



HAL
open science

A spectral dominance approach to large random matrices: part II

Charles Bertucci, Jean-Michel Lasry, Pierre Louis Lions

► **To cite this version:**

Charles Bertucci, Jean-Michel Lasry, Pierre Louis Lions. A spectral dominance approach to large random matrices: part II. 2024. hal-04474892v1

HAL Id: hal-04474892

<https://hal.science/hal-04474892v1>

Preprint submitted on 23 Feb 2024 (v1), last revised 19 Mar 2024 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

A SPECTRAL DOMINANCE APPROACH TO LARGE RANDOM MATRICES: PART II

CHARLES BERTUCCI ^A, JEAN-MICHEL LASRY^B, PIERRE-LOUIS LIONS^{B,C}

ABSTRACT. This paper is the second of a series devoted to the study of the dynamics of the spectrum of large random matrices. We study general extensions of the partial differential equation arising to characterize the limit spectral measure of the Dyson Brownian motion. We show that several results of part I extend to cases in which there is no spectral dominance property. We also provide several modeling extensions of such models. Finally we establish new regularizing results for the case of the Dyson Brownian motion.

Keywords: Random matrix, Mean field limits, Partial differential equations.

MSC: 60B20 35D40 60F99

CONTENTS

1. Introduction	1
2. Notation and results of the first part	2
3. Corrections on the first part	4
4. General extensions of the Dyson equation	4
5. Some regularizing properties of the Dyson flow	13
6. Complements on modeling	19
Acknowledgments	23
References	24

1. INTRODUCTION

In this paper, we precise and extend some results and proofs introduced in part I [3]. Our main interest is the study of the partial differential equation (PDE)

$$(1.1) \quad \begin{aligned} \partial_t m + \partial_x(mH[m]) &= 0 \text{ in } (0, \infty) \times \mathbb{R}, \\ m|_{t=0} &\in \mathcal{P}(\mathbb{R}), \end{aligned}$$

and of some of its extensions. We shall call (1.1) the Dyson equation in the rest of the paper. In (1.1), $H[m]$ denotes the Hilbert transform of m and $\mathcal{P}(\mathbb{R})$ denotes the set of probability measures on \mathbb{R} . This equation naturally arises in the theory of random matrices as the limit equation for the spectral measure of the Dyson Brownian motion. We refer to [1] for a comprehensive and detailed introduction to the theory of random matrices.

For the moment, most of the literature on random matrices on equations of the form of (1.1) has been focused on the exact nonlinearity $mH[m]$. In this paper we provide significant improvements on the results obtained in this case. Furthermore, following part I, we also present results on other nonlinearities of the same form. Let us insist upon the fact that we believe that, for practical applications, it is fundamental to develop techniques which are robust to small changes in (1.1).

The rest of the paper is organized as follows. In Section 2, we recall notations and results of the first part which shall be used here. We then precise and correct some results and proofs of the first part I in Section 3. In Section 4, we extend the results of part I to more general equations, before presenting new regularizing properties of the pure equation (1.1) in Section 5. We then conclude this paper by indicating some extensions which are natural from a modeling perspective in Section 6.

2. NOTATION AND RESULTS OF THE FIRST PART

2.1. Notation. The set of Borel probability measures on \mathbb{R} is $\mathcal{P}(\mathbb{R})$, when not specified otherwise, it will be endowed with the weak $*$ convergence. For any $1 \leq p \leq \infty$, $u \in L^p(\mathbb{R})$, we denote by $H[u]$ its Hilbert transform, that is the linear anti-self adjoint operator on L^p defined on smooth function with the formula

$$H[u](x) = \int_{\mathbb{R}} \frac{u(y) - u(x)}{x - y} dy.$$

Recall that for $m \in \mathcal{P}(\mathbb{R})$, $H[m]$ is the distribution $p.v.(\frac{1}{x}) * m$, defined against smooth test function ϕ through

$$\left\langle p.v.(\frac{1}{x}) * m, \phi \right\rangle = - \int_{\mathbb{R}} H[\phi] dm.$$

We denote by A_0 the half-Laplacian operator, defined by $A_0 = \frac{d}{dx} H$. On smooth functions, it is thus given by

$$A_0[u](x) = \int_{\mathbb{R}} \frac{u(x) - u(y)}{(x - y)^2} dy.$$

Let us also define for $\delta > 0$ the operators A_δ and $A_{-\delta}$ with

$$A_\delta[u](x) = \int_{|x-y| \geq \delta} \frac{u(x) - u(y)}{(x - y)^2} dy, \quad A_{-\delta}[u](x) = \int_{|x-y| < \delta} \frac{u(x) - u(y)}{(x - y)^2} dy.$$

For a locally bounded function u on a subset Ω of \mathbb{R}^d , we define for all $x \in \Omega$

$$u_*(x) = \liminf_{y \rightarrow x} u(y), \quad u^*(x) = \limsup_{y \rightarrow x} u(y).$$

2.2. Characterization of solutions of the Dyson equation. As in part I, we are going to say that m is a solution of (1.1) if $u(t, x) := m_t((-\infty, x])$ is a viscosity solution of the primitive PDE

$$(2.1) \quad \partial_t u + \partial_x u A_0[u] = 0 \text{ in } (0, \infty) \times \mathbb{R},$$

which is simply obtained by integrating (1.1) with respect to x . Naturally, the previous equation is associated with boundary conditions

$$\forall t \geq 0, \quad u(t, -\infty) = 0, \quad u(t, \infty) = 1,$$

and the initial condition $u_0 := m_0((-\infty, x])$, which is thus non-decreasing. We showed in part I that, in this context, this equation is well suited for the theory of viscosity solutions since it has a comparison principle. Moreover, this comparison principle does not rely on the exact form of A_0 (hence of $H[\cdot]$), but rather on the fact that it is an elliptic operator. Hence, it extends naturally to more general equations, namely PDE of the form

$$(2.2) \quad \partial_t u + \partial_x u c(x) A_0[u] + b \partial_x u = 0 \text{ in } (0, \infty) \times \mathbb{R},$$

where $c : \mathbb{R} \rightarrow \mathbb{R}$ is a non-negative continuous function and $b : \mathbb{R} \rightarrow \mathbb{R}$ is continuous. We now give a precise definition of viscosity solutions for (2.2).

Definition 1. *An usc (resp. lsc) function $u : \mathbb{R}_+ \times \mathbb{R}$ is a viscosity sub-solution of (2.2) (resp. a super-solution) if for any $T > 0$, for any $\phi \in C_b^{1,1}$, $0 < \delta \leq \infty$ $(t_0, x_0) \in (0, T] \times \mathbb{R}$ point of maximum (resp. minimum) of $u - \phi$,*

$$\begin{aligned} & \partial_t \phi(t_0, x_0) + \partial_x \phi(t_0, x_0) c(x_0) (A_{-\delta}[\phi(t_0)])(x_0) + A_\delta[u(t_0)](x_0) \\ & + b(x_0) \partial_x \phi(t_0, x_0) \leq 0 \text{ (resp. } \geq 0). \end{aligned}$$

Definition 2. *A viscosity solution of (2.2) is a locally bounded function such that u^* is a viscosity super-solution and u_* is viscosity sub-solution of (2.2).*

Below we shall also use the notion of viscosity solutions for slightly more involved PDE and in those cases, the adequate notion of viscosity solution is the natural extension of this one. In part I, several results were introduced on equations of the form of (2.2), namely the propagation of Lipschitz estimates as well as comparison results for viscosity solutions.

Let us remark that equations of the same form as (2.2) have been considered independently of random matrices, namely because of their applications in dislocations problems. We refer the reader to [6, 7, 5] for more detailed and such equations and their uses.

2.3. Derivation of the Dyson equation. We briefly recall the usual derivation of (1.1), using the so-called Dyson Brownian motion. Consider a collection $(B^i)_{i \in \mathbb{N}}$ of independent Brownian motions and a sequence $(\lambda^N)_{N \in \mathbb{N}}$ such that for all $N \geq 1$, $\lambda^N \in \mathbb{R}^N$ with $\lambda_i^N < \lambda_j^N$ as soon as $i < j$. It is well known (see e.g. [4, 11, 1]) that when, in law,

$$\lim_{N \rightarrow \infty} N^{-1} \sum_{i=1}^N \delta_{\lambda_i^N} = m_0 \in \mathcal{P}(\mathbb{R}),$$

then for all $t > 0$, $N^{-1} \sum_{i=1}^N \delta_{\lambda_{i,t}^N}$ converges in law toward $m_t \in \mathcal{P}(\mathbb{R})$, where the processes $((\lambda_{i,t}^N)_{t \geq 0})_{1 \leq i \leq N}$ are the unique strong solutions of

$$(2.3) \quad d\lambda_{i,t}^N = \frac{1}{N} \sum_{j \neq i} \frac{1}{\lambda_{i,t}^N - \lambda_{j,t}^N} dt + \frac{\sqrt{2}}{\sqrt{N}} dB_t^i$$

and $(m_t)_{t \geq 0} \in C(\mathbb{R}_+, \mathcal{P}(\mathbb{R}))$ is the unique solution of the Dyson equation (1.1) with initial condition m_0 .

3. CORRECTIONS ON THE FIRST PART

The first correction we make is on the convergence result presented in Theorem 3 in [3]. At the beginning of the proof, we considered an element F given as a limit point of the sequence $(F_N)_{N \geq 2}$ through a compactness argument. If this sequence is indeed relatively-compact almost surely, the problem is that the limit point depends on $\omega \in \Omega$. The exact same argument can be made correct by simply considering instead

$$F^*(t, x) = \limsup_{N \rightarrow \infty, x_N \rightarrow x, t_N \rightarrow t} F_N(t_N, x_N).$$

The exact same computations we presented in [3] then establishes that, almost surely, F^* is a viscosity sub-solution of the equation. The same argument can be made to show that F_* , the lim inf of $(F_N)_{N \geq 2}$, is almost surely a viscosity supersolution. We conclude as usual in viscosity solution theory, using the comparison result (Proposition 4.4 in [3]) to establish that almost surely $F^* \leq F_*$ and thus the result follows.

Corollary 2.2 of [3] is proved and detailed in Section 4.4 below.

The equations (4.31) and (4.32) of [3] should read differently, as a negative sign should be in front of the constant C . In (2.43) and (4.52), it lacks the condition $x \geq y$.

4. GENERAL EXTENSIONS OF THE DYSON EQUATION

In this Section, we present various results of well-posedness for equations of the form of (2.2).

4.1. Propagation of Lipschitz regularity for non-parabolic equations. We come back here on Theorem 7 in [3], namely to make precise how to adapt Proposition 4.2 of [3] to the case of an equation of the form

$$(4.1) \quad \partial_t u + \partial_x u c(x) A_0[u] + B(x; u) \partial_x u = 0 \text{ in } (0, \infty) \times \mathbb{R},$$

where $c : \mathbb{R} \rightarrow (0, \infty)$ is a smooth function such that $\inf_x c(x) > 0$ and B is given by

$$B(x; u) = \int_{\mathbb{R}} \beta(x, y) u(y) dy,$$

for a smooth $\beta : \mathbb{R}^2 \rightarrow \mathbb{R}$ which is smooth but not necessary non-negative. We recall that this type of equation models large interacting systems, similar to (2.3), in which the pairwise interaction is given by $\frac{f(\lambda_i, \lambda_j)}{\lambda_i - \lambda_j}$. In such a case, we obtain that $c(x) = f(x, x)$ and

$$\beta(x, y) = \frac{f(x, y) + (x - y) \partial_y f(x, y) - f(x, x)}{(x - y)^2},$$

which is smooth as soon as f is smooth. In [3], we explained in details the case in which (4.1) has a comparison principle. We now investigate in details the case in which such a comparison does not hold. In this situation, to overcome the lack of parabolicity of the equation, we have to use some regularity. Namely, we show that a Lipschitz bound

propagates for (4.1), and that we can apply a Grönwall like result on Lipschitz viscosity solutions to show a uniqueness result.

Let us recall the usual $\dot{H}^{\frac{1}{2}}(\mathbb{R})$ Sobolev semi norm defined by

$$(4.2) \quad \|u\|_{\dot{H}^{\frac{1}{2}}}^2 := \int_{\mathbb{R}} u(x) A_0[u](x) dx.$$

The Fourier representation of this semi-norm is

$$\|u\|_{\dot{H}^{\frac{1}{2}}}^2 = \int_{\mathbb{R}} |\xi| |\hat{u}(\xi)|^2 d\xi.$$

We have for any $u, v \in \dot{H}^{\frac{1}{2}}$

$$(4.3) \quad \int_{\mathbb{R}} u \partial_x v \leq \|u\|_{\dot{H}^{\frac{1}{2}}} \|v\|_{\dot{H}^{\frac{1}{2}}}, \quad \int_{\mathbb{R}} u A_0[v] \leq \|u\|_{\dot{H}^{\frac{1}{2}}} \|v\|_{\dot{H}^{\frac{1}{2}}}.$$

We shall use the following (standard) Lemma several times.

Lemma 4.1. *Let u and v be a bounded functions such that $\|u\|_{\dot{H}^{\frac{1}{2}}} + \|v\|_{\dot{H}^{\frac{1}{2}}} < \infty$. Then*

$$\|uv\|_{\dot{H}^{\frac{1}{2}}}^2 \leq 2(\|u\|_{\infty}^2 \|v\|_{\dot{H}^{\frac{1}{2}}}^2 + \|u\|_{\dot{H}^{\frac{1}{2}}}^2 \|v\|_{\infty}^2),$$

where C depends on v .

Proof. It suffices to verify it with the Fourier transform.

$$\begin{aligned} \int_{\mathbb{R}} |\xi| \left| \int_{\mathbb{R}} \hat{u}(\xi - z) \hat{v}(z) dz \right|^2 d\xi &= \int_{\mathbb{R}} \left| \int_{\mathbb{R}} \hat{u}(\xi - z) \sqrt{|\xi|} \hat{v}(z) dz \right|^2 d\xi \\ &\leq \int_{\mathbb{R}} \left| \int_{\mathbb{R}} \hat{u}(\xi - z) (\sqrt{|\xi - z|} + \sqrt{|z|}) \hat{v}(z) dz \right|^2 d\xi \\ &\leq \int_{\mathbb{R}} \left(\left| \int_{\mathbb{R}} \hat{u}(\xi - z) \sqrt{|\xi - z|} \hat{v}(z) dz \right| + \left| \int_{\mathbb{R}} \hat{u}(\xi - z) \hat{v}(z) \sqrt{|z|} dz \right| \right)^2 d\xi \\ &\leq 2(\|u\|_{\infty}^2 \|u\|_{\dot{H}^{\frac{1}{2}}}^2 + \|u\|_{\infty}^2 \|v\|_{\dot{H}^{\frac{1}{2}}}^2). \end{aligned}$$

□

Before proving the propagation of a Lipschitz estimate, we state a regularity regime for the nonlinear coupling B . The requirements are stated on β with the convention that x is the first argument of β and y its second. Furthermore, the operator A_0 is always used on the x first argument. We assume that β is smooth and that

$$(4.4) \quad \beta, \partial_x \beta, \partial_{xx} \beta \in L_x^\infty(L_y^1).$$

$$(4.5) \quad A_0 \beta, A_0 \partial_x \beta, A_0 \partial_{xx} \beta \in L_{x,y}^1.$$

The first condition (4.4) ensures that $B(\cdot; u)$, $\partial_x B(\cdot; u)$ and $\partial_{xx} B(\cdot; u)$ are bounded functions as soon as $u \in L^\infty$. The second condition implies that all those functions have a bounded $\dot{H}^{\frac{1}{2}}$ semi-norm depending only on $\|u\|_\infty$. This last fact simply follows from (4.2).

Proposition 4.2. *Assume that (4.4) and (4.5) hold. Let $u : [0, \infty) \times \mathbb{R}$ be a smooth function solution of (4.1) such that for all $t \geq 0$, $u(t, \cdot)$ is non-decreasing, bounded, and $u(-\infty) = 0, u(\infty) = 1$. Then, for any $T > 0$, there exists $C > 0$ depending only T, c, B and $\|u(0, \cdot)\|_{C^{1,\alpha}}$ such that $\|\partial_x u(T, \cdot)\|_\infty \leq C$.*

Proof. The proof follows the same idea as in Proposition 4.2 in [3]: ultimately, the result follows from a maximum principle like-estimate on the PDE satisfied by $\partial_x u$, which itself depends on an estimate on $\partial_t u$ obtained in a similar fashion.

Step 1: A fist estimate on the singular term. We start by multiplying (4.1) by $A_0[u]$ and by integrating over space, we then obtain

$$(4.6) \quad \begin{aligned} & \frac{d}{dt} \frac{1}{2} \int_{\mathbb{R}} u(t, x) A_0[u(t)](x) dx + \int_{\mathbb{R}} c(x) \partial_x u(t, x) (A_0[u(t)](x))^2 dx \\ & = - \int_{\mathbb{R}} A_0[B(\cdot; u(t)) \partial_x u(t, \cdot)](x) u(t, x) dx \end{aligned}$$

Let us now remark that

$$\begin{aligned} \int_{\mathbb{R}} A_0[B(\cdot; u(t)) \partial_x u(t, \cdot)](x) u(t, x) dx &= \int_{\mathbb{R}} \partial_x (A_0[u(t) B(\cdot; u(t))](x)) u(t, x) dx \\ &\quad - \int_{\mathbb{R}} u(t, x) A_0[\partial_x B(\cdot; u(t)) u(t)](x) dx, \end{aligned}$$

from which we obtain

$$\begin{aligned} 2 \int_{\mathbb{R}} A_0[B(\cdot; u(t)) \partial_x u(t, \cdot)](x) u(t, x) dx &= \int_{\mathbb{R}} A_0[B(\cdot; u(t)) \partial_x u(t, \cdot)](x) u(t, x) dx \\ &\quad + \int_{\mathbb{R}} \partial_x (A_0[u(t) B(\cdot; u(t))](x)) u(t, x) dx \\ &\quad - \int_{\mathbb{R}} u(t, x) A_0[\partial_x B(\cdot; u(t)) u(t)](x) dx, \\ &= \int_{\mathbb{R}} \partial_x u(t, x) \left(A_0[u(t) B(\cdot; u(t))](x) - B(x, u(t)) A_0[u(t)](x) \right) dx \\ &\quad - \int_{\mathbb{R}} u(t, x) A_0[\partial_x B(\cdot; u(t)) u(t)](x) dx. \end{aligned}$$

From Lemma 4.1,

$$\begin{aligned} \left| \int_{\mathbb{R}} u(t, x) A_0[\partial_x B(\cdot; u(t)) u(t)](x) dx \right| &\leq \|\partial_x B u\|_{\dot{H}^{\frac{1}{2}}} \|u\|_{\dot{H}^{\frac{1}{2}}} \\ &\leq C(1 + \|u\|_{\dot{H}^{\frac{1}{2}}}^2). \end{aligned}$$

Let us now compute

$$\begin{aligned}
\Gamma(x) &:= B(x)A_0[u(t)](x) - A_0[Bu](x) \\
&= \int_{\mathbb{R}} \frac{B(x)(u(x) - u(x+z)) - (B(x)u(x) - B(x+z)u(x+z))}{z^2} dz \\
&= \int_{\mathbb{R}} \frac{(B(x+z) - B(x))u(x+z)}{z^2} dz \\
&= \partial_x B(x) \int_{\mathbb{R}} \frac{u(x+z) - u(x)}{z} dz + \int_{\mathbb{R}} \Theta(x, z)u(x+z) dz,
\end{aligned}$$

where Θ is a smooth function, given by the Taylor expansion of $B(\cdot; u)$. In the previous, we lost the dependence of B in u to lighten the notation. The second term of the right side is smooth and has a finite $\dot{H}^{\frac{1}{2}}$ seminorm. The first one is the product of a smooth integrable function and $H[u]$. Since H is a bounded operator in $H^{\frac{1}{2}}$, we can deduce from the Lemma 4.1 that $\|\Gamma\|_{\dot{H}^{\frac{1}{2}}}^2 \leq C(1 + \|u\|_{\dot{H}^{\frac{1}{2}}}^2)$. Hence, returning to (4.6), we deduce that

$$\frac{d}{dt} \frac{1}{2} \|u(t)\|_{\dot{H}^{\frac{1}{2}}}^2 + \int_{\mathbb{R}} c(x) \partial_x u(t, x) (A_0[u(t)](x))^2 dx \leq C(1 + \|u(t)\|_{\dot{H}^{\frac{1}{2}}}^2).$$

Thus, from Grönwall's Lemma, we obtain that there exists $C > 0$ such that

$$\sup_{t \leq T} \|u(t)\|_{\dot{H}^{\frac{1}{2}}} + \int_0^T \int_{\mathbb{R}} c(x) \partial_x u(t, x) (A_0[u(t)](x))^2 dx dt \leq C.$$

Step 2: Estimate on the time derivative of the solution. Let us remark that since for all $t \geq 0$, $u(t)$ is non-decreasing and bounded, we deduce that $\partial_x u$ is bounded in $L_t^\infty(L_x^1)$. Hence, since B is uniformly bounded, we also deduce that $(t, x) \rightarrow B(x; u(t)) \partial_x u(t, x)$ is bounded in $L_t^\infty(L_x^1)$. On the other hand, we have that for all $t \geq 0, x \in \mathbb{R}$

$$|c(x) \partial_x u(t, x) A_0[u(t)](x)| = \sqrt{\partial_x u(t, x)} \sqrt{c(x) \partial_x u(t, x)} |A_0[u(t)](x)|.$$

Remark now that Hölder's inequality yields that $(t, x) \rightarrow c(x) \partial_x u(t, x) A_0[u(t)](x)$ is bounded in $L_t^2(L_x^1)$. Recalling the bound on $B \partial_x u$, we deduce from the equation (4.1) that $\partial_t u$ is also bounded in $L_t^2(L_x^1)$. Applying ∂_t to (4.1) and introducing $v = \partial_t u$, we find

$$\partial_t v + c(x) \partial_x u A_0[v(t)](x) + (c(x) A_0[u(t)] + B(x; u(t))) \partial_x v(t, x) = -B(x, v(t)) \partial_x u(t, x) \text{ in } (0, \infty) \times \mathbb{R}.$$

Hence, from the maximum principle (recall that A_0 preserves comparison results and that both c and $\partial_x u$ are non-negative), we obtain that

$$\frac{d}{dt} \|v(t)\|_\infty \leq \|\partial_x u(t, x)\|_\infty \left| \int_{\mathbb{R}} \beta(x, z) v(t, z) dz \right|.$$

Then, since v is bounded in $L_t^2(L_x^1)$, it follows that $(t, x) \rightarrow B(x, v(t))$ is bounded in $L_t^2(L_x^\infty)$. Hence we obtain from Grönwall's Lemma that

$$\|v(t)\|_\infty \leq C + M(t) \|\partial_x u(t, x)\|_\infty,$$

where M is a bounded function of $L^2((0, \infty), \mathbb{R})$. Note that C here depends on $\|v(0)\|_\infty$ which depends on $\|c \partial_x u_0 A_0[u_0]\|_\infty$, which can be bounded by $\|u_0\|_{C^{1, \alpha}}$.

Step 3: Bound on $\partial_x u$. Then, from (4.1), we obtain

$$(4.7) \quad \|c(x)\partial_x u A_0[u]\|_\infty \leq C + M(t)\|\partial_x u(t, x)\|_\infty.$$

We now apply ∂_x to (4.1) and, with the notation $w = \partial_x u$, we find

$$\begin{aligned} & \partial_t w + \partial_x w (c(x)A_0[u(t)] + B(x; u(t))) + c(x)w(t, x)A_0[w(t)] \\ & = -c'(x)w(t, x)A_0[u(t)](x) - \partial_x B(x; u(t))w(t, x) \text{ in } (0, \infty) \times \mathbb{R}. \end{aligned}$$

Recall now that since $\inf_x c(x) > 0$ and c' is bounded, we can use the bound (4.7) to obtain with Grönwall's Lemma that w is bounded which proves the claim. \square

In fact, we have even proven more regularity as the next result explains.

Proposition 4.3. *Under the assumptions of the previous result, there exists $C > 0$ which depends only on $T, c, B, \|u(0, \cdot)\|_{C^{1,\alpha}}$ and $\eta \in (0, 1)$ such that the norm of u in $C^\eta([0, T], C^\eta(\mathbb{R}))$ is bounded by C .*

Proof. Recalling the previous proof, there exists $C > 0$ such that

$$\|\partial_t u\|_\infty + \|\partial_x u\|_\infty \leq C.$$

Hence, since $\|u\|_\infty \leq \|u|_{t=0}\|_\infty$, we deduce from the so-called Aubin-Lions Lemma the required result. \square

4.2. Uniqueness of Lipschitz continuous viscosity solutions. We now establish a uniqueness result for viscosity solutions of (4.1) which satisfy the regularity established in the previous Section.

Theorem 4.4. *Assume that (4.4) holds. Consider $T > 0$, u_1 and u_2 two viscosity solutions of*

$$\partial_t u + c(x)\partial_x u A_0[u] + B(x; u)\partial_x u = 0 \text{ in } (0, T) \times \mathbb{R},$$

such that $\partial_x u_1, \partial_x u_2 \in L^\infty$ and such that u_1 and u_2 are both continuous, bounded and satisfy $u_1(0, x) = u_2(0, x)$ for all $x \in \mathbb{R}$. Then $u_1 = u_2$.

Proof. Let us first remark that (4.4) is clearly sufficient to show that there exists $C > 0$ such that for $v, w \in L^\infty$,

$$\|B(\cdot; v) - B(\cdot; w)\|_\infty \leq C\|v - w\|_\infty.$$

We denote by $M(t) := \|u_2(t) - u_1(t)\|_\infty$ and, for $\epsilon, r > 0$, we consider

$$M_{\epsilon, r}(s, t, x, y) := u_1(t, x) - u_2(s, y) - \frac{1}{2\epsilon}((x - y)^2 + (t - s)^2) - r\psi(x),$$

where $\psi(x) = (x^2 + 1)^{\frac{1}{3}}$. We want to show that $M(t) = 0$ for all $t \geq 0$. In order to do so, we are going to prove that $M(t) \leq C \int_0^t M(s) ds$ for some $C > 0$, which is sufficient to establish the claim since $M(0) = 0$. Recall that from the time regularity of u_1 and u_2 , $M(\cdot)$ is continuous. Assume that the previous inequality does not hold, hence, up to exchanging u_1 and u_2 , there exists $\kappa, \gamma, T > 0 > 0$ such that for any $\epsilon, r > 0$ sufficiently small,

$$\sup \left\{ M_{\epsilon, r} - \gamma(t + s) - C_0 \int_0^t M(s) ds \mid x, y \in \mathbb{R}, t, s \in [0, T] \right\} \geq \kappa,$$

where C_0 is a constant to be chosen later on. Since u_1 and u_2 are bounded, there exists a point of maximum (t_*, s_*, x_*, y_*) of $M_{\epsilon, r}(t, s, x, y) - \gamma(t + s) - C_0 \int_0^t M(s) ds$. We are going to assume that $s_*, t_* > 0$. The case in which one of the two is 0 can be treated by standard arguments quite easily.

Since u_1 is a viscosity solution of (4.1), it is in particular a viscosity sub-solution of this equation. Hence, we obtain that for any $\delta > 0$

$$c(x_*) \left(\frac{x_* - y_*}{\epsilon} + r\psi'(x_*) \right) \left(\int_{\{|z| \leq \delta\}} \frac{\varphi_1(x_*) - \varphi_1(x_* + z)}{z^2} dz + \int_{\{|z| > \delta\}} \frac{u_1(t_*, x_*) - u_1(t_*, x_* + z)}{z^2} dz \right) + \frac{t_* - s_*}{\epsilon} + \gamma + C_0 M(t_*) + B(x_*; u_1(t_*)) \frac{x_* - y_*}{\epsilon} \leq 0,$$

where $\varphi_1(x) = \frac{1}{2\epsilon}(x - y_*)^2 + r\psi(x)$. Similarly, since u_2 is a viscosity super-solution of (4.1), we also deduce that

$$c(y_*) \frac{x_* - y_*}{\epsilon} \left(\int_{\{|z| \leq \delta\}} \frac{\varphi_2(y_*) - \varphi_2(y_* + z)}{z^2} dz + \int_{\{|z| > \delta\}} \frac{u_2(s_*, y_*) - u_2(s_*, y_* + z)}{z^2} dz \right) + \frac{t_* - s_*}{\epsilon} - \gamma + B(y_*; u_2(s_*)) \frac{x_* - y_*}{\epsilon} \geq 0,$$

where $\varphi_2(y) = -\frac{1}{2\epsilon}(x_* - y)^2$. Combining the two relations yields

$$\begin{aligned} 2\gamma + c(y_*) \frac{x_* - y_*}{\epsilon} & \left(\int_{\{|z| \leq \delta\}} \frac{\varphi_1(x_*) - \varphi_2(y_*) - \varphi_1(x_* + z) + \varphi_2(y_* + z)}{z^2} dz \right. \\ & \left. + \int_{\{|z| > \delta\}} \frac{u_1(t_*, x_*) - u_2(s_*, y_*) - u_1(t_*, x_* + z) + u_2(s_*, y_* + z)}{z^2} dz \right) \\ & + \left(r\psi'(x_*)c(x_*) + \frac{(c(x_*) - c(y_*))(x_* - y_*)}{\epsilon} \right) \left(\int_{\{|z| \leq \delta\}} \frac{\varphi_1(x_*) - \varphi_1(x_* + z)}{z^2} dz \right. \\ & \left. + \int_{\{|z| > \delta\}} \frac{u_1(t_*, x_*) - u_1(t_*, x_* + z)}{z^2} dz \right) \\ & + \frac{x_* - y_*}{\epsilon} (B(x_*; u_1(t_*)) - B(y_*; u_2(s_*))) + C_0 M(t_*) \leq 0 \end{aligned}$$

Using the definition of φ_1 and φ_2 , we obtain that

$$\int_{\{|z| \leq \delta\}} \frac{\varphi_1(x_*) - \varphi_2(y_*) - \varphi_1(x_* + z) + \varphi_2(y_* + z)}{z^2} dz \leq C\delta(r + \epsilon^{-1}).$$

From the definition of (t_*, s_*, x_*, y_*) we deduce that

$$\begin{aligned} \int_{\{|z| > \delta\}} \frac{u_1(t_*, x_*) - u_2(s_*, y_*) - u_1(t_*, x_* + z) + u_2(s_*, y_* + z)}{z^2} dz & \leq r \int_{\{|z| > \delta\}} \frac{\psi(x_*) - \psi(x_* + z)}{z^2} dz \\ & \leq r(1 + \delta^{-1}). \end{aligned}$$

From the same argument, it also holds true that

$$\int_{\{|z| \leq \delta\}} \frac{\varphi_1(x_*) - \varphi_1(x_* + z)}{z^2} dz + \int_{\{|z| > \delta\}} \frac{u_1(t_*, x_*) - u_1(t_*, x_* + z)}{z^2} dz \leq C\delta(r + \epsilon^{-1}) + C\delta^{-1}$$

Finally, thanks to the regularity of B ,

$$|B(x_*; u_1(t_*)) - B(y_*; u_2(s_*))| \leq C_1(|x_* - y_*| + \|u_1(t_*) - u_2(s_*)\|_\infty).$$

Using the four previous estimate, we deduce that

$$\begin{aligned} 2\gamma + C_0 M(t_*) &\leq C \left(\frac{|x_* - y_*|}{\epsilon} + r + \omega(\epsilon) \right) (\delta(r + \epsilon^{-1}) + r(1 + \delta^{-1})) \\ &\quad + C_1(|x_* - y_*| + \|u_1(t_*) - u_2(s_*)\|_\infty) \left| \frac{x_* - y_*}{\epsilon} \right|. \end{aligned}$$

As usual in this kind of arguments, we always have $(t_* - s_*)^2 + (x_* - y_*)^2 \leq \omega(\epsilon)\epsilon$, where $\omega(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$. Let us take $K > 0$ such that u_1 is, uniformly in t , K Lipschitz in x . For such a K , $|x_* - y_*| \leq K\epsilon$. Moreover, using the time regularity of u_1 , we can compute

$$\|u_1(t_*) - u_2(s_*)\|_\infty \leq M(t_*) + \|u_2(t_*) - u_2(s_*)\|_\infty \leq M(t_*) + \tilde{\omega}(|t_* - s_*|),$$

for a modulus of continuity ω , which depends only on u_2 . Hence, using those relations, we arrive at

$$2\gamma + C_0 M(t_*) \leq C(K + \omega(\epsilon) + r)(r(1 + \delta^{-1}) + \delta\epsilon^{-1}) + C_1(K\epsilon + M(t_*) + C\tilde{\omega}(\sqrt{\epsilon}))$$

Hence, taking $C_0 > C_1$ and simplifying by $C_1 M(t_*)$, we arrive at a contradiction by taking first the limit $r \rightarrow 0$, and then setting $\delta = \epsilon^2$ and taking the limit $\epsilon \rightarrow 0$. The previous contradiction implies that $M(t) \leq C_0 \int_0^t M(s) ds$, from which the result follows. \square

Putting together Propositions 4.2 and 4.3 and Theorem 4.4, we then arrive at the following.

Theorem 4.5. *Assume that (4.4) and (4.5) hold and that u_0 is a non-decreasing function such that $\|u_0\|_{C^{1,\alpha}} < \infty$. Then, there exists a unique viscosity solution u of (4.1) which is continuous and such that $\partial_x u$ is a bounded function, locally in time.*

4.3. Comments on the cases in which $c(\cdot)$ vanishes. Let us note that the case in which $c(\cdot)$ vanishes can lead to more complex situations. To justify our claim, we recall the so-called Wishart case. In this situation, the PDE at interest is given by

$$(4.8) \quad \partial_t u + x \partial_x u \int_{\mathbb{R}_+} \frac{u(x) - u(y)}{(x - y)^2} dy + (\eta - 1) \partial_x u + b(t, x) \partial_x u = 0 \text{ in } (0, \infty) \times (0, \infty).$$

In the previous, $\eta \geq 1$ is a parameter of the model, b is a standard drift term and the previous equation is associated to the Dirichlet boundary condition

$$u(t, 0) = 0 \text{ for all } t \geq 0.$$

When b is simply given by $b(t, x) = -x$, there are explicit stationary solutions of the previous equation. There are given by means of the so-called Marcenko-Pastur distributions, which we recall here. Consider the density $m : (0, \infty) \rightarrow \mathbb{R}_+$ given by

$$m(x) = \frac{\eta \sqrt{(\lambda_+ - x)(x - \lambda_-)}}{2\pi x} \mathbb{1}_{[\lambda_-, \lambda_+]}(x),$$

where $\lambda_{\pm} = (1 \pm \sqrt{\eta^{-1}})^2$. Then, define $F(x) = \int_0^x m(y)dy$. This function F satisfies for all $x > 0$

$$\partial_x F(x) \left(\eta - 1 - x + x \int_{\mathbb{R}_+} \frac{F(x) - F(y)}{(x-y)^2} dy \right) = 0.$$

Hence, F is a stationary solution of (4.8) when $b(t, x) = -x$.

Note that for any $\eta > 1$, this stationary solution is Lipschitz continuous, but that this property fails for $\eta = 1$, because F is only $C^{\frac{1}{2}}$. A singularity is then present at $x = 0$, precisely the point where both $c(\cdot)$ and $b(\cdot)$ vanish. In our opinion, this singularity is a strong argument in favor of the non-propagation of Lipschitz regularity. Even though, we insist upon the fact that even if: i) F is the unique stationary solution of this equation, ii) it is the limit in long time of any solution of (4.8), it is still not sufficient to establish the fact that we cannot propagate Lipschitz regularity as it could simply deteriorate in time.

In this case, because the singularity happens at the boundary of the domain of interest, it is fairly immediate to overcome this possible singularity at the boundary by using the Dirichlet boundary conditions. However, in general, if $c(\cdot)$ vanishes inside of the domain, the situation seems to require new arguments.

4.4. The case of a singular drift. We explain in this Section how the previous study naturally extends to the case of

$$(4.9) \quad \partial_t u + c(x)\partial_x u A_0[u] + b(x)\partial_x u = 0 \text{ in } (0, \infty) \times \mathbb{R},$$

where c is still a smooth function satisfying $\inf c > 0$ but now $b : \mathbb{R} \rightarrow \mathbb{R}$ is only a measurable function on which we impose that there exists $C_b > 0$ such that

$$(4.10) \quad \begin{aligned} |b(x)| &\leq C_b, \text{ a. e. for } x \in \mathbb{R}, \\ b + C_b Id &\text{ is a non-decreasing function.} \end{aligned}$$

The last assumption on b implies in particular that for any $x \in \mathbb{R}$, one has $b(x^-) := \lim_{y \rightarrow x^-, y \neq x} b(y) \leq \lim_{y \rightarrow x^+, y \neq x} b(y) =: b(x^+)$. With a slight abuse of notation, we shall denote the interval $[b(x^-), b(x^+)] = b(x)$. We can then reformulate the second assumption on b with

$$(4.11) \quad \forall x, y \in \mathbb{R}, p \in b(x), q \in b(y) \quad (p - q)(x - y) \geq -C(x - y)^2,$$

$$(4.12) \quad \partial_x b \geq -C.$$

We refer the interested reader to [8] for a systematic study of transport equation with drifts presenting this kind of regularity. In our framework, we adapt slightly the notion of viscosity solution we already gave for this case. Namely, we are concerned with the following Definition.

Definition 3. *An usc function $u : \mathbb{R}_+ \times \mathbb{R}$ is a viscosity sub-solution of (2.2) if for any $T > 0$, for any $\phi \in C_b^{1,1}$, $0 < \delta \leq \infty$ $(t_0, x_0) \in (0, T] \times \mathbb{R}$ point of maximum (resp. minimum) of $u - \phi$, there exists $p_0 \in b(x_0)$ such that*

$$\begin{aligned} \partial_t \phi(t_0, x_0) + \partial_x \phi(t_0, x_0) c(x_0) (A_{-\delta}[\phi(t_0)](x_0) + A_{\delta}[u(t_0)](x_0)) \\ + p_0 \partial_x \phi(t_0, x_0) \leq 0 \text{ (resp. } \geq 0). \end{aligned}$$

We can provide the following result.

Theorem 4.6. *Given a non-decreasing bounded initial condition u_0 , there exists a unique bounded, non-decreasing in x , viscosity solution of (4.9).*

The uniqueness part relies essentially on those two Lemmas.

Lemma 4.7. *Let u_1 and u_2 be two bounded functions, non-decreasing in x for all $t \geq 0$, which are respectively viscosity sub and super solutions of (4.9). Assume that one of them is, uniformly in time, Lipschitz continuous in x . If for all $x \in \mathbb{R}$, $u_1(0, x) \leq u_2(0, x)$, then for all $t \geq 0, x \in \mathbb{R}$, $u_1(t, x) \leq u_2(t, x)$.*

Proof. The proof of this Lemma is a straightforward extension of Lemma 4.1 in [3]. Indeed, the localization argument can be done exactly as in the proof of Theorem 4.4, namely by the addition of the term in $r\psi$. The terms involving b will simply be $\epsilon^{-1}(b(x_*) - b(y_*))(x_* - y_*)$, where $b(x_*)$ and $b(y_*)$ could be replaced by any element in those sets. By the assumptions on b , this term is greater than $-C_b\epsilon^{-1}(x_* - y_*)^2$ which converges toward 0 as $\epsilon \rightarrow 0$. Hence the proof can be carried on in the same manner. \square

Lemma 4.8. *Let $u_0 : \mathbb{R} \rightarrow \mathbb{R}$ be a non-decreasing bounded smooth function. Then, there exists a viscosity solution u of (4.9) such that $u \in L_{loc}^\infty(\mathbb{R}_+, W^{1,\infty}(\mathbb{R}))$.*

Proof. Fix a time horizon $T > 0$. The proof of this result is once again an extension of the case $b = 0$. The main argument consists in proving an a priori estimate on the propagation of Lipschitz regularity for solutions of (4.9). This reduces to proving an estimation on the supremum of non-negative solutions v (which is in fact $v = \partial_x u$) of (4.13)

$$\partial_t v + c(x)A_0[u]\partial_x v + c(x)vA_0[v] + b(x)\partial_x v + c'(x)vA_0[v] + \partial_x b(x)v = 0 \text{ in } (0, \infty) \times \mathbb{R},$$

with bounded initial condition. Note that in the previous equation, the first four terms are clearly compatible with maximum principle like arguments. From the assumptions made on b and the non-negativity of v , we know that

$$\partial_x b(x)v \geq -Cv \text{ in } (0, \infty) \times \mathbb{R}.$$

Concerning the term in $c'(\cdot)$, arguing as in Proposition 4.2 in [3], we can obtain an estimate by bounding $\partial_t u = w$. Indeed, taking the time derivative of (4.9) and denoting $w = \partial_t u$, we arrive at

$$\partial_t w + c(x)\partial_x w A_0[u] + c(x)\partial_x u A_0[w] + b(x)\partial_x w = 0 \text{ in } (0, \infty) \times \mathbb{R},$$

which clearly yields comparison with constants. Hence, we can bound $\|\partial_t u\|$ with its initial value which is $\|c\partial_x u_0 A_0[u_0] + b\partial_x u_0\|_\infty$. From this bound on $\partial_t u$, we obtain a bound on $c\partial_x u A_0[u] + b\partial_x u$. We now use that c is uniformly bounded from below and that b is bounded to obtain

$$\|c'(x)vA_0[v]\|_\infty \leq C_{u_0,b}(1 + \|v\|_\infty).$$

Coming back to (4.13), we are now able to conclude using Gronwall's inequality, that there exists $C > 0$ depending only on u_0 and b such that $\|\partial_x u\|_\infty \leq C$.

Now that this a priori estimate is established, the existence of a solution satisfying such requirements is classical and we do not detail it here. \square

With the help of those two results, the uniqueness part of Theorem 4.6 is proven exactly by following the lines of Theorem 6 in [3]. Concerning the existence part, the proof now follows.

Proof. Consider $u_{0,\epsilon}$ and u_0^ϵ two Lipschitz non-decreasing functions such that for all $\epsilon > \epsilon' > 0$, $u_{0,\epsilon} \leq u_{0,\epsilon'} \leq u_0 \leq u_0^{\epsilon'} \leq u_0^\epsilon$ together with $u_0^\epsilon - u_{0,\epsilon} \leq \epsilon$. For any $\epsilon > 0$, denote by u_ϵ (resp. u^ϵ) the solution of (4.9) given by Lemma 4.8 with initial condition $u_{0,\epsilon}$ (resp. u_0^ϵ). From Lemma 4.7, we know that $(u_\epsilon)_{\epsilon>0}$ (resp. $(u^\epsilon)_{\epsilon>0}$) is a non-decreasing (resp. non increasing) sequence. Hence it converges toward some function u_* (resp. u^*). We clearly have $u_* \leq u^*$. On the other hand, thanks to Lemma 4.9 below, we know that the lsc regularization of u_* is a viscosity super solution of (4.9) and that the usc regularization of u^* is a viscosity sub solution of (4.9), hence we deduce from the comparison principle that $u_* \geq u^\epsilon - \epsilon$ since the inequality holds at the initial time. Passing to the limit $\epsilon \rightarrow 0$, we deduce that $u_* = u^*$ is a viscosity solution of (4.9) with initial condition u_0 . \square

Lemma 4.9. *Let $(u_n)_{n \geq 0}$ be a bounded sequence of bounded viscosity sub solutions of (4.9) (resp. viscosity super solutions). The function $u^* := \limsup_{n \rightarrow \infty} u_n$ (resp. $u^* = \liminf_{n \rightarrow \infty} u_n$) is a viscosity sub solution (resp. a viscosity super solution) of (4.9).*

Proof. Let $\phi \in C_b^{1,1}$ such that $u^* - \phi$ has a point of maximum at (t_*, x_*) . Without loss of generality, we can assume that this is a point of strict maximum. Once again, without loss of generality, we can assume that $u_n - \phi$ has a point of maximum at some (t_n, x_n) . Remark that $(t_n, x_n) \rightarrow (t_*, x_*)$ as $n \rightarrow \infty$. Since u_n is a viscosity sub solution of (4.9), it holds that for all $n \geq 0$, there exists $p_n \in b(x_n)$ such that

$$\partial_t \phi(t_n, x_n) + c(x_n) \partial_x \phi(t_n, x_n) A_0[\phi(t_n, \cdot)](x_n) + b(x_n) \partial_x \phi(t_n, x_n) \leq 0.$$

Thanks to the regularity of b , we deduce that, extracting a subsequence if necessary, $p_n \rightarrow p_* \in b(x_*)$ as $n \rightarrow \infty$. Passing to the limit in the equation, we deduce that u^* is viscosity sub solution of (4.9). The case of viscosity super solutions follows the same argument. \square

5. SOME REGULARIZING PROPERTIES OF THE DYSON FLOW

In this Section, we prove various results around the so-called Dyson flow, i.e. the equation

$$(5.1) \quad \partial_t m + \partial_x (mH[m]) = 0 \text{ in } (0, \infty) \times \mathbb{R},$$

when the initial condition m_0 is in $\mathcal{P}(\mathbb{R})$. In each of the following cases, the strategy of proof is the same: we show an a priori estimate, or a certain property, for smooth solutions of (5.1); we then proceed by a regularizing argument to show that the associated a priori estimate, or property, is also valid for solutions m characterized by the fact that $u(t, x) = m(t, (-\infty, x])$ is a viscosity solution of (2.1)

5.1. Regularizing effect in L^∞ . We show a L^∞ regularizing effect of the Dyson flow. We begin with the following.

Proposition 5.1. *Let m be a C^1 function, C^2 in space, and globally Lipschitz continuous, classical solution of (1.1). Then, for any $t > 0$, $m(t, \cdot)$ belongs to L^∞ and*

$$\|m(t, \cdot)\|_\infty \leq \frac{1}{t}.$$

Proof. For any $t > 0$, consider a point \bar{x} of local maximum of $m(t, \cdot)$. From the regularity of m , such a point exists. Evaluating (1.1) at this point, we obtain that

$$\partial_t m(t, \bar{x}) + m(t, \bar{x})(H[\partial_x m(t, \bar{x})]) = 0.$$

Hence,

$$\partial_t m(t, \bar{x}) + m(t, \bar{x}) \int_{\mathbb{R}} \frac{m(t, \bar{x}) - m(t, y)}{(\bar{x} - y)^2} dy = 0.$$

Let us compute for $\delta > 0$

$$\begin{aligned} \int_{\mathbb{R}} \frac{m(t, \bar{x}) - m(t, y)}{(\bar{x} - y)^2} dy &\geq \int_{|\bar{x} - y| > \delta} \frac{m(t, \bar{x}) - m(t, y)}{(\bar{x} - y)^2} dy \\ &\geq m(t, \bar{x}) \int_{|\bar{x} - y| > \delta} \frac{1}{(\bar{x} - y)^2} dy - \int_{|\bar{x} - y| > \delta} \frac{m(t, y)}{\delta^2} dy \\ &\geq 2m(t, \bar{x})\delta^{-1} - \delta^{-2}. \end{aligned}$$

Choosing $\delta = m(t, \bar{x})^{-1}$, we obtain that

$$\partial_t m(t, \bar{x}) \leq -m(t, \bar{x})^{-2},$$

from which the required estimate follows since it implies

$$(5.2) \quad \frac{d}{dt} \|m(t, \cdot)\|_{\infty} \leq -\|m(t, \cdot)\|_{\infty}^2.$$

□

It now remains to justify that the previous estimate is valid for non-smooth m .

Lemma 5.2. *Let $\epsilon > 0$ and m_0 a C_b^2 non-negative function¹ in $L^1(\mathbb{R})$. Then there exists a (unique) solution of*

$$(5.3) \quad \partial_t m - \epsilon \partial_{xx} m + \partial_x(mH[m]) = 0 \text{ in } (0, \infty) \times \mathbb{R},$$

in $C_b^{1,2}$ which satisfies the initial condition $m|_{t=0} = m_0$.

Proof. Fix a time horizon $T > 0$. We start by assuming that such a smooth function exists and we want to show an a priori estimate (depending on $\epsilon > 0$ obviously). We denote by $C > 0$ a constant which depends only on m_0 and ϵ . First, since the term in ϵ does not perturb the proof of the L^∞ a priori estimate above, there exists $C > 0$ such that $\|m\|_{\infty} \leq C$. Since $m(t, \cdot)$ is in L^1 for all time $t \geq 0$, we also deduce that it is in all the L^p , for $1 \leq p \leq \infty$. Multiplying (5.3) by m and integrating over space and time, we then obtain

$$\frac{d}{dt} \int_{\mathbb{R}} m^2 + \epsilon \int_0^t \int_{\mathbb{R}} (\partial_x m)^2 = \int_0^t \int_{\mathbb{R}} (\partial_x m) m H[m].$$

Since $m \in L^\infty(L^2)$, there exists $C > 0$ such that $\|m(t)H[m(t)]\|_{L^1} \leq C$. Moreover, $mH[m] \in L^\infty$. Indeed, recall that $\|m(t)\|_{\infty} \leq C$. Furthermore, by simply splitting the integral defining $H[m]$ around the singularity, we also have that for all $\kappa > 0$, there

¹ C_b^2 is the subspace of C^2 of bounded functions with bounded first and second order derivatives

exists $C_\kappa > 0$ such that for all μ , $\|H[\mu]\|_\infty \leq \kappa\|\mu\|_{C^\alpha} + C_\kappa\|\mu\|_{L^1}$, for $\alpha \in (0, \frac{1}{2})$. Hence, recalling the energy estimate, we obtain

$$\frac{d}{dt} \int_{\mathbb{R}} m^2 + \epsilon \int_0^t \int_{\mathbb{R}} (\partial_x m)^2 \leq \|m\|_{L^2(H^1)} (\kappa\|m\|_{L^2(C^\alpha)} + C_\kappa\|m_0\|_{L^1}).$$

From the usual embedding of H^1 into C^α , we deduce that $\|m\|_{L^2(H^1)} \leq C$.

Further regularity can be obtained by looking at the equation satisfied by $w = \partial_x m$. Multiplying this relation by w and integrating yields

$$\frac{d}{dt} \int_{\mathbb{R}} w^2 + \epsilon \int_0^t \int_{\mathbb{R}} (\partial_x w)^2 = -2 \int_0^t \int_{\mathbb{R}} w^2 H[w] - \int_0^t \int_{\mathbb{R}} \partial_x w (H[m]w - H[mw]).$$

Arguing as in the previous case to bound $\|H[mw]\|_\infty$, we deduce that $\|w\|_{L^\infty(L^2)} + \|w\|_{L^2(H^1)} \leq C$. Further regularity can then be obtained in a similar fashion by bootstrapping techniques.

Once regularity is obtained, the existence of a solution of (5.3) with this regularity is classical and we do detail it here. Furthermore, in this situation, the uniqueness of such a solution is immediate. \square

We can now establish the main result of this Section.

Proposition 5.3. *For any $m_0 \in \mathcal{P}(\mathbb{R})$, the unique solution m of (5.1) satisfies that for any $t > 0$, $m(t)$ has a density with respect to the Lebesgue measure, which is bounded by t^{-1} .*

Proof. Take $\delta, \epsilon > 0$, and consider the $u_{\delta, \epsilon}$, the unique viscosity solution of

$$\partial_t u_{\delta, \epsilon} - \epsilon \partial_{xx} u_{\delta, \epsilon} + (\partial_x u_{\delta, \epsilon}) A_0[u_{\delta, \epsilon}] = 0 \text{ in } (0, \infty) \times \mathbb{R},$$

with initial condition $u_{\delta, \epsilon}(0, x) = m_\delta((-\infty, x])$, where m_δ is a smooth approximation of m_0 . Thanks to Lemma 5.2, this viscosity solution is in fact smooth. Moreover, from Proposition 5.1, it is (uniformly in ϵ, δ) Lipschitz continuous in space. Taking the limit $\epsilon, \delta \rightarrow 0$, we deduce that the sequence $(u_{\delta, \epsilon})_{\delta, \epsilon > 0}$ converges toward the unique viscosity solution of

$$\partial_t u + (\partial_x u) A_0[u] = 0 \text{ in } (0, \infty) \times \mathbb{R},$$

with initial condition $u(0, x) = m_0((-\infty, x])$. Hence, we deduce the result. \square

5.2. Decay of L^p norms. We prove here the following statement.

Proposition 5.4. *For any $m_0 \in \mathcal{P}_2(\mathbb{R})$, the unique solution of (5.1) with initial condition m_0 is such that for any $0 \leq t \leq s$, $1 \leq p \leq \infty$*

$$\|m(t)\|_p \geq \|m(s)\|_p.$$

Proof. We only prove the statement for smooth solutions, the general results can be obtained by approximation like the previous one, provided some sort of continuity in time for the narrow convergence of probability measures. Such a continuity holds since $m_0 \in \mathcal{P}_2(\mathbb{R})$. The relation is clearly true for $p = 1$ and, recalling the proof of Proposition

5.1, it is also true for $p = \infty$. Hence, it only remains to prove it for $1 < p < \infty$. Multiplying (5.1) by m^{p-1} and integrating we obtain

$$\begin{aligned}
\frac{d}{dt} \|m(t)\|_p^p &= \int_{\mathbb{R}} (\partial_x m) m m^{p-1} H[m] \\
&= \frac{1}{p} \int_{\mathbb{R}} \partial_x (m^p) H[m] \\
&= -\frac{1}{p} \int_{\mathbb{R}} m^p A_0[m] \\
&= -\frac{1}{p} \int_{\mathbb{R}} m^p(x) \int_{\mathbb{R}} \frac{m(x) - m(y)}{(x-y)^2} dy dx \\
&= -\frac{1}{2p} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{(m(x) - m(y))(m^p(x) - m^p(y))}{(x-y)^2} dx dy.
\end{aligned}$$

The last term being clearly non-positive since $m \geq 0$, the result follows. \square

5.3. Around the free entropy. We introduce the function $E : \mathcal{P}_2(\mathbb{R}) \rightarrow \mathbb{R} \cup \{-\infty\}$ which is often called the free entropy of the physical system associated to the Dyson flow. It is defined by

$$E(\mu) = \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \log(|x-y|) \mu(dx) \mu(dy).$$

For sufficiently smooth μ , we have the following

$$\begin{aligned}
\nabla_{\mu} E(\mu, x) &= \int_{\mathbb{R}} \log(|x-y|) \mu(dy), \\
D_{\mu} E(\mu, x) &= H[\mu](x).
\end{aligned}$$

Hence, this quantity is deeply linked with (5.1). We shall prove later on that it is increasing and continuous along the Dyson flow. We begin by remarking that E can be easily expressed in terms of the Fourier transform. Indeed, denote by \hat{f} the normalized Fourier transform of a distribution f and by $g(x) = \log(|x|)$. Recall the classical relation

$$\hat{g}(\xi) = -\frac{1}{2} f.p. \left(\frac{1}{|\xi|} \right) - \gamma \delta_0(\xi),$$

where $f.p. \left(\frac{1}{|\xi|} \right)$ denotes the finite part of $\xi \rightarrow |\xi|^{-1}$ and γ is the Euler constant. Hence, since $E(m) = \frac{1}{2} \langle m * g, m \rangle_{L^2}$, it holds that

$$E(m) = -\frac{1}{4} \left\langle f.p. \left(\frac{1}{|\xi|} \right), \hat{m}^2 \right\rangle_{L^2} - \gamma.$$

This remark leads us to the following result.

- Proposition 5.5.**
- Take $m \in \mathcal{P}_2$, then $E(m) > -\infty$ if and only if $m \in H^{-\frac{1}{2}}$.
 - For any $m \in \mathcal{P}_2$ such that $E(m) > -\infty$, there exists a sequence $(m^n)_{n \geq 0}$, valued in \mathcal{P}_2 , with smooth bounded densities, such that $m^n \rightharpoonup m$ and $E(m^n) \rightarrow E(m)$ as $n \rightarrow \infty$.
 - The function $E(\cdot)$ is upper semi-continuous for the narrow topology.

Proof. Since $m \in \mathcal{P}_2$, it follows that \hat{m} belongs to $C_b^2(\mathbb{R})$. Hence

$$\left\langle f.p. \left(\frac{1}{|\xi|} \right), \hat{m}^2 \right\rangle_{L^2} < \infty \iff \|m\|_{H^{-\frac{1}{2}}}^2 := \int_{\mathbb{R}} \frac{\hat{m}(\xi)^2}{\sqrt{1+|\xi|^2}} < \infty.$$

This proves the first part of the claim. For the second part, it suffices to remark that the result is true if we can choose $(m^n)_{n \geq 0}$ such that

$$\frac{1}{4} \left\langle f.p. \left(\frac{1}{|\xi|} \right), \hat{m}^{n^2} \right\rangle_{L^2} \xrightarrow{n \rightarrow \infty} \frac{1}{4} \left\langle f.p. \left(\frac{1}{|\xi|} \right), \hat{m}^2 \right\rangle_{L^2},$$

which is true since $m \in \mathcal{P}_2$ and $E(m) > -\infty$. The third part of the claim is obtained by remarking that $(x, y) \rightarrow \log(|x - y|)$ is upper semi continuous. \square

The first part of this result states that $E(\cdot)$ can indeed be used to measure some regularity while the second one shall be helpful later on. We now pass to the main result of this Section.

Proposition 5.6. *Let $(m_t)_{t \geq 0}$ be a smooth solution of (5.1) valued in \mathcal{P}_2 such that $E(m_0) > -\infty$. Then for any $0 \leq t' \leq t$*

$$E(m_t) = E(m_{t'}) + \int_{t'}^t \int_{\mathbb{R}} H[m_s]^2 m_s ds.$$

Proof. We start by proving the claim in the case of smooth solutions. We can compute

$$\begin{aligned} \frac{d}{dt} E(m_t) &= \int_{\mathbb{R}} \nabla_{\mu} E(m_t, x) \partial_t m(t, x) dx \\ (5.4) \quad &= \int_{\mathbb{R}} D_{\mu} E(m_t, x) H[m_t](x) m(t, x) dx. \end{aligned}$$

From which we deduce

$$\frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \log(|x-y|) m_t(dx) m_t(dy) = \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \log(|x-y|) m_{t'}(dx) m_{t'}(dy) + \int_{t'}^t \int_{\mathbb{R}} H[m_s]^2 m_s ds.$$

It remains to justify the relation in the case of non smooth solutions. Let $m_0 \in \mathcal{P}_2$ and assume first that $m_0 \in L^{\infty}$. Take a sequence $(\rho_{\epsilon})_{\epsilon > 0}$ of smooth functions which converges in L_{loc}^p toward \log , for any $p \in [1, \infty)$, which is of the form $\rho_{\epsilon} = \log * \tilde{\rho}_{\epsilon}$. It then follows that

$$\frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \rho_{\epsilon}(|x-y|) m_t(dx) m_t(dy) = \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \rho_{\epsilon}(|x-y|) m_{t'}(dx) m_{t'}(dy) + \int_{t'}^t \int_{\mathbb{R}} H[\tilde{\rho}_{\epsilon} * m_s] H[m_s] m_s ds.$$

Hence, the claim is proved for $m_0 \in L^{\infty} \cap \mathcal{P}_2$ if we can pass to the limit $\epsilon \rightarrow 0$ in the previous inequality. The passage to the limit is indeed valid in the first two terms from $m_t, m_{t'} \in L^{\infty}$ and the fact that $m_t, m_{t'} \in \mathcal{P}_2(\mathbb{R})$. For the third terms, it suffices to recall that H is continuous in any L^p , $1 < p < \infty$ and that $\tilde{\rho}_{\epsilon} * m_s \rightarrow m_s$ as $\epsilon \rightarrow 0$, uniformly in s , in L^p for any $1 < p < \infty$.

Consider now a general $m_0 \in \mathcal{P}_2(\mathbb{R})$ with $E(m_0) > -\infty$ and a sequence (m_0^n) as in Proposition 5.5. For all $n \geq 0$, $t \geq 0$, we have

$$E(m_t^n) = E(m_0^n) + \int_0^t \int_{\mathbb{R}} H[m_s^n]^2 m_s^n ds.$$

Passing to the limit $n \rightarrow \infty$, we deduce that for all $t \geq 0$

$$E(m_t) - E(m_0) \geq \int_0^t \int_{\mathbb{R}} H[m_s]^2 m_s ds.$$

Hence, $(E(m_t))_{t \geq 0}$ is indeed increasing in time. Since it is upper semi continuous for the weak topology, we deduce that it is continuous at $t = 0$. On the other hand, thanks again to the L^∞ regularizing effect, we have for any $0 < t' \leq t$.

$$E(m_t) = E(m_{t'}) + \int_{t'}^t \int_{\mathbb{R}} H[m_s]^2 m_s ds.$$

Thus we finally obtain the result by passing to the limit $t' \rightarrow 0$. \square

Remark 5.1. *In this case, the approximation is made on the log and not on the solution of the equation itself.*

Corollary 5.1. *Let $m_0 \in \mathcal{P}_2$ be such that $E(m_0) > -\infty$. Then $(E(m_t))_{t \geq 0}$ is a continuous increasing function.*

Remark 5.2. *Let us insist upon the fact that the continuity was not obtained for the L^p norm in the previous subsection.*

5.4. Contraction in the Wasserstein space. We also recall the following result, of which we provide a new proof in the case $p = 2$.

Proposition 5.7. *Let $1 \leq p \leq \infty$, and $(\mu_t)_{t \geq 0}, (\nu_t)_{t \geq 0}$ be two solutions of (1.1) valued in \mathcal{P}_p , then for any $t \leq s$, $W_p(\mu_s, \nu_s) \leq W_p(\mu_t, \nu_t)$.*

We start by recalling Cotlar's identity for smooth functions $u : \mathbb{R} \rightarrow \mathbb{R}$,

$$H[u]^2 = \pi^2 u^2 + 2H[uH[u]].$$

Multiplying by u and integrating, we obtain

$$\begin{aligned} \int_{\mathbb{R}} H[u]^2 u &= \pi^2 \int_{\mathbb{R}} u^3 + 2 \int_{\mathbb{R}} H[u]^2 u \\ &= \frac{\pi^2}{3} \int_{\mathbb{R}} u^3. \end{aligned}$$

In particular, for a probability measure $m \in \mathcal{P}$, $H[m] \in L^2(m)$ if and only if $m \in L^3$.

Proof. We start with the case in which $(\mu_t)_{t \geq 0}$ and $(\nu_t)_{t \geq 0}$ are valued in L^3 . Consider an optimal coupling γ between μ_0 and ν_0 for 2-Wasserstein distance and (X_0, Y_0) a couple of real random variables on a standard probability space such that $\mathcal{L}((X_0, Y_0)) = \gamma$. Because the processes are valued in L^3 , we deduce from Cotlar's identity that $H[\mathcal{L}(X_0)](X_0)$ is a squared integrable random variable. Consider now the ODEs

$$\frac{d}{dt} X_t = H[\mathcal{L}(X_t)](X_t) \quad ; \quad \frac{d}{dt} Y_t = H[\mathcal{L}(Y_t)](Y_t).$$

By construction, for any $t \geq 0$, $\mathcal{L}(X_t) = \mu_t$ and $\mathcal{L}(Y_t) = \nu_t$. It then follows that for all $t \geq 0$

$$\frac{d}{dt} W_2^2(\mu_t, \nu_t)|_{t=0} \leq \frac{d}{dt} \mathbb{E}[|X_t - Y_t|^2]|_{t=0}.$$

We now compute

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \mathbb{E}[|X_t - Y_t|^2]_{t=0} &= \mathbb{E}[(X_0 - Y_0)(H[\mathcal{L}(X_0)](X_0) - H[\mathcal{L}(Y_0)](Y_0))] \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} (x - y) \left(\frac{1}{x - x'} - \frac{1}{y - y'} \right) \gamma(dx, dy) \gamma(dx', dy') \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{-(x - y)^2 + (x - y)(x' - y')}{(x - x')(y - y')} \gamma(dx, dy) \gamma(dx', dy')
\end{aligned}$$

Recall that since γ is an optimal coupling, $(x - x')(y - y') \geq 0$ holds $\gamma \times \gamma$ almost everywhere. Using $2(x - x')(y - y') \leq (x - y)^2 + (x' - y')^2$, we obtain that

$$\frac{1}{2} \frac{d}{dt} \mathbb{E}[|X_t - Y_t|^2]_{t=0} \leq \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{(x' - y')^2 - (x - y)^2}{(x - x')(y - y')} \gamma(dx, dy) \gamma(dx', dy').$$

The term on the right side vanishes by symmetry, hence $\frac{d}{dt} \mathbb{E}[|X_t - Y_t|^2]_{t=0} \leq 0$ from which the result follows since the same argument can be made for any time.

In the general case, thanks to the L^∞ regularizing property, we deduce that for any $\epsilon > 0$ such that $t + \epsilon \leq s$, we have $W_2(\mu_s, \nu_s) \leq W_2(\mu_{t+\epsilon}, \nu_{t+\epsilon})$. Hence, by continuity of the Dyson flow in (\mathcal{P}_2, W_2) , we obtain the required result. \square

Remark 5.3. *A similar approach could have been used in the case $p \neq 2$ but we do not present it here.*

5.5. A conjecture. We end this Section on several properties of the Dyson flow by mentioning the following property, that we are only able to conjecture at the moment.

Conjecture 1. *For any initial condition $m_0 \in \mathcal{P}(\mathbb{R})$, the unique solution m of (5.1) is such that for all $t > 0$, $m(t, \cdot) \in C^{\frac{1}{2}}(\mathbb{R})$.*

Such a property would clearly be a maximal type regularizing property of the Dyson flow since it is precisely the regularity we obtain in the case $m_0 = \delta_0$.

6. COMPLEMENTS ON MODELING

We now present various independent extensions of the Dyson model.

6.1. A bulk of eigenvalues interacting with spikes. If the $N \times N$ matrix from which derives the spectral measure m is perturbed by a matrix of rank one, it has been observed in [2], that we can identify an outlier of the spectrum. That is an eigenvalue which corresponds to the perturbation. We explain the effect of a similar perturbation in this dynamical setting. Consider that the usual $N \times N$ Dyson model is perturbed by the addition of the matrix $a_t e_1 \otimes e_1$, for $(a_t)_{t \geq 0}$ is a given smooth process. Since the rank of this perturbation is only one, it does not alter the limit spectral measure, which is then given by the usual Dyson equation. However, if we can identify the "outlier" associated to this perturbation, its evolution shall naturally depend on the rest of the

spectral measure. This leads us to the system

$$(6.1) \quad \begin{aligned} d\lambda_t &= H[m_t](\lambda_t)dt + da_t, \\ \partial_t m + \partial_x(mH[m]) &= 0 \text{ in } (0, T) \times \mathbb{R}, \\ \lambda_0 &= a_0 + H[m_0](\lambda_0), \quad m|_{t=0} = m_0. \end{aligned}$$

Note that the main mathematical difficulty in this system lies in the fact that we do not know in general if $H[m_t]$ is sufficiently smooth to characterize $(\lambda_t)_{t \geq 0}$ as the unique solution of the first equation. Indeed, even if the Conjecture 1 were true, we would only have a Hölder estimate on $H[m_t]$ which would give existence but not necessary uniqueness of such a solution.

A case which is of particular importance in our opinion is the one in which the special eigenvalue λ is greater than all the other one, more specifically when m_t is supported on a set of the form $(-\infty, R_t]$ for some $R_t \in \mathbb{R}$ and when $\lambda_t > R_t$. Indeed, in such a context, $H[m_t]$ is locally Lipschitz and decreasing around λ_t as the next elementary result states.

Lemma 6.1. *Let $m \in \mathcal{P}(\mathbb{R})$ be supported on $(-\infty, R_0]$. Then, $H[m]$ is decreasing on $[R_0, \infty)$ and it is C Lipschitz continuous on $[R_0 + \alpha, \infty)$ and C depends only $\alpha > 0$.*

Proof. Let m be such a measure. Take $\lambda \geq \lambda' \geq R_0$ and compute

$$H[m](\lambda) - H[m](\lambda') = (\lambda' - \lambda) \int_{\mathbb{R}} \frac{1}{(\lambda - x)(\lambda' - x)} m(dx).$$

Hence the result follows since for all x in the support of m , $x \leq R_0 \leq \lambda' \leq \lambda$. \square

This implies that if we can guarantee that λ stays strictly outside the support of m , then the ODE satisfied by λ is well-posed under standard assumptions on $(a_t)_{t \geq 0}$. Note that in several situations, this can be checked by an explicit computation. Indeed, consider for instance the case in which m_0 is supported on $(-\infty, R_0]$. Then, thanks to Proposition 4.4 in [3], we can compare the cumulative distribution functions of the solutions of (5.1) associated to m_0 and δ_{R_0} . The latter is given by the semi-circular law. Hence we know that for all $t \geq 0$, m_t is supported on $(-\infty, R_0 + \sqrt{t}]$. Hence, if $\lambda_0 > R_0$ and $da_t \geq \frac{1}{2\sqrt{t}}$, we know that for all $t > 0$, $\lambda_t > R_t$.

An interesting feature is that in general, the fact that $(\lambda_t)_{t \geq 0}$ stays outside of the support of m_t does not simply follow from the fact that this holds at the initial time as the next result shows.

Proposition 6.2. *Consider $m_0 = \delta_0$ and $\lambda_0 > 0$ and assume that $(a_t)_{t \geq 0}$ is constant with $a_0 > 0$. The system (6.1) is locally well posed in time and there exists $t_0 > 0$ such that $\lambda_{t_0} = \sqrt{t_0}$.*

Proof. Since $\lambda_0 > 0$ the system is well posed at least locally in time. Introduce $Z_t := \lambda_t^2 - t$. We want to show that there exists t_0 such that $Z_{t_0} = 0$. Since m_0 is Dirac mass, the solution of the Dyson equation is given by $m_t(dx) = \frac{2}{\pi t} \sqrt{t - x^2} \mathbb{1}_{[-\sqrt{t}, \sqrt{t}]}(x) dx$ and thus for any $x > \sqrt{t}$

$$H[m_t](x) = \frac{x - \sqrt{x^2 - t}}{2t}.$$

Hence we deduce that

$$\begin{aligned}\frac{d}{dt}Z_t &= \lambda_t \frac{\lambda_t - \sqrt{\lambda_t^2 - t}}{t} - 1 \\ &= \frac{Z_t - \lambda_t \sqrt{Z_t}}{t} \\ &= \frac{Z_t - \sqrt{Z_t + t} \sqrt{Z_t}}{t}\end{aligned}$$

Hence (Z_t) is decreasing, while it is positive. Remark that it cannot have a positive limit. Moreover, as it gets closer to 0, the previous ODE as a similar behaviour as $\dot{X} \leq \sqrt{X} \sqrt{t}^{-1}$, which reaches 0 in finite time, hence the result. \square

6.2. An extension to other non-linearities. Consider the equation

$$(6.2) \quad \partial_t m + \partial_x(m\sigma(m)H[m]) = 0 \text{ in } (0, \infty) \times \mathbb{R},$$

where $\sigma : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a continuous function. Then we claim that an analysis of this equation by means of the viscosity solutions of the PDE

$$(6.3) \quad \partial_t u + \sigma(\partial_x u) \partial_x u H[\partial_x u] = 0 \text{ in } (0, \infty) \times \mathbb{R}$$

can be done following the lines of [3]. Indeed, the function σ does not perturb the proof of the comparison of a viscosity super-solution and a viscosity sub-solution. Note that this is due in large part to the fact that, since we are considering here the exact Hilbert transform, and not some other singular operators, we do not need to show that the Lipschitz regularity propagates in (6.3). We can provide the following result.

Proposition 6.3. *Let u_1 be a viscosity super-solution of (6.3) and u_2 a viscosity sub-solution of the same equation. Assume moreover that both are non-decreasing in x . Then if for all $x \in \mathbb{R}$, $u_1(0, x) \geq u_2(0, x)$, it holds that for all $t \geq 0, x \in \mathbb{R}$, $u_1(t, x) \geq u_2(t, x)$. In particular, there exists at most one viscosity solution of (6.3) given an initial condition.*

Proof. We only sketch the main lines of the proof. For $\epsilon > 0$, consider a point (t_*, s_*, x_*, y_*) of maximum of

$$(t, s, x, y) \rightarrow u_1(t, x) - u_2(s, y) - \frac{1}{2\epsilon}((t-s)^2 + (x-y)^2),$$

and assume that $t_*, s_* > 0$. Using that u_1 is a viscosity sub-solution and u_2 a viscosity super-solution, we obtain that for all $\delta > 0$

$$\epsilon^{-1}(t_* - s_*) + \sigma(\epsilon^{-1}(x_* - y_*))\epsilon^{-1}(x_* - y_*) (A_{-\delta}[\phi_\epsilon](x_*) + A_\delta[u_1(t_*)](x_*)) \leq 0,$$

$$\epsilon^{-1}(t_* - s_*) + \sigma(\epsilon^{-1}(x_* - y_*))\epsilon^{-1}(x_* - y_*) (A_{-\delta}[\psi_\epsilon](y_*) + A_\delta[u_2(s_*)](y_*)) \geq 0,$$

where $\phi_\epsilon(x) = \frac{1}{2\epsilon}(x - y_*)^2$ and $\psi_\epsilon(y) = -\frac{1}{2\epsilon}(y - x_*)^2$. Since $u_1(t_*, \cdot)$ is non-decreasing we deduce as usual that $x_* \geq y_*$. Recall that from the definition of the point of maximum, $A_\delta[u_1(t_*)](x_*) - A_\delta[u_2(s_*)](y_*) \leq 0$. Hence, taking the difference of the two relations leads to

$$C\sigma(\epsilon^{-1}(x_* - y_*))\epsilon^{-1}(x_* - y_*)\delta \leq 0,$$

where we used the regularity of ϕ_ϵ and ψ_ϵ to estimate the terms in $A_{-\delta}$. The previous is the main argument to prove a comparison principle and the rest of the argument follows

what we did in the proof of Theorem 4.4, namely to localize the point of maximum or to arrive at a contradiction if it not reached for $t_*s_* = 0$. \square

6.3. Two systems in interaction. Another natural extension concerns the interaction of two systems governed by two equations of the type of (1.1). Namely, consider the system

$$(6.4) \quad \partial_t m_i + \partial_x(m_i H[m_i]) + \partial_x(b_i[m_1, m_2]m_i) = 0 \text{ in } (0, \infty) \times \mathbb{R},$$

for $i \in \{1; 2\}$. This system models two systems who are each driven by the Dyson Brownian motion but who also feels an additional force (b_i for the equation i) which depends on the state of the two systems. The following holds.

Proposition 6.4. *Assume that $b_1, b_2 : \mathcal{P}(\mathbb{R})^2 \rightarrow C^{0,1}(\mathbb{R})$ are smooth and bounded operators, and consider two probability measures $m_{0,1}$ and $m_{0,2}$ in $\mathcal{P}_2(\mathbb{R})$. Then, there exists a unique solution (m_1, m_2) of (6.4) which is characterized as being the derivative of the couple (F_1, F_2) , unique viscosity solutions of*

$$\partial_t F_i + \partial_x F_i H[\partial_x F_i] + b_i(\partial_x F_1, \partial_x F_2) \partial_x F_i = 0 \text{ in } (0, \infty) \times \mathbb{R},$$

with initial condition $F_i(0, x) = m_{0,i}((-\infty, x])$.

Proof. The existence follows easily from a compactness argument. Fix $T > 0$ and denote by $B := b_1(\mathcal{P}(\mathbb{R})) \cup b_2(\mathcal{P}(\mathbb{R}))$. Take $b : \mathbb{R}_+ \rightarrow B$ and consider the solution m^b of

$$\partial_t m^b + \partial_x(m^b H[m^b]) + \partial_x(bm^b) = 0 \text{ in } (0, T) \times \mathbb{R}.$$

Take a time $t \in [0, T]$ and an optimal coupling (X_t, Y_t) for the squared Wasserstein distance between $m^{b'}(t)$ and $m^b(t)$. Consider now the SDE

$$\begin{aligned} dX_s &= H[\mathcal{L}(X_s)](X_s)ds + b'(s, X_s)ds \\ dY_s &= H[\mathcal{L}(Y_s)](Y_s)ds + b(s, Y_s)ds. \end{aligned}$$

Then it follows from the computation of Proposition 5.7 that

$$\begin{aligned} \frac{d}{ds} \mathbb{E}[|X_s - Y_s|^2] &\leq \mathbb{E}[(X_s - Y_s)(b'(s, X_s) - b(s, Y_s))] \\ &= \mathbb{E}[(X_s - Y_s)(b(s, X_s) - b(s, Y_s))] + \mathbb{E}[(X_s - Y_s)(b'(s, X_s) - b(s, X_s))] \\ &\leq C \mathbb{E}[|X_s - Y_s|^2] + \frac{1}{2} \|b - b'\|_\infty^2. \end{aligned}$$

From Grönwall's Lemma, we obtain that there exists C such that

$$W_2^2(m^b(t), m^{b'}(t)) \leq (e^{Ct} - 1) \|b - b'\|_\infty^2.$$

Since there exists $C > 0$ such that for all $b \in B$,

$$W_2^2(m^b(t), m^b(s)) \leq C \sqrt{|t - s|},$$

we deduce the existence of a solution of the system from standard fixed point theorems.

The uniqueness part follows the line of the Grönwall estimate used in the proof of Theorem 4.4. Denote by (m_1, m_2) and $(\tilde{m}_1, \tilde{m}_2)$ two solutions of the system (6.4). We define $u_1(t, x) = \int_{-\infty}^x m(t, y) dy$ and similarly for u_2, \tilde{u}_1 and \tilde{u}_2 . Define $M(t) := \|(u_1(t), u_2(t)) - (\tilde{u}_1(t), \tilde{u}_2(t))\|_\infty$. From the regularity we assumed on b_1 and b_2 , we

deduce (from the proof of Theorem 4.4) that there exists $C > 0$ such that for $i = 1, 2$, $t \geq 0$

$$\|u_i(t) - \tilde{u}_i(t)\|_\infty \leq C \int_0^t M(s) ds.$$

Hence, we obtain the uniqueness of a solution of (6.4) \square

6.4. Reflexion at the boundary. We explain how we can model the reflexion of eigenvalues at a certain boundary. This classical question in stochastic analysis is now well understood, especially since the seminal works [9, 10]. Namely, we want to explain why the reflexion of the eigenvalues of the matrix on a certain maximum level R_0 does not perturb the mathematical analysis. We do not provide the full study of such a phenomenon but rather explain why this kind of reflexion does not create a singularity near R_0 . We follow the approach of [10].

To make the matter more precise, consider an element $m_0 \in \mathcal{P}(\mathbb{R})$ with bounded density, still denoted m_0 and consider for $\epsilon > 0$ the following equation

$$(6.5) \quad \partial_t m + \partial_x(mH[m]) + \partial_x \left(\frac{1}{\epsilon} (x - R_0)_+ m(t, x) \right) = 0 \text{ in } (0, \infty) \times \mathbb{R}.$$

This kind of penalization is well known to approximate the reflexion at R_0 , or equivalently the Neumann Boundary condition at R_0 . We can prove the following result.

Lemma 6.5. *For all $\epsilon > 0$, the solution m of (6.5) satisfies for all $t \geq 0, x \geq 0$ $m(t, x) \leq \|m_0\|_\infty$.*

Proof. The result follows from a standard a priori estimate once again. It is immediate to check that, if m is smooth, for any $t \geq 0$, at any point of local maximum \bar{x} of $m(t, \cdot)$,

$$\partial_t m(t, \bar{x}) + m(t, \bar{x}) A_0[m(t, \cdot)](\bar{x}) + \frac{1}{\epsilon} \mathbb{1}_{\{\bar{x} \geq R_0\}} m(t, \bar{x}) = 0.$$

Which implies the result since the same kind of argument as in Lemma 5.2 can be carried on to justify this computation. \square

With the help of this Lemma, it is now entirely classical to establish the following.

Theorem 6.6. *Let $u_0 \in W^{1,\infty}(\mathbb{R})$. The sequence $(u_\epsilon)_{\epsilon > 0}$ of solution of (6.5) converges locally uniformly toward the unique viscosity solution of*

$$\begin{aligned} \partial_t u + \partial_x u A_0[u] &= 0 \text{ in } (0, \infty) \times (-\infty, R_0), \\ \partial_x u(t, R_0) &= 0 \text{ for all } t > 0. \end{aligned}$$

Moreover, this solution satisfies

$$u(t, x) = 1 \text{ for all } t > 0, x \geq R_0.$$

ACKNOWLEDGMENTS

The authors have been partially supported by the Chair FDD/FIME (Institut Louis Bachelier). The first author has been partially supported by the Lagrange Mathematics and Computing Research Center.

REFERENCES

- [1] Greg W Anderson, Alice Guionnet, and Ofer Zeitouni. *An introduction to random matrices*. Number 118. Cambridge university press, 2010.
- [2] Jinho Baik, Gérard Ben Arous, and Sandrine Péché. Phase transition of the largest eigenvalue for nonnull complex sample covariance matrices. *Annals of Probability*, pages 1643–1697, 2005.
- [3] Charles Bertucci, Mérouane Debbah, Jean-Michel Lasry, and Pierre-Louis Lions. A spectral dominance approach to large random matrices. *Journal de Mathématiques Pures et Appliquées*, 164:27–56, 2022.
- [4] Terence Chan. The wigner semi-circle law and eigenvalues of matrix-valued diffusions. *Probability theory and related fields*, 93(2):249–272, 1992.
- [5] Nicolas Forcadel, Cyril Imbert, and Régis Monneau. Homogenization of some particle systems with two-body interactions and of the dislocation dynamics. *Discrete and continuous dynamical systems-series A*, 23(3):pp–785, 2009.
- [6] Cyril Imbert and Régis Monneau. Homogenization of first-order equations with-periodic hamiltonians. part i: Local equations. *Archive for Rational Mechanics and Analysis*, 187(1):49–89, 2008.
- [7] Cyril Imbert, Régis Monneau, and Elisabeth Rouy. Homogenization of first order equations with (u/ε) -periodic hamiltonians part ii: Application to dislocations dynamics. *Communications in Partial Differential Equations*, 33(3):479–516, 2008.
- [8] Pierre-Louis Lions. Cours au college de france. Available at www.college-de-france.fr, 2021-2022.
- [9] Pierre-Louis Lions and Alain-Sol Sznitman. Stochastic differential equations with reflecting boundary conditions. *Communications on pure and applied Mathematics*, 37(4):511–537, 1984.
- [10] Pierre-Louis Lions, Jose Luis Menaldi, and Alain-Sol Sznitman. Construction de processus de diffusion réfléchis par pénalisation du domaine. *Comptes Rendus de l’Académie des Sciences de Paris*, 292:559–562, 1981.
- [11] Leonard CG Rogers and Zhan Shi. Interacting brownian particles and the wigner law. *Probability theory and related fields*, 95(4):555–570, 1993.

^A CMAP, ECOLE POLYTECHNIQUE, UMR 7641, 91120 PALAISEAU, FRANCE,, ^B UNIVERSITÉ PARIS-DAUPHINE, PSL RESEARCH UNIVERSITY, UMR 7534, CEREMADE, 75016 PARIS, FRANCE,, ^B COLLÈGE DE FRANCE, 3 RUE D’ULM, 75005, PARIS, FRANCE.