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# Explicit lower bounds for the height in Galois extensions of number fields \*

Jonathan Jenvrin

## Abstract

Amoroso and Masser proved that for every real  $\epsilon > 0$ , there is a constant  $c(\epsilon) > 0$ , with the property that, for every algebraic number  $\alpha$  such that  $\mathbb{Q}(\alpha)/\mathbb{Q}$  is a Galois extension, the height of  $\alpha$  is either 0 or at least  $c(\epsilon)[\mathbb{Q}(\alpha) : \mathbb{Q}]^{-\epsilon}$ . In this article, we establish an explicit version of this theorem.

## 1 Introduction

In this article, we let  $\overline{\mathbb{Q}}$  be a fixed algebraic closure of  $\mathbb{Q}$ . For an algebraic number  $\alpha$ , we denote by  $h(\alpha) \geq 0$  its absolute logarithmic Weil height.

It's well known by Kronecker's theorem that  $h(\alpha) = 0$  if and only if  $\alpha = 0$  or  $\alpha$  is a root of unity. Lehmer's conjecture predicts the existence of a positive constant  $c$  such that

$$h(\alpha) \geq cd^{-1}$$

whenever  $\alpha \neq 0$  has degree  $d$  over  $\mathbb{Q}$  and is not a root of unity.

This is obviously true for  $\alpha$  not a unit with  $c = \log(2)$ . The conjecture has been established for various classes of  $\alpha$ . For  $\alpha$  non-reciprocal (which is always the case for  $d$  odd) the result holds true with  $c = 3h(\theta) = \log(\theta)$ , where  $\theta$  is the real root  $> 1$  of  $X^3 - X - 1$  (see [Sch73]). If  $\alpha$  belongs to an abelian extension and  $h(\alpha) \neq 0$ , we have  $h(\alpha) \geq \frac{\log(5)}{12}$  (see [AD00]). For  $\alpha$  in a CM-field with  $|\alpha| \neq 1$ , we have  $h(\alpha) \geq \frac{1}{2} \log\left(\frac{1+\sqrt{5}}{2}\right)$  (see [Sch73]). The best unconditional result we have to Lehmer's conjecture is Dobrowolski's result

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[Dob79], which implies that for any  $\epsilon > 0$ , there is  $c(\epsilon) > 0$  such that either  $h(\alpha) = 0$  or  $h(\alpha) \geq c(\epsilon)d^{-1-\epsilon}$ .

Amoroso and David proved Lehmer's conjecture when  $\mathbb{Q}(\alpha)/\mathbb{Q}$  is a Galois extension, see [AD99, Corollaire 1.7]. Later, Amoroso and Masser [AM16] proved the following stronger result.

**Theorem 1.1.** *For any  $\epsilon > 0$ , there is a positive effective constant  $c(\epsilon)$  with the following property. If  $\alpha \in \overline{\mathbb{Q}}^*$  is of degree  $d$  over  $\mathbb{Q}$ ,  $\alpha$  is not a root of unity and  $\mathbb{Q}(\alpha)/\mathbb{Q}$  is Galois, then*

$$h(\alpha) \geq c(\epsilon)d^{-\epsilon}.$$

The goal of our article is to give an explicit version of this theorem. Our main result is the following:

**Theorem 1.2.** *Suppose  $\alpha \in \overline{\mathbb{Q}}^*$  is of degree  $d$  over  $\mathbb{Q}$ . If  $\alpha$  is not a root of unity and  $\mathbb{Q}(\alpha)/\mathbb{Q}$  is Galois, then*

$$h(\alpha) \geq \exp(-500 \log(3d)^{3/4} \log(\log(3d))).$$

Theorem 1.2 is proved in Section 3. Our proof strategy relies on that of Amoroso and Masser in [AM16, Theorem 3.3]. There will be two cases. They are related to the comparison of the multiplicative rank  $\rho$  of the conjugates of  $\alpha$ , and  $d$ . If  $\rho$  is big in terms of  $d$ , then they originally use the main result of [AD99]. It says that for any  $\alpha_1, \dots, \alpha_r$  multiplicatively independent algebraic number and any  $\epsilon > 0$ , there exists  $C(\epsilon)$  such that

$$\max_i h(\alpha_i) \geq C(\epsilon)D^{-1/r-\epsilon}$$

where  $D = [\mathbb{Q}(\alpha_1, \dots, \alpha_r) : \mathbb{Q}]$ . We will instead use a more precise and explicit result from [AV12] (see Theorem 2.1). If  $\rho$  is small in terms of  $d$ , then they use the relative Dobrowolski lower bound of [AZ10]. Again, we will instead apply a more precise and explicit result from [AD07] (see Theorem 2.2). We moreover need an explicit estimate for the Euler Totient function (see Lemma 2.2).

## 2 Auxiliary Results

We state two important results that will be the key ingredients in our proof. The first was proved by Amoroso and Viada in [AV12, Corollary 1.6], where,

more generally, they give an explicit version of a generalized Dobrowolski result on Lehmer's problem.

**Theorem 2.1** ([AV12, Corollary 1.6]). *Let  $\alpha_1, \dots, \alpha_n$  be multiplicatively independent algebraic numbers in a number field  $K$ . Then*

$$h(\alpha_1) \dots h(\alpha_n) \geq [K : \mathbb{Q}]^{-1} (1050n^5 \log(3[K : \mathbb{Q}]))^{-n^2(n+1)^2}.$$

The second result was proved by Amoroso and Delsinne in [AD07, Théorème 1.3]. We recall a special case of this theorem, which is enough for our purposes.

**Theorem 2.2** ([AD07, Théorème 1.3]). *Let  $\alpha \in \overline{\mathbb{Q}}^*$  be not a root of unity and let  $L$  be a number field. Then if  $L/\mathbb{Q}$  is a finite abelian extension and  $D = [L(\alpha) : L]$ , we have*

$$h(\alpha) \geq D^{-1} \frac{\log \log(5D)^3}{\log(2D)^4}.$$

The following lemma is a general version of [AM16, Lemma 2.2], with an improvement on the value of  $n(\rho)$ .

**Lemma 2.1.** *Let  $F/K$  be a finite Galois extension and  $\alpha \in F \setminus \{0\}$ . Let  $\rho$  be the multiplicative rank of the conjugates  $\alpha_1, \dots, \alpha_d$  of  $\alpha$  over  $K$ , and suppose  $\rho \geq 1$ . Then there exists a subfield  $L \subset F$  which is Galois over  $K$  of degree  $[L : K] \leq n(\rho)$  and an integer  $k \geq 1$ , such that  $F$  contains a primitive  $k$ -th root of unity  $\zeta_k$  and  $\alpha^k \in L$ . We can take*

$$n(\rho) = \rho!2^\rho \text{ for } \rho = 1, 3, 5 \text{ and } \rho > 10.$$

*Otherwise we can take*

$$n(\rho) = 135\rho!2^{\rho-1}.$$

*Proof.* Let  $k$  be the order of the group of roots of unity in  $F$ , in particular  $F$  contains  $K(\zeta_k)$ . Define  $\beta_i = \alpha_i^k$  for  $1 \leq i \leq d$  and  $L = K(\beta_1, \dots, \beta_d)$ . We have  $L \subset F$  because  $F/K$  is Galois, and we easily check that  $L/K$  is Galois. The  $\mathbb{Z}$ -module

$$M = \{\beta_1^{a_1} \dots \beta_d^{a_d} \mid a_1, \dots, a_d \in \mathbb{Z}\}$$

is torsion free by the choice of  $k$  and so, by the Classification Theorem for abelian group, is free, of rank  $\rho$ . This shows that the action of  $\text{Gal}(L/K)$

over  $M$  defines an injective representation from  $\text{Gal}(L/K)$  to  $\text{GL}_\rho(\mathbb{Z})$ , hence  $\text{Gal}(L/K)$  identifies with a finite subgroup of  $\text{GL}_\rho(\mathbb{Z})$ . To conclude, we use a theorem stated by Feit in 1996, and proved by Rémond in [Rémond20, Théorème 7.1], which, in particular, gives an explicit bound for the order of the maximal finite subgroups of  $\text{GL}_\rho(\mathbb{Z})$ .  $\square$

We end this section with the following two lemmas.

**Lemma 2.2.** *Let  $\phi$  be the Euler's totient function. For every positive integer  $n$ , we have <sup>1</sup>*

$$\frac{n}{\phi(n)} \leq \frac{3 \log(\log(3n))}{2 \log(\log(3))}$$

and

$$\phi(n) \geq \sqrt{\frac{n}{2}}.$$

*Proof.* For  $n \geq 3$ , let

$$a_n = \frac{\phi(n) \log(\log(n))}{n}.$$

We are going to show that  $a_n \geq a_3$  for all  $n \geq 3$ . It's easy to check that  $a_4 \geq a_3$ , so we can suppose  $n \geq 5$ . We have (see [BS96, Theorem 8.8.7]) that for  $k \geq 3$

$$\phi(k) > \frac{k}{e^\gamma \log(\log(k)) + \frac{3}{\log(\log(k))}}$$

where  $\gamma$  is the Euler's constant. Thus for all  $k \geq 3$  we have

$$a_k \geq \frac{\log(\log(k))}{e^\gamma \log(\log(k)) + \frac{3}{\log(\log(k))}}.$$

Consider the function

$$f(x) = \frac{\log(\log(x))}{e^\gamma \log(\log(x)) + \frac{3}{\log(\log(x))}}.$$

Its derivative is

$$f'(x) = \frac{6 \log(\log(x))}{x \log(x) (e^\gamma (\log(\log(x)))^2 + 3)^2}.$$

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<sup>1</sup>The factor 3 in  $\log(\log(3n))$  is here for the cases  $n = 1$  and  $n = 2$ .

So the function  $f$  is increasing on the interval  $[3, +\infty[$ . We can check that  $f(5) \geq \frac{2 \log(\log(3))}{3}$ . As a result, for  $n \geq 5$

$$a_n \geq f(n) \geq f(5) \geq \frac{2 \log(\log(3))}{3} = a_3.$$

Hence, for  $n \geq 3$  we have

$$\frac{n}{\phi(n)} \leq \frac{3 \log(\log(n))}{2 \log(\log(3))}.$$

The second inequality is well known, and, for instance, it can be deduce easily from the previous one for  $n \geq 503$ , and checked for small values of  $n$ .  $\square$

**Lemma 2.3.** *For every positive integer  $n$ , we have*

$$\frac{2^n n!}{n^{n-5}} \leq \frac{2^{18} 18!}{18^{18-5}} < 80601.$$

*Proof.* One can check that the result hold true for  $n \leq 18$ . For  $n \geq 18$ , let  $U_n = \frac{2^n n!}{n^{n-5}}$ . Then

$$\begin{aligned} \frac{U_{n+1}}{U_n} &= 2 \left( \frac{n+1}{n} \right)^5 \left( \frac{n}{n+1} \right)^n \\ &\leq 2 \left( \frac{19}{18} \right)^5 \left( \frac{n}{n+1} \right)^n \\ &\leq \left( \frac{19}{18} \right)^5 \left( \frac{18}{19} \right)^{18} \\ &< 1, \end{aligned}$$

where we used the fact that  $\left( \left( \frac{n}{n+1} \right)^n \right)_{n \geq 1}$  is a decreasing sequence. Hence the sequence  $(U_n)_{n \geq 18}$  is decreasing, which proves the result.  $\square$

### 3 Proof of Theorem 1.2

We can now establish the proof of Theorem 1.2. Firstly, we will demonstrate a lower bound for  $h(\alpha)$  depending on the multiplicative rank  $\rho$  of the conjugates of  $\alpha$ , and on the degree  $d$  of  $\mathbb{Q}(\alpha)$  over  $\mathbb{Q}$ .

**Lemma 3.1.** *Let  $\alpha \in \overline{\mathbb{Q}}^*$  of degree  $d$  over  $\mathbb{Q}$  and not a root of unity. Let  $\rho$  be the multiplicative rank of the conjugates of  $\alpha$ . Define*

$$g_1(\rho, d) = \min_{1 \leq r \leq \rho} \left( d^{1/r} (1050r^5 \log(3d))^{r(r+1)^2} \right)$$

and

$$g_2(\rho, d) = 10^{20} \rho^\rho \log(\log(6d^2))^5.$$

Then:

$$h(\alpha) \geq \min(g_1(\rho, d), g_2(\rho, d))^{-1}.$$

*Proof.* By Theorem 2.1 applied to  $\mathbb{Q}(\alpha)$ , and given that all conjugates of  $\alpha$  over  $\mathbb{Q}$  are in  $\mathbb{Q}(\alpha)$  (since  $\mathbb{Q}(\alpha)/\mathbb{Q}$  is Galois), we have, for every  $1 \leq r \leq \rho$ , that

$$h(\alpha) \geq d^{-1/r} (1050r^5 \log(3d))^{-r(r+1)^2},$$

so  $h(\alpha) \geq g_1(\rho, d)^{-1}$ .

By Lemma 2.1, there exists  $L \subset \mathbb{Q}(\alpha)$  and  $k \geq 1$  such that  $L/\mathbb{Q}$  is Galois,  $[L : \mathbb{Q}] \leq 135\rho!2^{\rho-1}$ ,  $\mathbb{Q}(\zeta_k) \subset \mathbb{Q}(\alpha)$  and  $\alpha^k \in L$ . Set  $M = \mathbb{Q}(\zeta_k)$ . We have  $M(\alpha) = \mathbb{Q}(\alpha) = L(\alpha)$ . Notice that

$$[M(\alpha) : M] = [L(\alpha) : L] \frac{[L : \mathbb{Q}]}{[M : \mathbb{Q}]}.$$

Since  $\alpha$  is a root of  $X^k - \alpha^k \in L[X]$ , we have  $[L(\alpha) : L] \leq k$ . We also have  $\phi(k) = [M : \mathbb{Q}]$ . Therefore, we obtain

$$[M(\alpha) : M] \leq k \frac{[L : \mathbb{Q}]}{[M : \mathbb{Q}]} \leq \frac{k}{\phi(k)} [L : \mathbb{Q}].$$

By Lemma 2.2, we have

$$\frac{k}{\phi(k)} \leq \frac{3 \log(\log(3k))}{2 \log(\log(3))},$$

so

$$[M(\alpha) : M] \leq 405 \frac{\log(\log(3k))}{\log(\log(3))} \rho! 2^{\rho-2}.$$

We notice that  $\phi(k) \leq d$ . By the last statement of Lemma 2.3, we have  $\sqrt{k/2} \leq \phi(k)$ . Hence, we have  $k \leq 2d^2$ , and so

$$[M(\alpha) : M] \leq 405 \frac{\log(\log(6d^2))}{\log(\log(3))} D \rho! 2^{\rho-2} \leq 10^4 \rho! 2^\rho \log(\log(6d^2)).$$

By Lemma 2.3 we have

$$[M(\alpha) : M] \leq 10^9 \rho^{\rho-5} \log(\log(6d^2)).$$

By Theorem 2.2, applied to  $\mathbb{Q}(\zeta_k)/\mathbb{Q}$ , we have

$$h(\alpha) \geq [M(\alpha) : M]^{-1} \frac{\log(\log(5[M(\alpha) : M]))^3}{\log(2[M(\alpha) : M])^4}.$$

Since the function

$$f(x) = \frac{1}{x} \frac{\log(\log(5x))^3}{\log(2x)^4}$$

is decreasing on the interval  $[1, +\infty[$ , setting

$$X(\rho, d) = 10^9 \rho^{\rho-5} \log(\log(6d^2))$$

we deduce that

$$h(\alpha)^{-1} \leq X(\rho, d) \log(2X(\rho, d))^4 \leq 10^{20} \rho^\rho \log(\log(6d^2))^5.$$

Hence  $h(\alpha) \geq g_2(\rho, d)^{-1}$ . This proves the lemma.  $\square$

*Proof of Theorem 1.2.* Now, we are ready to prove Theorem 1.2. Suppose, first, that  $\rho \leq \log(3d)^{1/4}$ . By Lemma 3.1, we get

$$\begin{aligned} h(\alpha)^{-1} &\leq g_2(\rho, d) = 10^{20} (\log(3d)^{1/4})^{\log(3d)^{1/4}} \log(\log(6d^2))^5 \\ &= \exp(20 \log(10) + \log(3d)^{1/4} \log(\log(3d)^{1/4}) + 5 \log(\log(\log(6d^2))))). \end{aligned}$$

So, we have

$$h(\alpha)^{-1} \leq \exp(20 \log(10) + 6 \log(3d)^{1/4} \log(\log(3d))). \quad (1)$$

Now, we suppose that  $\rho \geq \log(3d)^{1/4}$ . We choose  $r \in \mathbb{Z}$  such that  $\log(3d)^{1/4} + 1 > r \geq \log(3d)^{1/4}$ . In particular  $r \leq \rho$ . Then, thanks again to Lemma 3.1, we obtain

$$\begin{aligned} h(\alpha)^{-1} &\leq g_1(\rho, d) = d^{1/r} (1050 r^5 \log(3d))^{r(r+1)^2} \\ &\leq d^{1/\log(3d)^{1/4}} (1050 (\log(3d)^{1/4} + 1)^5 \log(3d))^{(\log(3d)^{1/4} + 1)(\log(3d)^{1/4} + 2)^2} \\ &\leq d^{1/\log(3d)^{1/4}} (2^{16} \log(3d)^{9/4})^{2^3 \log(3d)^{3/4}} \\ &= \exp(\log(3d)^{3/4} + 2^3 \log(3d)^{3/4} \log(2^{16} \log(3d)^{9/4})). \end{aligned}$$



So, we have

$$h(\alpha)^{-1} \leq \exp(\log(3d)^{3/4} + 400 \log(3d)^{3/4} \log(\log(3d))). \quad (2)$$

From (1) and (2), we obtain

$$h(\alpha)^{-1} \leq \exp(500 \log(3d)^{3/4} \log(\log(3d))).$$

□

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