



HAL
open science

Approximation and perturbations of stable solutions to a stationary mean field game system

Jules Berry, Olivier Ley, Francisco J Silva

► **To cite this version:**

Jules Berry, Olivier Ley, Francisco J Silva. Approximation and perturbations of stable solutions to a stationary mean field game system. 2024. hal-04475153v3

HAL Id: hal-04475153

<https://hal.science/hal-04475153v3>

Preprint submitted on 2 Apr 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Approximation and perturbations of stable solutions to a stationary mean field game system

Jules Berry

Univ Rennes, INSA, CNRS, IRMAR - UMR 6625, Rennes F-35000, France
jules.berry@insa-rennes.fr

Olivier Ley

Univ Rennes, INSA, CNRS, IRMAR - UMR 6625, Rennes F-35000, France
olivier.ley@insa-rennes.fr

Francisco J. Silva

Univ Limoges, Faculté des Sciences et Techniques, XLIM-DMI, UMR-CNRS 7252, 87060 Limoges, France
francisco.silva@unilim.fr

April 2, 2024

Abstract

This work introduces a new general approach for the numerical analysis of stable equilibria to second order mean field games systems in cases where the uniqueness of solutions may fail. For the sake of simplicity, we focus on a simple stationary case. We propose an abstract framework to study these solutions by reformulating the mean field game system as an abstract equation in a Banach space. In this context, stable equilibria turn out to be regular solutions to this equation, meaning that the linearized system is well-posed. We provide three applications of this property: we study the sensitivity analysis of stable solutions, establish error estimates for their finite element approximations, and prove the local convergence of Newton's method in infinite dimensions.

1 Introduction

Mean Field Games (MFG for short) were introduced independently by Lasry-Lions [37, 35, 36] and Huang-Caines-Malhamé [34, 33]. The goal of this theory is to study (stochastic) differential games with a large number of interchangeable players. We refer the reader to [3, 18, 5, 31, 40] for general references on this topic.

The numerical analysis of MFG systems introduced in [37, 35, 36] has been extensively studied under a monotonicity assumption also introduced by Lasry and Lions, see [3, Chapter 4], [1], and the references therein. Indeed, the latter provides a sufficient condition for the uniqueness of solutions to MFG systems, which allows to show the convergence of numerical methods in [2] and error estimates in [8]. In the absence of this monotonicity assumption, uniqueness may fail (see [13, 4, 20, 19]) and the study of MFG systems with several solutions is delicate both from the theoretical and numerical points of view. In [13], Briani and Cardaliaguet define a particular notion of solution for second order potential

2020 *Mathematics subject classification*. 35J47, 35Q89, 65N20

Key words and phrases. Mean field games, Newton method, finite element method, numerical methods, non-uniqueness of solutions.

MFG systems, the so-called *stable solutions* (see also [10] for a related notion in the context of first order mean field games). These solutions may not be unique but the authors show in [13] that they have some interesting properties motivating their name: stable solutions are isolated and the fictitious play algorithm, introduced in Cardaliaguet-Hadikhhanloo [16], converges locally to these solutions. We also mention the recent work by Tang-Song [46], where the authors implement a smoothed policy iteration method to locally approximate stable solutions. In this paper, we provide new results in this direction, which reinforce the importance of the notion of stable solutions. We are going to prove that stable solutions are indeed stable under perturbations and that local convergence holds for their approximations by finite element methods and Newton iterations.

This paper is the first in a series of works dealing with the numerical analysis of stable equilibria to MFG models. Our goal is to introduce a general framework that covers different types of MFG systems under fairly general assumptions. In order to convey our main ideas, we focus in this paper on the following stationary MFG system

$$\begin{cases} -\Delta u + \frac{1}{2}|Du|^2 + \lambda u = f(m) & \text{in } \mathbb{T}^d, \\ -\Delta m - \operatorname{div}(mDu) + \lambda m = \lambda m_0 & \text{in } \mathbb{T}^d, \end{cases} \quad (1)$$

where $\lambda > 0$ is a given constant and $m_0: \mathbb{T}^d \rightarrow \mathbb{R}$ and $f: \mathbb{R} \rightarrow \mathbb{R}$ are given functions. This system has been introduced in the monograph by Bensoussan-Frehse-Yam [5, Chapter 7] and has been furtherly studied in [25, 30]. We briefly recall its interpretation in Section 2.

In this work we give a definition of stable solutions for system (1) (Definition 2.5 below) and prove that, for every $\lambda > 0$ large enough, every solution to (1) is stable. Furthermore, if the coupling f is monotone, then the unique solution to (1) is stable for every $\lambda > 0$. As in [13], we prove that stable solutions are isolated and we are able to slightly improve the result in [13] by considering weaker norms.

Let us now present the main contribution of this paper. We reformulate system (1) in the form

$$F(u, m) = 0,$$

where $F: X \rightarrow X$ is a nonlinear mapping, defined on a Banach space X , having the form

$$F = I + T \circ G.$$

More precisely, given a suitable Banach space Z , we choose $T: Z \rightarrow X$ as the linear operator defined by $T(f, g) = (v, \rho)$, where (v, ρ) solves

$$\begin{cases} -\Delta v + \lambda v = f & \text{in } \mathbb{T}^d, \\ -\Delta \rho + \lambda \rho = g & \text{in } \mathbb{T}^d, \end{cases}$$

and $G: X \rightarrow Z$ the nonlinear mapping defined by

$$G(v, \rho) = \left(\frac{1}{2}|Dv|^2 - f(\rho), -\operatorname{div}(\rho Du) - \lambda m_0 \right).$$

We refer the reader to Section 3 for the details of this reformulation. In the case where the mapping G is differentiable, the mapping F is also differentiable and the stability of a solution (u, m) to (1) is equivalent to the injectivity of the differential

$$dF[u, m] = I + T \circ dG[u, m].$$

Thus, if T is a compact operator and (u, m) is a stable solution to (1), then $dF[u, m]$ is an injective perturbation of the identity by a compact linear operator. Henceforth, by the Fredholm alternative, we deduce that $dF[u, m]$ is an isomorphism on X . This isomorphism property will be rigorously established below for Banach spaces of the form

$$X = C^{2,\gamma}(\mathbb{T}^d) \times C^{2,\beta}(\mathbb{T}^d) \quad \text{and} \quad X = W^{1,p}(\mathbb{T}^d) \times L^q(\mathbb{T}^d),$$

but many other choices are possible depending on the application in mind.

We now describe three applications of the above isomorphism property for stable solutions.

A first and straightforward application, which follows from the implicit function theorem, concerns the sensitivity analysis of stable solutions to (1) under perturbations of the coupling function f and the distribution m_0 (Proposition 5.1 below).

In the second application, we make use of the Brezzi-Rappaz-Raviart theory on the approximation of nonlinear problems (see [12, 29, 21, 14]) to obtain existence and error estimates for finite element approximations of stable solutions (Theorem 5.4 below). The finite element approximation of a MFG system similar to (1) has been studied by Osborne and Smears in [43] (see [44] for a parabolic MFG system), where the convergence of the approximations is established by using compactness arguments which do not provide error estimates. While finishing this paper, we have learnt about the recent work [45] by the same authors addressing this issue. Regarding the convergence analysis, the results in [43, 45] deal only with the case where the coupling term f is (strongly) monotone, which ensures the uniqueness of the solution to the MFG system. In contrast, our results apply locally around any stable solution without requiring any monotonicity of the coupling and rely on a completely different approach.

In our last application, we provide convergence rates for the iterates of Newton's method in infinite dimension applied to (1) in various functional spaces (Theorems 5.8, 5.10 and 5.11 below). Let us also mention that the analysis of Newton's method in infinite dimensions to approximate the solution to a time-dependent MFG system, with monotone couplings, has been recently addressed by Camilli and Tang [15] using different techniques. Compared with their approach, our result follows directly from classical convergence results of Newton's iterates in function spaces (see [32, 48, 22]) which allow us to deal with non-monotone couplings and to obtain convergence rates in stronger norms.

The paper is structured as follows. In Section 2 we study the well-posedness of (1) and provide some useful estimates on the solutions. We then introduce the definition of stable solution to (1), which is similar to the one proposed in [13], and give sufficient conditions for the existence of such solutions. In Section 3 we reformulate solutions to (1) as zeros of a well-chosen nonlinear mapping F and prove the isomorphism property of the differential of F at stable solutions under the assumption that F is continuously differentiable. This assumption is rigorously justified in Section 4 in the case of Hölder and Sobolev spaces. Finally, applications to the sensitivity analysis of (1), to its finite element approximation, and to the convergence of Newton's method, are studied in Section 5.

Notations For $k \in \mathbb{N}$ and $\alpha \in (0, 1]$, we write $C^{k,\alpha}(\mathbb{T}^d)$ the usual Hölder space on \mathbb{T}^d , i.e.

$$C^{k,\alpha}(\mathbb{T}^d) = \{u \in C^k(\mathbb{T}^d) : \|u\|_{C^{k,\alpha}} < +\infty\},$$

where

$$\|u\|_{C^{k,\alpha}} = \sum_{|j| \leq k} \|\partial^j u\|_{L^\infty} + \sum_{|j|=k} [\partial^j u]_\alpha,$$

with

$$[u]_\alpha = \sup_{\substack{y, x \in \mathbb{T}^d \\ x \neq y}} \frac{|u(x) - u(y)|}{|x - y|^\alpha}.$$

Similarly, $C_{\text{loc}}^{k,\alpha}(\mathbb{R})$ is the space of locally Hölder continuous functions on \mathbb{R} , i.e. $f \in C_{\text{loc}}^{k,\alpha}(\mathbb{R})$ if $f \in C^{k,\alpha}(\Omega)$ for every bounded open subset $\Omega \subset \mathbb{R}$. We also write $C_b^k(\mathbb{R})$ the set of k times continuously differentiable functions f on \mathbb{R} such that the derivatives $f^{(j)}$, for $0 \leq j \leq k$, are bounded. We denote by $\mathcal{P}(\mathbb{T}^d)$ the space on probability measures over \mathbb{T}^d and we always identify a measure $m \in \mathcal{P}(\mathbb{T}^d)$ with its density, which we also denote m , provided that the latter exists. For $1 < p \leq \infty$ the dual of the Sobolev space $W^{1,p}(\mathbb{T}^d)$ is denoted by $W^{-1,p'}(\mathbb{T}^d)$, where $1/p + 1/p' = 1$, and we reserve the notation $H^{-1}(\mathbb{T}^d)$ for the dual of $H^1(\mathbb{T}^d)$.

For Banach spaces X and Y and a mapping $\Phi: X \rightarrow Y$, we write $d\Phi[x]$ the Fréchet differential of Φ at $x \in X$, when it exists. We also use the notation $Y \hookrightarrow X$ when Y is continuously embedded in X .

By convention, we do not specify integration domains when integrals are considered on \mathbb{T}^d , i.e.,

$$\int f \, dx := \int_{\mathbb{T}^d} f \, dx.$$

2 The mean field game system

In this section we establish some properties of the MFG system (1). We first state existence of solutions to (1) as well as a uniqueness result under the Lasry-Lions monotonicity condition on the coupling f . We then define stable solutions to (1) following [13] and prove that they are isolated. Finally, we provide sufficient conditions to ensure that any classical solution to (1) is stable.

Let us begin by describing the mean field game interpretation of system (1). We consider a typical player whose dynamics is governed by the following controlled stochastic differential equation

$$\begin{cases} dX_t^\alpha = \alpha(X_t^\alpha)dt + \sqrt{2}dB_t & \text{for all } t > 0, \\ X_0^\alpha = x, \end{cases}$$

where B_t is a d -dimensional Brownian motion and α is a feedback control. Assume that the player forecasts $\hat{\rho}$ as being the evolution of the distribution of players, from which it is possible to compute the weighted averaged density \hat{m} defined by

$$\hat{m}(x) = \lambda \int_0^{+\infty} \hat{\rho}(t, x) e^{-\lambda t} \, dt \quad (2)$$

for some given $\lambda > 0$. Then this player aims to minimize the following cost

$$J(x, \alpha) = \mathbb{E} \left[\int_0^{+\infty} \left(\frac{|\alpha(X_t^\alpha)|^2}{2} + f(\hat{m}(X_t^\alpha)) \right) e^{-\lambda t} \, dt \right].$$

This yields the Hamilton-Jacobi equation for the value function $u(x) := \inf_\alpha J(x, \alpha)$:

$$-\Delta u + \frac{|Du|^2}{2} + \lambda u = f(\hat{m}).$$

Since all the players are assumed to be interchangeable, they should all play according to the optimal strategy provided by the Hamilton-Jacobi equation, i.e. $\alpha^*(x) = -Du(x)$ (see [26, 27, 47]). The player must then update the forecasted density by solving the Fokker-Planck equation

$$\begin{cases} \partial_t \rho - \Delta \rho - \text{div}(\rho Du) = 0, \\ \rho(0) = m_0, \end{cases} \quad (3)$$

where m_0 is the initial distribution of agents. This yields an updated averaged density m through (2). Using integration by parts in (2) one can easily derive the following equation on m

$$-\Delta m - \operatorname{div}(mDu) + \lambda m = \lambda m_0. \quad (4)$$

As usual we define an MFG equilibrium as being a fix point of this procedure which corresponds to a solution to (1). Notice that once we have a solution to (1) it is possible to recover the probability density function ρ by solving the Fokker-Planck equation (3).

In all of this paper we make the assumption that

$$m_0 \in \mathcal{P}(\mathbb{T}^d) \cap C^{0,\alpha}(\mathbb{T}^d) \quad \text{for some } \alpha \in (0, 1). \quad (5)$$

2.1 Well-posedness

The following result was established in [5, Chapter 7] by approximation. For the sake of self-containedness we provide a proof in Appendix A based on the classical method introduced by Lasry-Lions.

Theorem 2.1. *Assume that $f \in W^{1,\infty}(\mathbb{R})$. Then there exists a classical solution $(u, m) \in C^{2,\alpha}(\mathbb{T}^d) \times C^{2,\alpha}(\mathbb{T}^d)$ to (1), where the constant α is fixed in (5). Furthermore, if $f' \geq 0$, then this solution is unique.*

We now turn to technical results, which will be used throughout the paper. The first one recalls some properties of Fokker-Planck type equations.

Lemma 2.2. *Let $b \in L^\infty(\mathbb{T}^d; \mathbb{R}^d)$, $f \in L^2(\mathbb{T}^d)$, and $g \in L^2(\mathbb{T}^d; \mathbb{R}^d)$. There exists a weak solution $m \in H^1(\mathbb{T}^d)$ to*

$$-\Delta m - \operatorname{div}(mb) + \lambda m = f + \operatorname{div}(g) \quad \text{in } \mathbb{T}^d.$$

Moreover,

(i) *m is the only element in $L^2(\mathbb{T}^d)$ such that*

$$\int (-\Delta \varphi + b \cdot D\varphi + \lambda \varphi) m \, dx = \int f \varphi - g \cdot D\varphi \, dx \quad \text{for every } \varphi \in H^2(\mathbb{T}^d).$$

(ii) *If $f \geq 0$ and $\operatorname{div}(g) = 0$, then, either $f = 0$ and $m = 0$, or $m > 0$.*

(iii) *If $p \geq 2$, $g \in L^p(\mathbb{T}^d; \mathbb{R}^d)$, and $f \in L^q(\mathbb{T}^d)$, with*

$$\begin{cases} q = dp/(d+p) & \text{if } p \neq d \text{ and } d \geq 2, \\ q > d/2 & \text{if } p = d \text{ and } d \geq 2, \\ q = 1 & \text{if } d = 1, \end{cases}$$

then $m \in W^{1,p}(\mathbb{T}^d)$ and there exists a positive constant $C = C(\|b\|_{L^\infty}, \lambda, d, p, q)$ such that

$$\|m\|_{W^{1,p}} \leq C (\|m\|_{L^1} + \|g\|_{L^p} + \|f\|_{L^q}). \quad (6)$$

(iv) *If $g \in L^p(\mathbb{T}^d; \mathbb{R}^d)$ and $f \in L^q(\mathbb{T}^d)$ for $p > d$ and $q > d/2$, then there exists a positive constant $C = C(\|b\|_{L^\infty}, \|f\|_{L^{p/2}}, \|g\|_{L^p}, \lambda, d, p, q)$ such that*

$$\|m\|_{L^\infty} \leq C. \quad (7)$$

Proof. The existence of a weak solution $m \in H^1(\mathbb{T}^d)$ (and its uniqueness in $H^1(\mathbb{T}^d)$) is standard and follows from the Lax-Milgram theorem (see [24, 28]).

For (i) let $\xi \in L^2(\mathbb{T}^d)$ and let $v \in H^2(\mathbb{T}^d)$ be the unique solution (again see [24, 28]) to

$$-\Delta v + b \cdot Dv + \lambda v = \xi \quad \text{in } \mathbb{T}^d.$$

Then using m as a test function for the equation satisfied by v we have

$$\int \xi m \, dx = \int (-\Delta v + b \cdot Dv + \lambda v)m \, dx = \int f v - g \cdot Dv \, dx.$$

If $f, g = 0$, and since ξ is arbitrary, we obtain $m = 0$ which proves (i). When $\xi \geq 0$, $\operatorname{div}(g) = 0$ and $f \geq 0$ we obtain $m \geq 0$ since, in this case, $v \geq 0$ by maximum principle (see [28, Theorem 8.1]). The strict positivity when $m \neq 0$ is then a consequence of the Harnack inequality [28, Theorem 8.20]. This proves (ii)

The $W^{1,p}$ estimate (iii) is a direct consequence of [6, Theorem 1.7.4] while (iv) follows from De Giorgi-Nash-Moser estimates [28, Theorem 8.17]. \square

Remark 2.3. We recall from [38, Theorem 10.41] that, for $1 < r \leq \infty$, any element $h \in W^{-1,r'}(\mathbb{T}^d)$, where $1/r + 1/r' = 1$, can be identified with $g_1 + \operatorname{div}(g_2)$ for $g_1 \in L^r(\mathbb{T}^d)$, $g_2 \in L^r(\mathbb{T}^d; \mathbb{R}^d)$ and $\|h\|_{W^{-1,r'}} = (\|g_1\|_{L^r}^r + \|g_2\|_{L^r}^r)^{1/r}$. In particular the conclusions of Lemma 2.2 can be extended to equations with right-hand side in $W^{-1,r'}(\mathbb{T}^d)$ for appropriate values of r .

The following proposition contains a priori estimates on classical solutions to (1).

Proposition 2.4. *Assume $f \in C_b^0(\mathbb{R})$. Then there exists a positive constant $K = K(\|f\|_{L^\infty}, d)$ such that for every classical solution (u, m) to (1) it holds that*

$$\|u\|_{L^\infty} \leq \frac{\|f\|_{L^\infty}}{\lambda}, \quad (8)$$

$$\|Du\|_{L^\infty} \leq K, \quad (9)$$

and

$$\|\Delta u\|_{L^\infty} \leq 2\|f\|_{L^\infty} + \frac{K^2}{2} =: M. \quad (10)$$

Furthermore, if $\lambda > M$, then

$$\|m\|_{L^\infty} \leq \frac{\lambda}{\lambda - M} \|m_0\|_{L^\infty}. \quad (11)$$

Proof. Inequality (8) is a direct consequence of the comparison principle for u and (9) is given by [39, Theorem 1.1], since any classical solution is also a continuous viscosity solution. The estimate (10) directly follows from the equation satisfied by u . For the last inequality (11), notice that we may rewrite the second equation in (1) as

$$-\Delta m - Du \cdot Dm + (\lambda - \Delta u)m = \lambda m_0 \quad \text{in } \mathbb{T}^d.$$

Therefore when $\lambda > M$ we deduce from (10) that $\lambda - \Delta u > 0$ and therefore the Fokker-Planck equation satisfies a comparison principle from which the inequality follows. \square

2.2 Stable solutions

The following definition is taken from [13].

Definition 2.5 (Stable solutions). Let $f \in C_b^1(\mathbb{R})$ and let (u, m) be a classical solution to (1). We say that (u, m) is stable if $(v, \rho) = (0, 0)$ is the unique classical solution to

$$\begin{cases} -\Delta v + Du \cdot Dv + \lambda v = f'(m)\rho & \text{in } \mathbb{T}^d, \\ -\Delta \rho - \operatorname{div}(\rho Du) + \lambda \rho = \operatorname{div}(mDv) & \text{in } \mathbb{T}^d. \end{cases} \quad (12)$$

Definition 2.6. Let $f \in C_b^1(\mathbb{R})$ and let (u, m) be a classical solution to (1). A pair $(v, \rho) \in H^1(\mathbb{T}^d) \times L^2(\mathbb{T}^d)$ is a weak solution to (12) if it satisfies

$$\int Dv \cdot D\varphi + Du \cdot Dv\varphi + \lambda v\varphi \, dx = \int f'(m)\rho\varphi \, dx \quad \text{for every } \varphi \in H^1(\mathbb{T}^d) \quad (13)$$

and

$$\int (-\Delta\psi + Du \cdot D\psi + \lambda\psi)\rho \, dx = - \int mDv \cdot D\psi \, dx \quad \text{for every } \psi \in H^2(\mathbb{T}^d). \quad (14)$$

Lemma 2.7. Let $f \in C_b^1(\mathbb{R}) \cap C_{\text{loc}}^{1,1}(\mathbb{R})$ and $(u, m) \in C^{2,\alpha}(\mathbb{T}^d) \times C^{2,\alpha}(\mathbb{T}^d)$ be a classical solution to (1) and $(v, \rho) \in H^1(\mathbb{T}^d) \times L^2(\mathbb{T}^d)$ be a weak solution to (12). Then $(v, \rho) \in C^{2,\alpha}(\mathbb{T}^d) \times C^{2,\alpha}(\mathbb{T}^d)$ and is a classical solution to (12).

Proof. Notice first that if $\rho \in C^{0,\alpha}(\mathbb{T}^d)$, then by Schauder estimates [28, Corollary 6.3] we have $v \in C^{2,\alpha}(\mathbb{T}^d)$ and then also $\rho \in C^{2,\alpha}(\mathbb{T}^d)$. It is therefore enough to prove that if $(v, \rho) \in H^1(\mathbb{T}^d) \times L^2(\mathbb{T}^d)$ is a weak solution to (12) then $\rho \in C^{0,\alpha}(\mathbb{T}^d)$. For this, let $1 < p < \infty$, depending on d and α , be such that Morrey's inequality yields $W^{1,p}(\mathbb{T}^d) \hookrightarrow C^{0,\alpha}(\mathbb{T}^d)$. It is now enough to prove that $v \in W^{1,p}(\mathbb{T}^d)$ which implies $\rho \in W^{1,p}(\mathbb{T}^d)$ by Lemma 2.2 (iii).

Since $\rho \in L^2(\mathbb{T}^d)$ and f' is bounded we deduce from standard elliptic regularity that $v \in H^2(\mathbb{T}^d)$. Therefore in the case $d = 1, 2$ the fact that $v \in W^{1,p}(\mathbb{T}^d)$ directly follows from Sobolev's inequality. In the rest of the proof we assume $d \geq 3$.

Since $v \in H^2(\mathbb{T}^d)$ we have that $\operatorname{div}(mDv) = Dm \cdot Dv + m\Delta v$ is an element of $L^2(\mathbb{T}^d)$. It is then well known that the second equation in (12) has a weak solution $\tilde{\rho} \in H^2(\mathbb{T}^d)$. Using the property (i) in Lemma 2.2, we deduce that in fact $\tilde{\rho} = \rho$ and therefore $\rho \in H^2(\mathbb{T}^d)$. In the case where $d = 3, 4$ we have $\rho \in L^r(\mathbb{T}^d)$ for every $2 \leq r < \infty$ and in particular $\rho \in L^p(\mathbb{T}^d)$. Injecting this information in the equation satisfied by v we conclude that $v \in W^{2,p}(\mathbb{T}^d)$ (see [28, Theorem 9.11]) and the conclusion follows. We may therefore assume that $d \geq 5$ and in this case Sobolev's inequality yields $\rho \in L^{q_1}(\mathbb{T}^d)$, where $q_1 = \frac{2d}{d-4}$ and, arguing as above, that $v \in W^{2,q_1}(\mathbb{T}^d)$.

We now know that $\operatorname{div}(mDv)$ belongs to $L^{q_1}(\mathbb{T}^d)$. One may keep this bootstrap argument going and conclude that either we can obtain that $\rho \in W^{2,d/2}(\mathbb{T}^d)$ after a finite number of steps, or there exists a sequence of real numbers $2 \leq q_n < d/2$, defined by

$$\begin{cases} q_{n+1} = \frac{dq_n}{d-2q_n}, \\ q_0 = 2, \end{cases}$$

and such that $\rho \in W^{2,q_n}(\mathbb{T}^d)$ for every $n \geq 1$.

We claim that the latter case cannot happen. Indeed, if it were the case, we notice that $\frac{q_{n+1}}{q_n} = \frac{d}{d-2q_n} > 1$ so that the sequence is increasing. In particular $q_n \geq 2$ for every n and therefore $\frac{q_{n+1}}{q_n} > \frac{d}{d-4}$. It follows that $\frac{d}{2} > q_n \geq 2 \left(\frac{d}{d-4}\right)^n$ which yields a contradiction.

We must therefore have $\rho \in W^{2,d/2}(\mathbb{T}^d)$ after a finite number of steps and hence, also, $\rho \in L^s(\mathbb{T}^d)$ for every $s \in [1, \infty)$. Using elliptic regularity one more time we have $v \in W^{1,p}(\mathbb{T}^d)$. This concludes the proof according to the discussion at the beginning of the argument. \square

The following proposition states that stable solutions to (1), although not unique in general, are isolated. The result is a straightforward adaptation of [13, Proposition 4.2] with weaker norms. We provide the proof in Appendix B.

Proposition 2.8 (Stable solutions are isolated). *Let $f \in C_b^1(\mathbb{R})$, let (u, m) be a stable solution to (1), and let $p = \max\{2, d\}$. Then there exists $R > 0$ such that, if $(\tilde{u}, \tilde{m}) \neq (u, m)$ is another classical solution to (1), then*

$$\|u - \tilde{u}\|_{W^{1,p}} + \|m - \tilde{m}\|_{L^2} > R.$$

The next result provides two sufficient conditions for the stability of solutions to (1).

Theorem 2.9. *Let $f \in C_b^1(\mathbb{R})$ and (u, m) be a classical solution to (1). Then (u, m) is stable provided that one of the following conditions holds:*

- (i) (Monotonicity of the coupling) $f' \geq 0$.
- (ii) (Large discount factor) $\lambda > \Lambda$, where

$$\Lambda := \max \left\{ 2 \|m_0\|_{L^\infty}, \frac{K^2}{2} + \|m_0\|_{L^\infty} \|f'\|_{L^\infty} \right\} \quad (15)$$

and the constant K is given in Proposition 2.4.

Proof. Let (v, ρ) be a weak solution to (12). Using Lemma 2.7 we may assume that $(v, \rho) \in C^2(\mathbb{T}^d) \times C^2(\mathbb{T}^d)$. Using ρ as a test-function in (13) and v as a test-function in (14) we get

$$\int Dv \cdot D\rho + \rho Du \cdot Dv + \lambda v \rho \, dx = \int f'(m) \rho^2 \, dx, \quad (16)$$

and

$$\int D\rho \cdot Dv + \rho Du \cdot Dv + \lambda \rho v \, dx = \int -m |Dv|^2 \, dx. \quad (17)$$

Subtracting (17) from (16) we obtain

$$\int m |Dv|^2 \, dx = - \int f'(m) |\rho|^2 \, dx. \quad (18)$$

In the case where $f' \geq 0$, and since m is nonnegative (see Lemma 2.2), this implies that $Dv = 0$ on the set where $m > 0$, in particular $mDv = 0$ on \mathbb{T}^d . Using Lemma 2.2, we deduce that $\rho = 0$ on \mathbb{T}^d . Using the uniqueness of the solution to the equation satisfied by v , we conclude that $v = 0$.

We now assume that $\lambda > \Lambda$. Notice that this and Proposition 2.4 yield $\|m\|_{L^\infty} \leq 2 \|m_0\|_{L^\infty}$. Then, using ρ as a test function in (14), it follows from (9), the positivity of m and (18), that

$$\begin{aligned} \int |D\rho|^2 + \lambda |\rho|^2 \, dx &\leq \|Du\|_{L^\infty} \|\rho\|_{L^2} \|D\rho\|_{L^2} + \|D\rho\|_{L^2} \|mDv\|_{L^2} \\ &\leq K \|\rho\|_{L^2} \|D\rho\|_{L^2} + \|m\|_{L^\infty}^{1/2} \|D\rho\|_{L^2} \left\| m^{1/2} Dv \right\|_{L^2} \\ &\leq K \|\rho\|_{L^2} \|D\rho\|_{L^2} + \|m\|_{L^\infty}^{1/2} \|D\rho\|_{L^2} \left(- \int f'(m) |\rho|^2 \right)^{1/2} \\ &\leq \|D\rho\|_{L^2}^2 + \frac{K^2}{2} \|\rho\|_{L^2}^2 + \frac{1}{2} \|f'\|_{L^\infty} \|m\|_{L^\infty} \|\rho\|_{L^2}^2 \\ &\leq \|D\rho\|_{L^2}^2 + \left(\frac{K^2}{2} + \|m_0\|_{L^\infty} \|f'\|_{L^\infty} \right) \|\rho\|_{L^2}^2. \end{aligned}$$

From our assumption on λ , we obtain $\|\rho\|_{L^2} = 0$ and the conclusion follows as in the first case. \square

3 Reformulation of the MFG system

We are now going to reformulate (1) as an abstract equation $F(u, m) = 0$. In order to do this let us introduce an additional assumption which will be useful to state some results in a general form.

(H) X and Z are Banach spaces such that

$$C^{2,\alpha}(\mathbb{T}^d) \times C^{2,\alpha}(\mathbb{T}^d) \subset X \subset H^1(\mathbb{T}^d) \times L^2(\mathbb{T}^d) \quad \text{and} \quad Z \subset L^1(\mathbb{T}^d) \times (W^{1,\infty}(\mathbb{T}^d))'. \quad (19)$$

with continuous embeddings. For every $(v, \rho) \in X$, we have

$$G(v, \rho) := \left(\frac{1}{2} |Dv|^2 - f(\rho), -\operatorname{div}(\rho Du) - \lambda m_0 \right) \quad \text{belongs to } Z \quad (20)$$

and, for every $(\xi, \zeta) \in Z$, the equation

$$\begin{cases} -\Delta v + \lambda v = \xi & \text{in } \mathbb{T}^d, \\ -\Delta \rho + \lambda \rho = \zeta & \text{in } \mathbb{T}^d, \end{cases} \quad (21)$$

admits a unique distributional solution $T(\xi, \zeta)$ that belongs to X .

Assumption (H) allows to define two mappings $G: X \rightarrow Z$ and $T: Z \rightarrow X$ such that, at least formally, (u, m) solves (1) if and only

$$F(u, m) := (u, m) + T(G(u, m)) = 0. \quad (22)$$

This is made rigorous in the following result.

Proposition 3.1. *Assume that $f \in W^{1,\infty}(\mathbb{R})$ and that (H) holds. A pair $(u, m) \in C^{2,\alpha}(\mathbb{T}^d) \times C^{2,\alpha}(\mathbb{T}^d)$ solves (1) if and only if it satisfies (22).*

Proof. Assume that $(u, m) \in C^{2,\alpha}(\mathbb{T}^d) \times C^{2,\alpha}(\mathbb{T}^d)$ is a solution to (1). We can rewrite (1) as

$$\begin{cases} -\Delta u + \lambda u = f(m) - \frac{1}{2} |Du|^2 & \text{in } \mathbb{T}^d, \\ -\Delta m + \lambda m = \lambda m_0 + \operatorname{div}(m Du) & \text{in } \mathbb{T}^d, \end{cases}$$

or, with an obvious abuse of notation,

$$(-\Delta + \lambda I)(u, m) = -G(u, m). \quad (23)$$

We can apply $T = (-\Delta + \lambda I)^{-1}$ on both sides of (23) and use the linearity of T to obtain that

$$(u, m) = -T(G(u, m)).$$

Conversely if (u, m) satisfies (22) we can apply the operator $(-\Delta + \lambda I)$ to get (23). \square

Let now (u, m) be a classical solution to (1) and X and Z be Banach spaces such that (H) holds and $(u, m) \in X$. Formally, the Fréchet differential of the mapping F defined in (22) at (u, m) is given by

$$dF[u, m](v, \rho) := (v, \rho) + T(dG[u, m](v, \rho)) \quad (24)$$

where

$$dG[u, m](v, \rho) = (Du \cdot Dv - f'(m)\rho, -\operatorname{div}(\rho Du) - \operatorname{div}(mDv)). \quad (25)$$

The rigorous proof of Fréchet differentiability will be made in Section 4 below. Notice that (v, ρ) solves (12) if and only if

$$dF[u, m](v, \rho) = 0. \quad (26)$$

Theorem 3.2 (Isomorphism property of stable solutions). *Let $f \in C_b^1(\mathbb{R})$ and let X and Z be Banach spaces such that (H) holds and let Y be a Banach space such that*

$$Y \subset X \subset H^1(\mathbb{T}^d) \times L^2(\mathbb{T}^d) \quad \text{with} \quad Y \subset X \text{ compact.}$$

Assume also that the mapping $G: X \rightarrow Z$ defined by (20) is continuously differentiable and that $T \in \mathcal{L}(Z, Y)$. Then the mapping $F: X \rightarrow X$ defined by (22) is continuously differentiable with $dF = I + T \circ dG$, and, for every stable solution $(u, m) \in C^{2,\alpha}(\mathbb{T}^d) \times C^{2,\alpha}(\mathbb{T}^d)$ to (1), the linear operator $dF[u, m]$ is an isomorphism on X .

Proof. The differentiability of F is a direct consequence of the chain rule and the differentiability of G . If $(u, m) \in C^{2,\alpha}(\mathbb{T}^d) \times C^{2,\alpha}(\mathbb{T}^d)$ is a stable solution to (1), then $(I + T \circ dG[u, m])$ is injective on $C^{2,\alpha}(\mathbb{T}^d) \times C^{2,\alpha}(\mathbb{T}^d)$. By Lemma 2.7 we know that any weak solution $(v, \rho) \in H^1(\mathbb{T}^d) \times L^2(\mathbb{T}^d)$ to (12) belongs to $C^{2,\alpha}(\mathbb{T}^d) \times C^{2,\alpha}(\mathbb{T}^d)$. Hence the operator $(I + T \circ dG[u, m])$ is also injective on $H^1(\mathbb{T}^d) \times L^2(\mathbb{T}^d)$, in particular it is injective on X .

Notice now that since $dG[u, m] \in \mathcal{L}(X, Z)$ and $T \in \mathcal{L}(Z, Y)$ we have that $T \circ dG[u, m] \in \mathcal{L}(X, Y)$ and since $Y \subset X$ is compact we deduce that the operator $T \circ dG[u, m] \in \mathcal{L}(X)$ is compact. Using Fredholm's alternative, we conclude that $dF[u, m]$ is an isomorphism on X . \square

We now provide two concrete examples of Banach spaces X , Y , and Z , satisfying (H) and the assumptions of Theorem 3.2. These examples will constitute the building blocks of the applications studied in Section 5.

Example 3.3. Let $f \in C_b^1(\mathbb{R})$, let $0 < \gamma < \beta < \alpha$ and set

$$X = C^{2,\beta}(\mathbb{T}^d) \times C^{2,\gamma}(\mathbb{T}^d), \quad Y = C^{2,\alpha}(\mathbb{T}^d) \times C^{2,\beta}(\mathbb{T}^d) \quad \text{and} \quad Z = C^{0,\alpha}(\mathbb{T}^d) \times C^{0,\beta}(\mathbb{T}^d)$$

where α is fixed in (5).

Let us check that (H) and the assumptions of Theorem 3.2 are satisfied in this case. From the Arzela-Ascoli theorem we have the compact embedding of Y into X . It is also clear that $X \subset H^1(\mathbb{T}^d) \times L^2(\mathbb{T}^d)$. We fix some stable solution $(u, m) \in C^{2,\alpha}(\mathbb{T}^d) \times C^{2,\alpha}(\mathbb{T}^d)$ to (1). From the Schauder estimates [28, Corollary 6.3] we have $T \in \mathcal{L}(Z, Y)$. It remains to prove that $G: X \rightarrow Z$ is well-defined. First notice that

$$\left\| |Dv|^2 \right\|_{C^{0,\alpha}} \leq 2 \|Dv\|_{C^{0,\alpha}}^2 \leq C \|v\|_{C^{2,\beta}}^2.$$

Since f is Lipschitz continuous we also have

$$\|f(\rho)\|_{C^{0,\alpha}} \leq \|f\|_{W^{1,\infty}} \|\rho\|_{C^{0,\alpha}} \leq C \|f\|_{W^{1,\infty}} \|\rho\|_{C^{2,\gamma}}$$

Finally, witting

$$\operatorname{div}(\rho Dv) = D\rho \cdot Dv + \rho \Delta v,$$

we obtain that

$$\|\operatorname{div}(\rho Dv)\|_{C^{0,\beta}} \leq 2 \|D\rho\|_{C^{0,\beta}} \|Dv\|_{C^{0,\beta}} + 2 \|\rho\|_{C^{0,\beta}} \|\Delta v\|_{C^{0,\beta}} \leq C \|\rho\|_{C^{2,\gamma}} \|v\|_{C^{2,\beta}}.$$

In conclusion we have

$$\|G(v, \rho)\|_Z \leq C \left(\|v\|_{C^{2,\beta}}^2 + \|\rho\|_{C^{2,\gamma}}^2 \right) \leq C \|(v, \rho)\|_X^2,$$

and the mapping G is therefore well defined from X to Z .

In the case where G is also continuously differentiable we have $dG: X \rightarrow \mathcal{L}(X, Z)$ and we can apply Theorem 3.2 to conclude that $dF[u, m]$ is an isomorphism on $C^{2,\beta}(\mathbb{T}^d) \times C^{2,\gamma}(\mathbb{T}^d)$.

Example 3.4. Let $f \in C_b^1(\mathbb{R})$ and

$$X = W^{1,p}(\mathbb{T}^d) \times L^q(\mathbb{T}^d), \quad Y = W^{2,p/2}(\mathbb{T}^d) \times W^{1,r}(\mathbb{T}^d) \quad \text{and} \quad Z = L^{p/2}(\mathbb{T}^d) \times W^{-1,r'}(\mathbb{T}^d)$$

where $d < p, q < \infty, r, r' > 1$ are such that $1/r \geq 1/p + 1/q, 1/r + 1/r' = 1$ and we have a compact embedding $W^{1,r}(\mathbb{T}^d) \hookrightarrow L^q(\mathbb{T}^d)$.

We now verify that **(H)** and the assumptions in Theorem 3.2 are satisfied. Since we assume $p > d$ the Rellich-Kondrachov theorem gives the compact embedding $W^{2,p/2}(\mathbb{T}^d) \hookrightarrow W^{1,p}(\mathbb{T}^d)$. It follows that there is a compact embedding from Y into X . Using $W^{2,p}$ estimates [28, Theorem 9.11] and the $W^{1,p}$ estimate (6) from Lemma 2.2, and using Remark 2.3, we have that $T \in \mathcal{L}(Z, W^{2,p/2}(\mathbb{T}^d) \times W^{1,r}(\mathbb{T}^d))$. Therefore $T \in \mathcal{L}(Z, Y) \subset \mathcal{L}(Z, X)$. Moreover it is easy to check that $G: X \rightarrow Z$ is well defined.

Therefore in the case where G is continuously differentiable we obtain from Theorem 3.2 that $dF[u, m]$ is an isomorphism on $W^{1,p}(\mathbb{T}^d) \times L^q(\mathbb{T}^d)$.

We now turn to the particular case of a monotone coupling and prove that the unique classical solution to (1), which we also know to be stable by Theorem 2.9, satisfies a stronger isomorphism property than the one resulting from the direct application of Theorem 3.2.

Theorem 3.5 (Isomorphism property for monotone couplings). *Let $f \in C_b^1(\mathbb{R}) \cap C_{loc}^{1,1}(\mathbb{R})$. Assume that $f' \geq 0$ and let $(u, m) \in C^{2,\alpha}(\mathbb{T}^d) \times C^{2,\alpha}(\mathbb{T}^d)$, where α is fixed in (5), be the unique classical solution to (1). Then **(H)** holds for $X = C^{2,\alpha}(\mathbb{T}^d) \times C^{2,\alpha}(\mathbb{T}^d)$ and $Z = C^{0,\alpha}(\mathbb{T}^d) \times C^{0,\alpha}(\mathbb{T}^d)$. Moreover, assume that G in **(H)** is continuously differentiable and define F according to (22). Then $dF[u, m]$ is an isomorphism on X .*

Proof. According to Theorem 2.9, (u, m) is a stable solution to (1). Therefore the injectivity of $dF[u, m]$ follows from the definition of stable solutions.

We now prove its surjectivity. More precisely we prove that for every $(w, \mu) \in C^{2,\alpha}(\mathbb{T}^d) \times C^{2,\alpha}(\mathbb{T}^d)$ there exists $(v, \rho) \in C^{2,\alpha}(\mathbb{T}^d) \times C^{2,\alpha}(\mathbb{T}^d)$ such that

$$dF[u, m](v, \rho) = (w, \mu), \tag{27}$$

which is equivalent to showing that there exists a unique solution $(v, \rho) \in C^{2,\alpha}(\mathbb{T}^d) \times C^{2,\alpha}(\mathbb{T}^d)$ to

$$\begin{cases} -\Delta v + Du \cdot Dv + \lambda v = f'(m)\rho - \Delta w + \lambda w & \text{in } \mathbb{T}^d, \\ -\Delta \rho - \operatorname{div}(\rho Du) + \lambda \rho = \operatorname{div}(mDv) - \Delta \mu + \lambda \mu & \text{in } \mathbb{T}^d. \end{cases} \tag{28}$$

The argument relies on the Leray-Schauder fixed point theorem and is adapted from [17, Lemma 3.4]. Let us first define the mapping for which we will find a fixed point. Fix some $\rho \in L^2(\mathbb{T}^d)$. Then there exists a solution $v \in H^2(\mathbb{T}^d)$ to

$$-\Delta v + Du \cdot Dv + \lambda v = f'(m)\rho - \Delta w + \lambda w \quad \text{in } \mathbb{T}^d,$$

and a unique weak solution $\tilde{\rho} \in H^1(\mathbb{T}^d)$ to

$$-\Delta \tilde{\rho} - \operatorname{div}(\tilde{\rho} Du) + \lambda \tilde{\rho} = \operatorname{div}(mDv) - \Delta \mu + \lambda \mu \quad \text{in } \mathbb{T}^d.$$

This allows us to define a mapping $\Phi: L^2(\mathbb{T}^d) \rightarrow L^2(\mathbb{T}^d)$ by setting $\Phi(\rho) = \tilde{\rho}$. In order to apply the Leray-Schauder fixed point theorem [28, Theorem 11.3] we have to prove that the set of solutions $\rho \in L^2(\mathbb{T}^d)$ to $\rho = \sigma\Phi(\rho)$, where $\sigma \in [0, 1]$, is uniformly bounded in $L^2(\mathbb{T}^d)$. This amounts to prove a uniform bound on the solutions to

$$\begin{cases} -\Delta v + Du \cdot Dv + \lambda v = \sigma [f'(m)\rho - \Delta w + \lambda w] & \text{in } \mathbb{T}^d, \\ -\Delta \rho - \operatorname{div}(\rho Du) + \lambda \rho = \operatorname{div}(mDv) - \sigma [\Delta \mu - \lambda \mu] & \text{in } \mathbb{T}^d. \end{cases}$$

Using ρ as a test function for the equation satisfied by v we obtain that

$$\int Dv \cdot D\rho + \rho Du \cdot Dv + \lambda v \rho \, dx = \sigma \int f'(m)\rho^2 + \rho(\lambda w - \Delta w) \, dx. \quad (29)$$

Similarly, using v as a test function for the equation satisfied by ρ we obtain that

$$\int D\rho \cdot Dv + \rho Du \cdot Dv + \lambda \rho v \, dx = \int -m|Dv|^2 + \sigma(\lambda v \mu + Dv \cdot D\mu) \, dx. \quad (30)$$

Subtracting (30) to (29) we get

$$\int m|Dv|^2 \, dx = \sigma \int -f'(m)\rho^2 - \rho(\lambda w - \Delta w) + \lambda v \mu + Dv \cdot D\mu \, dx.$$

The positivity of f' then yields

$$\int m|Dv|^2 \, dx \leq C(\|\rho\|_{L^2} \|w\|_{H^2} + \|v\|_{H^1} \|\mu\|_{H^1}). \quad (31)$$

Let now $\xi \in H^{-1}(\mathbb{T}^d)$ and $z \in H^1(\mathbb{T}^d)$ be the unique weak solution to

$$-\Delta z + Du \cdot Dz + \lambda z = \xi \quad \text{in } \mathbb{T}^d.$$

Using z as a test function for the equation satisfied by ρ and recalling that $m \geq 0$ (see Lemma 2.2-(ii)) we get

$$\begin{aligned} \langle \xi, \rho \rangle_{H^{-1}, H^1} &= \int mDv \cdot Dz + \sigma(\lambda z \mu + D\mu \cdot Dz) \, dx \\ &\leq C \left(\|m^{1/2} Dz\|_{L^2} \|m^{1/2} Dv\|_{L^2} + \|z\|_{H^1} \|\mu\|_{H^1} \right) \\ &\leq C \|z\|_{H^1} \left((\|\rho\|_{L^2} \|w\|_{H^2} + \|v\|_{H^1} \|\mu\|_{H^1})^{1/2} + \|\mu\|_{H^1} \right) \\ &\leq C \|\xi\|_{H^{-1}} \left((\|\rho\|_{L^2} \|w\|_{H^2} + \|v\|_{H^1} \|\mu\|_{H^1})^{1/2} + \|\mu\|_{H^1} \right) \end{aligned}$$

where we used (31) to obtain the second inequality. Since ξ is arbitrary, we deduce by duality that

$$\|\rho\|_{H^1} \leq C \left((\|\rho\|_{L^2} \|w\|_{H^2} + \|v\|_{H^1} \|\mu\|_{H^1})^{1/2} + \|\mu\|_{H^1} \right),$$

and hence, by Young's inequality, we get

$$\|\rho\|_{H^1} \leq C \left(\|w\|_{H^2} + \|v\|_{H^1}^{1/2} \|\mu\|_{H^1}^{1/2} + \|\mu\|_{H^1} \right).$$

From elliptic regularity we know that

$$\|v\|_{H^2} \leq C(\|\rho\|_{L^2} + \|w\|_{H^2}) \leq C \left(\|w\|_{H^2} + \|v\|_{H^1}^{1/2} \|\mu\|_{H^1}^{1/2} + \|\mu\|_{H^1} \right),$$

and hence, after another application of Young's inequality, we get a uniform bound on v in $H^2(\mathbb{T}^d)$

$$\|v\|_{H^2} \leq C (\|w\|_{H^2} + \|\mu\|_{H^1}).$$

Using standard H^1 estimates we can now deduce a uniform on ρ in $H^1(\mathbb{T}^d)$

$$\|\rho\|_{H^1} \leq C (\|w\|_{H^2} + \|\mu\|_{H^1}).$$

The compactness on Φ then follows from the Rellich-Kondrachov theorem and we can therefore apply the Leray-Schauder fixed point theorem to obtain a pair $(v; \rho) \in H^2(\mathbb{T}^d) \times H^1(\mathbb{T}^d)$ solving (12).

Using a bootstrap argument similar to the one used in Lemma 2.7, we can obtain that $(v, \rho) \in C^{2,\alpha}(\mathbb{T}^d) \times C^{2,\alpha}(\mathbb{T}^d)$. □

4 Differentiability of the mapping F

In this section we prove the differentiability of the mapping F defined by (22) in both Hölder and Sobolev spaces. This corresponds to the situations considered in Examples 3.3 and 3.4.

4.1 Differentiability in Hölder spaces

The following proposition summarizes the differentiability properties of the Nemytskii operator on Hölder spaces which we will use to prove the differentiability of the mapping G .

Proposition 4.1. *Assume that $h \in C^2(\mathbb{R})$ and let $\beta \in (0, 1]$. Then the Nemytskii operator $H : C^{0,\beta}(\mathbb{T}^d) \rightarrow C^{0,\beta}(\mathbb{T}^d)$ defined by*

$$H[u](x) = h(u(x)) \quad \text{for } x \in \mathbb{T}^d$$

is continuously differentiable and $dH[u](v) = h'(u)v$. In particular, for every $\gamma \in (0, 1]$ the mapping

$$C^{2,\gamma}(\mathbb{T}^d) \ni u \mapsto H[u] \in C^{0,\beta}(\mathbb{T}^d)$$

is also continuously differentiable. Furthermore, if $h \in C_{\text{loc}}^{2,1}(\mathbb{R})$, then dH is locally Lipschitz continuous in $\mathcal{L}(C^{2,\gamma}(\mathbb{T}^d), C^{0,\beta}(\mathbb{T}^d))$.

Proof. The first statement is proved in [42, Theorem 4.1] (this is where we need the C^2 assumption on f). For the second one let us write $J \in \mathcal{L}(C^{2,\gamma}(\mathbb{T}^d), C^{0,\beta}(\mathbb{T}^d))$ for the natural injection of $C^{2,\gamma}(\mathbb{T}^d)$ into $C^{0,\beta}(\mathbb{T}^d)$. It is enough to notice that $H \circ J$ is continuously differentiable from the chain rule.

For the local Lipschitz continuity let $v \in C^{2,\gamma}(\mathbb{T}^d)$. We have

$$\|(h'(u_1) - h'(u_2))v\|_{C^{0,\beta}} \leq \|v\|_{C^{0,\beta}} \|h'(u_1) - h'(u_2)\|_{C^{0,\beta}} \leq \|v\|_{C^{2,\gamma}} \|h'(u_1) - h'(u_2)\|_{C^{0,\beta}}$$

so that

$$\|dH[u_1] - dH[u_2]\|_{\mathcal{L}(C^{2,\gamma}, C^{0,\beta})} \leq \|h'(u_1) - h'(u_2)\|_{C^{0,\beta}}.$$

Therefore dH is locally Lipschitz continuous as soon as

$$C^{2,\gamma}(\mathbb{T}^d) \ni u \mapsto h'(u) \in C^{0,\beta}(\mathbb{T}^d)$$

is locally Lipschitzian, which is the case if $h' \in C_{\text{loc}}^{1,1}$ according to [42, Theorem 3.1]. □

We are now going to prove the differentiability of the mapping F , defined by (22), in the situations described in Example 3.3 and Theorem 3.5.

Lemma 4.2. *Let $f \in C_b^1(\mathbb{R}) \cap C^2(\mathbb{R})$ and $0 < \gamma \leq \beta \leq \alpha$, where α is set in (5). Set $X = C^{2,\beta}(\mathbb{T}^d) \times C^{2,\gamma}(\mathbb{T}^d)$ and $Z = C^{0,\alpha}(\mathbb{T}^d) \times C^{0,\beta}(\mathbb{T}^d)$. Then the mapping $G: X \rightarrow Z$, defined by (20), is continuously differentiable and, for every $(u, m) \in X$, we have*

$$dG[u, m](v, \rho) = (Du \cdot Dv - f'(m)\rho, -\operatorname{div}(\rho Du) - \operatorname{div}(mDv)).$$

Furthermore, if $f \in C_{\text{loc}}^{2,1}(\mathbb{R})$, then dG is locally Lipschitz continuous.

Proof. From Proposition 4.1, we have that the differentials of

$$C^{2,\beta}(\mathbb{T}^d) \ni u \mapsto \frac{|Du|^2}{2} \in C^{0,\beta}(\mathbb{T}^d)$$

and

$$C^{2,\gamma}(\mathbb{T}^d) \ni m \mapsto f(m) \in C^{0,\beta}(\mathbb{T}^d),$$

are given by

$$C^{2,\beta}(\mathbb{T}^d) \ni v \mapsto Du \cdot Dv \in C^{0,\beta}(\mathbb{T}^d)$$

and

$$C^{2,\gamma}(\mathbb{T}^d) \ni \rho \mapsto f'(m)\rho \in C^{0,\beta}(\mathbb{T}^d),$$

respectively. The second component of dG being continuous and bilinear, its differentiability is also easy to check. The remaining conclusions follow from the assumptions on f and last assertion in Proposition 4.1. \square

Then fact that F is continuously differentiable when $f \in C_b^1(\mathbb{R}) \cap C^2(\mathbb{R})$ is then a direct consequence of Lemma 4.2 and the chain rule.

Proposition 4.3. *Let $f \in C_b^1(\mathbb{R}) \cap C^2(\mathbb{R})$ and let $0 < \gamma \leq \beta \leq \alpha$, where α is fixed in (5). Set $X = C^{2,\beta}(\mathbb{T}^d) \times C^{2,\gamma}(\mathbb{T}^d)$ and $Z = C^{0,\alpha}(\mathbb{T}^d) \times C^{0,\beta}(\mathbb{T}^d)$. Then the mapping $F: X \rightarrow X$ defined by (22) is continuously differentiable with dF given by (24). Furthermore, if $f \in C_{\text{loc}}^{2,1}(\mathbb{R})$, then dF is locally Lipschitz continuous in $\mathcal{L}(X)$.*

4.2 Differentiability in Sobolev spaces

We begin with a preliminary result on the differentiability of the Nemytskii operator on Lebesgue spaces.

Proposition 4.4. *Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space and let $1 < p, q, r < \infty$ with $q < p$ and $1/r = 1/q - 1/p$. Consider a function $h \in C^1(\mathbb{R})$ such that*

$$|h'(x)| \leq C \left(1 + |x|^{p/r}\right) \quad (32)$$

and define the Nemytskii operator $H: L^p(\Omega, \mu) \rightarrow L^q(\Omega, \mu)$ by

$$H[u](x) = h(u(x)) \quad \text{for } x \in \Omega \quad \text{for } \mu\text{-a.e. } x \in \Omega.$$

Then H is continuously differentiable with $dH[u](v) = h'(u)v$.

Proof. Let $u, v \in L^p(\Omega, \mu)$. Since $h \in C^1(\mathbb{R})$, we have

$$h(u(x) + v(x)) - h(u(x)) = v(x) \int_0^1 h'(u(x) + tv(x)) dt$$

for μ -a.e. $x \in \Omega$. Therefore, it follows from Holder's and Jensen's inequalities as well as Fubini's theorem that

$$\begin{aligned} \|H[u+v] - H[u] - h'(u)v\|_{L^q}^q &= \left\| v \int_0^1 h'(u+tv) - h'(u) dt \right\|_{L^q}^q \\ &\leq \|v\|_{L^p}^q \left\| \int_0^1 h'(u+tv) - h'(u) dt \right\|_{L^r}^q \\ &\leq \|v\|_{L^p}^q \int_0^1 \|h'(u+tv) - h'(u)\|_{L^r}^q dt. \end{aligned}$$

It follows that H is Fréchet differentiable at u if we are able to prove that

$$\lim_{n \rightarrow \infty} \sup_{t \in [0,1]} \|h'(u+tv_n) - h'(u)\|_{L^r} = 0. \quad (33)$$

for every sequence v_n in $L^p(\Omega, \mu)$ converging to 0. Let v_n be such a sequence, we may assume that the convergence also holds in the μ -almost everywhere sense. Using the uniform continuity of h' on the set $\bar{B}(u(x), \sup_n |v_n(x)|)$ we obtain that

$$\lim_{n \rightarrow \infty} \sup_{t \in [0,1]} |h'(u(x) + tv_n(x)) - h'(u(x))|^r = 0 \quad \text{for } \mu\text{-a.e. } x \in \Omega. \quad (34)$$

For every $\sigma > 1$ we denote by c_σ the smallest positive constant such that

$$(a+b)^\sigma \leq c_\sigma(a^\sigma + b^\sigma) \quad \text{for every } a, b \geq 0.$$

Using (32) we have, for every $t \in [0, 1]$,

$$\begin{aligned} |h'(u+tv_n) - h'(u)|^r &\leq c_r (|h'(u+tv_n)|^r + |h'(u)|^r) \\ &\leq c_r \left(\left(C(1 + |u+tv_n|^{p/r}) \right)^r + \left(C(1 + |u|^{p/r}) \right)^r \right) \\ &\leq c_r^2 C^r (2 + |u+tv_n|^p + |u|^p) \\ &\leq c_r^2 C^r (2 + c_p |v_n|^p + (1 + c_p) |u|^p) \end{aligned}$$

so that

$$\sup_{t \in [0,1]} |h'(u+tv_n) - h'(u)|^r \leq c_r^2 C^r (2 + c_p |v_n|^p + (1 + c_p) |u|^p) \quad \mu\text{-a.e.} \quad (35)$$

Clearly the right-hand side in (35) converges in $L^1(\Omega; \mu)$ and is therefore uniformly integrable according to the Lebesgue-Vitali theorem [7, Theorem 4.5.4]. This proves that the left-hand side of (35) is uniformly integrable. Combining this last point with (34), we can use the other implication in the Lebesgue-Vitali theorem to establish

$$\lim_{n \rightarrow \infty} \int \sup_{t \in [0,1]} |h'(u+tv_n) - h'(u)|^r \mu(dx) = 0.$$

Since

$$\sup_{t \in [0,1]} \int |h'(u+tv_n) - h'(u)|^r \mu(dx) \leq \int \sup_{t \in [0,1]} |h'(u+tv_n) - h'(u)|^r \mu(dx)$$

we deduce (33). This proves the differentiability of H .

For the continuity of dH , let u_n be a sequence in $L^p(\Omega, \mu)$ converging to $u \in L^p(\Omega, \mu)$. Notice that

$$\|dH[u_n] - dH[u]\|_{\mathcal{L}(L^p, L^q)} \leq \|h'(u_n) - h'(u)\|_{L^r}.$$

The continuity can then be proven using the Lebesgue-Vitali theorem similarly to what we did for differentiability. \square

We now come back to the differentiability of the mapping F for the situation described in Example 3.4.

Proposition 4.5. *Let $f \in C_b^1(\mathbb{R}) \cap C_{\text{loc}}^{1,1}(\mathbb{R})$, $X = W^{1,p}(\mathbb{T}^d) \times L^q(\mathbb{T}^d)$, and $Z = L^{p/2}(\mathbb{T}^d) \times W^{-1,r'}(\mathbb{T}^d)$, where $d \leq p, q < \infty$, $q > p/2$, and $1/r \geq 1/p + 1/q$. Assume also that $W^{1,r}(\mathbb{T}^d) \hookrightarrow L^q(\mathbb{T}^d)$. Then the mapping $F: X \rightarrow X$ defined by (22) is continuously differentiable.*

Proof. Under these assumptions, we recall from Example 3.4 that **(H)** holds. From Proposition 4.4 we have that the mappings

$$L^q(\mathbb{T}^d) \ni m \mapsto f(m) \in L^{p/2}(\mathbb{T}^d)$$

and

$$W^{1,p}(\mathbb{T}^d) \ni u \mapsto |Du|^2 \in L^{p/2}(\mathbb{T}^d)$$

are continuously differentiable. Moreover

$$W^{1,p}(\mathbb{T}^d) \times L^q(\mathbb{T}^d) \ni (u, m) \mapsto \text{div}(mDu) \in W^{-1,r}(\mathbb{T}^d)$$

is also continuously differentiable as a continuous bilinear operator. The differentiability of F then follows from the chain rule. \square

5 Applications

In this section we provide three applications of the isomorphism property of stable solutions to (1).

5.1 Stability under perturbations of the MFG system

In this section, our goal is to study perturbations of the mean field game system (1), or equivalently (22). In the case of stable solutions, the isomorphism property obtained in Theorems 3.2 and 3.5 motivates the use of the implicit function theorem, allowing us to consider a large class of perturbations. In what follows we provide a simple example of this idea. Namely, we consider the system

$$\begin{cases} -\Delta u + \frac{1}{2}|Du|^2 + \lambda u = f(m) + \epsilon \hat{f}(m) & \text{in } \mathbb{T}^d, \\ -\Delta m - \text{div}(mDu) + \lambda m = \lambda((1-\epsilon)m_0 + \epsilon m_1) & \text{in } \mathbb{T}^d, \end{cases} \quad (36)$$

where $\epsilon > 0$ is a small parameter and $\hat{f} \in C^2((0, +\infty))$ and $m_1 \in C^{0,\alpha}(\mathbb{T}^d) \cap \mathcal{P}(\mathbb{T}^d)$ are perturbations of $f \in C_b^1(\mathbb{R}) \cap C^2(\mathbb{R})$ and m_0 , respectively. We recall that α is fixed in (5).

Let $(u, m) \in C^{2,\alpha}(\mathbb{T}^d) \times C^{2,\alpha}(\mathbb{T}^d)$ be a stable solution to (1) and set $0 < \gamma < \beta < \alpha$. Since $m_0 \neq 0$ we also have $m > 0$ from Lemma 2.2 (ii). In particular, there exists a bounded neighborhood \mathcal{O} of m in $C^{2,\gamma}(\mathbb{T}^d)$ and $\eta > 0$ such that, for every $\tilde{m} \in \mathcal{O}$, we have $\tilde{m} \geq \eta$. Notice that if we define

$$E_{\mathcal{O}} := \{ \tilde{m}(x) : \tilde{m} \in \mathcal{O}, x \in \mathbb{T}^d \},$$

then we have that $E_{\mathcal{O}}$ is bounded in \mathbb{R} and $\inf E_{\mathcal{O}} \geq \eta > 0$. Since $\hat{f} \in C^2((0, +\infty))$ is Lipschitz continuous on $E_{\mathcal{O}}$, we deduce from Proposition 4.1 that the mapping $\mathcal{O} \ni \tilde{m} \mapsto \hat{f}(\tilde{m}) \in C^{0,\beta}(\mathbb{T}^d)$ is well-defined and continuously differentiable.

Let $X = C^{2,\beta}(\mathbb{T}^d) \times C^{2,\gamma}(\mathbb{T}^d)$, $Z = C^{0,\beta}(\mathbb{T}^d) \times C^{0,\gamma}(\mathbb{T}^d)$ and consider the mappings G and T , defined by (20) and (21), respectively. We recall that it was checked in Example 3.3 that **(H)** holds in this case. We introduce the mapping

$$\hat{G}: C^{2,\beta}(\mathbb{T}^d) \times \mathcal{O} \times \mathbb{R}_+ \rightarrow C^{0,\beta}(\mathbb{T}^d) \times C^{0,\gamma}(\mathbb{T}^d)$$

defined by

$$\hat{G}(u, m, \epsilon) = G(u, m) - \epsilon(\hat{f}(m), \lambda(m_1 - m_0)).$$

Then, setting

$$\hat{F}(u, m, \epsilon) = (\hat{I} + T \circ \hat{G})(u, m, \epsilon),$$

where T is defined by (21) and $\hat{I}(u, m, \epsilon) = (u, m)$, it holds that $(u_\epsilon, m_\epsilon) \in X$ solves (36) if and only if

$$\hat{F}(u_\epsilon, m_\epsilon, \epsilon) = 0.$$

Arguing as in the proof of Proposition 4.3, we have that \hat{F} is continuously differentiable on $C^{2,\beta}(\mathbb{T}^d) \times \mathcal{O} \times [0, +\infty)$. Moreover, it follows from Theorem 3.2 and Example 3.3 that $d_{(u,m)}\hat{F}[u, m, 0] = dF[u, m]$ is an isomorphism on X . We can apply the implicit function theorem to obtain the following result.

Proposition 5.1 (Sensitivity analysis). *Let $f \in C_b^1(\mathbb{R}) \cap C^2(\mathbb{R})$, $\hat{f} \in C^2((0, +\infty))$, $m_1 \in C^{0,\alpha}(\mathbb{T}^d) \cap \mathcal{P}(\mathbb{T}^d)$, $0 < \gamma < \beta < \alpha$, where α is fixed in (5), and $(u, m) \in C^{2,\alpha}(\mathbb{T}^d) \times C^{2,\alpha}(\mathbb{T}^d)$ be a stable solution to (1). Then, for some $\epsilon_0 > 0$ and every $\epsilon \in [0, \epsilon_0)$, there exists $(u_\epsilon, m_\epsilon) \in C^{2,\beta}(\mathbb{T}^d) \times C^{2,\gamma}(\mathbb{T}^d)$ such that*

$$\hat{F}(u_\epsilon, m_\epsilon, \epsilon) = 0,$$

with

$$(u_\epsilon, m_\epsilon) = (u, m) - \epsilon dF(u, m)^{-1} T(\hat{f}(m), \lambda(m_1 - m_0)) + o(\epsilon),$$

and

$$\|u_\epsilon - u\|_{C^{2,\beta}} + \|m_\epsilon - m\|_{C^{2,\gamma}} = O(\epsilon).$$

Furthermore, (u_ϵ, m_ϵ) is a stable solution to (36).

Remark 5.2. We may choose $\alpha = \beta = \gamma$ in Proposition 5.1 under the additional assumption that $f' \geq 0$.

5.2 Finite Element approximation of the MFG system

Our goal here is to obtain error estimates for the finite element approximation of a stable classical solution to (1) by applying the following result of Brezzi-Rappaz-Raviart [12] (see also [29, Section IV.3]).

Theorem 5.3 ([29, Theorem 3.3 and Remark 3.5]). *Let V, W be Banach spaces, let $T, T_h \in \mathcal{L}(W, V)$, for every $h > 0$, and let $G: V \rightarrow W$ be a continuously differentiable mapping such that dG locally Lipschitz continuous in $\mathcal{L}(V, W)$. Set $F = I + T \circ G$ and let $x \in V$ be such that $F(x) = 0$ and $dF[x]$ is an isomorphism on V . If*

$$\lim_{h \rightarrow 0} \|T - T_h\|_{\mathcal{L}(W, V)} = 0, \tag{37}$$

then there exists $h_0 > 0$ and a neighborhood \mathcal{O} of x in V such that, for every $0 < h \leq h_0$, there exists $x_h \in V$ such that

$$F_h(x_h) := (I + T_h \circ G)(x_h) = 0. \tag{38}$$

Furthermore the following properties hold

- (i) $dF_h[x_h]$ is an isomorphism on V ,
- (ii) $x_h \in \mathcal{O}$ for every $0 < h \leq h_0$ and there is no other solution to (38) in \mathcal{O} ,
- (iii) there exists a constant $K > 0$, independent of h , such that

$$\|x - x_h\|_V \leq K \|(T - T_h)G(x)\|_V. \tag{39}$$

We fix $d \leq 3$ and consider the following Banach spaces

$$X = W^{1,p}(\mathbb{T}^d) \times L^q(\mathbb{T}^d) \quad \text{and} \quad Z = L^{p/2}(\mathbb{T}^d) \times H^{-1}(\mathbb{T}^d), \quad (40)$$

where $p, q \in (3, 6)$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$. Notice that under these assumptions we have $q > p/2$. Moreover the Rellich-Kondrachov theorem gives the compact embedding $H^1(\mathbb{T}^d) \hookrightarrow L^q(\mathbb{T}^d)$ since $q < 6$. Therefore all the assumptions in Example 3.4 are satisfied (with $r = 2$). In particular, the linear operator $T \in \mathcal{L}(Z, X)$, defined by (21), and the mapping $G: X \rightarrow Z$, given by (20), satisfy **(H)**. We set $r := \frac{dp}{p+d} < \min\{p, d\}$ and we notice that Sobolev's inequality implies that $W^{2,r}(\mathbb{T}^d) \hookrightarrow W^{1,p}(\mathbb{T}^d)$.

For every $h > 0$, let \mathcal{T}_h be a quasi-uniform family of periodic triangulations of $[0, 1]^d$ (see [9, Definition 4.4.13]). Let also $V_h \subset W^{1,\infty}(\mathbb{T}^d)$ be the associated finite element space induced by \mathbb{P}^1 Lagrange finite elements. We define $S_h \in \mathcal{L}(H^{-1}(\mathbb{T}^d), H^1(\mathbb{T}^d))$ by $S_h \xi = v_h$ where v_h is the unique element in V_h such that

$$\int_{\mathbb{T}^d} Dv_h \cdot D\phi_h + \lambda v_h \phi_h \, dx = \langle \xi, \phi_h \rangle_{H^{-1}, H^1} \quad \text{for every } \phi_h \in V_h.$$

In addition, we denote by $S \in \mathcal{L}(H^{-1}(\mathbb{T}^d), H^1(\mathbb{T}^d))$ the linear operator defined by $S\xi = v$, where $v \in H^1(\mathbb{T}^d)$ is the unique weak solution to

$$-\Delta v + \lambda v = \xi \quad \text{in } \mathbb{T}^d. \quad (41)$$

These linear operators are known to be well defined through the Lax-Milgram theorem and we have

$$\|S\|_{\mathcal{L}(H^{-1}, H^1)}, \|S_h\|_{\mathcal{L}(H^{-1}, H^1)} \leq \frac{1}{\min\{1, \lambda\}}. \quad (42)$$

We also have from [23, Theorem 3.16, Theorem 3.18] that

$$\|(S - S_h)\xi\|_{L^2} \leq Ch \|(S - S_h)\xi\|_{H^1} \leq Ch \|S\xi\|_{H^1} \leq Ch \|\xi\|_{H^{-1}}, \quad (43)$$

where the constant C is independent of h .

Let $\xi \in L^r(\mathbb{T}^d)$. According to [23, Theorem 3.21] and [9, Theorem 8.5.3], there exists $h_0 > 0$ and a positive constant C such that, for every $h \leq h_0$, it holds that

$$\|S_h \xi\|_{W^{1,p}} \leq C \|S\xi\|_{W^{1,p}}. \quad (44)$$

Using the continuous embedding $W^{2,r}(\mathbb{T}^d) \hookrightarrow W^{1,p}(\mathbb{T}^d)$ and $W^{2,p}$ estimates [28, Theorem 9.11], we have

$$\|S\xi\|_{W^{1,p}} \leq C \|S\xi\|_{W^{2,r}} \leq C \|\xi\|_{L^r}. \quad (45)$$

Combining (44) and (45), for every $h \leq h_0$, we have $S_h \in \mathcal{L}(L^r(\mathbb{T}^d), W^{1,p}(\mathbb{T}^d))$, where $\|S_h\|_{\mathcal{L}(L^r, W^{1,p})}$ is bounded from above by a constant which is independent of h . Moreover, up to the choice of a smaller h_0 , from [23, Theorem 3.21, Corollary 3.23] and [9, Theorem 8.5.3] we also have, for every $h \leq h_0$, that $S_h \in \mathcal{L}(L^p(\mathbb{T}^d), W^{1,p}(\mathbb{T}^d))$ with

$$\|(S - S_h)\xi\|_{W^{1,p}} \leq Ch \|S\xi\|_{W^{2,p}} \leq Ch \|\xi\|_{L^p}, \quad (46)$$

where the constant C is independent of h .

We can now define the linear operator

$$T_h \in \mathcal{L}(L^r(\mathbb{T}^d) \times H^{-1}(\mathbb{T}^d), W^{1,p}(\mathbb{T}^d) \times H^1(\mathbb{T}^d)),$$

with range in $X_h := V_h \times V_h$, by setting

$$T_h(\xi, \zeta) = (S_h \xi, S_h \zeta) \quad \text{for every } (\xi, \zeta) \in L^r(\mathbb{T}^d) \times H^{-1}(\mathbb{T}^d).$$

Notice that from (21), (43) and (44), we also have

$$T_h \in \mathcal{L}(L^p(\mathbb{T}^d) \times H^{-1}(\mathbb{T}^d), W^{1,p}(\mathbb{T}^d) \times L^2(\mathbb{T}^d))$$

with

$$\|(T - T_h)(\xi, \zeta)\|_{W^{1,p} \times L^2} \leq Ch \|(\xi, \zeta)\|_{L^p \times H^{-1}}. \quad (47)$$

In particular, using Sobolev inequalities ([11, Theorem 9.9, Corollary 9.11, Theorem 9.12], we have obtained that

$$(T - T_h) \in \mathcal{L}(L^r(\mathbb{T}^d) \times H^{-1}(\mathbb{T}^d), W^{1,p}(\mathbb{T}^d) \times L^s(\mathbb{T}^d)) \cap \mathcal{L}(L^p(\mathbb{T}^d) \times H^{-1}(\mathbb{T}^d), W^{1,p}(\mathbb{T}^d) \times L^2(\mathbb{T}^d)),$$

where s is the critical Sobolev exponent for the continuous embedding $H^1(\mathbb{T}^d) \hookrightarrow L^s(\mathbb{T}^d)$, namely

$$\begin{cases} s = 6 & \text{if } d = 3, \\ s \in [6, \infty) & \text{if } d = 2, \\ s = \infty & \text{if } d = 1. \end{cases}$$

We set $\theta = \frac{p-d}{p}$ and $\theta^* = \frac{(p-2)s-2p}{(s-2)p}$ so that

$$\frac{2}{p} = \frac{1-\theta}{r} + \frac{\theta}{p}$$

and

$$\frac{1}{q} = \frac{p-2}{2p} = \frac{1-\theta^*}{s} + \frac{\theta^*}{2}.$$

Using complex interpolation (see [41, Chapter 2]) we have that

$$L^{p/2}(\mathbb{T}^d) = [L^r(\mathbb{T}^d), L^p(\mathbb{T}^d)]_\theta \quad \text{and} \quad L^q(\mathbb{T}^d) = [L^s(\mathbb{T}^d), L^2(\mathbb{T}^d)]_{\theta^*}.$$

It follows from [41, Theorem 2.6] together with (44), (45), (46), (42) and (43) that that

$$\|S - S_h\|_{\mathcal{L}(L^{p/2}, W^{1,p})} \leq \|S - S_h\|_{\mathcal{L}(L^r, W^{1,p})}^{1-\theta} \|S - S_h\|_{\mathcal{L}(L^p, W^{1,p})}^\theta \leq Ch^\theta$$

and

$$\|S - S_h\|_{\mathcal{L}(H^{-1}, L^q)} \leq \|S - S_h\|_{\mathcal{L}(H^{-1}, L^s)}^{1-\theta^*} \|S - S_h\|_{\mathcal{L}(H^{-1}, L^2)}^{\theta^*} \leq Ch^{\theta^*}.$$

Noticing that $T - T_h = (S - S_h, S - S_h)$, we deduce that

$$\begin{aligned} \|T - T_h\|_{\mathcal{L}(L^{p/2} \times H^{-1}, W^{1,p} \times L^q)} &\leq \|S - S_h\|_{\mathcal{L}(L^{p/2}, W^{1,p})} + \|S - S_h\|_{\mathcal{L}(H^{-1}, L^q)} \\ &\leq C \left(h^\theta + h^{\theta^*} \right). \end{aligned}$$

Since

$$\begin{cases} \theta^* = (p-3)/p & \text{if } d = 3, \\ \theta^* \in [(p-3)/p, (p-2)/p] & \text{if } d = 2, \\ \theta^* = (p-2)/p & \text{if } d = 1, \end{cases}$$

we have $\sigma := \min\{\theta, \theta^*\} = \theta^*$. Therefore we obtain that

$$T_h, (T - T_h) \in \mathcal{L}(Z, X)$$

with

$$\|T - T_h\|_{\mathcal{L}(Z, X)} \leq Ch^\sigma, \quad (48)$$

for every $h \leq h_0 \leq 1$.

We can now apply Theorem 5.3 to obtain the following result.

Theorem 5.4 (Local convergence of finite element approximations). *Let $f \in W^{2,\infty}(\mathbb{R})$ and let (u, m) be a stable solution to (1). Let X, Z be defined according to (40) with $3 < p, q < 6$ and $1/2 = 1/p + 1/q$ and let T_h be defined as above. There exists $h_0 \in (0, 1]$ and a neighborhood \mathcal{O} of the origin in X such that, for every $0 < h \leq h_0$, there exists a solution $(u_h, m_h) \in X_h$ to*

$$F_h(u_h, m_h) := (u_h, m_h) + T_h(G(u_h, m_h)) = 0$$

satisfying

- (i) $(u, m) - (u_h, m_h) \in \mathcal{O}$,
- (ii) (u_h, m_h) is the unique solution to $F_h(u_h, m_h) = 0$ in $(u, m) + \mathcal{O}$,
- (iii) There exists a positive constant $C > 0$ such that

$$\|(u - u_h, m - m_h)\|_X \leq Ch^\sigma,$$

where

$$\begin{cases} \sigma = (p-3)/p & \text{if } d = 3, \\ \sigma \in [(p-3)/p, (p-2)/p] & \text{if } d = 2, \\ \sigma = (p-2)/p & \text{if } d = 1, \end{cases}$$

- (iv) $dF_h[u_h, m_h] \in \mathcal{L}(X)$ is an isomorphism.

Proof. Using Proposition 4.5 we have that F is continuously differentiable and from Example 3.4 we know that $dF[u, m]$ is an isomorphism on X . Note that the assumption $f \in W^{2,\infty}(\mathbb{R})$ ensures that dG is locally Lipschitz continuous in $\mathcal{L}(X, Z)$. Moreover, from (48) we deduce that (37) is satisfied. We can therefore apply Theorem 5.3 to obtain the conclusion. \square

As a direct consequence of Theorem 5.4 (iv) we deduce the local convergence of Newton's method for the discretized problem.

Corollary 5.5 (Local convergence of the discrete Newton method). *Under the assumptions of Theorem 5.4, let $(u_h, m_h) \in X_h$ be a solution to*

$$F_h(u_h, m_h) = 0. \quad (49)$$

Then there exists a neighborhood \mathcal{O} of (u_h, m_h) such that, if $(u_h^0, m_h^0) \in \mathcal{O}$, then the sequence (u_h^k, m_h^k) given by Newton's method applied to (49), i.e.,

$$(u_h^{k+1}, m_h^{k+1}) + T_h(G(u_h^k, m_h^k) + dG[u_h^k, m_h^k](u_h^{k+1} - u_h^k, m_h^{k+1} - m_h^k)) = 0, \quad (50)$$

converges quadratically to (u_h, m_h) in X_h .

Proof. Since $X_h \subset X$ is finite dimensional, and since $dF_h[u_h, m_h] \in \mathcal{L}(X_h)$ is injective on X_h by Theorem 5.4-(iv), we have that $dF_h[u_h, m_h]$ is also an isomorphism on X_h . We conclude using standard results on Newton's method (see Theorem 5.7 below). \square

Remark 5.6. Relation (50) amounts to finding $(v_h, \rho_h) \in X_h$ such that

$$\int Dv_h \cdot D\phi + Du_h^k \cdot Dv_h \phi + \lambda v_h \phi \, dx = \int f'(m_h^k)(\rho_h - m_h^k)\phi + |Du_h^k|^2 \phi \, dx, \quad (51)$$

$$\int D\rho_h \cdot D\psi + \rho_h Du_h^k \cdot D\psi + \lambda \rho_h \psi \, dx = \int m_h^k (Du_h^k - Dv_h) \cdot D\psi \, dx, \quad (52)$$

for every $(\phi, \psi) \in X_h$.

5.3 Newton's method

We recall here the convergence results Newton's method, see [32, Corollary 2.1 p.120], [48, Proposition 5.1], and [22, Theorem 6E.2] for instance.

Theorem 5.7 (Classical Newton method). *Let X and Y be Banach spaces and $F: X \rightarrow Y$ be continuously differentiable. Let $\bar{x} \in X$ be such that $F(\bar{x}) = 0$ and $dF[\bar{x}] \in \mathcal{L}(X, Y)$ is an isomorphism. Then there exists a neighborhood \mathcal{O} of \bar{x} in X such that the sequence defined by*

$$\begin{cases} x_0 \in \mathcal{O}, \\ F(x_k) + dF[x_k](x_{k+1} - x_k) = 0, \end{cases} \quad (53)$$

converges superlinearly to \bar{x} . Furthermore, if dF is locally Lipschitz continuous in $\mathcal{L}(X, Y)$, then the convergence is quadratic.

Let $0 < \gamma < \beta < \alpha$, where α is given in (5), and set

$$X = C^{2,\beta}(\mathbb{T}^d) \times C^{0,\gamma}(\mathbb{T}^d) \quad \text{and} \quad Z = C^{0,\alpha}(\mathbb{T}^d) \times C^{0,\beta}(\mathbb{T}^d)$$

so that (H) is satisfied and consider the mapping T , G , and F defined by (21), (20), and (22), respectively. From Proposition 4.3 we know that the mapping F is continuously differentiable with dF given by (24). Moreover, if we fix a stable solution $(u, m) \in C^{2,\alpha}(\mathbb{T}^d) \times C^{2,\alpha}(\mathbb{T}^d)$, then we know from Example 3.3 that $dF[u, m]$ is an isomorphism on X .

A direct application of Theorem 5.7 yields the following theorem.

Theorem 5.8. *Consider $f \in C_b^1(\mathbb{R}) \cap C^2(\mathbb{R})$, let $0 < \gamma < \beta < \alpha$ and $(u, m) \in C^{2,\alpha}(\mathbb{T}^d) \times C^{2,\alpha}(\mathbb{T}^d)$ be a stable solution to (1). Then there exists a neighborhood \mathcal{O} of (u, m) in $C^{2,\beta}(\mathbb{T}^d) \times C^{2,\gamma}(\mathbb{T}^d)$ such that, if $(u_0, m_0) \in \mathcal{O}$, then the sequence (u_k, m_k) generated by Newton's method applied to (22), i.e.,*

$$(u_{k+1}, m_{k+1}) + T(G(u_k, m_k) + dG[u_k, m_k](u_{k+1} - u_k, m_{k+1} - m_k)) = 0, \quad (54)$$

converges super-linearly to (u, m) in $C^{2,\beta}(\mathbb{T}^d) \times C^{2,\gamma}(\mathbb{T}^d)$. Furthermore, if we also assume $f \in C_{\text{loc}}^{2,1}(\mathbb{R})$, then the convergence is quadratic.

Remark 5.9. At each iteration, the relation (54) amounts to solving the linear system

$$\begin{cases} -\Delta v + Du_k \cdot Dv + \lambda v = f'(m_k)(\rho - m_k) - |Du_k|^2 & \text{in } \mathbb{T}^d, \\ -\Delta \rho - \text{div}(\rho Du_k) + \lambda \rho = \text{div}(m_k Dv) - \text{div}(m_k Du_k) & \text{in } \mathbb{T}^d. \end{cases} \quad (55)$$

In the case where $f' \geq 0$, we may use Theorem 3.5 instead of Example 3.3 to obtain a slightly better result than Theorem 5.8.

Theorem 5.10. *Let $f \in C_b^1(\mathbb{R}) \cap C^2(\mathbb{R})$ with $f' \geq 0$. Let $(u, m) \in C^{2,\alpha}(\mathbb{T}^d) \times C^{2,\alpha}(\mathbb{T}^d)$ be a stable solution to (1). Then there exists a neighborhood \mathcal{O} of (u, m) in $C^{2,\alpha}(\mathbb{T}^d) \times C^{2,\alpha}(\mathbb{T}^d)$ such that if $(u_0, m_0) \in \mathcal{O}$ then the sequence (u_k, m_k) generated by Newton's method (54) converges super-linearly to (u, m) in $C^{2,\alpha}(\mathbb{T}^d) \times C^{2,\alpha}(\mathbb{T}^d)$. Furthermore, if we also assume $f \in C_{\text{loc}}^{2,1}(\mathbb{R})$, then the convergence is quadratic.*

Finally we may also set $X = W^{1,p}(\mathbb{T}^d) \times L^q(\mathbb{T}^d)$ and $Z = L^{p/2}(\mathbb{T}^d) \times W^{-1,r'}(\mathbb{T}^d)$, where $d < p, q < \infty$ and $r > 1$ is such that $1/r = 1/p + 1/q$ and large enough so that there is a compact embedding $W^{1,r}(\mathbb{T}^d) \hookrightarrow L^q(\mathbb{T}^d)$. Then, we may replace Proposition 4.3 and Example 3.3 by Proposition 4.5 and Example 3.4, respectively, in the discussion above to obtain the convergence of Newton's method in Sobolev spaces. The point being that in this case the neighborhood for the initial guess is expected to be less restrictive.

Theorem 5.11. *Let $f \in C_b^1(\mathbb{R}) \cap C_{\text{loc}}^{1,1}(\mathbb{R})$, let X and Z be as above and $(u, m) \in C^{2,\alpha}(\mathbb{T}^d) \times C^{2,\alpha}(\mathbb{T}^d)$ be a stable solution to (1). Then there exists a neighborhood \mathcal{O} of (u, m) in $W^{1,p}(\mathbb{T}^d) \times L^q(\mathbb{T}^d)$ such that, if $(u_0, m_0) \in \mathcal{O}$, then the sequence (u_k, m_k) generated by Newton's method (54) converges superlinearly to (u, m) in $W^{1,p}(\mathbb{T}^d) \times L^q(\mathbb{T}^d)$. Furthermore, if $f \in W^{2,\infty}(\mathbb{R})$, then the convergence is quadratic.*

A Proof of Theorem 2.1

We are going to apply Schauder's fixed point theorem in $C^{0,\beta}(\mathbb{T}^d)$ for some $\beta \in (0, \alpha]$ to be determined.

Fix some $m \in C^{0,\beta}(\mathbb{T}^d)$. Since f is assumed to be Lipschitz continuous, we have that $f(m) \in C^{0,\beta}(\mathbb{T}^d)$. From the standard theory of elliptic equation (see [28, Theorem 15.12], using the gradient bound in Proposition 2.4, for the result with Dirichlet boundary conditions) we know that there exists a unique classical solution $u \in C^{2,\beta}(\mathbb{T}^d)$ to

$$-\Delta v + \frac{1}{2}|Dv|^2 + \lambda v = f(m) \quad \text{in } \mathbb{T}^d. \quad (56)$$

Then from standard Schauder theory (see [28, Corollary 6.3]) and Lemma 2.2, we also have a unique solution $\tilde{m} \in C^{2,\beta}(\mathbb{T}^d)$ to

$$-\Delta \rho - \text{div}(\rho Du) + \lambda \rho = \lambda m_0 \quad \text{in } \mathbb{T}^d. \quad (57)$$

This defines a mapping $\Phi: C^{0,\beta}(\mathbb{T}^d) \rightarrow C^{0,\beta}(\mathbb{T}^d)$ by setting $\Phi(m) = \tilde{m}$.

We now prove that Φ is continuous. Let $(m_n)_{n \geq 0}$ be a sequence in $C^{0,\beta}(\mathbb{T}^d)$ converging to some m in $C^{0,\beta}(\mathbb{T}^d)$. In particular, this sequence is bounded in $C^{0,\beta}(\mathbb{T}^d)$. Using the fact that f is Lipschitz continuous it follows that $f(m_n)$ is also bounded in $C^{0,\beta}(\mathbb{T}^d)$. From the inequality (9) in Proposition 2.4, we have the existence of a positive constant K such any solution classical solution u_n to (56), with m replaced by m_n , satisfies

$$\|Du_n\|_{L^\infty} \leq K$$

and the constant K depends on the right-hand side of (56) only through $\|f\|_{L^\infty}$ (and hence is independent of n). Then, from [28, Theorem 13.6] we deduce that there exist constants $\gamma \in (0, 1)$ and $\tilde{K} > 0$, depending on K and independent of n , such that

$$\|Du_n\|_{C^{0,\gamma}} \leq \tilde{K}.$$

Using Schauder estimates [28, Corollary 6.3], one has that the sequence u_n of solutions to (56) associated to m_n is bounded in $C^{2,\beta}(\mathbb{T}^d)$ for $\beta = \min\{\alpha, \gamma\}$. From the Arzela-Ascoli theorem, it admits a

subsequence converging in $C^2(\mathbb{T}^d)$ to a solution u to (56) associated to m . Since this solution is unique the whole sequence must converge to this limit u . Then, using again Schauder estimates we also have that $\Phi(m_n)$ is bounded in $C^{2,\beta}(\mathbb{T}^d)$ by a constant depending on K , and a similar argument shows that it must converge to $\Phi(m)$.

We now claim the $\Phi(C^{0,\beta}(\mathbb{T}^d))$ is bounded in $C^{0,\gamma}(\mathbb{T}^d)$ for $\gamma \in (\beta, 1)$. Indeed let us choose $1 < p < \infty$, depending only on d and γ , such that $W^{1,p}(\mathbb{T}^d) \hookrightarrow C^{0,\beta}(\mathbb{T}^d)$. From the $W^{1,p}$ estimates (6) in Lemma 2.2, we have that $\Phi(C^{0,\beta}(\mathbb{T}^d))$ is bounded in $W^{1,p}(\mathbb{T}^d)$ and therefore also in $C^{0,\gamma}(\mathbb{T}^d)$. Using the Arzela-Ascoli theorem we deduce that $\Phi(C^{0,\beta}(\mathbb{T}^d))$ is compact in $C^{0,\beta}(\mathbb{T}^d)$.

We can now apply Schauder's fixed point theorem [28, Corollary 11.2] to obtain a classical solution to (1).

The argument for uniqueness is a straightforward adaptation of the one introduced in [37].

B Proof of Proposition 2.8

The argument is adapted from [13, Proposition 4.2]. We only prove the case $d > 2$, the other ones being simpler, in particular in what follows we have $p = d$. Assume that the conclusion is false, then there exists a sequence (u_n, m_n) of classical solutions to (1) converging to (u, m) in $W^{1,p}(\mathbb{T}^d) \times L^2(\mathbb{T}^d)$. Note that we may assume that the convergence also holds in the almost everywhere sense. We then set

$$\delta_n = \|(u_n, m_n) - (u, m)\|_{W^{1,p} \times L^2}$$

and

$$v_n = \delta_n^{-1}(u_n - u), \quad \rho_n = \delta_n^{-1}(m_n - m).$$

Then for every n the pair (v_n, ρ_n) is a classical solution to

$$\begin{cases} -\Delta v_n + \lambda v_n = g_n & \text{in } \mathbb{T}^d, \\ -\Delta \rho_n + \lambda \rho_n = \operatorname{div}(h_n) & \text{in } \mathbb{T}^d, \end{cases} \quad (58)$$

where

$$g_n = \delta_n^{-1} \left(f(m_n) - f(m) + \frac{1}{2} |Du|^2 - \frac{1}{2} |Du_n|^2 \right)$$

and

$$h_n = \delta_n^{-1} (m_n Du_n - m Du).$$

We then notice that

$$\|g_n\|_{L^2} \leq \delta_n^{-1} (\|f'\|_{L^\infty} \|m_n - m\|_{L^2} + K \|Du_n - Du\|_{L^2}) \leq \|f'\|_{L^\infty} + K$$

and

$$\|h_n\|_{L^2} \leq \delta_n^{-1} (K \|m_n - m\|_{L^2} + \|m\|_{L^\infty} \|Du_n - Du\|_{L^2}) \leq C,$$

where K is given in Proposition 2.4 and C is a constant independent of n . Applying standard elliptic regularity to (58) it follows that (v_n, ρ_n) is bounded in $H^2(\mathbb{T}^d) \times H^1(\mathbb{T}^d)$. From the Rellich-Kondrachov theorem we deduce that it converges, up to a subsequence, to some (v, ρ) in $W^{1,q}(\mathbb{T}^d) \times L^2(\mathbb{T}^d)$ for every $2 \leq q < \frac{2d}{d-2}$. We may also assume that (v_n, ρ_n) converges weakly to (v, ρ) in $H^2(\mathbb{T}^d) \times H^1(\mathbb{T}^d)$. Moreover v_n is bounded in $W^{1,d}(\mathbb{T}^d)$ (recall that we consider the case $p = d$) so that we may also assume that it converges weakly to v in $W^{1,d}(\mathbb{T}^d)$. We now claim that the following fact holds

$$\begin{cases} \delta_n^{-1} (f(m_n) - f(m)) \rightarrow f'(m)\rho & \text{in } L^2(\mathbb{T}^d), \\ (2\delta_n)^{-1} (|Du|^2 - |Du_n|^2) \rightarrow Du \cdot Dv & \text{in } L^2(\mathbb{T}^d), \\ \delta_n^{-1} (m_n Du_n - m Du) \rightharpoonup \rho Du + m Dv & \text{weakly in } L^{\frac{2d}{d+2}}(\mathbb{T}^d; \mathbb{R}^d). \end{cases} \quad (59)$$

Let us prove the first line of (59). Since $f \in C_b^1(\mathbb{R})$ we have that

$$\delta_n^{-1}(f(m_n) - f(m)) = \rho_n \int_0^1 f'(\lambda m_n + (1 - \lambda)m) d\lambda \quad \text{a.e. in } \mathbb{T}^d. \quad (60)$$

We then decompose

$$\begin{aligned} & \left\| \int_0^1 f'(\lambda m_n + (1 - \lambda)m) \rho_n - f'(m) \rho d\lambda \right\|_{L^2} \\ & \leq \left\| \rho_n \int_0^1 f'(\lambda m_n + (1 - \lambda)m) - f'(m) d\lambda \right\|_{L^2} + \|f'(m)(\rho_n - \rho)\|_{L^2}. \end{aligned} \quad (61)$$

Since we assume that f' is bounded, the convergence to zero of the second term in the right-hand side of the last inequality follows from the convergence of ρ_n to ρ in $L^2(\mathbb{T}^d)$. Recalling that we assume that $d > 2$, it follows from Sobolev's inequality that $H^1(\mathbb{T}^d)$ embeds continuously in $L^{\frac{2d}{d-2}}(\mathbb{T}^d)$. Using Hölder's inequality we can write

$$\begin{aligned} & \left\| \rho_n \int_0^1 f'(\lambda m_n + (1 - \lambda)m) - f'(m) d\lambda \right\|_{L^2} \\ & \leq C \|\rho_n\|_{H^1} \left\| \int_0^1 f'(\lambda m_n + (1 - \lambda)m) - f'(m) d\lambda \right\|_{L^d}. \end{aligned} \quad (62)$$

Using Jensen's inequality and Fubini's theorem we have

$$\left\| \int_0^1 f'(\lambda m_n + (1 - \lambda)m) - f'(m) d\lambda \right\|_{L^d}^d \leq \int_0^1 \|f'(\lambda m_n + (1 - \lambda)m) - f'(m)\|_{L^d}^d d\lambda. \quad (63)$$

Using the fact that f' is bounded and continuous and that m_n converges a.e. to m one may apply Lebesgue's convergence theorem to obtain

$$\lim_{n \rightarrow \infty} \int_0^1 \|f'(\lambda m_n + (1 - \lambda)m) - f'(m)\|_{L^d}^d d\lambda = 0.$$

This, together with (60), (61), (62), and (63), proves the first line in (59).

For the second line of (59), we write

$$\begin{aligned} & \left\| (2\delta_n)^{-1} \left(|Du|^2 - |Du_n|^2 \right) - Du \cdot Dv \right\|_{L^2} = \frac{1}{2} \|(Du + Du_n) \cdot Dv_n - 2Du \cdot Dv\|_{L^2} \\ & \leq \frac{1}{2} \|Du \cdot (Dv_n - Dv)\|_{L^2} + \frac{1}{2} \|Du_n \cdot Dv_n - Du \cdot Dv\|_{L^2} \\ & \leq \frac{1}{2} \|Du\|_{L^\infty} \|Dv_n - Dv\|_{L^2} + \frac{1}{2} \|Du_n \cdot (Dv_n - Dv)\|_{L^2} + \frac{1}{2} \|(Du_n - Du) \cdot Dv\|_{L^2} \\ & \leq K \|Dv_n - Dv\|_{L^2} + \frac{1}{2} \|(Du_n - Du) \cdot Dv\|_{L^2}, \end{aligned}$$

where the constant K is given in Proposition 2.4. Using Hölder's inequality we have

$$\|(Du_n - Du) \cdot Dv\|_{L^2} \leq \|Du_n - Du\|_{L^d} \|Dv\|_{L^{\frac{2d}{d-2}}}.$$

It follows that

$$\left\| (2\delta_n)^{-1} \left(|Du|^2 - |Du_n|^2 \right) - Du \cdot Dv \right\|_{L^2} \leq K \|Dv_n - Dv\|_{L^2} + \|Du_n - Du\|_{L^d} \|Dv\|_{L^{\frac{2d}{d-2}}}.$$

Since $v \in H^2(\mathbb{T}^d) \hookrightarrow W^{1, \frac{2d}{d-2}}(\mathbb{T}^d)$ the right-hand side converges to 0.

For the third line of (59), we first notice that

$$h_n = \delta_n^{-1}(m_n Du_n - m Du) = m_n Dv_n + \rho_n Du.$$

Let $\varphi \in L^{\frac{2d}{d+2}}(\mathbb{T}^d; \mathbb{R}^d) = \left(L^{\frac{2d}{d+2}}(\mathbb{T}^d; \mathbb{R}^d)\right)'$. From convergence of m_n to m in $L^2(\mathbb{T}^d)$ and Hölder's inequality it is easy to see that $m_n \varphi$ converges to $m \varphi$ in $L^{\frac{d}{d-1}}(\mathbb{T}^d; \mathbb{R}^d) = (L^d(\mathbb{T}^d; \mathbb{R}^d))'$. Since v_n converges weakly to v in $W^{1,d}(\mathbb{T}^d)$, we have that Dv_n converges weakly to Dv in $L^d(\mathbb{T}^d; \mathbb{R}^d)$. It follows that

$$\begin{aligned} \lim_{n \rightarrow \infty} \langle \varphi, m_n Dv_n \rangle_{(L^{\frac{2d}{d+2}})', L^{\frac{2d}{d+2}}} &= \lim_{n \rightarrow \infty} \int m_n \varphi \cdot Dv_n \, dx = \lim_{n \rightarrow \infty} \langle m_n \varphi, Dv_n \rangle_{(L^d)', L^d} \\ &= \langle m \varphi, Dv \rangle_{(L^d)', L^d} = \langle \varphi, m Dv \rangle_{(L^{\frac{2d}{d+2}})', L^{\frac{2d}{d+2}}}. \end{aligned}$$

Since ρ_n converges to ρ in $L^{\frac{2d}{d-2}}(\mathbb{T}^d)$ we have that $\rho_n Du$ converges to ρDu in $L^{\frac{2d}{d-2}}(\mathbb{T}^d)$, and because $\frac{2d}{d-2} \geq \frac{2d}{d+2}$, we also have the strong (and therefore also weak) convergence of $\rho_n Du$ to ρDu in $L^{\frac{2d}{d+2}}(\mathbb{T}^d)$. This proves the third line of (59) and concludes the proof of the claim.

We can now pass to the limit in the weak formulation of (58) to obtain that $(v, \rho) \in H^1(\mathbb{T}^d) \times L^2(\mathbb{T}^d)$ is a weak solution to (12). Indeed, for $\varphi \in H^1(\mathbb{T}^d)$ we have

$$\lim_{n \rightarrow \infty} \int Dv_n \cdot D\varphi + \lambda v_n \varphi \, dx = \int Dv \cdot D\varphi + \lambda v \varphi \, dx$$

and

$$\lim_{n \rightarrow \infty} \int g_n \varphi \, dx = \int f'(m) \rho \varphi - Du \cdot Dv \varphi \, dx,$$

so that

$$\int Dv \cdot D\varphi + Du \cdot Dv \varphi + \lambda v \varphi \, dx = \int f'(m) \rho \varphi \, dx \quad \text{for every } \varphi \in H^1(\mathbb{T}^d).$$

Let now $\psi \in H^2(\mathbb{T}^d)$ and recall that this implies that $D\psi \in L^{\frac{2d}{d-2}}(\mathbb{T}^d; \mathbb{R}^d)$. Then we have

$$\lim_{n \rightarrow \infty} \int (-\Delta \psi + \lambda \psi) \rho_n \, dx = \int (-\Delta \psi + \lambda \psi) \rho \, dx$$

and

$$\lim_{n \rightarrow \infty} \int h_n D\psi \, dx = \int (\rho Du + m Dv) \cdot D\psi \, dx,$$

so that

$$\int (-\Delta \psi + Du \cdot D\psi + \lambda \psi) \rho \, dx = \int m Dv \cdot D\psi \, dx \quad \text{for every } \psi \in H^2(\mathbb{T}^d).$$

This proves that (v, ρ) is a weak solution to (12). According to Lemma 2.7, (v, ρ) is a classical solution and, since (u, m) is assumed to be stable, we must have $(v, \rho) = (0, 0)$. This contradicts the fact that $\|(v, \rho)\|_{W^{1,p} \times L^2} = 1$ and concludes the proof.

Acknowledgement. The authors would like to thank Y. A. P. Osborne and I. Smears (University College London) for pointing out an error in a preliminary version of this paper.

This work was partially supported by the ANR (Agence Nationale de la Recherche) through the COSS project ANR-22-CE40-0010 and the Centre Henri Lebesgue ANR-11-LABX-0020-01. The third author was partially supported by KAUST through the subaward agreement ORA-2021-CRG10-4674.6.

References

- [1] Y. Achdou. “Finite Difference Methods for Mean Field Games”. In: Y. Achdou et al. *Hamilton-Jacobi Equations: Approximations, Numerical Analysis and Applications*. Vol. 2074. Lecture Notes in Mathematics. Berlin, Heidelberg: Springer Berlin Heidelberg, 2013, pp. 1–47.
- [2] Y. Achdou, F. Camilli, and I. Capuzzo-Dolcetta. “Mean Field Games: Convergence of a Finite Difference Method”. In: *SIAM Journal on Numerical Analysis* 51.5 (Jan. 2013), pp. 2585–2612.
- [3] Y. Achdou et al. *Mean Field Games: Cetraro, Italy 2019*. Ed. by P. Cardaliaguet and A. Porretta. Vol. 2281. Lecture Notes in Mathematics. Cham: Springer International Publishing, 2020.
- [4] M. Bardi and M. Fischer. “On non-uniqueness and uniqueness of solutions in finite-horizon mean field games”. In: *ESAIM Control Optim. Calc. Var.* 25 (2019), Paper No. 44, 33.
- [5] A. Bensoussan, J. Frehse, and P. Yam. *Mean Field Games and Mean Field Type Control Theory*. SpringerBriefs in Mathematics. New York, NY: Springer New York, 2013.
- [6] V. Bogachev et al. *Fokker-Planck-Kolmogorov Equations*. Vol. 207. Mathematical Surveys and Monographs. Providence, Rhode Island: American Mathematical Society, Dec. 17, 2015.
- [7] V. I. Bogachev. *Measure Theory*. Berlin, Heidelberg: Springer Berlin Heidelberg, 2007.
- [8] J. F. Bonnans, K. Liu, and L. Pfeiffer. “Error Estimates of a Theta-Scheme for Second-Order Mean Field Games”. In: *ESAIM: Mathematical Modelling and Numerical Analysis* 57.4 (July 2023), pp. 2493–2528.
- [9] S. C. Brenner and L. R. Scott. *The Mathematical Theory of Finite Element Methods*. Red. by J. E. Marsden, L. Sirovich, and S. S. Antman. Vol. 15. Texts in Applied Mathematics. New York, NY: Springer New York, 2008.
- [10] A. Bressan and K. T. Nguyen. “Generic properties of first-order mean field games”. In: *Dyn. Games Appl.* 13.3 (2023), pp. 750–782.
- [11] H. Brezis. *Functional analysis, Sobolev spaces and partial differential equations*. Universitext. Springer, New York, 2011, pp. xiv+599.
- [12] F. Brezzi, J. Rappaz, and P. A. Raviart. “Finite Dimensional Approximation of Nonlinear Problems: Part I: Branches of Nonsingular Solutions”. In: *Numerische Mathematik* 36.1 (Mar. 1980), pp. 1–25.
- [13] A. Briani and P. Cardaliaguet. “Stable Solutions in Potential Mean Field Game Systems”. In: *Nonlinear Differential Equations and Applications NoDEA* 25.1 (Feb. 2018), p. 1.
- [14] G. Caloz and J. Rappaz. “Numerical analysis for nonlinear and bifurcation problems”. In: *Handbook of numerical analysis, Vol. V*. Vol. V. Handb. Numer. Anal. North-Holland, Amsterdam, 1997, pp. 487–637.
- [15] F. Camilli and Q. Tang. “A Convergence Rate for the Newton’s Method for Mean Field Games with Non-Separable Hamiltonians”. In: *arXiv preprint arXiv:2311.05416* (2023).
- [16] P. Cardaliaguet and S. Hadikhanloo. “Learning in Mean Field Games: The Fictitious Play”. In: *ESAIM: Control, Optimisation and Calculus of Variations* 23.2 (Apr. 2017), pp. 569–591.
- [17] P. Cardaliaguet et al. *The Master Equation and the Convergence Problem in Mean Field Games*. Princeton University Press, Aug. 13, 2019.
- [18] R. Carmona and F. Delarue. *Probabilistic Theory of Mean Field Games with Applications I-II*. Probability Theory and Stochastic Modelling 83-84. Springer Cham, 2018.
- [19] M. Cirant. “On the existence of oscillating solutions in non-monotone mean-field games”. In: *Journal of Differential Equations* 266.12 (2019), pp. 8067–8093.

- [20] M. Cirant and D. Tonon. “Time-dependent focusing mean-field games: the sub-critical case”. In: *J. Dynam. Differential Equations* 31.1 (2019), pp. 49–79.
- [21] M. Crouzeix and J. Rappaz. *On numerical approximation in bifurcation theory*. Vol. 13. Recherches en Mathématiques Appliquées [Research in Applied Mathematics]. Masson, Paris; Springer-Verlag, Berlin, 1990, pp. x+165.
- [22] A. L. Dontchev and R. T. Rockafellar. *Implicit Functions and Solution Mappings: A View from Variational Analysis*. Springer Series in Operations Research and Financial Engineering. New York, NY: Springer New York, 2014.
- [23] A. Ern and J.-L. Guermond. *Theory and Practice of Finite Elements*. Red. by S. S. Antman, J. E. Marsden, and L. Sirovich. Vol. 159. Applied Mathematical Sciences. New York, NY: Springer New York, 2004.
- [24] L. Evans. *Partial Differential Equations*. Second. Vol. 19. Graduate Studies in Mathematics. Providence, Rhode Island: American Mathematical Society, Mar. 2, 2010.
- [25] R. Ferreira and D. Gomes. “Existence of weak solutions to stationary mean-field games through variational inequalities”. In: *SIAM J. Math. Anal.* 50.6 (2018), pp. 5969–6006.
- [26] W. H. Fleming and R. W. Rishel. *Deterministic and Stochastic Optimal Control*. Vol. 1. Springer Science & Business Media, 2012.
- [27] W. H. Fleming and H. M. Soner. *Controlled Markov Processes and Viscosity Solutions*. Vol. 25. Springer Science & Business Media, 2006.
- [28] D. Gilbarg and N. S. Trudinger. *Elliptic Partial Differential Equations of Second Order*. Vol. 224. Classics in Mathematics. Berlin, Heidelberg: Springer Berlin Heidelberg, 2001.
- [29] V. Girault and P.-A. Raviart. *Finite Element Methods for Navier-Stokes Equations*. Vol. 5. Springer Series in Computational Mathematics. Berlin, Heidelberg: Springer Berlin Heidelberg, 1986.
- [30] D. A. Gomes, H. Mitake, and K. Terai. “The selection problem for some first-order stationary mean-field games”. In: *Netw. Heterog. Media* 15.4 (2020), pp. 681–710.
- [31] D. A. Gomes and J. Saúde. “Mean Field Games Models—A Brief Survey”. In: *Dynamic Games and Applications* 4.2 (June 2014), pp. 110–154.
- [32] M. Hinze et al. *Optimization with PDE Constraints*. Vol. 23. Mathematical Modelling: Theory and Applications. Dordrecht: Springer Netherlands, 2009.
- [33] M. Huang, P. E. Caines, and R. P. Malhamé. “Large-Population Cost-Coupled LQG Problems With Nonuniform Agents: Individual-Mass Behavior and Decentralized ε -Nash Equilibria”. In: *IEEE Transactions on Automatic Control* 52.9 (Sept. 2007), pp. 1560–1571.
- [34] M. Huang, R. P. Malhamé, and P. E. Caines. “Large Population Stochastic Dynamic Games: Closed-Loop McKean-Vlasov Systems and the Nash Certainty Equivalence Principle”. In: *Communications in Information and Systems* 6.3 (2006), pp. 221–252.
- [35] J.-M. Lasry and P.-L. Lions. “Jeux à champ moyen. I – Le cas stationnaire”. In: *Comptes Rendus Mathématique* 343.9 (Nov. 2006), pp. 619–625.
- [36] J.-M. Lasry and P.-L. Lions. “Jeux à champ moyen. II – Horizon fini et contrôle optimal”. In: *Comptes Rendus Mathématique* 343.10 (Nov. 2006), pp. 679–684.
- [37] J.-M. Lasry and P.-L. Lions. “Mean Field Games”. In: *Japanese Journal of Mathematics* 2.1 (Mar. 2007), pp. 229–260.
- [38] G. Leoni. *A first course in Sobolev spaces*. Vol. 105. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2009, pp. xvi+607.

- [39] O. Ley and V. D. Nguyen. “Lipschitz Regularity Results for Nonlinear Strictly Elliptic Equations and Applications”. In: *Journal of Differential Equations* 263.7 (Oct. 2017), pp. 4324–4354.
- [40] P.-L. Lions. “Cours Au College de France”. In: *Available at www.college-de-france.fr* (2007).
- [41] A. Lunardi. *Interpolation Theory*. Pisa: Scuola Normale Superiore, 2018.
- [42] R. Nugari. “Further Remarks on the Nemitskii Operator in Hölder Spaces”. In: *Commentationes Mathematicae Universitatis Carolinae* 34.1 (1993), pp. 89–95.
- [43] Y. A. P. Osborne and I. Smears. “Analysis and Numerical Approximation of Stationary Second-Order Mean Field Game Partial Differential Inclusions”. In: *SIAM J. Numer. Anal.* 62.1 (2024), pp. 138–166.
- [44] Y. A. P. Osborne and I. Smears. *Finite element approximation of time-dependent Mean Field Games with non-differentiable Hamiltonians*. 2023.
- [45] Y. A. P. Osborne and I. Smears. *Near and full quasi-optimality of finite element approximations of stationary second-order mean field games*. 2024.
- [46] Q. Tang and J. Song. “Learning Optimal Policies in Potential Mean Field Games: Smoothed Policy Iteration Algorithms”. In: *SIAM J. Control Optim.* 62.1 (2024), pp. 351–375.
- [47] J. Yong and X. Y. Zhou. *Stochastic Controls*. New York, NY: Springer New York, 1999.
- [48] E. Zeidler. *Nonlinear Functional Analysis and Its Applications: Fixed-point Theorems*. Springer-Verlag, 1993.