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Stability and instability of the quasilinear Gross–Pitaevskii dark solitons

Erwan Le Quiniou*

Abstract

We study a quasilinear Schrödinger equation with nonzero conditions at infinity. In previous works, we obtained a continuous branch of traveling waves, given by dark solitons indexed by their speed. Neglecting the quasilinear term, one recovers the Gross–Pitaevskii equation, for which the branch of dark solitons is stable. Moreover, Z. Lin showed that the Vakhitov–Kolokolov (VK) stability criterion (in terms of the momentum of solitons) holds for general semilinear equations with nonvanishing conditions at infinity.

In the quasilinear case, we prove that the VK stability criterion still applies, by generalizing Lin’s arguments. Therefore, we deduce that the branch of dark solitons is stable for weak quasilinear interactions. For stronger quasilinear interactions, a cusp appears in the energy-momentum diagram, implying the stability of fast waves and the instability of slow waves.

Keywords: Quasilinear Schrödinger equation, Gross–Pitaevskii equation, traveling waves, dark solitons, nonzero conditions at infinity, orbital stability.

2020 *MSC*: 35Q55, 35C07, 35C08, 35J62, 35B35, 34L40, 34L05, 35Q60, 82D50, 34B40.

1 Introduction

We consider the following quasilinear Gross–Pitaevskii equation in one dimension

$$i\partial_t\Psi + \partial_{xx}\Psi + \Psi(1 - |\Psi|^2) + \kappa\Psi\partial_{xx}(1 - |\Psi|^2) = 0, \quad \text{in } \mathbb{R} \times \mathbb{R}, \quad (1)$$

where $\kappa \in \mathbb{R}$ is a parameter measuring the strength of the quasilinear term, and $\Psi : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{C}$, satisfies the nonzero conditions at infinity

$$\lim_{|x| \rightarrow \infty} |\Psi(x, \cdot)| = 1. \quad (2)$$

Equation (1) was introduced as a generalization of the case $\kappa = 0$ also known as the Gross–Pitaevskii equation, which is a standard model in nonlinear optics and quantum hydrodynamics [17, 13, 16]. For instance, in the context of optical fibers in [19], they obtained (1) with $\kappa \geq 0$ to model weak nonlocal interactions in the refracting index of the beam of light. We refer to [18], for other relevant weakly nonlocal models. Equation (1) was also obtained with $\kappa \leq 0$ in [20] to describe superfluid films with surface tension.

Equation (1) is a particular case of the following quasilinear Schrödinger equation in \mathbb{R}^d

$$i\partial_t\Psi + \Delta\Psi + \Psi F(|\Psi|^2) - \kappa\Psi h'(|\Psi|^2)\Delta h(|\Psi|^2) = 0, \quad \text{in } \mathbb{R}^d \times \mathbb{R}, \quad (3)$$

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with $F(y) = 1 - y$ and $h(y) = 1 - y$, $\kappa \in \mathbb{R}$. To our knowledge, (3) was only considered with vanishing conditions by the mathematical community. Studying (3) with $d = 1$, I. D. Iliev and K. P. Kirchev obtained a continuous branch of solitary-wave solutions of the form $\Psi(x, t) = u_\omega(x)e^{i\omega t}$, and showed that the wave is stable whenever $\partial_\omega \|u_\omega\|_{L^2(\mathbb{R})}^2 > 0$ and unstable when $\partial_\omega \|u_\omega\|_{L^2(\mathbb{R})}^2 < 0$. In arbitrary space dimension, M. Colin, L. Jeanjean, and M. Squassina obtained the existence and stability or instability by blowup of the ground state solitary-wave for (3) with power law nonlinearities. In the semilinear regime ($\kappa = 0$ in (3)), we recover

$$i\partial\Psi + \Delta\Psi + F(|\Psi|^2)\Psi = 0, \quad \text{in } \mathbb{R}^d \times \mathbb{R}. \quad (4)$$

With vanishing conditions at infinity, M. I. Weinstein showed in [24] the stability of the solitary wave solution to (4) satisfying a ground state property using Lyapunov functionals. Assuming $F(\rho_0) = 0$ and that Ψ satisfies the nonzero conditions at infinity

$$\lim_{|x| \rightarrow \infty} |\Psi(x, t)| = \rho_0, \quad \text{for all } t \in \mathbb{R}, \quad (5)$$

Z. Lin investigated in [21] in one dimension of space, the stability of nonvanishing traveling-wave solutions to (4) of the form,

$$\Psi_c(x, t) = u_c(x - ct), \quad (6)$$

by performing the analysis on the hydrodynamical variables $\eta, \partial_x \theta$ with $\Psi = \sqrt{1 - \eta}e^{i\theta}$. In particular, he rigorously proves the VK criterion [23] using the framework of abstract Hamiltonian systems presented in the celebrated work of M. Grillakis, J. Shatah, and W. Strauss [14]. We also refer to the work of D. Chiron in [8] which constitutes, to our knowledge, the most complete description of the existing orbital stability results for the nontrivial traveling-waves solutions of equation (4) and in particular deals with solutions vanishing at some point. In this work, we show that the ideas of Lin can be generalized to investigate the stability of traveling-wave solutions to the quasilinear equation (1).

The quasilinear Gross–Pitaevskii equation (1) is Hamiltonian and formally preserves the energy

$$E_\kappa(\Psi(\cdot, t)) = \int_{\mathbb{R}} |\partial_x \Psi(x, t)|^2 dx + \frac{1}{2} \int_{\mathbb{R}} (1 - |\Psi(x, t)|^2)^2 dx - \frac{\kappa}{2} \int_{\mathbb{R}} (\partial_x |\Psi(x, t)|^2)^2 dx, \quad (7)$$

as well as the (renormalized) momentum, whenever $\inf_{x \in \mathbb{R}} |\Psi(x, t)| > 0$

$$P(\Psi(\cdot, t)) = - \int_{\mathbb{R}} \Re(i\partial_x \Psi(x, t) \overline{\Psi(x, t)}) \left(1 - \frac{1}{|\Psi(x, t)|^2}\right) dx. \quad (8)$$

Because of the nonzero conditions at infinity (2), the natural energy space to study (1) is (see [12])

$$\mathcal{X}(\mathbb{R}) = \{u \in H_{\text{loc}}^1(\mathbb{R}; \mathbb{C}) : u' \in L^2(\mathbb{R}), 1 - |u|^2 \in L^2(\mathbb{R})\}. \quad (9)$$

We endow $\mathcal{X}(\mathbb{R})$ with two metrics, the natural energy distance, which writes, for $\psi, \tilde{\psi} \in \mathcal{X}(\mathbb{R})$

$$d_{\mathcal{X}}(\psi, \tilde{\psi}) = \|\partial_x \psi - \partial_x \tilde{\psi}\|_{L^2(\mathbb{R})} + \||\psi| - |\tilde{\psi}|\|_{L^2(\mathbb{R})} + |\psi(0) - \tilde{\psi}(0)|, \quad (10)$$

and the hydrodynamical distance defined for nonvanishing functions in $\mathcal{X}(\mathbb{R})$ i.e. for $\psi = \sqrt{1 - \eta}e^{i\theta}$ and $\tilde{\psi} = \sqrt{1 - \tilde{\eta}}e^{i\tilde{\theta}}$ in $\mathcal{X}(\mathbb{R})$ with $\inf_{\mathbb{R}} |\psi| > 0$ and $\inf_{\mathbb{R}} |\tilde{\psi}| > 0$, we define

$$d_{\text{hy}}(\psi, \tilde{\psi}) = \|\eta - \tilde{\eta}\|_{H^1(\mathbb{R})} + \|\partial_x \theta - \partial_x \tilde{\theta}\|_{L^2(\mathbb{R})} + \text{Arg} \left(\frac{\psi(0)}{\tilde{\psi}(0)} \right), \quad (11)$$

where Arg is a continuous determination of the argument of complex numbers defined near 1. In a previous work with A. de Laire [12], we were interested in the classification and stability

properties of the so-called finite energy traveling-wave solutions to (1). Namely, studying solutions to (1) of the form (6) which represents a traveling wave with profile $u_c = u_{c,\kappa} \in \mathcal{X}(\mathbb{R})$ propagating at speed $c \in \mathbb{R}$. Hence, the profile $u_{c,\kappa}$ satisfies

$$-icu'_{c,\kappa} + u''_{c,\kappa} + u_{c,\kappa}(1 - |u_{c,\kappa}|^2) - \kappa u_{c,\kappa} (|u_{c,\kappa}|^2)'' = 0. \quad (\text{TW}(c, \kappa))$$

Taking the complex conjugate of equation (TW(c, κ)), we can assume without loss of generality that $c \geq 0$. We obtained a continuous branch of $\mathcal{C}^2(\mathbb{R})$ -solutions for every $(c, \kappa) \in \mathcal{D} = [0, \sqrt{2}) \times (-\infty, 1/2)$. A (partial) version of our result writes

Theorem 1.1 (Theorem 1, Proposition 3.8 in [12]). *Let $\kappa \in (-\infty, 1/2)$ and $c \geq 0$.*

- (i) *If $c \geq \sqrt{2}$ and $u_{c,\kappa} \in \mathcal{X}(\mathbb{R}) \cap \mathcal{C}^2(\mathbb{R})$ satisfies (TW(c, κ)), then $u_{c,\kappa}$ is trivial, i.e. there exists $\varphi \in \mathbb{R}$ such that*

$$u_{c,\kappa}(x) = e^{i\varphi}, \quad \text{for all } x \in \mathbb{R}.$$

- (ii) *If $0 \leq c < \sqrt{2}$ so that $(c, \kappa) \in \mathcal{D}$, then there exists a nonconstant function $u_{c,\kappa} \in \mathcal{X}(\mathbb{R}) \cap \mathcal{C}^2(\mathbb{R})$ called dark soliton that satisfies (TW(c, κ)). Moreover, this solution is unique up to translation and phase change, i.e. any other solution $\phi \in \mathcal{X}(\mathbb{R}) \cap \mathcal{C}^2(\mathbb{R})$ satisfies for some $(\tau, \varphi) \in \mathbb{R} \times \mathbb{R}$*

$$\phi(x) = u_{c,\kappa}(x - \tau)e^{i\varphi}, \quad \text{for all } x \in \mathbb{R}.$$

Additionally, the function $(c, \kappa, x) \mapsto u_{c,\kappa}(x)$ is smooth in $\mathcal{D} \times \mathbb{R}$ (where the derivative at $c = 0$ is in the sense of Dini) and satisfies the following decay estimate: for every multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3$ and $D^\alpha = \partial_c^{\alpha_1} \partial_\kappa^{\alpha_2} \partial_x^{\alpha_3}$, there exist $A, C > 0$ continuously depending on (c, κ) such that

$$|D^\alpha \partial_x u_{c,\kappa}| + |D^\alpha (1 - |u|^2)| \leq Ae^{-Cx}, \quad \text{for all } x \in \mathbb{R}. \quad (12)$$

Also $|u_{c,\kappa}|$ is even with $|u_{c,\kappa}'| > 0$ in $(0, \infty)$. Finally we have the bounds $1 > |u_{c,\kappa}(x)| \geq c/\sqrt{2}$ for all $x \in \mathbb{R}$ with equality at $x = 0$, so that $\inf_{\mathbb{R}} |u_{c,\kappa}| > 0$ if and only if $c > 0$.

In the case $c = 0$, the traveling-wave $u_{0,\kappa}$ vanishes at $x = 0$ and $u_{0,\kappa}$ is also called black soliton or kink soliton [7, 4]. Showing the stability of the black soliton requires other techniques (see [5, 8]) since the differentiable momentum (8) is not real-valued in that case [10]. Theorem 1 in [12] also show the existence of supersonic solution with $(c, \kappa) \in (\sqrt{2}, \infty) \times (1/2, \infty)$, but in that case, the linearized Hamiltonian operator around the traveling wave has essential spectrum in $(-\infty, \lambda)$ for some $\lambda > 0$, thus the analysis presented in this work, which is based on [21], is not suited to study the stability of these waves. In its weak formulation, equation (TW(c, κ)) can be recast as

$$c \frac{d}{ds} P(u + sh)|_{s=0} = \frac{d}{ds} E_\kappa(u + sh)|_{s=0},$$

which are the Euler–Lagrange equations associated with the minimization problem

$$\mathcal{E}_\kappa(\mathfrak{q}) = \inf \{ E_\kappa(u) : u \in \mathcal{X}(\mathbb{R}), \inf_{x \in \mathbb{R}} |u(x)| > 0, P(u) = \mathfrak{q} \}. \quad (13)$$

When $\kappa \leq 0$ so that (13) defines nonnegative function, we showed stability and a ground state property for the dark soliton using a concentration-compactness argument, generalizing the approach used for the Gross–Pitaevski equation (equation (1) with $\kappa = 0$) by F. Bethuel P. Gravejat and J. C. Saut in [4] (see also [11]).

Theorem 1.2 (Theorem 1.10–1.11 and Proposition 6.12 in [12]). *Let $\kappa \in \mathbb{R}$ and \mathcal{E}_κ be given by (13), then the following statements hold.*

- (i) If $\kappa > 0$, then $\mathcal{E}_\kappa(\mathbf{q}) = -\infty$, for all $\mathbf{q} \in \mathbb{R}$.
- (ii) If $\kappa \leq 0$, then there exist constants $\mathbf{q}_\kappa^* > 0$, $c_\kappa^* \in [0, \sqrt{2})$ and a decreasing map $\mathbf{c} : [0, \mathbf{q}_\kappa^*) \rightarrow (c_\kappa^*, \sqrt{2}]$ satisfying $P(u_{\mathbf{c}(\mathbf{q}), \kappa}) = \mathbf{q}$. Then also, for every $\mathbf{q} \in (-\mathbf{q}_\kappa^*, \mathbf{q}_\kappa^*)$ the infimum in (13) is attained, with $\mathcal{E}_\kappa(\mathbf{q}) = E_\kappa(u_{\mathbf{c}(|\mathbf{q}|), \kappa})$, the dark soliton $u_{\mathbf{c}(|\mathbf{q}|), \kappa}$ is the unique minimizer up to invariances and for every $|\mathbf{q}| > \mathbf{q}_\kappa^*$, we have $\mathcal{E}_\kappa(\mathbf{q}) = E_\kappa(u_{0, \kappa})$ and this infimum is not attained. Finally, for every $c \in (c_\kappa^*, \sqrt{2})$ the dark soliton $u_{c, \kappa}$ is orbitally stable for $d_\mathcal{X}$.

Theorem 1.2 implies that the dark solitons $u_{c, \kappa}$ with $0 < c < c_\kappa^*$ does not appear as a global minimum of the energy at fixed momentum. Thus other methods must be employed to ensure stability. We display the energy-momentum of the dark soliton parametrized by c in the left panel of Figure 1–2, it coincides with the function $\mathcal{E}_\kappa(\cdot)$ up to \mathbf{q}_κ^* . By orbital stability for a distance d , we mean stability with respect to the orbit described by translations and rotations of the dark soliton $u_{c, \kappa}$.

Definition 1.3. (i) We say that $u_{c, \kappa}$ is orbitally stable for d if for all $\varepsilon > 0$ if there exists $\delta > 0$ with the following property: for every $\phi_0 \in H^s(\mathbb{R})$ with $s > 2 + 1/2$, such that $d(u_{c, \kappa}, u_{c, \kappa} + \phi_0) \leq \delta$, then, denoting $T = T_{\phi_0}$ the maximal time of existence of the solution $\Psi(x, t) = u_{c, \kappa}(x) + \phi(x, t)$ to (1) with $\phi \in \mathcal{C}((-T, T); H^s(\mathbb{R}))$, we have

$$\sup_{t \in (-T, T)} \inf_{(s, \varphi) \in \mathbb{R}^2} d(u_{c, \kappa}(\cdot - s)e^{i\varphi}, u_{c, \kappa}(\cdot) + \phi(\cdot, t)) \leq \varepsilon. \quad (14)$$

- (ii) Assuming furthermore that the solution is global, we say that $u_{c, \kappa}$ is orbitally unstable for d if there exists $\varepsilon_0 > 0$ such that for every $\delta > 0$ we can find $\phi_0 \in H^s(\mathbb{R})$ with $s > 2 + 1/2$ so that $d(u_{c, \kappa}, u_{c, \kappa} + \phi_0) \leq \delta$ and the solution $\Psi(x, t) = u_{c, \kappa}(x) + \phi(x, t)$ to (1) with $\phi \in \mathcal{C}(\mathbb{R}; H^s(\mathbb{R}))$ satisfies the following: there exists a constant $t = t(\delta, \varepsilon_0)$ such that

$$\inf_{(s, \varphi) \in \mathbb{R}^2} d(u_{c, \kappa}(\cdot - s)e^{i\varphi}, u_{c, \kappa}(\cdot) + \phi(\cdot, t)) > \varepsilon_0. \quad (15)$$

To our knowledge, the results concerning the well-posedness of (1) are limited to local well-posedness (see Section 2). Therefore to make sense of orbital instability, we place ourselves in the case where blowup does not occur in the sense that the solution Ψ to (1) starting at time $t = 0$ close to the dark soliton in Definition 1.3 (ii) exists for all $t \in \mathbb{R}$.

Our main result is the rigorous justification of the VK stability criterion for equation (1).

Theorem 1.4. Let $\kappa < 1/2$ and $0 < c < \sqrt{2}$. Assume that $u_{c, \kappa} \in \mathcal{X}(\mathbb{R}) \cap \mathcal{C}^2(\mathbb{R})$ is, up to invariances, the unique solution to (TW(c, κ)) given by Theorem 1.1. Then

$$c \mapsto P(u_{c, \kappa}) \in \mathcal{C}^1((0, \sqrt{2})),$$

and the traveling-wave solution $u_{c, \kappa}$ is orbitally stable for $d_\mathcal{X}$ if

$$\frac{dP(u_{c, \kappa})}{dc} > 0, \quad (16)$$

whereas it is unstable for $d_\mathcal{X}$ if

$$\frac{dP(u_{c, \kappa})}{dc} < 0. \quad (17)$$

As explained in Section 2, conditions (16) and (17) amount to $d''(c) > 0$ and $d''(c) < 0$ respectively, where $d : c \mapsto E_\kappa(u_{c, \kappa}) - cP(u_{c, \kappa})$ is the scalar function characterizing the stability in [14]. In Section 5 in [12], we were able to compute explicitly the function $(c, \kappa) \mapsto P(u_{c, \kappa})$ and analyze its variations. Hence, we can restate the stability result in Theorem 1.4 in terms of the propagation speed $c \in (0, \sqrt{2})$ of the dark soliton follows.

Corollary 1.5. *There exists $\kappa_0 < 0$ with $\kappa_0 \approx -3.636$ such that the two following statements hold*

- (i) *If $\kappa \in [\kappa_0, 1/2)$, then $u_{c,\kappa}$ is orbitally stable for every $c \in (0, \sqrt{2})$.*
- (ii) *If $\kappa < \kappa_0$, then there exists $\tilde{c}_\kappa \in (0, \sqrt{2})$ such that $u_{c,\kappa}$ is orbitally unstable if $c \in (0, \tilde{c}_\kappa)$, whereas it is orbitally stable if $c \in (\tilde{c}_\kappa, \sqrt{2})$.*

When $\kappa_0 \leq \kappa < 1/2$ we recover that the whole branch is stable as in the semilinear case $\kappa = 0$ (see Theorem 4 in [4]). In the left and center panels of Figure 1 and Figure 2 respectively, we displayed the momentum of the dark soliton as a function of c which yielded the stability conjecture proven in Corollary 1.5. Figure 2 also depicts the value of c_κ which satisfies $\partial_c P(u_{\tilde{c}_\kappa, \kappa}) = 0$ and that we approach using Newton's method.

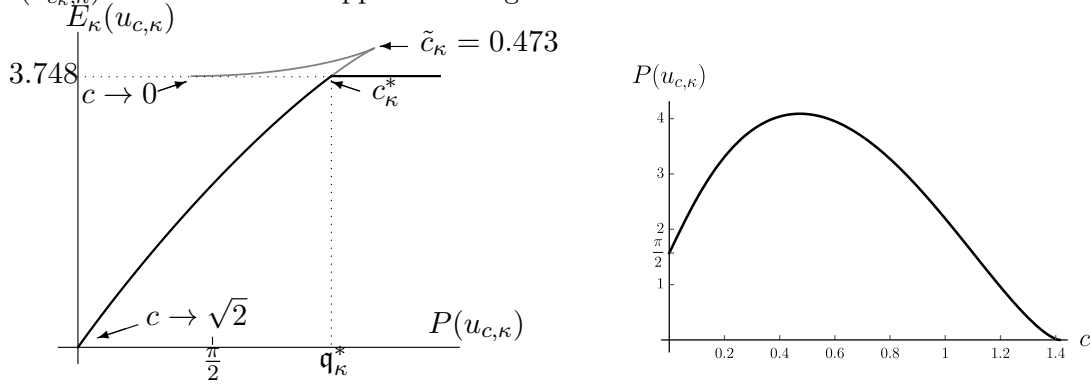


Figure 1: Left panel depicts in grey the energy-momentum diagram of dark solitons $u_{c,\kappa}$ with $\kappa = -50$ it also displays in black the curve $\mathcal{E}_\kappa(\cdot)$. Right panel displays the momentum of the dark soliton $u_{c,\kappa}$ with $\kappa = -50$ as a function of the speed c .

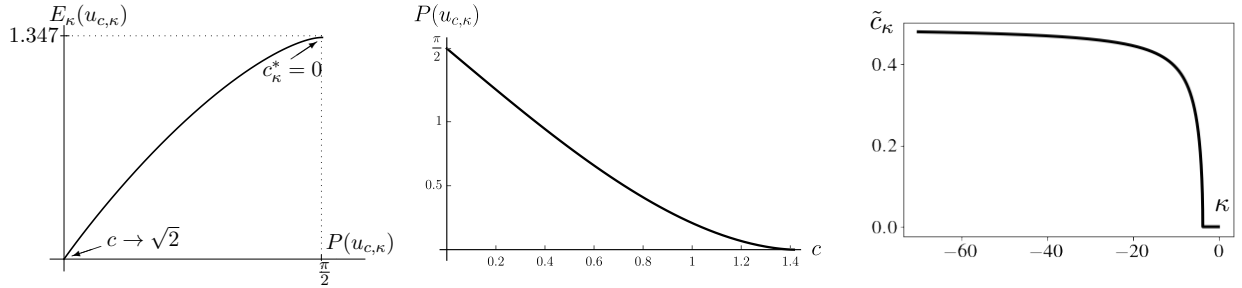


Figure 2: Left panel depicts the energy-momentum diagram of the dark soliton $u_{c,\kappa}$ with $\kappa = -3$. The center panel represents the momentum of the dark soliton $u_{c,\kappa}$ with $\kappa = -3$ as a function of the speed c . The right panel displays the value of \tilde{c}_κ depending on κ , we put $\tilde{c}_\kappa = 0$ for $\kappa \geq \kappa_0$.

From now on, we fix $\kappa < 1/2$ and remove this variable from the subscripts when no confusion can arise. The analysis will be conducted on the hydrodynamical formulation of (1). Namely, writing $\Psi = \sqrt{1 - \eta} e^{i\theta}$ with $|\Psi| > 0$ in \mathbb{R} , then η and $v = \theta_x$ formally satisfy the following hydrodynamical system with quantum pressure

$$\begin{cases} \eta_t = \partial_x(2v(1 - \eta)), \\ v_t = -\partial_x \left(\frac{\eta_{xx}}{2(1 - \eta)} + \frac{(\eta_x)^2}{4(1 - \eta)^2} + v^2 - \eta - \kappa \eta_{xx} \right), \end{cases} \quad (18)$$

We will check in Section 2 that in its weak form, we can recast (18) as the Hamiltonian system

$$\partial_t(\eta, v) = JE'(\eta, v), \quad \text{where } J = \begin{pmatrix} \partial_x & 0 \\ 0 & \partial_x \end{pmatrix}, \quad (19)$$

in particular J is not onto as explained in Remark 2.5. Let us rewrite the energy and momentum in terms of (η, v) and recall other useful quantities formally conserved in time by the solutions to (18)

$$E_\kappa(\sqrt{1-\eta}e^{i\theta}) = E((\eta, v)) = \int_{\mathbb{R}} \frac{(\eta_x)^2}{4(1-\eta)} + v^2(1-\eta) + \frac{1}{2}(\eta^2 + \kappa(\eta_x)^2), \quad \text{with } v = \theta_x \quad (\text{energy}), \quad (20)$$

$$P(\sqrt{1-\eta}e^{i\theta}) = P((\eta, v)) = - \int_{\mathbb{R}} \eta v, \quad \text{with } v = \theta_x \quad (\text{momentum}), \quad (21)$$

$$N((\eta, v)) = - \int_{\mathbb{R}} \eta, \quad (\text{number of particles}), \quad (22)$$

$$\Theta((\eta, v)) = \int_{\mathbb{R}} v, \quad (\text{twisting angle}). \quad (23)$$

When $c > 0$, since the traveling wave $u_c(x-ct) = \sqrt{1-\eta_c(x-ct)}e^{i\theta_c(x-ct)}$ is smooth and rapidly decaying (in (x, t)), all the computations (18)–(23) become rigorous and integrating in space (18), we get for all $x \in \mathbb{R}$

$$\begin{cases} -c\eta_c(x) = 2v_c(x)(1-\eta_c(x)), \\ -cv_c(x) = - \left(\frac{\partial_{xx}\eta_c(x)}{2(1-\eta_c(x))} + \frac{(\partial_x\eta_c(x))^2}{4(1-\eta_c(x))^2} + v_c^2 - \eta_c(x) - \kappa\partial_{xx}\eta_c(x) \right), \end{cases} \quad (24)$$

where $v_c = \theta_x$. System (24) in its weak form can also be written as

$$cP'((\eta_c, v_c)) = E'((\eta_c, v_c)), \quad \text{in } H^{-1}(\mathbb{R}) \times L^2(\mathbb{R}), \quad (25)$$

so that solving (24) amounts to find critical points of the Hamiltonian $E - cP$.

The outline of this work follows that of [21]. In Section 2 we check that (18) is a Hamiltonian system fitting the framework of [14] so that stability criterion (16) holds for the hydrodynamical distance d_{hy} using their result. We deal with the instability result in Section 3. Since J is not onto, the instability result of [14] does not apply, we instead follow the construction of the energy-decreasing direction developed in [21]. In Section 4, we deduce the stability and instability for the energy distance $d_{\mathcal{X}}$ in Theorem 1.4 and we show Corollary 1.5. In Appendix A, we discuss some extensions of Theorem 1.4.

2 Hamiltonian system framework and orbital stability

Let $X = H^1(\mathbb{R}; \mathbb{R}) \times L^2(\mathbb{R}; \mathbb{R})$ with the natural scalar product $\langle (\eta_1, v_1), (\eta_2, v_2) \rangle_X = \langle \eta_1, \eta_2 \rangle_{H^1(\mathbb{R})} + \langle v_1, v_2 \rangle_{L^2(\mathbb{R})}$. We also introduce the dual space $X^* = H^{-1}(\mathbb{R}; \mathbb{R}) \times L^2(\mathbb{R}; \mathbb{R})$ with duality product

$$\langle \Lambda((\eta_1, v_1), (\eta_2, v_2)) \rangle = \langle \eta_1, v_1 \rangle, (\eta_2, v_2) \rangle_X, \quad \text{for all } (\eta_1, v_1), (\eta_2, v_2) \in X,$$

where $\Lambda = (1 - \partial_{xx}, 1)$ is an isometric isomorphism between X and X^* , but we do not identify X to X^* . Regarding (20), the space X should be interpreted as the hydrodynamical counterpart of the energy space $\mathcal{X}(\mathbb{R})$. We have the variational triple $X \subset L^2(\mathbb{R}; \mathbb{R})^2 \subset X^*$ and we denote by $(\cdot, \cdot)_{L^2(\mathbb{R})}$ the natural scalar product on $L^2(\mathbb{R})$. For notational convenience, we shall use these products for $(\eta, v) \in X$ or solely in terms of $\eta \in H^1(\mathbb{R})$ or $v \in L^2(\mathbb{R})$, assuming the other coordinate to be zero. In this setting, straightforward computations implies that E and P are \mathcal{C}^2 -functional on X with, in X^* ,

$$E'(\eta, v) = \left(-\frac{\partial_x \eta_x}{2(1-\eta)} - \frac{\eta_x^2}{4(1-\eta)^2} - v^2 + \eta + \kappa \partial_x \eta_x, \quad 2v(1-\eta) \right), \quad (26)$$

so that (18) can be recast as the Hamiltonian system (19). For every $(\delta\eta, \delta v) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$, we get in X^*

$$E''(\eta, v)(\delta\eta, \delta v) = 2\left(-\partial_x \left(\frac{1-2\kappa+2\kappa\eta}{4(1-\eta)}(\delta\eta)_x\right) - \partial_x \left(\frac{\eta_x}{4(1-\eta)^2}\right)\delta\eta + \frac{\eta_x^2}{4(1-\eta)^3}\delta\eta - v\delta v, \right. \quad (27)$$

$$\left. (1-\eta)\delta v - v\delta\eta\right), \quad (28)$$

$$P'(\eta, v) = -(v, \eta), \quad \text{and} \quad P''(\eta, v)(\delta\eta, \delta v) = -(\delta v, \delta\eta). \quad (29)$$

Let τ denote the unitary group of translation on X

$$\tau(s)(\eta, v) = (\eta, v)(x-s), \quad (30)$$

so that

$$\tau'(0) = \begin{pmatrix} -\partial_x & 0 \\ 0 & -\partial_x \end{pmatrix}, \quad \text{and} \quad \tau'(s) = \tau(s)\tau'(0) = \tau'(0)\tau(s),$$

are closed operator in X with domain $D(\tau'(0)) = H^1(\mathbb{R}) \times H^1(\mathbb{R})$. Also, letting

$$B = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \in \mathcal{L}(X; X^*), \quad (31)$$

we get $P((\eta, v)) = \langle B(\eta, v), (\eta, v) \rangle / 2$ and JB is an extension of $\tau'(0)$. We call bound state a solution to (18) of the form

$$(\eta, v)(x, t) = \tau(ct)(\eta_c(x), v_c(x)), \quad \text{for all } (x, t) \in \mathbb{R} \times \mathbb{R}, \quad (32)$$

such that $(\eta_c, v_c) \in X$ are critical points of $E - cP$ i.e.

$$cP'((\eta_c, v_c)) = E'((\eta_c, v_c)).$$

It will also be of importance to perform spectral analysis on the Hessian

$$H_c = E''((\eta_c, v_c)) - cP''((\eta_c, v_c)), \quad (33)$$

in this regard, a demonstration of every abstract result we use can be found either in this textbook by C. Cheverry and N. Raymond [6] or in Appendix B of the book by J. Angulo Pava [1]. Let us recall the three assumptions in [14] that ensure their stability theorem.

Assumption 1 (Existence of solutions). For each $(\eta_0, v_0) \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$, there exist $T > 0$ and a solution $(\eta, v) \in \mathcal{C}((-T, T); H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R}))$ to (18) such that $(\eta, v)(\cdot, 0) = (\eta_0, v_0)$ and the energy and momentum are conserved in time

$$E_\kappa((\eta, v)(\cdot, t)) = E_\kappa((\eta_0, v_0)), \quad \text{and} \quad P((\eta, v)(\cdot, t)) = P((\eta_0, v_0)). \quad (34)$$

If (η_c, v_c) is a bound state, it suffices to have the above for (η_0, v_0) in a neighborhood of (η_c, v_c) in $(H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R}), \|\cdot\|_X)$

Assumption 2 (Existence of bound states). There exist constants $c_1 < c_2$ and a $\mathcal{C}^1((c_1, c_2); X)$ map $c \mapsto (\eta_c, v_c)$ with the following properties.

(i) $E'(\eta_c, v_c) = cP'(\eta_c, v_c)$.

(ii) $(\eta_c, v_c) \in D(\tau'(0)^3) \cap D(J\Lambda\tau'(0)^2)$, where $\Lambda = (1 - \partial_{xx}, 1)$ is the isomorphism between X and X^* .

(iii) $\tau'(0)(\eta_c, v_c) \neq 0$.

Assumption 3 (Spectral decomposition of H_c). For every $c \in (c_1, c_2)$ and H_c given by (33), we assume

(i) There exists $\chi \in X$ such that

$$\langle H_c \chi, \chi \rangle < 0. \quad (35)$$

(ii) There exists a closed subspace $P_c \subset X$, and $C > 0$ such that

$$\langle H_c p, p \rangle \geq C \|p\|^2, \quad \text{for all } p \in P_c. \quad (36)$$

(iii) For every $u \in X$, there exist unique $(a, b) \in \mathbb{R}^2$ and $p \in P_c$ such that

$$u = a\chi + b\tau'(0)(\eta_c, v_c) + p. \quad (37)$$

Remark 2.1. To our knowledge, when $\kappa \neq 0$, there are no result for the IVP associated with (18) with initial data in X , thus *Assumption 1* agrees with our definition of orbital stability with respect to $H^s(\mathbb{R})$ -perturbations (14). Also, it is worth mentioning that *Assumption 3* corresponds *Assumption 3B* in [14] i.e. the decomposition (37) is not necessarily the orthogonal decomposition; this enables them to handle Hamiltonian systems in Banach spaces.

The system (18) can be rewritten as an Euler–Korteweg system for which the local well-posedness was investigated by S. Benzoni-Gavage, R. Danchin, and S. Descombes in [3]. Applying their result to (1) for $\kappa < 1/2$ we deduce the following result (see [12] for details), from which *Assumption 1* follows in a neighborhood of (η_c, v_c) by Lemma 4.1.

Theorem 2.2 (Theorem 5.1 in [3]). *Let $s > 1/2 + 2$, $\kappa < 1/2$ and $c > 0$ and*

$$J_\kappa = \begin{cases} (0, \infty) & \text{if } \kappa \leq 0, \\ (0, 1/\sqrt{2\kappa}), & \text{if } 0 < \kappa < 1/2. \end{cases}$$

If $\Psi_0 \in u_{c,\kappa} + H^s(\mathbb{R})$ satisfies $|\Psi_0(x)| \in J_\kappa$ for every $x \in \mathbb{R}$, then there exists $T_{\Psi_0} > 0$, the maximal time of existence such that for every $T \in (0, T_{\Psi_0})$, (1) has a unique solution Ψ on $\mathbb{R} \times [0, T]$, satisfying $\Psi(\cdot, 0) = \Psi_0$ and

$$\begin{cases} \Psi \in \mathcal{C}([0, T]; u_{c,\kappa} + H^s(\mathbb{R})), \\ |\Psi|(\mathbb{R} \times [0, T]) \subset\subset J_\kappa. \end{cases}$$

Moreover, the flow map is continuous in a neighborhood of Ψ_0 in $u_{c,\kappa} + H^s(\mathbb{R})$, and the energy and momentum are conserved, i.e. (34) holds for all $t \in [0, T]$. Also writing $\Psi = \sqrt{1 - \eta}e^{i\theta}$ we deduce that η and $v = \partial_x \theta$ satisfy (18) with $(\eta, v) \in \mathcal{C}([0, T]; H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R}))$.

For *Assumption 2*, take $c \in (0, \sqrt{2})$, and let u_c be the dark soliton given by Theorem 1.1 so that $\inf_{x \in \mathbb{R}} |u_c| > 0$. Then we can write $u_c = \sqrt{1 - \eta_c}e^{i\theta_c}$ with $(\eta_c, \partial_x \theta_c)$ satisfying (24) pointwisely and *Assumption 2* (1) follows. By the smoothness of $u_c(x)$ with respect to c and the decay estimate (12) and using the identity

$$u'_c = \left(\frac{\eta'_c}{2\sqrt{1 - \eta}} + i\sqrt{1 - \eta}\theta'_c \right) e^{i\theta}, \quad (38)$$

it is easy to check that $c \mapsto (\eta_c, \partial_x \theta_c)$ belongs in $\mathcal{C}^\infty((0, \sqrt{2}), X)$. Once again appealing to the smoothness and decay of u_c , we recover $(\eta_c, \partial_x \theta_c) \in H^\infty(\mathbb{R})^2$ which implies *Assumption 2* (2). Finally, since u_c is nonconstant, *Assumption 2* (3) follows.

To check *Assumption 3*, it remains to compute $\langle H_c(\delta\eta, \delta v), (\delta\eta, \delta v) \rangle$. Using (27)–(29), we get

$$\begin{aligned} \langle H_c(\delta\eta, \delta v), (\delta\eta, \delta v) \rangle &= \langle -\partial_x \left(\frac{1-2\kappa+2\kappa\eta}{2(1-\eta)} \partial_x \right) \delta\eta, \delta\eta \rangle - \langle \partial_x \left(\frac{\eta_x}{2(1-\eta)^2} \right) \delta\eta, \delta\eta \rangle + \int_{\mathbb{R}} \frac{\eta_x^2}{2(1-\eta)^3} \delta\eta^2 + \delta\eta^2 \\ &\quad + \int_{\mathbb{R}} -4v\delta v\delta\eta + 2(1-\eta)\delta v^2 + 2c\delta v\delta\eta, \\ &= \langle -\partial_x \left(\frac{1-2\kappa+2\kappa\eta}{2(1-\eta)} \partial_x \right) \delta\eta, \delta\eta \rangle - \langle \partial_x \left(\frac{\eta_x}{2(1-\eta)^2} \right) \delta\eta, \delta\eta \rangle + \int_{\mathbb{R}} \frac{\eta_x^2}{2(1-\eta)^3} \delta\eta^2 + \delta\eta^2 \\ &\quad - \int_{\mathbb{R}} \frac{(c-2v)^2}{2(1-\eta)} \delta\eta^2 + \int_{\mathbb{R}} 2(1-\eta_c) \left(\delta v + \frac{c-2v}{2(1-\eta)} \delta\eta \right)^2. \end{aligned} \quad (39)$$

Using the first line in (24) and taking $(\eta, v) = (\eta_c, v_c)$ in (39), we can rewrite the last line in (39) to obtain

$$\langle H_c(\delta\eta, \delta v), (\delta\eta, \delta v) \rangle = \langle L_c \delta\eta, \delta\eta \rangle + \int_{\mathbb{R}} 2(1-\eta_c) \left(\delta v + \frac{c}{2(1-\eta_c)^2} \delta\eta \right)^2, \quad (40)$$

where L_c is an self-adjoint unbounded operator on $L^2(\mathbb{R}; \mathbb{R})$ with domain $H^2(\mathbb{R}; \mathbb{R})$ given by

$$L_c = -\partial_x \left(\frac{1-2\kappa+2\kappa\eta_c}{2(1-\eta_c)} \partial_x \right) - \frac{\partial_{xx}\eta_c}{2(1-\eta_c)^2} - \frac{(\partial_x\eta_c)^2}{2(1-\eta_c)^3} + 1 - \frac{c^2}{2(1-\eta_c)^3}. \quad (41)$$

Since L_c is semi-bounded from below, integrating by part the first term in $\langle L_c \delta\eta, \delta\eta \rangle$, we infer that L_c is associated with a quadratic form on $H^1(\mathbb{R}; \mathbb{R})$ by the Lax–Milgram theorem (Theorem 2.89 in [6]).

Since $0 < \eta_c \leq 1 - c^2/2$ and $0 < c < \sqrt{2}$, we have $1 - \eta_c > 0$ and $1 - 2\kappa + 2\kappa\eta_c > 0$ in \mathbb{R} . Then, using ideas similar to the ones in section 6.3.2 of [6] and Lemma 6.13 in [6], we have in $L^2(\mathbb{R}; \mathbb{R})$

$$\sigma_{\text{ess}} \left(-\partial_x \left(\frac{1-2\kappa+2\kappa\eta_c}{2(1-\eta_c)} \partial_x \right) \right) = [0, \infty). \quad (42)$$

We infer that the other terms in $L_c - \lambda$ are compact operators for some $\lambda > 0$. Indeed, notice that

$$\lim_{|x| \rightarrow \infty} -\frac{\partial_{xx}\eta_c(x)}{2(1-\eta_c(x))^2} - \frac{(\partial_x\eta_c(x))^2}{2(1-\eta_c(x))^3} + 1 - \frac{c^2}{2(1-\eta_c(x))^3} = 1 - \frac{c^2}{2} > 0. \quad (43)$$

Lemma 2.3. *Let $0 < c < \sqrt{2}$ and fix $\lambda = 1 - c^2/2 > 0$. Let also $T_c \in \mathcal{L}(H^2(\mathbb{R}; \mathbb{R}), L^2(\mathbb{R}; \mathbb{R}))$ be given by*

$$T_c = -\frac{\partial_{xx}\eta_c}{2(1-\eta_c)^2} - \frac{(\partial_x\eta_c)^2}{2(1-\eta_c)^3} + 1 - \frac{c^2}{2(1-\eta_c)^3} \quad (44)$$

then $T_c - \lambda$ is compact i.e. for every sequence $(u_n) \subset H^2(\mathbb{R})$ with $u_n \rightharpoonup 0$ we can extract a subsequence such that $(T_c - \lambda)(u_{n_k}) \rightarrow 0$ in $L^2(\mathbb{R})$.

Proof. Let $\varepsilon > 0$ and $R = R_\varepsilon > 0$. by Kato–Rellich theorem (Theorem 4.20 in [6]) there exists $v \in H^1((-R, R))$ and a subsequence $u_{n_k} \rightarrow v$ in $H^1((-R, R))$. Of course $v = 0$ by the uniqueness of the weak limit. Then denoting $I_R = \mathbb{R} \setminus (-R, R)$ we split $\|(T_c - \lambda)(u_{n_k})\|_{L^2(\mathbb{R})}^2$ as

$$\|(T_c - \lambda)(u_{n_k})\|_{L^2(\mathbb{R})}^2 = \|(T_c - \lambda)(u_{n_k})\|_{L^2(I_R)}^2 + \|(T_c - \lambda)(u_{n_k})\|_{L^2((-R, R))}^2. \quad (45)$$

We deduce, using (43) that taking $R > 0$ big enough yield

$$\|T_c - \lambda\|_{\mathcal{L}(H^2(I_R); L^2(I_R))}^2 \leq \varepsilon / (2 \sup_{k \in \mathbb{N}} \|u_{n_k}\|_{H^2(\mathbb{R})}^2),$$

letting us bound the first term in (45) by $\varepsilon/2$. Once R is fixed, we take k big enough so that considering $T_c - \lambda$ as an operator in $\mathcal{L}(H^2((-R, R)); L^2((-R, R)))$, we have $\|(T_c - \lambda)(u_{n_k})\|_{L^2((-R, R))}^2 \leq \varepsilon/2$. This shows that $(T_c - \lambda)(u_{n_k}) \rightarrow 0$ in $L^2(\mathbb{R})$ ending the proof. \square

Lemma 2.3, enables us to compute $\sigma_{\text{ess}}(L_c) = \sigma_{\text{ess}}(\partial_x((1 - 2\kappa + 2\kappa\eta_c)/(2 - 2\eta_c)\partial_x) + \lambda + T_c - \lambda)$ which yields, using a generalized version of the Weyl essential spectrum theorem (see Theorem B.48 in [1])

$$\sigma_{\text{ess}}(L_c) = [\lambda, \infty) = [1 - c^2/2, \infty). \quad (46)$$

The operator L_c being self-adjoint, the rest of its spectrum $\sigma_{\text{fred}}(L_c) = \sigma(L_c) \setminus \sigma_{\text{ess}}(L_c)$ lies in $(-\infty, 1 - c^2/2)$ and every $\mu \in \sigma_{\text{fred}}(L_c)$ is an isolated eigenvalue of finite multiplicity (see Section 6.3 in [6]). Using the first line in (24) we can rewrite second equation in (24) only in terms of η_c

$$-\left(\frac{1 - 2\kappa + 2\kappa\eta_c}{2(1 - \eta_c)}\partial_x\partial_x\eta_c\right) - \frac{(\partial_x\eta_c)^2}{4(1 - \eta)^2} + \eta_c - c^2\frac{\eta^2 - 2\eta}{4(1 - \eta)^2}, \quad \text{in } \mathbb{R}. \quad (47)$$

Differentiating (47) with respect to x yields $\partial_x\eta_c \in \ker(L_c)$. Since $|u_c|' > 0$ in $(0, \infty)$ and $|u_c|$ is even, we deduce that $\partial_x\eta_c = \partial_x(1 - |u_c|^2)$ vanishes once at $x = 0$, so that, by the Sturm-Liouville oscillation theorem (see Appendix B.5 in [1]), there exists one and only one negative eigenvalue $\mu_- < 0$ of multiplicity one with eigenvector $\chi_1 \in H^2(\mathbb{R})$. The eigenvalues μ_- and 0 being simple and isolated, by the min-max principle (see Section 6.4 in [6]), we infer that there exists $\mu_1 > 0$ such that for every $z \in H^1(\mathbb{R})$ satisfying

$$(z, \chi_1)_{L^2(\mathbb{R})} = (z, \partial_x\eta_c)_{L^2(\mathbb{R})} = 0, \quad (48)$$

we have $\langle L_c z, z \rangle \geq \mu_1 \|z\|_{L^2(\mathbb{R})}^2$. It is also clear that L_c satisfies some kind of Gårding inequality so that proceeding along the same lines as in Lemma 4.1 in [21] yields the existence of $\mu > 0$ such that $\langle L_c z, z \rangle \geq \mu \|z\|_{H^1(\mathbb{R})}^2$. Setting

$$\chi_- = \left(\chi_1, -\frac{c}{2(1 - \eta)^2}\chi_1\right), \quad (49)$$

and using (40), we get $\langle H_c \chi_-, \chi_- \rangle = \langle L_c \chi_1, \chi_1 \rangle < 0$. Letting

$$P_c = \{(p_1, p_2) \in X : (p_1, \chi_1)_{L^2(\mathbb{R})} = (p_1, \partial_x\eta_c)_{L^2(\mathbb{R})} = 0\},$$

and considering the $L^2(\mathbb{R})$ -orthogonal decomposition of $\eta = a\chi_1 + b\partial_x\eta + p_1$, we perform the following decomposition of $(\eta, v) \in X$: $(\eta, v) = a\chi_- + b\partial_x(\eta_c, v_c) + (p_1, p_2)$, with $p_2 = v - ac\chi_1/(2 - 2\eta) - b\partial_x v_c$. The uniqueness of the decomposition of (η, v) follows from the uniqueness of the orthogonal decomposition of η . Using Lemma 4.2 in [21] (see also [15]) we deduce that there exists $C > 0$ satisfying $\langle H_c p, p \rangle > C\|p\|_X^2$ for all $p \in P_c$ so that *Assumption 3* follows.

Let $d(c) = E((\eta_c, v_c)) - cP((\eta_c, v_c))$ since $c \mapsto (\eta_c, v_c)$ in $\mathcal{C}^1((0, \sqrt{2}))$ we can compute

$$d''(c) = (\langle E'((\eta_c, v_c)) - cP'((\eta_c, v_c)), \partial_c(\eta_c, v_c) \rangle - P((\eta_c, v_c)))' = -\frac{d}{dc}P((\eta_c, v_c)). \quad (50)$$

Assuming (16), we get $d''(c) > 0$ and we can apply Theorem 3 in [14]. The only remaining thing to check is that for any sequence $(\eta^n, v^n) = (\eta, v)_n \subset X$ with $E((\eta, v)_n) \rightarrow E((\eta_c, v_c))$ and $P((\eta, v)_n) \rightarrow P((\eta_c, v_c))$, we can construct $(u_n) \subset X$ such that, up to extraction, we have $\|u_n - (\eta, v)_n\|_X \rightarrow 0$ and $P(u_n) = P((\eta_c, v_c))$. By the first line in (24), we have $P((\eta_c, v_c)) > 0$ for all $c \in (0, \sqrt{2})$, thus, up to a subsequence extraction, we assume $P((\eta, v)_n) > 0$ for all $n \in \mathbb{N}$. Hence, $\alpha_n = P((\eta_c, v_c))/P((\eta, v)_n)$ is well defined with $\alpha_n \rightarrow 1$ so that $u_n = (\eta_n, \alpha_n v_n)$

satisfies $P(u_n) = P((\eta_c, v_c))$ and we can check $\|u_n - (\eta_n, v_n)\|_X \rightarrow 0$. For $\psi = \sqrt{1 - \eta}e^{i\theta}$ with $\inf_{\mathbb{R}} |\psi| > 0$, notice that we can always choose $\varphi \in \mathbb{R}$ so that $\text{Arg}(\psi(0)/u_c(0)) = 0$ thus

$$\inf_{\varphi \in \mathbb{R}} d_{\text{hy}}(\psi, u_c e^{i\varphi}) = \|(\eta, v) - (\eta_c, v_c)\|_X,$$

so that the stability of u_c for d_{hy} follows from the orbital stability of the bound state (η_c, v_c) in X .

Theorem 2.4 (Theorem 3 in [14]). *If Assumptions 1–3 hold, then the following statement is satisfied. If $d''(c) > 0$ then $E(\mathbf{u})$ is minimized at $u = (\eta_c, v_c)$ for \mathbf{u} in a neighborhood of (η_c, v_c) in X with $P(\mathbf{u}) = P((\eta_c, v_c))$. Moreover the dark soliton u_c is orbitally stable for d_{hy} .*

Remark 2.5. Theorem 3 in [14] is stated as an equivalence between the variation of $d(\cdot)$ and the stability. However, we cannot apply the converse statement since in our case the skew-symmetric operator J in (19) is not onto. For instance $(\eta_c, v_c) \notin \text{Range}(J)$ because $\eta_c > 0$ in \mathbb{R} and hence $C + \int_{-\infty}^x \eta_c \notin H^{-1}(\mathbb{R})$ for every $C \in \mathbb{R}$.

3 Instability

Regarding the instability, arguments similar to the ones used by Lin in [21] yield the result. We explain how to modify the first steps that consist of creating an energy-decreasing direction in X while referring to [21] for detailed demonstrations. From now on, we assume that $c \in (0, \sqrt{2})$ is such that (17) holds and we denote by $(\eta_c, v_c) \in X$ the dark soliton satisfying Assumption 2 we also fix $s > 2 + 1/2$ the regularity of the perturbations in Definition 14 (ii). Firstly, we have $\chi_1 \in H^\infty(\mathbb{R})$ by a bootstrap argument since it satisfies $\chi_1 \in H^2(\mathbb{R})$ and $L_c(\chi_1) = \mu_- \chi_1$, with L_c given by (41). Thus as in [21], we can assume that $\chi_- \in X_s \subset X$ with

$$X_s = (H^s(\mathbb{R}) \times H^s(\mathbb{R})) \cap (L^1(\mathbb{R}, (1 + |x|)dx) \times L^1(\mathbb{R}, (1 + |x|)dx)), \quad (51)$$

where χ_- is the negative direction of H_c given by (49). We also get $(\eta_c, v_c) \in X_s$ using the smoothness and rapid decay of the dark solitons established in Theorem 1.1. Using the implicit function theorem on $(q, l) \mapsto (\eta_q, v_q) + l\chi_-$, we can find a \mathcal{C}^1 -map, $l(q)$ and a curve $\Psi(q)$ defined in the neighborhood of c with

$$\Psi(q) = (\eta_q, v_q) + l(q)\chi_-, \quad (52)$$

satisfies $P(\Psi(q)) \equiv P((\eta_c, v_c))$. This allows us to build a negative direction for the Hessian H_c given by (33) with the following properties.

Lemma 3.1 (Corollary 5.1 in [21]). *There exists $y_0 \in X_s$ such that $\langle By_0, (\eta_c, v_c) \rangle$ and $\langle H_c y_0, y_0 \rangle < 0$, where B is given by (31).*

We want to build a negative direction as in Lemma 3.1 for which the number of particles (22) and twisting angle (23) are zero. We use the following.

Lemma 3.2. *For all $\eta \in H^2(\mathbb{R})$ and $\nu \in \mathbb{R}$, there exists a sequence $(u_n) \subset X_s$ such that $\int_{\mathbb{R}} u_n = \nu$, $u_n \rightarrow 0$ in $H^1(\mathbb{R})$ and $\langle u_n, r \rangle_X = 0$.*

Proof. The proof proceeds as in [21]; let $\varphi \in C_c^\infty(\mathbb{R})$ with $\int_{\mathbb{R}} \varphi = \nu$. First if $\eta \equiv 0$, then taking $u_n(\cdot) = \varphi(\cdot/n)/n$ for any φ defined above suffices. Assuming that $\eta \neq 0$, then by a duality and density argument there exists $\psi \in C_c^\infty(\mathbb{R})$ such that $\langle \eta_x, \psi \rangle_{H^1(\mathbb{R})} \neq 0$. Indeed, otherwise we would have $\eta_x = 0$ but since $\eta \in L^2(\mathbb{R})$ this contradicts $\eta \neq 0$. Letting $u_n(\cdot) = \varphi(\cdot/n)/n + \alpha_n \psi_x$, we readily get $u_n \in X_s$ and $\int_{\mathbb{R}} u_n = \nu$. Then, setting $\alpha_n = \langle \varphi(\cdot/n)/n, \eta \rangle_{H^1(\mathbb{R})} / \langle \eta_x, \psi \rangle_{H^1(\mathbb{R})}$, we obtain $\langle \eta, u_n \rangle_{H^1(\mathbb{R})} = 0$. Finally, we deduce after rescaling that $\alpha_n \rightarrow 0$ using $|\alpha_n| |\langle \eta_x, \psi \rangle_{H^1(\mathbb{R})}| \leq \|\eta\|_{H^1(\mathbb{R})} \|\varphi(\cdot/n)/n\|_{H^1(\mathbb{R})}$ and also $u_n \rightarrow 0$ in $H^1(\mathbb{R})$. \square

Since $s > 2 + 1/2$ we can apply Lemma 3.2 to each coordinate of $y_0 = (\eta_0, v_0)$ where $y_0 \in X_s$ is given by Lemma 3.1.

Lemma 3.3 (Lemma 5.3 in [21]). *For all $s > 2 + 1/2$, there exists $y = (\eta_1, v_1) \in X_s$ such that $\langle By, (\eta_c, v_c) \rangle = 0$, $\langle H_c y, y \rangle < 0$ and*

$$\int_{\mathbb{R}} \eta_1 = \int_{\mathbb{R}} v_1 = 0.$$

Following along Section 5 in [21] we can deduce that (η_c, v_c) is not a minimizer of the energy at fixed momentum and that it is unstable. More precisely, let $(\eta, v) = \Psi(q)$ where $\Psi(\cdot)$ is given by (52) and q is in the vicinity of c . Taking $\psi = \sqrt{1 - \eta} e^{i\theta}$ with $\theta(x) = \theta_c(0) + \int_0^x v$, we can prove that ψ is arbitrarily close to $u_{c,\kappa}$ for d_{hy} as $q \rightarrow c$ with $q \neq c$. This implies that eventually as $q \rightarrow c$ we have $|\psi|(\mathbb{R}) \subset\subset J_\kappa$ and $\psi \in u_c + H^s$ so that there exists a solution to (1) starting at ψ at $t = 0$. The instability of u_c for d_{hy} follows as in [21].

Theorem 3.4 (Theorem 5.1 in [21]). *Let $c \in (0, \sqrt{2})$, $s > 2 + 1/2$ and assume that $(\eta_c, v_c) \in X_s$ is the dark soliton. If $\partial_c P((\eta_c, v_c)) > 0$, then the curve $\Psi(\cdot)$ given by (52) defined in the vicinity of c satisfies $P(\Psi(\cdot)) \equiv P((\eta_c, v_c))$ and $E(\Psi(q)) < E((\eta_c, v_c))$, when $q \neq c$. Moreover the dark soliton u_c is unstable for d_{hy} .*

4 Stability in the energy distance $d_{\mathcal{X}}$

Let $\kappa < 1/2$ and $c \in (0, \sqrt{2})$, we first show that the stability and instability results stated in terms of d_{hy} given by (11) in Theorem 2.4 and in Theorem 3.4 respectively, are equivalent to the stability and instability result in Theorem 1.4 in terms of the distance $d_{\mathcal{X}}$ given by (10) respectively; thus proving Theorem 1.4. For deeper insights on this equivalence we refer to Lemma 9–10 in [8]).

Lemma 4.1. *Let $\kappa < 1/2$, $c \in (0, \sqrt{2})$ and $u_{c,\kappa}$ be a dark soliton given by Theorem 1.1. Then there exists K and $\varepsilon > 0$ depending only on $u_{c,\kappa}$ such that for every $\Psi \in \mathcal{X}(\mathbb{R})$, if Ψ satisfies $d_{\mathcal{X}}(\Psi, u_{c,\kappa}) < \varepsilon$, then we can write $\Psi = \sqrt{1 - \eta} e^{i\theta}$ with $(\eta, \partial_x \theta) \in X$ and we have*

$$\frac{1}{K} d_{\text{hy}}(\Psi, u_{c,\kappa}) \leq d_{\mathcal{X}}(\Psi, u_{c,\kappa}) \leq K d_{\text{hy}}(\Psi, u_{c,\kappa}). \quad (53)$$

Proof. For clarity's sake, we recall the main arguments of the proof in [9]. We remove κ in the subscript of $u_{c,\kappa} = u_c$ and write $u_c = \sqrt{1 - \eta_c} e^{i\theta_c}$. Since u_c does not vanish, taking $\varepsilon > 0$ small enough then, for any $\Psi \in \mathcal{X}(\mathbb{R})$ satisfying $d_{\mathcal{X}}(\Psi, u_c) \leq \varepsilon$, we get $\|\Psi - |u_c|\|_{L^\infty(\mathbb{R})} \leq \inf_{x \in \mathbb{R}} |u_c(x)|/2$ so that

$$\inf_{x \in \mathbb{R}} |\Psi(x)| \geq \inf_{x \in \mathbb{R}} |u_c(x)|/2. \quad (54)$$

We can express $\partial_x \Psi$ in terms of η and θ ,

$$\partial_x \Psi = e^{i\theta} \left(\frac{\eta_x}{2\sqrt{1 - \eta}} + i\sqrt{1 - \eta} \theta_x \right), \quad (55)$$

and similarly for u_c . We also need to bound $\|\Psi\|_{L^\infty(\mathbb{R})}$ in terms of ε , $\|u_c\|_{L^\infty(\mathbb{R})}$ and $\|\partial_x u_c\|_{L^2(\mathbb{R})}$. To do so, notice that for any constants $y < x$, we have

$$|\Psi(x)| \leq |\Psi(y)| + \int_y^x |\partial_x \Psi| \leq |u_c(y)| + \|\Psi(y) - |u_c(y)|\| + \sqrt{x - y} \|\partial_x \Psi\|_{L^2(\mathbb{R})}, \quad (56)$$

where we used Cauchy–Schwarz inequality to bound the integral. Taking $0 < x - y < 1$ then integrating (56) with respect to y from $x-1$ to x we get $|\Psi(x)| \leq |u_c| + \int_{\mathbb{R}} |u_c| - |\Psi| + \|\partial_x \Psi\|_{L^2(\mathbb{R})}$, so that

$$\|\Psi\|_{L^\infty(\mathbb{R})} \leq K_0(u_c, \varepsilon). \quad (57)$$

We show that $d_{\text{hy}}(\Psi, u_c) \leq K d_{\mathcal{X}}(\Psi, u_c)$. First notice that $|\Psi|^2 \theta_x = \Im(\Psi_x \bar{\Psi})$, thus

$$\partial_x \theta - \partial_x \theta_c = \Im(\partial_x u_c \bar{u}_c) \left(\frac{1}{|\Psi|^2} - \frac{1}{|u_c|^2} \right) + \frac{1}{|\Psi|^2} (\Im((\partial_x \Psi - \partial_x u_c) \bar{\Psi}) - \Im(\partial_x u_c (\bar{u}_c - \bar{\Psi}))),$$

hence

$$\begin{aligned} \|\partial_x \theta - \partial_x \theta_c\|_{L^2(\mathbb{R})} &\leq \frac{\Im(\partial_x u_c \bar{u}_c) (\|\Psi\|_{L^\infty(\mathbb{R})} + \|u_c\|_{L^\infty(\mathbb{R})})}{\inf_{\mathbb{R}} |\Psi|^2 \inf_{\mathbb{R}} |u_c|^2} \|\Psi - |u_c|\|_{L^2(\mathbb{R})} \\ &\quad + \frac{1}{\inf_{\mathbb{R}} |\Psi|^2} (\|\Psi\|_{L^\infty(\mathbb{R})} \|\partial_x \Psi - \partial_x u_c\|_{L^2(\mathbb{R})} + \|\partial_x u_c (\bar{u}_c - \bar{\Psi})\|_{L^2(\mathbb{R})}). \end{aligned} \quad (58)$$

Since $\Psi(x) - u_c(x) = \Psi(0) - u_c(0) + \int_0^x \partial_x \Psi - \partial_x u_c$ for all $x \in \mathbb{R}$, appealing the Cauchy–Schwarz inequality, we get

$$|\partial_x u_c(x)| |\Psi(x) - u_c(x)| \leq |\partial_x u_c(x)| |\Psi(0) - u_c(0)| + |\partial_x u_c(x)| \sqrt{|x|} \|\partial_x \Psi - \partial_x u_c\|_{L^2(\mathbb{R})}, \quad \text{for all } x \in \mathbb{R}. \quad (59)$$

From the decay estimate (12) satisfied by $\partial_x u_c$, we can bound the last term in (58), therefore using also (54) and (57), we deduce that there exists $K_1 = K_1(\varepsilon, u_c)$ such that

$$\|\partial_x \theta - \partial_x \theta_c\| \leq K_1 d_{\text{hy}}(\Psi, u_c). \quad (60)$$

Similarly, we have $2\Re(\Psi_x \bar{\Psi}) = \eta_x$, thus

$$\begin{aligned} \|\partial_x \eta - \partial_x \eta_c\|_{L^2(\mathbb{R})} &\leq 2\|\Psi\|_{L^\infty(\mathbb{R})} \|\partial_x \Psi - \partial_x u_c\|_{L^2(\mathbb{R})} + 2\|\partial_x u_c (\bar{u}_c - \bar{\Psi})\|_{L^2(\mathbb{R})}, \\ &\leq 2K_0 d_{\text{hy}}(\Psi, u_c) + 2\|\partial_x u_c\|_{L^2(\mathbb{R})} \|\Psi(0) - |u_c(0)|\| + \|\sqrt{x} \partial_x u_c\|_{L^2(\mathbb{R})} \|\partial_x \Psi - \partial_x u_c\|_{L^2(\mathbb{R})}, \\ &\leq K_2 d_{\text{hy}}(\Psi, u_c). \end{aligned} \quad (61)$$

Then, we also have $\|\eta - \eta_c\|_{L^2(\mathbb{R})} = \|(|\Psi| + |u_c|)(|\Psi| - |u_c|)\|_{L^2(\mathbb{R})} \leq (\|\Psi\|_{L^\infty(\mathbb{R})} + \|u_c\|_{L^\infty(\mathbb{R})}) \|\Psi - |u_c|\|_{L^2(\mathbb{R})}$. We bound the last term $|\text{Arg}(\Psi(0)/u_c(0))|$. Letting Log be the principal determination of the logarithm defined in $\mathbb{C} \setminus (-\infty, 0]$ and taking $\varepsilon > 0$ small enough, we get

$$|\text{Arg}(\Psi(0)/u_c(0))| = |\Im(\text{Log}(\Psi(0)/u_c(0)))| \leq \sup_{|z-1| \leq \varepsilon/u_c(0)} |1/z| \left| \frac{\Psi(0) - u_c(0)}{u_c(0)} \right|,$$

which ends to prove that $d_{\text{hy}}(\Psi, u_c) \leq K d_{\mathcal{X}}(\Psi, u_c)$. Ideas along the same lines as in Lemma 10 in [8] show that $d_{\mathcal{X}}(\Psi, u_c) \leq K d_{\text{hy}}(\Psi, u_c)$. \square

To prove Corollary 1.5, we shall recall some properties of the function

$$P_\kappa(c) = P(u_{c,\kappa}), \quad \text{for all } c \in (0, \sqrt{2}). \quad (62)$$

Lemma 4.2 (Proposition 5.6 and Lemma 5.7 in [12]). *There exists $\kappa_0 < 0$ with $\kappa_0 \approx -3.636$ such that the two following statements hold.*

- (i) *If $\kappa \in [\kappa_0, 1/2)$, then $P'_\kappa < 0$ in $(0, \sqrt{2})$.*
- (ii) *If $\kappa < \kappa_0$, then there exists \tilde{c}_κ such that $P'_\kappa > 0$ in $(0, \tilde{c}_\kappa)$ while $P'_\kappa < 0$ in $(\tilde{c}_\kappa, \sqrt{2})$.*

A Extensions of the stability result

Finally, we briefly describe the Lyapunov functional approach to the stability of the dark solitons (see [8]). For the sake of conciseness, we chose not to use this method in this work to avoid tedious computations. As pointed out by C. A. Stuart in Section 4–5 in [22], in the Hamiltonian system framework of [14], the stability criterion (16) together with *Assumptions 1–3* allow us to construct a Lyapunov function invariant by translation by taking K large enough in

$$V(\eta, v) = E(\eta, v) - cP(\eta, v) - E(u_c) + cP(u_c) + K(P(\eta, v) - P(u_c))^2, \quad (63)$$

giving a quantitative measure of stability. The heuristic is the following, since V is constant on the flow of solutions, to ensure the stability, we want V to satisfy, for some $\alpha > 0$

$$V(\eta, v) \geq \alpha \|(\eta, v) - (\eta_c, v_c)\|_X^2,$$

for some $\varepsilon > 0$ and all η, v satisfying $\inf_{s \in \mathbb{R}} \|(\eta, v) - (\eta_c, v_c)(\cdot - s)\|_X < \varepsilon$. This problem reduces to the positive definiteness of $V''(\eta_c, v_c)$ which writes

$$\langle V''(\eta_c, v_c)(\delta\eta, \delta v), (\delta\eta, \delta v) \rangle = \langle H_c(\delta\eta, \delta v), (\delta\eta, \delta v) \rangle + 2K(\langle P'(u_c), (\delta\eta, \delta v) \rangle)^2 \geq C \|(\delta\eta, \delta v)\|_X^2,$$

for some $C > 0$. We readily compute $-P'(u_c) = \partial_c(E' - cP')(u_c)$ and since $E'(u_c) - cP'(u_c) = 0$ for every $c \in (0, \sqrt{2})$, we get $\partial_c(E'(u_c) - cP'(u_c)) = H_c \partial_c u_c - P'(u_c) = 0$ by the chain rule. Testing this linear form against $\partial_c(\eta_c, v_c) \in X$, if (16) holds, we deduce that,

$$\langle H_c \partial_c(\eta_c, v_c), \partial_c(\eta_c, v_c) \rangle = \frac{d}{dc} P(\eta_c, v_c) < 0. \quad (64)$$

By Lemma 5.3 in [22], considering *Assumption 3*, we can assume the decomposition to be the orthogonal decomposition

$$u = a \partial_c(\eta_c, v_c) + b \tau'(0)(\eta_c, v_c) + p, \quad \text{for all } u \in X, \quad (65)$$

where $P = \{\partial_c(\eta_c, v_c), \tau'(0)(\eta_c, v_c)\}^\perp$. Then we can show that taking K big enough in (63) yield the coercivity of the quadratic form V'' on $\text{span}\{\partial_c(\eta_c, v_c)\}$ which upon further computations leads to (see Lemma 4.3 in [22])

$$\langle V''(\eta_c, v_c)(\delta\eta, \delta v), (\delta\eta, \delta v) \rangle \geq C \|\Pi(\delta\eta, \delta v)\|_X^2, \quad (66)$$

for some $C > 0$ and where Π is the orthogonal projection on $\{\tau'(0)(\eta_c, v_c)\}^\perp$. Finally if $(\delta\eta, \delta v) = (\eta - \eta_c, v - v_c)$ then we can show that (see Lemma 2.1 in [22])

$$\|\Pi(\delta\eta, \delta v)\| = \inf_{s \in \mathbb{R}} \|(\eta, v) - (\eta_c, v_c)(\cdot - s)\|_X. \quad (67)$$

For any solution (η, v) to (18), one can readily bound from above $V((\eta, v))(t)$ for all t by the quantity (67) at $t = 0$. Therefore a solution starting close to the orbit of traveling waves cannot leave it, ensuring stability. This approach is implemented for the cubic-quintic nonlinear Schrödinger equation with nonzero conditions at infinity by I. V. Barashenkov in [2].

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