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# Spectral correctness of the continuous finite element stabilized approximation of the first-order form of Maxwell's equations\*

Alexandre Ern<sup>†</sup>      Jean-Luc Guermond<sup>‡</sup>

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## Abstract

Maxwell's equations in first-order form with smooth permeability and smooth permittivity are approximated using continuous finite elements. The method is stabilized using the continuous interior penalty technique. Using an existence result of approximation operators commuting with the curl operator established in Boffi et al. [5], it is proved herein that the approximation is spectrally correct. The result is numerically illustrated.

**Keywords.** Curl-curl problem, duality argument, involution, spectral approximation, finite elements, Maxwell's equations

**MSC.** 65N25, 35Q61, 65N15, 65N30, 35L05, 35P15

## 1 Introduction

The goal of the paper is to study the spectral correctness of the approximation of Maxwell's eigenvalue problem in first-order form using continuous finite elements. Assuming that the material properties are smooth, we prove spectral correctness under two conditions: (i) the finite element approximation is properly stabilized; (ii) the underlying triangulation has a two-scale structure obtained by suitably splitting an initial triangulation (see below for more details). We emphasize that our long-term goal is not to approximate the spectrum of the Maxwell operator per se. The work presented in the paper is part of a larger research project on the approximation of nonlinear conservation equations, such as the Euler–Maxwell equations or the equations of ideal magnetohydrodynamics. Our goal is to construct methods that are both invariant-domain and involution preserving. In this context, the involutions are that the magnetic field and the electric field remain in the image of the curl operator (in the absence of free charges). At the continuous level, involutions yield compactness of the solution operator (see, e.g., Boillat [6], Dafermos [17]). Preserving the involutions at the discrete level is key to establishing that the approximation remains

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well-behaved over long simulation times. It is for all these reasons that we consider the first-order formulation of the Maxwell eigenvalue problem.

Let  $D \subset \mathbb{R}^d$ ,  $d \in \{2, 3\}$ , be an open, bounded, connected polyhedron with Lipschitz boundary. The Maxwell eigenvalue problem in first-order form consists of seeking a complex number  $\vartheta \in \mathbb{C}$  and nonzero vector fields  $(\mathbf{H}, \mathbf{E}) : D \rightarrow \mathbb{R}^d \times \mathbb{R}^d$  such that

$$\nabla \times \mathbf{H} = \vartheta \epsilon \mathbf{E}, \quad -\nabla \times \mathbf{E} = \vartheta \mu \mathbf{H}, \quad \mathbf{H}|_{\partial D} \times \mathbf{n}_D = \mathbf{0}, \quad (1)$$

where  $\mathbf{n}_D$  is the unit outward normal to the boundary  $\partial D$  of  $D$ . As mentioned above, the material properties  $\mu$  and  $\epsilon$  are assumed to be smooth. The domain  $D$  can have a general topology; in particular,  $D$  can be multiply connected and  $\partial D$  can have several connected components. We are only interested in nonzero eigenvalues, i.e.,  $\vartheta \neq 0$ . This implies that  $\mu \mathbf{H}$  is in the image of the curl operator and  $\epsilon \mathbf{E}$  is in the image of the curl operator acting on vector fields with zero tangential trace at the boundary. Using the terminology employed in [6, 17], we refer to these properties as involutions. In addition to implying Gauss's laws, the involutions imply that the solutions to (1) have some smoothness. For instance it is well established in the literature that there exists  $s > \frac{1}{2}$  so that the solutions to (1) belong to the Sobolev space  $\mathbf{H}^s(D) \times \mathbf{H}^s(D)$  when the material properties are smooth. As the material properties are smooth, the fields  $\mu \mathbf{H}$  and  $\epsilon \mathbf{E}$  are loosely speaking in the same smoothness class as  $\mathbf{H}$  and  $\mathbf{E}$ , respectively, and it is therefore legitimate to consider continuous finite elements for the approximation.

In the paper, we establish that the Galerkin approximation of the first-order system (1) using stabilized continuous finite elements maintains a discrete version of the involutions that is strong enough to imply that the approximation of (1) is spectrally correct, i.e., the approximation is free of spurious eigenvalues. The stabilization hinges on the continuous interior penalty (CIP) technique (a.k.a. edge stabilization) from Douglas and Dupont [21] and Burman and Hansbo [13]. The proof of spectral correctness extends to CIP-stabilized continuous finite elements the arguments introduced in [27, 28] for discontinuous Galerkin methods. In particular, the convergence of the approximation in the operator norm is established by proving a deflated inf-sup condition and invoking a duality argument. One important assumption required in the context of continuous finite elements is that the meshes have some special structure. For instance, one can use piecewise linear Lagrange elements on Powell–Sabin splits and quadratic Lagrange elements on Alfeld (or Clough–Tocher) splits in two space dimensions. This allows us to invoke the commuting interpolation operator devised in Boffi et al. [5] using previous results from Fu et al. [29], Guzmán et al. [31].

To our knowledge, the approximation of the Maxwell eigenvalue problem in first-order form using stabilized continuous finite elements and involving no mixed formulation has not yet been explored in the literature. In contrast, the eigenvalue problem in second-order form has received considerable attention. For instance, methods using weighted grad-div regularization on general triangulations have been proposed in Costabel and Dauge [16]; see also Bonito and Guermond [7]. Methods using a mixed formulation (including a Lagrange multiplier for the divergence constraint) have been proposed in Buffa et al. [11], Du and Duan [22], Duan et al. [23], and [7] on general triangulations. A method based on the first-order form of the Maxwell's equations using a mixed formulation without regularization, but on specific mesh structures, has been proposed in Boffi et al. [5]. The method proposed in the paper uses the same mesh structures as in [5].

The paper is organized as follows. In Section 2, we briefly recall from [27, 28] the functional setting that is used to formulate the eigenvalue problem (1). In Section 3, we present the discrete setting for CIP-stabilized continuous finite elements. In Section 4, we state our key assumption (see Assumption 4.1) on the existence of a commuting quasi-interpolation operator, that is known to hold true whenever the mesh has some special structure. Moreover, we establish a discrete Poincaré–Steklov inequality (see Lemma 4.3) and a consistency bound on the CIP stabilization (see Lemma 4.4). In Section 5, we prove our main result, Theorem 5.7. Finally, we present

numerical results illustrating the theory in Section 6. The present results can be extended to other types of stabilization, like Galerkin Least-Squares (GaLS), Local Projection Stabilization (LPS), Orthogonal Subscale Stabilization (OSS), and Subgrid Viscosity (SGV), among others, since the CIP-stabilization is only used to achieve some weak control on the curl in the deflated inf-sup condition (see Lemma 5.1). These extensions can be done, for instance, by invoking the techniques described in Chapters 57-59 from [26]. We omit the details for brevity.

## 2 Continuous setting

In this section, we briefly recall from [27, 28] the functional setting that is used to formulate the eigenvalue problem (1) and the corresponding boundary-value problem. Most of the results can be found in the literature; see e.g., Amrouche et al. [2], Birman and Solomyak [3], Bramble and Pasciak [10], Costabel [15], Dautray and Lions [19], Girault and Raviart [30], Hiptmair [32].

### 2.1 Material properties and reference quantities

The magnetic permeability,  $\mu$ , and the electric permittivity,  $\epsilon$ , take values in  $\mathbb{R}^{d \times d}$  and are assumed to be in  $W^{1,\infty}(D; \mathbb{R}^{d \times d})$ . The magnetic permeability and electric permittivity of vacuum are denoted  $\mu_0$  and  $\epsilon_0$ , respectively. To be dimensionally consistent, we introduce a length scale,  $\ell_D$ , associated with  $D$ . For instance, this length scale can be the diameter of  $D$ . Recalling that  $c := (\mu_0 \epsilon_0)^{-\frac{1}{2}}$  is the speed of light, we introduce the quantity  $\omega := c \ell_D^{-1}$  which scales as the reciprocal of a time scale.

In the paper, for any positive real numbers  $A$  and  $B$ , we write  $A \lesssim B$  to mean that there exists a non-dimensional constant  $C$  so that  $A \leq CB$ . The value of  $C$  may change at each occurrence provided it is independent of the parameters  $\mu_0$ ,  $\epsilon_0$ ,  $\ell_D$  and any fields involved in the inequality. For simplicity, the value of  $C$  can depend on  $D$ , the ratios  $\operatorname{ess\,inf} \frac{\|\mu\|}{\mu_0}$ ,  $\operatorname{ess\,sup} \frac{\|\mu\|}{\mu_0}$ ,  $\operatorname{ess\,inf} \frac{\|\epsilon\|}{\epsilon_0}$ , and  $\operatorname{ess\,sup} \frac{\|\epsilon\|}{\epsilon_0}$ . Moreover, in arguments involving the mesh family  $(\mathcal{T}_h)_{h \in \mathcal{H}}$ , the value of  $C$  is independent of the mesh size  $h$ , but can depend on the shape-regularity parameter of the mesh sequence and on the polynomial degree of the approximation.

### 2.2 Functional spaces

We use standard notation to denote Lebesgue and Sobolev spaces, which are all considered to be vector spaces over  $\mathbb{C}$ . Vector fields in  $\mathbb{C}^d$  and functional spaces composed of such fields are denoted with boldface fonts. The space composed of Lebesgue integrable vector fields that are square integrable is denoted  $\mathbf{L}^2(D)$ , and the corresponding canonical inner product is denoted  $(\mathbf{u}, \mathbf{v})_{\mathbf{L}^2(D)} := \int_D \sum_{k \in \{1:d\}} u_k(\mathbf{x}) \bar{v}_k(\mathbf{x}) \, dx$ , where  $(u_k)_{k \in \{1:d\}}, (v_k)_{k \in \{1:d\}}$  are the Cartesian components of  $\mathbf{u}$  and  $\mathbf{v}$ , respectively. We introduce

$$\mathbf{H}(\mathbf{curl}; D) := \{\mathbf{h} \in \mathbf{L}^2(D) \mid \nabla \times \mathbf{h} \in \mathbf{L}^2(D)\}, \quad (2a)$$

$$\mathbf{H}_0(\mathbf{curl}; D) := \{\mathbf{h} \in \mathbf{H}(\mathbf{curl}; D) \mid \gamma_{\partial D}^c(\mathbf{h}) = \mathbf{0}\}, \quad (2b)$$

where  $\gamma_{\partial D}^c : \mathbf{H}(\mathbf{curl}; D) \rightarrow \mathbf{H}^{-\frac{1}{2}}(\partial D)$  is the tangential trace operator (see, e.g., [25, Thm. 4.15]). These two spaces are equipped with the Hilbertian structure induced by the natural graph norm  $\|\mathbf{h}\|_{\mathbf{H}(\mathbf{curl}; D)}^2 := \|\mathbf{h}\|_{\mathbf{L}^2(D)}^2 + \ell_D^2 \|\nabla \times \mathbf{h}\|_{\mathbf{L}^2(D)}^2$ .

The two differential operators involved in Maxwell's equations are

$$\nabla \times : \mathbf{H}(\mathbf{curl}; D) \ni \mathbf{e} \mapsto \nabla \times \mathbf{e} \in \mathbf{L}^2(D), \quad (3a)$$

$$\nabla_0 \times : \mathbf{H}_0(\mathbf{curl}; D) \ni \mathbf{h} \mapsto \nabla_0 \times \mathbf{h} \in \mathbf{L}^2(D). \quad (3b)$$

These two operators are adjoint to each other since for all  $\mathbf{h} \in \mathbf{H}_0(\mathbf{curl}; D)$  and for all  $\mathbf{e} \in \mathbf{H}(\mathbf{curl}; D)$ , we have  $(\nabla_0 \times \mathbf{h}, \mathbf{e})_{\mathbf{L}^2(D)} = (\mathbf{h}, \nabla \times \mathbf{e})_{\mathbf{L}^2(D)}$ . As the involution property for the problem (1) is  $\mu \mathbf{H} \in \text{im}(\nabla \times)$  and  $\epsilon \mathbf{E} \in \text{im}(\nabla_0 \times)$ , and since  $\text{im}(\nabla \times) = (\ker(\nabla_0 \times))^\perp$  and  $\text{im}(\nabla_0 \times) = (\ker(\nabla \times))^\perp$ , we introduce the following closed subspaces of  $\mathbf{H}(\mathbf{curl}; D)$ :

$$\mathbf{X}_{\mu 0} := \{\mathbf{h} \in \mathbf{H}_0(\mathbf{curl}; D) \mid \mu \mathbf{h} \in \mathbf{H}_0(\mathbf{curl} = \mathbf{0}; D)^\perp\}, \quad (4a)$$

$$\mathbf{X}_\epsilon := \{\mathbf{e} \in \mathbf{H}(\mathbf{curl}; D) \mid \epsilon \mathbf{e} \in \mathbf{H}(\mathbf{curl} = \mathbf{0}; D)^\perp\}, \quad (4b)$$

where  $\mathbf{H}(\mathbf{curl} = \mathbf{0}; D) := \ker(\nabla \times)$  and  $\mathbf{H}_0(\mathbf{curl} = \mathbf{0}; D) := \ker(\nabla_0 \times)$ , and the symbol  $^\perp$  denotes the orthogonality in  $\mathbf{L}^2(D)$ . We also set

$$\mathbf{X}_0 := \mathbf{X}_{\mu 0}, \quad \mathbf{X} := \mathbf{X}_{\epsilon 0}, \quad (5)$$

to simplify the notation. The following lemma is classical (see [27, Lem. 2.8], Amrouche et al. [2], Birman and Solomyak [3], Bonito et al. [8], Costabel [15], Jochmann [33], Weber [36]).

**Lemma 2.1.** (i) *The following operators are isomorphisms:*

$$\nabla \times : \mathbf{X} \rightarrow \mathbf{H}_0(\mathbf{curl} = \mathbf{0}; D)^\perp, \quad \nabla_0 \times : \mathbf{X}_0 \rightarrow \mathbf{H}(\mathbf{curl} = \mathbf{0}; D)^\perp, \quad (6a)$$

$$\nabla \times : \mathbf{X}_\epsilon \rightarrow \mathbf{H}_0(\mathbf{curl} = \mathbf{0}; D)^\perp, \quad \nabla_0 \times : \mathbf{X}_{\mu 0} \rightarrow \mathbf{H}(\mathbf{curl} = \mathbf{0}; D)^\perp. \quad (6b)$$

(ii) *There exists a real number  $s \in (\frac{1}{2}, 1]$  such that the following estimates hold true for all  $(\boldsymbol{\eta}, \boldsymbol{\varepsilon}) \in \mathbf{X}_0 \times \mathbf{X}$  and for all  $(\boldsymbol{\eta}, \boldsymbol{\varepsilon}) \in \mathbf{X}_{\mu 0} \times \mathbf{X}_\epsilon$ :*

$$|\boldsymbol{\eta}|_{\mathbf{H}^s(D)} \lesssim \ell_D^{1-s} \|\nabla_0 \times \boldsymbol{\eta}\|_{\mathbf{L}^2(D)}, \quad |\boldsymbol{\varepsilon}|_{\mathbf{H}^s(D)} \lesssim \ell_D^{1-s} \|\nabla \times \boldsymbol{\varepsilon}\|_{\mathbf{L}^2(D)}. \quad (7)$$

In order to state a weak form of Poincaré–Steklov (in)equalities associated with the operators  $\nabla_0 \times$  and  $\nabla \times$ , we introduce the dual norms

$$\|\nabla \times \mathbf{e}\|_{\mathbf{X}'_0} := \sup_{\boldsymbol{\eta} \in \mathbf{X}_0} \frac{|(\mathbf{e}, \nabla_0 \times \boldsymbol{\eta})_{\mathbf{L}^2(D)}|}{\ell_D \|\nabla_0 \times \boldsymbol{\eta}\|_{\mathbf{L}^2(D)}}, \quad \|\nabla_0 \times \mathbf{h}\|_{\mathbf{X}'} := \sup_{\boldsymbol{\varepsilon} \in \mathbf{X}} \frac{|(\mathbf{h}, \nabla \times \boldsymbol{\varepsilon})_{\mathbf{L}^2(D)}|}{\ell_D \|\nabla \times \boldsymbol{\varepsilon}\|_{\mathbf{L}^2(D)}}. \quad (8)$$

**Corollary 2.2** (Weak Poincaré–Steklov equalities). *The following holds true:*

$$\|\mathbf{e}\|_{\mathbf{L}^2(D)} = \ell_D \|\nabla \times \mathbf{e}\|_{\mathbf{X}'_0}, \quad \forall \mathbf{e} \in \mathbf{H}(\mathbf{curl} = \mathbf{0}; D)^\perp, \quad (9a)$$

$$\|\mathbf{h}\|_{\mathbf{L}^2(D)} = \ell_D \|\nabla_0 \times \mathbf{h}\|_{\mathbf{X}'}, \quad \forall \mathbf{h} \in \mathbf{H}_0(\mathbf{curl} = \mathbf{0}; D)^\perp. \quad (9b)$$

Next, we consider the  $\mathbf{L}^2$ -orthogonal projections onto the kernels of  $\nabla \times$  and  $\nabla_0 \times$

$$\boldsymbol{\Pi} : \mathbf{L}^2(D) \rightarrow \mathbf{H}(\mathbf{curl} = \mathbf{0}; D) := \ker(\nabla \times), \quad (10a)$$

$$\boldsymbol{\Pi}_0 : \mathbf{L}^2(D) \rightarrow \mathbf{H}_0(\mathbf{curl} = \mathbf{0}; D) := \ker(\nabla_0 \times). \quad (10b)$$

As  $\ker(\boldsymbol{\Pi}) = \ker(\nabla \times)^\perp$  and  $\ker(\boldsymbol{\Pi}_0) = \ker(\nabla_0 \times)^\perp$ , the following characterizations of the spaces  $\mathbf{X}_{\mu 0}$  and  $\mathbf{X}_\epsilon$  hold true:

$$\mathbf{X}_{\mu 0} = \{\mathbf{h} \in \mathbf{H}_0(\mathbf{curl}; D) \mid \boldsymbol{\Pi}_0(\mu \mathbf{h}) = \mathbf{0}\}, \quad (11a)$$

$$\mathbf{X}_\epsilon = \{\mathbf{e} \in \mathbf{H}(\mathbf{curl}; D) \mid \boldsymbol{\Pi}(\epsilon \mathbf{e}) = \mathbf{0}\}. \quad (11b)$$

**Remark 2.3** (Topology of  $D$ ). Let  $\mathbf{H}(\operatorname{div}; D) := \{\mathbf{v} \in \mathbf{L}^2(D) \mid \nabla \cdot \mathbf{v} \in L^2(D)\}$  and  $\mathbf{H}_0(\operatorname{div}; D) := \{\mathbf{v} \in \mathbf{H}(\operatorname{div}; D) \mid \gamma_{\partial D}^d(\mathbf{v}) = 0\}$ , where  $\gamma_{\partial D}^d : \mathbf{H}(\operatorname{div}; D) \rightarrow H^{-\frac{1}{2}}(\partial D)$  is the usual normal trace operator. We have

$$\mathbf{H}_0(\operatorname{curl} = \mathbf{0}; D)^\perp \subseteq \mathbf{H}(\operatorname{div} = 0; D) := \{\mathbf{v} \in \mathbf{H}(\operatorname{div}; D) \mid \nabla \cdot \mathbf{v} = 0\}, \quad (12a)$$

$$\mathbf{H}(\operatorname{curl} = \mathbf{0}; D)^\perp \subseteq \mathbf{H}_0(\operatorname{div} = 0; D) := \{\mathbf{v} \in \mathbf{H}_0(\operatorname{div}; D) \mid \nabla_0 \cdot \mathbf{v} = 0\}, \quad (12b)$$

and equality holds true in (12a) if  $\partial D$  is connected, and it holds true in (12b) if  $D$  is simply connected.

**Remark 2.4** (Involutions). Remark 2.3 shows that the orthogonality properties  $\mathbf{\Pi}(\epsilon \mathbf{E}) = \mathbf{0}$ ,  $\mathbf{\Pi}_0(\mu \mathbf{H}) = \mathbf{0}$  imply Gauss's laws, i.e.,  $\nabla \cdot (\epsilon \mathbf{E}) = 0$  (in the absence of free charges) and  $\nabla \cdot (\mu \mathbf{H}) = 0$ . There is equivalence between Gauss's laws and the orthogonality properties only when the topology of  $D$  is trivial.

### 2.3 Boundary-value and eigenvalue problems

Given sources  $(\mathbf{f}, \mathbf{g})$  in  $L := \mathbf{L}^2(D) \times \mathbf{L}^2(D)$ , the boundary-value problem associated with (1) consists of seeking a pair  $(\mathbf{H}, \mathbf{E})$  in  $\mathbf{H}_0(\operatorname{curl}; D) \times \mathbf{H}(\operatorname{curl}; D)$  so that  $-\nabla \times \mathbf{E} = \omega \mu \mathbf{f}$  and  $\nabla_0 \times \mathbf{H} = \omega \epsilon \mathbf{g}$ , where the factor  $\omega$  is introduced for dimensional consistency. Of course, this problem does not make sense unless  $\mu \mathbf{f} \in \operatorname{im}(\nabla \times) = (\ker(\nabla_0 \times))^\perp = \ker(\mathbf{\Pi}_0)$  and  $\epsilon \mathbf{g} \in \operatorname{im}(\nabla_0 \times) = (\ker(\nabla \times))^\perp = \ker(\mathbf{\Pi})$ . Hence, we must reformulate the above problem and seek a pair  $(\mathbf{H}, \mathbf{E})$  in  $\mathbf{H}_0(\operatorname{curl}; D) \times \mathbf{H}(\operatorname{curl}; D)$  such that  $-\nabla \times \mathbf{E} = \omega(\mathbf{I} - \mathbf{\Pi}_0)(\mu \mathbf{f})$  and  $\nabla_0 \times \mathbf{H} = (\mathbf{I} - \mathbf{\Pi})(\epsilon \mathbf{g})$ . But, again, in the light of (6b) and referring to Remark 2.4 for the terminology, for the above problem to have a unique solution, one must require that the pair  $(\mathbf{H}, \mathbf{E})$  satisfy the involution properties  $\mathbf{\Pi}_0(\mu \mathbf{H}) = \mathbf{0}$  and  $\mathbf{\Pi}(\epsilon \mathbf{E}) = \mathbf{0}$ . In conclusion, the above discussion leads us to define the operator

$$T : L \rightarrow \mathbf{X}_{\mu 0} \times \mathbf{X}_\epsilon \subset L, \quad (13)$$

such that, for all  $(\mathbf{f}, \mathbf{g}) \in L$ , the pair  $(\mathbf{H}, \mathbf{E}) := T(\mathbf{f}, \mathbf{g})$  is the unique solution in  $\mathbf{X}_{\mu 0} \times \mathbf{X}_\epsilon$  to the following boundary-value problem:

$$-\nabla \times \mathbf{E} = \omega(\mathbf{I} - \mathbf{\Pi}_0)(\mu \mathbf{f}), \quad (14a)$$

$$\nabla_0 \times \mathbf{H} = \omega(\mathbf{I} - \mathbf{\Pi})(\epsilon \mathbf{g}). \quad (14b)$$

To be dimensionally consistent, we equip the space  $L$  with the norm

$$\|(\mathbf{f}, \mathbf{g})\|_L := \left\{ \|\mu^{\frac{1}{2}} \mathbf{f}\|_{\mathbf{L}^2(D)}^2 + \|\epsilon^{\frac{1}{2}} \mathbf{g}\|_{\mathbf{L}^2(D)}^2 \right\}^{\frac{1}{2}}. \quad (15)$$

We recall the following result from [27, Lem. 2.4].

**Lemma 2.5** (Well-posedness and stability). (i) *There is  $s \in (\frac{1}{2}, 1]$  such that the solution  $(\mathbf{H}, \mathbf{E})$  to (14) satisfies the following a priori estimates:*

$$\ell_D (\mu_0^{\frac{1}{2}} \|\nabla_0 \times \mathbf{H}\|_{\mathbf{L}^2(D)} + \epsilon_0^{\frac{1}{2}} \|\nabla \times \mathbf{E}\|_{\mathbf{L}^2(D)}) \lesssim \|(\mathbf{f}, \mathbf{g})\|_L, \quad (16a)$$

$$\|\mathbf{H}\|_{\mathbf{H}^s(D)} \lesssim \ell_D^{1-s} \|\nabla_0 \times \mathbf{H}\|_{\mathbf{L}^2(D)}, \quad \|\mathbf{E}\|_{\mathbf{H}^s(D)} \lesssim \ell_D^{1-s} \|\nabla \times \mathbf{E}\|_{\mathbf{L}^2(D)}, \quad (16b)$$

(ii) *The operator  $T : L \rightarrow L$  is compact.*

Let us now characterize the spectrum of  $T$ . Let  $(\lambda, (\mathbf{H}, \mathbf{E})) \in \mathbb{C} \times L$  be an eigenpair of the operator  $T$ , i.e.,  $T(\mathbf{H}, \mathbf{E}) = \lambda(\mathbf{H}, \mathbf{E})$  (notice that the eigenvalue  $\lambda$  is nondimensional). Whenever the eigenvalue  $\lambda$  is nonzero, this pair satisfies

$$-\nabla \times \mathbf{E} = \frac{\omega}{\lambda} \mu \mathbf{H}, \quad \nabla_0 \times \mathbf{H} = \frac{\omega}{\lambda} \epsilon \mathbf{E}. \quad (17)$$

Hence, we recover the eigenvalue problem (1). The identities (17) imply that any eigenvector  $(\mathbf{H}, \mathbf{E})$  of the operator  $T$  associated with a nonzero eigenvalue is in  $\mathbf{X}_{\mu 0} \times \mathbf{X}_\epsilon$ , i.e., we have  $\mathbf{\Pi}_0(\mu \mathbf{H}) = \mathbf{\Pi}(\epsilon \mathbf{E}) = \mathbf{0}$ .

### 3 Discrete setting

We introduce in this section the discrete setting we are going to use to approximate the eigenvalue problem (17).

#### 3.1 Mesh, jumps and edge stabilization

Recall that we assume that  $D$  is an open, bounded, connected polyhedron with Lipschitz boundary. Let  $(\mathcal{T}_h)_{h \in \mathcal{H}}$  be a shape-regular family of affine simplicial meshes such that each mesh covers  $D$  exactly. We assume that the meshes are quasi-uniform, but, to stay general, we perform the analysis under the broader assumption of a shape-regular mesh family, and we indicate where the quasi-uniformity assumption is invoked.

For any mesh cell  $K$  in  $\mathcal{T}_h$ , we denote by  $h_K$  and  $\mathbf{n}_K$  the diameter of  $K$  and the outward unit normal at the boundary of  $K$ . We set  $h := \max_{K \in \mathcal{T}_h} h_K$ . The set of mesh faces is denoted by  $\mathcal{F}_h$ . The set of interfaces is denoted by  $\mathcal{F}_h^\circ$ . The set of boundary faces is denoted by  $\mathcal{F}_h^\partial$ . Each interface is oriented using a unit normal vector  $\mathbf{n}_F$ , and letting  $K_l, K_r$  be the two cells sharing the interface  $F$ , we adopt the convention that  $\mathbf{n}_F$  points from  $K_l$  to  $K_r$ . Every boundary face  $F \in \mathcal{F}_h^\partial$  is oriented by the unit normal  $\mathbf{n}_F := \mathbf{n}_D$ .

Let  $k \geq 1$  be an integer which we henceforth call polynomial degree. Let  $\mathbb{P}_{k,d}$  be the vector space over  $\mathbb{C}$  composed of the  $d$ -variate polynomials of total degree at most  $k$ . We set  $\mathbf{P}_{k,d} := [\mathbb{P}_{k,d}]^d$ . Consider the  $\mathbf{H}^1$ -conforming finite element spaces

$$\mathbf{P}_k^g(\mathcal{T}_h) := \{\mathbf{w}_h \in \mathbf{H}^1(D) \mid \mathbf{w}_h|_{\hat{K}} \in \mathbf{P}_{k,d} \forall K \in \mathcal{T}_h\}, \quad (18a)$$

$$\mathbf{P}_{kT}^g(\mathcal{T}_h) := \{\mathbf{w}_h \in \mathbf{P}_k^g(\mathcal{T}_h) \mid \mathbf{w}_h|_F \times \mathbf{n}_F = \mathbf{0} \forall F \in \mathcal{F}_h^\partial\}. \quad (18b)$$

Notice that  $\mathbf{P}_k^g(\mathcal{T}_h) \subset \mathbf{H}(\mathbf{curl}; D)$  and  $\mathbf{P}_{kT}^g(\mathcal{T}_h) \subset \mathbf{H}_0(\mathbf{curl}; D)$ . We are also going to make use of the following broken polynomial space and broken Sobolev space:

$$\mathbf{P}_k^b(\mathcal{T}_h) := \{\mathbf{w}_h \in \mathbf{L}^2(D) \mid \mathbf{w}_h|_{\hat{K}} \in \mathbf{P}_{k,d} \forall K \in \mathcal{T}_h\}, \quad (19a)$$

$$\mathbf{H}^{b,s}(\mathcal{T}_h) := \{\mathbf{w} \in \mathbf{L}^2(D) \mid \mathbf{w}|_{\hat{K}} \in \mathbf{H}^s(\hat{K}) \forall K \in \mathcal{T}_h\}, \quad s \in (\frac{1}{2}, 1]. \quad (19b)$$

For all  $K \in \mathcal{T}_h$ , all  $F \in \mathcal{F}_h$  with  $F \subset \partial K$ , and all  $\mathbf{w} \in \mathbf{H}^{b,s}(\mathcal{T}_h)$ , we define the local trace operators such that  $\gamma_{K,F}^g(\mathbf{w})(\mathbf{x}) := \lim_{K \ni \mathbf{y} \rightarrow \mathbf{x}} \mathbf{w}|_K(\mathbf{y})$ , and  $\gamma_{K,F}^c(\mathbf{w})(\mathbf{x}) := \lim_{K \ni \mathbf{y} \rightarrow \mathbf{x}} \mathbf{w}|_K(\mathbf{y}) \times \mathbf{n}_F$  for a.e.  $\mathbf{x} \in F$ . For all  $F \in \mathcal{F}_h^\circ$  and all  $\mathbf{w} \in \mathbf{H}^{b,s}(\mathcal{T}_h)$ , we define the jump of the tangential component of  $\mathbf{w}$  across  $F = \partial K_l \cap \partial K_r$  as

$$\llbracket \mathbf{w} \rrbracket_F^c := \gamma_{K_l, F}^c(\mathbf{w})(\mathbf{x}) - \gamma_{K_r, F}^c(\mathbf{w})(\mathbf{x}). \quad (20)$$

We also set  $[[\mathbf{w}]]_F^c := \gamma_{K,F}^c(\mathbf{w})(\mathbf{x})$  for all  $F = \partial K \cap \partial D \in \mathcal{F}_h^\partial$ . We adopt similar definitions for  $[[\mathbf{w}]]_F^g$ . We define the continuous interior penalty sesquilinear forms such that for all  $\mathbf{H}_h, \mathbf{h}_h \in \mathbf{P}_{k_T}^g(\mathcal{T}_h)$  and all  $\mathbf{E}_h, \mathbf{e}_h \in \mathbf{P}_k^g(\mathcal{T}_h)$ ,

$$s_h^H(\mathbf{H}_h, \mathbf{h}_h) := \sum_{F \in \mathcal{F}_h^\circ} h_F^2 ( [[\nabla_0 \times \mathbf{H}_h]]_F^c, [[\nabla_0 \times \mathbf{h}_h]]_F^c )_{L^2(F)}, \quad (21a)$$

$$s_h^E(\mathbf{E}_h, \mathbf{e}_h) := \sum_{F \in \mathcal{F}_h} h_F^2 ( [[\nabla \times \mathbf{E}_h]]_F^c, [[\nabla \times \mathbf{e}_h]]_F^c )_{L^2(F)}. \quad (21b)$$

These two sesquilinear forms induce the following seminorms:

$$|\mathbf{h}_h|_J^H := s_h^H(\mathbf{h}_h, \mathbf{h}_h)^{\frac{1}{2}}, \quad |\mathbf{e}_h|_J^E := s_h^E(\mathbf{e}_h, \mathbf{e}_h)^{\frac{1}{2}}. \quad (22)$$

**Remark 3.1** (Jumps). *The normal component of  $\nabla_0 \times \mathbf{h}_h$  and  $\nabla \times \mathbf{e}_h$  is continuous across every mesh interface, and the normal component of  $\nabla_0 \times \mathbf{h}_h$  (but not necessarily that of  $\nabla \times \mathbf{e}_h$ ) vanishes at every mesh boundary face. Therefore, the jumps across the mesh interfaces in (21) can be implemented as jumps of the Cartesian components. This is not the case for the summation over the boundary mesh faces in (21b).*

### 3.2 Stabilized Galerkin formulation and discrete involutions

Our goal now is to construct a discrete formulation of the eigenvalue problem (1). We start by defining the discrete space

$$L_h := \mathbf{P}_{k_T}^g(\mathcal{T}_h) \times \mathbf{P}_k^g(\mathcal{T}_h). \quad (23)$$

Then, we introduce the discrete sesquilinear form  $a_h : L_h \times L_h \rightarrow \mathbb{C}$  defined by

$$\begin{aligned} a_h((\mathbf{H}_h, \mathbf{E}_h), (\mathbf{h}_h, \mathbf{e}_h)) &:= -(\nabla \times \mathbf{E}_h, \mathbf{h}_h)_{L^2(D)} + (\nabla_0 \times \mathbf{H}_h, \mathbf{e}_h)_{L^2(D)} \\ &\quad + \kappa_H s_h^H(\mathbf{H}_h, \mathbf{h}_h) + \kappa_E s_h^E(\mathbf{E}_h, \mathbf{e}_h), \end{aligned} \quad (24a)$$

with  $\kappa_H := \mu_0 \omega \ell_D = (\mu_0 / \epsilon_0)^{\frac{1}{2}}$  and  $\kappa_E := \epsilon_0 \omega \ell_D = (\epsilon_0 / \mu_0)^{\frac{1}{2}}$ ; notice that  $\kappa_H \kappa_E = 1$ .

We want to find discrete eigenpairs  $(\lambda_h, (\mathbf{H}_h, \mathbf{E}_h)) \in \mathbb{C} \times L_h$ ,  $\lambda_h \neq 0$ , such that, for all  $(\mathbf{h}_h, \mathbf{e}_h) \in L_h$

$$a_h((\mathbf{H}_h, \mathbf{E}_h), (\mathbf{h}_h, \mathbf{e}_h)) = \frac{\omega}{\lambda_h} ((\mu \mathbf{H}_h, \epsilon \mathbf{E}_h), (\mathbf{h}_h, \mathbf{e}_h))_{L^2(D) \times L^2(D)}. \quad (25)$$

In order to identify the involution associated with (25), we introduce the operator  $A_h : L_h \rightarrow L_h$  so that  $(A_h(\mathbf{H}_h, \mathbf{E}_h), (\mathbf{h}_h, \mathbf{e}_h))_{L^2(D) \times L^2(D)} := a_h((\mathbf{H}_h, \mathbf{E}_h), (\mathbf{h}_h, \mathbf{e}_h))$ , for all  $(\mathbf{H}_h, \mathbf{E}_h) \in L_h$  and all  $(\mathbf{h}_h, \mathbf{e}_h) \in L_h$ . Then, (25) can be rewritten as

$$A_h(\mathbf{H}_h, \mathbf{E}_h) = \frac{\omega}{\lambda_h} (\mu \mathbf{H}_h, \epsilon \mathbf{E}_h). \quad (26)$$

Hence, the discrete counterpart of the involution property identified in §2.3 consists of stating that the pair  $(\mu \mathbf{H}_h, \epsilon \mathbf{E}_h)$  is in the range of  $A_h$ , i.e., the involution-preserving discrete fields are all the members of  $\ker(A_h^T)^\perp \cap L_h$ . Thus, we have to characterize the kernel of the adjoint operator  $A_h^T$ . To this purpose, we set

$$\mathbf{P}_{k_T}^g(\mathbf{curl} = \mathbf{0}; \mathcal{T}_h) := \mathbf{P}_{k_T}^g(\mathcal{T}_h) \cap \ker(\nabla_0 \times), \quad (27a)$$

$$\mathbf{P}_k^g(\mathbf{curl} = \mathbf{0}; \mathcal{T}_h) := \mathbf{P}_k^g(\mathcal{T}_h) \cap \ker(\nabla \times). \quad (27b)$$



**Lemma 3.2** (Kernel of adjoint operator). *We have*

$$\ker(A_h^\top) = \mathbf{P}_{k_T}^g(\mathbf{curl} = \mathbf{0}; \mathcal{T}_h) \times \mathbf{P}_k^g(\mathbf{curl} = \mathbf{0}; \mathcal{T}_h). \quad (28)$$

*Proof.* (1) Let  $(\mathbf{h}_h, \mathbf{e}_h) \in \ker(A_h^\top)$ . Then, for all  $(\mathbf{H}_h, \mathbf{E}_h) \in L_h$ , the pair  $(\mathbf{h}_h, \mathbf{e}_h)$  satisfies  $a_h((\mathbf{H}_h, \mathbf{E}_h), (\mathbf{h}_h, \mathbf{e}_h)) = 0$ . Since this implies that  $a_h((\mathbf{h}_h, \mathbf{e}_h), (\mathbf{h}_h, \mathbf{e}_h)) = 0$ , we infer from (24a) that  $|\mathbf{h}_h|_J^H = |\mathbf{e}_h|_J^E = 0$ . Recalling (21), this gives

$$\nabla_0 \times \mathbf{h}_h \in \mathbf{P}_k^g(\mathcal{T}_h), \quad \nabla \times \mathbf{e}_h \in \mathbf{P}_{k_T}^g(\mathcal{T}_h).$$

(Notice that the tangential component of  $\nabla \times \mathbf{e}_h$  vanishes on the boundary faces since  $s_h^E$  contains a contribution from the boundary faces.) This means that the pair  $(\mathbf{H}_h, \mathbf{E}_h) = (\nabla \times \mathbf{e}_h, \nabla_0 \times \mathbf{h}_h)$  is a legitimate test function in  $L_h$ . This readily gives  $\nabla_0 \times \mathbf{h}_h = \nabla \times \mathbf{e}_h = \mathbf{0}$ . Hence,  $(\mathbf{h}_h, \mathbf{e}_h) \in \mathbf{P}_{k_T}^g(\mathbf{curl} = \mathbf{0}; \mathcal{T}_h) \times \mathbf{P}_k^g(\mathbf{curl} = \mathbf{0}; \mathcal{T}_h)$ .

(2) Conversely, let  $(\mathbf{h}_h, \mathbf{e}_h) \in \mathbf{P}_{k_T}^g(\mathbf{curl} = \mathbf{0}; \mathcal{T}_h) \times \mathbf{P}_k^g(\mathbf{curl} = \mathbf{0}; \mathcal{T}_h)$ . Two integrations by parts immediately give  $a_h((\mathbf{H}_h, \mathbf{E}_h), (\mathbf{h}_h, \mathbf{e}_h)) = 0$ , for all  $(\mathbf{H}_h, \mathbf{E}_h) \in L_h$ . Hence,  $(\mathbf{h}_h, \mathbf{e}_h) \in \ker(A_h^\top)$ . Since  $(\mathbf{h}_h, \mathbf{e}_h) \in L_h$  as well, the assertion is proved.  $\square$

As satisfying the involutions is equivalent to being a member of  $\ker(A_h^\top)^\perp \cap L_h$ , where we recall that  $^\perp$  means orthogonality in  $L^2(D) \times L^2(D)$ , we now define the  $L^2$ -orthogonal projections

$$\mathbf{\Pi}_{h_T} : L^2(D) \rightarrow \mathbf{P}_{k_T}^g(\mathbf{curl} = \mathbf{0}; \mathcal{T}_h), \quad (29a)$$

$$\mathbf{\Pi}_h : L^2(D) \rightarrow \mathbf{P}_k^g(\mathbf{curl} = \mathbf{0}; \mathcal{T}_h). \quad (29b)$$

Notice that  $\ker(\mathbf{\Pi}_{h_T}) = \mathbf{P}_{k_T}^g(\mathbf{curl} = \mathbf{0}; \mathcal{T}_h)^\perp$  and  $\ker(\mathbf{\Pi}_h) = \mathbf{P}_k^g(\mathbf{curl} = \mathbf{0}; \mathcal{T}_h)^\perp$ . It is therefore natural to introduce the following discrete counterparts of the spaces  $\mathbf{X}_{\mu 0}$  and  $\mathbf{X}_\epsilon$ :

$$\mathbf{X}_{\mu T, h} = \{\mathbf{h}_h \in \mathbf{P}_{k_T}^g(\mathcal{T}_h) \mid \mu \mathbf{h}_h \in \mathbf{P}_{k_T}^g(\mathbf{curl} = \mathbf{0}; \mathcal{T}_h)^\perp\}, \quad (30a)$$

$$\mathbf{X}_{\epsilon, h} = \{\mathbf{e}_h \in \mathbf{P}_k^g(\mathcal{T}_h) \mid \epsilon \mathbf{e}_h \in \mathbf{P}_k^g(\mathbf{curl} = \mathbf{0}; \mathcal{T}_h)^\perp\}. \quad (30b)$$

In conclusion, the pair  $(\mathbf{H}_h, \mathbf{E}_h) \in L_h$  satisfies the discrete involution property iff

$$\mathbf{\Pi}_{h_T}(\mu \mathbf{H}_h) = \mathbf{\Pi}_h(\epsilon \mathbf{E}_h) = \mathbf{0} \iff (\mathbf{H}_h, \mathbf{E}_h) \in \mathbf{X}_{\mu T, h} \times \mathbf{X}_{\epsilon, h}. \quad (31)$$

In the paper, we show that, under the conditions stated in Assumption 4.1, the discrete involution (31) is strong enough to guarantee strong convergence of the approximation in  $L$ .

### 3.3 Discrete boundary-value problem

We are now in a position to introduce the discrete counterpart of the operator  $T : L \rightarrow L$  defined in (13). We call

$$T_h : L \rightarrow \mathbf{X}_{\mu T, h} \times \mathbf{X}_{\epsilon, h} \subset L_h \subset L \quad (32)$$

the operator in question. Specifically, for all  $(\mathbf{f}, \mathbf{g}) \in L$ , we define  $(\mathbf{H}_h, \mathbf{E}_h) := T_h(\mathbf{f}, \mathbf{g})$  to be the unique pair in  $\mathbf{X}_{\mu T, h} \times \mathbf{X}_{\epsilon, h}$  so that the following holds true for all  $(\mathbf{h}_h, \mathbf{e}_h) \in (\ker(\mathbf{\Pi}_{h_T}) \times \ker(\mathbf{\Pi}_h)) \cap L_h$ :

$$a_h((\mathbf{H}_h, \mathbf{E}_h), (\mathbf{h}_h, \mathbf{e}_h)) = \omega((\mathbf{I} - \mathbf{\Pi}_{h_T})(\mu \mathbf{f}), \mathbf{h}_h)_{L^2(D)} + \omega((\mathbf{I} - \mathbf{\Pi}_h)(\epsilon \mathbf{g}), \mathbf{e}_h)_{L^2(D)}. \quad (33)$$

It is shown in Lemma 5.1(ii) that the above definition is meaningful.

Since (33) trivially holds true also for every test function  $(\mathbf{h}_h, \mathbf{e}_h) \in \mathbf{P}_{k_T}^g(\mathbf{curl} = \mathbf{0}; \mathcal{T}_h) \times \mathbf{P}_k^g(\mathbf{curl} = \mathbf{0}; \mathcal{T}_h)$ , one can request that  $(\mathbf{H}_h, \mathbf{E}_h) := T_h(\mathbf{f}, \mathbf{g})$  be such that (33) holds true for all  $(\mathbf{h}_h, \mathbf{e}_h) \in L_h$ . Hence, (33) can be rewritten as follows:

$$A_h(\mathbf{H}_h, \mathbf{E}_h) = \omega((\mathbf{I} - \mathbf{\Pi}_{h_T})(\mu\mathbf{f}), (\mathbf{I} - \mathbf{\Pi}_h)(\epsilon\mathbf{g})). \quad (34)$$

This shows that, for all  $\lambda_h \in \mathbb{C} \setminus \{0\}$ ,  $(\lambda_h, (\mathbf{H}_h, \mathbf{E}_h))$  is an eigenpair of  $T_h$  iff  $(\frac{1}{\lambda_h}, (\mathbf{H}_h, \mathbf{E}_h))$  is an eigenpair of  $A_h$ . Therefore, proving that the spectrum of  $T_h$  is pollution-free is equivalent to proving that the spectrum of  $A_h$  is pollution-free. We do so in Theorem 5.7 by establishing that  $\lim_{h \rightarrow 0} \|T - T_h\|_{\mathcal{L}(L;L)} = 0$ .

## 4 Preliminary results

In this section, we establish two preliminary results that are used in the convergence proof reported in §5: discrete Poincaré–Steklov inequalities and a consistency bound on the stabilization. But, before deriving these two results, we list for the reader's convenience the quasi-interpolation operators invoked in the paper and recall their main properties.

### 4.1 Quasi-interpolation operators

We are going to invoke three types of quasi-interpolation operators in the paper.

The first ones are the averaging operators  $\mathcal{J}_{h_T}^{g,av} : \mathbf{P}_k^b(\mathcal{T}_h) \rightarrow \mathbf{P}_{k_T}^g(\mathcal{T}_h)$  and  $\mathcal{J}_h^{g,av} : \mathbf{P}_k^b(\mathcal{T}_h) \rightarrow \mathbf{P}_k^g(\mathcal{T}_h)$  analyzed in [24, §4] (see also [25, §22.2]). These operators are used in the proof of the deflated inf-sup condition (Lemma 5.1). They have the following approximation property: For all  $\mathbf{v}_h \in \mathbf{P}_k^b(\mathcal{T}_h)$  and all  $h \in \mathcal{H}$ ,

$$\|\mathbf{v}_h - \mathcal{J}_{h_T}^{g,av}(\mathbf{v}_h)\|_{\mathbf{L}^2(D)}^2 \lesssim \sum_{F \in \mathcal{F}_h^\circ} h_F \|\llbracket \mathbf{v}_h \rrbracket_F^g\|_{\mathbf{L}^2(F)}^2 + \sum_{F \in \mathcal{F}_h^\partial} h_F \|\llbracket \mathbf{v}_h \rrbracket_F^c\|_{\mathbf{L}^2(F)}^2, \quad (35a)$$

$$\|\mathbf{v}_h - \mathcal{J}_h^{g,av}(\mathbf{v}_h)\|_{\mathbf{L}^2(D)}^2 \lesssim \sum_{F \in \mathcal{F}_h^\circ} h_F \|\llbracket \mathbf{v}_h \rrbracket_F^g\|_{\mathbf{L}^2(F)}^2. \quad (35b)$$

These  $\mathbf{L}^2$ -estimates can be localized to every mesh cell  $K \in \mathcal{T}_h$  by considering the subset of the mesh faces sharing at least one vertex with  $K$ .

The second family of operators is defined by composing the above averaging operators with the  $\mathbf{L}^2$ -orthogonal projection onto  $\mathbf{P}_k^b(\mathcal{T}_h)$ . This leads to the quasi-interpolation operators  $\mathcal{I}_{h_T}^{g,av} : \mathbf{L}^1(D) \rightarrow \mathbf{P}_{k_T}^g(\mathcal{T}_h)$  and  $\mathcal{I}_h^{g,av} : \mathbf{L}^1(D) \rightarrow \mathbf{P}_k^g(\mathcal{T}_h)$  described [24, §5] (see also [25, §22.3]). These operators have optimal local approximation properties and are  $\mathbf{L}^1$ -stable. They are used in establishing a consistency bound on the stabilization (Lemma 4.4) and the deflated inf-sup condition (Lemma 5.1).

The third family of operators has been introduced in Boffi et al. [5]. Their key property is to commute with  $\nabla_0 \times$  and  $\nabla \times$ . These operators are used in establishing discrete Poincaré–Steklov inequalities (Lemma 4.3) and bounding the divergence-conformity error (Lemma 5.5). We formulate the existence of these operators as an assumption as their existence relies on specific mesh structures. We introduce the (infinite-dimensional) spaces

$$\mathbf{Z}_T(\mathcal{T}_h) := \ker(\nabla_0 \times) + \mathbf{P}_{k_T}^g(\mathcal{T}_h), \quad (36a)$$

$$\mathbf{Z}(\mathcal{T}_h) := \ker(\nabla \times) + \mathbf{P}_k^g(\mathcal{T}_h), \quad (36b)$$

where the sums are not necessarily direct.

**Assumption 4.1** (Commuting quasi-interpolation operators). *The mesh sequence  $(\mathcal{T}_h)_{h \in \mathcal{H}}$  is such that there is  $s \in (\frac{1}{2}, 1]$  so that, for every  $h \in \mathcal{H}$ , there exists an operator  $\mathcal{P}_{h\tau}^g : \mathbf{Z}_\tau(\mathcal{T}_h) \rightarrow \mathbf{P}_{k\tau}^g(\mathcal{T}_h)$  that satisfies*

$$\nabla_0 \times (\mathcal{P}_{h\tau}^g(\mathbf{h})) = \nabla_0 \times \mathbf{h} \quad (37a)$$

$$\|\mathbf{h} - \mathcal{P}_{h\tau}^g(\mathbf{h})\|_{\mathbf{L}^2(D)} \lesssim h^s (|\mathbf{h}|_{\mathbf{H}^s(D)} + \ell_D^{1-s} \|\nabla_0 \times \mathbf{h}\|_{\mathbf{L}^2(D)}), \quad (37b)$$

for all  $\mathbf{h} \in \mathbf{Z}_\tau(\mathcal{T}_h)$  and all  $\mathbf{h} \in \mathbf{Z}_\tau(\mathcal{T}_h) \cap \mathbf{H}^s(D)$ , respectively, and there exists an operator  $\mathcal{P}_h^g : \mathbf{Z}(\mathcal{T}_h) \rightarrow \mathbf{P}_k^g(\mathcal{T}_h)$  that satisfies

$$\nabla \times (\mathcal{P}_h^g(\mathbf{e})) = \nabla \times \mathbf{e}, \quad (38a)$$

$$\|\mathbf{e} - \mathcal{P}_h^g(\mathbf{e})\|_{\mathbf{L}^2(D)} \lesssim h^s (|\mathbf{e}|_{\mathbf{H}^s(D)} + \ell_D^{1-s} \|\nabla \times \mathbf{e}\|_{\mathbf{L}^2(D)}), \quad (38b)$$

for all  $\mathbf{e} \in \mathbf{Z}(\mathcal{T}_h)$  and all  $\mathbf{e} \in \mathbf{Z}(\mathcal{T}_h) \cap \mathbf{H}^s(D)$ , respectively.

The Assumption 4.1 is shown to hold on various mesh structures in Boffi et al. [5]. Notice that the operators  $\mathcal{P}_{h\tau}^g$  and  $\mathcal{P}_h^g$  are projections, but this property is never invoked in the paper.

**Remark 4.2** (Localization). *The approximation properties (37b) and (38b) can be localized, but this is of little use because later in the paper we are going to bound the right-hand sides in (37b) and (38b) by using the compactness properties (7) which involve global norms.*

## 4.2 Discrete Poincaré–Steklov inequalities

In this section, we establish discrete Poincaré–Steklov inequalities that are discrete counterparts of (9).

**Lemma 4.3** (Discrete Poincaré–Steklov inequalities). *For all  $\mathbf{H}_h \in \mathbf{X}_{\mu\tau,h}$ , all  $\mathbf{E}_h \in \mathbf{X}_{\epsilon,h}$ , and all  $h \in \mathcal{H}$ , we have*

$$\|\mu^{\frac{1}{2}} \mathbf{H}_h\|_{\mathbf{L}^2(D)} \lesssim \mu_0^{\frac{1}{2}} \ell_D (\|\nabla_0 \times \mathbf{H}_h\|_{\mathbf{X}'} + (h/\ell_D)^s \|\nabla_0 \times \mathbf{H}_h\|_{\mathbf{L}^2(D)}), \quad (39a)$$

$$\|\epsilon^{\frac{1}{2}} \mathbf{E}_h\|_{\mathbf{L}^2(D)} \lesssim \epsilon_0^{\frac{1}{2}} \ell_D (\|\nabla \times \mathbf{E}_h\|_{\mathbf{X}'_0} + (h/\ell_D)^s \|\nabla \times \mathbf{E}_h\|_{\mathbf{L}^2(D)}). \quad (39b)$$

*Proof.* Let  $\mathbf{H}_h \in \mathbf{X}_{\mu\tau,h}$ . Let  $\boldsymbol{\xi} := \mathbf{H}_h - \boldsymbol{\Pi}_0(\mathbf{H}_h)$ . Observe that  $\boldsymbol{\xi} \in \mathbf{X}_0$ . Hence, using (9b) along with the equality  $\nabla_0 \times \boldsymbol{\xi} = \nabla_0 \times \mathbf{H}_h$  (since  $\boldsymbol{\Pi}_0(\mathbf{H}_h)$  is curl-free by construction), we obtain

$$\|\boldsymbol{\xi}\|_{\mathbf{L}^2(D)} = \ell_D \|\nabla_0 \times \boldsymbol{\xi}\|_{\mathbf{X}'} = \ell_D \|\nabla_0 \times \mathbf{H}_h\|_{\mathbf{X}'}$$

Since, by construction,  $\boldsymbol{\xi} \in \mathbf{Z}_\tau(\mathcal{T}_h)$  (see (36)), we can use the commuting projection  $\mathcal{P}_{h\tau}^g$  introduced in Assumption 4.1. Invoking (37a) gives  $\nabla_0 \times (\mathcal{P}_{h\tau}^g(\boldsymbol{\xi}) - \mathbf{H}_h) = \nabla_0 \times \boldsymbol{\xi} - \nabla_0 \times \mathbf{H}_h = \mathbf{0}$ , so that  $\mathcal{P}_{h\tau}^g(\boldsymbol{\xi}) - \mathbf{H}_h \in \mathbf{P}_{k\tau}^g(\mathbf{curl} = \mathbf{0}; \mathcal{T}_h)$ . Hence, using that  $\mu \mathbf{H}_h \in \mathbf{P}_{k\tau}^g(\mathbf{curl} = \mathbf{0}; \mathcal{T}_h)^\perp$  together with the triangle inequality and the estimate (37b), we obtain

$$\begin{aligned} \|\mu^{\frac{1}{2}} \mathbf{H}_h\|_{\mathbf{L}^2(D)}^2 &= (\mu \mathbf{H}_h, \mathbf{H}_h + (\mathcal{P}_{h\tau}^g(\boldsymbol{\xi}) - \mathbf{H}_h))_{\mathbf{L}^2(D)} = (\mu \mathbf{H}_h, \mathcal{P}_{h\tau}^g(\boldsymbol{\xi}))_{\mathbf{L}^2(D)} \\ &\lesssim \mu_0^{\frac{1}{2}} \|\mu^{\frac{1}{2}} \mathbf{H}_h\|_{\mathbf{L}^2(D)} (\|\boldsymbol{\xi}\|_{\mathbf{L}^2(D)} + \|\boldsymbol{\xi} - \mathcal{P}_{h\tau}^g(\boldsymbol{\xi})\|_{\mathbf{L}^2(D)}) \\ &\lesssim \mu_0^{\frac{1}{2}} \|\mu^{\frac{1}{2}} \mathbf{H}_h\|_{\mathbf{L}^2(D)} (\|\boldsymbol{\xi}\|_{\mathbf{L}^2(D)} + h^s (|\boldsymbol{\xi}|_{\mathbf{H}^s(D)} + \ell_D^{1-s} \|\nabla_0 \times \boldsymbol{\xi}\|_{\mathbf{L}^2(D)})). \end{aligned}$$

Simplifying by  $\|\mu^{\frac{1}{2}} \mathbf{H}_h\|_{\mathbf{L}^2(D)}$  and recalling that  $\|\boldsymbol{\xi}\|_{\mathbf{L}^2(D)} = \ell_D \|\nabla_0 \times \mathbf{H}_h\|_{\mathbf{X}'}$  and  $\nabla_0 \times \boldsymbol{\xi} = \nabla_0 \times \mathbf{H}_h$ , we infer that

$$\|\mu^{\frac{1}{2}} \mathbf{H}_h\|_{\mathbf{L}^2(D)} \lesssim \mu_0^{\frac{1}{2}} (\ell_D \|\nabla_0 \times \mathbf{H}_h\|_{\mathbf{X}'} + h^s (|\boldsymbol{\xi}|_{\mathbf{H}^s(D)} + \ell_D^{1-s} \|\nabla_0 \times \mathbf{H}_h\|_{\mathbf{L}^2(D)})).$$

Furthermore, as  $\boldsymbol{\xi} \in \mathbf{X}_0$ , we can invoke the smoothness estimate (7). Since  $\nabla_0 \times \boldsymbol{\xi} = \nabla_0 \times \mathbf{H}_h$ , we infer that

$$\|\mu^{\frac{1}{2}} \mathbf{H}_h\|_{\mathbf{L}^2(D)} \lesssim \mu_0^{\frac{1}{2}} \ell_D (\|\nabla \times \mathbf{H}_h\|_{\mathbf{X}'} + (h/\ell_D)^s \|\nabla_0 \times \mathbf{H}_h\|_{\mathbf{L}^2(D)}).$$

The second assertion is proved similarly.  $\square$

### 4.3 Consistency bound on stabilization

Now, only assuming the regularity available in  $\mathbf{X}_{\mu_0} \times \mathbf{X}_\epsilon$ , we show that the consistency error induced by the stabilization sesquilinear forms  $s_h^{\mathbf{H}}$  and  $s_h^{\mathbf{E}}$  vanishes as the meshsize goes to zero. To our knowledge, the result established in Lemma 4.4 does not seem to be known in the literature; see Remark 4.5.

**Lemma 4.4** (Consistency bound on stabilization). *The following consistency estimates hold for all pairs  $(\boldsymbol{\eta}, \boldsymbol{\epsilon}) \in \mathbf{X}_{\mu_0} \times \mathbf{X}_\epsilon$ :*

$$|\mathcal{I}_{h\mathbf{T}}^{\mathbf{g},\text{av}}(\boldsymbol{\eta})|_{\mathbf{J}}^{\mathbf{H}} \lesssim (h/\ell_D)^{s-\frac{1}{2}} \ell_D^{\frac{1}{2}} \|\nabla_0 \times \boldsymbol{\eta}\|_{\mathbf{L}^2(D)}, \quad |\mathcal{I}_{h\mathbf{T}}^{\mathbf{g},\text{av}}(\boldsymbol{\epsilon})|_{\mathbf{J}}^{\mathbf{E}} \lesssim (h/\ell_D)^{s-\frac{1}{2}} \ell_D^{\frac{1}{2}} \|\nabla \times \boldsymbol{\epsilon}\|_{\mathbf{L}^2(D)}. \quad (40)$$

*Proof.* Let  $\boldsymbol{\eta} \in \mathbf{X}_{\mu_0}$  and let us set, for all  $F \in \mathcal{F}_h^\circ$ ,  $\mathcal{T}(F) := K_l \cup K_r$ ,  $\bar{\boldsymbol{\eta}}_K := \frac{1}{|K|} \int_K \boldsymbol{\eta} \, dx$  for all  $K \in \mathcal{T}(F)$ , and  $\boldsymbol{\eta}_h := \mathcal{I}_{h\mathbf{T}}^{\mathbf{g},\text{av}}(\boldsymbol{\eta})$ . Using inverse inequalities gives

$$\begin{aligned} h_F \|\llbracket \nabla_0 \times \boldsymbol{\eta}_h \rrbracket_F^{\mathbf{g}}\|_{\mathbf{L}^2(F)} &\lesssim \sum_{K \in \mathcal{T}(F)} h_F^{\frac{1}{2}} \|\nabla_0 \times \boldsymbol{\eta}_h\|_{\mathbf{L}^2(K)} = \sum_{K \in \mathcal{T}(F)} h_F^{\frac{1}{2}} \|\nabla_0 \times (\boldsymbol{\eta}_h - \bar{\boldsymbol{\eta}}_K)\|_{\mathbf{L}^2(K)} \\ &\lesssim \sum_{K \in \mathcal{T}(F)} h_F^{-\frac{1}{2}} \|\boldsymbol{\eta}_h - \bar{\boldsymbol{\eta}}_K\|_{\mathbf{L}^2(K)} \lesssim \sum_{K \in \mathcal{T}(F)} h_F^{-\frac{1}{2}} (\|\boldsymbol{\eta}_h - \boldsymbol{\eta}\|_{\mathbf{L}^2(K)} + \|\boldsymbol{\eta} - \bar{\boldsymbol{\eta}}_K\|_{\mathbf{L}^2(K)}). \end{aligned}$$

Invoking the local approximation properties of  $\mathcal{I}_{h\mathbf{T}}^{\mathbf{g},\text{av}}$ , the local fractional Poincaré–Steklov inequality in  $\mathbf{H}^s(K)$ , and the shape-regularity of the mesh family shows that

$$h_F \|\llbracket \nabla_0 \times \boldsymbol{\eta}_h \rrbracket_F^{\mathbf{g}}\|_{\mathbf{L}^2(F)} \lesssim \sum_{K \in \mathcal{T}(F)} h_K^{s-\frac{1}{2}} |\boldsymbol{\eta}|_{\mathbf{H}^s(K)}.$$

Summing over the mesh interfaces and reasoning similarly for the mesh boundary faces, we infer that

$$|\mathcal{I}_{h\mathbf{T}}^{\mathbf{g},\text{av}}(\boldsymbol{\eta})|_{\mathbf{J}}^{\mathbf{H}} \lesssim h^{s-\frac{1}{2}} |\boldsymbol{\eta}|_{\mathbf{H}^s(D)} \lesssim (h/\ell_D)^{s-\frac{1}{2}} \ell_D^{\frac{1}{2}} \|\nabla_0 \times \boldsymbol{\eta}\|_{\mathbf{L}^2(D)},$$

where the last bound follows from the regularity estimate (7) (recall that  $\boldsymbol{\eta} \in \mathbf{X}_{\mu_0}$  by assumption). The proof of the second estimate is similar.  $\square$

**Remark 4.5** (Consistency of interior penalty). *The proof of Lemma 4.4 shows that the generic continuous interior penalty sesquilinear form involving the jumps of the normal derivative of a scalar-valued function (see [21, Eq. 1.7] and [13]) satisfies the following consistency property: For all  $\eta \in H^s(D)$ ,  $s \in (\frac{1}{2}, 1]$ ,*

$$\left\{ \sum_{F \in \mathcal{F}_h^\circ} h_F^2 \|\llbracket \partial_n (\mathcal{I}_h^{\mathbf{g},\text{av}}(\eta)) \rrbracket\|_{\mathbf{L}^2(F)}^2 \right\}^{\frac{1}{2}} \lesssim h^{s-\frac{1}{2}} |\eta|_{H^s(D)}.$$

#### 4.4 Examples based on mesh splits

Given an initial simplicial triangulation  $\mathcal{T}_h^0$  of  $D$ , its Alfeld split, say  $\mathcal{T}_h^{\text{AL}}$ , is constructed by connecting the barycenter of every mesh cell  $T \in \mathcal{T}_h^0$  to each vertex of  $T$ . This construction is used for the (cubic) Clough–Tocher finite elements in two dimensions [14] and the (quintic) Alfeld elements in three dimensions [1]. Moreover, the Powel–Sabin split of  $\mathcal{T}_h^0$ , say  $\mathcal{T}_h^{\text{PS}}$ , is constructed as follows in the two-dimensional case. First, adjoin the barycenter of every mesh cell  $T \in \mathcal{T}_h^0$  to each vertex of  $T$  (i.e., construct the Alfeld split of  $\mathcal{T}_h^0$ ). Next, connect the barycenters of every pair of cells in  $\mathcal{T}_h^0$  sharing an edge and connect the barycenter of every cell in  $\mathcal{T}_h^0$  having a boundary edge to the midpoint of that edge.

Properties on the mesh sequence  $(\mathcal{T}_h)_{h \in \mathcal{H}}$  that are sufficient for Assumption 4.1 to hold are given in Boffi et al. [5]. In particular, it is shown therein that Powell–Sabin splits are legitimate candidates for all polynomial degrees  $k \geq 1$ . Alfeld splits are also legitimate for all polynomial degrees  $k \geq 2$ .

### 5 Error analysis

The objective of this section is to prove Theorem 5.7 which is the main result of the paper. As in [28], the convergence proof hinges on two cornerstones: a deflated inf-sup condition established in §5.1 and a duality argument à la Aubin–Nitsche established in §5.2.

#### 5.1 Deflated inf-sup condition

Recall the norm  $\|(\mathbf{f}, \mathbf{g})\|_L^2 := \|\mu^{\frac{1}{2}} \mathbf{f}\|_{\mathbf{L}^2(D)}^2 + \|\epsilon^{\frac{1}{2}} \mathbf{g}\|_{\mathbf{L}^2(D)}^2$  defined in (15). We also equip  $L_h$  with the following mesh-dependent norm: For all  $(\mathbf{h}_h, \mathbf{e}_h) \in L_h$ ,

$$\begin{aligned} \|(\mathbf{h}_h, \mathbf{e}_h)\|_{b,h} &:= \omega^{\frac{1}{2}} \|(\mathbf{h}_h, \mathbf{e}_h)\|_L + \kappa_{\mathbb{H}}^{\frac{1}{2}} |\mathbf{h}_h|_{\mathbb{J}}^{\mathbb{H}} + \kappa_{\mathbb{E}}^{\frac{1}{2}} |\mathbf{e}_h|_{\mathbb{J}}^{\mathbb{E}} \\ &\quad + \kappa_{\mathbb{H}}^{\frac{1}{2}} \|\tilde{h}^{\frac{1}{2}} \nabla_0 \times \mathbf{h}_h\|_{\mathbf{L}^2(D)} + \kappa_{\mathbb{E}}^{\frac{1}{2}} \|\tilde{h}^{\frac{1}{2}} \nabla \times \mathbf{e}_h\|_{\mathbf{L}^2(D)}. \end{aligned} \quad (41)$$

Here,  $\tilde{h} := \mathcal{J}_h^{\text{g,av}}(\bar{h})$ , where  $\mathcal{J}_h^{\text{g,av}}$  is the averaging operator with polynomial degree set to  $k := 1$  (see Section 4.1) and  $\bar{h}$  is the piecewise constant mesh-size function such that  $\bar{h}|_K = h_K$  for all  $K \in \mathcal{T}_h$ .

**Lemma 5.1** (Deflated inf-sup condition and well-posedness). *The following holds for every discrete pair  $(\mathbf{H}_h, \mathbf{E}_h) \in \mathbf{X}_{\mu\tau,h} \times \mathbf{X}_{\epsilon,h}$ :*

$$\omega^{\frac{1}{2}} \|(\mathbf{H}_h, \mathbf{E}_h)\|_{b,h} \lesssim \sup_{(\mathbf{h}_h, \mathbf{e}_h) \in L_h} \frac{|a_h((\mathbf{H}_h, \mathbf{E}_h), (\mathbf{h}_h, \mathbf{e}_h))|}{\|(\mathbf{h}_h, \mathbf{e}_h)\|_L}. \quad (42)$$

(ii) *The discrete boundary-value problem (33) is well-posed, and its solution satisfies the a priori estimate*

$$\|(\mathbf{H}_h, \mathbf{E}_h)\|_{b,h} \lesssim \omega^{\frac{1}{2}} \|(\mathbf{f}, \mathbf{g})\|_L. \quad (43)$$

*Proof.* (1) Deflated inf-sup condition. We decompose the proof of (42) in two steps. The arguments in the first step are somewhat standard (see, e.g., Burman and Ern [12] where the trial and the test norms are identical), whereas the arguments in the second step crucially hinge on the discrete Poincaré–Steklov inequalities from Lemma 4.3 (see [28, Lem. 4.2] for a similar argument in the context of discontinuous Galerkin). Let  $(\mathbf{H}_h, \mathbf{E}_h) \in \mathbf{X}_{\mu\tau,h} \times \mathbf{X}_{\epsilon,h}$  and let  $\mathbb{S}$  denote the right-hand side of (42).

(1a) Using the pair  $(\mathbf{H}_h, \mathbf{E}_h)$  as test function gives

$$\kappa_{\mathbb{H}}(|\mathbf{H}_h|_{\mathbb{J}}^{\mathbb{H}})^2 + \kappa_{\mathbb{E}}(|\mathbf{E}_h|_{\mathbb{J}}^{\mathbb{E}})^2 = a_h((\mathbf{H}_h, \mathbf{E}_h), (\mathbf{H}_h, \mathbf{E}_h)) \leq \mathbb{S} \|(\mathbf{H}_h, \mathbf{E}_h)\|_L. \quad (44)$$

Now we set  $\mathbf{h}_h := \mathcal{J}_{h\mathbb{T}}^{\mathbb{S}, \text{av}}(\nabla \times \mathbf{E}_h)$  and consider the test function  $(-\kappa_{\mathbb{E}} \tilde{h} \mathbf{h}_h, \mathbf{0})$ . Using the identity  $\kappa_{\mathbb{H}} \kappa_{\mathbb{E}} = 1$  gives

$$\begin{aligned} \kappa_{\mathbb{E}} \|\tilde{h}^{\frac{1}{2}} \nabla \times \mathbf{E}_h\|_{\mathbf{L}^2(D)}^2 &= a_h((\mathbf{H}_h, \mathbf{E}_h), (-\kappa_{\mathbb{E}} \tilde{h} \mathbf{h}_h, \mathbf{0})) + s_h^{\mathbb{H}}(\mathbf{H}_h, \tilde{h} \mathbf{h}_h) \\ &\quad + \kappa_{\mathbb{E}} (\tilde{h}^{\frac{1}{2}} \nabla \times \mathbf{E}_h, \tilde{h}^{\frac{1}{2}} (\nabla \times \mathbf{E}_h - \mathcal{J}_{h\mathbb{T}}^{\mathbb{S}, \text{av}}(\nabla \times \mathbf{E}_h)))_{\mathbf{L}^2(D)}. \end{aligned} \quad (45)$$

We have  $|a_h((\mathbf{H}_h, \mathbf{E}_h), (-\kappa_{\mathbb{E}} \tilde{h} \mathbf{h}_h, \mathbf{0}))| \leq \kappa_{\mathbb{E}}^{\frac{1}{2}} \mathbb{S} \|\mu^{\frac{1}{2}} \kappa_{\mathbb{E}}^{\frac{1}{2}} \tilde{h} \mathbf{h}_h\|_{\mathbf{L}^2(D)}$ , and  $\|\mu^{\frac{1}{2}} \kappa_{\mathbb{E}}^{\frac{1}{2}} \tilde{h} \mathbf{h}_h\|_{\mathbf{L}^2(D)}$  is bounded as follows:

$$\|\mu^{\frac{1}{2}} \kappa_{\mathbb{E}}^{\frac{1}{2}} \tilde{h} \mathbf{h}_h\|_{\mathbf{L}^2(D)} \lesssim (\mu_0 \ell_D \kappa_{\mathbb{E}})^{\frac{1}{2}} \|\tilde{h}^{\frac{1}{2}} \nabla \times \mathbf{E}_h\|_{\mathbf{L}^2(D)} = \omega^{-\frac{1}{2}} \|\tilde{h}^{\frac{1}{2}} \nabla \times \mathbf{E}_h\|_{\mathbf{L}^2(D)},$$

where we used that  $\tilde{h} \leq \ell_D$ ,  $\ell_D \mu_0 \kappa_{\mathbb{E}} = \omega^{-1}$ , and the  $\mathbf{L}^2$ -stability of  $\mathcal{J}_{h\mathbb{T}}^{\mathbb{S}, \text{av}}$  (which follows from (35a), a triangle inequality, and a discrete trace inequality). Invoking similar arguments shows that  $|s_h^{\mathbb{H}}(\mathbf{H}_h, \tilde{h} \mathbf{h}_h)| \leq |\mathbf{H}_h|_{\mathbb{J}}^{\mathbb{H}} |\tilde{h} \mathbf{h}_h|_{\mathbb{J}}^{\mathbb{H}} \lesssim |\mathbf{H}_h|_{\mathbb{J}}^{\mathbb{H}} \|\tilde{h}^{\frac{1}{2}} \nabla \times \mathbf{E}_h\|_{\mathbf{L}^2(D)}$ . Moreover, still owing to (35a), we infer that  $\|\tilde{h}^{\frac{1}{2}} (\nabla \times \mathbf{E}_h - \mathcal{J}_{h\mathbb{T}}^{\mathbb{S}, \text{av}}(\nabla \times \mathbf{E}_h))\|_{\mathbf{L}^2(D)} \lesssim |\mathbf{E}_h|_{\mathbb{J}}^{\mathbb{E}}$ . Using the Cauchy–Schwarz inequality and  $\kappa_{\mathbb{H}} \kappa_{\mathbb{E}} = 1$ , (45) gives

$$\kappa_{\mathbb{E}} \|\tilde{h}^{\frac{1}{2}} \nabla \times \mathbf{E}_h\|_{\mathbf{L}^2(D)}^2 \lesssim (\omega^{-\frac{1}{2}} \mathbb{S} + \kappa_{\mathbb{H}}^{\frac{1}{2}} |\mathbf{H}_h|_{\mathbb{J}}^{\mathbb{H}} + \kappa_{\mathbb{E}}^{\frac{1}{2}} |\mathbf{E}_h|_{\mathbb{J}}^{\mathbb{E}}) \kappa_{\mathbb{E}}^{\frac{1}{2}} \|\tilde{h}^{\frac{1}{2}} \nabla \times \mathbf{E}_h\|_{\mathbf{L}^2(D)}.$$

Therefore, we conclude that

$$\begin{aligned} \kappa_{\mathbb{E}}^{\frac{1}{2}} \|\tilde{h}^{\frac{1}{2}} \nabla \times \mathbf{E}_h\|_{\mathbf{L}^2(D)} &\lesssim \omega^{-\frac{1}{2}} \mathbb{S} + \kappa_{\mathbb{H}}^{\frac{1}{2}} |\mathbf{H}_h|_{\mathbb{J}}^{\mathbb{H}} + \kappa_{\mathbb{E}}^{\frac{1}{2}} |\mathbf{E}_h|_{\mathbb{J}}^{\mathbb{E}} \\ &\lesssim \omega^{-\frac{1}{2}} \mathbb{S} + \mathbb{S}^{\frac{1}{2}} \|(\mathbf{H}_h, \mathbf{E}_h)\|_L^{\frac{1}{2}}, \end{aligned}$$

where the second bound follows from (44). Furthermore, considering the test function  $(\mathbf{0}, \kappa_{\mathbb{H}} \tilde{h} \mathcal{J}_h^{\mathbb{S}, \text{av}}(\nabla_0 \times \mathbf{H}_h))$  gives a similar bound. Altogether, we have

$$\kappa_{\mathbb{H}}^{\frac{1}{2}} \|\tilde{h}^{\frac{1}{2}} \nabla_0 \times \mathbf{H}_h\|_{\mathbf{L}^2(D)} + \kappa_{\mathbb{E}}^{\frac{1}{2}} \|\tilde{h}^{\frac{1}{2}} \nabla \times \mathbf{E}_h\|_{\mathbf{L}^2(D)} \lesssim \omega^{-\frac{1}{2}} \mathbb{S} + \mathbb{S}^{\frac{1}{2}} \|(\mathbf{H}_h, \mathbf{E}_h)\|_L^{\frac{1}{2}}. \quad (46)$$

(1b) The second step in the proof of (42) consists of controlling  $\|(\mathbf{H}_h, \mathbf{E}_h)\|_L$ . This is done by invoking the discrete Poincaré–Steklov inequalities from Lemma 4.3. To do so, we need to estimate the weak norms  $\|\nabla_0 \times \mathbf{H}_h\|_{\mathbf{X}'}$  and  $\|\nabla \times \mathbf{E}_h\|_{\mathbf{X}'}$ . To estimate  $\|\nabla_0 \times \mathbf{H}_h\|_{\mathbf{X}'}$ , we consider an arbitrary field  $\boldsymbol{\varepsilon} \in \mathbf{X}$  and set  $\boldsymbol{\varepsilon}_h := \mathcal{I}_h^{\mathbb{S}, \text{av}}(\boldsymbol{\varepsilon})$ . Notice that we have

$$\|\boldsymbol{\varepsilon}_h\|_{\mathbf{L}^2(D)} \lesssim \|\boldsymbol{\varepsilon}\|_{\mathbf{L}^2(D)} + h^s |\boldsymbol{\varepsilon}|_{\mathbf{H}^s(D)} \lesssim \ell_D \|\nabla \times \boldsymbol{\varepsilon}\|_{\mathbf{L}^2(D)}, \quad (47)$$

where the first bound follows from the triangle inequality and the approximation properties of  $\mathcal{I}_h^{\mathbb{S}, \text{av}}$ , and the second bound follows from  $h \leq \ell_D$ , the Poincaré–Steklov inequality  $\|\boldsymbol{\varepsilon}\|_{\mathbf{L}^2(D)} \lesssim \ell_D \|\nabla \times \boldsymbol{\varepsilon}\|_{\mathbf{L}^2(D)}$ , and the regularity property (7) since  $\boldsymbol{\varepsilon} \in \mathbf{X}$ . Using the definition of the sesquilinear form  $a_h$  gives

$$\begin{aligned} (\mathbf{H}_h, \nabla \times \boldsymbol{\varepsilon})_{\mathbf{L}^2(D)} &= (\nabla_0 \times \mathbf{H}_h, \boldsymbol{\varepsilon})_{\mathbf{L}^2(D)} \\ &= (\nabla_0 \times \mathbf{H}_h, \boldsymbol{\varepsilon}_h)_{\mathbf{L}^2(D)} + (\nabla_0 \times \mathbf{H}_h, \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_h)_{\mathbf{L}^2(D)} \\ &= a_h((\mathbf{H}_h, \mathbf{E}_h), (\mathbf{0}, \boldsymbol{\varepsilon}_h)) + (\nabla_0 \times \mathbf{H}_h, \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_h)_{\mathbf{L}^2(D)} - \kappa_{\mathbb{E}} s_h^{\mathbb{E}}(\mathbf{E}_h, \boldsymbol{\varepsilon}_h). \end{aligned}$$

We now bound the three terms on the right-hand side. We have

$$|a_h((\mathbf{H}_h, \mathbf{E}_h), (\mathbf{0}, \boldsymbol{\varepsilon}_h))| \leq \mathbb{S} \|\epsilon^{\frac{1}{2}} \boldsymbol{\varepsilon}_h\|_{\mathbf{L}^2(D)} \lesssim \epsilon_0^{\frac{1}{2}} \mathbb{S} \|\boldsymbol{\varepsilon}_h\|_{\mathbf{L}^2(D)} \lesssim \epsilon_0^{\frac{1}{2}} \ell_D \mathbb{S} \|\nabla \times \boldsymbol{\varepsilon}\|_{\mathbf{L}^2(D)},$$

where the last bound follows from (47). Furthermore, using the Cauchy–Schwarz inequality, the approximation properties of  $\mathcal{I}_h^{\mathbb{E}, \text{av}}$ , and the regularity property (7) gives

$$\begin{aligned} |(\nabla_0 \times \mathbf{H}_h, \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_h)_{\mathbf{L}^2(D)}| &\lesssim \|\nabla_0 \times \mathbf{H}_h\|_{\mathbf{L}^2(D)} h^s |\boldsymbol{\varepsilon}|_{\mathbf{H}^s(D)} \\ &\lesssim (h/\ell_D)^{s-\frac{1}{2}} \ell_D^{\frac{1}{2}} \|\tilde{h}^{\frac{1}{2}} \nabla_0 \times \mathbf{H}_h\|_{\mathbf{L}^2(D)} \|\nabla \times \boldsymbol{\varepsilon}\|_{\mathbf{L}^2(D)}. \end{aligned}$$

Notice that we used the quasi-uniformity of the mesh sequence to insert the factor  $\tilde{h}^{\frac{1}{2}}$  in front of  $\nabla_0 \times \mathbf{H}_h$ . Finally, invoking the consistency estimate on the stabilization (see Lemma 4.4) gives

$$|s_h^{\mathbb{E}}(\mathbf{E}_h, \boldsymbol{\varepsilon}_h)| \leq |\mathbf{E}_h|_{\mathbb{J}}^{\mathbb{E}} |\boldsymbol{\varepsilon}_h|_{\mathbb{J}}^{\mathbb{E}} \lesssim (h/\ell_D)^{s-\frac{1}{2}} \ell_D^{\frac{1}{2}} |\mathbf{E}_h|_{\mathbb{J}}^{\mathbb{E}} \|\nabla \times \boldsymbol{\varepsilon}\|_{\mathbf{L}^2(D)}.$$

Putting everything together and recalling the definition of the  $\|\cdot\|_{\mathbf{X}'}$ -norm, we infer that

$$\|\nabla_0 \times \mathbf{H}_h\|_{\mathbf{X}'} \lesssim \epsilon_0^{\frac{1}{2}} \mathbb{S} + (h/\ell_D)^{s-\frac{1}{2}} \ell_D^{-\frac{1}{2}} (\|\tilde{h}^{\frac{1}{2}} \nabla_0 \times \mathbf{H}_h\|_{\mathbf{L}^2(D)} + \kappa_{\mathbb{E}} |\mathbf{E}_h|_{\mathbb{J}}^{\mathbb{E}}).$$

Using the discrete Poincaré–Steklov inequality (39) from Lemma 4.3, which is legitimate since  $\mathbf{H}_h \in \mathbf{X}_{\mu_{\text{T}}, h}$ , and the above estimate on  $\|\nabla_0 \times \mathbf{H}_h\|_{\mathbf{X}'}$ , we infer that

$$\begin{aligned} \|\mu^{\frac{1}{2}} \mathbf{H}_h\|_{\mathbf{L}^2(D)} &\lesssim \mu_0^{\frac{1}{2}} \ell_D \{ \|\nabla_0 \times \mathbf{H}_h\|_{\mathbf{X}'} + (h/\ell_D)^s \|\nabla_0 \times \mathbf{H}_h\|_{\mathbf{L}^2(D)} \} \\ &\lesssim \mu_0^{\frac{1}{2}} \ell_D \{ \|\nabla_0 \times \mathbf{H}_h\|_{\mathbf{X}'} + (h/\ell_D)^{s-\frac{1}{2}} \ell_D^{-\frac{1}{2}} \|\tilde{h}^{\frac{1}{2}} \nabla_0 \times \mathbf{H}_h\|_{\mathbf{L}^2(D)} \} \\ &\lesssim \mu_0^{\frac{1}{2}} \ell_D \{ \epsilon_0^{\frac{1}{2}} \mathbb{S} + (h/\ell_D)^{s-\frac{1}{2}} \ell_D^{-\frac{1}{2}} (\|\tilde{h}^{\frac{1}{2}} \nabla_0 \times \mathbf{H}_h\|_{\mathbf{L}^2(D)} + \kappa_{\mathbb{E}} |\mathbf{E}_h|_{\mathbb{J}}^{\mathbb{E}}) \}. \end{aligned}$$

Notice that we used again the quasi-uniformity of the mesh sequence to insert the factor  $\tilde{h}^{\frac{1}{2}}$  in front of  $\nabla_0 \times \mathbf{H}_h$ . Since  $\mu_0^{\frac{1}{2}} \ell_D \epsilon_0^{\frac{1}{2}} = \omega^{-1}$ ,  $\mu_0 \ell_D = \omega^{-1} \kappa_{\mathbb{H}}$  and  $\kappa_{\mathbb{H}} \kappa_{\mathbb{E}} = 1$ , we obtain

$$\|\mu^{\frac{1}{2}} \mathbf{H}_h\|_{\mathbf{L}^2(D)} \lesssim \omega^{-1} \mathbb{S} + (h/\ell_D)^{s-\frac{1}{2}} \omega^{-\frac{1}{2}} (\kappa_{\mathbb{H}}^{\frac{1}{2}} \|\tilde{h}^{\frac{1}{2}} \nabla_0 \times \mathbf{H}_h\|_{\mathbf{L}^2(D)} + \kappa_{\mathbb{E}}^{\frac{1}{2}} |\mathbf{E}_h|_{\mathbb{J}}^{\mathbb{E}}).$$

We proceed similarly to establish that

$$\|\epsilon^{\frac{1}{2}} \mathbf{E}_h\|_{\mathbf{L}^2(D)} \lesssim \omega^{-1} \mathbb{S} + (h/\ell_D)^{s-\frac{1}{2}} \omega^{-\frac{1}{2}} ((\kappa_{\mathbb{E}}^{\frac{1}{2}} \|\tilde{h}^{\frac{1}{2}} \nabla \times \mathbf{E}_h\|_{\mathbf{L}^2(D)} + \kappa_{\mathbb{H}}^{\frac{1}{2}} |\mathbf{H}_h|_{\mathbb{J}}^{\mathbb{H}}).$$

Putting together the above two bounds, invoking (44) and (46) with  $h \leq \ell_D$  gives

$$\|(\mathbf{H}_h, \mathbf{E}_h)\|_{\mathbf{L}} \lesssim \omega^{-1} \mathbb{S} + \omega^{-\frac{1}{2}} \mathbb{S}^{\frac{1}{2}} \|(\mathbf{H}_h, \mathbf{E}_h)\|_{\mathbf{L}}^{\frac{1}{2}}.$$

We conclude that  $\omega^{\frac{1}{2}} \|(\mathbf{H}_h, \mathbf{E}_h)\|_{\mathbf{L}} \lesssim \omega^{-\frac{1}{2}} \mathbb{S}$  by using Young's inequality. Recalling the definition (41) of the  $\|\cdot\|_{\mathbf{b}, h}$ -norm, we obtain (42) by substituting this bound into (44) and (46) and putting everything together.

(2) Well-posedness and a priori estimate. A dimensional argument shows that the discrete boundary-value problem (33) amounts to a square linear system since  $\dim(\mathbf{X}_{\mu_{\text{T}}, h}) = \dim(\ker(\mathbf{\Pi}_{h_{\text{T}}}) \cap \mathbf{P}_{k_{\text{T}}}^{\mathbb{E}}(\mathcal{T}_h))$  and  $\dim(\mathbf{X}_{\epsilon, h}) = \dim(\ker(\mathbf{\Pi}_h) \cap \mathbf{P}_k^{\mathbb{E}}(\mathcal{T}_h))$ . The deflated inf-sup condition (42) implies uniqueness of the solution, and therefore also existence. Finally, the a priori estimate (43) follows from the Cauchy–Schwarz inequality and (42).  $\square$

## 5.2 Duality argument

Let  $(\mathbf{f}, \mathbf{g}) \in L$  and let  $(\mathbf{H}, \mathbf{E}) \in \mathbf{X}_{\mu_0} \times \mathbf{X}_\epsilon$  be the solution to the continuous problem (14). Let  $(\mathbf{H}_h, \mathbf{E}_h) \in \mathbf{X}_{\mu\tau, h} \times \mathbf{X}_{\epsilon, h}$  be the solution to the discrete problem (33). We define the errors

$$\delta \mathbf{h} := \mathbf{H} - \mathbf{H}_h, \quad \delta \mathbf{e} := \mathbf{E} - \mathbf{E}_h. \quad (48)$$

Let  $(\boldsymbol{\eta}, \boldsymbol{\varepsilon}) \in \mathbf{X}_0 \times \mathbf{X}$  be the unique solution to the dual problem

$$\nabla \times \boldsymbol{\varepsilon} = \omega(\mathbf{I} - \boldsymbol{\Pi}_0)(\mu \delta \mathbf{h}), \quad (49a)$$

$$-\nabla_0 \times \boldsymbol{\eta} = \omega(\mathbf{I} - \boldsymbol{\Pi})(\epsilon \delta \mathbf{e}). \quad (49b)$$

As shown in [28], the dual solution satisfies the a priori estimate

$$\ell_D(\mu_0^{\frac{1}{2}} \|\nabla_0 \times \boldsymbol{\eta}\|_{\mathbf{L}^2(D)} + \epsilon_0^{\frac{1}{2}} \|\nabla \times \boldsymbol{\varepsilon}\|_{\mathbf{L}^2(D)}) \lesssim \|(\delta \mathbf{h}, \delta \mathbf{e})\|_L. \quad (50)$$

Moreover, there is  $s \in (\frac{1}{2}, 1]$  so that

$$|\boldsymbol{\eta}|_{\mathbf{H}^s(D)} \lesssim \ell_D^{1-s} \|\nabla_0 \times \boldsymbol{\eta}\|_{\mathbf{L}^2(D)}, \quad |\boldsymbol{\varepsilon}|_{\mathbf{H}^s(D)} \lesssim \ell_D^{1-s} \|\nabla \times \boldsymbol{\varepsilon}\|_{\mathbf{L}^2(D)}. \quad (51)$$

**Lemma 5.2** (Regularity of  $\nabla \times \boldsymbol{\varepsilon}$  and  $\nabla_0 \times \boldsymbol{\eta}$ ). *We have  $\nabla \times \boldsymbol{\varepsilon} \in \mathbf{X}_0$  and  $\nabla_0 \times \boldsymbol{\eta} \in \mathbf{X}$ , so that  $\nabla \times \boldsymbol{\varepsilon} \in \mathbf{H}^s(D)$  and  $\nabla_0 \times \boldsymbol{\eta} \in \mathbf{H}^s(D)$ .*

*Proof.* The equality (49a) implies that  $\gamma_{\partial D}^s(\nabla \times \boldsymbol{\varepsilon}) = \mathbf{0}$  and  $\boldsymbol{\Pi}_0(\nabla \times \boldsymbol{\varepsilon}) = \mathbf{0}$ . It also implies that  $\nabla_0 \times (\nabla \times \boldsymbol{\varepsilon}) = \omega \nabla_0 \times (\mu \delta \mathbf{h}) \in \mathbf{L}^2(D)$  since  $\mu \delta \mathbf{h} \in \mathbf{H}_0(\mathbf{curl}; D)$  because  $\mu$  is smooth. Hence,  $\nabla \times \boldsymbol{\varepsilon} \in \mathbf{X}_0$ , and it readily follows from the regularity estimate (7) that  $\nabla \times \boldsymbol{\varepsilon} \in \mathbf{H}^s(D)$ . The assertion on  $\boldsymbol{\eta}$  is proved similarly.  $\square$

**Lemma 5.3** ( $L^2$ -error representation). *We have*

$$\omega \|(\delta \mathbf{h}, \delta \mathbf{e})\|_L^2 = \theta_{\text{gal}} + \theta_{\text{div}}, \quad (52)$$

where  $\theta_{\text{gal}}, \theta_{\text{div}}$  are defined by

$$\theta_{\text{gal}} := -(\nabla \times \delta \mathbf{e}, \boldsymbol{\eta})_{\mathbf{L}^2(D)} + (\nabla_0 \times \delta \mathbf{h}, \boldsymbol{\varepsilon})_{\mathbf{L}^2(D)} \quad (53a)$$

$$\theta_{\text{div}} := \omega \{(\delta \mathbf{h}, \boldsymbol{\Pi}_0(\mu \delta \mathbf{h}))_{\mathbf{L}^2(D)} + (\delta \mathbf{e}, \boldsymbol{\Pi}(\epsilon \delta \mathbf{e}))_{\mathbf{L}^2(D)}\}. \quad (53b)$$

*Proof.* Using (49) and integration by parts, we infer that

$$\begin{aligned} \omega \|(\delta \mathbf{h}, \delta \mathbf{e})\|_L^2 &= \omega \{ \|\mu^{\frac{1}{2}} \delta \mathbf{h}\|_{\mathbf{L}^2(D)}^2 + \|\epsilon^{\frac{1}{2}} \delta \mathbf{e}\|_{\mathbf{L}^2(D)}^2 \} \\ &= \omega \{ (\delta \mathbf{h}, (\mathbf{I} - \boldsymbol{\Pi}_0)(\mu \delta \mathbf{h}))_{\mathbf{L}^2(D)} + (\delta \mathbf{e}, (\mathbf{I} - \boldsymbol{\Pi})(\epsilon \delta \mathbf{e}))_{\mathbf{L}^2(D)} \} + \theta_{\text{div}} \\ &= -(\delta \mathbf{e}, \nabla_0 \times \boldsymbol{\eta})_{\mathbf{L}^2(D)} + (\delta \mathbf{h}, \nabla \times \boldsymbol{\varepsilon})_{\mathbf{L}^2(D)} + \theta_{\text{div}} \\ &= -(\nabla \times \delta \mathbf{e}, \boldsymbol{\eta})_{\mathbf{L}^2(D)} + (\nabla_0 \times \delta \mathbf{h}, \boldsymbol{\varepsilon})_{\mathbf{L}^2(D)} + \theta_{\text{div}}. \end{aligned}$$

This completes the proof.  $\square$

**Lemma 5.4** (Bound on  $\theta_{\text{gal}}$ ). *We have*

$$|\theta_{\text{gal}}| \lesssim (h/\ell_D)^{s-\frac{1}{2}} \omega \|(\mathbf{f}, \mathbf{g})\|_L \|(\delta \mathbf{h}, \delta \mathbf{e})\|_L. \quad (54)$$



*Proof.* Let us set  $\boldsymbol{\eta}_h := \mathcal{I}_{h_T}^{\text{g,av}}(\boldsymbol{\eta})$  and  $\boldsymbol{\varepsilon}_h := \mathcal{I}_h^{\text{g,av}}(\boldsymbol{\varepsilon})$ . The definition of  $\theta_{\text{gal}}$  gives

$$\begin{aligned} \theta_{\text{gal}} &= -(\nabla \times \boldsymbol{\delta e}, \boldsymbol{\eta} - \boldsymbol{\eta}_h)_{\mathbf{L}^2(D)} + (\nabla_0 \times \boldsymbol{\delta h}, \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_h)_{\mathbf{L}^2(D)} \\ &\quad - (\nabla \times \boldsymbol{\delta e}, \boldsymbol{\eta}_h)_{\mathbf{L}^2(D)} + (\nabla_0 \times \boldsymbol{\delta h}, \boldsymbol{\varepsilon}_h)_{\mathbf{L}^2(D)}. \end{aligned}$$

Let  $\theta_{\text{gal},1}$  be composed of the first two terms on the right-hand side, and  $\theta_{\text{gal},2}$  be composed of the two remaining terms.

(1) To bound  $\theta_{\text{gal},1}$ , we observe that

$$\begin{aligned} |(\nabla \times \boldsymbol{\delta e}, \boldsymbol{\eta} - \boldsymbol{\eta}_h)_{\mathbf{L}^2(D)}| &\lesssim \|\nabla \times \boldsymbol{\delta e}\|_{\mathbf{L}^2(D)} h^s |\boldsymbol{\eta}|_{\mathbf{H}^s(D)} \\ &\lesssim (\|\nabla \times \mathbf{E}\|_{\mathbf{L}^2(D)} + \|\nabla \times \mathbf{E}_h\|_{\mathbf{L}^2(D)}) (h/\ell_D)^s \ell_D \|\nabla_0 \times \boldsymbol{\eta}\|_{\mathbf{L}^2(D)} \\ &\lesssim (\|\nabla \times \mathbf{E}\|_{\mathbf{L}^2(D)} + \|\nabla \times \mathbf{E}_h\|_{\mathbf{L}^2(D)}) (h/\ell_D)^s \mu_0^{-\frac{1}{2}} \|(\boldsymbol{\delta h}, \boldsymbol{\delta e})\|_L, \end{aligned}$$

where we used the Cauchy–Schwarz inequality and the approximation properties of  $\mathcal{I}_{h_T}^{\text{g,av}}$  on the first line, the triangle inequality and the regularity property (51) on the second line, and the a priori estimate (50) on the dual solution on the third line. Since  $h \leq \ell_D$  and  $s \in (\frac{1}{2}, 1]$ , the a priori estimate (16a) gives

$$(h/\ell_D)^s \|\nabla \times \mathbf{E}\|_{\mathbf{L}^2(D)} \leq (h/\ell_D)^{s-\frac{1}{2}} \|\nabla \times \mathbf{E}\|_{\mathbf{L}^2(D)} \lesssim (h/\ell_D)^{s-\frac{1}{2}} \epsilon_0^{-\frac{1}{2}} \ell_D^{-1} \|(\mathbf{f}, \mathbf{g})\|_L,$$

whereas the a priori estimate (43) yields

$$\begin{aligned} (h/\ell_D)^s \|\nabla \times \mathbf{E}_h\|_{\mathbf{L}^2(D)} &\lesssim (h/\ell_D)^{s-\frac{1}{2}} \ell_D^{-\frac{1}{2}} \|\tilde{h}^{\frac{1}{2}} \nabla \times \mathbf{E}_h\|_{\mathbf{L}^2(D)} \\ &\lesssim (h/\ell_D)^{s-\frac{1}{2}} \ell_D^{-\frac{1}{2}} \kappa_{\text{H}}^{-\frac{1}{2}} \omega^{\frac{1}{2}} \|(\mathbf{f}, \mathbf{g})\|_L. \end{aligned}$$

Notice that we used the quasi-uniformity of the mesh sequence to insert the factor  $\tilde{h}^{\frac{1}{2}}$  in front of  $\nabla \times \mathbf{E}_h$ . Putting the above bounds together and since  $(\epsilon_0 \mu_0)^{-\frac{1}{2}} \ell_D^{-1} = \omega$  and  $(\ell_D \kappa_{\text{H}} \mu_0)^{-\frac{1}{2}} = \omega^{\frac{1}{2}}$ , we infer that

$$|(\nabla \times \boldsymbol{\delta e}, \boldsymbol{\eta} - \boldsymbol{\eta}_h)_{\mathbf{L}^2(D)}| \lesssim (h/\ell_D)^{s-\frac{1}{2}} \omega \|(\mathbf{f}, \mathbf{g})\|_L \|(\boldsymbol{\delta h}, \boldsymbol{\delta e})\|_L.$$

Similar arguments give the same bound on  $|(\nabla_0 \times \boldsymbol{\delta h}, \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_h)_{\mathbf{L}^2(D)}|$ , and altogether this gives

$$|\theta_{\text{gal},1}| \lesssim (h/\ell_D)^{s-\frac{1}{2}} \omega \|(\mathbf{f}, \mathbf{g})\|_L \|(\boldsymbol{\delta h}, \boldsymbol{\delta e})\|_L.$$

(2) To bound  $\theta_{\text{gal},2}$ , we observe that

$$\begin{aligned} \theta_{\text{gal},2} &= \omega((\mathbf{I} - \boldsymbol{\Pi}_0)(\mu \mathbf{f}), \boldsymbol{\eta}_h)_{\mathbf{L}^2(D)} + \omega((\mathbf{I} - \boldsymbol{\Pi})(\mu \mathbf{g}), \boldsymbol{\varepsilon}_h)_{\mathbf{L}^2(D)} \\ &\quad - a_h((\mathbf{H}_h, \mathbf{E}_h), (\boldsymbol{\eta}_h, \boldsymbol{\varepsilon}_h)) + \kappa_{\text{H}} s^{\text{H}}(\mathbf{H}_h, \boldsymbol{\eta}_h) + \kappa_{\text{E}}^{\text{E}}(\mathbf{E}_h, \boldsymbol{\varepsilon}_h) \\ &= -\omega((\boldsymbol{\Pi}_0 - \boldsymbol{\Pi}_{h_T})(\mu \mathbf{f}), \boldsymbol{\eta}_h)_{\mathbf{L}^2(D)} - \omega((\boldsymbol{\Pi} - \boldsymbol{\Pi}_h)(\mu \mathbf{g}), \boldsymbol{\varepsilon}_h)_{\mathbf{L}^2(D)} \\ &\quad + \kappa_{\text{H}} s^{\text{H}}(\mathbf{H}_h, \boldsymbol{\eta}_h) + \kappa_{\text{E}}^{\text{E}}(\mathbf{E}_h, \boldsymbol{\varepsilon}_h). \end{aligned}$$

Let  $\theta_{\text{gal},2a}$  and  $\theta_{\text{gal},2b}$  be composed of the first two terms and the last two terms on the right-hand side, respectively. Since  $\boldsymbol{\Pi}_0(\boldsymbol{\eta}) = \mathbf{0}$  and  $\boldsymbol{\Pi}_{h_T}(\boldsymbol{\eta}) = \boldsymbol{\Pi}_{h_T}(\boldsymbol{\Pi}_0(\boldsymbol{\eta})) = \mathbf{0}$ , we infer that

$$\begin{aligned} \omega|((\boldsymbol{\Pi}_{h_T} - \boldsymbol{\Pi}_0)(\mu \mathbf{f}), \boldsymbol{\eta}_h)_{\mathbf{L}^2(D)}| &= \omega|((\boldsymbol{\Pi}_{h_T} - \boldsymbol{\Pi}_0)(\mu \mathbf{f}), \boldsymbol{\eta}_h - \boldsymbol{\eta})_{\mathbf{L}^2(D)}| \\ &\lesssim \omega \mu_0^{\frac{1}{2}} \|\mu^{\frac{1}{2}} \mathbf{f}\|_{\mathbf{L}^2(D)} \|\boldsymbol{\eta}_h - \boldsymbol{\eta}\|_{\mathbf{L}^2(D)} \\ &\lesssim \omega \mu_0^{\frac{1}{2}} \|\mu^{\frac{1}{2}} \mathbf{f}\|_{\mathbf{L}^2(D)} (h/\ell_D)^s \ell_D \|\nabla_0 \times \boldsymbol{\eta}\|_{\mathbf{L}^2(D)} \\ &\lesssim (h/\ell_D)^s \omega \|(\mathbf{f}, \mathbf{g})\|_L \|(\boldsymbol{\delta h}, \boldsymbol{\delta e})\|_L, \end{aligned}$$

where we used the Cauchy–Schwarz inequality and the  $L^2$ -stability of  $\mathbf{\Pi}_{h_T}$  and  $\mathbf{\Pi}_0$  on the second line, the approximation properties of  $\mathcal{I}_{h_T}^{\mathbf{g}, \text{av}}$  and the regularity estimate (51) on the third line, and the a priori estimate (50) on the dual solution on the fourth line. We estimate  $\omega |((\mathbf{\Pi} - \mathbf{\Pi}_h)(\mu \mathbf{g}), \varepsilon_h)_{L^2(D)}|$  similarly and obtain

$$\begin{aligned} |\theta_{\text{gal}, 2a}| &\lesssim (h/\ell_D)^s \omega \|(\mathbf{f}, \mathbf{g})\|_L \|(\delta \mathbf{h}, \delta \mathbf{e})\|_L \\ &\lesssim (h/\ell_D)^{s-\frac{1}{2}} \omega \|(\mathbf{f}, \mathbf{g})\|_L \|(\delta \mathbf{h}, \delta \mathbf{e})\|_L, \end{aligned}$$

since  $h \leq \ell_D$  and  $s \in (\frac{1}{2}, 1]$ . We use the estimate (40) from Lemma 4.4 to estimate  $\theta_{\text{gal}, 2b}$ . This gives

$$\begin{aligned} |\theta_{\text{gal}, 2b}| &\lesssim (h/\ell_D)^{s-\frac{1}{2}} \ell_D^{\frac{1}{2}} (\kappa_{\text{H}}^{\frac{1}{2}} |\mathbf{H}_h|_{\text{J}}^{\text{H}} \kappa_{\text{H}}^{\frac{1}{2}} \|\nabla_0 \times \boldsymbol{\eta}\|_{L^2(D)} + \kappa_{\text{E}}^{\frac{1}{2}} |\mathbf{E}_h|_{\text{J}}^{\text{E}} \kappa_{\text{E}}^{\frac{1}{2}} \|\nabla \times \boldsymbol{\varepsilon}\|_{L^2(D)}) \\ &\lesssim (h/\ell_D)^{s-\frac{1}{2}} \omega^{\frac{1}{2}} \|(\mathbf{f}, \mathbf{g})\|_L \ell_D^{\frac{1}{2}} (\kappa_{\text{H}}^{\frac{1}{2}} \|\nabla_0 \times \boldsymbol{\eta}\|_{L^2(D)} + \kappa_{\text{E}}^{\frac{1}{2}} \|\nabla \times \boldsymbol{\varepsilon}\|_{L^2(D)}) \\ &\lesssim (h/\ell_D)^{s-\frac{1}{2}} \omega \|(\mathbf{f}, \mathbf{g})\|_L \|(\delta \mathbf{h}, \delta \mathbf{e})\|_L, \end{aligned}$$

where the second bound follows from the a priori estimate (43), and the third bound follows from the a priori estimate (50) on the dual solution and the identities  $\ell_D^{-1} \mu_0^{-1} \kappa_{\text{H}} = \ell_D^{-1} \epsilon_0^{-1} \kappa_{\text{E}} = \omega$ .  $\square$

**Lemma 5.5** (Bound on  $\theta_{\text{div}}$ ). *We have*

$$|\theta_{\text{div}}| \leq (h/\ell_D)^{s-\frac{1}{2}} \omega \|(\mathbf{f}, \mathbf{g})\|_L \|(\delta \mathbf{h}, \delta \mathbf{e})\|_L. \quad (55)$$

*Proof.* The proof relies on the commuting quasi-interpolation operators  $\mathcal{P}_{h_T}^{\mathbf{g}}$  and  $\mathcal{P}_h^{\mathbf{g}}$  introduced in Assumption 4.1. Let us define  $\boldsymbol{\xi} := \mathbf{H}_h - \mathbf{\Pi}_0(\mathbf{H}_h)$ . We observe that  $\boldsymbol{\xi} \in \mathbf{H}_0(\text{curl}; D)$  and  $\nabla_0 \times \boldsymbol{\xi} = \nabla_0 \times \mathbf{H}_h$ . Hence,  $\boldsymbol{\xi} \in \mathbf{Z}_T(\mathcal{T}_h)$  (see (36)). We then have  $\nabla_0 \times (\mathcal{P}_{h_T}^{\mathbf{g}}(\boldsymbol{\xi}) - \mathbf{H}_h) = \mathbf{0}$  owing to (38a), so that  $\mathcal{P}_{h_T}^{\mathbf{g}}(\boldsymbol{\xi}) - \mathbf{H}_h \in \mathbf{P}_{k_T}^{\mathbf{g}}(\text{curl} = \mathbf{0}; \mathcal{T}_h)$ . Moreover, we have  $\mathbf{\Pi}_{h_T}(\mathbf{\Pi}_0(\mu \delta \mathbf{h})) = -\mathbf{\Pi}_{h_T}(\mathbf{\Pi}_0(\mu \mathbf{H}_h)) = \mathbf{0}$  owing to Lemma 5.6, so that  $\mathbf{\Pi}_0(\mu \delta \mathbf{h}) \in \mathbf{P}_{k_T}^{\mathbf{g}}(\text{curl} = \mathbf{0}; \mathcal{T}_h)^\perp$ . This implies that

$$(\mathcal{P}_{h_T}^{\mathbf{g}}(\boldsymbol{\xi}) - \mathbf{H}_h, \mathbf{\Pi}_0(\mu \delta \mathbf{h}))_{L^2(D)} = 0.$$

Now we observe that  $\mathbf{\Pi}_0(\mathbf{H}) \in \mathbf{Z}_T(\mathcal{T}_h)$  and  $\nabla_0 \times \mathcal{P}_{h_T}^{\mathbf{g}}(\mathbf{\Pi}_0(\mathbf{H})) = \mathbf{0}$ . This implies that  $\mathcal{P}_{h_T}^{\mathbf{g}}(\mathbf{\Pi}_0(\mathbf{H})) \in \mathbf{P}_{k_T}^{\mathbf{g}}(\text{curl} = \mathbf{0}; \mathcal{T}_h)$ . Hence, using that  $\mathbf{\Pi}_0(\boldsymbol{\xi}) = \mathbf{0}$ , we infer that

$$\begin{aligned} (\delta \mathbf{h}, \mathbf{\Pi}_0(\mu \delta \mathbf{h}))_{L^2(D)} &= (\mathbf{H} - \mathbf{H}_h, \mathbf{\Pi}_0(\mu \delta \mathbf{h}))_{L^2(D)} \\ &= (\mathbf{H} - \mathbf{H}_h - (\mathcal{P}_{h_T}^{\mathbf{g}}(\boldsymbol{\xi}) - \mathbf{H}_h), \mathbf{\Pi}_0(\mu \delta \mathbf{h}))_{L^2(D)} \\ &= (\mathbf{H} - \mathcal{P}_{h_T}^{\mathbf{g}}(\boldsymbol{\xi}), \mathbf{\Pi}_0(\mu \delta \mathbf{h}))_{L^2(D)} \\ &= (\mathbf{\Pi}_0(\mathbf{H}) - \mathcal{P}_{h_T}^{\mathbf{g}}(\mathbf{\Pi}_0(\mathbf{H})) + \boldsymbol{\xi} - \mathcal{P}_{h_T}^{\mathbf{g}}(\boldsymbol{\xi}), \mathbf{\Pi}_0(\mu \delta \mathbf{h}))_{L^2(D)}. \end{aligned}$$

Invoking the triangle and the Cauchy–Schwarz inequality gives

$$\begin{aligned} |(\delta \mathbf{h}, \mathbf{\Pi}_0(\mu \delta \mathbf{h}))_{L^2(D)}| &\lesssim \|(\mathbf{I} - \mathcal{P}_{h_T}^{\mathbf{g}})(\mathbf{\Pi}_0(\mathbf{H}))\|_{L^2(D)} \\ &\quad + \|(\mathbf{I} - \mathcal{P}_{h_T}^{\mathbf{g}})(\boldsymbol{\xi})\|_{L^2(D)} \mu_0^{\frac{1}{2}} \|\mu^{\frac{1}{2}} \delta \mathbf{h}\|_{L^2(D)}. \end{aligned}$$

We now bound  $\|(\mathbf{I} - \mathcal{P}_{h_T}^{\mathbf{g}})(\mathbf{\Pi}_0(\mathbf{H}))\|_{L^2(D)}$  and  $\|(\mathbf{I} - \mathcal{P}_{h_T}^{\mathbf{g}})(\boldsymbol{\xi})\|_{L^2(D)}$ . We observe that  $|\mathbf{\Pi}_0(\mathbf{H})|_{\mathbf{H}^s(D)} \leq |\mathbf{H}|_{\mathbf{H}^s(D)} + |\mathbf{H} - \mathbf{\Pi}_0(\mathbf{H})|_{\mathbf{H}^s(D)} \lesssim \ell_D^{1-s} \|\nabla_0 \times \mathbf{H}\|_{L^2(D)}$  owing to the regularity estimate (7) since  $\mathbf{H} \in \mathbf{X}_{\mu_0}$  and  $\mathbf{H} - \mathbf{\Pi}_0(\mathbf{H}) \in \mathbf{X}_0$  with  $\nabla_0 \times (\mathbf{H} - \mathbf{\Pi}_0(\mathbf{H})) = \nabla_0 \times \mathbf{H}$ . Using the approximation property (37b) together with  $\nabla_0 \times \mathbf{\Pi}_0(\mathbf{H}) = \mathbf{0}$ , and recalling the a priori estimate (16a), this gives

$$\begin{aligned} \|(\mathbf{I} - \mathcal{P}_{h_T}^{\mathbf{g}})(\mathbf{\Pi}_0(\mathbf{H}))\|_{L^2(D)} &\lesssim h^s |\mathbf{\Pi}_0(\mathbf{H})|_{\mathbf{H}^s(D)} \\ &\lesssim (h/\ell_D)^s \ell_D \|\nabla_0 \times \mathbf{H}\|_{L^2(D)} \\ &\lesssim (h/\ell_D)^s \mu_0^{-\frac{1}{2}} \|(\mathbf{f}, \mathbf{g})\|_L. \end{aligned}$$

Moreover,  $\mathbf{\Pi}_0(\boldsymbol{\xi}) = \mathbf{0}$  implies that  $\boldsymbol{\xi} \in \mathbf{X}_0$ ; hence,  $|\boldsymbol{\xi}|_{\mathbf{H}^s(D)} \lesssim \ell_D^{1-s} \|\nabla_0 \times \mathbf{H}_h\|_{\mathbf{L}^2(D)}$  owing to (7). Using the approximation property (37b) together with  $\nabla_0 \times \boldsymbol{\xi} = \nabla_0 \times \mathbf{H}_h$ , and recalling the a priori estimate (43), this implies that

$$\begin{aligned} \|\boldsymbol{\xi} - \mathcal{P}_{h_T}^g(\boldsymbol{\xi})\|_{\mathbf{L}^2(D)} &\lesssim h^s (|\boldsymbol{\xi}|_{\mathbf{H}^s(D)} + \ell_D^{1-s} \|\nabla_0 \times \boldsymbol{\xi}\|_{\mathbf{L}^2(D)}) \\ &\lesssim (h/\ell_D)^s \ell_D \|\nabla_0 \times \mathbf{H}_h\|_{\mathbf{L}^2(D)} \\ &\lesssim (h/\ell_D)^{s-\frac{1}{2}} \ell_D^{\frac{1}{2}} \|\tilde{h}^{\frac{1}{2}} \nabla_0 \times \mathbf{H}_h\|_{\mathbf{L}^2(D)} \\ &\lesssim (h/\ell_D)^{s-\frac{1}{2}} \kappa_H^{-\frac{1}{2}} \omega^{\frac{1}{2}} \ell_D^{\frac{1}{2}} \|(\mathbf{f}, \mathbf{g})\|_L. \end{aligned}$$

Notice that we used once again the quasi-uniformity of the mesh sequence to insert the factor  $\tilde{h}^{\frac{1}{2}}$  in front of  $\nabla_0 \times \mathbf{H}_h$ . Combining the above results and using the identity  $\ell_D \mu_0 \kappa_H^{-1} = \omega^{-1}$  gives altogether

$$\omega |(\boldsymbol{\delta h}, \mathbf{\Pi}_0(\mu \boldsymbol{\delta h}))_{\mathbf{L}^2(D)}| \lesssim (h/\ell_D)^{s-\frac{1}{2}} \omega \|(\mathbf{f}, \mathbf{g})\|_L \|(\boldsymbol{\delta h}, \boldsymbol{\delta e})\|_L.$$

Invoking similar arguments gives the same bound on  $\omega |(\boldsymbol{\delta e}, \mathbf{\Pi}(\mu \boldsymbol{\delta e}))_{\mathbf{L}^2(D)}|$ . This completes the proof.  $\square$

**Lemma 5.6.** *We have  $\mathbf{\Pi}_{h_T} \circ \mathbf{\Pi}_0 = \mathbf{\Pi}_{h_T}$  and  $\mathbf{\Pi}_h \circ \mathbf{\Pi} = \mathbf{\Pi}_h$ .*

*Proof.* Let  $\mathbf{h} \in \mathbf{L}^2(D)$ . Then,  $\mathbf{\Pi}_0(\mathbf{h}) - \mathbf{h} \in \mathbf{H}_0(\mathbf{curl} = \mathbf{0}; D)^\perp$  so that the following holds for all  $\mathbf{z}_h \in \mathbf{P}_{k_T}^g(\mathbf{curl} = \mathbf{0}; \mathcal{T}_h) \subset \mathbf{H}_0(\mathbf{curl} = \mathbf{0}; D)$ :

$$(\mathbf{\Pi}_{h_T}(\mathbf{\Pi}_0(\mathbf{h}) - \mathbf{h}), \mathbf{z}_h)_{\mathbf{L}^2(D)} = (\mathbf{\Pi}_0(\mathbf{h}) - \mathbf{h}, \mathbf{z}_h)_{\mathbf{L}^2(D)} = 0.$$

This proves the first assertion. The second assertion is proved similarly.  $\square$

### 5.3 Main result

We are now ready to state the main result of the paper which establishes the convergence of  $T_h$  to  $T$  as  $h \rightarrow 0$  in  $\mathcal{L}(L; L)$ . We equip the functional space  $L$  with the  $\|\cdot\|_L$ -norm defined in (15). This norm is equivalent to the canonical norm in  $\mathbf{L}^2(D) \times \mathbf{L}^2(D)$ . Owing to standard spectral approximation results (see, e.g., Bramble and Osborn [9, Lem. 2.2], Osborn [35, Thm. 3&4], Descloux et al. [20, Thm. 2], Boffi [4, Prop. 7.4]), Theorem 5.7 proves the spectral correctness of the approximation technique considered in the paper.

**theorem 5.7** (Convergence). *Assume that  $D$ ,  $\mu$ ,  $\epsilon$  meet the assumptions stated in §2.1. Assume that the mesh sequence  $(\mathcal{T}_h)_{h \in \mathcal{H}}$  is quasi-uniform and that Assumption 4.1 is met. Then,  $\lim_{h \rightarrow 0} \|T - T_h\|_{\mathcal{L}(L; L)} = 0$ .*

*Proof.* Invoking the error estimate (52) together with the bounds (54) and (55) implies that that for all  $h \in \mathcal{H}$ ,  $\|T - T_h\|_{\mathcal{L}(L; L)} \lesssim (h/\ell_D)^{s-\frac{1}{2}}$ . The assertion follows from  $s > \frac{1}{2}$ .  $\square$

## 6 Numerical experiments

In this section, we illustrate the method described in the paper.

## 6.1 Numerical details

We solve the eigenvalue problem (1) with continuous finite elements of degree 1, 2, and 3 in dimension 2 on simplicial meshes. We set  $\mu = 1$ ,  $\varepsilon = 1$ , and  $\omega = 1$ . We consider both Cartesian meshes (obtained by splitting the squares of a Cartesian mesh along the main diagonal) and unstructured, quasi-uniform Delaunay meshes; unless specified otherwise, the meshes are of Delaunay type. The tests with  $k = 1$  are done on Powell–Sabin splits (PS). The tests with  $k = 2$  and  $k = 3$  are done on Powell–Sabin and Alfeld (Clough–Tocher) splits (ACT). The eigenpairs are computed using the software Arpack documented in Lehoucq et al. [34].

As the space dimension is 2, the variable  $\mathbf{H}$  is a vector field in  $\mathbb{C}^2$  and  $\mathbf{E}$  is a scalar field in  $\mathbb{C}$ . Setting  $\mathbf{H} = (H_1, H_2)$  and  $\mathbf{E} = E$ , we have  $\nabla \times \mathbf{H} = \partial_{x_1} H_2 - \partial_{x_2} H_1$  and  $\nabla \times \mathbf{E} = (\partial_{x_2} E, -\partial_{x_1} E)$ .

Given an exact eigenvalue  $\lambda$  and its approximation  $\lambda_h$  obtained with the technique described in the paper, we use the definition  $\text{Error}(\lambda) := 1 - \frac{\lambda}{\lambda_h}$  to compute the relative error on  $\lambda$ . Given a list  $\mathcal{N} := \{1:N\}$  of eigenvalues, we use the notation  $\text{GErr}(\mathcal{N}) := \left( \sum_{n \in \mathcal{N}} \left( \frac{1}{\lambda_n} - \frac{1}{\lambda_{h,n}} \right)^2 / \sum_{n \in \mathcal{N}} \frac{1}{\lambda_n^2} \right)^{\frac{1}{2}}$ . In all the tables, the symbol  $I$  refers to the number of grid points in each mesh.

## 6.2 Unit square

We consider the domain  $D := (0, 1)^2$ . The eigenvalues are  $\lambda_{n,m} = i(\pi(n^2 + m^2))^{\frac{1}{2}}$ ,  $n, m \in \mathbb{N}$  with  $n + m \geq 1$  and  $i^2 = -1$ . The associated eigenvectors are

$$\mathbf{H}_{nm}(x_1, x_2) = \frac{\pi}{\lambda_{nm}} (m \cos(n\pi x_1) \sin(m\pi x_2), -n \sin(n\pi x_1) \cos(m\pi x_2))^{\top}, \quad (56a)$$

$$E_{nm}(x_1, x_2) = \cos(n\pi x_1) \sin(m\pi x_2). \quad (56b)$$

We show in Table 1 convergence tests on the first 40 eigenvalues (counting the algebraic multiplicity) for  $\mathbb{P}_1$  elements on Powell–Sabin splits. We test Cartesian and Delaunay meshes. We observe that the convergence rate is approximately equal to 2 for both types of meshes.

Table 1: Unit square.  $\mathbb{P}_1$  elements. Powell–Sabin splits. Left: Delaunay meshes. Right: Cartesian meshes

$I$	GErr(40)	rate	$I$	GErr(40)	rate
1651	1.06E-02	–	641	3.00E-02	–
2877	5.90E-03	2.09	2481	7.24E-03	2.10
11237	1.45E-03	2.06	9761	1.79E-03	2.04
44625	3.60E-04	2.02	38721	4.43E-04	2.02
100165	1.59E-04	2.01	154241	1.10E-04	2.01

We show in Table 2 convergence tests on the first 40 eigenvalues (counting the algebraic multiplicity) for  $\mathbb{P}_2$  and  $\mathbb{P}_3$  elements on Powell–Sabin and Alfeld splits using Cartesian meshes. The same tests using Delaunay meshes are shown in Table 3. We observe that the convergence rate for the  $\mathbb{P}_2$  approximation using Powell–Sabin splits is close to 5, whereas it seems to be close to 4.5 with Alfeld splits. The convergence rate for the  $\mathbb{P}_3$  approximation is 6 for both Powell–Sabin and Alfeld splits. The above observations for  $\mathbb{P}_2$  and  $\mathbb{P}_3$  approximations hold for both the Cartesian meshes and the Delaunay meshes.

## 6.3 L-shaped domain

We finish with tests on the two-dimensional L-shaped domain  $D := (-1, 1)^2 \setminus (0, 1) \times (-1, 0)$  with homogeneous tangential boundary conditions on  $\mathbf{H}$ . The first eigenfunction is not smooth; it is in  $\mathbf{H}^s(D) \times C^\infty(D)$  for all  $s < \frac{2}{3}$ .

Table 2: Unit square.  $\mathbb{P}_2$  and  $P_3$  elements. Powell–Sabin and Alfeld splits. Cartesian meshes

$\mathbb{P}_2$ , PS			$\mathbb{P}_2$ , ACT			$\mathbb{P}_3$ , PS			$\mathbb{P}_3$ , ACT		
$I$	GlErr(40)	rate	$I$	GlErr(40)	rate	$I$	GlErr(40)	rate	$I$	GlErr(40)	rate
1233	1.24E-04	–	1241	2.34E-04	–	1411	9.87E-06	–	1009	5.30E-05	–
2481	2.22E-05	4.92	2761	4.51E-05	4.12	3553	6.42E-07	5.92	2761	2.44E-06	6.12
5521	3.00E-06	5.00	4881	1.34E-05	4.25	5521	1.73E-07	5.95	3961	7.96E-07	6.21
9761	7.15E-07	5.03	19361	6.27E-07	4.45	9283	3.91E-08	5.73	7009	1.40E-07	6.08
21841	9.50E-08	5.01	49409	7.22E-08	4.61	14017	1.60E-08	4.33	10921	3.91E-08	5.76

Table 3: Unit square.  $\mathbb{P}_2$  and  $P_3$  elements. Powell–Sabin and Alfeld splits. Delaunay meshes

$\mathbb{P}_2$ , PS			$\mathbb{P}_2$ , ACT			$\mathbb{P}_3$ , PS			$\mathbb{P}_3$ , ACT		
$I$	GlErr(40)	rate	$I$	GlErr(40)	rate	$I$	GlErr(40)	rate	$I$	GlErr(40)	rate
881	2.84E-04	–	441	1.74E-03	–	976	4.59E-05	–	976	4.59E-05	–
2937	1.11E-05	5.38	1469	1.24E-04	4.39	3274	8.69E-07	6.55	3274	8.69E-07	6.55
4201	4.24E-06	5.37	3241	2.10E-05	4.49	4690	2.85E-07	6.21	4690	2.85E-07	6.21
6481	1.34E-06	5.33	5673	5.94E-06	4.52	7246	7.67E-08	6.03	7246	7.67E-08	6.03
9241	5.34E-07	5.17	22313	2.41E-07	4.68	10342	2.84E-08	5.58	12703	1.83E-08	5.11

We compute the first ten eigenvalues. Very accurate approximations of the square of these values are available at [18]. We show in Table 4 the error for each of these eigenvalues using  $\mathbb{P}_1$  finite elements on Powell–Sabin splits using quasi-uniform Delaunay meshes. We only show the errors for  $\lambda_1, \lambda_3, \lambda_5, \lambda_7, \lambda_9$ , as  $\lambda_1 = -\lambda_2, \lambda_3 = -\lambda_4, \lambda_5 = -\lambda_6, \lambda_7 = -\lambda_8, \lambda_9 = -\lambda_{10}$ . We observe that the first eigenvalue converges with a rate close to  $\frac{4}{3} = 2 \times \frac{2}{3}$ . The other eigenvalues converge with rate 2.

Table 4: L-shaped domain. First 10 eigenvalues.  $\mathbb{P}_1$  elements, Powell–Sabin splits.

$I$	Error( $\lambda_1$ )	rate	Error( $\lambda_3$ )	rate	Error( $\lambda_5$ )	rate	Error( $\lambda_7$ )	rate	Error( $\lambda_9$ )	rate
2211	6.41E-03	–	2.23E-04	–	6.19E-04	–	6.17E-04	–	7.10E-04	–
8459	2.38E-03	1.48	5.52E-05	2.08	1.53E-04	2.08	1.54E-04	2.07	1.76E-04	2.08
33471	1.33E-03	0.84	1.36E-05	2.04	3.76E-05	2.04	3.78E-05	2.04	4.35E-05	2.03
132833	5.36E-04	1.32	3.22E-06	2.09	9.37E-06	2.02	9.46E-06	2.01	1.08E-05	2.02
530571	1.94E-04	1.47	9.45E-07	1.77	2.21E-06	2.08	2.41E-06	1.97	2.62E-06	2.05

Finally, we show in Table 5 convergence results for the first eigenvalue using  $\mathbb{P}_2$  and  $\mathbb{P}_3$  finite elements. We test both the Powell–Sabin and Alfeld splits. We observe that the method converges, as expected, with a rate close to  $\frac{4}{3}$ . Tests on large clusters of eigenvalue, not reported here for brevity, show that the approximation is spurious-free and that the method converges in this case as well.

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Table 5: L-shape domain. First (and second) eigenvalue. Col. 1-3:  $\mathbb{P}_2$ , Powell–Sabin split. Col. 4-6:  $\mathbb{P}_2$ , Alfeld split. Col. 7-9:  $\mathbb{P}_3$ , Powell–Sabin split. Col. 10-12:  $\mathbb{P}_3$ , Alfeld split.

$\mathbb{P}_2$ , PS			$\mathbb{P}_2$ , ACT			$\mathbb{P}_3$ , PS			$\mathbb{P}_3$ , ACT		
$I$	Error( $\lambda_1$ )	rate	$I$	Error( $\lambda_1$ )	rate	$I$	Error( $\lambda_1$ )	rate	$I$	Error( $\lambda_1$ )	rate
2361	5.66E-03	–	4341	4.91E-03	–	19411	1.13E-03	–	9706	2.23E-03	–
8681	2.31E-03	1.38	16745	1.87E-03	1.43	43291	6.40E-04	1.41	37555	8.52E-04	1.42
33489	8.80E-04	1.43	66621	1.03E-03	0.87	75109	4.30E-04	1.44	149656	4.70E-04	0.86
133241	4.84E-04	0.86	265025	4.14E-04	1.32	146527	3.26E-04	0.83	233422	3.41E-04	1.45
406409	2.21E-04	1.41	500837	2.66E-04	1.40	315157	2.19E-04	1.04	335224	2.49E-04	1.74

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