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INVERSE SOURCE PROBLEM FOR DISCRETE HELMHOLTZ EQUATION

ROMAN NOVIKOV AND BASANT LAL SHARMA

ABSTRACT. We consider multi-frequency inverse source problem for the discrete Helmholtz operator on the square lattice \mathbb{Z}^d , $d \geq 1$. We consider this problem for the cases with and without phase information. We prove uniqueness results and present examples of non-uniqueness for this problem for the case of compactly supported source function. Relations with inverse scattering problem for the discrete Schrödinger operators in the Born approximation are also provided.

Keywords: discrete Helmholtz operators, multi-frequency inverse source problem, phase retrieval, monochromatic inverse scattering in the Born approximation

1. INTRODUCTION

We consider the discrete Helmholtz equation

$$\Delta\psi(x) - \lambda\psi(x) = f(x), \quad x \in \mathbb{Z}^d, d \geq 1, \quad (1)$$

where Δ is the discrete Laplacian defined by

$$\Delta\psi(x) = \sum_{|x'-x|=1} \psi(x'), \quad x, x' \in \mathbb{Z}^d, \quad (2)$$

and f is a scalar source term with sufficient decay at infinity. For example, our considerations are focused on the case when

$$\text{supp } f \text{ is compact.} \quad (3)$$

We assume that

$$\lambda \in S := [-2d, 2d] \setminus S_0, \quad (4)$$

where

$$S_0 := \begin{cases} \{\pm 4n \text{ when } d \text{ is even,} \\ \pm 2(2n + 1) \text{ when } d \text{ is odd, } n \in \mathbb{Z} \text{ and } 2n \leq d\}. \end{cases} \quad (5)$$

Discrete Helmholtz equations are closely related to discrete Schrödinger equations which appear naturally in the tight-binding model of the electrons in crystals [1, 2, 3]. Similar equations also appear in case of studies involving time harmonic elastic waves in lattice models of crystals [4, 5, 6]; see, for example, [7, 8] specially in the case $d = 2$. In the present work, we consider the simplest cubic lattice and the simplest Laplacian Δ defined by (2). But our considerations can be extended to more complicated lattices and Laplacians.

For Eq. (1), we consider the following solutions:

$$\psi^\pm(x) = \lim_{\nu \rightarrow \lambda \pm i0} (\mathcal{R}_\nu f)(x), \quad x \in \mathbb{Z}^d, \quad (6)$$

where

$$\mathcal{R}_\nu = (\Delta - \nu)^{-1} : l^2(\mathbb{Z}^d) \rightarrow l^2(\mathbb{Z}^d), \quad \text{Im } \nu \neq 0. \quad (7)$$

To recall some properties of ψ^\pm , we consider the surface

$$\Gamma(\lambda) = \{k : k \in T^d, \phi(k) = \lambda\}, \quad \lambda \in [-2d, 2d], \quad (8)$$

where

$$\phi(k) = 2 \sum_{i=1}^d \cos k_i, \quad k = (k_1, \dots, k_d), \quad T^d = \mathbb{R}^d / 2\pi\mathbb{Z}^d. \quad (9)$$

In particular, if

$$\lambda \in \Lambda = \{\zeta \in \mathbb{R} : 2d - 4 < |\zeta| < 2d\}, \quad (10)$$

then $\Gamma(\lambda)$ is smooth strictly convex with non-zero principal curvatures, and there is a unique point

$$\kappa = \kappa(\omega, \lambda) \in \Gamma(\lambda), \quad \omega \in \mathbb{S}^{d-1}, \quad (11)$$

where

$$\frac{\nabla\phi(\kappa)}{|\nabla\phi(\kappa)|} = \omega. \quad (12)$$

Remark 1.1. *One can consider $\Gamma(\lambda)$ to be symmetric with respect to the origin $\mathbf{0}$ in \mathbb{R}^d for $\lambda > 0$, and to be symmetric with respect to the point $\mathbf{0}_\pi$ in \mathbb{R}^d for $\lambda < 0$, where all coordinates of $\mathbf{0}_\pi$ are equal to π . In this case, when $2d - 4 < \lambda < 2d$ ($-2d < \lambda < -2d + 4$), the surface $\Gamma(\lambda)$ is located strictly inside the cube $[-\pi, \pi]^d$ ($[0, 2\pi]^d$, respectively); see Lemma 2 in [9], and also Remark 1.1 in [10].*

Under conditions (3), (10), the following asymptotic formula holds (see, for example, [9]):

$$\psi^\pm(x) = \frac{e^{\pm i\mu(\omega, \lambda)|x|}}{|x|^{\frac{d-1}{2}}} a^\pm(\omega, \lambda) + O\left(\frac{1}{|x|^{\frac{d+1}{2}}}\right), \quad (13)$$

$$\omega = \hat{x} := x/|x|, \quad |x| \rightarrow \infty, \quad x \in \mathbb{Z}^d,$$

where

$$\mu(\omega, \lambda) = \kappa(\omega, \lambda) \cdot \omega, \quad (14)$$

$a^\pm(\omega, \lambda)$ are smooth, and the remainder can be estimated uniformly in $\omega \in \mathbb{S}^{d-1}$.

Remark 1.2. *The far-field amplitudes $a^\pm(\omega, \lambda)$ arise in (13) for $\omega \in \Omega$, where Ω is the countable, everywhere dense subset of \mathbb{S}^{d-1} , defined by*

$$\Omega = \{\theta : \theta = x/|x| \text{ for some } x \in \mathbb{Z}^d\}. \quad (15)$$

However, we can consider $a^\pm(\omega, \lambda)$ for $\omega \in \mathbb{S}^{d-1}$ taking into account that a^\pm are continuous, and even analytic, at least, under assumptions (3), (10).

Remark 1.3. *Eq. (1) can be re-written as*

$$-(\Delta - 2d)\psi(x) - (2d - \lambda)\psi(x) = -f(x), \quad x \in \mathbb{Z}^d, d \geq 1, \quad (16)$$

where Δ is the discrete Laplacian defined by (2). In addition, for the continuous case, the related equation can be written as

$$-\Delta\Psi(x) - E\Psi(x) = F(x), \quad x \in \mathbb{R}^d, d \geq 1, E > 0, \quad (17)$$

where Δ is the usual continuous Laplacian and F is the source function.

Therefore, for λ close to $2d$, under assumption (4), the solution ψ^- in (6), (13) corresponds to the physical (outgoing radiation) solution Ψ^+ of (17) for E close to 0.

In this connection, the related limit from discrete to continuous case is justified in [7, 34, 35] for $d = 2$; see, for more details, Section 5.3 of [7] and Section 4.3 of [34].

In the present work we consider the inverse source problem for Eq. (1) consisting in recovering the source f from the far-field data a^- (or a^+), for simplicity, under assumption (10).

We show that the far-field amplitude a^- (or a^+), given in a neighbourhood of any fixed pair ω, λ , where $\omega \in \mathbb{S}^{d-1}$ and $\lambda \in \Lambda$ with Λ defined in (10), uniquely determines f under condition (3); see Theorem 2.1. In addition, we show that the far-field amplitude a^- (or a^+), given on \mathbb{S}^{d-1} for finite but arbitrarily large number of the spectral parameter $\lambda \in \Lambda$, fails to determine f under condition (3); see Theorem 2.2.

We also give uniqueness results on the phaseless inverse source problem for the discrete Helmholtz equation (1) with background information; see Theorems 2.9, 2.10, 2.11.

In addition, we show similarities of the multi-frequency inverse source problem for discrete Helmholtz equations and the monochromatic inverse scattering problem for discrete Schrödinger operators in the Born approximation. Our Theorems A.4, A.7, A.11, A.12, given in Appendix A for the latter case, are similar to Theorems 2.1, 2.2, 2.10, 2.11 mentioned above.

Remark 1.4. *If $d \geq 3$ then the case*

$$|\lambda| < 2d - 4, \quad \lambda \in S, \quad (18)$$

is also possible. In this case, the asymptotic formula (13) should be replaced by multi-term formula (5) in [9] which involves several far-field amplitudes a^\pm . In addition, formula (32) in our proofs should be replaced by formula (13) in [9] for the aforementioned far-field amplitudes. It is not difficult to extend the results of the present work to the case when λ satisfies (18). However, the considerations become rather cumbersome and are not discussed in the present article.

Remark 1.5. *To our knowledge, formulas (13), (32), and more general formulas, mentioned in Remark 1.4, are available in the literature under assumption (3) only. However, these formulas, apparently, remain valid for f exponentially decaying at infinity and also for f decaying at infinity as $O(|x|^{-N})$ for sufficiently large N depending on d . It seems that $N > d$ is already sufficient with $o(|x|^{-(d-1)/2})$ in place of $O(|x|^{-(d+1)/2})$ in (13), by analogy with the continuous case. In this connection we also give Remarks 2.7 and 2.8.*

Note that many important results are given in the literature for the inverse source problem for Helmholtz equation in the continuous case; see, for example, [11], [12], [13], [14], [15], [16], [17], [18] and references therein. To our knowledge, however, the inverse source problem for the discrete Helmholtz equation (1) has not been considered before the present work. For some discrete transport equation the inverse source problem was considered in [19]. Potential applications of the inverse source problem for discrete Helmholtz equations, including the simplest case of Eq. (1), can be similar to the continuous case.

On the other hand, inverse scattering for discrete Schrödinger equations was studied already in many works; see, for example, [20], [21], [22], [23], [24], [25], [26], [10] and references therein. In particular, the studies of this problem for the phaseless case were initiated recently [10].

As about the inverse scattering problem for the Schrödinger equation in the continuous case, see, for example, [27, 28, 29] and references therein.

Note also that studies on inverse scattering and inverse source problems for both continuous and discrete cases use results on properties of Green's function for related Helmholtz operators. In connection with these properties for the discrete case, see, for example, [30, 31, 32, 33, 9, 24, 26, 34, 36, 37, 38, 39] and references therein.

The main results of the present article on the inverse source problem for Eq. (1) are given in detail in Section 2 and are proved in Section 3. Similar results on monochromatic inverse scattering in discrete framework in the Born approximation are given in Appendix A.

2. MAIN RESULTS

We start with the inverse source problem for Eq. (1) for the phased case, that is for the case when complex values of a^- (or a^+) are given.

Theorem 2.1. *Let f satisfy (3). Then the far-field amplitude $a^- = a^-(\omega, \lambda)$ (or a^+) arising in (13) and given in an open nonempty neighbourhood \mathcal{N} of any fixed pair ω, λ , where $\omega \in \mathbb{S}^{d-1}$ and $\lambda \in \Lambda$ with Λ defined in (10), uniquely determines f .*

Theorem 2.1 is proved in Section 3.

Let \mathcal{F} denote the discrete Fourier transform defined by the formula

$$\hat{u}(k) = \mathcal{F}u(k) = (2\pi)^{-d/2} \sum_{x \in \mathbb{Z}^d} u(x) e^{-ik \cdot x}, \quad k \in T^d, \quad (19)$$

where u is a test function on \mathbb{Z}^d and $k \cdot x := \sum_{i=1}^d k_i x_i$.

We recall that

$$u(x) = \mathcal{F}^{-1}\hat{u}(x) = (2\pi)^{-d/2} \int_{T^d} \hat{u}(k) e^{ik \cdot x} dk, \quad x \in \mathbb{Z}^d. \quad (20)$$

In particular, we consider the discrete Fourier transform \hat{f} of the source f in (1).

Theorem 2.2. *Let u be a compactly supported function on \mathbb{Z}^d and $\hat{u} = \mathcal{F}u$. Let $f = \mathcal{F}^{-1}\hat{f}$, where $\hat{f}(k) = \hat{u}(k) \prod_{j=1}^J (\phi(k) - \lambda_j)$, ϕ is defined by (9), λ_j satisfy (10), $k \in T^d$, and J is a positive integer. Then f satisfies (3) and its far-field amplitude $a^- = a^-(\omega, \lambda)$ (and a^+) arising in (13) vanishes identically on \mathbb{S}^{d-1} for each $\lambda = \lambda_j$, $j = 1, \dots, J$.*

Remark 2.3. *Under the assumptions of Theorem 2.2, the order of zero of $a^\pm = a^\pm(\omega, \lambda)$ on \mathbb{S}^{d-1} for $\lambda = \lambda_j$ is equal or greater than the multiplicity of λ_j as a root of the polynomial $\prod_{j=1}^J (z - \lambda_j)$ in z , where we do not assume that the numbers λ_j are distinct.*

Remark 2.4. *The function f in Theorem 2.2 can be also written as*

$$f = \left(\prod_{j=1}^J (\Delta - \lambda_j) \right) u. \quad (21)$$

Theorem 2.2 is proved in Section 3.

One can see that Theorem 2.1 is a uniqueness result on multi-frequency inverse source problem for Eq. (1), whereas Theorem 2.2 gives a large class of non-uniqueness examples for this problem. However, there is no contradiction between these two theorems: the first theorem involves infinitely many frequencies in contrast to the second one.

Remark 2.5. *Theorems 2.1 and 2.2 have analogues for the continuous case. In this case, Eq. (1) is considered for $x \in \mathbb{R}^d$, where Δ is the standard Laplacian, $\lambda < 0$, and $\phi(k) = -k^2$, $k \in \mathbb{R}^d$. The analogue of Theorem 2.1 for the continuous case is well-known. In the continuous analogue of Theorem 2.2, the function u should be also sufficiently smooth on \mathbb{R}^d in order to ensure a regularity of the function f on \mathbb{R}^d ; see [11, 15] for the continuous case at least at a single frequency.*

Remark 2.6. *It is also interesting to compare Theorem 2.2 with results on recovering point sources for the continuous Helmholtz equation at a single frequency. The latter problem is uniquely solvable (see [14] and references therein) in contrast with Theorem 2.2.*

Remark 2.7. *Theorem 2.1 remains valid for f exponentially decaying at infinity, i.e.*

$$|f(x)| \leq \beta e^{-\alpha|x|}, \quad x \in \mathbb{Z}^d, \quad (22)$$

for some $\alpha, \beta > 0$, under the assumption that formulas (13) and (32) remain valid; see Remark 1.5.

Remark 2.8. *Theorem 2.1 is not valid, in general, for f decaying at infinity as $O(|x|^{-\infty})$ even if formulas (13) and (32) remain valid. Counter-examples can be constructed using formulas (20) with f in place of u , where \hat{u} is infinitely smooth on T^d and identically zero on $\bigcup_{(\theta, \zeta) \in \mathcal{N}} \kappa(\theta, \zeta)$ mentioned in formula (34) in the proof of Theorem 2.1.*

We also consider the inverse source problem for Eq. (1) for the phaseless case, that is for the case when only values of $|a^\pm|^2$ are given. We deal with this case using the method of background information in phaseless inverse problems; see [40, 41, 28, 42, 17] and references therein.

Like in [28] for the continuous case, we consider f_0 and f on \mathbb{Z}^d , assuming that

$$\text{supp } f_0 \subset D_0, \quad \text{supp } f \subset D, \quad (23)$$

where

$$\begin{aligned} D_0, D \text{ are open convex bounded domains in } \mathbb{R}^d, \\ D_0 \cap \mathbb{Z}^d \neq \emptyset, \quad D \cap \mathbb{Z}^d \neq \emptyset. \end{aligned} \quad (24)$$

Let

$$\text{diam } D = \sup_{x, y \in D} |x - y|, \quad (25)$$

$$-D := \{-x : x \in D\}, \quad (26)$$

$$D_1 + D_2 := \{x + y : x \in D_1, y \in D_2\}, \quad (27)$$

where D, D_1, D_2 are bounded sets in \mathbb{R}^d .

Theorem 2.9. *Let f_0, f satisfy (23), (24), where $f_0 \not\equiv 0$. Then the following formulas hold:*

$$\mathcal{F}f(p) := (\overline{\mathcal{F}f_0(p)})^{-1} \mathcal{F}q(p), \quad p \in T^d, \quad (28)$$

$$q(x) := \chi_{D-D_0}(x) (u(x) - (2\pi)^{-d/2} \sum_{y \in D_0 \cap \mathbb{Z}^d} f_0(x+y) \overline{f_0(y)} dy), \quad (29)$$

$$u(x) := \mathcal{F}^{-1}(|\mathcal{F}(f+f_0)|^2)(x) \text{ if } \text{dist}(D, D_0) > \text{diam } D, \quad (30)$$

$$u(x) := \mathcal{F}^{-1}(|\mathcal{F}(f+f_0)|^2)(x) - \mathcal{F}^{-1}(|\mathcal{F}f|^2)(x) \text{ if } \text{dist}(D, D_0) > 0, \quad (31)$$

where $x \in \mathbb{Z}^d$.

Note that, under our assumptions, formula (28) is defined correctly taking into account that $\text{Meas } \mathcal{Z} = 0$ in T^d , where $\mathcal{Z} = \{p \in T^d : \mathcal{F}f_0(p) = 0\}$.

Note also that Theorem 2.9 is just a result on phase retrieval for the Fourier transform \mathcal{F} defined by (19). For known results on phase retrieval for different Fourier transforms, see, for example, [43, 44, 28, 42] and references therein. Using Theorem 2.9, we obtain the following results on the phaseless inverse source problem for Eq. (1).

Theorem 2.10. *Let f_0, f satisfy (23), (24), where $f_0 \neq 0$, $\text{dist}(D, D_0) > \text{diam } D$. Let a_1^- be the far-field amplitude for Eq. (1) with f replaced by $f + f_0$. Then the intensity $|a_1^-|^2 = |a_1^-(\omega, \lambda)|^2$ given in an open nonempty neighbourhood of any fixed pair ω, λ , where $\omega \in \mathbb{S}^{d-1}$ and $\lambda \in \Lambda$ with Λ defined in (10), uniquely determines f , assuming that f_0 is known a-priori.*

Theorem 2.11. *Let f_0, f satisfy (23), (24), where $f_0 \neq 0$, $\text{dist}(D, D_0) > 0$. Let a, a_1^- be the far-field amplitudes for Eq. (1) and for Eq. (1) with f replaced by $f + f_0$, respectively. Then the intensities $|a^-|^2 = |a^-(\omega, \lambda)|^2$ and $|a_1^-|^2 = |a_1^-(\omega, \lambda)|^2$ given in an open nonempty neighbourhood of any fixed pair ω, λ , where $\omega \in \mathbb{S}^{d-1}$ and $\lambda \in \Lambda$ with Λ defined in (10), uniquely determines f , assuming that f_0 is known a-priori.*

Theorems 2.10 and 2.11 are also valid with a^-, a_1^- replaced by a^+, a_1^+ .

Theorems 2.9, 2.10 and 2.11 are proved in Section 4.

Remark 2.12. *For the continuous case, the analogues of Theorems 2.9, 2.10, 2.11 are given in [28]; see also [17] and references therein.*

3. PROOFS OF THEOREMS 2.1 AND 2.2

We consider ϕ defined by (9), $\Gamma(\lambda)$ defined by (8), and $\hat{f} = \mathcal{F}f$ defined by (19).

For compactly supported f , under assumption (10), due to formula (13) from [9], we have that

$$a^\pm(\omega, \lambda) = \frac{\sqrt{2\pi} \hat{f}(\kappa(\pm\omega, \lambda)) e^{i(\sigma+2)\frac{\pi}{4}}}{\sqrt{|K(\omega, \lambda)| |\nabla\phi(\kappa(\omega, \lambda))|}}, \quad \omega \in \mathbb{S}^{d-1}, \quad (32)$$

where $a^\pm = a^\pm(\omega, \lambda)$ are the far-field amplitudes in (13), $\kappa(\omega, \lambda)$ is the point in (11), $K(\omega, \lambda)$ is the total curvature (i.e. the product of principal curvatures) of $\Gamma(\lambda)$ at the point $\kappa(\omega, \lambda)$, and $\sigma = d - 1$. More precisely, formula (32) is mentioned in [9] for a^+ , but the case of a^- is similar. In formula (32), we also use the symmetries of Γ mentioned in Remark 1.1.

Note that if f is compactly supported on \mathbb{Z}^d , then

$$\hat{f} \text{ is real-analytic on } T^d. \quad (33)$$

Note also that if (ω, λ) is a fixed pair as in Theorem 2.1, and \mathcal{N} is its open nonempty neighbourhood in $\mathbb{S}^{d-1} \times \Lambda$, where Λ is defined in (10), then there is an open nonempty neighbourhood \mathcal{K}^\pm of $\kappa(\pm\omega, \lambda)$ in T^d such that

$$\mathcal{K}^\pm \subseteq \bigcup_{(\theta, \zeta) \in \mathcal{N}} \kappa(\pm\theta, \zeta). \quad (34)$$

Theorem 2.1 is proved as follows.

Due to formulas (32), (34), the function a^\pm , given in a neighbourhood of any fixed pair (ω, λ) as in Theorem 2.1, determines \hat{f} in a neighbourhood of $\kappa(\pm\omega, \lambda)$ in T^d . In turn, in view of (33), \hat{f} in a neighbourhood of $\kappa \in T^d$ uniquely determines \hat{f} on T^d via analytic continuation, and then determines f using (20).

This completes the proof of Theorem 2.1.

Let

$$\delta(x) = \begin{cases} 1 & \text{for } x = 0, \\ 0 & \text{for } x \in \mathbb{Z}^d \setminus \{0\}. \end{cases} \quad (35)$$

We also recall that

$$(2\pi)^{d/2} \mathcal{F}^{-1}(\hat{u}_1 \hat{u}_2)(x) = u_1 * u_2(x) = \sum_{y \in \mathbb{Z}^d} u_1(x-y) u_2(y), \quad x \in \mathbb{Z}^d, \quad (36)$$

where $\hat{u}_j = \mathcal{F}u_j, \quad j = 1, 2,$

where u_1, u_2 are test functions on \mathbb{Z}^d .

Theorem 2.2 is proved as follows.

In fact, it is sufficient to prove that f is compactly supported on \mathbb{Z}^d . The rest follows from the definition of \hat{f} in Theorem 2.2, formula (32), definition of $\Gamma(\lambda)$ by formulas (8), (9), and the property that $\kappa(\pm\omega, \lambda) \in \Gamma(\lambda)$ for $\omega \in \mathbb{S}^{d-1}$.

We have that

$$\begin{aligned} \mathcal{F}^{-1}\phi(x) &= (2\pi)^{d/2} \sum_{|x'|=1} \delta(x-x'), \quad x \in \mathbb{Z}^d, \\ \mathcal{F}^{-1}\lambda(x) &= (2\pi)^{d/2} \lambda\delta(x), \quad x \in \mathbb{Z}^d, \\ f_\lambda(x) &:= \mathcal{F}^{-1}(\phi - \lambda)(x) = (2\pi)^{d/2} \left(\sum_{|x'|=1} \delta(x-x') - \lambda\delta(x) \right), \quad x \in \mathbb{Z}^d, \end{aligned}$$

and f_λ is compactly supported on \mathbb{Z}^d . (37)

One can see that

$$\mathcal{F}f_\lambda(k) = \phi(k) - \lambda, \quad k \in T^d. \quad (38)$$

The property that f is compactly supported on \mathbb{Z}^d follows from the definition of \hat{f} , formula (36), property (37), formula (38), and the property that the convolution of compactly supported functions is compactly supported.

Theorem 2.2 is proved.

4. PROOFS OF THEOREMS 2.9, 2.10 AND 2.11

The proof of Theorem 2.9 is similar to the proof of Theorems 3.1 and 3.2 in [28] for the case of continuous Fourier transform.

To prove Theorem 2.9, we use formula (36) and the formulas

$$\overline{\mathcal{F}u} = \mathcal{F}\tilde{u}, \quad \tilde{u}(x) = \overline{u(-x)}, \quad (39)$$

$$(2\pi)^{-d/2} \mathcal{F}(u_1 * u_2) = \mathcal{F}u_1 \mathcal{F}u_2, \quad (40)$$

where u, u_1, u_2 are test functions.

We have that

$$\begin{aligned} \Sigma &= (2\pi)^{d/2} \mathcal{F}^{-1}(|\mathcal{F}(f + f_0)|^2) = (f + f_0) * (\tilde{f} + \tilde{f}_0) \\ &= \sum_{y \in \mathbb{Z}^d} (f(x-y) + f_0(x-y)) \overline{(f(-y) + f_0(-y))} \\ &= \sum_{y \in \mathbb{Z}^d} f(x-y) \overline{f(-y)} dy + \sum_{y \in \mathbb{Z}^d} f_0(x-y) \overline{f(-y)} + \\ &+ \sum_{y \in \mathbb{Z}^d} f(x-y) \overline{f_0(-y)} + \sum_{y \in \mathbb{Z}^d} f_0(x-y) \overline{f_0(-y)} =: \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4, \end{aligned} \quad (41)$$

where $x \in \mathbb{Z}^d$.

Let

$$B_r = \{x \in \mathbb{R}^d : |x| < r\}. \quad (42)$$

Note that

$$\Sigma_1(x) = \sum_{y \in -D \cap \mathbb{Z}^d} f(x-y) \overline{f(-y)}, \quad (43)$$

$$\text{supp } \Sigma_1 \subset B_r, \quad r = \text{diam } D, \quad (44)$$

$$\Sigma_2(x) = \sum_{y \in -D \cap \mathbb{Z}^d} f_0(x-y) \overline{f_0(-y)}, \quad (45)$$

$$\text{supp } \Sigma_2 \subset D_0 - D, \quad (46)$$

$$\Sigma_3(x) = \sum_{-y \in D_0 \cap \mathbb{Z}^d} f(x-y) \overline{f_0(-y)}, \quad (47)$$

$$\text{supp } \Sigma_3 \subset D - D_0, \quad (48)$$

$$\Sigma_4(x) = \sum_{-y \in D_0 \cap \mathbb{Z}^d} f_0(x-y) \overline{f_0(-y)}, \quad (49)$$

$$\text{supp } \Sigma_4(x) \subset B_r, \quad r = \text{diam } D_0, \quad (50)$$

where $x \in \mathbb{Z}^d$, and $D_0 - D, D - D_0, B_r$ are defined according to (26), (27), (42).

Similar to [28], we have that:

$$(2\pi)^{-d/2} \mathcal{F} \Sigma_3 = \mathcal{F} f \mathcal{F} \tilde{f}_0; \quad (51)$$

$$\Sigma_3(x) = \chi_{D-D_0}(x) (\Sigma(x) - \Sigma_4(x)), \quad \text{if } \text{dist}(D, D_0) > \text{diam } D; \quad (52)$$

$$\Sigma_3(x) = \chi_{D-D_0}(x) (\Sigma(x) - \Sigma_1(x) - \Sigma_4(x)), \quad \text{if } \text{dist}(D, D_0) > 0. \quad (53)$$

Theorem 2.9 follows from formulas (41), (51), (52), (53), and (39).

Theorems 2.10 and 2.11 are proved as follows.

For compactly supported f , under assumption (10), as a corollary of formula (32), we have that

$$|a^\pm(\omega, \lambda)|^2 = \frac{2\pi |\hat{f}(\kappa(\pm\omega, \lambda))|^2}{|K(\omega, \lambda)| |\nabla \phi(\kappa(\omega, \lambda))|^2}, \quad \omega \in \mathbb{S}^{d-1}. \quad (54)$$

For compactly supported f , as a corollary of (33), we have that

$$|\hat{f}|^2 = \hat{f} \overline{\hat{f}} \text{ is real-analytic on } T^d. \quad (55)$$

Due to formulas (34), (54), the function $|a^\pm|^2$, given in a neighbourhood of any fixed pair (ω, λ) as in Theorem 2.11, determines $|\hat{f}|^2$ in a neighbourhood of $\kappa(\pm\omega, \lambda)$ in T^d . Similarly, the function $|a_1^\pm|^2$ given in a neighbourhood of any fixed pair (ω, λ) as in Theorems 2.10 and 2.11, determines $|\hat{f} + \hat{f}_0|^2$ in a neighbourhood of $\kappa(\pm\omega, \lambda)$ in T^d .

In turn, in view of (55), $|\hat{f}|^2$ in a neighbourhood of $\kappa \in T^d$ uniquely determines $|\hat{f}|^2$ on T^d via analytic continuation. Similarly, $|\hat{f} + \hat{f}_0|^2$ in a neighbourhood of $\kappa \in T^d$ uniquely determines $|\hat{f} + \hat{f}_0|^2$ on T^d .

Finally, the use of Theorem 2.9 completes the proof of Theorems 2.10 and 2.11.

APPENDIX A. INVERSE SCATTERING FOR DISCRETE SCHRÖDINGER OPERATORS IN THE BORN APPROXIMATION

We consider the discrete Schrödinger equation

$$\Delta \psi(x) + v(x)\psi(x) - \lambda \psi(x) = 0, \quad x \in \mathbb{Z}^d, d \geq 1, \quad (56)$$

where Δ is the discrete Laplacian defined by (2), v is scalar potential such that

$$\text{supp } v \subset D, \quad (57)$$

D is an open bounded domain in \mathbb{R}^d , $D \cap \mathbb{Z}^d \neq \emptyset$, and $\lambda \in S$ as in (4).

For Eq. (56) we consider the scattering solutions

$$\psi^-(x, k) = \psi_0 + \psi_{sc}^-, \quad (58)$$

where

$$\psi_0(x, k) = e^{ik \cdot x}, \quad k \in \Gamma(\lambda), \quad x \in \mathbb{Z}^d, \quad (59)$$

and $\psi_{sc}^-(x, k)$ is the outgoing solution for the non-homogenous equation

$$(\Delta + v)\psi_{sc} - \lambda\psi_{sc} = -v\psi_0, \quad (60)$$

obtained using the limiting absorption principle, i.e. via formulas (6) and (7) with sign $-$, where $f = -v\psi_0$ and Δ is replaced by $\Delta + v$; see [9] and Remark 1.3.

We recall that, under assumption (10), ψ^- has the asymptotic expansion

$$\begin{aligned} \psi^-(x, k) &= e^{ik \cdot x} + \frac{e^{-i\mu(\omega, \lambda)|x|}}{|x|^{\frac{d-1}{2}}} A(k, \omega) + O\left(\frac{1}{|x|^{\frac{d+1}{2}}}\right) \\ &\text{as } |x| \rightarrow \infty, \quad \omega = \frac{x}{|x|}, \quad x \in \mathbb{Z}^d, \end{aligned} \quad (61)$$

where μ is as in (13), (14) and the coefficient $A(k, \omega)$ is smooth and the remainder can be estimated uniformly in ω ; see [20, 9]. The coefficient A is the scattering amplitude for (56).

Remark A.1. *In our previous article [10], Eq. (56) is misprinted with $-\Delta$ in place of Δ ; see Eq. (1) and Eq. (10) in [10]. However, there exists some disagreement regarding the standard conventions to present the discrete Schrödinger operators on \mathbb{Z}^d ; compare, for example, the conventions of [9] and [24].*

Remark A.2. *In the present article, we define μ exactly as in [9]. This μ differs in sign from that defined in our previous article if $2d - 4 < \lambda < 2d$. See also Remark 1.2 in [10].*

In this appendix, we consider the inverse scattering problem for Eq. (56) consisting in recovering the potential v from the scattering amplitude A , for simplicity, under assumption (10).

As we mentioned already in introduction, many important results are given in the literature on inverse scattering for Eq. (56). Below in this appendix, we discuss relations between inverse scattering for Eq. (56) in the Born approximation for small v and the inverse source problem for Eq. (1), for simplicity, under assumption (10).

In the Born approximation for small v , under assumption (10), we have that

$$A(k, \omega) = -\frac{\sqrt{2\pi}\hat{v}(\kappa(-\omega, \lambda) - k) e^{i(\sigma+2)\frac{\pi}{4}}}{\sqrt{|K(\omega, \lambda)|} |\nabla\phi(\kappa(\omega, \lambda))|}, \quad k \in \Gamma(\lambda), \quad \omega \in \mathbb{S}^{d-1}, \quad (62)$$

where A is the scattering amplitude in (61), \hat{v} is the Fourier transform of v , defined according to (19), K, ϕ, κ, σ are the same as in (32).

Formula (62) follows from (58)-(60), where in (60) the product $v\psi_{sc}$ is replaced by 0 in the Born approximation, and from formula (32) with $f = -v\psi_0$.

In connection with formula (62), we also consider \hat{v} on

$$\mathcal{B}(\lambda) := \{p = \kappa - k : \kappa, k \in \Gamma(\lambda)\}, \quad (63)$$

where $\Gamma(\lambda)$ is as in Remark 1.1, and $\mathcal{B}(\lambda)$ can be considered as a subset of \mathbb{R}^d as well as a subset of T^d .

Remark A.3. *For comparisons with the continuous case, in a way similar to Remark 1.3, note that $\Gamma(\lambda) \approx \{k \in \mathbb{R}^d : k^2 = 2d - \lambda\}$, $\phi(k) \approx 2d - k^2$ in a small neighbourhood of $\Gamma(\lambda)$, $\kappa(-\omega, \lambda) \approx \sqrt{2d - \lambda} \omega$ as $\lambda \rightarrow 2d$. In view of these approximate equalities, the structure of (62) is similar to the structure of formula*

for the scattering amplitude in the Born approximation for the continuous case; see, for e.g., [29].

The multi-frequency inverse source problem for Eq. (1) and monochromatic inverse scattering for Eq. (56) in the Born approximation for $d \geq 2$ have considerable similarities following from the similarities of formulas (32) and (62). In particular, Theorems A.4 and A.7 given below for the latter case are analogues of Theorems 2.1 and 2.2 for the former.

Theorem A.4. *Let v satisfy (57), λ satisfy (10), and $d \geq 2$. Then the scattering amplitude A for Eq. (56), arising in (62) and given at fixed λ in an open non-empty neighbourhood \mathcal{N} of any fixed pair k, ω , where $k \in \Gamma(\lambda)$ and $\omega \in \mathbb{S}^{d-1}$, uniquely determines v .*

Under our assumptions, Theorem A.4 follows from the statements:

- (i) The function A is real-analytic on $\Gamma(\lambda) \times \mathbb{S}^{d-1}$. Therefore, A on \mathcal{N} uniquely determines A on $\Gamma(\lambda) \times \mathbb{S}^{d-1}$.
- (ii) A on $\Gamma(\lambda) \times \mathbb{S}^{d-1}$ uniquely determines \hat{v} on $\mathcal{B}(\lambda)$ in (63) via (62).
- (iii) \hat{v} is real-analytic on T^d . Therefore, \hat{v} on $\mathcal{B}(\lambda)$ uniquely determines \hat{v} on T^d , and v on \mathbb{Z}^d .

Remark A.5. *Theorem A.4 remains valid under assumption (10) but without the assumption that v is small. In this case, this result follows from the statement (i) as in the proof of Theorem A.4 above and from the uniqueness for inverse scattering from full A on $\Gamma(\lambda) \times \mathbb{S}^{d-1}$ proved in [26].*

Remark A.6. *Similar to Remarks 2.7, 2.8, our Theorem A.4 remains valid for v exponentially decaying at infinity, and is not valid, in general, for v decaying at infinity as $O(|x|^{-\infty})$.*

Theorem A.7. *Let u be a compactly supported function on \mathbb{Z}^d , $\hat{u} = \mathcal{F}u$, and λ satisfy (10). Let $v = \mathcal{F}^{-1}\hat{v}$, where $\hat{v}(p) = \hat{u}(p) \prod_{j=1}^J (\phi(p + k_j) - \lambda)$, ϕ is defined by (9), $p \in T^d$, $k_j \in \Gamma(\lambda)$, and J is a positive integer. Then v is compactly supported on \mathbb{Z}^d and the scattering amplitude $A = A(k, \omega)$ arising in (62) vanishes identically on \mathbb{S}^{d-1} for each $k = k_j$, $j = 1, \dots, J$.*

Remark A.8. *The function v in Theorem A.7 can be also written as*

$$v = \left(\prod_{j=1}^J (L_{k_j} - \lambda) \right) u, \quad (64)$$

where $L_{k_j}u(x) = e^{-ik_jx} \Delta(e^{ik_jx}u(x))$, $x \in \mathbb{Z}^d$.

The proof of Theorem A.7 is similar to the proof of Theorem 2.2.

Remark A.9. *Theorem A.7 and Remark A.8 admit straightforward polychromatic versions involving several λ .*

Remark A.10. *Theorems A.4 and A.7 have analogues for the continuous case. In this case, Eq. (56) is considered for $x \in \mathbb{R}^d$, where Δ is replaced by the standard Laplacian Δ , $\phi(k)$ replaced by $\Phi(k) = -k^2$, $k \in \mathbb{R}^d$, and $\lambda < 0$. The analogue of Theorem A.4 for the continuous case is well-known. In the continuous analogue of Theorem A.7, the function u should be also sufficiently smooth on \mathbb{R}^d in order to ensure a regularity of the potential v on \mathbb{R}^d , and $\Gamma(\lambda)$ should be replaced by $\mathbb{S}^{\frac{d-1}{\sqrt{|\lambda|}}} = \{k \in \mathbb{R}^d : -k^2 = \lambda\}$; see [27] for the continuous monochromatic case.*

Additionally, Theorems A.11 and A.12, given below for the phaseless inverse scattering for Eq. (56) in Born approximation, are similar to Theorems 2.10 and 2.11 for phaseless inverse source problem for Eq. (1).

Theorem A.11. *Let v_0, v satisfy assumptions (23), (24) in place of f_0, f , where $v_0 \neq 0$, $\text{dist}(D, D_0) > \text{diam } D$. Let A_1 be the scattering amplitude for Eq. (56) arising in the Born approximation in (62), under assumption (10), with v replaced by $v + v_0$. Then the intensity $|A_1|^2 = |A_1(k, \omega)|^2$ given in an open nonempty neighbourhood \mathcal{N} of any fixed pair k, ω , where $k \in \Gamma(\lambda)$ and $\omega \in \mathbb{S}^{d-1}$, uniquely determines v , assuming that v_0 is known a-priori.*

Theorem A.12. *Let v_0, v satisfy assumptions (23), (24) in place of f_0, f , where $v_0 \neq 0$, $\text{dist}(D, D_0) > 0$. Let A and A_1 be the scattering amplitudes for v and $v + v_0$, respectively, arising in (62), under assumption (10). Then the intensities $|A|^2 = |A(k, \omega)|^2$ and $|A_1|^2 = |A_1(k, \omega)|^2$ given in an open nonempty neighbourhood \mathcal{N} of any fixed pair k, ω , where $k \in \Gamma(\lambda)$ and $\omega \in \mathbb{S}^{d-1}$, uniquely determines v , assuming that v_0 is known a-priori.*

The proofs of Theorems A.11 and A.12 are similar to the proofs of Theorems 2.10 and 2.11.

Remark A.13. *In the present work, we consider reconstruction from far-field measurements. Reconstruction from near-field measurements can be reduced in many cases to reconstruction from far-field measurements. For discrete problems, results in this direction can be found in [26, 10]. In particular, [10] gives formulas for finding phased far-field amplitude A from phaseless measurements of ψ^- for the discrete Schrödinger equation (56).*

Remark A.14. *Open questions include extensions of the results of the present work to the case of Helmholtz and Schrödinger equations on more complicated lattices, for example, as in [45, 37].*

REFERENCES

- [1] Bloch, F. (1929). Über die Quantenmechanik der Elektronen in Kristallgittern. *Z. Physik* 52, 555–600
- [2] Slater, J.C. and Koster, G.F. (1954). Simplified LCAO Method for the Periodic Potential Problem, *Phys. Rev.* 94, 1498
- [3] Harrison, W.A. (1989). *Electronic Structure and the Properties of Solids: The Physics of the Chemical Bond*, Dover
- [4] Brillouin, L. (1953). *Wave propagation in periodic structures; electric filters and crystal lattices*. Dover Publications, New York
- [5] Maradudin, A.A., Montroll, E.W., Weiss, G.H., and Ipatova, I.P., (1971). *Theory of Lattice Dynamics in the Harmonic Approximation*, 2nd ed., Academic Press, New York
- [6] Lifshitz, I. M., Kosevich, A. M. (1966) The dynamics of a crystal lattice with defects, *Rep. Prog. Phys.* 29, pp. 217–254.
- [7] Sharma, B.L. (2015). Diffraction of waves on square lattice by semi-infinite crack. *SIAM Journal on Applied Mathematics* 75, 1171–1192
- [8] Sharma, B.L. (2015). Diffraction of waves on square lattice by semi-infinite rigid constraint. *Wave Motion* 59, 52–68
- [9] Shaban, W., and Vainberg, B. (2001). Radiation conditions for the difference Helmholtz operators. *Applicable Analysis* 80.3-4: 525–556
- [10] Novikov, R., Sharma, B.L. (2023). Phase recovery from phaseless scattering data for discrete Schrödinger operators. *Inverse Problems* 39, 125006 (12pp)
- [11] Devaney, A., Sherman, G. (1982). Nonuniqueness in inverse source and scattering problems. *IEEE Transactions on Antennas and Propagation*, 30(5), 1034-1037.
- [12] Bao, G., Lin, J., Triki, F. (2010). A multi-frequency inverse source problem, *Journal of Differential Equations*, 249, 3443–3465.
- [13] El Badia, A., El Hajj, A. (2012) Hölder stability estimates for some inverse pointwise source problems, *C. R. Acad. Sci. Paris, Ser. I* 350 1031–1035
- [14] Bao, G., Liu, Y., Triki, F. (2021) Recovering point sources for the inhomogeneous Helmholtz equation, *Inverse Problems* 37 095005
- [15] Burov, V.A., Romyantseva, O.D. (2016). Inverse wave problems of acoustic tomography. Part 1. Inverse problems of radiation in acoustics, Moscow, LENAND, 2016

- [16] Hohage, T, Raumer, H-G, Spehr, C.. (2020). Uniqueness of an inverse source problem in experimental aeroacoustics. *Inverse Problems* 36: 075012.
- [17] Sun, F., Wang, X. (2023). Reconstruction of acoustic sources from multi-frequency phaseless far-field data. *Journal of Inverse and Ill-posed Problems*, 31(2), 177-189.
- [18] Eller, M., Valdivia, N. P. (2009). Acoustic source identification using multiple frequency information. *Inverse Problems*, 25(11), 115005.
- [19] Fokas, A. S., Novikov, R. G. (1991). Discrete analogues of $\bar{\partial}$ -equation and of Radon transform. *Comptes rendus de l'Académie des sciences. Série 1, Mathématique*, 313(2), 75-80.
- [20] Eskina, M.S. (1966). The direct and the inverse scattering problem for a partial-difference equation. *Soviet Math. Doklady*, 7(1), 193-197
- [21] Case, K. M., Kac, M. (1973). A discrete version of the inverse scattering problem. *Journal of Mathematical Physics*, 14(5), 594-603.
- [22] Zakhariiev, B. N., Suzko, A. A. (1990). Direct and inverse problems. Potentials in quantum scattering, Translated from the Russian by G. Pontecorvo. Springer-Verlag, Berlin.
- [23] Guseinov, G. S. (1995). On the inverse scattering problem for a discrete one-dimensional Schrödinger equation. *Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics*, 44.
- [24] Isozaki, H., and Korotyaev, E. (2012). Inverse problems, trace formulae for discrete Helmholtz operators. *Annales Henri Poincaré*. Vol. 13. No. 4. Basel: SP Birkhäuser Verlag Basel
- [25] Ando, K. (2013). Inverse scattering theory for discrete Schrödinger operators on the hexagonal lattice. In *Annales Henri Poincaré* (Vol. 14, pp. 347-383). SP Birkhäuser Verlag Basel.
- [26] Isozaki, H., Morioka, H. (2015). Inverse scattering at a fixed energy for discrete Schrödinger operators on the square lattice. In *Annales de l'institut Fourier* (Vol. 65, No. 3, pp. 1153-1200).
- [27] Devaney, A. J. (1978). Nonuniqueness in the inverse scattering problem. *Journal of Mathematical Physics*, 19(7), 1526-1531.
- [28] Novikov, R. G., Sivkin, V. N. (2021). Phaseless inverse scattering with background information. *Inverse Problems*, 37(5), 055011.
- [29] Novikov, R. G. (2022). Multidimensional inverse scattering for the Schrödinger equation. In *ISAAC Congress (International Society for Analysis, its Applications and Computation)* (pp. 75-98). Cham: Springer International Publishing
- [30] Morita, T., Horiguchi, T. (1971). Calculation of the lattice Green's function for the bcc, fcc, and rectangular lattices. *Journal of Mathematical Physics*, 12(6), 986-992.
- [31] Horiguchi, T. (1971). Lattice Green's function for the simple cubic lattice. *Journal of the Physical Society of Japan*, 30(5), 1261-1272.
- [32] Morita, T., Horiguchi, T. (1972). Analytic properties of the lattice Green function. *Journal of Physics A: General Physics*, 5(1), 67.
- [33] Abe, Y., Katsura, S. (1973). Lattice Green's function for the simple cubic and tetragonal lattices at arbitrary points. *Annals of Physics*, 75(2), 348-380.
- [34] Sharma, B.L. (2015). Near-tip field for diffraction on square lattice by crack. *SIAM Journal on Applied Mathematics* 75, 1915-1940
- [35] Sharma, B. L. (2017). Continuum limit of discrete Sommerfeld problems on square lattice. *Sādhanā*, 42, 713-728.
- [36] Sharma, B.L. (2015). Near-tip field for diffraction on square lattice by rigid constraint. *Zeitschrift für Angewandte Mathematik und Physik* 66, 2719-2740
- [37] Sharma, B.L. (2016), Edge diffraction on triangular and hexagonal lattices: Existence, uniqueness, and finite section, *Wave Motion* 65, 55-78
- [38] Sharma, B.L. (2023). Scattering of surface waves by inhomogeneities in crystalline structures. [arXiv:2309.08552](https://arxiv.org/abs/2309.08552)
- [39] Garnier, J., Sharma, B.L. (2023). Effective dynamics in lattices with random mass perturbations. [arXiv:2309.03090](https://arxiv.org/abs/2309.03090)
- [40] Perutz, M.F. (1963). X-ray Analysis of Hemoglobin: The results suggest that a marked structural change accompanies the reaction of hemoglobin with oxygen, *Science*, 140(3569), 863-869
- [41] Aktosun, T., Sacks, P. E. (1998). Inverse problem on the line without phase information, *Inverse Problems* 14, 211-224
- [42] Hohage, T., Novikov, R., Sivkin, V. (2022). Phase retrieval and phaseless inverse scattering with background information. <https://hal.science/hal-03806616/>
- [43] Klibanov, M. V., Sacks, P. E., Tikhonravov, A. V. (1995). The phase retrieval problem. *Inverse problems*, 11(1), 1.
- [44] Barnett, A. H., Epstein, C. L., Greengard, L. F., Magland, J. F. (2020). Geometry of the phase retrieval problem. *Inverse Problems*, 36(9), 094003.

- [45] Ando, K., Hiroshi I., and Hisashi M. (2016). Spectral properties of Schrödinger operators on perturbed lattices. In *Annales Henri Poincaré*, vol. 17, pp. 2103–2171. Springer International Publishing.

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