



HAL
open science

Long time asymptotic behavior of a self-similar fragmentation equation

Gaetano Agazzotti, Madalina Deaconu, Antoine Lejay

► **To cite this version:**

Gaetano Agazzotti, Madalina Deaconu, Antoine Lejay. Long time asymptotic behavior of a self-similar fragmentation equation. 2024. hal-04477123

HAL Id: hal-04477123

<https://hal.science/hal-04477123v1>

Preprint submitted on 26 Feb 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Long time asymptotic behavior of a self-similar fragmentation equation

Gaetano Agazzotti* Madalina Deaconu† Antoine Lejay‡

February 26, 2024

Abstract

Using the Mellin transform, we study a self-similar fragmentation equations whose breakage rate follows the power law distribution, and a particle is split into a fixed number of smaller particles. First, we show how to extend the solution of such equations to measure-valued initial conditions, by a closure argument on the Mellin space. Second, we use appropriate series representations to give a rigorous proof to the asymptotic behavior of the moments, completing some results known through heuristic derivations.

1 Introduction

Fragmentation is a phenomenon that appears in a variety of domains in physics, geophysics, biology, ... Although ruled by an apparently simple equation, a wide variety of behaviors can be reached. The self-similar fragmentation is itself a subclass where the solution asymptotically enjoys some scaling property. This class is itself rich enough and can be studied from several point of views, by studying functional analysis [5, 6, 15, 23, 27, 34], probability [7, 8, 28], looking for specific solutions [11, 40], dimensional analysis [16], ... or a mix [9].

In this article, we consider the fragmentation equation

$$\partial_t u(t, x) = \lambda^{2+\alpha} x^\alpha u(t, \lambda x) - x^\alpha u(t, x), \quad (1)$$

*ENSMN – École Nationale Supérieure des Mines de Nancy, France

†Université de Lorraine, CNRS, IECL, Inria, F-54000 Nancy, France; Madalina.Deaconu@inria.fr

‡Université de Lorraine, CNRS, IECL, Inria, F-54000 Nancy, France; ORCID: 0000-0003-0406-9550; Antoine.Lejay@univ-lorraine.fr

where $\alpha \geq 0$ and $\lambda > 1$. We associate with (1) the companion equation

$$\partial_t v(t, x) = \lambda^{1+\alpha} x^\alpha v(t, \lambda x) - x^\alpha v(t, x) \quad (2)$$

with $v(t, x) := x \cdot u(t, x)$. Here, $u(t, x)$ represents the concentration of the particles at time t at the point x , while $v(t, x)$ represents the mass. The idea is that when a breakage occurs, a particle breaks into λ daughter's of equal size.

We rewrite (2) as

$$\partial_t v(t, x) = -F(x)v(t, x) + \int_x^{+\infty} \kappa\left(\frac{x}{y}\right) F(y)v(t, y) \frac{dy}{y}$$

with

$$F(x) := x^\alpha \text{ and } \kappa(dz) := \lambda \delta_{1/\lambda}(dz),$$

where κ is a scaled Dirac delta mass at $1/\lambda$.

This corresponds to a homogeneous fragmentation, whose solution has some asymptotic self-similar properties. This form is the one found in [19, 20]. Most often, results related to self-similar fragmentation equations use for κ a measure

With tools from functional analysis [4], we show that there exists an analytic, positive semi-group, and then a mild solution to (1) provided that the initial solution belongs to some weighted L^1 space.

We identify functions with a support contained in $[0, 1]$ as an invariant subset of functions. In particular, using some tools from q -calculus, we give a series expansion of the semi-group that used the q -binomial coefficients.

Owing to the particular form of the operator, with

$$C(t, \sigma) := \int_0^{+\infty} x^{\sigma-1} u(t, x) dx \text{ for } \sigma \geq 1,$$

the moment (or Mellin transform) of $u(t, \cdot)$, the first main result (Theorem 4) of this article is to extend (1) as a measure-valued solution when the initial condition is a measure with support in $[0, 1]$. Our approach differs from the one in [19] as we only consider solutions in the metric space given by the weak convergence of measures. The idea is to pass to the limit in the space of Mellin transforms, and then to carry back this extension in the original space.

When the initial condition is δ_1 , the Dirac mass at 1, then C solves a pantograph equation. We then use this particular property to prove an asymptotic results on C .

In [16, Chapter 3], the statistical aspects of homogeneous fragmentation are discussed from the physical point of view. In particular, an *ansatz* states that

$$u(t, x) \approx \frac{1}{s(t)^2} \phi\left(\frac{x}{s(t)}\right), \text{ as } t \rightarrow \infty,$$

where $s'(t)s(t)^{-(1+\alpha)} = -\omega$ where $\omega > 0$ is a separation constant. In the present case, we take

$$s(t) := \frac{1}{t^{1/\alpha}}, \quad \omega := \frac{-1}{\alpha}, \quad (3)$$

while ϕ is the solution to

$$\omega(2\phi(\xi) + \xi\phi'(\xi)) = \lambda^{2+\alpha}\xi^\alpha\phi(\lambda\xi) - \xi^\alpha\phi(\xi), \quad (4)$$

where $\xi = x/s$. The function ϕ is called a *self-similar profile*.

In [23, Theorem 3.1], it is shown¹, that a self-similar profile exists and is unique. In addition [23, Theorem 3.2] states that

$$\lim_{t \rightarrow \infty} \int_0^{+\infty} y |u(t, x) - (1+t)^{2/\alpha} \phi(yt^{1/\alpha})| dy = 0.$$

Another result on the existence of a self-similar profile is given in [9], yet when the measure κ has a density.

A series of works (see *e.g.* [14, 15, 18, 21, 26, 27, 34]) establish exponential rate of convergence toward self-similar solution, yet for the growth-fragmentation equation. In this setting, the main tools are the ones from the spectral analysis of the problem and the rate of convergence is due to a spectral gap. The presence of a “growth term”, equivalent to a transport term, appears crucial to establish a spectral gap.

We establish the long-time asymptotics of the moments when the initial condition is a Dirac mass. Using the above choices (3) and (4), we obtain that

$$C(t, \sigma) = \int_0^{+\infty} x^{\sigma-1} c(t, x) dx \approx s(t)^{\sigma-2} \int_0^{+\infty} \xi^{\sigma-1} \phi(\xi) d\xi.$$

Therefore,

$$C(t, \sigma) \approx B(\sigma) t^{\frac{2-\sigma}{\alpha}} \text{ for } B(\sigma) := \int_0^{+\infty} \xi^{\sigma-1} \phi(\xi) d\xi.$$

As the mass $m = \int_0^{+\infty} x \cdot u(t, x) dx$ is preserved in the time, we select ϕ so that $B(2) = \int_0^{+\infty} \xi \cdot \phi(\xi) d\xi = m$.

¹Actually, this article considers the equation (2), but the results can be adapted to (1).

We also note that

$$\int_0^{+\infty} x^{\sigma-1} t^{2/\alpha} \phi(xt^{1/\alpha}) dx = t^{(2-\sigma)/\alpha} B(\sigma).$$

Using appropriate series representation of the Mellin transform, we give a formal proof of this result (Corollary 5 and Propositions 14). In particular, we see that there are different regimes: for $\sigma < 2$, the solution is increasing to infinity, for $\sigma = 2$, $C(t, 2)$ is constant (this is mass conservation). For $\sigma > 2$, the solution decreases to 0. We also establish that for $\alpha > 1/2$, the self-similar profile is uniquely determined by its moments.

Our tools are based on complex analysis, q -calculus, appropriate series representation of holomorphic functions, as well as the Karamata theorem. Although Mellin transforms are common to study self-similar fragmentation equations (see *e.g.* [12, 13, 16, 19, 20]), and some works use the pantograph equation, [12, 13], we present alternative series representations that are suitable for finding the appropriate asymptotic behaviors. Besides, our extension to measure-valued solutions by relying on a limiting argument on the moments is also new.

Outline. In Section 2, we construct the solution of the self-similar fragmentation equation using the theory of semi-group, and we identify invariant subspaces. In Section 3, we introduce the equation for the evolution of the moments. In Section 4, we show how to extend the notion of solutions to measure-valued solutions. In Section 5, we study the qualitative behavior of the moments. Finally, in Section 6, we give the asymptotic behavior of the moments.

2 Solutions of the fragmentation equation

In a first time, we solve the fragmentation equation through semi-groups.

Notation 1 (Space of solutions). We fix some $p \geq 0$. We set

$$\mathbb{L}_p := L^1(\mathbb{R}_+, w_p(x) dx) \text{ with } w_p(x) = 1 + x^p, \quad x \geq 0.$$

We denote by $\|\cdot\|_p$ the norm

$$\|f\|_p := \int_{\mathbb{R}_+} |f(x)| w_p(x) dx.$$

We define $\mathbb{L}_p^+ := \{f \in \mathbb{L}_p \mid f \geq 0\}$, the cone of non-negative functions in \mathbb{L}_p .

Notation 2 (Operators). We set

$$\begin{aligned} \mathcal{D}_\lambda f(x) &:= f(\lambda x), \quad \mathcal{F}_\alpha f(x) = x^\alpha f(x) \\ \text{and } \mathcal{G}_\alpha f(x) &:= \lambda^2 \mathcal{D}_\lambda \mathcal{F}_\alpha f(x) = \lambda^{2+\alpha} x^\alpha f(\lambda x). \end{aligned}$$

We also define $F(x) := x^\alpha$.

Since $\lambda \leq 1$ and w_p is non-decreasing,

$$\|\mathcal{F}_\alpha f\|_p \leq \|f\|_{p+\alpha} \text{ for } f \in \mathbb{L}_{p+\alpha}, \quad (5)$$

$$\|\mathcal{D}_\lambda f\|_p = \frac{1}{\lambda} \int |f(x)| w_p\left(\frac{x}{\lambda}\right) dx \leq \frac{1}{\lambda} \|f\|_p \text{ for } f \in \mathbb{L}_p, \quad (6)$$

$$\text{and } \|\mathcal{G}_\alpha f\|_p \leq \|\mathcal{D}_\lambda \mathcal{F}_\alpha f\|_p = \lambda^{1+\alpha} \|\mathcal{F}_\alpha f\|_p \text{ for } f \in \mathbb{L}_{p+\alpha}. \quad (7)$$

From (6), \mathcal{D}_λ is a bounded operator on \mathbb{L}_p . Besides, \mathcal{D}_λ is invertible with inverse $\mathcal{D}_{\lambda^{-1}}$.

2.1 Solving the fragmentation equation when $\alpha = 0$

We first consider that $\alpha = 0$, so that (8) becomes

$$\partial_t u(t, x) = \lambda^2 u(t, \lambda x) - u(t, x). \quad (8)$$

For any $p \geq 0$, the operators \mathcal{D}_λ and \mathcal{G}_α are bounded in \mathbb{L}_p while $\mathcal{F}_\alpha = \text{Id}$, the identity operator, so that $\text{Dom}(\mathcal{D}_\lambda) = \text{Dom}(\mathcal{F}_\alpha) = \text{Dom}(\mathcal{G}_\alpha) = \mathbb{L}_p$.

Proposition 1. *Fix $p \geq 0$. For any given initial condition $f \in \mathbb{L}_p$, there exists a unique solution $u \in \mathcal{C}^1([0, T], \mathbb{L}_p)$ of (8).*

Proof. Since $\mathcal{G}_0 - \mathcal{F}_0 = \mathcal{G}_0 - \text{Id}$ is bounded, for any $p \geq 0$, the semi-group Q defined by $Q(t) := \exp(t(\mathcal{G}_0 - \text{Id}))$, $t \geq 0$ is analytic and generated by $(\mathcal{G}_0 - \text{Id}, \mathbb{L}^p)$. The mild solution of (8) with initial condition $f \in \mathbb{L}_p$ is given by $(t, x) \mapsto Q(t)f(x)$.

Following [33], this semi-group may be explicitly constructed through the following sequence: for $f \in \mathbb{L}_p$, we set

$$a_0 := f \text{ and } a_{k+1} := \mathcal{G}_0 a_k - a_k = \lambda^2 \mathcal{D}_\lambda a_k - a_k.$$

The series

$$Q(t)f = \sum_{k \geq 0} \frac{t^k}{k!} a_k = \sum_{k \geq 0} \frac{t^k}{k!} (\mathcal{G}_0 - \text{Id})^k f = \sum_{k \geq 0} \frac{t^k}{k!} (\lambda^2 \mathcal{D}_\lambda - \text{Id})^k f$$

converges in \mathbb{L}_p , and gives the semi-group generated by $(\mathcal{G}_0 - \text{Id}, \mathbb{L}^p)$. Moreover, $t \mapsto Q(t)f$ is of class \mathcal{C}^1 , with $\partial_t Q(t)f = (\mathcal{G}_0 - \text{Id})Q(t)f$ for any $t \geq 0$.

Regarding uniqueness, since (8) is linear, it is sufficient to consider that $f = 0$. In this case,

$$\|u(t, \cdot)\|_p \leq (\lambda - 1) \int_0^t \|u(s, \cdot)\|_p \, ds$$

so that the Gronwall inequality implies that $\|u(t, \cdot)\|_p = 0$ for any $t \geq 0$. This yields the uniqueness. \square

2.2 Solving the fragmentation equation when $\alpha > 0$

We now consider $\alpha > 0$.

Unlike Section 2.1, the operators \mathcal{F}_α and \mathcal{G}_α defined in Notation 2 are not bounded. We define

$$\text{Dom}(\mathcal{F}_\alpha) := \{f \in \mathbb{L}_p \mid \mathcal{F}_\alpha f \in \mathbb{L}_p\} \text{ and } \text{Dom}(\mathcal{G}_\alpha) := \{f \in \mathbb{L}_p \mid \mathcal{G}_\alpha f \in \mathbb{L}_p\}.$$

With (5) and (7),

$$\text{Dom}(\mathcal{F}_\alpha) = \text{Dom}(\mathcal{G}_\alpha) = \mathbb{L}_{p+\alpha}. \quad (9)$$

The proof of the next result is straightforward.

Lemma 1. *The operator $(\mathcal{F}_\alpha, \text{Dom}(\mathcal{F}_\alpha))$ is closed, densely defined and injective.*

The proof of the next result relies on the Hille-Yoshida theorem and [4, Examples 2.8 and 3.9, p. 74, Proposition 2.2, p. 13].

Notation 3. For $\mu \in \mathbb{R}$, we set $\mathcal{J}_\mu f := \mu f$ for any function f .

Lemma 2. *For any $\mu \geq 0$, the operator $(-\mathcal{F}_\alpha - \mathcal{J}_\mu, \text{Dom}(\mathcal{F}_\alpha))$ generates the strongly continuous, contractive semi-group $(P(t))_{t \geq 0}$ with $P(t)f(x) = e^{-\mu t - tF(x)}f(x)$. Besides, \mathbb{L}_p^+ is invariant by $P(t)$ for any $t \geq 0$.*

Since \mathbb{L}_p is stable under \mathcal{D}_λ and \mathcal{D}_λ is invertible, $\text{Dom}(\mathcal{G}_\alpha) = \text{Dom}(\mathcal{F}_\alpha)$. Besides, $\mathcal{G}_\alpha(\mathbb{L}_p^+) \subset \mathbb{L}_p^+$.

Fix $g \in \mathbb{L}_p$ and $\mu > 0$. Setting $f(x) := \int_0^{+\infty} \exp(-\mu t - tF(x))g(x) \, dt$, we obtain by an integration that for any $x > 0$,

$$(\mu + \mathcal{F}_\alpha)f(x) = g(x).$$

In addition, since $F(x) \geq 0$,

$$\|f\|_p \leq \int_0^{+\infty} \int_0^{+\infty} \exp(-\mu t) |g(x)| w_p(x) dx dt \leq \frac{1}{\mu} \|g\|_p.$$

Therefore, $\mu + \mathcal{F}_\alpha$ is invertible with a bounded inverse for any $\mu > 0$. From (7), \mathcal{G}_α is \mathcal{F}_α -bounded.

Lemma 3. Fix $p > 1$ and $\mu \geq \lambda - 1$. There exists an operator $(\mathcal{K}_{\alpha,\mu}, \text{Dom}(\mathcal{K}_{\alpha,\mu}))$ that extends of $(\mathcal{G}_\alpha - \mathcal{F}_\alpha - \mathcal{J}_\mu, \text{Dom}(\mathcal{F}_\alpha))$ and that generates a strongly continuous semi-group of contractions $(Q_\mu(t))_{t \geq 0}$ which satisfies the Duhamel equation

$$Q_\mu(t)f = P(t)f + \int_0^t Q_\mu(t-s) \mathcal{G}_\alpha P(s)f ds, \quad \forall f \in \text{Dom}(\mathcal{F}_\alpha), \quad \forall t \geq 0. \quad (10)$$

Proof. For $f \in \text{Dom}(\mathcal{G}_\alpha) = \text{Dom}(\mathcal{F}_\alpha)$, since $\mathbb{L}_p \subset \mathbb{L}_1$,

$$\int_{\mathbb{R}_+} \mathcal{G}_\alpha f(x) dx = \lambda^2 \int_{\mathbb{R}_+} \mathcal{D}_\lambda \mathcal{F}_\alpha f(x) dx = \lambda \int_{\mathbb{R}_+} \mathcal{F}_\alpha f(x) dx.$$

Similarly,

$$\int_{\mathbb{R}_+} \mathcal{G}_\alpha f(x) x^p dx = \lambda^{1-p} \int_{\mathbb{R}_+} \mathcal{F}_\alpha f(x) x^p dx.$$

When $p > 1$, $\lambda^{1-p} < 1$ since $\lambda > 1$. For $f \in \mathbb{L}_p^+ \cap \text{Dom}(\mathcal{F}_\alpha)$,

$$\begin{aligned} & \int_{\mathbb{R}_+} (\mathcal{G}_\alpha f(x) - \mathcal{F}_\alpha f(x) - \mathcal{J}_\mu f(x)) w_p(x) dx \\ &= (\lambda - 1) \int_{\mathbb{R}_+} \mathcal{F}_\alpha f(x) dx + (\lambda^{1-p} - 1) \int_{\mathbb{R}_+} \mathcal{F}_\alpha f(x) x^p dx - \mu \int_{\mathbb{R}_+} f(x) w_p(x) dx \\ &\leq (\lambda - 1) \int_{\mathbb{R}_+} \mathcal{F}_\alpha f(x) dx - \mu \int_{\mathbb{R}_+} f(x) w_p(x) dx. \quad (11) \end{aligned}$$

On the other hand, for $p \geq \alpha$,

$$\int_{\mathbb{R}_+} \mathcal{F}_\alpha f(x) dx \leq \int_{\mathbb{R}_+} f(x) w_p(x) dx.$$

Therefore, for $\mu \geq \lambda - 1$,

$$\int_{\mathbb{R}_+} (\mathcal{G}_\alpha f(x) - \mathcal{F}_\alpha f(x) - \mathcal{J}_\mu f(x)) w_p(x) dx \leq 0.$$

The result follows from [4, Corollary 5.17, p. 147] and [4, Corollary 5.8, p. 139]. \square

From the Duhamel equation (10), we have that $Q_\mu(t)f \geq 0$ whenever $f \in \text{Dom}(\mathcal{K}_{\alpha,\mu}) \cap \mathbb{L}_p^+$.

Proposition 2. *For $\mu \geq \lambda - 1$, the semi-group $(Q_\mu(t))_{t \geq 0}$ is positive.*

Proof. Fix $\nu > 0$. We have that $(\nu + \mathfrak{F}_\alpha + \mu)^{-1} = \frac{1}{\mu + \nu + x^\alpha}$ so that

$$\mathfrak{G}_\alpha(\nu + \mathfrak{F}_\alpha + \mu)^{-1}f(x) = \lambda^2(\lambda x)^\alpha \frac{f(\lambda x)}{\mu + \nu + (\lambda x)^\alpha}.$$

Therefore,

$$\begin{aligned} & \int_{\mathbb{R}_+} |\mathfrak{G}_\alpha(\nu + \mathfrak{F}_\alpha + \mu)^{-1}f(x)|w_p(x) \, dx \\ & \leq \lambda^2 \int_{\mathbb{R}_+} |f(\lambda x)| \frac{(\lambda x)^\alpha}{\mu + \nu + (\lambda x)^\alpha} \frac{w_p(x)}{w_p(\lambda x)} w_p(\lambda x) \, dx \leq \int_{\mathbb{R}_+} |f(x)| \frac{(\lambda x)^\alpha}{\psi(x)} w_p(x) \, dx \end{aligned}$$

with

$$\psi(x) := \lambda^{1-p} \frac{x^\alpha}{\mu + \nu + (\lambda x)^\alpha} \frac{\lambda^p + x^p}{1 + x^p}.$$

The function ψ is continuous on \mathbb{R}_+ with $\lim_{x \rightarrow 0} \psi(x) = 0$ and $\lim_{x \rightarrow +\infty} \psi(x) = \lambda^{1-p} < 1$ when $p > 1$. From this, we easily obtain that for ν large enough, $\sup_x \psi(x) < 1$. The result then follows from [4, Lemma 5.12, p. 143]. \square

Proposition 3. *For $\mu \geq \lambda - 1$, the operator $(\mathcal{K}_{\alpha,\mu}, \text{Dom}(\mathcal{K}_{\alpha,\mu}))$ generates an analytic semi-group $(Q_\mu(t))_{t \geq 0}$.*

Proof. The semi-group $(P(t))_{t \geq 0}$ generated by $(-\mathfrak{F}_\alpha - \mu, \text{Dom}(\mathfrak{F}_\alpha))$ is analytic [22, Theorem 4.6, p. 101]: Since for any $\nu \in \mathbb{C}$, $(\nu + \mu + \mathfrak{F}_\alpha)^{-1}f(x) = f(x)/(\nu + \mu + x^\alpha)$ for $x > 0$, and $|\nu + \mu + x^\alpha| \geq \mathfrak{S}(\nu)$, this implies that

$$\int_{\mathbb{R}_+} |(\nu + \mu + \mathfrak{F}_\alpha)^{-1}f(x)|w_p(x) \, dx \leq \frac{\|f\|_p}{\mathfrak{S}(\nu)}.$$

By Proposition 2, the resolvent of $(\mathfrak{G}_\alpha - \mathfrak{F}_\alpha - \mu, \text{Dom}(\mathfrak{F}_\alpha))$ is positive so that by the Arendt-Rhandi theorem ([3] or [5, Theorem 1.1]), $(Q_\mu(t))_{t \geq 0}$ is an analytic semi-group. \square

We may now remove the restriction on μ .

Lemma 4. *For any $r \in \mathbb{R}$ and $\mu \geq \lambda - 1$, set $Q_r(t) := \exp((\mu - r)t)Q_\mu(t)$ for $t \geq 0$. Then $(Q_r(t))_{t \geq 0}$ is generated by $(\mathcal{K}_{\alpha,r}, \text{Dom}(\mathcal{K}_{\alpha,r}))$. This semi-group is positive and for $f \in \mathbb{L}_p$,*

$$\|Q_r(t)f\|_p \leq \exp((\lambda - 1 - r)t)\|f\|_p, \quad \forall t \geq 0. \quad (12)$$

Notation 4 (Semi-group Q). The semi-group $(Q_0(t))_{t \geq 0}$ is denoted by $(Q(t))_{t \geq 0}$.

The next proposition summarizes our results.

Proposition 4. *For $f \in \mathbb{L}_p$ and $T > 0$, there exists a solution $u \in \mathcal{C}([0, T], \mathbb{L}_p) \cap \mathcal{C}^1((0, T], \text{Dom}(\mathcal{F}_\alpha))$ to (1), and then to (8). If $f \in \mathbb{L}_p^+$, then $u(t, \cdot)$ is non-negative for any $t \in [0, T]$.*

Proof. From [4, Theorem 2.29, p. 39 and Theorem 2.40, p. 40], for any $t > 0$, $t \mapsto Q_\mu(t)f$ is infinitely differentiable for any $f \in \mathbb{L}_p$. Thus, there exists a measurable function $u(t, x)$ on $[\epsilon, T] \times \mathbb{R}_+^*$ such that for any $k \geq 0$, $\partial_t^k u(t, x)$ exists almost everywhere, is absolutely continuous and $\partial_t^k u(t, \cdot) = \partial_t^k Q_\mu(t)f$ for any $t \in [0, T]$. \square

2.3 Scaling invariance

The self-similar fragmentation presents an invariant behavior.

Proposition 5 (Invariance property). *Let u be the solution to (1) with initial condition $f \in \mathbb{L}_p$. Then $v(t, x) := u(t\gamma^\alpha, x\gamma)$ also solves (1) with initial condition $x \mapsto f(\gamma x)$.*

2.4 Invariant subspaces

We now exhibit two invariant subspaces.

2.4.1 A Fréchet space

For $0 < p < q$, an immediate computation shows that $\|f\|_p \leq 2\|f\|_q$ for any $f \in \mathbb{L}_q$, so that $\mathbb{L}_q \subset \mathbb{L}_p$. We define

$$\mathbb{F} := \bigcap_{q>0} \mathbb{L}_q.$$

When p is fixed, with (9), $\text{Dom}(\mathcal{F}_\alpha^k) = \mathbb{L}_{p+k\alpha}$. Since the spaces \mathbb{L}_p are decreasing with respect to inclusion,

$$\bigcap_{k \geq 0} \text{Dom}(\mathcal{F}_\alpha^k) = \bigcap_{k \geq 0} \mathbb{L}_{p+k\alpha} = \mathbb{F}. \quad (13)$$

Fix $\gamma > 0$. We equip \mathbb{F} with the norm

$$\|f\|_{\mathbb{F},\gamma} := \sum_{k \geq 0} \gamma^k \int_0^{+\infty} x^{k\alpha} |f(x)| dx = \sum_{k \geq 0} \gamma^k \int_0^{+\infty} |\mathcal{F}_\alpha^k f(x)| dx.$$

This norm is induced by the Fréchet space \mathbb{F} equipped with the norms

$$f \mapsto \|f\|_0 + \|\mathcal{F}_\alpha^k f\|_0, \text{ for } k = 0, 1, 2, \dots$$

For any $k \geq 0$, the norm $\|\cdot\|_0 + \|\mathcal{F}_\alpha^k \cdot\|_0$ is equivalent to $\|\cdot\|_{k\alpha}$.

Lemma 5. For any $f \in \mathbb{F}$,

$$\|\mathcal{G}_\alpha f\|_{\mathbb{F},\gamma} \leq \frac{\lambda}{\gamma} \|f\|_{\mathbb{F},\gamma}, \quad (14)$$

$$\|\mathcal{F}_\alpha f\|_{\mathbb{F},\gamma} \leq \frac{1}{\gamma} \|f\|_{\mathbb{F},\gamma}. \quad (15)$$

Proof. For any $x > 0$,

$$\mathcal{F}_\alpha^k \mathcal{G}_\alpha f(x) = \lambda^{2+\alpha} x^{(k+1)\alpha} f(\lambda x).$$

With a change of variable,

$$\begin{aligned} \|\mathcal{G}_\alpha f\|_{\mathbb{F},\gamma} &= \lambda^{2+\alpha} \sum_{k \geq 0} \gamma^k \int_0^{+\infty} x^{(k+1)\alpha} |f(\lambda x)| dx \\ &= \lambda^{1+\alpha} \sum_{k \geq 0} \gamma^k \int_0^{+\infty} \frac{x^{(k+1)\alpha}}{\lambda^{(k+1)\alpha}} |f(x)| dx = \sum_{k \geq 0} \frac{\gamma^{k+1}}{\gamma \lambda^{k\alpha-1}} \int_0^{+\infty} x^{(k+1)\alpha} |f(x)| dx. \end{aligned}$$

Since $\lambda > 1$, for $\gamma > \lambda$, $\sup_{k \geq 0} 1/\gamma \lambda^{k\alpha-1} < \lambda/\gamma < 1$. This leads to (14). Besides,

$$\|\mathcal{F}_\alpha f\|_{\mathbb{F},\gamma} = \sum_{k \geq 0} \gamma^k \int_0^{+\infty} x^{(k+1)\alpha} |f(\lambda x)| dx \leq \frac{1}{\gamma} \|\phi\|_{\mathbb{F},\gamma},$$

which is (15). □

Proposition 6. Fix $p \geq 0$ and $\gamma > 0$. The space $\mathbb{F} \subset \mathbb{L}_p$ is invariant under the semi-group Q . In addition, $Q(t)f \in \mathbb{F}$ for any $f \in \mathbb{L}_p$ and any $t > 0$. Finally, the semi-group Q is continuous on $(\mathbb{F}, \|\cdot\|_{\mathbb{F},\gamma})$.

Proof. Not only \mathbb{F} is invariant under the semi-group Q , but as Q is analytic, for any $p \geq 0$, $Q(f) \in \mathbb{F}$ for any $f \in \mathbb{L}_p$ and any $t > 0$.

Besides, (14) and (15) show that $\mathcal{G}_\alpha - \mathcal{F}_\alpha$ is bounded on $(\mathbb{F}, \|\cdot\|_{\mathbb{F},\gamma})$. This implies that Q is continuous on $(\mathbb{F}, \|\cdot\|_{\mathbb{F},\gamma})$. □

This space \mathbb{F} is contained in $\text{Dom}(\mathcal{F}_\alpha^k)$ for any $k \geq 0$. On the other hand, we can also construct a scale of spaces by extrapolation to extend the solution to distributions. As we focus on another set of spaces, we do not enter the details of such a procedure that may be found *e.g.* in [22, Sect. 5, Chap. 2].

2.4.2 Functions with compact support

The construction of the above invariant subspace \mathbb{F} arises in a general manner in the theory of semi-groups. We now exhibit another invariant space of functions which is specific to the kind of equation we look at.

Notation 5 (Support). When h is a function f or a measure μ , its *support* is denoted by $\text{Supp}(h)$.

Using the scaling invariance 2.3, there is no restriction to consider that the support is contained in $[0, 1]$.

Notation 6. We define \mathbb{V} as the set of bounded, measurable functions from \mathbb{R}_+ to \mathbb{R} with $\text{Supp}(f) \subset [0, 1]$. We also define \mathbb{V}^+ the subset of non-negative functions in \mathbb{V} .

Proposition 7. *The space \mathbb{V} is invariant by \mathcal{F}_α and \mathcal{G}_α with $\mathcal{F}_\alpha(\mathbb{V}) \subset \mathbb{V}$ and $\mathcal{G}_\alpha(\mathbb{V}) \subset \mathbb{V}$. Besides, for any $p \geq 0$,*

$$\begin{aligned} \|\mathcal{F}_\alpha f\|_p &\leq \|f\|_p, & \|\mathcal{G}_\alpha f\|_p &\leq \lambda^{1+\alpha} \|f\|_p, \\ \|\mathcal{F}_\alpha f\|_\infty &\leq \|f\|_\infty, & \|\mathcal{G}_\alpha f\|_\infty &\leq \lambda^{2+\alpha} \|f\|_\infty. \end{aligned}$$

Proof. Let $f \in \mathbb{V}$. Clearly, $\|f\|_p \leq 2\|f\|_\infty$. Since $f(x) = 0$ for $x > 1$, $\mathcal{F}_\alpha f(x) = x^\alpha f(x) = 0$. Since $\lambda > 1$, $\mathcal{G}_\alpha f(x) = \lambda^{2+\alpha} x^\alpha f(\lambda x) = 0$ when $x > 1$. In addition, $\mathcal{F}_\alpha f(x) \leq f(x)$ since $x^\alpha \leq 1$. With (7),

$$\|\mathcal{F}_\alpha f\|_p = \int_0^1 (1 + x^p) \mathcal{F}_\alpha f(x) dx \leq \|\mathcal{F}_\alpha\|_p f \text{ and } \|\mathcal{G}_\alpha f\|_p \leq \lambda^{1+\alpha} \|f\|_p.$$

The control of the supremum norm $\|\cdot\|_\infty$ is also easy to obtain. □

Since $\mathcal{G}_\alpha - \mathcal{F}_\alpha$ is a bounded operator, the next result is immediate.

Corollary 1. *For $f \in \mathbb{V}$, $Q(t)f \in \mathbb{V}$ for any $t \geq 0$ with*

$$\|Q(t)f\|_p \leq \exp((1 + \lambda^{1+\alpha})t) \|f\|_p \text{ and } \|Q(t)f\|_\infty \leq \exp((1 + \lambda^{2+\alpha})t) \|f\|_\infty.$$

Corollary 2. *The semi-group $Q(t)$ is positive on $(\mathbb{V}, \|\cdot\|_\infty)$.*

Proof. We identify continuous functions in \mathbb{V} with $\mathcal{C}([0, 1])$, since the support of $f \in \mathbb{V}$ is contained in $[0, 1]$. The subset $\mathcal{C}([0, 1])$ is dense in \mathbb{V} with respect to the uniform norm so that we have only to show that $Q(t)f \geq 0$ for any non-negative, continuous function f .

If f is continuous, then $\mathcal{F}_\alpha f$ and $\mathcal{G}_\alpha f$ are continuous. With Proposition 7, $\mathcal{G}_\alpha - \mathcal{F}_\alpha$ is a bounded operator on $(\mathcal{C}([0, 1]), \|\cdot\|_\infty)$. Consider $f \in \mathcal{C}([0, 1])$ and $f \geq 0$. Let x be such that $f(x) = 0$. Then

$$\mathcal{G}_\alpha f(x) - \mathcal{F}_\alpha f(x) = \lambda^{2+\alpha} x^\alpha f(\lambda x) \geq 0.$$

From [2, Theorem 1.3, p. 124], it follows that $Q(t)f \geq 0$ for any $t \geq 0$. \square

Remark 1. From the above proof, we have also that $Q(t)$ maps $\mathcal{C}([0, 1])$ to itself.

2.5 A monotonic property with respect to α

Let $(Q(t; \alpha))_{t \geq 0}$ be the semi-group generated by $\mathcal{G}_\alpha - \mathcal{F}_\alpha$ when the parameter is α .

Proposition 8. *For any $f \in \mathbb{V}^+$ and any $t \geq 0$, $\alpha \mapsto Q(t; \alpha)f$ is non-increasing.*

Proof. We set $\mathcal{K}_\alpha := \mathcal{G}_\alpha - \mathcal{F}_\alpha$. From the properties of the semi-group,

$$Q(t; \alpha)f(x) = f(x) + \int_0^t Q(s; \alpha)\mathcal{K}_\alpha f(x) ds.$$

Fix $\beta > \alpha$ and write $\gamma = \beta - \alpha$. Then

$$\begin{aligned} (\mathcal{G}_\beta - \mathcal{F}_\beta)f &= \lambda^{2+\beta} x^\beta f(\lambda x) - x^\beta f(x) = \lambda^{2+\alpha} x^\alpha (\lambda x)^\beta f(\lambda x) - x^\gamma x^\alpha f(x) \\ &= \mathcal{K}_\alpha \mathcal{F}_\gamma f. \end{aligned}$$

Then for $f \in \mathbb{V}^+$,

$$\begin{aligned} Q(t; \beta)f - Q(t; \alpha)f &= \int_0^t Q(t-s; \beta)(\mathcal{K}_\beta - \mathcal{K}_\alpha)Q(s; \alpha)f ds \\ &= \int_0^t Q(t-s; \beta)\mathcal{K}_\alpha(\mathcal{F}_\gamma - \mathcal{I}_1)Q(s; \alpha)f ds. \end{aligned}$$

The supports of f and $Q(s; \alpha)f$ are contained in $[0, 1]$. For a non-negative function g with support in $[0, 1]$,

$$\mathcal{F}_\gamma g(x) = x^\gamma g(x) \leq g(x)$$

and then $(\mathcal{F}_\gamma - \mathcal{I}_1)g(x) \leq 0$. Hence, $Q(t; \beta)f \leq Q(t; \alpha)f$. \square

2.6 Expansion of the semi-group

Since $(Q(t; \alpha))_{t \geq 0}$ is an analytic semi-group, it admits formally the expansion

$$Q(t; \alpha) = \sum_{k \geq 0} \frac{t^k}{k!} (\mathcal{G}_\alpha - \mathcal{F}_\alpha)^k. \quad (16)$$

With (13), when f belongs to \mathbb{F} , $\mathcal{F}_\alpha^k f$ and $\mathcal{G}_\alpha^k f$ also belong to \mathbb{F} for any $k \geq 0$. With Lemma 5, such expansion is valid for any $f \in \mathbb{F}$ and any $t > 0$, using the norm $\|\cdot\|_{\mathbb{F}, \gamma}$.

In this section, we give an explicit expression involving \mathcal{D}_λ , \mathcal{F}_α , as well as q -binomial coefficients.

Definition 1 (see e.g. [31, 35]). For integers k, n with $0 \leq k \leq n$, and $q \in (0, 1)$, the q -binomial coefficient (or *Gaussian polynomial*) is

$$\binom{n}{k}_q := \frac{(1 - q^n) \cdots (1 - q^{n-k+1})}{(1 - q) \cdots (1 - q^k)},$$

where we use the convention that an empty product is equal to 1. This means that for $n = 0$, $k = 0$ or $k = n$, $\binom{n}{k}_q := 1$.

Definition 2. The q -Pochhammer symbol is for $a \in \mathbb{R}$ and $q \in (0, 1)$,

$$(a; q)_k := \prod_{i=0}^{k-1} (1 - aq^i) \text{ for } k \geq 1 \text{ and } (a; q)_0 := 1.$$

Theorem 1 ([35, Theorems 3.2.3 and 3.2.4, pp. 79 and 80]). *The Pascal identity holds*

$$\binom{n+1}{k}_q = \binom{n}{k}_q + q^{n-k+1} \binom{n}{k-1}_q. \quad (17)$$

Moreover, for any $a \in \mathbb{R}$,

$$(a; q)_n = \sum_{k=0}^n q^{n(n-1)/2} \binom{n}{k}_q a^k.$$

We note the relation

$$\mathcal{G}_\alpha \mathcal{F}_\alpha f = \lambda^\alpha \mathcal{F}_\alpha \mathcal{G}_\alpha f \quad (18)$$

for any $f \in \text{Dom}(\mathcal{F}_\alpha^2)$.

Proposition 9. For any $k \geq 0$,

$$(\mathcal{G}_\alpha - \mathcal{F}_\alpha)^k = \sum_{i=0}^k (-1)^{k-i} \binom{k}{i}_{\lambda^{-\alpha}} \mathcal{G}_\alpha^i \mathcal{F}_\alpha^{k-i}. \quad (19)$$

Proof. We proceed by induction. For $k = 0, 1$, (19) is easily verified. Let us assume that (19) is true for $k \geq 1$. Then

$$\begin{aligned} (\mathcal{G}_\alpha - \mathcal{F}_\alpha)^{k+1} &= (\mathcal{G}_\alpha - \mathcal{F}_\alpha) \sum_{i=0}^k (-1)^{k-i} \binom{k}{k-i}_{\lambda^{-\alpha}} \mathcal{G}_\alpha^i \mathcal{F}_\alpha^{k-i} \\ &= \sum_{i=0}^k (-1)^{k-i} \binom{k}{k-i}_{\lambda^{-\alpha}} \mathcal{G}_\alpha^{i+1} \mathcal{F}_\alpha^{k-i} - \sum_{i=0}^k (-1)^{k-i} \binom{k}{k-i}_{\lambda^{-\alpha}} \mathcal{F}_\alpha \mathcal{G}_\alpha^i \mathcal{F}_\alpha^{k-i}. \end{aligned}$$

Applying i times the commutation relation (18),

$$\mathcal{F}_\alpha \mathcal{G}_\alpha^i \mathcal{F}_\alpha^{k-i} = \lambda^{-i\alpha} \mathcal{G}_\alpha^i \mathcal{F}_\alpha^{k+1-i}.$$

Reindexing the sum,

$$\sum_{i=0}^k (-1)^{k-i} \binom{k}{i}_{\lambda^{-\alpha}} \mathcal{G}_\alpha^{i+1} \mathcal{F}_\alpha^{k-i} = \sum_{i=1}^{k+1} (-1)^{k+1-i} \binom{k}{k-i+1}_{\lambda^{-\alpha}} \mathcal{G}_\alpha^i \mathcal{F}_\alpha^{k+1-i}.$$

Using the Pascal identity (17),

$$(\mathcal{G}_\alpha - \mathcal{F}_\alpha)^{k+1} = \sum_{i=1}^k (-1)^{k-i} \binom{k}{i}_{\lambda^{-\alpha}} \mathcal{G}_\alpha^i \mathcal{F}_\alpha^{k-i} + \mathcal{G}_\alpha^{k+1} + (-1)^{k+1} \mathcal{F}_\alpha^{k+1}.$$

This completes the proof. \square

The proof of the next lemma is easily obtained by induction.

Lemma 6. For any $\lambda > 0$, $\mathcal{D}_\lambda \mathcal{F}_\alpha = \lambda^\alpha \mathcal{F}_\alpha \mathcal{D}_\lambda$ and for any $k \geq 1$,

$$\mathcal{G}_\alpha^k := \lambda^{\frac{2k+k(k+1)\alpha-\alpha}{2}} \mathcal{F}_\alpha^k \mathcal{D}_\lambda^k = \lambda^{\frac{2k+k(k-1)\alpha-\alpha}{2}} \mathcal{D}_\lambda^k \mathcal{F}_\alpha^k.$$

Corollary 3. The expansion (16) is also given by

$$Q(t; \alpha)_{\mathbb{F}} = \sum_{k \geq 0} \frac{t^k}{k!} \mathcal{L}_k \mathcal{F}_\alpha^k \quad (20)$$

with

$$\mathcal{L}_k := \sum_{i=0}^k (-1)^{k-i} \binom{k}{k-i}_{\lambda^{-\alpha}} \lambda^{2i+\alpha \frac{i(i-1)}{2} - \alpha} \mathcal{D}_\lambda^i.$$

The q -calculus is sometimes seen as a “deformation” of the ordinary calculus. It is also a way to give a meaning to some summation formula for $q < 1$ and then letting q converging to 1. In contradistinction, $q = 0$ is equivalent to $\alpha = 0$, so that the situation is much simpler. We illustrate here that our semi-group converges as $\alpha \rightarrow 0$ to the semi-group generated by $\alpha = 0$.

Lemma 7. *For any $t \geq 0$ and any $f \in \mathbb{F}$, $Q(t; \alpha)f$ converges to $Q(t; 0)f$ as $\alpha \rightarrow 0$, where $Q(t; 0)$ is given by (2.1).*

Proof. It is known that

$$\lim_{q \rightarrow 1} \binom{k}{i}_q = \binom{k}{i}.$$

Hence, when $\alpha \rightarrow 0$, $\lambda^{-\alpha} \rightarrow 1$ and then

$$\lim_{\alpha \rightarrow 0} \mathcal{L}_k = \sum_{i=0}^k (-1)^{k-i} \binom{k}{k-i} \lambda^{2i} \mathcal{D}_\lambda^i = (\lambda^2 \mathcal{D}_\lambda - \text{Id})^k.$$

On the other hand, it is easily seen that \mathcal{F}_α converges to Id on \mathbb{F} . Therefore, for any $t \geq 0$,

$$\lim_{\alpha \rightarrow 0} Q(t; \alpha) = \sum_{k \geq 0} \frac{t^k}{k!} (\lambda^2 \mathcal{D}_\lambda - \text{Id})^k = \exp(t(\lambda^2 \mathcal{D}_\lambda - \text{Id})).$$

This proves the result. □

3 Evolution of the moments

3.1 Mellin transform

Definition 3 (Mellin transform). The Mellin transform of f is

$$\mathcal{M}[f](\sigma) = \int_0^{+\infty} x^{\sigma-1} \cdot f(x) dx$$

for any $\sigma \in \mathbb{C}$ such that the integral is well defined.

Fix $p > 0$ and define the vertical strip

$$\mathbb{V}(a, b) := \{\sigma \in \mathbb{C} \mid a < \text{Re}(\sigma) < b\} \text{ for } a, b \in \mathbb{R}, a < b.$$

Let $f \in \mathbb{L}_p^+$. Then $\sigma \mapsto \mathcal{M}[f](\sigma)$ is well defined on the closure of the vertical strip $\text{Cl}\mathbb{V}(1, p+1)$, and is analytic for any $\sigma \in \mathbb{V}(1, p+1)$. In particular, for $f \in \mathbb{F}$, $\mathcal{M}[f](\sigma)$ is well defined for any σ with $\text{Re}(\sigma) = 1$ and is analytic in the half-plane \mathbb{H}_1 .

Since $Q(t)f \in \mathbb{F}$ for any $t > 0$, it holds that $\sigma \mapsto \mathcal{M}[Q(t)f](\sigma)$ is well defined for any $t > 0$ and any $\sigma \in \text{Cl}\mathbb{H}_1$. Besides, it is analytic on \mathbb{H}_1 .

Lemma 8. *Assume $f \in \mathbb{L}_p^+$. Then for any $0 < r \leq t$,*

$$\int_r^t \mathcal{M}[\mathcal{F}Q(s)f](\sigma) ds = \int_r^t \mathcal{M}[Q(s)f](\sigma + \alpha) ds, \quad (21)$$

$$\int_r^t \mathcal{M}[\mathcal{G}Q(s)f](\sigma) ds = \lambda^{2-\sigma} \int_r^t \mathcal{M}[Q(s)f](\sigma + \alpha) ds. \quad (22)$$

If $f \in \mathbb{L}_{p+\alpha}$, then one may set $r = 0$ in (21)-(22).

Proof. Let u be the function given by Proposition 4, so that $u(t, x) = Q(t)f(x)$ x -almost everywhere. Then

$$\begin{aligned} \int_0^{+\infty} x^{\sigma-1} \int_r^t \mathcal{F}_\alpha Q(s)f(x) ds dx &= \int_0^{+\infty} x^{\sigma-1} \int_r^t x^\alpha Q(s)f(x) ds dx \\ &= \int_0^{+\infty} x^{\alpha+\sigma-1} \int_r^t Q(s)f(x) ds dx = \int_0^{+\infty} x^{\alpha+\sigma-1} \int_r^t u(s, x) ds dx. \end{aligned}$$

By Fubini's theorem,

$$\int_0^{+\infty} x^{\alpha+\sigma-1} \int_r^t u(s, x) ds dx = \int_r^t \mathcal{M}[u(s, \cdot)](\sigma + \alpha) ds = \int_r^t \mathcal{M}[Q(s)f](\sigma + \alpha) ds \quad (23)$$

Similarly,

$$\int_0^{+\infty} x^{\sigma-1} \int_0^t \mathcal{G}_\alpha Q(s)f(x) ds dx = \lambda^{2-\sigma} \int_0^t \mathcal{M}[Q(s)f](\sigma + \alpha) ds. \quad (24)$$

This leads to (21) and (22).

If $f \in \mathbb{L}_{p+\alpha} = \text{Dom}(\mathcal{F}_\alpha)$, then $\mathcal{F}_\alpha Q(s)f \in \mathbb{L}_p$ and is bounded for small s . Therefore, $\int_0^t \mathcal{F}_\alpha Q(s)f(x) ds dx$ is well defined and (21) and (22) are well defined for $r = 0$. \square

Notation 7. We set for $\sigma \in \mathbb{C}$ and a measurable function $f \in \mathbb{L}_p$,

$$C(t, \sigma; f, \lambda) := \mathcal{M}[Q(t)f](\sigma),$$

whenever this quantity is well defined (*e.g.*, for any σ with $\text{Re}(\sigma) \geq 1$ if $t > 0$, and $1 \leq \text{Re}(\sigma) \leq p+1$ if $t = 0$).

The next lemma is immediate from the control over the semi-group $(Q(t))_{t \geq 0}$. It shows that the moments of order σ remain finite for any time $t > 0$. On this topic, see also [6].

Lemma 9. *If the initial conditions f belongs to $\mathbb{L}_{\text{Re}(\sigma)-1}$ with $\text{Re}(\sigma) \geq 1$, then*

$$|C(t, \sigma; f, \lambda)| \leq e^{(\lambda-1)t} \|f\|_{\text{Re}(\sigma)-1} \leq e^{(\lambda-1)t} \mathcal{M}[|f|](\text{Re}(\sigma)),$$

for any $t \geq 0$.

Proof. This follows from the inequality

$$|\mathcal{M}[g](\sigma)| \leq \int_0^{+\infty} x^{\text{Re}(\sigma)} |g|(x) dx \leq \|g\|_{\text{Re}(\sigma)}$$

and (12) applied to $g := Q(t)f$. □

Combining (21) and (22) leads to the following result.

Proposition 10. *Let $f \in \mathbb{L}_{p+\alpha}$. Then $(t, \sigma) \mapsto C(t, \sigma; f, \lambda)$ satisfies*

$$C(t, \sigma; f, \lambda) = \mathcal{M}[f](\sigma) + (\lambda^{2-\sigma} - 1) \int_r^t C(s, \sigma + \alpha; f, \lambda) ds \quad (25)$$

for any $t \geq 0$, $\sigma \geq p + \alpha + 1$.

Corollary 4 (Mass conservation). *When $\sigma = 2$, $C(t, \sigma; f, \lambda) = C(0, \sigma; f, \lambda)$ for any $t \geq 0$. This means that $t \mapsto \int_0^{+\infty} x Q(t)f(x) dx$ remains constant.*

3.2 A solution of the moment equation

We now give a setting to consider the solution to (25). This framework is restrictive so that most initial conditions cannot be considered within it, as the Mellin transform is unbounded in many situations. However, this framework will be used for initial conditions with compact support, and then to extend the solution to measure-valued solutions.

Fix $\sigma_0 > 0$. We set $\mathbb{S} = [\sigma_0, +\infty)$. Let $(\mathbb{U}, \|\cdot\|_{\mathbb{U}})$ be the space $\mathbb{U} := \mathcal{C}(\mathbb{S}, \mathbb{R})$ with

$$\|g\|_{\mathbb{U}} := \sup_{\sigma \geq 2} |g(\sigma)|.$$

We define the operator \mathcal{A} on \mathbb{U} by

$$\mathcal{A}g(\sigma) = (\lambda^{2-\sigma} - 1)g(\sigma + \alpha) \text{ for any } \sigma \geq 2.$$

Since $\lambda > 1$, \mathcal{A} is bounded with norm $\|\mathcal{A}\|_{\mathbb{U} \rightarrow \mathbb{U}} \leq \max\{1, \lambda^{2-\sigma_0} - 1\}$.

For any $g \in \mathbb{U}$, there exists a unique solution $C \in \mathcal{C}(\mathbb{R}_+ \times \mathbb{S}, \mathbb{R})$ to

$$C(t, \sigma) = g(\sigma) + \int_0^t \mathcal{A}C(s, \sigma) ds = g(\sigma) + (\lambda^{2-\sigma} - 1) \int_0^t C(s, \sigma + \alpha) ds. \quad (26)$$

This solution is given by

$$C(t, \sigma) = R(t)g(\sigma) \text{ with } R(t) := \sum_{k=0}^{+\infty} \frac{t^k}{k!} \mathcal{A}^k.$$

The family of linear operators $(R(t))_{t \geq 0}$ forms a continuous semigroup with the bounded infinitesimal generator $(\mathcal{A}, \mathbb{U})$. In addition,

$$\|R(t)\|_{\mathbb{U} \rightarrow \mathbb{U}} \leq \exp(\|\mathcal{A}\|_{\mathbb{U} \rightarrow \mathbb{U}} t) \text{ for any } t \geq 0.$$

4 Initial condition as measure with compact support

4.1 Extension of the semi-group

In this section, we consider expanding the notion of solution to the fragmentation equation (8) to deal with an initial condition which is a measure. The basic idea is to pass to the limit in the moment equation (26) with respect to the initial condition, and then then transfer this as a limit in the space of measures. This strategy requires the inversion of the Mellin transform, and in particular to be able to solve the moments problem. As this is not always possible, we restrict ourselves to measures with support contained in $[0, 1]$. Possibly, this strategy may be applied to a broader class of measures, yet with some restriction on the growth of the moments.

Notation 8 (Sets \mathbb{M} and \mathbb{K}). We define

$$\begin{aligned} \mathbb{M} &:= \{\mu \mid \mu \text{ is a probability measure, } \text{Supp}(\mu) \subset [0, 1], \mu(\{0\}) = 0\}, \\ \mathbb{K} &:= \left\{ f \in \mathbb{V} \mid \int_0^1 x f(x) dx = 1 \right\} \end{aligned}$$

and the map $\Phi : \mathbb{K} \rightarrow \mathbb{M}$ by

$$\Phi(f)(dx) := x f(x) dx \text{ for } f \in \mathbb{K}.$$

Recall that $\mathbb{V} \subset \mathbb{F}$ was defined in Notation 6.

We equip the set \mathbb{M} with the induced topology of weak convergence of measures.

Lemma 10. *Let $\text{Cl } \Phi(\mathbb{K})$ be the closure of $\Phi(\mathbb{K})$ in \mathbb{M} . Then $\mathbb{M} = \text{Cl } \Phi(\mathbb{K})$.*

Proof. For $a \in (0, 1]$, $\delta_a \in \mathbb{M}$. We define when $0 < \epsilon < a$ the function

$$f_\epsilon(x) = \frac{1}{\epsilon(a + \epsilon/2)} \mathbb{1}_{[a-\epsilon, a]}(x).$$

Therefore, $\Phi(f_\epsilon)$ converges weakly toward δ_a .

The set \mathbb{M} is stable by any barycentric combination. Thus, for any a_1, \dots, a_m with $0 < a_i$, $a_1 + \dots + a_m = 1$, the discrete measure $\sum_{i=1}^m \delta_{x_i}$ belongs to \mathbb{M} .

Let $\mathbb{M}_\epsilon := \{\mu \in \mathbb{M} \mid \mu([0, \epsilon)) = 0\}$. By identifying elements of \mathbb{M}_ϵ with measures on the compact $[\epsilon, 1]$, we know that any element of \mathbb{M}_ϵ may be approximated by discrete probability measures on $[\epsilon, 1]$ (See *e.g.* [10, Example 8.1.6, vol. 2]).

Hence, the closure of $\text{Cl } \Phi(\mathbb{K})$ contains $\bigcup_{0 < \epsilon < 1} \mathbb{M}_\epsilon$.

Finally, let $\mu \in \mathbb{M}$. For $\epsilon \in (0, 1)$, we set

$$\mu_\epsilon(dx) := \frac{1}{\mu([\epsilon, 1])} \mathbb{1}_{x \geq 1} \mu(dx) \in \mathbb{M}_\epsilon.$$

It is easily shown that μ_ϵ converges weakly to μ . This proves the result. \square

We define on \mathbb{M} the Mellin transform

$$\mathcal{N}[\mu](\sigma) := \int_{[0,1]} x^{\sigma-2} \mu(dx) \text{ for any } \mu \in \mathbb{M},$$

so that $\mathcal{N}[\mu](2) = 1$. We remark that $\mathcal{N}[\mu](\sigma)$ is well defined for any $\sigma \in \mathbb{C}$ with $\text{Re}(\sigma) \geq 2$, with $|\mathcal{N}[\mu](\sigma)| \leq 1$. For $f \in \mathbb{K}$,

$$\mathcal{N}[\Phi(f)](\sigma) = \mathcal{M}[f](\sigma) \text{ for any } \sigma \geq 1. \quad (27)$$

Lemma 11. *The set \mathbb{K} is invariant under the action of the semi-group Q .*

Proof. For $f \in \mathbb{K} \subset \mathbb{V} \subset \mathbb{F}$, $\mathcal{M}[f](2) = \int_0^1 x f(x) dx = 1$ and $\mathcal{M}[Q_0(t)f](2) = \mathcal{M}[f](2)$ for any $t \geq 0$ since the mass is preserved. Therefore, \mathbb{K} is invariant as $\mathbb{K} \subset \mathbb{V}$ and \mathbb{V} is itself invariant. \square

The space \mathbb{U} and the semi-group $(R(t))_{t \geq 0}$ have been defined in Section 3.2.

Lemma 12. For any $t \geq 0$, define $P(t) : \Phi(\mathbb{K}) \rightarrow \Phi(\mathbb{K}) \subset \mathbb{M}$ by

$$P(t)\Phi(f) = \Phi(Q(t)f) \text{ for any } f \in \mathbb{K}.$$

Then $P(0) = \text{Id}$, $P(t+s) = P(t)P(s)$ for any $s, t \geq 0$ and

$$\mathcal{N} \circ P(t) = R(t) \circ \mathcal{N}, \quad t \geq 0 \text{ on } \Phi(\mathbb{K}). \quad (28)$$

Finally, $\mathcal{N}(\Phi(\mathbb{K}))$ is invariant by $(R(t))_{t \geq 0}$.

Proof. We note first that $P(t) : \Phi(\mathbb{K}) \rightarrow \mathbb{M}$ is well defined, as from Lemma 11, $Q(t)(\mathbb{K}) \subset \mathbb{K}$ for any $t \geq 0$. Since

$$P(t)P(s)\Phi(f) = P(t)\Phi(Q(s)f) = \Phi(Q(t)Q(s)f) = \Phi(Q(t+s)f) = P(t+s)\Phi(f),$$

we have that P satisfies the semi-group property. In addition, $P(0)$ is the identity operator.

With (27), for $f \in \mathbb{K}$ and $t \geq 0$,

$$\mathcal{N}(P(t)\Phi(f)) = \mathcal{N}(\Phi(Q(t)f)) = \mathcal{M}(Q(t)f).$$

With Proposition 10,

$$\mathcal{M}(Q(t)f) = R(t) \mathcal{M}[f] = R(t) \mathcal{N}[\Phi(f)].$$

We have then established that for any $\mu \in \Phi(\mathbb{K})$,

$$\mathcal{N}[P(t)\mu] = R(t) \mathcal{N}[\mu], \quad t \geq 0 \text{ for any } \mu \in \Phi(\mathbb{K}), \quad (29)$$

which is (28).

Since $P(t)(\Phi(\mathbb{K})) \subset \Phi(\mathbb{K})$, it follows from (28) that $R(t)(\mathcal{N}(\Phi(\mathbb{K}))) \subset \mathcal{N}(\Phi(\mathbb{K}))$. \square

The idea is now to extend $P(t)$ to the closure of $\text{Cl } \Phi(\mathbb{K})$ of $\Phi(\mathbb{K})$ in \mathbb{M} by taking profit from the fact that \mathcal{N} is invertible from a suitable subset of \mathbb{M} to a suitable subset of \mathbb{U} , continuous with a continuous inverse. Besides, the continuity of $R(t)$ implies the strong continuity of $P(t)$, making $(P(t))_{t \geq 0}$ a non-linear semi-group.

Not all probabilities distributions μ can be recovered from their moments $\{\mathcal{N}[\mu](\sigma + k)\}_{k \geq 0}$. This is the Stieltjes (for distribution with positive support) or the Hamburger problem [32, 37].

Theorem 2 ([24, Example VIII.1.(d), p. 251]). *A probability measure μ with support in $[0, 1]$ is fully determined by its moments, meaning that the knowledge of all its moments is sufficient to characterize the distribution.*

This implies the following classical result on convergence [32, 36].

Theorem 3 (Fréchet-Shohat theorem [32]). *Let $\{\mu_n\}_{n \geq 0}$ be a sequence of measures in \mathbb{M} . Assume that there exists a sequence $\{m_k\}_{k \geq 0}$ for which*

$$\mathcal{M}[\mu_n](k+2) = \int_0^1 x^k \mu_n(dx) \xrightarrow{n \rightarrow \infty} m_k \text{ for any } k \geq 0. \quad (30)$$

Then there exists a measure $\mu \in \mathbb{M}$ such that $\{\mu_n\}_{n \geq 0}$ converges weakly to μ and $\mathcal{M}[\mu](k+2) = m_k$, $k \geq 0$.

On the converse, if $\{\mu_n\}_{n \geq 0}$ converges weakly to μ , then $\mathcal{M}[\mu_n](\sigma)$ converges to $\mathcal{M}[\mu](\sigma)$ for any $\sigma \geq 2$. In particular, (30) holds with $m_k = \mathcal{M}[\mu](k+2)$.

Lemma 13. *For $\mu \in \mathbb{M}$, $\sigma \in \mathbb{R} \mapsto \mathcal{M}[\mu](\sigma)$ is continuous on $[2, +\infty)$. Besides, this function is completely monotonic on $(2, +\infty)$. In particular, it is decreasing, convex and never vanishes. It can be extended analytically on the half-plane \mathbb{H}_2 .*

Remark 2. It follows from the proof that if $\mu(\{0\}) > 0$, then $\lim_{\sigma \rightarrow 2} \mathcal{M}[\mu](\sigma) < 1$ and then $\mathcal{M}[\mu]$ is discontinuous at $\sigma = 2$.

Proof. As for $\mu \in \mathbb{M}$, $\mu(\{0\})$, we obtain from the monotone convergence theorem that $\sigma \mapsto \mathcal{M}[\mu](\sigma)$ is continuous, decreasing on $(2, +\infty)$. Besides,

$$\lim_{\sigma \rightarrow 2, \sigma > 2} \mathcal{M}[\mu](\sigma) = \mu((0, 1]) = 1 = \mathcal{M}[\mu](2),$$

so that $\sigma \in \mathbb{R} \mapsto \mathcal{M}[\mu](\sigma)$ is continuous on $[2, +\infty)$.

In addition, for any $\sigma > 2$,

$$\partial_\sigma^k \mathcal{M}[\mu](\sigma) = \int_0^1 \ln(x)^k x^{\sigma-2} \mu(dx) \text{ for any } k \geq 0.$$

Since $x \mapsto x^\gamma \ln(x)^k$ is bounded on $[0, 1]$ for any $\gamma > 0$, $k \geq 1$, it holds that for some constant C_γ ,

$$|\partial_\sigma^k \mathcal{M}[\mu](\sigma)| \leq C_\gamma \|f\|_\infty \text{ for any } \sigma \geq 2 + \gamma.$$

Besides, $(-1)^k \partial_\sigma^k \mathcal{M}[\mu](\sigma) \geq 0$. This means that $\mathcal{M}[\mu](\sigma)$ is completely monotonic on $(2, +\infty)$. The properties of $\mathcal{M}[\mu]$ then follow from [37, Theorem IV.3a and Theorem IV.3a, p. 146-147]. \square

Lemma 14. *Let $\{\mu_n\}_{n \geq \mathbb{N}}$ with $\mu_n \in \mathbb{M}$. It holds that:*

- (i) *If μ_n converges weakly to $\mu \in \mathbb{M}$ then $\mathcal{N}[\mu_n]$ converges uniformly to $\mathcal{N}[\mu]$ on $[2, +\infty)$.*

(ii) If $\mathcal{N}[\mu_n]$ converges uniformly to some function g on $[2, +\infty)$ then there exists $\mu \in \mathbb{M}$ such that $g = \mathcal{N}[\mu]$ and μ_n converges weakly to μ .

Proof. We assume the weak convergence of $\mu_n \in \mathbb{M}$ to $\mu \in \mathbb{M}$. Since $\text{Supp}(\mu_n) \in [0, 1]$, and $\mu(\{0\}) = 0$, $\mathcal{M}[\mu_n](\sigma)$ converges to $\mathcal{N}[\mu](\sigma)$ for any $\sigma \geq 2$.

We prove the uniform convergence in two steps: We establish the uniform convergence of $\mathcal{N}[\mu_n]$ on $[\sigma_0, +\infty)$ with $\sigma_0 > 2$, and then on $[2, +\infty)$.

Set $F_n(x) := \mu_n([0, x])$ and $F(x) := \mu([0, x])$. For some sequence $\{a_n\}_{n \geq 0}$ in $[0, 1]$,

$$\begin{aligned} & \left| \int_0^1 x^{\sigma-2} \mu_n(dx) - \int_0^1 x^{\sigma-2} \mu(dx) \right| \\ & \leq \left| \int_0^{a_n} x^{\sigma-2} \mu_n(dx) - \int_0^{a_n} x^{\sigma-2} \mu(dx) \right| + |\mu_n((a_n, 1]) - \mu((a_n, 1])|. \end{aligned}$$

Since $F(1) = F_n(1) = 1$,

$$|\mu_n((a_n, 1]) - \mu((a_n, 1])| = |F_n(a_n) - F(a_n)|.$$

Fix $\sigma_0 > 2$. For any $\sigma > \sigma_0$,

$$\left| \int_0^{a_n} x^{\sigma-2} \mu_n(dx) - \int_0^{a_n} x^{\sigma-2} \mu(dx) \right| \leq a_n^{\sigma-2} \leq a_n^{\sigma_0-2}.$$

Since $\mu(\{0\}) = 0$, F is continuous at 0 so that there exists some $\gamma > 0$ such that F is continuous on $[0, \gamma]$. Therefore, as $a_n \rightarrow 0$, $\lim_{n \rightarrow \infty} F(a_n) = 0$. On the other hand, $F_n(a_n) \leq F_n(\eta)$ when $a_n \leq \eta < \gamma$ and $\lim_{n \rightarrow \infty} F_n(\eta) = F(\eta)$ since η is a point of continuity of F . Therefore, for any $\epsilon > 0$, there exists η such that $F(\eta) \leq \epsilon$, $F_n(\eta) - F(\eta) \leq \epsilon$ for $n \geq n_0$ with n_0 large enough. Thus, $\lim_{n \rightarrow \infty} F_n(a_n) = 0$. The choice of η and n_0 do not depend on σ .

Letting $\{a_n\}_{n \geq 0}$ converging to 0 leads to the uniform convergence on $\mathcal{N}[\mu_n]$ to $\mathcal{N}[\mu]$ on any interval $[\sigma_0, +\infty)$ with $\sigma_0 > 2$.

Since $\sigma \mapsto \mathcal{N}[\mu_n](\sigma)$ is non-increasing for each n , and $\sigma \mapsto \mathcal{N}[\mu](\sigma)$ is continuous, it follows from the Dini theorem that the convergence of $\mathcal{N}[\mu_n]$ toward $\mathcal{N}[\mu]$ is uniform on $[0, \sigma_0]$.

Regarding (ii), the Fréchet-Shohat theorem implies the weak convergence of μ_n to some measure μ with compact support on $[0, 1]$. Since $\mathcal{N}[\mu_n] = 2$, then $g(2) = 1$ so that μ is a probability measure. It remains to prove that $\mu(\{0\}) = 0$. With Remark 2, this happens if and only if $\mathcal{N}[\mu]$ is right-continuous at $\sigma = 2$. Since $\{\mathcal{N}[\mu_n]\}_{n \geq 0}$ converges uniformly on $[2, +\infty)$, $\mathcal{N}[\mu]$ is continuous at $\sigma = 2$ and then $\mu(\{0\}) = 0$. \square

Lemma 15. *It holds that*

$$\text{Cl}\mathcal{N}[\Phi(\mathbb{K})] = \mathcal{N}[\text{Cl}\Phi(\mathbb{K})] = \mathcal{N}[\mathbb{M}],$$

and \mathcal{N} is invertible and bicontinuous from \mathbb{M} to $\mathcal{N}[\mathbb{M}]$. Finally, $\mathcal{N}[\mathbb{M}]$ is invariant under the semi-group $(R(t))_{t \geq 0}$.

Proof. Let $g \in \text{Cl}\mathcal{N}[\Phi(\mathbb{K})]$. This implies that there exists a sequence $\{f_n\}_{n \geq 0}$ such that $f_n \in \mathbb{K}$ and $g_n = \mathcal{N}[\Phi(f_n)]$ converges uniformly to g . With $\mu_n = \Phi(f_n)$ and Lemma 14 there exists $\mu \in \mathbb{M}$ such that μ_n converges weakly to μ and $\mathcal{N}[\mu] = g$. Thus, $g \in \mathcal{N}[\mathbb{M}]$.

Hence, $\text{Cl}\mathcal{N}[\Phi(\mathbb{K})] \subset \mathcal{N}[\mathbb{M}]$. Conversely, if $g = \mathcal{N}[\mu]$ with $\mu \in \mathbb{M}$, there exists a sequence $\{\mu_n\}_{n \geq 0}$ with $\mu_n \in \Phi(\mathbb{K})$ such that μ_n converges weakly to μ . By Lemma 14 again, this proves that $\mathcal{N}[\mathbb{M}] \subset \text{Cl}\mathcal{N}[\Phi(\mathbb{K})]$.

We then set $\mathcal{N}^{-1}(g) = \mu$. This measure is unique, otherwise, it contradicts Theorem 2 that states that elements of \mathbb{M} are uniquely determined by their moments.

From Lemma 12, $R(t)g_n \in \mathcal{N}[\Phi(\mathbb{K})]$ and by continuity of $R(t)$ in \mathbb{U} , $R(t)g \in \text{Cl}\mathcal{N}[\Phi(\mathbb{K})] = \mathcal{N}[\mathbb{M}]$. \square

Proposition 11. *We define the family $P = (P(t))_{t \geq 0}$ of operators from \mathbb{M} to \mathbb{M} by*

$$P(t) := \mathcal{N}^{-1} \circ R(t) \circ \mathcal{N} \text{ for any } t \geq 0.$$

Then P is a continuous non-linear semi-group on \mathbb{M} . Furthermore, the set $\Phi(\mathbb{K})$ is invariant under $P(t)$ for any $t \geq 0$.

By construction, since for any $t > 0$, $R(t)$ is continuous on \mathbb{U} , we obtain that $\mu \mapsto P(t)\mu$ is also continuous with respect to the weak topology. Since the space of measures is also a metric space, P is jointly continuous on $(0, +\infty) \times \mathbb{M}$ [17, Theorem 4].

4.2 Measure-valued solutions

If u is solution to (1) with the initial condition $f \in \mathbb{K}$, then $(t, x) \mapsto xu(t, x)$ is solution to (2) with initial condition $x \mapsto xf(x)$.

Let μ be a measure in \mathbb{M} . Set $\phi_\lambda(x) := x/\lambda$. Let $\phi_\lambda \# \mu$ be the pushforward of μ , that is the measure such that for any function $\psi \in \mathcal{C}([0, 1])$,

$$\int_0^1 \psi\left(\frac{x}{\lambda}\right) \mu(dx) = \int_0^1 \psi(x) \phi_\lambda \# \mu(dx).$$

When $\mu(dx) = f(x)dx$, it holds that $\phi_\lambda \# \mu(dx) = \lambda f(\lambda x) \mathbb{1}_{x \leq 1}$.

For a given measure $\mu \in \mathbb{M}$, we define $\mu_t = P(t)\mu$, which is formally a solution to the equation

$$\partial_t \mu_t(dx) = \lambda^\alpha x^\alpha \phi_\lambda \# \mu_t(dx) - x^\alpha \mu_t(dx). \quad (31)$$

For any $\psi \in \mathcal{C}([0, 1])$,

$$\partial_t \int_0^1 \psi(x) \mu_t(dx) = \int_0^1 \lambda^\alpha x^\alpha \psi\left(\frac{x}{\lambda}\right) \mu_t(dx) - \int_0^1 x^\alpha \psi(x) \mu_t(dx).$$

Clearly, we could extend by linearity (31) to any initial condition in the cone $\text{Cone}(\mathbb{M}) := \{\gamma\mu \mid \gamma > 0, \mu \in \mathbb{M}\}$ of non-negative measures generated by \mathbb{M} .

We also define

$$\mathcal{M}[\mu](\sigma) := \int_0^1 x^{\sigma-1} \mu(dx) = \mathcal{N}[\mu](\sigma + 1).$$

The difference with \mathcal{N} lies in the index. Let \mathbb{M}_{-1} be the set of measures $\mu \in \mathbb{M}$ such that $\mathcal{M}[\mu](1) < +\infty$ or equivalently, $\mathcal{N}[\mu](2) < +\infty$. This implies that $\sigma \mapsto \mathcal{M}[\mu](\sigma)$ is continuous on $[1, +\infty)$. The semi-group $(R(t))_{t \geq 0}$ may be extended to $\mathcal{C}([1, +\infty))$ so that $R(t) \mathcal{M}[\mu]$ is bounded for any $t \geq 0$.

For $\mu \in \mathbb{M}_{-1}$, we set $\mathcal{R}\mu(dx) := \frac{1}{x} \mu(dx)$. It is easily checked that

$$\phi_\lambda \# (\mathcal{R}\mu)(dx) = \frac{1}{\lambda} \mathcal{R} \phi_\lambda \# \mu(dx).$$

Theorem 4. *For $\mu_0 \in \text{Cone}(\mathbb{M}_{-1})$, there exists a unique measure-valued solution $\{\mu_t\}_{t \geq 0}$ to*

$$\partial_t \mathcal{R}\mu_t(dx) = \lambda^{1+\alpha} x^\alpha \phi_\lambda \# \mathcal{R}\mu_t(dx) - x^\alpha \mathcal{R}\mu_t(dx), \quad (32)$$

which is given by $\mu_t = R(t)\mu_0$. If $\mu_0 = f(x) dx$ with $f \in \mathbb{K}$, then $\mu_t = c(t, x) dx$ where c solves (1) with $c(0, \cdot) = f$.

Proof. Eq. (32) is a rewriting of (31). The uniqueness follows from the uniqueness of the moment equations. On the other hand, if $f \in \mathbb{K}$ and $\mu_0 = f(x) dx$, then with $\nu_0 = \Psi(f)$, the solution to (31) has a solution $\nu_t \in \Phi(\mathbb{K})$ so that $\mu_t := \mathcal{R}\nu_t \in \Phi(\mathbb{K})$. \square

5 Evolution of the moments - follow up

The above results show that the moment equation is an important tool to study the fragmentation equation. We now pursue our analysis of the moment equation (26).

5.1 A differential recursion

Hypothesis 1. For any integer k , we consider $g_k > 0$. For some k_0 , we consider β_k such that $\beta_k < 0$ for any $k > k_0$, and $\beta_k > 0$ for $k < k_0$, and $\beta_{k_0} \geq 0$.

We consider the differential recursion

$$C_k(t) = g_k + \beta_k \int_0^t C_{k+1}(s) ds. \quad (33)$$

Hypothesis 2. The family $\{C_k\}$ is such that each C_k is continuous and non-negative.

Lemma 16. *Under Hypotheses 1 and 2, for $k > k_0$, each C_k is infinitely differentiable at each $t \geq 0$ and a completely monotonic function on $[0, +\infty)$, that is*

$$(-1)^\ell \partial_t^\ell C_k(t) \geq 0 \text{ for any } t \geq 0.$$

Therefore, C_k is analytic, is convex and never vanishes.

Proof. For any $k \geq k_0$, since C_{k+1} is continuous, clearly C_k is \mathcal{C}^1 with

$$-\partial_t C_k(t) = -\beta_k C_{k+1}(t).$$

We then deduce that C_k is infinitely differentiable and

$$(-1)^\ell \partial_t^\ell C_k(t) = \left(\prod_{i=0}^{\ell-1} (-\beta_{k+i}) \right) C_{k+\ell}(t) \geq 0.$$

By Hypothesis 2, C_k is non-negative. If C_k vanishes at some time t_0 , then $C_k(t) = 0$ for any $t \geq t_0$ as the sign of its derivative does not change. Being analytic, C_k is identically zero. Yet $C_k(0) = g_k > 0$ from Hypothesis 1. Thus C_k never vanishes.

Since $\partial_t^2 C_k(t) = \beta_k \beta_{k+1} C_{k+2}(t)$, $t \geq 0$, and $\beta_k \beta_{k+1} > 0$, the function $C_k(t)$ is convex. \square

Under Hypothesis 2, C_k is non-decreasing if $\beta_k > 0$, C_k is non-increasing if $\beta_k < 0$ and $C_k = g_k$ if $\beta_k = 0$.

Lemma 17. *Under Hypotheses 1 and 2, for $k < k_0$, $\lim_{t \rightarrow \infty} C_{k+1}(t) > 0$. Moreover, C_k is increasing and $\lim_{t \rightarrow \infty} C_k(t) = +\infty$.*

Proof. Set $A = \lim_{t \rightarrow \infty} C_{k+1}(t)$. Then for some $t_0 \geq 0$, $C_{k+1}(t) \geq A/2$ for any $t \geq t_0$. It follows that

$$C_k(t) - C_k(t_0) \geq \frac{\beta_k}{A} 2(t - t_0).$$

Hence the result. \square

Lemma 18. *Assume Hypotheses 1 and 2. Then for any $k > k_0$, C_k is decreasing and $\lim_{t \rightarrow \infty} C_k(t) = 0$.*

Proof. Since C_{k+1} is non-negative, C_k is non-increasing for any $k > k_0$. By Lemma 16, C_k does not vanish for $k > k_0$ and thus C_k is decreasing.

Assume that for some $k > k_0$, $A := \lim_{t \rightarrow \infty} C_{k+1}(t) > 0$. Since $\beta_k \leq 0$, $C_k(t) \leq C_k(t_1) + \beta_k A(t - t_1)$, for any $t \geq t_1$. Hence, there exists t_2 such that $C_k(t_2) = 0$. This contradicts Lemma 18.

This implies that $\lim_{t \rightarrow \infty} C_k(t) = 0$ yet each C_k is positive. □

Combining Lemma 18 and Lemma 17, we obtain that

- for $k > k_0$, C_k decreases to 0, and is convex,
- for $k = k_0$, $C_k \geq g_k > 0$,
- for $k < k_0$, C_k increases to $+\infty$. Besides, if $k < k_0 - 1$, the C_k is convex.

Lemma 19. *Under Hypothesis 1, there exists at most a unique solution $\{C_k\}$ that satisfies Hypothesis 2. Besides, C_k is analytic for any $k \geq 0$.*

Proof. For $k > k_0$, C_k is analytic by Lemma 16. It is uniquely defined by its derivatives at 0, which are related to the $\{g_k\}$ and $\{\beta_k\}$.

For $k \leq k_0$, it is sufficient to iterate ℓ times the development

$$C_k(t) = g_k + \beta_k g_{k+1} t + \beta_k \beta_{k+1} \int_0^t \int_0^{t_1} C_{k+2}(t_2) dt_2 dt - 1,$$

where ℓ is such that $k + \ell > k_0$. As the integral of analytic functions are analytic, C_k is analytic. □

5.2 Behavior of the moments with respect to time

Notation 9. We set

$$\mathbb{M}_{\mathbb{F}} := \{\mu \mid \mu(dx) = f(x) dx \text{ with } f \in \mathbb{F}^+\}.$$

Let $\mu \in \text{Cone } \mathbb{M}_{-1}$ or $\mu \in \mathbb{M}_{\mathbb{F}}$. Then for any $\sigma \geq 1$, let $C(t, \sigma; \mu, \lambda)$ be the solution of the moment equation (26) (resp. (25)) with $g := \mathcal{M}[\mu](\sigma)$.

Fix some $\sigma \geq 1$. With $C_k(t) := C(t, \sigma + k\alpha; \mu, \lambda)$ and $\beta(\lambda) := \lambda^{2-\sigma} - 1$, $\{C_k\}_{k \geq 0}$ solves the differential recursion equation.

Moreover, $\beta(2) = 0$. We summarize the above results.

Proposition 12. For any $\sigma \geq 1$, $C(\cdot, \sigma; \mu, \lambda)$ is analytic. In addition, when $\alpha > 0$,

- for $\sigma > 2$, $C(\cdot, \sigma; \mu, \lambda)$ decreases to 0, and is convex,
- for $\sigma = 2$, $C(t, 2; \mu, \lambda) = \mathcal{M}[\mu](2)$ for $t \geq 0$.
- for $\sigma < 2$, $C(\cdot, \sigma; \mu, \lambda)$ increases to $+\infty$. Besides, for $\sigma + \alpha < 2$, $C(\cdot, \sigma; \mu, \lambda)$ is convex and for $2 - \alpha < \sigma < 2$, $C(\cdot, \sigma; \mu, \lambda)$ is concave,
- for $\sigma = 2 - \alpha$, $C(t, \sigma; \mu, \lambda) = \mathcal{M}[\mu](1) + t\mathcal{M}[\mu](2)$, for $t \geq 0$.

When $\alpha = 0$, (26) is an ordinary differential equation, with

$$C(t, \sigma; \mu, \lambda) = \exp(t(\lambda^{2-\sigma} - 1)) \mathcal{M}[\mu](\sigma).$$

The long-time behavior of C is exponentially decreasing for $\sigma > 2$ and exponentially increasing for $\sigma < 2$.

More precise asymptotic behavior are given below in Corollary 5 and Proposition 14.

5.3 Expansion of the moments

Since $C(\cdot, \sigma; \mu, \lambda)$ is analytic, it has a power series representation. We could apply the Mellin transform to use (20). However, a direct expansion to (25) leads to a power series in a simple form.

Notation 10. The Pochhammer symbol is given in Definition 2. We write

$$\varpi_k(\sigma, \lambda) := (-1)^k (\lambda^{2-\sigma}; \lambda^{-\alpha})_k, \quad k \geq 1 \text{ and } \varpi_0(\sigma, \lambda) := 1.$$

From a straightforward computation,

$$\varpi_k(\sigma + \alpha, \lambda) = \frac{\varpi_{k+1}(\sigma, \lambda)}{\varpi_1(\sigma, \lambda)}, \quad (34)$$

We deduce from (34) that

$$\varpi_1(\sigma, \lambda) \varpi_1(\sigma + \alpha, \lambda) \cdots \varpi_1(\sigma + k\alpha, \lambda) = \varpi_{k+1}(\sigma, \lambda).$$

If $n = (2 - \sigma)/\alpha \in \mathbb{N}$, then $\varpi_m(\sigma, \lambda) = 0$ for any $m \geq n$. In addition, computing $\partial_\lambda \log \varpi_k(\sigma, \lambda)$, we obtain

$$\partial_\lambda \varpi_k(\sigma, \lambda) = \varpi_k(\sigma, \lambda) D_k(\sigma, \alpha, \lambda) \quad (35)$$

$$\text{with } D_k(\sigma, \alpha, \lambda) := \sum_{i=0}^{k-1} (2 - \sigma - i\alpha) \frac{\lambda^{1-\sigma-i\alpha}}{1 - \lambda^{2-\sigma-i\alpha}}, \quad (36)$$

unless $2 - \sigma - i\alpha \neq 0$ for some $i \in \{0, \dots, k-1\}$. In the latter case, $\varpi_k(\sigma, \lambda) = 0$ and then $\partial_\lambda \varpi(\sigma, \lambda) = 0$.

Proposition 13. *Let $\mu \in \text{Cone } \mathbb{M}_{-1}$ or $\mu \in \mathbb{M}_{\mathbb{F}}$, Then*

$$C(t, \sigma; \mu, \lambda) = \sum_{k=0}^{+\infty} \varpi_k(\sigma, \lambda) \frac{t^k}{k!} \mathcal{M}[\mu](\sigma + k\alpha) \text{ for } t \geq 0, \sigma \in \mathbb{H}_1. \quad (37)$$

Let us end this section with a result regarding the scaling by invariance.

Lemma 20 (Scaling for initial Dirac measure). *For any $\sigma > 0$, $t \geq 0$ and $a \in (0, 1]$,*

$$C(t, \sigma; \delta_a, \lambda) = a^{\sigma-1} C(ta^\alpha, \sigma; \delta_1, \lambda). \quad (38)$$

Proof. For the Dirac mass, $\mathcal{M}[\delta_a](\sigma) = a^{\sigma-1}$,

$$\begin{aligned} C(t, \sigma; \delta_a, \lambda) &= \sum_{k=0}^{+\infty} \varpi_k(\sigma, \lambda) \frac{t^k}{k!} a^{\sigma+k\alpha-1} \\ &= a^{\sigma-1} \sum_{k=0}^{+\infty} \varpi_k(\sigma, \lambda) \frac{(ta^\alpha)^k}{k!} = a^{\sigma-1} C(ta^\alpha, \sigma; \delta_1, \lambda) \end{aligned}$$

as $\mathcal{M}[\delta_1](\sigma) = 1$. □

6 Asymptotic behavior of the moments

We extend the operator \mathcal{F}_α to measures by $\mathcal{F}_\alpha \mu(dx) := x^\alpha \mu(dx)$. Thus, $\mathcal{M}[\mathcal{F}_\alpha \mu](\sigma) = \mathcal{M}[\mu](\sigma + \alpha)$ for any $\sigma \geq 1$ when $\mu \in \text{Cone } \mathbb{M}_{-1}$.

Lemma 21. *For $\mu \in \text{Cone } \mathbb{M}_{-1}$ or $\mu \in \mathbb{M}_{\mathbb{F}}$,*

$$\partial_t C(t, \sigma; \mu, \lambda) = \lambda^{2-\sigma} C(\lambda^{-\alpha} t, \sigma; \mathcal{F}_\alpha \mu, \lambda) - C(t, \sigma; \mathcal{F}_\alpha \mu, \lambda), \quad (39)$$

for $t \geq 0$, $\sigma \geq 1$.

Proof. From (37),

$$C(t, \sigma; \mathcal{F}_\alpha \mu, \lambda) = \sum_{k=0}^{+\infty} \varpi_k(\sigma, \lambda) \frac{t^k}{k!} \mathcal{M}[\mu](\sigma + (k+1)\alpha).$$

Besides,

$$\partial_t C(t, \sigma; \mu, \lambda) = \sum_{k=0}^{+\infty} \varpi_{k+1}(\sigma, \lambda) \frac{t^k}{k!} \mathcal{M}[\mu](\sigma + (k+1)\alpha).$$

With (34),

$$\partial_t C(t, \sigma; \mu, \lambda) = \varpi_1(\sigma, \lambda) \sum_{k=0}^{+\infty} \varpi_k(\sigma, \lambda) \frac{t^k}{k!} \mathcal{M}[\mu](\sigma + (k+1)\alpha) = \varpi_1(\sigma, \lambda) C(t, \sigma; \mathcal{F}_\alpha \mu, \lambda).$$

Since $\varpi_1(\sigma, \lambda) = \lambda^{2-\sigma} - 1$, we obtain (39) and then the result. \square

Lemma 22. *There exists a family of constants $D_{i,k}(\lambda, \sigma, \alpha)$ with $0 \leq i \leq k$ such that*

$$\partial_t^k C(t, \sigma; f) = \sum_{i=0}^k D_{i,k}(\lambda, \sigma, \alpha) C(\lambda^{-i\alpha} t, \sigma; \mathcal{F}^k f), \quad (40)$$

for any $t \geq 0, k \geq 0$. These constants are given by $D_{0,1} = -1, D_{1,1} = \lambda^{2-\sigma}$, and

$$D_{0,k+1} = -D_{0,k} \text{ and } D_{i,k} = (-1)^{k-i} \lambda^{i(2-\sigma) - \binom{i}{2}\alpha} \binom{k}{i}_{\lambda^{-\alpha}}. \quad (41)$$

Proof. We proceed by recursion. The first step $k = 1$ follows from Lemma 21. Let assume that (41) holds for $k \geq 1$. By using Lemma 21 on the time derivative of (40),

$$\begin{aligned} \partial_t^{k+1} C(t, \sigma; f) &= \sum_{i=0}^k D_{i,k}(\lambda, \sigma, \alpha) \lambda^{2-\sigma-i\alpha} C(\lambda^{-(i+1)\alpha} t, \sigma; \mathcal{F}^{k+1} f) \\ &\quad - \sum_{i=0}^k D_{i,k}(\lambda, \sigma, \alpha) \lambda^{-i\alpha} C(\lambda^{-i\alpha} t, \sigma; \mathcal{F}^{k+1} f). \end{aligned} \quad (42)$$

Using the recursion hypothesis after some reindexing and since $\binom{i}{2} = \binom{i-1}{2} + i - 1$,

$$\begin{aligned} &\lambda^{2-\sigma-(i-1)\alpha} D_{i-1,k}(\lambda, \sigma, \alpha) + \lambda^{-i\alpha} D_{i,k}(\lambda, \sigma, \alpha) \\ &= (-1)^{k+1-i} \lambda^{i(2-\sigma) - \binom{i}{2}\alpha} \left[\binom{k}{i-1}_{\lambda^{-\alpha}} + \lambda^{-i\alpha} \binom{k}{i}_{\lambda^{-\alpha}} \right]. \end{aligned}$$

From the Pascal's identity (17),

$$\lambda^{2-\sigma-(i-1)\alpha} D_{i-1,k}(\lambda, \sigma, \alpha) + \lambda^{-i\alpha} D_{i,k}(\lambda, \sigma, \alpha) = D_{i,k+1}(\lambda, \sigma, \alpha), \quad (43)$$

where for all $1 \leq i \leq k$:

$$D_{i+1,k}(\lambda, \sigma, \alpha) := (-1)^{k+1-i} \lambda^{i(2-\sigma) - \binom{i}{2}\alpha} \binom{k+1}{i}_{\lambda^{-\alpha}}. \quad (44)$$

By computing the term in $i = 0$ and $i = k + 1$, we have easily the desired formula. \square

A direct consequence that $\mathcal{F}\delta_1 = \delta_1$ in (39) is that $C(\cdot, \sigma; \delta_1, \lambda)$ solves a *pantograph equation*.

Lemma 23. *For any $t \geq 0$, $\sigma \geq 1$,*

$$\partial_t C(t, \sigma; \delta_1, \lambda) = \lambda^{2-\sigma} C(\lambda^{-\alpha} t, \sigma; \delta_1, \lambda) - C(t, \sigma; \delta_1, \lambda) \quad (45)$$

with $C(0, \sigma; \delta_1, \lambda) = 1$.

The pantograph equation has given rise to a large number of studies, see *e.g.*, [25, 29, 30].

Remark 3. Using the scaling (38) or a direct computation shows that for any $a > 0$,

$$\partial_t C(t, \sigma; \delta_a, \lambda) = a^\alpha \lambda^{2-\sigma} C(\lambda^{-\alpha} t, \sigma; \delta_a, \lambda) - a^\alpha C(t, \sigma; \delta_a, \lambda) \text{ for } t > 0. \quad (46)$$

This is why we focus only on the case where the initial measure δ_1 .

We now focus on the long time behavior of $t \mapsto C(t, \sigma; \delta_1, \lambda)$, for which a first classification was given in Section 5.2. As this function is analytic, its asymptotic behavior is given by power series, or via Tauberian theorems.

When $\sigma \neq 2$, the solution to (45) is given by the power series [25, Eq. (1.5)]

$$C(t, \sigma; \delta_1, \lambda) = \sum_{k \geq 0} \frac{(\lambda^{2-\sigma}; \lambda^{-\alpha})_k}{k!} (-t)^k, \quad (47)$$

which is uniformly convergent for any $t \in \mathbb{C}$. The article [38] gives some nice probabilistic interpretations of this formula both in term of Brownian motion and Point Processes.

Remark 4. Assume that for some integer n , $\sigma + n\alpha = 2$. In particular, $\sigma < 2$. Then

$$C(t, \sigma; \delta_1, \lambda) = \sum_{k=0}^n \frac{(\lambda^{2-\sigma}; \lambda^{-\alpha})_k}{k!} (-t)^k,$$

as $(\lambda^{2-\sigma}; \lambda^{-\alpha})_k = 0$ for any $k \geq n$. Hence,

$$C(t, \sigma; \delta_1, \lambda) \sim_{t \rightarrow \infty} (-1)^n \frac{(\lambda^{2-\sigma}; \lambda^{-\alpha})_n}{n!} t^{\frac{2-\sigma}{\alpha}}.$$

This could have also been deduce from the moment equation.

An alternative form to (47) is given by [25, Eq. (1.6)] or [39, Proposition 2.1]

$$C(t, \sigma; \delta_1, \lambda) = (\lambda^{2-\sigma}; \lambda^{-\alpha})_\infty \sum_{k \geq 0} \frac{\lambda^{k(2-\sigma)}}{(\lambda^{-\alpha}; \lambda^{-\alpha})_k} \exp(-\lambda^{-k\alpha} t), \quad (48)$$

which is convergent for any $t \in \mathbb{C}$ when $\sigma > 2$, and thus $\lambda^{2-\sigma} < 1$.

Notation 11 (Laplace transform). The Laplace transform of C is denoted by

$$\widehat{C}(s) := \int_0^{+\infty} \exp(-st) C(t, \sigma; \mu, \lambda) dt \text{ for } s > 0.$$

Since $C(\cdot, \sigma; \mu, \lambda)$ is non-negative, \widehat{C} is completely monotone and thus analytic. For $\sigma > 2$, C is bounded so that \widehat{C} exists for any $s > 0$. Since the series (47) is uniformly convergent for any $t \geq 0$, we may invert integrals and series so that the Laplace transform \widehat{C} exists also when $\sigma < 2$.

Tauberian theorems relate the large-time behavior of $C(\cdot, \sigma; \delta_1, \lambda)$ with the behavior of $\widehat{C}(s)$ for s close to 0.

Taking the Laplace transform of (47) leads to

$$\widehat{C}(s) = \sum_{k \geq 0} (\lambda^{2-\sigma}; \lambda^{-\alpha})_k \frac{(-1)^k}{s^{k+1}}. \quad (49)$$

This is not sufficient to study the asymptotic behavior of $C(t, \sigma; \delta_1, \lambda)$ for large t . Taking the Laplace transform of (48) leads to

$$\widehat{C}(s) = K \sum_{k \geq 0} \frac{\lambda^{k(2-\sigma)}}{(\lambda^{-\alpha}; \lambda^{-\alpha})_k} \frac{1}{s + \lambda^{-k\alpha}}. \quad (50)$$

Again, there is no clear interpretation. We are then looking for an alternative representation.

Applying the Laplace transform to both sides of (45) leads to

$$(1 + s)\widehat{C}(s) = 1 + \lambda^{2-\sigma+\alpha}\widehat{C}(\lambda^\alpha s) \text{ for } s > 0,$$

since $C(0, \sigma; \delta_1, \lambda) = 1$. We rewrite this equation as

$$(1 + qs)\widehat{C}(qs) = 1 + \lambda^{2-\sigma+\alpha}\widehat{C}(s) \text{ with } q = \lambda^{-\alpha} \in (0, 1). \quad (51)$$

Lemma 24. *Assume $\sigma \geq 1$, $\sigma \neq 2 + \alpha$. Then for any $s > 0$,*

$$\widehat{C}(s) = A(\sigma) \frac{1}{s^{(2-\sigma)/\alpha+1}} \sum_{k \geq 0} \frac{\lambda^{-\alpha k(k+1)/2}}{(\lambda^{-\alpha}; \lambda^{-\alpha})_k} s^k + \sum_{k \geq 0} \left(\frac{-\lambda^{-\alpha}}{1 - \lambda^{2-\sigma+\alpha}} \right)^k s^k,$$

for some constant $A(\sigma)$.

Proof. We use the results of [1]. The solution \widehat{C} to (51) is written in a generic way as

$$\widehat{C}(s) = a\phi(s)P(s) + Q(s),$$

for a constant a , where $\phi(qs) = \phi(s)$, and P and Q are solutions to the homogeneous and inhomogeneous equations

$$\begin{aligned} a_0(qs)P(qs) + a_1(s)P(s) &= 0, \\ a_0(qs)Q(qs) + a_1(s)Q(s) &= 1 \end{aligned}$$

with

$$a_0(s) = 1 + s \text{ and } a_1(s) = -\lambda^{2-\sigma+\alpha}.$$

Since P and Q are given by power series, and \widehat{C} is analytic, ϕ is analytic and it is easily checked that ϕ is necessarily constant.

Define $\theta := (2 - \sigma - \alpha)/\alpha$, so that

$$a_0(0)\lambda^{-\alpha\theta} = \lambda^{2-\sigma-\alpha} = -a_1(0).$$

Hence,

$$q^\theta = \lambda^{2-\sigma-\alpha}. \tag{52}$$

A solution of the homogeneous equation is given by

$$P(s) = s^\theta(p_0 + p_1s + p_2s^2 + \dots)$$

with $p_0 := 1$ (this constant is arbitrary) and

$$p_k := \frac{q^{k(k+1)/2}}{(q; q)_k}. \tag{53}$$

Due to (51), this choice of p_0 is arbitrary. Identifying the coefficients of $s^{k+\theta}$, we get

$$(p_k + p_{k-1})q^{\theta+k} = \lambda^{2-\sigma+\alpha}p_k.$$

Owing to (52), this leads to

$$p_k = \frac{q^k}{1 - q^k}p_{k-1}$$

and then to (53).

Assume $a_1 \neq -1$, that is $\sigma \neq 2 + \alpha$. A solution of the inhomogeneous equation is given by

$$Q(s) = q_0 + q_1s + q_2s^2 + \dots$$

with

$$q_0 := \frac{1}{1+a_1} \text{ and } q_{k+1} := \frac{-q}{1+a_1} q_k = \left(\frac{-q}{1-\lambda^{2-\sigma+\alpha}} \right)^k.$$

This is immediate by identifying the coefficients.

The same work can be done where P and Q are power series in $1/s$. In this case, we obtain (49). \square

The next corollary precises the results given in Section 5.2. It gives the asymptotic behavior of $C(t, \sigma; \delta_1, \lambda)$ for large t is an immediate consequence of a Tauberian theorem [37, Theorem 4.3, p. 192 and Corollary 4.4b, p. 194].

Corollary 5. *Assume $\alpha > 0$ and $\sigma < 2 + \alpha$. Then*

$$C(t, \sigma; \delta_1, \lambda) \sim_{t \rightarrow \infty} B(\sigma) t^{\frac{2-\sigma}{\alpha}} \quad (54)$$

with

$$B(\sigma) := \frac{\frac{2-\sigma}{\alpha} + 1}{\Gamma\left(\frac{2-\sigma}{\alpha} + 2\right)} A(\sigma), \quad (55)$$

as $t \rightarrow \infty$.

Since $C(t, 2; \delta_1, \lambda) = 1$, it holds that $A(2) = B(2) = 1$. Since $C(t, 2-\alpha; \delta_1, \lambda) = 1+t$, we have also that $B(2-\alpha) = 1$ and $A(2-\alpha) = 1/2$.

Remark 5. Since $\sigma \mapsto C(t, \sigma; \delta_1, \lambda)$ is analytic for any $t \geq 0$, it follows easily that \widehat{C} is also analytic in σ and then A and B are analytic. In particular, A and B are bounded on any finite interval.

An alternative form for $C(t, \sigma; \delta_1, \lambda)$ is given by [39, Theorem 1.2].

Proposition 14. *Assume that $\alpha > 0$ and $\sigma > 2$. Then there exists a constant $B(\sigma)$ such that*

$$C(t, \sigma; \delta_1, \lambda) \sim_{t \rightarrow \infty} B(\sigma) t^{\frac{2-\sigma}{\alpha}}.$$

Proof. It holds that

$$\begin{aligned} C(t, \sigma; \delta_1, \lambda) &= \frac{1}{\alpha \ln \lambda} \frac{(\lambda^{\sigma-2}; \lambda^{-\alpha})_{\infty}}{(\lambda^{-\alpha}; \lambda^{-\alpha})_{\infty}} \frac{1}{t^{(\sigma-2)/\alpha}} \\ &\quad \times \sum_{k \geq 0} \frac{\lambda^{-\alpha k(k+1)/2}}{(\lambda^{-\alpha}; \lambda^{-\alpha})_k} \psi \left(\frac{\sigma-2}{\alpha} + k, \frac{\pi^2}{\alpha \ln(\lambda)}, -\frac{\ln t}{\alpha \ln \lambda} \right) \frac{(-1)^k}{t^k} \end{aligned} \quad (56)$$

with

$$\psi(u, v, w) := \sum_{k \geq 0} \Gamma(u + 2i\pi vk) \exp(2ik\pi w).$$

This concludes the proof. \square

Proposition 15. For $\sigma \neq 2$,

$$B(\sigma) = \alpha \frac{\lambda^{2-\sigma} - 1}{2 - \sigma} B(\sigma + \alpha). \quad (57)$$

Proof. Eq. (57) follows from the L'Hospital rule applied to $t \mapsto C(t, \sigma; \delta_1, \lambda)$ and $t \mapsto t^{(2-\sigma)/\alpha}$, since $\partial_t C(t, \sigma) = (\lambda^{2-\sigma} - 1)C(t, \sigma + \alpha)$. \square

The relation (57) is not sufficient to reconstruct the function B . Actually, let P be any α -periodic function, then $\sigma \mapsto B(\sigma)P(\sigma)$ also solves (57).

Lemma 25. For any $\sigma > 2$,

$$B(\sigma + k\alpha) \sim_{k \rightarrow \infty} \frac{\Gamma(k)}{(\lambda^{2-\sigma}; \lambda^{-\alpha})_k} k^{\frac{\sigma-2}{\alpha}} \frac{B(\sigma)}{\Gamma(\frac{\sigma-2}{\alpha})}. \quad (58)$$

In particular, $\lim_{\sigma \rightarrow \infty} B(\sigma) = +\infty$.

Corollary 6. When $\alpha > 1/2$, there exists a unique measure μ such that

$$B(\sigma) = \int_0^{+\infty} x^{\sigma-1} \mu(dx).$$

The support of this measure is not bounded.

Proof. If $\alpha > 1/2$, the growth of B is such that the moments problem has a unique solution (see e.g., [32, Theorem 2]).

With (57), the value of $(B(\sigma))_{\sigma > s}$ can be computed from the knowledge of $(B(\sigma))_{\sigma \in [s-\alpha, s]}$. As seen in Remark 5, the later function is bounded.

For any integer m , there exists an integer $k = \lfloor (m - \sigma)/\alpha \rfloor$ and $\sigma \in [2 - \alpha, 2]$ such that $\sigma + k\alpha = m$. By (58) and since the q -Pochhammer symbol $(\lambda^{2-\sigma}; \lambda^{-\alpha})_k$ converges as $k \rightarrow \infty$,

$$\log \frac{1}{m} B(m)^{1/2m} \sim \frac{1}{2m} \log \Gamma \left(\left\lfloor \frac{m - \sigma}{\alpha} \right\rfloor \right) - \log m.$$

With the Stirling formula, when $\alpha \neq 1/2$,

$$\log \frac{1}{m} B(m)^{1/2m} \sim \left(\frac{1}{2\alpha} - 1 \right) \log m.$$

If $\alpha = 1/2$, $\log \frac{1}{m} B(m)^{1/2m} \sim -1$. If $\alpha > 1/2$, this proves that

$$\limsup_{m \rightarrow \infty} m^{-1} B(m)^{1/2m} < +\infty,$$

which is sufficient to ensure that $B(m)$ is the moment of a measure μ which is determined by its moments.

If the self-similar measure μ has a support included in $[0, K]$, then $\mathcal{M}[\mu](\sigma) \leq CK^{\sigma-2}$. From (58), the growth of B is faster so that the support cannot be contained in a compact. \square

Notation 12 (Rising factorial or Pochhammer symbol). We set

$$(x)_n := \prod_{j=0}^{n-1} (x + j),$$

that is $(x)_n$ is the *rising factorial*, or *Pochhammer symbol*. It satisfies

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} = \frac{\Gamma(n)}{\text{Beta}(x, n)},$$

where Beta is the *Beta function*.

Proof. When $\sigma \neq 2$, we deduce from (57) that

$$B(\sigma) = B(\sigma + k\alpha) \left(\frac{\alpha}{\sigma - 2} \right)^k (\lambda^{2-\sigma}; \lambda^{-\alpha})_k \left(\prod_{j=0}^{k-1} \left(1 + j \frac{\alpha}{\sigma - 2} \right) \right)^{-1}.$$

We set $q := \frac{\sigma-2}{\alpha}$. With the above notations,

$$B(\sigma) = B(\sigma + k\alpha) \frac{(\lambda^{2-\sigma}; \lambda^{-\alpha})_k}{(q)_k}.$$

If $\sigma > 2$, $((\lambda^{2-\sigma}; \lambda^{-\alpha})_k)_{k \geq 0}$ has a limit $(\lambda^{2-\sigma}; \lambda^{-\alpha})_\infty$ and never vanishes. Therefore,

$$B(\sigma + k\alpha) = \frac{(q)_k}{(\lambda^{2-\sigma}; \lambda^{-\alpha})_k} B(\sigma) = \frac{\Gamma(k)}{\text{Beta}(q, k)} \times \frac{B(\sigma)}{(\lambda^{2-\sigma}; \lambda^{-\alpha})_k}.$$

Using the asymptotic $\text{Beta}(q, k) \sim_{k \rightarrow \infty} \Gamma(q)k^{-q}$, we obtain (58). \square

Corollary 7. *Using the scaling property (Lemma 20), it holds that*

$$C(t, \sigma; \delta_a, \lambda) \sim_{t \rightarrow \infty} aB(\sigma)t^{\frac{2-\sigma}{\alpha}}.$$

References

- [1] C. R. Adams. “On the Linear Ordinary q -Difference Equation”. In: *The Annals of Mathematics* 30.1/4 (1928), p. 195. DOI: 10.2307/1968274.
- [2] W. Arendt, A. Grabosch, G. Greiner, U. Groh, H. P. Lotz, U. Moustakas, R. Nagel, F. Neubrander, and U. Schlotterbeck. *One-parameter semigroups of positive operators*. Vol. 1184. Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1986, pp. x+460. DOI: 10.1007/BFb0074922.
- [3] W. Arendt and A. Rhandi. “Perturbation of positive semigroups”. In: *Arch. Math. (Basel)* 56.2 (1991), pp. 107–119. DOI: 10.1007/BF01200341.
- [4] J. Banasiak and L. Arlotti. *Perturbations of positive semigroups with applications*. Springer Monographs in Mathematics. Springer-Verlag London, Ltd., London, 2006.
- [5] J. Banasiak and W. Lamb. “Analytic fragmentation semigroups and continuous coagulation-fragmentation equations with unbounded rates”. In: *J. Math. Anal. Appl.* 391.1 (2012), pp. 312–322. DOI: 10.1016/j.jmaa.2012.02.002.
- [6] J. Banasiak and W. Lamb. “On the existence of moments of solutions to fragmentation equations”. In: *J. Math. Anal. Appl.* 413.2 (2014), pp. 1017–1029. DOI: 10.1016/j.jmaa.2013.12.040.
- [7] J. Bertoin. “Different aspects of a random fragmentation model”. In: *Stochastic Processes and their Applications* 116.3 (Mar. 2006), 345–369. DOI: 10.1016/j.spa.2005.11.001.
- [8] J. Bertoin and A. Gnedin. “Asymptotic Laws for Nonconservative Self-similar Fragmentations”. In: *Electronic Journal of Probability* 9 (Jan. 2004). DOI: 10.1214/ejp.v9-215.
- [9] W. Biedrzycka and M. Tyran-Kamińska. “Self-similar solutions of fragmentation equations revisited”. In: *Discrete & Continuous Dynamical Systems - B* 23.1 (2018), 13–27. DOI: 10.3934/dcdsb.2018002.
- [10] V. I. Bogachev. *Measure theory. Vol. I, II*. Springer-Verlag, Berlin, 2007, Vol. I: xviii+500 pp., Vol. II: xiv+575. DOI: 10.1007/978-3-540-34514-5.
- [11] G. Breschi and M. A. Fontelos. “A note on the self-similar solutions to the spontaneous fragmentation equation”. In: *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences* 473.2201 (May 2017), 201607401–. DOI: 10.1098/rspa.2016.0740.
- [12] B. van Brunt and G. C. Wake. “A Mellin transform solution to a second-order pantograph equation with linear dispersion arising in a cell growth model”. In: *European Journal of Applied Mathematics* 22.2 (Jan. 2011), 151–168. DOI: 10.1017/s0956792510000367.

- [13] B. van Brunt, A. A. Zaidi, and T. Lynch. “Cell Division And The Pantograph Equation”. In: *ESAIM: Proceedings and Surveys* 62 (2018). Ed. by M. Dauhoo, L. Dumas, P. Gabriel, and P. Lafitte, 1587. DOI: 10.1051/proc/201862158.
- [14] J. A. Cañizo, P. Gabriel, and H. Yoldaş. “Spectral gap for the growth-fragmentation equation via Harris’s theorem”. In: *SIAM J. Math. Anal.* 53.5 (2021), pp. 5185–5214. DOI: 10.1137/20M1338654.
- [15] M. J. Cáceres, J. A. Cañizo, and S. Mischler. “Rate of convergence to self-similarity for the fragmentation equation in L^1 spaces”. In: *Commun. Appl. Ind. Math.* 1.2 (2010), pp. 299–308.
- [16] Z. Cheng and S. Redner. “Kinetics of fragmentation”. In: *J. Phys. A: Math. Gen.* 23.7 (Apr. 1990), pp. 1233–1258. DOI: 10.1088/0305-4470/23/7/028. (Visited on 01/12/2023).
- [17] P. Chernoff and J. Marsden. “On continuity and smoothness of group actions”. In: *Bull. Amer. Math. Soc.* 76 (1970), pp. 1044–1049. DOI: 10.1090/S0002-9904-1970-12552-6.
- [18] M. J. Cáceres, J. A. Cañizo, and S. Mischler. “Rate of convergence to an asymptotic profile for the self-similar fragmentation and growth-fragmentation equations”. In: *Journal de Mathématiques Pures et Appliquées* 96.4 (Oct. 2011), 334–362. DOI: 10.1016/j.matpur.2011.01.003.
- [19] M. Doumic, M. Escobedo, and M. Tournus. *An inverse problem: recovering the fragmentation kernel from the short-time behaviour of the fragmentation equation*. 2021. eprint: arXiv:2112.10423.
- [20] M. Doumic, M. Escobedo, and M. Tournus. “Estimating the division rate and kernel in the fragmentation equation”. en. In: *Annales de l’Institut Henri Poincaré (C) Non Linear Analysis* 35.7 (2018), p. 1847. DOI: 10.1016/j.anihpc.2018.03.004.
- [21] M. Doumic Jauffret and P. Gabriel. “Eigenelements of a general aggregation-fragmentation model”. In: *Mathematical Models and Methods in Applied Sciences* 20.05 (May 2010), pp. 757–783. DOI: 10.1142/s021820251000443x.
- [22] K.-J. Engel and R. Nagel. *One-parameter semigroups for linear evolution equations*. Vol. 194. Graduate Texts in Mathematics. With contributions by S. Brendle, M. Campiti, T. Hahn, G. Metafune, G. Nickel, D. Pallara, C. Perazzoli, A. Rhandi, S. Romanelli and R. Schnaubelt. Springer-Verlag, New York, 2000.
- [23] M. Escobedo, S. Mischler, and M. Rodriguez Ricard. “On self-similarity and stationary problem for fragmentation and coagulation models”. In: *Annales de l’Institut Henri Poincaré C, Analyse non linéaire* 22.1 (Feb. 2005), 99–125. DOI: 10.1016/j.anihpc.2004.06.001.

- [24] W. Feller. *An introduction to probability theory and its applications. Vol. II.* Second. John Wiley & Sons, Inc., New York-London-Sydney, 1971, pp. xxiv+669.
- [25] L. Fox, D. F. Mayers, J. R. Ockendon, and A. B. Tayler. “On a Functional Differential Equation”. In: *IMA Journal of Applied Mathematics* 8.3 (1971), 271–307. DOI: 10.1093/imamat/8.3.271.
- [26] P. Gabriel and F. Salvarani. “Exponential relaxation to self-similarity for the superquadratic fragmentation equation”. In: *Applied Mathematics Letters* 27 (Jan. 2014), 74–78. DOI: 10.1016/j.aml.2013.08.001.
- [27] P. Gabriel. “Long-time asymptotics for nonlinear growth-fragmentation equations”. In: *Communications in Mathematical Sciences* 10.3 (2012), 787–820. DOI: 10.4310/cms.2012.v10.n3.a4.
- [28] B HAAS. “Regularity of formation of dust in self-similar fragmentations”. In: *Annales de l’Institut Henri Poincaré (B) Probability and Statistics* 40.4 (Aug. 2004), 411–438. DOI: 10.1016/j.anihpb.2003.11.002.
- [29] A. J. Hall and G. C. Wake. “A functional-differential equation arising in modelling of cell growth”. In: *J. Austral. Math. Soc. Ser. B* 30.4 (1989), pp. 424–435. DOI: 10.1017/S0334270000006366.
- [30] T. Kato and J. B. McLeod. “The functional-differential equation $y'(x) = ay(\lambda x) + by(x)$ ”. In: *Bull. Amer. Math. Soc.* 77 (1971), pp. 891–937. DOI: 10.1090/S0002-9904-1971-12805-7.
- [31] J. Konvalina. “A Unified Interpretation of the Binomial Coefficients, the Stirling Numbers, and the Gaussian Coefficients”. In: *The American Mathematical Monthly* 107.10 (Dec. 2000), 901–910. DOI: 10.1080/00029890.2000.12005290.
- [32] G. D. Lin. “Recent developments on the moment problem”. In: *Journal of Statistical Distributions and Applications* 4.1 (July 2017). DOI: 10.1186/s40488-017-0059-2.
- [33] Z. Melzak. “A scalar transport equation”. In: *Trans. Amer. Math. Soc.* 85 (1957), pp. 547–560.
- [34] S. Mischler and J. Scher. “Spectral analysis of semigroups and growth-fragmentation equations”. In: *Ann. Inst. H. Poincaré C Anal. Non Linéaire* 33.3 (2016), pp. 849–898. DOI: 10.1016/j.anihpc.2015.01.007.
- [35] B. E. Sagan. *Combinatorics: the art of counting.* Vol. 210. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, [2020] ©2020, pp. xix+304. DOI: 10.1090/gsm/210.
- [36] L. Takacs. “A Moment Convergence Theorem”. In: *The American Mathematical Monthly* 98.8 (Oct. 1991), p. 742. DOI: 10.2307/2324428.

- [37] D. V. Widder. *The Laplace Transform*. Vol. vol. 6. Princeton Mathematical Series. Princeton University Press, Princeton, NJ, 1941, pp. x+406.
- [38] M. Wiśniewolski. “On the probabilistic representations of solutions of pantograph equations and triangle coefficients”. In: *Journal of Differential Equations* 379 (Jan. 2024), 600–625. DOI: 10.1016/j.jde.2023.10.041.
- [39] C. Zhang. “Analytical study of the pantograph equation using Jacobi theta functions”. In: *Journal of Approximation Theory* 296 (Dec. 2023), p. 105974. DOI: 10.1016/j.jat.2023.105974.
- [40] R. M. Ziff. “New solutions to the fragmentation equation”. In: *Journal of Physics A: Mathematical and General* 24.12 (June 1991), 2821–2828. DOI: 10.1088/0305-4470/24/12/020.