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# A short perspective on a posteriori error control and adaptive discretizations

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## Abstract

Error control by means of a posteriori error estimators or indicators and adaptive discretizations, such as adaptive mesh refinement, have emerged in the late seventies. Since then, numerous theoretical developments and improvements have been made, as well as the first attempts to introduce them into real-life industrial applications. The present introductory chapter provides an overview of the subject, highlights some of the achievements to date and discusses possible perspectives.

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**Keywords :** error control ; a posteriori error estimators ; adaptive discretizations ; mesh refinement ; applications.

## 1 Introduction

The numerical solution of partial differential equations is a crucial aspect of computational science and engineering, finding applications in diverse fields of

physics and engineering. However, achieving accurate and efficient solutions to partial differential equations is often challenging and a posteriori error estimators are crucial tools that evaluate the accuracy and enhance computational efficiency of numerical methods. Unlike a priori error estimators, which are not computable, these estimators assess solutions after computation, providing a pragmatic evaluation of accuracy.

A posteriori estimators work as evaluative tools, systematically analysing error distributions to offer insight into numerical approximation fidelity. This shift from anticipatory to retrospective assessment enables practitioners to iteratively refine solutions to align them more closely with the intricate details of the underlying mathematical models.

This article explores the conceptual landscape of a posteriori error estimators, examining their theoretical foundations, applications, and consequential impact on partial differential equations problem-solving. The discussion invites an exploration of numerical accuracy, navigating the terrain where mathematical abstraction converges with computational reality.

## 2 A few basic notions

Before entering the core of the topic, let us describe a few useful basic notions, in a very general setting and without entering into details. For reviews about a posteriori error estimation, see, e.g., Eriksson and Johnson (1985); Carstensen and Merdon (2010); Nochetto and Veiser (2012); Chamoin and Legoll (2023).

### 2.1 Numerical approximation of a mathematical model

For several decades, computers have been used in routine calculations to provide an approximate solution to the sophisticated mathematical models in current engineering practice, which are, for example, (linear or non-linear) partial differential equations supplemented by (linear or non-linear) boundary conditions and by initial conditions, and possibly coupled to other equations, and now, to data (even in real-time).

Let us call  $u$  the exact solution of this mathematical model, which belongs to a set  $S$  of admissible solutions. There are now many techniques for transforming the original equation through a sophisticated pipeline, so that, in the end, a concrete approximation to the solution  $u$  is provided by a computer in the form of a collection of numbers  $(U_1, \dots, U_N) \in \mathbb{R}^N$  which can be used to recover a function  $u_{\text{computer}}(U_1, \dots, U_N)$  (hopefully) close enough to  $u$ .

Figure 1 and Figure 2 below provide an overview of the process of mathematical modelling.

### 2.2 Galerkin methods and the discretization error

To be illustrative, we consider (Petrov-)Galerkin methods, that include a broad class of numerical approximation techniques based on the weak form of the

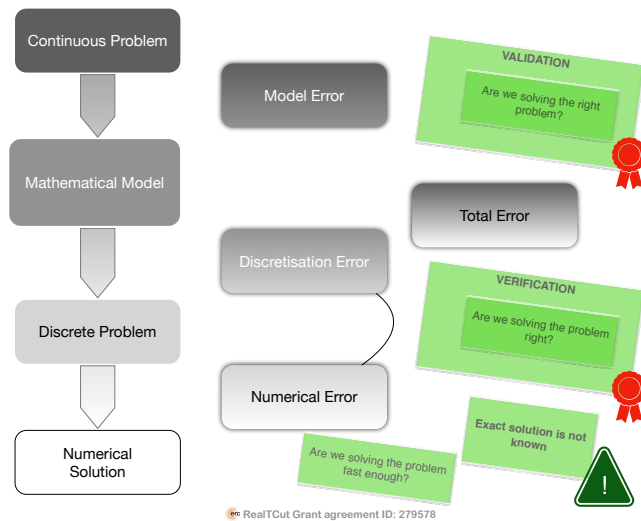


Figure 1: Mathematical modelling and sources of error. This figure showcases the steps involved in mathematical modelling, which is a process used to represent real-world phenomena using mathematical equations. The diagram highlights the different stages of mathematical modelling, including problem formulation, model development, parameter estimation, model validation, and prediction. Additionally, the figure identifies potential sources of error that may impact the accuracy and reliability of the mathematical model. These sources of error can arise from various factors, such as measurement uncertainties, assumptions made during model development, limitations in data availability, simplifications in model assumptions, and uncertainties in parameter estimation. The figure serves as a visual representation of the complexities and challenges associated with mathematical modelling, emphasising the need for careful consideration of potential sources of errors to ensure the robustness and validity of the model's results. It underscores the importance of thorough validation and verification processes to enhance the accuracy and reliability of mathematical models, which are crucial for decision-making, prediction, and understanding complex systems in various fields of science, engineering, and beyond.

mathematical model. This includes particularly the case of well-known Finite Element Methods (Lagrange, mixed, etc), see, e.g., Brenner and Scott (2008); Ciarlet (2002); Ern and Guermond (2021); Quarteroni and Valli (1994); Szabó and Babuška (2021) but also many other new methods, among which:

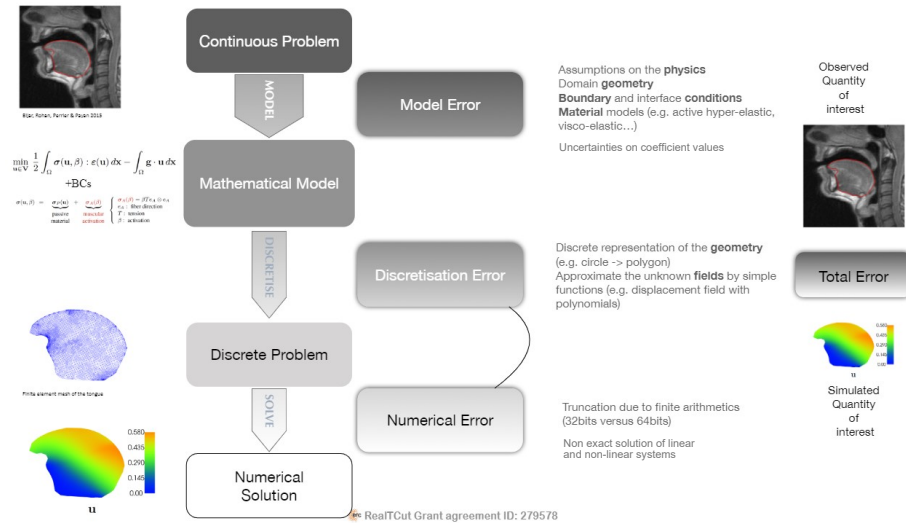


Figure 2: This figure showcases the process of mathematical modelling along with the identification and estimation of potential sources of error. The diagram illustrates the steps involved in mathematical modelling, including problem formulation, model development, parameter estimation, model validation, and prediction. Furthermore, the figure highlights the importance of error estimation in the modelling process. It identifies potential sources of error, such as measurement uncertainties, assumptions made during model development, limitations in data availability, simplifications in model assumptions, and uncertainties in parameter estimation. The figure emphasises the need to account for and quantify these sources of error in order to assess the reliability and accuracy of the mathematical model. The figure serves as a visual representation of the comprehensive approach to mathematical modelling, which includes not only model development but also thorough error estimation to enhance the robustness and validity of the model’s results. It underscores the significance of error estimation in improving the quality of mathematical models and their applicability in various fields of science, engineering, and beyond.

- discontinuous Galerkin (DG) methods, see for instance the pioneering works Arnold (1982); Lesaint and Raviart (1974) and the recent monograph Di Pietro and Ern (2012);
- recent polytopal methods such as Conforming Polygonal Finite Elements, see, e.g., Sukumar and Tabarraei (2004); Hybrid Discontinuous Galerkin

(HDG), see, e.g., Cockburn et al. (2016), Hybrid High Order (HHO) methods, see, e.g., Cicuttin et al. (2021); Cockburn et al. (2016); Di Pietro and Droniou (2020); Lemaire (2021), the Weak Galerkin Method, see, e.g., Dong and Ern (2022), the Virtual Element Method (VEM), see, e.g. Beirão da Veiga et al. (2013); Lemaire (2021); the Smooth Finite Element Method (SFEM), see, e.g., Liu et al. (2007); Nguyen-Xuan et al. (2008b); modified discontinuous Galerkin with static condensation, see, e.g., Lozinski (2019);

- Discontinuous Petrov Galerkin (DPG) methods, see, e.g., Demkowicz and Gopalakrishnan (2010); Demkowicz et al. (2012);
- IsoGeometric Analysis (IGA) and variants, see, e.g., Atroshchenko et al. (2018); Cottrell et al. (2009); Nguyen et al. (2015), motivated by the link between computer aided design and numerical simulation;
- unfitted finite elements or geometrically nonconforming finite elements, where the mesh boundary and the domain boundary do not match, as it occurs in fictitious domain methods, the eXtended Finite Element Method (XFEM) or the cut Finite Element Method (cutFEM): see, e.g., Bordas and Menk (2023); Burman et al. (2015); Duprez and Lozinski (2020); Glowinski et al. (1994); Haslinger and Renard (2009); Moës et al. (2006); Nguyen et al. (2008); Peskin (2002);
- reduced basis techniques, such as Reduced Order modelling (ROM) or Proper Orthogonal Decomposition (POD), see, e.g., Grepl et al. (2007); Hesthaven et al. (2016); Kerfriden et al. (2011);
- spectral methods, see, e.g., Bernardi and Maday (1997) or Canuto et al. (2006);
- wavelet-based discretization, see, e.g., Bertoluzza et al. (1994); Bertoluzza (1995); Cohen and Masson (2000); Cohen et al. (2001); Monasse and Perrier (1998).

Of course, the above list is far from exhaustive and the discussion can be extended, with appropriate modifications, to other classes of approximation techniques such as finite differences, finite volumes, collocation methods, etc, which are not based on a variational paradigm.

So, for a Petrov-Galerkin method, let us suppose the exact solution  $u$  satisfies (for instance) the weak problem

$$u \in V : a(u; v) = L(v) \quad \forall v \in W, \tag{1}$$

where  $V, W$  are some function spaces and  $L$  is a linear form. The form

$$a : V \times W \rightarrow \mathbb{R}$$

is possibly nonlinear in the first variable.

**Remark 1.** *The above formalism (1) does not encompass nonlinear nonsmooth problems related to variational inequalities for instance, such as contact, friction, plasticity. However this is not critical for the present discussion, and to see how extensions for variational inequalities can be carried out, see, e.g., Han (2005) or some contributions in these volumes, among which Bartels and Kaltenbach (2024); Gustafsson (2024); Repin (2024).*

A discrete Petrov-Galerkin method consists in approximating Problem (1) by a simpler problem in finite dimensional vector spaces:

$$u_N \in V_N : a_G(u_N; v_N) = L_G(v_N) \quad \forall v_N \in W_N, \quad (2)$$

where  $V_N$  and  $W_N$  are *finite dimensional spaces* of dimension  $N$ , and  $a_G$ , resp.  $L_G$ , is a form that mimics  $a$ , resp.  $L$  (in the simplest situations, we can take  $a_G = a$ , resp.  $L_G = L$ ).

**Remark 2.** *For many methods, the trial space  $V_N$  and the test space  $W_N$  are the same ( $V_N = W_N$ ) and this corresponds to standard Galerkin (or standard Ritz-Galerkin) methods. The case where the spaces differ is often associated with the terminology of Petrov-Galerkin methods, or non-standard (Ritz-)Galerkin methods, see Ern and Guermond (2004). For a historical perspective about this class of methods, see Gander and Wanner (2012) and references therein, which emphasise W. Ritz's outstanding contributions related to the approximation of the spectral biharmonic problem for Chladni figures.*

As a result,  $u_N$  can be represented in a basis of  $N$  functions, and a computer can provide the  $N$  values of the components when Problem (2) is solved.

The challenge now is to design the spaces  $V_N$  and  $W_N$  carefully enough to calculate a *discrete solution*  $u_N$  that represents the exact solution  $u$  as accurately as possible. To measure the difference between  $u$  and  $u_N$ , the concept of discretisation error  $\epsilon_N$  is introduced. A natural and simple definition can be

$$\epsilon_N(u, u_N) := \|u - u_N\|_V, \quad (3)$$

where  $\|\cdot\|_V$  is the natural norm associated with the Banach or Hilbert structure of  $V$  (Sobolev norm or energy norm). Of course, other possibilities exist motivated by practical applications, see paragraph 2.5 below.

Since  $N$  is linked to the available computing resources, a practitioner may wish to achieve the lowest possible value for the discretisation error  $\epsilon_N$ , while keeping  $N$  as small as possible. For example, and ideally, one would like  $\epsilon_N$  to be of the order of machine precision ( $\epsilon_N \simeq 10^{-16}$  for double precision arithmetic), and  $N$  to be such that the solution  $u_N$  is obtained in real time (a few milliseconds for example). Today, even with the enormous progress made in computing power, this objective remains a challenge, particularly for industrial applications where the geometry and mathematical model can be very complex. In general, a compromise has to be found between acceptable accuracy and acceptable use of computing resources, and this compromise is highly dependent on the

context and the targeted applications. To achieve acceptable accuracy, the main problem one encounters is that  $\epsilon_N$  is unknown, because the exact solution  $u$  itself is unknown.

Another issue is that the solution  $u_N$  to Problem (2) is not exactly the solution  $u_{\text{computer}}(U_1, \dots, U_N)$  delivered by a computer. Indeed, the code written to provide  $u_{\text{computer}}(U_1, \dots, U_N)$  also contains other approximations, that are mostly due to: numerical integration, iterative solvers for nonlinear and/or linear systems, and finite precision arithmetic, see, e.g., Ciarlet (2002); Ern and Guermond (2004) in the context of finite element methods. This fact introduces another layer of errors, *numerical errors*, that can be quantified as, for instance:

$$\|u_{\text{computer}}(U_1, \dots, U_N) - u_N\|_D$$

where  $\|\cdot\|_D$  is a convenient norm to measure this in finite dimensions (it does not have to be necessarily the same norm as for the discretization error).

In general, it is assumed that the numerical errors are of very small magnitude in comparison to discretization errors. However, in some situations they need to be taken into account: for instance a gradient conjugate solver can be stopped after a few iterations to save computation time. See for instance Ern and Vohralík (2013) for a technique that can resolve this issue.

Finally we can get back to Figure 1 and Figure 2 in which the discretization error and numerical error are depicted, and complemented with the model error that encompasses all the discrepancies between the actual physical system that needs to be modelled and the idealised mathematical model (this important topic is far beyond the scope of this short overview).

### 2.3 Error estimators and error control

Since the late 1970s, pioneering work within the finite element community has shown how it is possible to compute a quantity  $\eta_N (= \eta_N(u_N))$  that depends only of the discrete solution  $u_N$  and that is equivalent to the discretization error:

$$\eta_N(u_N) \simeq \epsilon_N(u, u_N). \quad (4)$$

Classical references are Babuška and Rheinboldt (1978), Ladeveze and Leguillon (1983), Eriksson and Johnson (1985), Bank and Weiser (1985), and the books Ainsworth and Oden (2000) and Verfürth (2013). For simple mathematical models, we can take

$$\eta_N(u_N) = \|L_G(\cdot) - a_G(u_N; \cdot)\|_{\star} \quad (5)$$

where  $L_G(\cdot) - a_G(u_N; \cdot)$  is the residual associated with Problem (2) and  $\|\cdot\|_{\star}$  is a dual norm. A problem with (5) comes from the norm  $\|\cdot\|_{\star}$ , that is not computable. A highly desirable property of an estimator is not only its computability from  $u_N$ , but it should allow for an implementation, such that the necessary time for computation is negligible in comparison to the time needed to solve Problem (2).



An important mathematical property to have a trustworthy estimator is its *reliability*: if it can not provide the exact discretization error, it should at least provides an *upper bound* of the discretization error, in the sense that, there exists  $C > 0$  independent of  $N$  and  $u_N$ , such that

$$\epsilon_N(u, u_N) \leq C \eta_N, \quad (6)$$

for all  $u \in V$  and  $u_N \in V_N$ . See for instance Verfürth (1999); Veese and Verfürth (2009, 2012) for a discussion and details in the case of the residual error estimate. If the constant  $C$  in (6) is equal to 1, we have an *exact guaranteed upper bound*:

$$\epsilon_N(u, u_N) \leq \eta_N, \quad (7)$$

which is of high practical interest since it ensures the discrete solution  $u_N$  is approximated with a degree of accuracy that is known. See for instance Neittaanmäki and Repin (2004); Vohralík (2007); Braess and Schöberl (2008) and a discussion in Verfürth (2009) about these “constant-free” estimates. This paves the way for *certified numerical methods* in which one can ensure that the discretization error is below a known threshold. Indeed, note that if (6) is satisfied only, the estimator  $\eta_N$  can underestimate the error for instance.

Last but not least, it is usual to define the effectivity index of an estimator as

$$\text{eff} := \frac{\eta_N(u_N)}{\epsilon_N(u, u_N)}. \quad (8)$$

The above definition is motivated by (4) and is meant to describe the accuracy of an estimator. Hopefully, it should not vary too much with  $N$  and  $u$ , and should be close to 1. First, if (6) holds, this implies that

$$\text{eff} \geq \frac{1}{C}. \quad (9)$$

Clearly, the boundedness of the effectivity index is equivalent to the lower bound

$$\eta_N \leq c \epsilon_N(u, u_N) \quad (10)$$

for some  $c > 0$  independent of  $u$  and  $N$ ; we then have  $\text{eff} \leq c$ . Lower bounds are often called effectivity and have been an important research topic, see Verfürth (2013). In the context of the finite element method, they generally even have a local form, see (17) .

Asymptotic exactness of an estimator refers to

$$\lim_{N \rightarrow \infty} \text{eff}_N = 1. \quad (11)$$

This property depends on  $u$ , the method to compute  $u_N$  and the sequence of spaces  $V_N$ . In special cases, estimators such as the Bank and Weiser type, are known to yield (11), see Babuška et al. (1992); Bank and Weiser (1985); Durán et al. (1991) for details.

## 2.4 Error estimators and mesh refinement

In addition to assess the accuracy of a simulation, error estimators are used to iteratively improve the numerical approximation. The typical form of the loop is

$$\mathbf{Solve} \rightarrow \mathbf{Estimate} \rightarrow \mathbf{Mark} \rightarrow \mathbf{Adapt} \rightarrow \dots$$

where **Solve** refers to solution of the discrete problem for given approximation spaces, see, e.g., Problem (2), and the estimator is then used to adapt the approximation spaces. The algorithm produces a sequence of meshes  $\mathcal{K}_\ell$  and associated spaces  $V_\ell$ , solutions  $u_\ell$  and estimators  $\eta_\ell$ . The global upper bound justifies the use of the estimator  $\eta_\ell$  as a stopping criterion.

In the finite element method, the approximation spaces are built from meshes and the **Adapt** step usually consists in local mesh-refinement. Since the spatial domain associated with the mathematical problem is subdivided into cells  $K$ , say triangles in two dimensions, the solution  $u_N$  is obtained by collecting over all the local cell contributions  $u_K$ , representing the approximation of the solution  $u$  on the cell  $K$ :

$$u_K = u_\ell|_K. \tag{12}$$

In order to be useful for local mesh refinement, the estimator  $\eta_\ell$  is supposed to have a similar local structure, i.e., it is the sum of local contributions  $\eta_K$  corresponding to  $K$ :

$$\eta_\ell = \left( \sum_K \eta_K^2 \right)^{\frac{1}{2}}. \tag{13}$$

The information encoded in the estimator map  $\eta_K$  is used in different ways to adapt or refine the mesh. The main idea is to split locally a cell  $K$ , if the local error indicator  $\eta_K$  is considered too large, see for instance Pelle et al. (1996) or Dörfler (1996). In a similar way one could use local coarsening, i.e., neighbouring cells are glued together to define a bigger cell where the local error indicator is small enough. The algorithm used to select cells for refinement (or coarsening) is called **Mark** and the step of mesh modification **Adapt**. The first natural question is about convergence of the adaptive algorithm (which does not follow from a priori error analysis). Under the condition of convergence, the second question is about the speed of convergence, measured in terms of number of unknowns.

Let us sketch a short argument for convergence. We consider the step from  $\ell$  to  $\ell + 1$ . For the sake of simplicity, we suppose that the boundary value problem (1) corresponds to the minimisation of a quadratic energy functional. Moreover, we suppose that the finite element spaces  $V_\ell$  form a sequence of conforming nested spaces ( $V_\ell \subset V_{\ell+1} \subset V$ ) and that there holds a Galerkin orthogonality  $(e_{\ell+1}, u_{\ell+1} - u_\ell)_V = 0$ , with  $(\cdot, \cdot)_V$  the inner product in  $V$ . Then the errors  $e_\ell := \|u - u_\ell\|_V$  are related by

$$e_{\ell+1}^2 = e_\ell^2 - \|u_{\ell+1} - u_\ell\|_V^2$$

and we get geometrical convergence if for some constant  $\beta > 0$

$$e_\ell \leq \beta \|u_{\ell+1} - u_\ell\|_V. \quad (14)$$

since then  $e_{\ell+1} \leq \rho e_\ell$  with  $\rho = \sqrt{1 - 1/\beta^2}$ .

In order to establish (14), it is convenient to assume the local lower bound

$$\eta_K \leq c \|u_{\ell+1} - u_\ell\|_K \quad (15)$$

for all cells  $K$  from the set of refined cells  $\mathcal{R}_\ell \subset \mathcal{K}_\ell$  and to ensure that, for some  $\theta > 0$ , the estimator linked to the cells in  $\mathcal{R}_\ell$  is at least equal to a proportion  $\theta$  of the total estimator:

$$\sum_{K \in \mathcal{R}_\ell} \eta_K^2 \geq \theta^2 \sum_{K \in \mathcal{K}_\ell} \eta_K^2. \quad (16)$$

Indeed, using (6) and (13), then (16) and finally (15), we get

$$e_\ell^2 \leq C^2 \sum_{K \in \mathcal{K}_\ell} \eta_K^2 \leq \frac{C^2}{\theta^2} \sum_{K \in \mathcal{R}_\ell} \eta_K^2 \leq \frac{C^2 c^2}{\theta^2} \sum_{K \in \mathcal{R}_\ell} \|u_{\ell+1} - u_\ell\|_K^2,$$

which implies (14) with  $\beta = Cc/\theta$ .

The assumed bound (15) looks similar to what is called *local efficiency*: there exists a constant  $c > 0$  independent of  $N$ ,  $u_N$  and  $K$  such that, for every mesh cell  $K$ , there holds:

$$\eta_K \leq c \|u - u_N\|_{\omega_K}, \quad (17)$$

where  $\omega_K$  is a collection of neighbouring cells of  $K$  (patch) and  $\|\cdot\|_{\omega_K}$  is an appropriate norm on  $\omega_K$ , see, e.g., Verfürth (2013). This property strictly translates the fact that if the local discretization error is small, then the local estimator must also be small. However, (17) does not imply (15) (the difficulty is not the presence of  $\omega_K$  instead of  $K$ ), and even (17) might not always hold. It might be thought that if an estimator does not satisfy this local efficiency property, mesh refinement could be misled. Indeed, regions where the discretization error is low could nevertheless be refined, as the local  $\eta_K$  would be free to overestimate the actual discretization error. However, it turns out in practice that an estimator with poor efficiency can still be useful to drive the mesh refinement algorithm (see the discussion in Carstensen and Merdon (2010)), and can even lead to optimal meshes.

Condition (16) states that a minimum percentage of the estimator contributions should give rise to refinement. It is usually called Dörfler marking or bulk chasing. Clearly, (16) allows for a large number of possibilities for selecting  $\mathcal{R}_\ell$  and one naturally tends to refine the cells with the largest contributions. This choice is related to our second question about the speed of convergence. Choosing the cells to be refined as the largest contributors allows to bound the number of additional cells by the difference in estimators. This idea basically leads to an estimate of the number of cells needed to produce a given accuracy, see Stevenson (2007) for the first proof.

The unavailability of lower bounds has led to the development of several more involved techniques, see section 3.

## 2.5 The goal-oriented paradigm

Many practical situations are focused on the approximation of a single quantity  $J$ , such as the drag- and lift-coefficients, average of heat transfer along a part of the boundary, stress intensity factors, a local stress or strain in elasticity, etc. However, we observe that the aforementioned theory heavily relies on the functional analytic setting and the corresponding norms are used to measure the error  $u - u_N$ . Instead one is interested in

$$|J(u) - J(u_N)|$$

and the objective of goal-oriented error estimation is to provide estimators for this quantity. The main idea is to compute an approximation to the solution  $z$  to a *dual* problem that involves  $J$  in its right-hand side. This dual solution then allows to evaluate locally the sensitivity of  $J$  with respect to the discretization error leading to an error estimator in the usual form, see for instance Becker and Rannacher (2001), and also Becker and Rannacher (1996); Giles and Süli (2002); González-Estrada et al. (2014); Han (2005); Prudhomme and Oden (1999); Rognes and Logg (2013).

## 3 Theory of adaptive finite element methods

The mathematical theory of adaptive finite elements started with the work of Ivo Babuška in the 1970s (see Babuška and Rheinboldt (1978)), and found increasing interest with the works of Claes Johnson (see Eriksson and Johnson (1985)), Mark Ainsworth and John Tinsley Oden (see Ainsworth and Oden (1993, 2000)), Pedro Morin, Ricardo Nochetto and Kunibert G. Siebert (see Morin et al. (2000)), and Rüdiger Verfürth (see Verfürth (1999)) in the 1990s. Since then, it has made important theoretical progress, that we summarise below.

### 3.1 Optimality of adaptive algorithms

New insights have been brought to the theory of adaptive algorithms. We first define what is meant by the notion of optimality. We assume a given procedure that, for any mesh  $\mathcal{K}$ , computes an approximation  $u_{\mathcal{K}}$  to the solution  $u$  of a given partial differential equation, usually by a finite element method based on approximation spaces  $V_{\mathcal{K}}$  whose dimensions are denoted  $N(\mathcal{K})$ . We further assume possible to approximate the solution  $u$  by elements of  $V_{\mathcal{K}}$  at a certain convergence rate  $s > 0$ , which mathematically translates into: there exists an absolute constant  $C > 0$  such that

$$\inf_{v_{\mathcal{K}} \in V_{\mathcal{K}}, N(\mathcal{K}) \leq N} \|u - v_{\mathcal{K}}\| \leq CN^{-s} \quad \forall N \in \mathbb{N},$$

in an appropriate norm. Then, an adaptive algorithm constructing a (in principle infinite) sequence of meshes  $(\mathcal{K}_{\ell})_{\ell=1,2,\dots}$  and consequent solutions  $(u_{\ell})_{\ell=1,2,\dots}$

is said to be (quasi-)optimal, if the sequence  $(u_\ell)_{\ell=1,2,\dots}$  has a similar error decay:

$$\|u - u_\ell\| \leq C (N(\mathcal{K}_\ell))^{-s} \quad \forall \ell \in \mathbb{N},$$

(with possibly a different constant  $C$ ). One should mention that such an optimality result is much stronger than the classical *a priori* finite element theory, since no direct assumption on the regularity of the solution is made.

This theory brings the adaptive finite element technology close to the nonlinear approximation theory of multi-resolution wavelet methods, see, e.g., Cohen et al. (2001) and Karaivanov and Petrushev (2003). It has been a very active research field, starting with the works of Dörfler (1996); Morin et al. (2000). The combination with nonlinear approximation theory has been achieved in Binev et al. (2004); Stevenson (2007), which lead to many further results for different methods/equations/algorithms, see for example Cascon et al. (2008); Becker and Mao (2008); Bonito and Nochetto (2010); Ferraz-Leite et al. (2010); Kreuzer and Georgoulis (2018); Buffa et al. (2022). Now it has attained a state of maturity in the context of second-order linear elliptic partial differential equations, certain finite element methods and mesh refinement algorithms. The appearance of an underlying structure of the available results has lead to the Axiom-paper of Carstensen et al. (2014). Despite the large number of articles about application of goal-oriented adaptivity, theoretical results concerning their optimality are still sparse. Mommer and Stevenson (2009) provides the first optimality result, using the product of primal and dual estimators as an upper bound to the functional error; see also Becker et al. (2011); Feischl et al. (2016).

### 3.2 Other recent achievements

Let us mention first that the described theory of adaptive finite elements has been generalised to nonlinear monotone equations by Feischl et al. (2014) and saddle-point systems as the Stokes equations, see Becker and Mao (2011); Bringmann and Carstensen (2017); Feischl (2019, 2022). Also progress has been made in the understanding of parabolic problems, see Ern et al. (2017).

Moreover, the theory has been generalised to control the overall work of the adaptive algorithm, and not just the dimension of the discrete spaces, see, e.g., Haberl et al. (2021); Becker et al. (2023); F evotte et al. (2024). This is particularly important from a practical viewpoint, since the various stopping criteria in the different nested loops can then be harmonised.

## 4 Current engineering practice(s)

Now let us turn to the practical side of numerical methods, when they are used in practical computations in physics, applied sciences and industrial applications. In this context, there have already been some initiatives to incorporate error control based on a posteriori error estimators as well as adaptive discretization techniques, see for instance the special volume Plewa et al. (2005) published

earlier. Nevertheless, the true potential of all these techniques seems to remain largely untapped.

#### 4.1 Accuracy vs. computational cost vs. complexity

For numerical simulation oriented towards real applications, such as in the technological and industrial sector, the challenges are, at least, threefold:

1. Simulation of complex systems, that may encompass, for instance, complex three-dimensional geometries, nonlinear partial differential equations with nonlinear boundary conditions, multi-physics or multi-scale phenomena, complex materials, etc.
2. Simulation with reduced computation time and/or limited computational resources may be of major importance for certain applications, with the possible goal to even provide predictions in real-time.
3. Certified numerical simulation, with trustworthy and accurate predictions.

It is clear that the third point is related to error control and adaptive discretizations. In fact, the notions of verification and validation of finite element simulations have always been of importance within the numerical simulation community, see, e.g., Babuska and Oden (2004); Babuška and Oden (2006); Babuška et al. (2007); Stein (2014); Szabó and Babuška (2021). However this may be balanced or even be in conflict with the first two points, and this partly explains why error control is sometimes an undertaken issue, because it is not the top one priority for some practitioners devoted to a specific goal. Our personal experience is that precision is often sacrificed for speed. One of our goals in these volumes is to show that these two requirements can be reconciled by the adaptivity and optimality provided by a posteriori error estimation.

#### 4.2 Some initiatives

Let us however mention a few examples in which a posteriori error estimations or control have been considered in an industrial context:

1. For the neutron diffusion equation in nuclear reactor cores, see Ciarlet et al. (2023) (and see also Conjungo Taumhas, Y. et al. (2023) for an evaluation of the modelling error).
2. For turbulent hydraulic and thermal-hydraulic simulations with a perspective of subsequent applications to the nuclear energy sector, see Nassredine et al. (2022, 2023) and Dakroub et al. (2023).
3. Still with applications inspired by the nuclear industry, Liu et al. (2017) address contact problems in elastostatics.
4. For real-time surgical simulation, see Bui et al. (2018b). as well as more details in section 5.1 below.

5. Still for perspectives in surgical simulation and modelling of soft tissue, see Duprez et al. (2020) and Bui et al. (2023).
6. In the aeronautical sector, viscous compressible flows around airfoils were investigated in Basile et al. (2022)

Remark that these works are a step toward certified numerical simulation and have been highly motivated by a context where safety issues are fundamental. Of course, while there are not many existing works, the above list is not exhaustive, and other illustrations are highlighted in some of the contributions to these volumes.

### 4.3 Some bottlenecks

According to our own experience, we can point out the following bottlenecks relative to error control and adaptive discretization in routine industrial computations:

1. For trustworthy certified numerical simulation, not only the discretization errors need to be controlled, but also modelling errors and uncertainties in parameter calibration. These latter can be of larger magnitude and it can be more involved to evaluate and reduce them.
2. The design of a posteriori error estimators is, to a large extent, specific to the model under consideration, and it is challenging and time-consuming to design a posteriori error estimators well-suited for a complex model related to an industrial application.
3. The efficient implementation of some error estimation techniques is not always straightforward in industrial or commercial codes.
4. Error control and adaptive techniques are not always part of training related to numerical simulation, and a large part of the literature on the topic is not easily accessible to outsiders.

Note finally that part of the current activity in the field is inspired from these bottlenecks. Just to mention an example, various recent works have been dedicated to explore the interplay between adaptive finite element meshing and uncertainty quantification technologies, see Bepalov et al. (2022); Bepalov and Silvester (2023); Eigel and Merdon (2016); Eigel et al. (2016); Guignard et al. (2016); Guignard and Nobile (2018); Oden et al. (2005).

## 5 Special topics and industrial applications

Error estimation for numerical methods plays an important role in quality assurance. It enables users to better control the quality of their simulations. So-called non-standard numerical methods have also seen some advances in terms of error

estimation. We will mention here a few numerical methods which offer alternatives to the standard finite element method based on Lagrange polynomials (see 2.2). We also discuss some of the most novel practical applications where a posteriori error estimates have been introduced, including surgical simulation and industrial scale fracture mechanics.

## 5.1 Real-time/interactive simulations

Because of the increasing need for interactive simulations, both for medical applications and robotic control and computer animations, but also in order to build digital twins of actual systems, the last 10 years or so have seen major progress towards real-time quality control of simulated solutions.

This poses a number of problems which will be discussed in forthcoming special issues of *Advances in Applied Mechanics*. The first difficulty lies in computing the error estimate fast enough for the solution to be usable within the constraints of the practical application. For example, in surgical simulation, surgical guidance and surgical training, a range from 50 (for visual feedback) to 500 (for haptic feedback) solutions per second are necessary. Moreover, it is necessary to refine the mesh locally, based on the error estimate, within the time constraints posed by the application at hand. The interested reader can refer to the work of Bui et al. (2018a,b, 2019), which show the first real-time error estimation techniques for interactive mechanics simulations of bodies undergoing large deformations.

Real-time error estimation for interactive deformable bodies still poses several challenges, especially as the demand for more realistic simulations in applications such as virtual surgery, computer animation, and virtual reality continues to grow. Here are some key challenges:

**Computational Efficiency** Developing error estimation techniques that can operate within the constraints of real-time simulation environments is crucial. These methods need to be computationally efficient to provide accurate estimates without significantly impacting the overall simulation performance.

**Integration with Simulation Frameworks** Integrating error estimation techniques seamlessly into existing simulation frameworks poses a challenge. These techniques need to be compatible with various simulation algorithms, such as finite element, meshfree, or position-based methods, to ensure widespread adoption across different applications.

**Sensitivity to Simulation Parameters** Error estimation methods should be robust and reliable across different simulation scenarios and parameter settings. They should account for uncertainties and variations in material properties, boundary conditions, and external forces to provide accurate estimates in diverse environments. In the context of biomechanics, recent progress was made in a series of papers which attempt to disentangle model error from discretization error Hauseux et al. (2017a,b, 2018). In



general, this connection between model and discretization error seems one of the richest direction of investigation, in particular when real-time data acquisition is possible and in the context of machine learning and artificial intelligence surrogate acceleration methods, e.g. Deshpande et al. (2022, 2023).

**User Interaction and Control** Providing intuitive interfaces for users to interactively control error estimation and refinement processes is essential. Users should have the flexibility to adjust parameters, set error tolerances, and guide adaptive refinement strategies in real time to achieve desired simulation outcomes. A closer interaction between the human user and the computer could allow to switch between total computer control and total human control depending on the error level.

**Guaranteed upper bounds** In real-time simulations of surgery, guaranteed upper bounds are crucial for ensuring the safety and effectiveness of the procedures. These bounds provide maximum limits on various parameters such as response time, computational complexity, and error rates. For instance, in robotic surgery, having guaranteed upper bounds on latency ensures that movements are executed within acceptable time frames to prevent accidents or errors. Similarly, computational complexity bounds ensure that simulations run efficiently without overwhelming hardware resources, allowing for smooth and accurate real-time feedback during surgical procedures. Overall, these guaranteed upper bounds are essential for maintaining reliability, accuracy, and safety in real-life applications like surgery simulations.

In short, one of the key remaining questions could be stated as: *“Given multi-modal experimental observations obtained, in the best case, in real-time, and an underlying model, how sufficient is the data acquired to simulate the system? How important and uncertain is the form of the model? How important are the parameters used? Are we better off simulating more scenarios (offline or online) or should we make more measurements?”*. This direction seems ripe for fruitful investigations, as summarised in a recent review Eftimie et al. (2023).

## 5.2 Strain smoothing

Smoothed finite element methods see e.g. a review Nguyen-Xuan et al. (2008a) are based on the idea of transforming derivatives of shape functions into products with outward normals of smoothing cells. This suppresses the need to use Jacobian transformations, enables polytopal meshes, makes it possible to compute on extremely distorted meshes, alleviates numerical locking and can, in certain conditions provide upper bounds for the system’s energy. Several approaches were developed, mostly based on the methods shown in Nguyen-Xuan et al. (2008a).

Little work has been done in the area of error estimates for strain smoothing stabilized finite elements. The main question lies in the choice of the number

of subcells. One subcell may provide equivalent methods to stress-based finite elements (dual), giving an upper bound to the energy. In the infinite limit, as the number of subcells goes to infinity, the standard displacement based (primal) finite element method is recovered. González-Estrada et al. (2013) presents ideas on how to obtain error indicators for smoothed finite element methods.

### 5.3 Partition of unity enrichment

Extended finite element methods are partition of unity methods, similar to the generalized finite element method and a few others, including hp-clouds. The idea of the method (Bordas and Menk (2023)) is to enrich the approximation by special functions which introduce known features of the exact solution. The idea is to improve the approximation property of the finite element (or meshfree) space by making it possible to reproduce the known features of the solution. See the book (Bordas and Menk (2023)) and the recent review on error estimates for enriched approximations in (González-Estrada et al. (2023)).

For example, to model a crack, a discontinuous function is introduced to take into account the jump across the crack surface. Asymptotic functions can be introduced to improve the accuracy of the solution close to the crack front. These methods were used in a wide variety of contexts, and called for specific treatments in terms of error estimation. Those are summarized in (Bordas and Menk (2023)). They were also implemented in a commercial code *MorfeoCrack*, based on the developments in the following seminal paper (Jin et al. (2017)).

In short, standard recovery based methods fail because they are based on smoothing (projection on a polynomial space), whilst the enrichment functions are usually non-polynomial, see, e.g., (Bordas and Duflot (2007)). This leads to smearing of the recovered solution, defeating its purpose to provide a higher quality solution to compare the raw finite element counterpart. Residual based methods have also to be adapted because of the extra terms present in the approximations, see (Gerasimov et al. (2012); Hild et al. (2009); Rüter et al. (2013)).

Several approaches exist for this, including subtracting the enrichment, see (Ródenas et al. (2008)), or projecting on enriched spaces which are able to take into account the special features present in the solution space, see (Bordas et al. (2008); Bordas and Duflot (2007); Duflot and Bordas (2008)). Other approaches were shown in the literature, such as (Panetier et al. (2010); Prange et al. (2012); Rüter et al. (2013)) and, recently, in (Bento et al. (2023)), higher-order methods were investigated.

The goal-oriented techniques mentioned in 2.5 can be extended, and a number of contributions exist in the field of goal-oriented error estimation for enriched approximations. The interested reader can refer to (González-Estrada et al. (2013, 2014, 2021, 2023)).

The above papers introduce goal-oriented error estimates (GOEE), which quantify and control local errors in quantities of interest (QoI) for advanced engineering applications, such as aerospace, see also Section 2.5. This includes a recovery-based error estimation technique for QoIs is presented, using an en-

hanced version of the Superconvergent Patch Recovery (SPR) technique, see Zienkiewicz and Zhu (1987). This approach provides nearly statically admissible stress fields, resulting in accurate estimations of local discretization error contributions to QoI. The technique requires reasonable computational cost and could be easily implemented into finite element codes or used as an independent postprocessing tool.

The error estimation in QoI relies on evaluating QoI through solving auxiliary problems. Energy estimates are used to relate errors in QoI to the initial problem and auxiliary problem solutions. Explicit and implicit residual-based approaches and smoothing techniques for energy estimates can be used, as well as the Zienkiewicz-Zhu estimate and SPR techniques. Bridging these approaches aims to obtain guaranteed upper bounds while retaining ease of implementation. The SPR-CX approach is an efficient and simple goal-oriented adaptivity procedure for linear QoI in elasticity problems, extended to handle singular elasticity problems.

#### **5.4 How important is it that the recovered (smoothed) solution satisfy the boundary conditions and the governing equations?**

We discuss here briefly the question of statical admissibility of recovered solutions, presented in the following: González-Estrada et al. (2012); Bordas and Menk (2023); González-Estrada et al. (2023). These contributions assesses the effect of statical admissibility and the ability of recovered solutions to represent singular solutions, along with the accuracy, local, and global effectivity of recovery-based error estimators for enriched finite element methods, such as the extended finite element method (XFEM). Two recovery techniques are studied: the superconvergent patch recovery procedure with equilibration and enrichment (SPR-CX) and the extended moving least squares recovery (XMLS). Both techniques enrich the recovered solutions, with SPR-CX enforcing equilibrium constraints.

*Numerical results highlight the necessity of extended recovery techniques in error estimators for this class of problems, with statically admissible recovered solutions yielding significant improvements in effectivities. This emphasizes the importance of both extended recovery procedures and statical admissibility for accurate assessment of the quality of enriched finite element approximations.*

#### **5.5 Meshfree/meshless methods**

Meshfree methods are numerical techniques used for solving partial differential equations without relying on a predefined mesh. Unlike traditional methods like finite element analysis, meshfree methods operate directly on scattered data points, making them particularly useful for problems with complex geometries or evolving domains. One notable advantage is their flexibility in handling irregular geometries and dynamic problems, which often pose challenges for mesh-based approaches.

A significant strength of meshfree methods lies in their ability to reproduce complex phenomena with high fidelity, especially in situations where traditional mesh-based methods struggle due to mesh distortion or excessive refinement requirements. Additionally, meshfree methods often exhibit good scalability and efficiency, particularly for problems involving large deformations or adaptive refinement.

However, these methods also have their limitations. Reproducibility can be challenging since the results depend on the distribution of nodes or data points, which may vary between different simulations. Furthermore, achieving smooth solutions can be difficult, especially in regions with sparse data or irregular distributions of nodes.

For a comprehensive understanding of meshfree methods, one can refer to review papers such as Nguyen et al. (2008), which provide an in-depth analysis of various aspects, including the theoretical foundations, implementation strategies, and applications of meshfree methods.

When it comes to error estimation in meshfree methods, several factors need to be taken into account. These include the choice of basis functions or shape functions, the accuracy of the numerical integration scheme, and the interpolation error associated with the scattered data points. Proper error estimation is crucial for assessing the reliability and accuracy of the computed solutions and guiding the refinement or adaptation strategies. Importantly, certain meshfree methods suffer from conditioning issues which may lead to algebraic errors, as discussed in this volume of *Advances in Applied Mechanics*, see Papež (2024). In particular this can apply to methods based on collocation, i.e., methods that, conversely to Galerkin techniques presented in 2.2, consist in writing the strong form directly at “quasi-arbitrary” sets of nodes within the domain, see Jacquemin et al. (2023) for a review of adaptive schemes in meshless collocation, also known as the smart cloud method.

It is important to note that in meshfree methods, Galerkin orthogonality, a property commonly satisfied in traditional finite element methods, may not hold on the boundary. This is because the test functions employed in meshfree formulations typically do not vanish on the boundary, leading to deviations from orthogonality. Understanding and managing such deviations are essential for ensuring the accuracy and stability of the numerical solutions, particularly near domain boundaries.

Additionally, meshless methods do not employ polynomial approximations, which makes Gauss quadrature inexact. It is therefore critical to contain integration errors. This can be done by an interplay between using more background subcells in regions of interest as well as increasing the number of integration points per subcell, see Racz and Bui (2012).

The interested reader can refer to the following papers for additional references on error analysis for meshfree methods: Arnold and Wendland (1983); Davydov and Oanh (2011); Dolbow and Belytschko (1998); Li et al. (2020); Orkisz and Milewski (2008); Park et al. (2003); Perazzo et al. (2008); Rabczuk and Belytschko (2005), which includes applications to shell elements, optimal point placement in collocation methods, a posteriori error estimation driving point

cloud adaptation, and applications to localized phenomena and large gradients.

## 5.6 Isogeometric analysis

Isogeometric analysis (IGA) is a computational technique that integrates the geometric design and analysis of structures or materials. Unlike traditional finite element methods, IGA employs the same basis functions to represent both the geometry and the solution field, typically using Non-Uniform Rational B-Splines (NURBS) or other spline-based representations. This seamless integration of geometric and simulation aspects offers several advantages over traditional methods.

One of the key strengths of IGA lies in its ability to precisely represent complex geometries using the same basis functions employed for the analysis, thereby eliminating the need for mesh generation and simplifying the workflow. This leads to significant reductions in pre-processing time and enables more accurate representations of geometric features.

Moreover, IGA facilitates the use of higher-order basis functions, which can lead to more accurate solutions, particularly for problems involving curved or irregular boundaries. By leveraging the smoothness properties of spline functions, IGA often produces smoother and more realistic solutions compared to traditional finite element methods.

However, despite these advantages, IGA also has its challenges. One notable limitation is the computational cost associated with the construction and manipulation of spline representations, especially for problems involving large-scale simulations or dynamic analyses. Additionally, the coupling between the geometry and analysis introduces complexities in the formulation and implementation of boundary conditions and geometric modifications, see, e.g., Hu et al. (2018); Nguyen et al. (2014).

For a deeper understanding of IGA, interested individuals can refer to review papers such as Nguyen et al. (2015), which provide comprehensive insights into the theoretical foundations, computational aspects, and applications of isogeometric analysis.

When it comes to error estimation in IGA, similar considerations apply as in traditional finite element methods, including the choice of basis functions, integration schemes, and interpolation errors. Proper error estimation is essential for assessing the accuracy and reliability of the computed solutions and guiding refinement strategies.

Isogeometric analysis offers a powerful framework for integrating geometric design and analysis, with advantages in accuracy, efficiency, and geometric flexibility. However, challenges remain in terms of computational cost and the formulation of boundary conditions, highlighting the ongoing research efforts in this field.

The key difficulties in adaptive isogeometric simulations lie in the tensor-product nature of the underlying shape functions (NURBS), which, if nothing is done, leads to spurious propagation of refinements throughout the domain. Alternatives such as the geometry independent field approximation techniques

avoid such issues by enabling the use of local refinement (through splines such as PHT splines, for example), whilst keeping the geometrical representation of the domain identical, and approximated by NURBS, see Atroshchenko et al. (2018); Jansari et al. (2022a); Videla et al. (2019, 2024).

Other exciting directions of research are provided in the following recent contributions including space-time analysis, as well as functional-type error estimates for isogeometric analysis, see Langer et al. (2016); Matculevich (2018); Langer et al. (2019).

## 6 Perspectives

In *Advances in Applied Mechanics (AAMS) Vol 58: Error control, adaptive discretizations, and applications, Part 1*, world-leading authors present cutting-edge research at the intersection of computational mechanics and applied mathematics, exploring innovative approaches to error control and adaptive discretizations across various fields.

### Chapters

1. ***hp* adaptive Discontinuous Galerkin strategies driven by a posteriori error estimation with application to aeronautical flow problems** (Chapelier et al., 2024) presents recent developments about  $h$ ,  $p$ , and  $hp$ -adaptive strategies driven by a posteriori error estimators using a high-order Discontinuous Galerkin finite element numerical framework, where  $h$  stands for the mesh size and  $p$  for the polynomial degree of the finite element approximation. A combination of error estimators tailored for the numerical method considered is presented along with smoothness indicators driving the decision to refine in  $h$  or  $p$ . The  $h$ ,  $p$ , and  $hp$ -adaptation strategies are described and applied to flow problems of aeronautical interest, including scale-resolving simulations of the transitional flow past a NACA0012 airfoil, scale-resolving simulations of the turbulent jet issued by a realistic nozzle geometry, and inviscid simulations of the transonic flow around a complex CRM aircraft geometry. For all cases considered, the interest of  $h$ ,  $p$ , and  $hp$ -adaptation is demonstrated for easing the meshing process and increasing the resolution in flow regions of interest, enabling a significant reduction of the total number of computational degrees of freedom compared to manual meshing techniques and classical lower-order numerical approximations.
2. **An anisotropic mesh adaptation method based on gradient recovery and optimal shape elements** (Fortin, 2024) The authors present a complete mesh adaptation strategy applicable for controlling the discretization error on a finite element solution of Lagrange type of any degree. The method is described in detail, outlining the process of constructing a more accurate solution from a finite element solution using a

gradient recovery method. The error is then estimated as the difference between the reconstructed and the initial solution. Subsequently, the mesh is modified using local operations to minimize the error on the gradient. Additionally, a number of numerical examples are provided to illustrate the effectiveness of the approach.

3. **Model reduction techniques for parametrized nonlinear partial differential equations** (Nguyen, 2024) The authors present model reduction techniques for parametrized nonlinear partial differential equations (PDEs). The main ingredients of their approach include reduced basis (RB) spaces to provide rapidly convergent approximations to the parametric manifold, Galerkin projection of the underlying PDEs onto the RB space to reduce the number of degrees of freedom, and empirical interpolation schemes to rapidly evaluate the nonlinear terms associated with the Galerkin projection. They devise a first-order empirical interpolation method to construct an inexpensive and stable interpolation of the nonlinear terms. Two different hyper-reduction strategies are considered: hyper-reduction followed by linearization, and linearization followed by hyper-reduction. The authors extend empirical interpolation to noninvasive model reduction and apply it to compressible flows in both supersonic and hypersonic regimes. They present numerical results to illustrate the accuracy, efficiency, and stability of the reduced-order models. The interested reader can refer to Kerfriden et al. (2014); Hoang et al. (2016); Goury et al. (2016); Agathos et al. (2020); Hoang et al. (2021, 2022); Chen et al. (2024) for adaptive (certified) model order reduction for non-linear problems and localised phenomena such as fracture, molecular dynamics and inverse problems.
4. **Adaptive finite elements for obstacle problems** (Gustafsson, 2024) The author summarize three applications of the obstacle problem to membrane contact, elastoplastic torsion, and cavitation modelling, demonstrating how the resulting models can be solved using mixed finite elements. He highlights the challenge of constructing fixed computational meshes for any inequality-constrained problem due to the unknown shape of the coincidence set. Consequently, he demonstrates how  $h$ -adaptivity can be utilized to resolve the unknown coincidence set. Additionally, the author discusses practical challenges that must be overcome in the application of the adaptive methods.
5. **A Posteriori Error Identities and Estimates of modelling Errors** (Repin, 2024) This Chapter discusses a posteriori estimation methods for mathematical models based on partial differential equations. The analysis is based on functional identities of a special kind, reflecting the most general relations that hold for deviations from the exact solution of a boundary value problem. These identities do not depend on special properties of approximations and contain no mesh-dependent constants, making them valid for any function in the admissible (energy) class. This universality

enables the control of accuracy for various numerical approximations and the comparison of solutions of mathematical models. The capabilities are demonstrated with the paradigm of modelling errors generated by simplifications of the original problem. Three groups of problems are discussed, where errors of simplification have different origins: errors arising from simplifying coefficients of the equation, errors associated with simplifying geometry, and dimension reduction errors. It is shown that in any of these cases, the desired error estimates follow from the general a posteriori identity after proper specification of the functional spaces and operators associated with the boundary value problem.

6. **Exact error control for variational problems via convex duality and explicit flux reconstruction** (Bartels and Kaltenbach, 2024) A posteriori error estimates serve as crucial tools for bounding discretization errors in terms of computable quantities, bypassing the need for establishing often challenging regularity conditions. However, for problems involving non-linearity, non-differentiability, jumping coefficients, or finite element methods with anisotropic triangulations, such estimates can often result in large factors, leading to sub-optimal error estimates. To address this issue, exact and explicit error representations are derived using convex duality arguments, effectively avoiding such effects.
7. **Algebraic error in numerical PDEs and its estimation** (Papež, 2024) The paper discusses error estimation for the Poisson model problem, focusing on residual-based and flux reconstruction error estimates. Residual-based estimates, while useful for estimating discretization errors, have limitations due to the lack of exact algebraic solutions. The paper proposes adjustments to these estimates but highlights drawbacks. Flux reconstruction offers local error indicators and guaranteed bounds but is computationally intensive. The paper emphasizes the importance of reducing algebraic error for accurate discretization error estimation and suggests further research on adaptive mesh refinement and algebraic solver stopping criteria. It also advocates for rigorous methods to mitigate algebraic errors in various numerical methods. It is worth noting that other methods including meshfree methods, enriched partition of unity methods, see Agathos et al. (2016b,a, 2018, 2019b,a, 2020); Bordas and Menk (2023), isogeometric analysis, see Langer et al. (2016); Yu et al. (2018); Matculevich (2018); Videla et al. (2019); Langer et al. (2019); Jansari et al. (2022b) and collocation methods, see Arnold and Wendland (1983); Jia et al. (2019), smart point clouds, see Perazzo et al. (2008); Jacquemin et al. (2023), are also concerned by algebraic errors due to ill-conditioning of the stiffness matrix. The interested reader can refer to the following papers.



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