to control flexible-joints robots.

In the context of port-Hamiltonian systems, some results on contraction theory are available but only for the case of constant metric, see Yaghmaei and Yazdanpanah (2017); Barabanov et al. (2019); Yaghmaei and Yazdanpanah (2023), and recently a new concept of incremental stability has been introduced on Dirac structure in Camlibel and van der Schaft (2013, 2022). To the best of the authors’ knowledge, none of the works on contractive port-Hamiltonian systems consider a nonlinear state-dependent metric, nor exploit the knowledge of the available Hamiltonian function. Additionally, there is no work on differential passivity in the context of port-Hamiltonian systems.

The aim of this work is to introduce the concept of dissipative differential Hamiltonian system, resulting in

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a weaker notion of contractive system. We then cast the contraction theory and differential passivity in the framework of port-Hamiltonian systems.

The paper is structured as follows. In section Sec. 2, we recall the concepts of dissipative, passive, and port-Hamiltonian systems. We also introduce in Sec. 3, the variational and prolonged systems to describe the concepts of contractive systems in Sec. 4 and differential dissipativity and passivity in Sec. 5. Then in Section 6, we provide our main result by introducing the definition of (dissipative) differential Hamiltonian system and extending it to the concept of differential passivity for port-Hamiltonian systems. We then provide an example in Sec. 7, before summarizing the paper content in Sec. 8.

*Nomenclature:* Given a real-valued function  $H : \mathbb{R}^n \rightarrow \mathbb{R}$ , we denote its gradient (in covector form) as  $\frac{\partial H}{\partial x} = \partial_x H$ , and for the sake of simplicity we write its column vector

form as  $\nabla H(x) = \frac{\partial H^\top}{\partial x}(x)$ . Its Hessian will then be

written as  $\nabla^2 H(x) = \frac{\partial}{\partial x} \cdot \frac{\partial H^\top}{\partial x}(x)$ . For a matrix-valued function  $J(x) \in \mathbb{R}^{n \times n}$ ,  $x \in \mathbb{R}^n$ , we consider, with some abuse of notation, that its time derivative (or Lie derivative with respect to the vector field  $\dot{x} = f(x)$ , in this case indicated as  $\mathcal{L}_{f(x)}J(x)$ ), to be written as  $\dot{J}(x) = \frac{\partial J(x)}{\partial x}\dot{x}$ ,

or, column-wise, we write  $\dot{J}_i(x) = \frac{\partial J_i}{\partial x}\dot{x}$ , where  $J_i(x)$  refers to the  $i$ -th column of  $J(x)$ . When we evaluate  $\dot{J}(x)\nabla H(x)$ , we write it in matrix form as  $\dot{J}(x)\nabla H(x) = \frac{\partial H}{\partial x} \frac{\partial^\top J}{\partial x}\dot{x}$ , meaning that the resulting vector will have as  $i$ -th element

$$\dot{J}_i \nabla H = \left( \dot{x}^\top \frac{\partial J_{i\top}}{\partial x} \right)^\top \nabla H(x) = \frac{\partial H}{\partial x} \frac{\partial^\top J_i}{\partial x} \dot{x}.$$

## 2. DISSIPATIVE AND PASSIVE SYSTEMS

Consider a nonlinear system, in the state space (local) coordinates of a smooth manifold  $\mathcal{X}$ , expressed by a model of the form

$$\Sigma : \begin{cases} \dot{x} = f(x, u), & x(0) = x_0 \\ y = h(x, u) \end{cases}$$

with the input value  $u \in \mathcal{U} \subseteq \mathbb{R}^m$ , the output values  $y \in \mathcal{Y} \subseteq \mathbb{R}^p$ , and the system's state  $x \in \mathcal{X} \subseteq \mathbb{R}^n$ , where  $\mathcal{U}, \mathcal{Y}$ , and  $\mathcal{X}$  are the input, output, and state spaces. With some abuse of notation  $\mathcal{U}$  refers, for the sake of brevity of exposition, also to the space of admissible inputs.

The definition of dissipative system for  $\Sigma$ , introduced in Willems (1972), employs two real-valued functions, i.e., the supply rate denoted by  $s : \mathcal{U} \times \mathcal{Y} \rightarrow \mathbb{R}$ , and the storage function  $S : \mathcal{X} \rightarrow \mathbb{R}$ .

*Definition 2.1.* (Dissipation inequality). System  $\Sigma$  is said to satisfy the *dissipative inequality* with respect to the supply rate  $s$  and storage function  $S$  if

$$S(x(t_2)) - S(x(t_1)) \leq \int_{t_1}^{t_2} s(u(t), y(t)) dt \quad (1)$$

for all  $t_1, t_2 \in \mathbb{R}$ , with  $t_2 \geq t_1$  and all admissible input-output-state trajectory  $(u(t), y(t), x(t)) \in \mathcal{U} \times \mathcal{Y} \times \mathcal{X}$ .

To be more precise, the tuple  $(u(t), y(t), x(t))$  should belong to the behavior of  $\Sigma$ , see Willems (2007) for more details.

The well-known and accepted interpretation of the dissipation inequality (1), is that (in general) not all the supplied quantity ( $\int s dt$ ) is stored (in  $S$ ), and thus the amount that is supplied but not stored is *dissipated*. This allows us to introduce the definition of dissipative and cyclo-dissipative systems.

*Definition 2.2.* (Cyclo-dissipative systems). A system  $\Sigma$  is said to be cyclo-dissipative with respect to the supply rate  $s : \mathcal{U} \times \mathcal{Y} \rightarrow \mathbb{R}$  if there exists a real-valued storage function  $S : \mathcal{X} \rightarrow \mathbb{R}$  such that the dissipation inequality (1) holds.

*Definition 2.3.* (Dissipative systems). A system  $\Sigma$  is said to be *dissipative* with respect to the supply rate  $s : \mathcal{U} \times \mathcal{Y} \rightarrow \mathbb{R}$  if it is cyclo-dissipative with a non-negative storage function  $S : \mathcal{X} \rightarrow \mathbb{R}_{\geq 0}$  satisfying the dissipation inequality (1).

As described in Willems (2007), the notion of storage function for dissipative systems is a generalization to open systems of the concept of Lyapunov function for closed systems.

A particular case of dissipative system is given when the supply rate takes the form of an inner product between the input and the output of the system, i.e.,  $s(u, y) = y^\top u$ . We then have the following definition.

*Definition 2.4.* ((Cyclo-)Passive system). A square (i.e.,  $m = p$ ) dynamical system  $\Sigma$  is said to be (cyclo-)passive if it is (cyclo-)dissipative with respect to the supply rate  $s(u, y) = y^\top u$ .

### 2.1 Port-Hamiltonian systems

The concept of passive system is explicitly expressed by the dynamics equation of a class of nonlinear systems, i.e., the so-called port-Hamiltonian systems. In particular, a nonlinear system  $\Sigma$  is port-Hamiltonian when its dynamical equation presents a predefined structure, of the form

$$\begin{aligned} \dot{x} &= (J(x) - R(x)) \nabla H(x) + g(x)u \\ y &= g(x)^\top \nabla H(x) \end{aligned} \quad (2)$$

where  $J : \mathcal{X} \rightarrow \mathbb{R}^{n \times n}$  is a skew-symmetric matrix that describes the underlying Dirac structure (i.e., the energy transfer between different physical domains),  $R : \mathcal{X} \rightarrow \mathbb{R}^{n \times n}$  is a positive semi-definite matrix describing the dissipative elements of the system, and  $H : \mathcal{X} \rightarrow \mathbb{R}$  is the Hamiltonian function associated to the system energy. The particular structure of  $y$  defines an output that is power-conjugated to the input  $u$ , i.e., their inner product provides an equivalent power quantity. We said that the passivity concept is explicitly expressed by the dynamics of a port-Hamiltonian system (2) because, via the energy balance, we retrieve the differential form of the dissipation inequality (1), i.e.,

$$\dot{H} = \nabla H^\top (J - R) \nabla H + \nabla^\top H g u \leq y^\top u.$$

Thus the system dynamics explicitly satisfies the dissipation inequality of a passive system if  $H$  is lower bounded, or of a cyclo-passive system otherwise.

### 3. VARIATIONAL AND PROLONGED SYSTEM

As introduced in Crouch and van der Schaft (1987) and exploited in Forni and Sepulchre (2013a), we show how a system  $\Sigma$ , defined on the smooth manifold  $\mathcal{X}$ , can be extended/prolonged (or ‘lifted’ as mentioned in Forni and Sepulchre (2013a)) to a system on the  $2n$ -dimensional tangent bundle  $\mathcal{T}\mathcal{X}$  of the manifold  $\mathcal{X}$ , with  $2m$  inputs and  $2p$  outputs.

Given an admissible input-output-state trajectory  $t \mapsto (u, y, x)$  for  $\Sigma$ , the *variational system* along such a trajectory is given by the following time-varying system

$$D\Sigma : \begin{cases} \dot{\xi} = \frac{\partial f}{\partial x}(x, u)\xi + \frac{\partial f}{\partial u}(x, u)\nu \\ \zeta = \frac{\partial h}{\partial x}(x, u)\xi + \frac{\partial h}{\partial u}(x, u)\nu \end{cases}$$

with state  $\xi \in \mathbb{R}^n$ , where  $\nu = (\nu_1, \dots, \nu_m)$  and  $\zeta = (\zeta_1, \dots, \zeta_p)$  denote the inputs and the outputs of the variational system. As highlighted in van der Schaft (2013), the terminology ‘variational’ system comes from considering the infinitesimal variations of a family of input-output-state trajectories of  $\Sigma$ , i.e.,  $(u(t, \varepsilon), y(t, \varepsilon), x(t, \varepsilon))$  parameterized by  $\varepsilon \in (-\delta, \delta)$ , with  $x(t, 0) = x(t)$ ,  $u(t, 0) = u(t)$ , and  $y(t, 0) = y(t)$ . Thus the infinitesimal variations

$$\xi(t) = \frac{\partial x}{\partial \varepsilon}(t, 0), \quad \nu(t) = \frac{\partial u}{\partial \varepsilon}(t, 0), \quad \zeta(t) = \frac{\partial y}{\partial \varepsilon}(t, 0),$$

satisfy the dynamical equations of  $D\Sigma$ . In other words, the input-output-state variational trajectory  $(\nu, \zeta, \xi)$  describes any ‘virtual displacement’ of the input-output-state trajectory of  $\Sigma$ .

The prolonged system of  $\Sigma$  thus corresponds to the augmented  $2n$ -dimensional system on the tangent bundle  $\mathcal{T}\mathcal{X}$  composed of  $\Sigma$  together with  $D\Sigma$ , i.e., the system dynamics

$$\mathcal{T}\Sigma : \begin{cases} \dot{x} = f(x, u), & x(0) = x_0 \\ y = h(x, u) \\ \dot{\xi} = \frac{\partial f}{\partial x}(x, u)\xi + \frac{\partial f}{\partial u}(x, u)\nu \\ \zeta = \frac{\partial h}{\partial x}(x, u)\xi + \frac{\partial h}{\partial u}(x, u)\nu. \end{cases}$$

Note that in the standard notation, as exploited e.g. in van der Schaft (2013),  $\xi = \delta x$ ,  $\nu = \delta u$ , and  $\zeta = \delta y$ . Furthermore, in Forni and Sepulchre (2013a), in the case of closed system, the variational state  $\xi = \delta x$  refers to the tangent vector to the parameterized curve connecting any two system trajectories. The infinitesimal variations  $\delta x(t)$  on the state  $x(t)$  can also be interpreted as being a generic ‘virtual displacement’ along any possible direction on the tangent space  $\mathcal{T}_x\mathcal{X}$ . However, if we specialize such a direction to be the one of the ‘state velocity’, i.e., we obtain the dynamics of  $\xi = \dot{x}$ , we describe the time evolution of the vectorfield  $f(x, u)$ . Where this time, we can properly determine the initial conditions of the variational system, i.e.,  $\xi(0) = f(x_0, u(0))$ . This choice of the variational system direction, recalls and extends the definition of Forward Contraction introduced in (Forni and Sepulchre, 2013a, sec. III.C). In our opinion, it can lead to different interpretations of the contraction theory approach to determine the stability of equilibrium points of nonlinear systems, which is however out of the scope of this paper and thus postponed to a future work.

### 4. CONTRACTION THEORY

Before discussing differential dissipative systems, we need to introduce the notion of contractive system.

The contraction theory approach was first introduced in books such as Demidovich (1967) and Willems (1970) (see also Pavlov et al. (2004)), and extended in Lohmiller and Slotine (1998); Forni and Sepulchre (2013a); Andrieu et al. (2016). It plays a crucial role in studying the stability and/or the attractiveness of invariant manifolds for nonlinear autonomous systems. In particular, one can determine the exponential convergence property by studying the properties of the variational dynamics. More specifically, this attractiveness property is equivalent to the existence of a positive definite quadratic form (called Lyapunov-Finsler metric in Forni and Sepulchre (2013a)) which is decreasing along the flow of the autonomous prolonged dynamics  $\mathcal{T}\Sigma$ .

*Definition 4.1.* (Contractive system). An autonomous system  $\dot{x} = f(x)$  on a smooth manifold  $\mathcal{X}$ , is said to be contractive if there exists a  $C^1$  function  $P : \mathcal{X} \rightarrow \mathbb{R}^{n \times n}$ , two strictly positive real numbers  $\underline{p}$  and  $\bar{p}$  such that  $P$  has a Lie derivative along the vectorfield  $f$ , i.e.,  $\mathcal{L}_{f(x)}P(x)$ , and it satisfies

$$\underline{p}I \leq P(x) \leq \bar{p}I, \\ \mathcal{L}_{f(x)}P(x) + P(x)\frac{\partial f}{\partial x}(x) + \frac{\partial f}{\partial x}^\top(x) < 0, \quad \forall x \in \mathcal{X}.$$

A strictly related and independent concept is the one of incremental stability, that is the exponential attractiveness between any two trajectories of a system dynamics.

*Definition 4.2.* An autonomous system  $\dot{x} = f(x)$  is said to be globally exponentially incrementally stable, if there exist two strictly positive real numbers  $k$  and  $\alpha$  such that any two trajectories  $x_1(t)$  and  $x_2(t)$ , originated in  $x_{01}$  and  $x_{02}$ , for all  $(x_{01}, x_{02}) \in \mathcal{X} \times \mathcal{X}$ , we have

$$|x_1(t) - x_2(t)| \leq k|x_{01} - x_{02}| \exp(-\alpha t), \quad \forall t \geq 0.$$

It has been shown in (Andrieu et al., 2016, Prop. 1) that if the vectorfield  $f(x)$  is globally Lipschitz with bounded second derivative, the two concepts of contractive and incrementally stable systems are *equivalent*. Additionally, they are both equivalent, see (Andrieu et al., 2016, Prop. 1), to the global exponential attractiveness of the manifold  $\mathcal{E} = \{(x, \xi) : \xi = 0\}$  for the prolonged systems of  $\dot{x} = f(x)$ , i.e.,

$$\dot{x} = f(x), \quad \dot{\xi} = \frac{\partial f}{\partial x}(x)\xi.$$

*A motivation:* By the characterization we gave to the virtual state  $\xi$  in the previous section, i.e., considering  $\xi = \dot{x}$ , we can additionally conclude that the manifold  $\mathcal{E}$ , as introduced in (Andrieu et al., 2016, Property P2), is the set of all equilibrium points of the autonomous system  $\dot{x} = f(x)$ . As a consequence, because the contraction property is assumed to hold globally in Andrieu et al. (2016), we can conclude that there is a single stable equilibrium point, i.e.,  $\mathcal{E} = \{(\bar{x}, 0)\}$  where  $\bar{x}$  is such that  $f(\bar{x}) = 0$ .

### 5. DIFFERENTIAL DISSIPATIVITY AND PASSIVITY

The theory of differential dissipativity was first developed in Forni and Sepulchre (2013b) in which the authors

introduce the concept of differential storage function  $\mathcal{S} : \mathcal{TX} \rightarrow \mathbb{R}$  and of differential supply rate  $\sigma : \mathcal{TU} \times \mathcal{T}\mathcal{Y} \rightarrow \mathbb{R}$ , so to ‘lift’ the standard dissipativity notion to the tangent bundle. We recall the definitions of differential dissipativity, differential passivity, respectively, given in Forni and Sepulchre (2013b) and Forni et al. (2013), here extended to the weaker notion of differential cyclo-dissipativity and cyclo-passivity.

*Definition 5.1.* (Diff. (cyclo-) dissipative system). The dynamical system  $\Sigma$  is said to be differentially (cyclo-) dissipative with respect to the differential supply rate  $\sigma : \mathcal{TU} \times \mathcal{T}\mathcal{Y} \times \mathcal{TX} \rightarrow \mathbb{R}$  if there exists a differential storage function ( $\mathcal{S} : \mathcal{TX} \rightarrow \mathbb{R}$ )  $\mathcal{S} : \mathcal{TX} \rightarrow \mathbb{R}_{\geq 0}$  such that

$$\mathcal{S}(x(t_1), \xi(t_1)) - \mathcal{S}(x(t_0), \xi(t_0)) \leq \int_{t_0}^{t_1} \sigma(x(t), u(t), \nu(t), y(t), \zeta(t)) dt \quad (3)$$

for all  $t_1, t_0 \in \mathbb{R}$ ,  $t_1 \geq t_0$ , and for any admissible input-output-state trajectory of  $\mathcal{T}\Sigma$ , i.e.,  $(u, \nu, y, \zeta, x, \xi) \in \mathcal{TU} \times \mathcal{T}\mathcal{Y} \times \mathcal{TX}$ .

By exploiting the assumption that  $\mathcal{S} \in C^1$ , (3) is equivalent to

$$\dot{\mathcal{S}}(x(t), \xi(t)) \leq \sigma(x(t), u(t), \nu(t), y(t), \zeta(t)) \quad (4)$$

along the flow of  $\mathcal{T}\Sigma$ .

*Remark 1.* It is worth noticing that the definition of dissipativity applied to the prolonged system  $\mathcal{T}\Sigma$  is not equivalent to differential passivity of  $\Sigma$ , because the variational state  $\xi$  is not involved in the supply rate.

Following the standard definition of passivity introduced above and lifted to the tangent bundle  $\mathcal{TX}$ , we have the following definition of differential passivity.

*Definition 5.2.* (Diff. (cyclo-) passivity). A square (i.e.,  $m = p$ ) dynamical system  $\Sigma$  is differentially (cyclo-) passive if it is differentially (cyclo-) dissipative with respect to the differential supply rate  $\sigma(\nu, \zeta) = \zeta^\top \nu$ .

Equivalently, with the additional assumption  $\mathcal{S} \in C^1$ , the differential passivity inequality can be written as

$$\dot{\mathcal{S}}(x(t), \xi(t)) \leq \zeta(t)^\top \nu(t). \quad (5)$$

It is worth noting the similarity (suggested by an anonymous reviewer) with the definition of strictly Krasovskii’s passivity in (Kawano et al., 2023, Def. 3.1), that however is a particular case of the differential passivity defined here and in Forni et al. (2013).

In the remainder of the paper, we focus our attention on the particular direction, on the tangent bundle, determined by the system vectorfield. In this case, the differential dissipativity notion earns a more concrete and tangible meaning, where  $\xi(t) = \dot{x}(t)$ ,  $\nu(t) = \dot{u}(t)$ , and  $\zeta(t) = \dot{y}(t)$ .

## 6. MAIN RESULT

The main result of this paper is twofold. We first show an original way to apply and exploit the contraction theory approach to dissipative Hamiltonian systems, i.e., closed port-Hamiltonian systems. We then extend this concept to (open) port-Hamiltonian systems with the related concept of differential passivity. Both results are associated to the employment of the Hamiltonian’s Hessian as contractive metric and to explicitly define a differential storage function.

### 6.1 Contractive and dissipative Hamiltonian systems

A dissipative Hamiltonian system is the autonomous version of a port-Hamiltonian system (or a ‘closed’ port-Hamiltonian system), and its dynamics on a manifold  $\mathcal{X}$  reads as

$$\dot{x} = (J(x) - R(x)) \nabla H(x) := f_H(x), x(0) = x_0 \quad (6)$$

where  $J(x) = -J(x)^\top$  and  $R(x) = R(x)^\top \geq 0$ , with  $x \in \mathcal{X}$  and such a dynamics is related to a  $C^2$  real function  $H(x)$  called the Hamiltonian of the system.

Here we introduce the notion of differential Hamiltonian system, when the variational dynamics of (6) has again the structure of a dissipative Hamiltonian system.

*Definition 6.1.* ((Diss.) Diff. Hamiltonian system). A dissipative Hamiltonian system (6) on the smooth manifold  $\mathcal{X}$ , is said differentially Hamiltonian if there exists a ‘lifted’ Hamiltonian function  $\mathcal{H} : \mathcal{TX} \rightarrow \mathbb{R}$ , a skew symmetric matrix  $J' : \mathcal{X} \rightarrow \mathbb{R}^{n \times n}$  and a symmetric matrix  $R' : \mathcal{X} \rightarrow \mathbb{R}^{n \times n}$ , such that the variational dynamics of (6) reads as

$$\dot{\xi} = (J'(x) - R'(x)) \nabla_\xi \mathcal{H}(x, \xi), \quad \xi \in \mathcal{TX}. \quad (7)$$

Furthermore, we say that it is *dissipative differentially Hamiltonian*, if additionally  $\mathcal{H}$  is lower bounded and  $\dot{\mathcal{H}} \leq 0$  along the flow of prolonged system (6)-(7).

We then provide the first result of the paper, concerning the conditions to have dissipative differential Hamiltonian systems.

*Theorem 6.1.* A dissipative Hamiltonian system (6) is dissipative differentially Hamiltonian with respect to the Hamiltonian function  $\mathcal{H}(x, \xi) = \frac{1}{2} \xi^\top \nabla^2 H(x) \xi$ , if and only if there exists a real  $q$  such that  $qI \leq \nabla^2 H$ , for all  $x \in \mathcal{X}$ , and  $\nabla^2 H$  is everywhere locally invertible and for all  $x \in \mathcal{X}$ ,

$$\mathcal{L}_{f_H} \nabla^2 H(x) - 2\nabla^2 H(x) R'(x) \nabla^2 H(x) \leq 0 \quad (8)$$

where

$$R' = R - \text{sym} \left[ \frac{\partial H}{\partial x} \left( \frac{\partial^\top J}{\partial x} - \frac{\partial^\top R}{\partial x} \right) \nabla^2 H^{-1}(x) \right]. \quad (9)$$

*Proof.* We first show that under the assumption of everywhere locally invertibility of Hamiltonian Hessian  $\nabla^2 H(x)$ , any dissipative Hamiltonian system (6) is differentially Hamiltonian with Hamiltonian function

$$\mathcal{H}(x, \xi) = \frac{1}{2} \xi^\top \nabla^2 H(x) \xi.$$

We thus compute the variational system of (6), i.e., by defining  $\xi = \dot{x}$ , we write its dynamics

$$\begin{aligned} \dot{\xi} &= \left( \dot{J}(x) - \dot{R}(x) \right) \nabla H(x) + (J(x) - R(x)) \dot{\nabla} H(x) \\ &= \frac{\partial H}{\partial x} \left[ \frac{\partial^\top J}{\partial x} - \frac{\partial^\top R}{\partial x} \right] \xi + (J(x) - R(x)) \nabla^2 H(x) \xi \\ &= (J'(x) - R'(x)) \nabla^2 H(x) \xi, \end{aligned}$$

where  $R'$  is as defined in (9) and

$$J' = J - \text{skew} \left[ \frac{\partial H}{\partial x} \left( \frac{\partial^\top J}{\partial x} - \frac{\partial^\top R}{\partial x} \right) \nabla^2 H^{-1}(x) \right].$$

We now show that the time derivative of  $\mathcal{H}(x, \xi) = \frac{1}{2} \xi^\top \nabla^2 H(x) \xi$  is nonpositive under condition (8), since it is given by

$$\dot{\mathcal{H}} = \frac{1}{2} \xi^\top (\mathcal{L}_{f_H} \nabla^2 H) \xi - \xi^\top \nabla^2 H R'(x) \nabla^2 H \xi$$

thus condition (8) provides the necessary and sufficient to guarantee  $\dot{\mathcal{H}} \leq 0$  along the prolonged system.  $\square$

From the above theorem, one can notice that every dissipative Hamiltonian system (6) is differentially Hamiltonian with a ‘natural’ Hamiltonian function  $\mathcal{H}(x, \xi)$ , but in order to be also dissipative, the structure matrix must satisfy the additional requirement (8).

In the remainder of this subsection, we link the introduced definition of dissipative differential Hamiltonian system to the notion of contractive dynamics. In order to do so, we exploit the Hessian of the Hamiltonian function  $H$  as a metric of contraction, and we thus introduce the following assumption.

*Assumption 1.* There exist positive real values  $\underline{q}$  and  $\bar{q}$ , such that

$$\underline{q}I \leq \nabla^2 H \leq \bar{q}I, \quad \forall x \in \mathcal{X}.$$

Then the following corollary is immediate.

*Corollary 1.* System (6) is contractive on  $\mathcal{X}$  with metric  $\nabla^2 H(x)$  under Assumption 1<sup>1</sup>, if and only if it is dissipative differentially Hamiltonian and (8) holds with strict inequality for all  $x \in \mathcal{X}$ .

The sufficient part of the proof is just an application of Theorem 6.1, while the necessary part comes from the definition of Contractive systems, we thus omit the proof of this corollary.

## 6.2 Differential Passivity for port-Hamiltonian systems

When talking about passivity we need to consider the inputs and outputs interfaces with the environment, such as the input-output ports in the port-Hamiltonian system dynamics (2). We extend this dynamics to obtain the prolonged system on the tangent bundle  $\mathcal{TX}$ ,

$$\begin{aligned} \dot{x} &= (J(x) - R(x)) \nabla H(x) + g(x)u := f_{H,u}(x), \\ y &= g(x)^\top \nabla H(x), \quad x(0) = x_0 \\ \dot{\xi} &= (J'(x) - R'(x)) \nabla^2 H(x) \xi + \dot{g}(x)u + g(x)\dot{u}, \\ \xi(0) &= (J(x_0) - R(x_0)) \nabla H(x_0) + g(x_0)u(0), \\ \dot{y} &= \dot{g}(x)^\top \nabla H(x) + g(x)^\top \nabla^2 H(x) \xi. \end{aligned} \quad (10)$$

To describe the differential passivity properties of (2), we focus our attention on the variational dynamics, i.e., the differential Hamiltonian system with the additional input vectorfield  $(g(x)u)$  variation, extended with the variation of the output terms as given in  $\dot{y}$ . We then have the following conditions to guarantee differential passivity for (2) with storage function  $\mathcal{H}(x, \xi) = \frac{1}{2} \xi^\top \nabla^2 H(x) \xi$ .

*Theorem 6.2.* Consider the port-Hamiltonian system (2) on the smooth manifold  $\mathcal{X}$ . It is differentially passive with storage function  $\mathcal{H} : \mathcal{TX} \rightarrow \mathbb{R}$ ,  $\mathcal{H}(x, \xi) = \frac{1}{2} \xi^\top \nabla^2 H(x) \xi$ , if and only if the input matrix  $g(x)$  is constant and the following inequality<sup>2</sup> holds for all  $x \in \mathcal{X}$

$$\mathcal{L}_{f_{H,u}} \nabla^2 H(x) - 2\nabla^2 H R'(x) \nabla^2 H \leq 0. \quad (11)$$

*Proof.* To prove passivity with storage function  $\mathcal{H}(x, \xi) = \frac{1}{2} \xi^\top \nabla^2 H(x) \xi$ , one can first notice that the variational dynamics with input-output-state trajectory  $(\dot{u}, \dot{y}, \xi)$  has a port-Hamiltonian structure with Hamiltonian function

<sup>1</sup> See the properties of the  $P$  in the definition of contractive system.

<sup>2</sup> Or equivalently one can check  $\mathcal{L}_{f_{H,u}} \nabla^2 H(x)^{-1} + 2R'(x) \geq 0$ .

$\mathcal{H}(x, \xi) = \frac{1}{2} \xi^\top \nabla^2 H(x) \xi$  and input-output conjugate pair if and only if  $\dot{g}(x) = 0$ . However, to guarantee differential passivity, a sufficient condition is that the equivalent differential Hamiltonian system is dissipative, i.e., inequality (11) holds, so that

$$\begin{aligned} \dot{\mathcal{H}} &= \frac{1}{2} \xi^\top [\mathcal{L}_{f_{H,u}} \nabla^2 H(x) - 2\nabla^2 H R'(x) \nabla^2 H] \xi + \\ &\quad + \xi^\top \nabla^2 H(x) g(x) \dot{u} \leq \dot{y}^\top \dot{u}. \end{aligned}$$

This proves the sufficient part.

To obtain the necessary part we find conditions to satisfy

$$\dot{\mathcal{H}} \leq \dot{y}^\top \dot{u}$$

for all admissible  $(x, \xi, u, \dot{u}, y, \dot{y}) \in \mathcal{TX} \times \mathcal{TU} \times \mathcal{TY}$ . Where the Lie derivative of  $\mathcal{H}(x, \xi)$  along the prolonged dynamics reads as

$$\begin{aligned} \dot{\mathcal{H}} &= \frac{1}{2} \xi^\top [\mathcal{L}_{f_{H,u}} \nabla^2 H(x) - 2\nabla^2 H R'(x) \nabla^2 H] \xi + \\ &\quad + \xi^\top \nabla^2 H(x) g(x) \dot{u} + \xi^\top \nabla^2 H(x) \dot{g}(x) u \\ &= \frac{1}{2} \xi^\top [\mathcal{L}_{f_{H,u}} \nabla^2 H(x) - 2\nabla^2 H R'(x) \nabla^2 H] \xi + \\ &\quad + \dot{y}^\top \dot{u} + \nabla H(x)^\top \dot{g}(x) \dot{u} + \xi^\top \nabla^2 H(x) \dot{g}(x) u \end{aligned}$$

that is

$$\begin{aligned} \frac{1}{2} \xi^\top [\mathcal{L}_{f_{H,u}} \nabla^2 H(x) - 2\nabla^2 H R'(x) \nabla^2 H] \xi + \\ + \nabla H(x)^\top \dot{g}(x) \dot{u} + \xi^\top \nabla^2 H(x) \dot{g}(x) u \leq 0 \end{aligned}$$

must hold for all admissible  $(x, \xi, u, \dot{u})$ , this implies that  $\dot{g}(x) = 0$  and that inequality (11) hold. Thus proving the theorem.  $\square$

With this approach, we let the differential Hamiltonian  $\mathcal{H}(x, \xi) = \frac{1}{2} \xi^\top \nabla^2 H(x) \xi$  play the role of a ‘natural’ differential storage function, denoted by  $\mathcal{S}$  in the previous sections, under appropriate conditions. We refer to this property to be a ‘differential’ passivity-preservation, since the role of  $\mathcal{H}$ , in the differential framework, is the analogous of the standard role played by the system Hamiltonian  $H$  for passivity in standard port-Hamiltonian systems. Moreover, when we compare our result to the conditions in (van der Schaft, 2013, eq. (16)), we notice that our conditions are in a sense weaker because we do not require the storage function  $\mathcal{S}$  to have a zero Lie derivative along the complete lift of the input vectorfield  $g(x)$ . On the other hand, some limitations on the input class might be considered so to satisfy the inequality (11). However, when the Hessian of  $H$  is constant, such a condition is easier to satisfy as shown in the following illustrative example.

## 7. EXAMPLE

We consider the dynamics of a nonlinear (in the resistor part) RL-circuit. The port-Hamiltonian dynamics with Hamiltonian function  $H(\phi) = \frac{\phi^2}{2L}$  reads

$$\dot{\phi} = -R \left( \frac{\phi}{L} \right) \frac{1}{L} \phi + u, \quad y = \frac{1}{L} \phi, \quad R(i) = i^4$$

with  $\phi$  being the magnetic flux of the inductor and  $u$  being the voltage input applied to the R-L series. Thus the variational dynamics read as

$$\dot{\xi} = - \left[ \frac{\partial R}{\partial i} \left( \frac{\phi}{L} \right) \frac{\phi}{L} + R \left( \frac{\phi}{L} \right) \right] \frac{\xi}{L} + \dot{u}, \quad \dot{y} = \frac{1}{L} \xi.$$

In this particular case, we have  $\dot{g} = 0$  and  $\nabla^2 H = 0$ , implying that inequality (11) is always satisfied, because  $R'(x) \geq 0$  for all  $x \in \mathbb{R}$ , i.e.,

$$\frac{\partial R}{\partial i} \left( \frac{\phi}{L} \right) \frac{\phi}{L} + R \left( \frac{\phi}{L} \right) = 4 \left( \frac{\phi}{L} \right)^3 \frac{\phi}{L} + \left( \frac{\phi}{L} \right)^4 = 5 \left( \frac{\phi}{L} \right)^4 \geq 0.$$

Hence, the system is passive and differentially passive with differential storage function  $\mathcal{S} = \mathcal{H} = H(\xi)$ .

## 8. CONCLUSIONS

In this paper, we introduce the concepts of contractive system for autonomous dynamics and variational dynamics for open dynamical systems. The latter allows us to describe the prolonged dynamics, lifting the system state space to its tangent bundle. Furthermore, such a prolonged (or extended) dynamics allows us to define the concept of differential dissipative system and in particular of differential passive system. We then recast the concept of contractive systems into that of dissipative Hamiltonian systems, thus providing the weaker notion of dissipative differential Hamiltonian systems; and we apply the differential passivity definition to the prolonged dynamics of port-Hamiltonian system. We thus provide necessary and sufficient conditions to exploit the knowledge of the system Hamiltonian Hessian, to define a ‘natural’ differential storage function to describe the differential passivity properties of the system. This is analogous to exploiting the system Hamiltonian as storage function to describe the passivity of the system property in the standard port-Hamiltonian framework.

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