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Adaptive input redundancy of polynomial parameter-dependent systems

Valessa V. Viana, Jérémie Kreiss, and Marc Jungers. *

Abstract

The concept of adaptive input redundancy is introduced in this paper to extend the notion of input redundancy to parameter-dependent systems. Sufficient conditions to verify the adaptive input redundancy of polynomial parameter-dependent systems are provided. It is important to point out that although the obtained conditions depend on the parameter, they can be practically verified in a computational algorithm. The developed strategy uses concepts from the geometric control theory, such as the notion of controlled invariant and output invisible subspaces. Another crucial tool to derive the results of this work is the Smith Normal form of a matrix. A numerical example is provided to illustrate the strategy.

1 Introduction

The output of a system might not be uniquely generated by one specific input. In contrast, two or more different input functions may produce the same output trajectory. This concept is known as input redundancy [1–3]. An input redundant system exhibits several degrees of freedom concerning the selection of inputs responsible for generating the desired output. These degrees of freedom can be leveraged to meet additional requirements beyond the main control task, as investigated in the context of the control allocation problems [4,5]. The system inputs can be allocated with the aim of, for example, minimizing the total energy required, strengthening the resistance to failure, or promoting one input over another to reduce power consumption [6].

The characterization of input redundancy of the system plays an important role in how the control allocation problem is defined. These techniques use the redundancy information of the system to state the control allocation problem. Therefore, an inadequate characterization of input redundancy can result in a control allocation strategy that will potentially demonstrate limitations due to the lack of full access to the redundancy of the system [3].

In this context, [1] proposed an input redundancy characterization for linear time-invariant (LTI) systems. The concepts of strong and weak input redundancy were introduced along with their characterization based on the null space dimension of the input matrices and the steady state plant transfer function matrix, respectively. Subsequently, [2] proposed a new characterization of these concepts under the assumption that the system is right-invertible, strictly proper,

and minimal. Geometric control theory tools began to be used in this context. More recently, [3] proposed a reformulation of the input redundancy definition based on the idea that an output can be generated by at least two different inputs from the same initial condition. A formal framework is proposed to support this idea, along with a tractable and complete characterization of this concept for LTI systems using geometric control tools. This paper aims to provide a unified definition among the various ones proposed in the literature. See [3] for more details on the differences between the mentioned works.

In the last decades, systems whose operation is influenced by parameters have gained significant importance in the field of control theory. These parameters that affect the system may be responsible for representing non-deterministic parameters, unmodeled dynamics, or non-linearities. Consequently, this representation is recognized as a more realistic model compared to the LTI representation. In the context of the characterization of input redundancy for parameter-dependent systems, [7] introduced the notion of robust input redundancy for systems subject to unknown parameters. This concept means that the input redundancy property and the degrees of freedom associated are not parameter-dependent. [7] also provided a characterization of this concept by using geometric control tools.

Differently from [7], in this paper, we deal with parameter-dependent systems in which the parameter is accessible for measurement, i.e., the parameter is known. In this context, an input redundancy that depends on the parameter can be considered. A parameter-dependent system might not be robust input redundant but still manifests some input redundancy that may depend on the parameter values. For instance, a system may present input redundancy properties for some values of parameters but not for others. Therefore, a more comprehensive notion of input redundancy for parameter-dependent systems emerges, in which the redundancy adapts according to the parameter.

In this paper, we aim to introduce a new definition of input redundancy in the context of parameter-dependent systems, which is called *adaptive input redundancy*. Obtaining a complete characterization of this concept is a difficult problem. Therefore, the core idea of the paper is to propose a practical approach to verify the adaptive input redundancy of linear parameter-dependent systems. The strategy is developed for systems with polynomial dependence on the parameter, where the parameter is subject to variations in time. The geometric concepts of controlled invariant subspaces are used to meet the results. In particular, the contributions of the article are: (i) the extension of the concept of input redundancy to a new one that ex-

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exploits the parameter to have a more general notion of input redundancy for parameter-dependent systems, (ii) the development of practically tractable sufficient conditions to check if a polynomial parameter-dependent system is adaptive input redundant.

The paper is organized as follows. In Section II, we present the system description and the adaptive input redundancy concept. We recall some tools from the geometric control theory for parameter-dependent systems in Section III. Section IV presents the Smith Normal form and some mathematical preliminaries related to parameter-dependent subspaces. A technical result and the main results related to the adaptive input redundancy are presented in Section V. Section VI provides a numerical example to highlight the applicability of our strategy. Finally, in Section VII, some concluding remarks are presented.

Notation. Let \mathbb{R} , \mathbb{R}^+ , \mathbb{N} , and \mathbb{N}^* be, respectively, the set of (i) real numbers, (ii) real positive numbers including zero, (iii) natural numbers including zero, and (iv) natural numbers without zero. For a matrix M , M^\top means its transpose. $0_{n \times m}$ stands for the zero matrix of n rows and m columns, and I_n stands for the identity matrix of dimension n . The notation $\text{diag}_{i=0}^N M_i$ denotes the block diagonal matrix where M_i is the i -th block. $\text{Im}\{V\}$ is the vector space spanned by the columns of matrix V . The inverse map of set \mathcal{V} by an application B is given by $B^{-1}\mathcal{V} := \{u \in \mathbb{R}^m; Bu \in \mathcal{V}\}$, where B is not necessarily invertible. The kernel of a parameter-dependent matrix is given by $\ker\{B(\theta)\} := \{u \in \mathbb{R}^m; B(\theta)u = 0\}$, which depends on the parameter θ . For polynomial functions $\beta_1(\cdot)$ and $\beta_2(\cdot)$, the notation $\beta_1 \mid \beta_2$ means that β_2 is divisible by β_1 . The term y_{u_a} denotes the output trajectory produced by the system when excited by an input trajectory u_a . For time functions, the notation $u_a \neq u_b$ means that $\int \|u_a(\tau, \theta(\tau)) - u_b(\tau, \theta(\tau))\| d\tau > 0$. In order not to make notations heavy, depending on the context, θ will denote sometimes a time function and sometimes just a real value.

2 System description and input redundancy

Consider a linear parameter-dependent system:

$$\begin{aligned} \dot{x}(t) &= A(\theta(t))x(t) + B(\theta(t))u(t), \\ y(t) &= C(\theta(t))x(t) + D(\theta(t))u(t), \end{aligned} \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the state with initial condition $x(0) := x_0 \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ is the input, $y(t) \in \mathbb{R}^p$ is the output of the system, $\theta(t) \in \Theta \subset \mathbb{R}$ is a known scalar time-varying parameter, and Θ a bounded set. The parameter $\theta(t)$ is piecewise continuous with a finite number of discontinuities in a finite length interval. Moreover, $A(\theta(t))$, $B(\theta(t))$, $C(\theta(t))$, $D(\theta(t))$ are matrices of appropriate dimension depending polynomially on the parameter $\theta(t)$. Since the system has linear dynamics on the state $x(t)$, the system matrices are polynomial on $\theta(t)$, Θ is bounded, and $t \mapsto \theta(t)$ is piecewise continuous, the existence and uniqueness of the solutions of system (1) are ensured [8].

In this work, we would like to investigate the input redundancy of parameter-dependent systems, represented by system (1). The parameter affects the input redundancy of the system, which means that the redundancy adapts according to the parameter. Therefore, based on the definitions given in [3] and [7], we introduce the definition of adaptive input redundancy.

Definition 1. *The parameter-dependent system (1) is adaptive input redundant if there exists an output y that can be produced by, at least, two distinct parameter-dependent inputs, for some function θ and some initial condition, i.e., there exist $\theta : t \rightarrow \theta(t)$, $x_0 \in \mathbb{R}^n$, and two distinct input functions $u_a \neq u_b$, where $u_a : (t, \theta) \rightarrow u_a(t, \theta)$ and $u_b : (t, \theta) \rightarrow u_b(t, \theta)$, such that $y_{u_a}(t, \theta(t), x_0) = y_{u_b}(t, \theta(t), x_0)$ for all $t \geq 0$.*

This paper aims to provide tractable conditions for which a system is adaptive input redundant, i.e., when the input redundancy of the system depends on the parameter.

3 Background on geometric control theory

The fundamental concepts for characterizing input redundancy originate from the geometric control theory [9, 10]. The controlled invariant and output invisible subspaces have been one of the main tools used in the problem [2, 3, 7]. Then, let us recall this concept of the geometric control theory for parameter-dependent systems. We can mention the definitions of generalized and generalized adaptively controlled invariant and output invisible subspaces [7, 11–13]. Both subspaces are independent of the parameters. The differences between them rely on the fact that the input trajectories making the subspace invariant and output invisible must be independent of the parameters in the generalized subspaces, whereas they are allowed to depend on them for the generalized adaptively ones. The generalized subspaces can be considered as a particular case of the generalized adaptively ones. Here we are interested in inputs that possibly depend on the parameter, so we focus on the generalized adaptively subspaces. The definition of such a subspace is presented in the following.

Definition 2. [13] *A subspace $\mathcal{V} \subseteq \mathbb{R}^n$ is a generalized adaptively controlled invariant and output invisible subspace if for any $x_0 \in \mathcal{V}$, there exists an input function $u(t, \theta(t))$, where $\theta : \mathbb{R}^+ \rightarrow \mathbb{R}^p$, such that $x_u(t, \theta(t), x_0) \in \mathcal{V}$ and $y_u(t, \theta(t), x_0) = 0$, for all $\theta(t) \in \mathbb{R}^p$ and $t \geq 0$.*

For system (1), a subspace $\mathcal{V} \subseteq \mathbb{R}^n$ is called generalized adaptively controlled invariant and output invisible if there exists a matrix $F : \Theta \rightarrow \mathbb{R}^{m \times n}$ such that, for all $\theta \in \Theta \subset \mathbb{R}$,

$$\begin{aligned} (A(\theta) + B(\theta)F(\theta))\mathcal{V} &\subseteq \mathcal{V}, \\ (C(\theta) + D(\theta)F(\theta))\mathcal{V} &= 0. \end{aligned} \quad (2)$$

The set of these subspaces admits a maximal element which is called here \mathcal{V}^* that is the largest generalized adaptively

controlled invariant and output invisible subspace. Characterizing this subspace is not an easy task. Some works have proposed practical techniques for computing this subspace [11, 12, 14, 15]. Recently, [13] proposed a practical way to compute this subspace for parameter-dependent systems depending polynomially on the parameter and under the assumption that the friend also depends polynomially on θ . This work also developed an iterative algorithm to compute \mathcal{V}^* and a procedure to compute a friend of this subspace. Note that a matrix $F(\theta)$ verifying (2) is not necessarily unique. Accordingly, we denote here $\mathbf{F}(\mathcal{V})$ as the set of all the friends, depending on θ , of the subspace \mathcal{V} .

For the remaining part of the paper, we will consider the following assumption.

Assumption 1. *We assume that the largest generalized adaptively controlled invariant and output invisible subspace is already known.*

Remark 1. *It is important to highlight that to use the technique from [13] to compute the largest subspace \mathcal{V}^* , Θ has to be a compact set since [13] develops its results based on the assumption that the parameters lie in a compact set.*

4 Preliminaries

This section presents some concepts and relations that will be useful to derive the results. First, we recall the Smith normal form. After, we present a practical way of computing the kernel of parameter-dependent matrices using the Smith normal form. To finish, we also present a practical way to compute the inverse map of a subspace \mathcal{V} by a parameter-dependent matrix.

4.1 Smith normal form

The Smith normal form can be derived for matrices with polynomial entries as matrices of system (1) [16].

Definition 3. [17] *Let $\theta \in \mathbb{R}$. Consider a matrix $\theta \mapsto W(\theta) \in \mathbb{R}^{n \times m}$ of normal rank r , this matrix is equivalent to a matrix in the following form:*

$$S(\theta) = \begin{bmatrix} \beta_1(\theta) & 0 & \cdots & 0 & 0 \\ 0 & \beta_2(\theta) & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \beta_r(\theta) & 0 \\ 0 & 0 & \cdots & 0 & 0_{n-r \times m-r} \end{bmatrix}, \quad (3)$$

where $\beta_i(\theta)$ for $i = \{1, \dots, r\}$ are non-null polynomial functions on θ such that $\beta_i \mid \beta_{i+1}, 1 \leq i \leq r-1$. $S(\theta)$ is called the Smith normal form of $W(\theta)$.

The Smith normal form of $W(\theta)$ can be obtained after a combination of a finite number of elementary operations applied over the rows and columns of $W(\theta)$. Moreover, non-singular applications $P : \mathbb{R} \rightarrow \mathbb{R}^{n \times n}$ and $Q : \mathbb{R} \rightarrow \mathbb{R}^{m \times m}$ can be defined, such that

$$P(\theta)W(\theta)Q(\theta) = S(\theta). \quad (4)$$

This form is valid for matrices that depend polynomially only on one scalar parameter, which allows to use Euclidian division of polynomials, corresponding with the framework of this paper, where $\Theta \subset \mathbb{R}$.

4.2 Kernel of a parameter-dependent matrix

Computing the kernel of a parameter-dependent matrix is not an easy task since we have an infinite number of values for the parameter, so we have to analyze the kernel of such a matrix for each parameter value. An alternative and more efficient way of computing such a kernel is to use the Smith normal form.

Lemma 1. *Consider a matrix $W(\theta) \in \mathbb{R}^{n \times m}$. Define its associated Smith normal form by $S(\theta)$ as in (3), $Q(\theta)$, and $P(\theta)$ such that (4) holds. Then,*

$$\ker\{W(\theta)\} = Q(\theta)\ker\{S(\theta)\}, \quad (5)$$

such that

$$\ker\{S(\theta)\} = \text{Im} \left\{ \begin{bmatrix} \text{diag}_{i=1}^r \Delta_i & 0_{r \times m-r} \\ 0_{n-r \times r} & I_{n-r \times m-r} \end{bmatrix} \right\}, \quad (6)$$

$$\Delta_i = \sum_{j_i=1}^{\bar{d}_i} \Delta_{i,j_i}, \quad (7)$$

and

$$\Delta_{i,j_i} = \begin{cases} 1, & \text{if } \theta = \delta_{i,j_i} \\ 0, & \text{otherwise,} \end{cases} \quad (8)$$

where δ_{i,j_i} are the roots of each $\beta_i(\theta)$ such that $\beta_i(\delta_{i,j_i}) = 0$ for $i = \{1, \dots, r\}$ and $j_i = \{1, \dots, \bar{d}_i\}$, with \bar{d}_i equals the degree of $\beta_i(\theta)$.

Proof. From the Smith normal form, $W(\theta) = P^{-1}(\theta)S(\theta)Q^{-1}(\theta)$, then

$$\ker\{W(\theta)\} = \{u \in \mathbb{R}^m, P^{-1}(\theta)S(\theta)Q^{-1}(\theta)u = 0\}. \quad (9)$$

Since $Q(\theta)$ is non-singular, by applying a change of variables $z = Q^{-1}(\theta)u$, we obtain the set $\{z \in \mathbb{R}^m, P^{-1}(\theta)S(\theta)z = 0\}$ which is equivalent to $\ker\{S(\theta)\} = \{z \in \mathbb{R}^m, S(\theta)z = 0\}$, then $Q^{-1}(\theta)u \in \ker\{S(\theta)\}$. Therefore, back in the original variable u , the kernel of the parameter-dependent matrix can be represented as

$$\ker\{W(\theta)\} = Q(\theta)\ker\{S(\theta)\}. \quad (10)$$

Furthermore, since matrix $S(\theta)$ has some diagonal terms and zeros elsewhere as shown in (3), its kernel can be obtained from the values of θ such that the diagonal term $\beta_i(\theta) = 0$ for $i = \{1, \dots, r\}$, which are the roots of each polynomial $\beta_i(\theta)$ in the diagonal. Therefore, the direction related to each non-zero diagonal will be activated when θ meets one of the roots of the polynomial $\beta_i(\theta)$, and the directions related to the zeros will be always active. Then, the kernel of $S(\theta)$ can be defined as in (6), where Δ_i represents a function dependent on θ that will be equal to one only

when θ meets one of the roots of the correspondent diagonal term, activating the direction of the kernel, otherwise is equal to zero. \square

Remark 2. Note that, in (8), δ_{i,j_i} is equal to θ only when δ_{i,j_i} is a real value root since $\theta \in \mathbb{R}$. Hence, when there is a complex value root δ_{i,j_i} , the total number of roots that can be equal to θ is less than the degree of $\beta_i(\theta)$. If β_i has two or more equal roots, then only one indicator (8) should be defined for this specific root value.

Remark 3. Since β_{i+1} is divisible by β_i , all β_j with $j \geq i$ depending on θ has a common root, then a common Δ_{i,j_i} . Thus, when $\theta = \delta_{i,j_i}$ such that $\Delta_{i,j_i} = 1$, all Δ_i 's in the kernel diagonal will be one.

Remark 4. It is important to highlight that although Δ_i is defined as a sum of terms as in (7), it can only be zero or one since when a Δ_{i,j_i} is one, all the others in the sum are zero.

4.2.1 Some properties of the parameter-dependent kernel

In order to analyze the kernel of a parameter-dependent matrix, let us define $L(\theta)$ such that $\text{Im}\{L(\theta)\} = \ker\{W(\theta)\}$. Note that obtaining such a basis is now systematically possible by using Lemma 1. One can show that the structure of such a matrix is given by

$$L(\theta) = [\Delta_1 L_1(\theta) \quad \cdots \quad \Delta_r L_r(\theta) \quad \tilde{L}(\theta)], \quad (11)$$

where $L_i(\theta)$, for $i = 1$ to $i = r$, and $\tilde{L}(\theta)$ depend polynomially on θ . All these matrices may be empty. It follows that both the dimension and directions of such a kernel may depend on the parameter θ . We can derive three different cases regarding the kind of dependency w.r.t. θ which are summarized in the three following points:

- *Independence:* Some directions are possibly independent of θ , i.e., constant directions with a constant dimension. In this case, $\tilde{L}(\theta)$ contains constant columns.
- *Isolated dimension drop:* When directions depend on the value of θ , such directions can be time-varying if θ varies in time. The respective basis adapts according to the parameter value. The directions depend continuously on the parameter θ . For some isolated values of θ , some directions may vanish or be a linear combination of the others, leading to drops in the kernel dimension. This is represented by the columns of matrix $\tilde{L}(\theta)$ that depend on θ .
- *Isolated dimension increase:* Differently from the second case, some directions are only active for a finite number of parameter values. It means that when θ takes some specific values, a direction appears in the kernel. In Lemma 1, this is represented by the indicator Δ_i in (7). Therefore, it influences the kernel dimension which can change when such directions appear or not. This is represented by the columns $\Delta_i L_i(\theta)$, for $i = 1$ to $i = r$, of $L(\theta)$ in (11).

To conclude, even with an unusual geometric structure due to the parameter dependence that leads to changes in the subspace dimension, the kernel of a parameter-dependent can be efficiently computed by obtaining the Smith normal form of the matrix and by defining the kernel of matrix $S(\theta)$ which is given in (6).

4.3 Parameter-dependent inverse map

In the following, for a subspace $\mathcal{V} \subseteq \mathbb{R}^n$ with V_\perp as an orthonormal basis of the orthogonal complement \mathcal{V}_\perp of \mathcal{V} (i.e. $\text{Im}\{V_\perp\} = \mathcal{V}_\perp$), we define

$$W_{\mathcal{V}}(\theta) = \begin{bmatrix} V_\perp B(\theta) \\ D(\theta) \end{bmatrix}. \quad (12)$$

Similar to the kernel of parameter-dependent matrices, computing the inverse map of a subspace by a parameter-dependent matrix ($B^{-1}(\theta)\mathcal{V}$), is not straightforward. Therefore, we also use the Smith normal form and the result from Lemma 1 to help us in computing it.

Lemma 2. Consider a subspace $\mathcal{V} \subseteq \mathbb{R}^n$. The following relation holds:

$$B^{-1}(\theta)\mathcal{V} \cap \ker\{D(\theta)\} = \ker\{W_{\mathcal{V}}(\theta)\}, \quad (13)$$

where $W_{\mathcal{V}}(\theta)$ is given by (12).

Proof. Let \mathcal{V}_\perp be the orthogonal complement of \mathcal{V} . Let V and V_\perp be a basis of \mathcal{V} and \mathcal{V}_\perp , respectively, so that $\text{Im}\{V\} = \mathcal{V}$ and $\text{Im}\{V_\perp\} = \mathcal{V}_\perp$. First, we know that $B^{-1}(\theta)\mathcal{V} = \{u \in \mathbb{R}^m, B(\theta)u \in \mathcal{V}\}$. Then, since V is a basis of \mathcal{V} , $B^{-1}(\theta)\mathcal{V} = \{u \in \mathbb{R}^m, \exists z \in \mathbb{R}^q, B(\theta)u = Vz\}$. Multiplying at the left the expression $B(\theta)u = Vz$ by V_\perp^\top , we obtain $V_\perp^\top B(\theta)u = V_\perp^\top Vz = 0$, then it follows that, as shown in [18, Property 3.1.3],

$$B^{-1}(\theta)\mathcal{V} = \{u \in \mathbb{R}^m, V_\perp^\top B(\theta)u = 0\} = \ker\{V_\perp^\top B(\theta)\}.$$

Therefore, $B^{-1}(\theta)\mathcal{V} \cap \ker\{D(\theta)\} = \ker\{V_\perp^\top B(\theta)\} \cap \ker\{D(\theta)\}$ which can be rewritten as $\ker\{W_{\mathcal{V}}(\theta)\}$, then we can conclude that relation (13) holds. \square

Therefore, the subspace $B(\theta)^{-1}\mathcal{V}^* \cap \ker\{D(\theta)\}$ can be practically computed since we can compute the kernel of a parameter-dependent matrix by Lemma 1 using the Smith normal form.

5 adaptive input redundancy

Here, we present a technical result that will be helpful for the adaptive input redundancy result.

Lemma 3. Let $\mathcal{V}^* \subseteq \mathbb{R}^n$, $F : \Theta \rightarrow \mathbb{R}^{m \times n} \in \mathbf{F}(\mathcal{V}^*)$, and $L : \Theta \rightarrow \mathbb{R}^{m \times q}$ such that

$$\text{Im}\{L(\theta)\} = Q(\theta)\ker\{S(\theta)\}, \quad (14)$$

where $Q(\theta)$ and $S(\theta)$ are the matrices of the Smith normal form of matrix $W_{\mathcal{V}^*}(\theta)$ defined in (12), and $\ker\{S(\theta)\}$ is

defined in (6). Let $x_0(\theta) \in \mathcal{V}^*$ and u be an input function of the form:

$$u(t, \theta) = F(\theta)x(t) + L(\theta)w(t), \quad (15)$$

for some function w , then the output resulting from u and x_0 is zero.

Proof. Consider the input in the form (15), the system trajectories have the following dynamics,

$$\begin{aligned} \dot{x}_u(t, \theta, x_0) &= (A(\theta) + B(\theta)F(\theta))x_u(t, \theta, x_0) + B(\theta)v(t, \theta), \\ y_u(t, \theta, x_0) &= (C(\theta) + D(\theta)F(\theta))x_u(t, \theta, x_0) + D(\theta)v(t, \theta), \end{aligned}$$

where $v(t, \theta) = L(\theta)w(t)$. Since $B^{-1}(\theta)\mathcal{V}^* \cap \ker\{D(\theta)\} = \ker\{W_{\mathcal{V}^*}(\theta)\}$ by Lemma 2 and $\ker\{W_{\mathcal{V}^*}(\theta)\} = Q(\theta)\ker\{S(\theta)\}$ by Lemma 1, then $\text{Im}\{L(\theta)\} = B^{-1}(\theta)\mathcal{V}^* \cap \ker\{D(\theta)\}$. It follows that

$$v(t, \theta) \in B^{-1}(\theta)\mathcal{V}^* \cap \ker\{D(\theta)\}. \quad (16)$$

Therefore, $B(\theta)v(t, \theta) \in \mathcal{V}^*$ and $D(\theta)v(t, \theta) = 0, \forall t \in \mathbb{R}^+$. Thus, we can conclude that $y_u(t, \theta, x_0) = C(\theta)x_u(t, \theta, x_0) + D(\theta)u(t) = 0, \forall t \in \mathbb{R}^+$. \square

Now, the main result of the paper is presented in Theorem 1, where we develop conditions to verify the adaptive input redundancy of parameter-dependent systems in the form (1).

Theorem 1. *The parameter-dependent system (1) is adaptive input redundant if there exists a parameter $\theta \in \Theta$, such that*

$$\dim(B^{-1}(\theta)\mathcal{V}^* \cap \ker\{D(\theta)\}) > 0, \quad (17)$$

or equivalently,

$$\dim(Q(\theta)\ker\{S(\theta)\}) > 0, \quad (18)$$

where $Q(\theta)$ and $S(\theta)$ are the matrices of the Smith normal form of matrix $W_{\mathcal{V}^*}(\theta)$ defined in (12), and $\ker\{S(\theta)\}$ is defined in (6).

Proof. Let us consider two trajectories of system (1), x_a and x_b , related to two inputs u_a and u_b , respectively, and a parameter $\theta(t) = \theta \in \Theta, \forall t \geq 0$. We consider two trajectories starting from the same initial condition $x_a(0, \theta(0)) = x_b(0, \theta(0)) = x_0 \in \mathbb{R}^n$. The error between the two trajectories satisfies the following dynamics:

$$\begin{aligned} \dot{x}_a(t, \theta) - \dot{x}_b(t, \theta) &= A(\theta)(x_a(t, \theta) - x_b(t, \theta)) + \\ &\quad B(\theta)(u_a(t, \theta) - u_b(t, \theta)), \\ y_a(t, \theta) - y_b(t, \theta) &= C(\theta)(x_a(t, \theta) - x_b(t, \theta)) + \\ &\quad D(\theta)(u_a(t, \theta) - u_b(t, \theta)), \end{aligned}$$

for all $t \geq 0$, where the initial condition $x_a(0, \theta(0)) - x_b(0, \theta(0)) = 0$. In view of input redundancy, we have to impose $y_a(t, \theta(t)) = y_b(t, \theta(t))$ for all $t \geq 0$, leading to $y_a(t, \theta(t)) - y_b(t, \theta(t)) = 0$ for all $t \geq 0$, which is the output of the error dynamics. Then, since $x_a(0, \theta(0)) - x_b(0, \theta(0)) = 0 \in \mathcal{V}^*$, applying Lemma 3, we have that the output trajectory of the error, $y_a(t, \theta(t)) - y_b(t, \theta(t))$, is zero if

$$u_a(t, \theta) - u_b(t, \theta) = F(\theta)(x_a(t, \theta) - x_b(t, \theta)) + L(\theta)w(t),$$

for some function w . Hence, if there exists a parameter θ such that $\text{Im}\{L(\theta)\}$ does not have a zero dimension thus system (1) is adaptive input redundant. By Lemma 2, and Lemma 3, $\text{Im}\{L(\theta)\} = Q(\theta)\ker\{S(\theta)\} = B(\theta)^{-1}\mathcal{V}^* \cap \ker\{D(\theta)\}$, then these subspaces do not have a zero dimension either, and conditions (17) and (18) are equivalent. \square

5.1 Some properties of adaptive input redundancy

Theorem 1 establishes conditions to verify if a parameter-dependent system is input redundant. This is represented by the dimension of $B(\theta)^{-1}\mathcal{V}^* \cap \ker\{D(\theta)\}$, sometimes called the input unobservability subspace, being greater than zero. This subspace is derived from the kernel of a parameter-dependent matrix. As discussed in Section 4.2, the kernel of a parameter-dependent matrix presents an unusual geometric structure that can be described by three distinct characteristics. The input unobservability subspace of an adaptive input redundant system follows the same geometric structure.

The input unobservability subspace plays a role not only in verifying if a system is adaptive input redundant but also in the ability to compute the invisible inputs through the $L(\theta)$ matrix. This matrix is related to unobservable inputs since the difference between the two distinct inputs leading to the same output is:

$$u_a(t) - u_b(t) = F(\theta)(x_a(t) - x_b(t)) + L(\theta)w(t).$$

Therefore, the different features in the input unobservability subspace lead us to a matrix $L(\theta)$ that has the same structure as the basis defined in (11). Hence, an adaptive input redundant system can have invisible inputs from the output that depend polynomially on the parameter or exist only for specific parameter values.

6 Numerical Example

Consider the following parameter-dependent system,

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} -1 & 0 \\ \theta^3(t) & -1 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0 & \theta(t) \\ \theta(t) & 2\theta(t) & -\theta^2(t) \end{bmatrix} u(t), \\ y(t) &= [0 \quad 1] x(t), \end{aligned} \quad (19)$$

where $0 \leq \theta(t) \leq 1$ for all $t \geq 0$. To verify the adaptive input redundancy of this system we need first to compute the largest generalized adaptively controlled invariant and output invisible subspace of this system, denoted \mathcal{V}^* . Since θ lies in a compact set, we can apply the strategy proposed in [13] to compute \mathcal{V}^* of system (19), as mentioned in Remark 1. Therefore, by applying the algorithm proposed in [13], the largest generalized adaptively subspace is given by $\mathcal{V}^* = [1 \ 0]^\top \mathbb{R}$ and one of the friends of the subspace is

$$F(\theta) = \begin{bmatrix} 0 & 0 & \theta \\ 0 & 0 & 0 \end{bmatrix}^\top.$$

Now, to verify the input redundancy by applying Theorem 1, we have to compute the subspace $Q(\theta)\ker\{S(\theta)\}$, where $Q(\theta)$ and $S(\theta)$ are the matrices of the Smith normal form of the matrix $W_{\mathcal{V}^*}(\theta)$ in (12), where V_{\perp} is the basis of the orthogonal complement of \mathcal{V}^* , and $\ker\{S(\theta)\}$ is defined in (6). Since $\mathcal{V}^* = [1 \ 0]^T \mathbb{R}$, a basis of this subspace is $V = [1 \ 0]^T$ and a basis of the orthogonal complement of \mathcal{V}^* is $V_{\perp} = [0 \ 1]^T$. Therefore,

$$W_{\mathcal{V}^*}(\theta) = \begin{bmatrix} \theta & 2\theta & -\theta^2 \\ 0 & 0 & 0 \end{bmatrix}.$$

The Smith normal form of $W_{\mathcal{V}^*}(\theta)$, is given by $P^{-1}S(\theta)Q^{-1}(\theta)$, where

$$S(\theta) = \begin{bmatrix} \theta & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, P = I_2, \text{ and } Q(\theta) = \begin{bmatrix} 1 & -2 & \theta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Now that we have the matrix in the Smith normal form, we can determine the kernel of $S(\theta)$. First, we should identify the roots of the polynomials in each diagonal component of $S(\theta)$. In this case, we have only one polynomial in the diagonal that is θ , then the root of this diagonal term is $\delta_{1,1} = 0$, and we can define the indicator $\Delta_1 = \Delta_{1,1}$, where

$$\Delta_{1,1} = \begin{cases} 1, & \text{if } \theta = 0 \\ 0, & \text{otherwise.} \end{cases} \quad (20)$$

Then, from (6),

$$\ker\{S(\theta)\} = \text{Im} \left\{ \begin{bmatrix} \Delta_1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\},$$

such that

$$Q(\theta)\ker\{S(\theta)\} = \text{Im} \left\{ \begin{bmatrix} \Delta_1 & -2 & \theta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}. \quad (21)$$

The subspace $Q(\theta)\ker\{S(\theta)\}$ has a dimension greater than zero, then system (19) is adaptive input redundant. This subspace presents constant, θ -dependent, and Δ_i -dependent directions. Hence, we can define a matrix

$$L(\theta) = \begin{bmatrix} \Delta_1 & -2 & \theta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

such that the first column is only different from zero when $\theta = 0$, the second column is constant, and the third column depends on θ .

In order to verify the adaptive input redundancy of system (19) concerning a parameter θ , let us present practical simulations. We are going to analyze two different scenarios where the parameter influences the input redundancy, which are given by the Δ_1 -dependent and θ -dependent directions of the subspace (21). The main objective is to understand how the parameter influences the input redundancy in each of these cases. Then, we define distinct inputs, a parameter function, and an initial condition $x_0 = [1 \ 1]^T$ in each scenario.

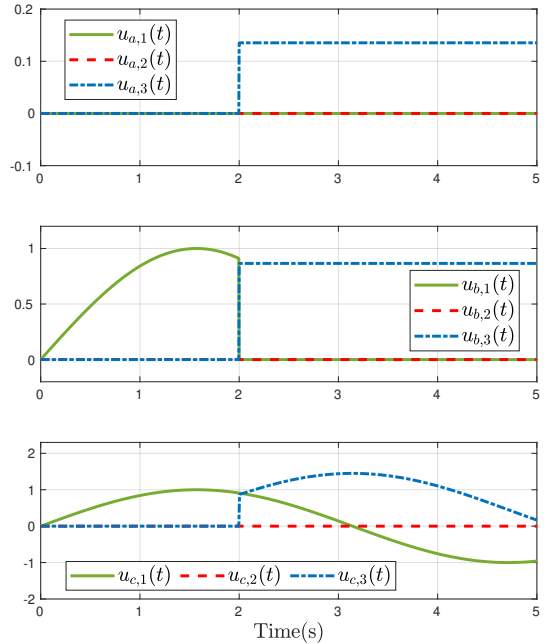


Figure 1: Input trajectories (23).

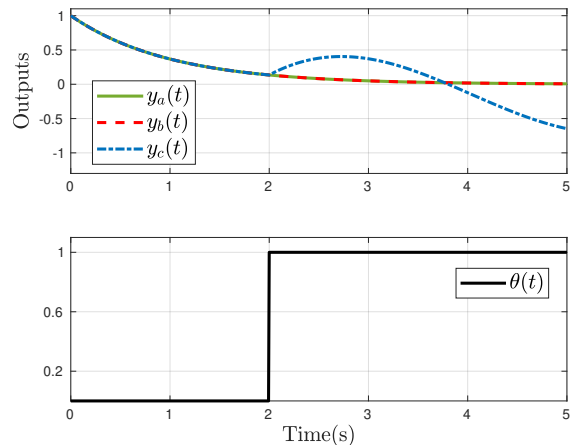


Figure 2: Output trajectories for inputs in given (23) and $\theta(t)$ given in (22).

For the first scenario, let us consider only the Δ_1 -dependent direction. Then, we define a parameter function:

$$\theta(t) = \begin{cases} 0, & \text{if } 0 \leq t < 2, \\ 1, & \text{if } t \geq 2, \end{cases} \text{ s.t. } \Delta_1(t) = \begin{cases} 1, & \text{if } 0 \leq t < 2, \\ 0, & \text{if } t \geq 2, \end{cases} \quad (22)$$

and three distinct inputs:

$$\begin{aligned} u_a(t) &= F(\theta)x_a(t), \\ u_b(t) &= F(\theta)x_b(t) + [\Delta_1 \ 0 \ 0]^T \sin(t), \\ u_c(t) &= F(\theta)x_c(t) + [1 \ 0 \ 0]^T \sin(t). \end{aligned} \quad (23)$$

For $0 \leq t < 2$, the term $[\Delta_1 \ 0 \ 0]^T \sin(t)$ in input $u_b(t)$ is equal to the term $[1 \ 0 \ 0]^T \sin(t)$ in $u_c(t)$, then inputs $u_b(t)$ and $u_c(t)$ are equal and different of $u_a(t)$, as verified in Fig 1. Since the term $[\Delta_1 \ 0 \ 0]^T \sin(t)$ is in the Δ_1 -dependent direction of (21), by Theorem 1, the outputs generated by

the three inputs should be equal for $0 \leq t < 2$, and this is verified in Fig 2. On the other hand, for $t \geq 2$, the parameter takes the value of one, so Δ_1 becomes zero. The term $[\Delta_1 \ 0 \ 0]^\top \sin(t)$ is not anymore equal to $[1 \ 0 \ 0]^\top \sin(t)$, thus input $u_b(t)$ is now different from $u_c(t)$ and it is equal to $u_a(t)$. Then, $y_a(t) = y_b(t)$ for $t \geq 2$, while $y_c(t)$ has a different trajectory, as we verify in Fig 2. This happens because $[1 \ 0 \ 0]^\top \sin(t)$ is not in the Δ_1 -dependent direction of (21) for a $\theta(t) \neq 0$. Thus, the two outputs $y_a(t) = y_b(t)$ for all $t \geq 0$.

For the second scenario, let us consider the θ -dependent direction. Then, we define a parameter function:

$$\theta(t) = \begin{cases} 0.2, & \text{if } 0 \leq t < 2, \\ 0.2(t-2) + 0.2, & \text{if } 2 \leq t < 6, \\ 1, & \text{if } t \geq 6, \end{cases} \quad (24)$$

and three distinct inputs:

$$\begin{aligned} u_a(t) &= F(\theta)x_a(t), \\ u_b(t) &= F(\theta)x_b(t) + [\theta \ 0 \ 1]^\top \sin(t), \\ u_c(t) &= F(\theta)x_c(t) + [0.2 \ 0 \ 1]^\top \sin(t). \end{aligned} \quad (25)$$

For $0 \leq t < 2$, the term $[\theta \ 0 \ 1]^\top \sin(t)$ in input $u_b(t)$ is equal to the term $[0.2 \ 0 \ 1]^\top \sin(t)$ in $u_c(t)$, then inputs $u_b(t)$ and $u_c(t)$ are equal and different of $u_a(t)$, as shown in Fig 3. The term $[\theta \ 0 \ 1]^\top \sin(t)$ is in the θ -dependent direction of (21). Hence, by Theorem 1, the outputs generated by the three inputs should be equal for $0 \leq t < 2$, as verified in Fig 4. On the other hand, for $t \geq 2$, the term $[\theta \ 0 \ 1]^\top \sin(t)$ is not anymore equal to $[0.2 \ 0 \ 1]^\top \sin(t)$. However, it is still in the θ -dependent direction of (21). Then, $y_a(t)$ still equals $y_b(t)$, but $y_c(t)$ has a different trajectory since $[0.2 \ 0 \ 1]^\top \sin(t)$ is not in one of the directions of (21) for a $\theta(t) \neq 0.2$. This is verified in Fig 4. In this case, we also have $y_a(t) = y_b(t)$ for $t \geq 0$.

To conclude, in the two presented scenarios, we verified that there are different inputs leading to the same output trajectory for some parameter $\theta(t)$ and for the same initial condition, confirming that the system is adaptive input redundant. Furthermore, we verify that the parameter influences the invisible inputs from the output so that an implicit or explicit dependence on the parameter may appear.

7 Conclusion

In this paper, we introduced the concept of adaptive input redundancy for parameter-dependent systems. In the context where the parameter of the system is known, the input redundancy of the system can be characterized as dependent on the parameter. Hence, we have provided a sufficient condition to verify the input redundancy of parameter-dependent systems. This condition is described by the dimension of the subspace $B^{-1}(\theta)\mathcal{V}^* \cap \ker\{D(\theta)\}$ being greater than zero. By considering that the system matrices depend polynomially on the parameter and using its Smith normal form, we provide a practical way to compute this parameter-dependent subspace. So, a tractable condition to verify

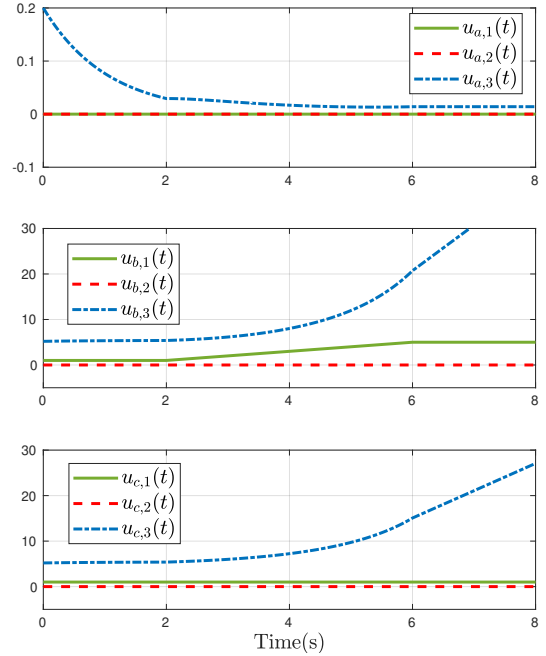


Figure 3: Input trajectories (25).

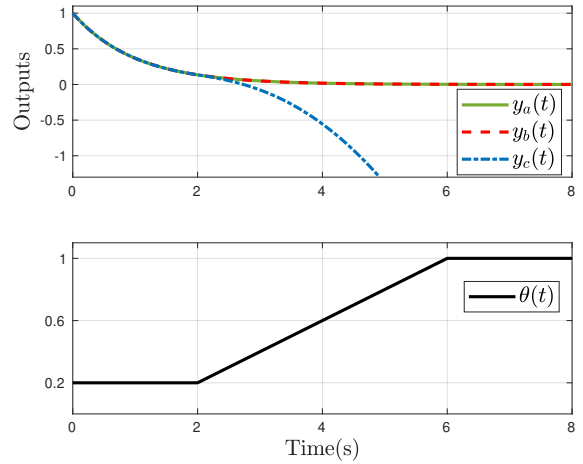


Figure 4: Output trajectories for inputs in given (25) and $\theta(t)$ given in (24).

adaptive input redundancy is obtained. It is important to highlight that since we consider subspaces that do not depend on the parameter, it restrains us from providing a complete characterization of input redundancy for this type of system. However, we can deal with all the previous results from the literature as particular cases. Future work should explore the possibility of considering parameter-dependent subspaces so that a complete characterization of adaptive redundancy can be provided.

References

- [1] L. Zaccarian, “Dynamic allocation for input redundant control systems,” *Automatica*, vol. 45, no. 6, pp. 1431–1438, 2009.

- [2] A. Serrani, “Output regulation for over-actuated linear systems via inverse model allocation,” in *51st IEEE conference on decision and control (CDC)*, 2012, pp. 4871–4876.
- [3] J. Kreiss and J.-F. Tréguët, “Input redundancy: Definitions, taxonomy, characterizations and application to over-actuated systems,” *Systems & Control Letters*, vol. 158, p. 105060, 2021.
- [4] O. Harkegard, “Efficient active set algorithms for solving constrained least squares problems in aircraft control allocation,” in *Proceedings of the 41st IEEE Conference on Decision and Control, 2002.*, vol. 2. IEEE, 2002, pp. 1295–1300.
- [5] M. Oppenheimer, D. Doman, and M. Bolender, *Control allocation*. in: W.S. Levine (Ed.), *The Control Handbook*, 2010, (Chapter 8).
- [6] T. A. Johansen and T. I. Fossen, “Control allocation—a survey,” *Automatica*, vol. 49, no. 5, pp. 1087–1103, 2013.
- [7] J. Kreiss and M. Jungers, “Robust input redundancy for uncertain systems,” *IFAC-PapersOnLine*, vol. 55, no. 25, pp. 157–162, 2022, 10th IFAC Symposium on Robust Control Design ROCOND 2022.
- [8] H. Khalil, “Nonlinear systems,” *3rd edition*, 2002.
- [9] H. L. Trentelman, A. A. Stoorvogel, and M. Hautus, *Control theory for linear systems*. Springer Science & Business Media, 2012.
- [10] W. M. Wonham, *Linear multivariable control: a geometric approach*. Springer-Verlag New York, 2012, vol. 10.
- [11] S. Bhattacharyya, “Generalized controllability, (A,B)-invariant subspaces and parameter invariant control,” *SIAM Journal on Algebraic Discrete Methods*, vol. 4, no. 4, pp. 529–533, 1983.
- [12] J. Bokor, Z. Szabó, and G. Stikkel, “Invariant subspaces for LPV systems and their applications,” in *Proceedings of the 10th Mediterranean Control Conference*, 2002.
- [13] V. V. Viana, J. Kreiss, and M. Jungers, “On the computation of controlled invariant and output invisible subspaces for parameter-dependent systems,” (*In revision*), 2023, <https://hal.science/hal-04329175>.
- [14] G. Balas, J. Bokor, and Z. Szabó, “Invariant subspaces for LPV systems and their applications,” *IEEE Transactions on Automatic Control*, vol. 48, no. 11, pp. 2065–2069, 2003.
- [15] G. Basile and G. Marro, “On the robust controlled invariant,” *Systems & control letters*, vol. 9, no. 3, pp. 191–195, 1987.
- [16] H. Bourlès, *Systèmes linéaires - De la modélisation à la commande*, 2006.
- [17] T. Kailath, *Linear systems*. Prentice-Hall Englewood Cliffs, NJ, 1980, vol. 156.
- [18] G. Basile and G. Marro, *Controlled and Conditioned Invariants in Linear System Theory*. Prentice Hall, 1992.