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# Local Definability of $\mathsf{HOD}$ in $L(\mathbb{R})$

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**Abstract**

We show that in $L(\mathbb{R})$, assuming large cardinals, $\mathsf{HOD}\|\eta^{+\mathsf{HOD}}$ is locally definable from $\mathsf{HOD}\|\eta$ for all $\mathsf{HOD}$-cardinals $\eta \in [\boldsymbol{\delta}^2_1, \Theta)$. This is a further elaboration of the statement "$\mathsf{HOD}^{L(\mathbb{R})}$ is a core model below $\Theta$" made by John Steel.

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# 1 Introduction

In this paper, we work in a $\mathsf{ZFC}$ universe $V$ with the following large cardinal assumption.

**Declaration 1.1.** We assume that there exists a measurable cardinal with $\omega$ Woodins below it. $\square$

There are two important consequences of this assumption that will play a central role in the paper.

**Theorem 1.2.** *The Axiom of Determinacy* $\mathsf{AD}$ *holds in* $L(\mathbb{R})$.

*Proof.* This is a classical result that came out of the work of Martin, Steel, and Woodin. Its full proof can be found in [Nee10]. $\square$

The other result, Theorem 1.3, concerns the existence and iterability of the mouse $\mathcal{M}_\omega^\sharp$. This theorem is a well-known fact in Inner Model Theory, but we were unable to pinpoint an exact reference. As a courtesy to the reader, we will briefly outline its proof. Regarding its notation, the mouse $\mathcal{M}_\omega^\sharp$ is defined in [Ste10, Definition 7.3]. We will state here its basic properties, but in later sections, we will be mostly concerned by the mouse $\mathcal{M}_\omega$, which is obtained from $\mathcal{M}_\omega^\sharp$ by iterating its top extender out of the universe. At the end of this introduction, we will review the notation concerning mice. We note here that all premice have the degree of soundness, which we denote by $k(-)$. Our convention is that $k(\mathcal{M}_\omega^\sharp) = 0$. This mouse projects to $\omega$ and it is sound.

**Theorem 1.3.** *Let $\lambda$ be the least limit of Woodins. Then $\mathcal{M}_\omega^\sharp$ exists and it has a unique $(\omega_1, \omega_1 + 1)$-iteration strategy $\Sigma$. The strategy $\Sigma{\restriction}H_{\omega_1}$ is coded by a $<\lambda$-universally Baire set.*

*Proof outline.* The existence of $\mathcal{M}_\omega^\sharp$ follows from [Ste10, Theorem 7.2]. To verify the uniqueness, we have to consider Q-structures, which will be reviewed in Section 2. Since $\mathcal{M}_\omega^\sharp$ projects to $\omega$, for all normal trees $\mathcal{T}$ on $\mathcal{M}_\omega^\sharp$ of countable limit length and for all cofinal wellfounded branches $b$ through $\mathcal{T}$, we have that $\mathcal{Q}(\mathcal{T}, b)$ exists. If $\mathcal{T}^\frown b$ is according to some $(\omega_1, \omega_1 + 1)$-iteration strategy, then $\mathcal{Q}(\mathcal{T}, b)$ is $(\omega_1 + 1)$-iterable. This means that any such strategy must pick the same branch through $\mathcal{T}$ (cf. [Ste10, Corollary 6.14]). This shows the uniqueness of the restriction of $\Sigma$ to normal trees. The uniqueness of the full strategy then follows from the normalization (cf. [Sch21]).

To verify that the strategy $\Psi := \Sigma{\restriction}H_{\omega_1}$ is $<\lambda$-universally Baire, fix an uncountable cardinal $\kappa < \lambda$. By doing the $K^c$ construction above $\kappa$, we reach $\mathcal{M}_\omega^\sharp$ and obtain the strategy for it from the construction. This construction is absolute between $V$ and any generic extension of $V$ by a poset of size $<\kappa$. In the generic extension, we also reach $\mathcal{M}_\omega^\sharp$ and obtain the strategy for it from the construction. Since these strategies come from the realizability into the construction (cf. [Ste10, Theorem 6.6]), it is easily seen that club many hulls are $<\kappa$-generically correct about $\Psi$. This implies that $\Psi$ is $<\kappa$-universally Baire (cf. [Ste09, Lemma 4.1]). $\square$

We denote by $\Theta$, $\mathsf{HOD}$, and $\mathsf{HOD}_x$ (for all $x \in \mathbb{R}$) the corresponding objects computed in $L(\mathbb{R})$. It was shown by Steel and Woodin (cf. [SW16]) that $\mathsf{HOD}\|\Theta$[1] is a premouse. They in fact represented this structure as a certain direct limit of mice, which we will describe in Section 6. The first paper in this direction was [Ste95], where the author said that "$\mathsf{HOD}^{L(\mathbb{R})}$ is a core model below $\Theta$". One usually expects of a core model more than just the structure of a premouse. For example, one might expect it to be locally definable[2]. Our work can be understood as a step in this direction. We show that for all $\mathsf{HOD}$-cardinals $\eta \in [\boldsymbol{\delta}^2_1, \Theta)$, the mouse $\mathsf{HOD}\|\eta^{+\mathsf{HOD}}$ is definable over a structure of the form

$$L_\chi((\mathsf{HOD}\|\eta)^\omega)[\mu_\eta],$$

with no additional parameters. Here, $\mu_\eta$ is the supercompactness measure[3] on $[\eta]^\omega$ in $L(\mathbb{R})$ and $\chi$ is a certain (minimal) admissible ordinal.

In Section 2, we review the basics of Q-structures. Under certain circumstances, these structures determine which branches should be picked by an iteration strategy. They play the crucial role of allowing us to approximate iteration strategies of certain mice inside $L(\mathbb{R})$. These mice will be called super-suitable and they will be our primary focus here. Since $L(\mathbb{R})$ cannot ascertain the super-suitability, we will need to extract some weaker properties from it. One such property is the suitability, introduced in Section 3. The super-suitability itself is introduced in Section 4, while Section 5 analyses one more weakening of it, the so-called short tree iterability. In Section 6, we describe the already mentioned result of Steel and Woodin on representing $\mathsf{HOD}\|\Theta$ as a direct limit of mice. This concludes the introductory part of the paper.

Sections 7-9 contain the main argument of the paper. In Section 7, we use known facts on the normalization of stacks of normal trees to see $\mathsf{HOD}$ as a normal iterate of super-suitable mice. The normal tree leading from a super-suitable mouse to $\mathsf{HOD}\|\Theta$ is unique and we analyze those trees. In Section 8, we use this analysis to give a definition of $\mathsf{HOD}\|\eta^{+\mathsf{HOD}}$ over a structure of the form

$$L_\kappa(\mathbb{R}, \mathcal{Q}_\eta),$$

where $\eta$ is a $\mathsf{HOD}$-cardinal from the interval $[\boldsymbol{\delta}^2_1, \Theta)$, $\mathcal{Q}_\eta$ is the shortest initial segment of $\mathsf{HOD}\|\Theta$ that knows that $\eta$ is not Woodin in $\mathsf{HOD}$, and $\kappa$ is a certain admissible ordinal. Finally, in Section 9, we show that $\mathcal{Q}_\eta$ is definable over a structure of the form

$$L_\chi((\mathsf{HOD}\|\eta)^\omega)[\mu_\eta]$$

and obtain the main result of the paper, Corollary 9.13.

[1] Whenever $W$ is a transitive model of $\mathsf{ZFC}$ and $\kappa$ is a cardinal in it, we denote by $W\|\kappa$ the model $H^W_\kappa$.

[2] This is elsewhere called "inductively definable"; see for example [Ste96, Theorem 6.15]

[3] We say *the* supercompactness measure since there exists a unique such measure (cf. Theorem 9.2).

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## Notation

We review the notation which we will be using. We will try to follow the notation of [Ste23] as closely as possible and note the differences when they arise. We will need to use J-hierarchy above some $x$, which is defined as follows:

$$\begin{aligned} J_1^E(x) &:= V_\omega \cup \mathsf{trcl}(\{x\}) \\ J_{\alpha+1}^E(x) &:= \mathsf{rud}_E(J_\alpha^E(x) \cup \{J_\alpha^E(x)\}) \\ J_\gamma^E(x) &:= \bigcup_{\xi<\gamma} J_\xi^E(x) \quad (\gamma \text{ limit}). \end{aligned}$$

An $x$-J-structure $M$ has the form

$$M = (J_\alpha^E(x), \in, E, A,\ y : y \in \mathsf{trcl}(\{x\})),$$

where $A$ is amenable to $J_\alpha^E(x)$. We denote

$$\hat{o}(M) := \alpha \text{ and } o(M) := \omega\alpha.$$

Following [Ste83], we define the first projectum of $M$ as below.

**Definition 1.4.** Suppose that

1. $x$ is a set,
2. $M = (J_\alpha^E(x), \dots)$ is an $x$-J-structure.

Then $\rho_1(M)$ is the least $\rho \leq \alpha$ such that there exists $A \in \mathbf{\Sigma}_1^M$ satisfying

$$A \cap J_\rho^E(x) \notin M.$$

□

Note that it can (and does) happen, according to our definition, that

$$\rho_1(M) = 1.$$

If $x = 0$, the usual definition would have the projectum be $\omega$, so this is a difference to keep in mind. If $\rho_1(M) > 1$, there are no discrepancies of this kind. Regarding the notions of the standard parameter and soundness, they have their usual description, but w.r.t. the language of $x$-J-structures. This comment about the language was not necessary when we were defining the

projectum: the set $A$ above needs only be boldface definable anyway. However, in the case of the standard parameter and soundness, it is important to know that the elements of $\mathsf{trcl}(\{x\})$ are always allowed to be used as constants.

All premice are MS-indexed and they have the soundness degree integrated into them. If $M$ is a premouse, then $k(M)$ denotes its soundness degree. For all $(\eta, l) \leq (\hat{o}(M), k(M))$, we denote

$$M|(\eta, l) := (J_\eta^{E^M}, \in, E^M \restriction \eta, E_\eta^M, l),$$

$$M|\eta := M|(\eta, 0),$$

$$M\|\eta := (J_\eta^{E^M}, \in, E^M \restriction \eta, \emptyset, 0).$$

An ordinal $\xi \leq o(M)$ is a *strong cutpoint* of $M$ iff for all extenders $E$ on the $M$-sequence, either $\mathsf{lh}(E) < \xi$ or $\mathsf{crit}(E) > \xi$. In the case that $M$ has a unique Woodin cardinal, that cardinal is denoted by $\delta(M)$. All of this notation relativizes to $x$-premice in the obvious way. If we end up talking about $r\Sigma_{\omega+1}$, the reader should understand this as simply talking about $r\Sigma_\omega$.

## 2 Q-structures

In this section, we review basic facts about Q-structures. For an introduction on this, the reader is invited to consult [Ste10]. Our definitions here are based on [MS21, Definition 3.3].

**Definition 2.1.** Suppose that

1. $M$ is a premouse,
2. $\delta \leq o(M)$,
3. $Q \trianglelefteq M$.

Q-structure of $M$ at $\delta$

Then $Q$ is a Q-structure of $M$ at $\delta$ iff both a and b hold, where:

a. either $\delta = o(Q)$ or $Q \models$"$\delta$ is Woodin";

b. one of the conditions i, ii, or iii is met, where:

   i. $\hat{o}(Q) < \hat{o}(M)$, $k(Q) = \omega$, and $\delta$ is not Woodin in $J_1(M)$[4];

   ii. $\hat{o}(Q) = \hat{o}(M)$, $\rho(Q) = \delta$, and there exists an $r\Sigma_{k(M)+1}$ subset of $\delta$ witnessing that $\delta$ is not Woodin in $J_1(M)$;

   iii. $\hat{o}(Q) = \hat{o}(M)$ and $\rho(Q) < \delta$. □

$\mathcal{Q}(M, \delta)$

**Notation 2.2.** In the setup of the previous definition, there there exists at most one $Q$ which is a Q-structure for $M$ at $\delta$. If such $Q$ does exist, we call it *the* Q-structure for $M$ at $\delta$ and we denote it by $\mathcal{Q}(M, \delta)$. To say that such $Q$ exists, we use the shorthand of saying "$\mathcal{Q}(M, \delta)$ exists". □

[4] Note that the universe of $J_1(M)$ is the same as the universe of $M\|(\hat{o}(M) + 1)$.

In an iteration tree of limit length which is not too complicated, cofinal branches come with a naturally assigned Q-structures. These structures can then be used to pick the right branch through that tree.

**Definition 2.3.** Suppose that

1. $M$ is a premouse,
2. $\mathcal{T}$ is a normal tree on $M$ of limit length,
3. $b$ is a cofinal well-founded branch of $\mathcal{T}$.

$\mathcal{Q}(\mathcal{T}, b)$

Then $\mathcal{Q}(\mathcal{T}, b)$ exists iff $\mathcal{Q}(\mathcal{M}_b^{\mathcal{T}}, \delta(\mathcal{T}))$ exists. In that case, we denote by $\mathcal{Q}(\mathcal{T}, b)$ the structure $\mathcal{Q}(\mathcal{M}_b^{\mathcal{T}}, \delta(\mathcal{T}))$. □

The Q-structures corresponding to two different cofinal branches are mutually in comparable, unless they are of a certain particular type. We isolate this exception in the following definition.

**Definition 2.4.** Suppose that

1. $M$ is a premouse,
2. $\mathcal{T}$ is a normal tree on $M$ of limit length,
3. $b$ is a cofinal well-founded branch of $\mathcal{T}$.

anomaly

Then $(M, \mathcal{T}, b)$ is an anomaly iff all of the following conditions are simultaneously met:

a. $M$ is not sound,

b. $b$ does not drop,

c. $\mathcal{Q}(\mathcal{T}, b)$ exists and is equal to $\mathcal{M}_b^{\mathcal{T}}$. □

**Proposition 2.5.** *Suppose that*

1. *$M$ is a premouse,*
2. *$\mathcal{T}$ is a normal tree of limit length on $M$,*
3. *$b \neq c$ are cofinal well-founded branches through $\mathcal{T}$,*
4. *$(M, \mathcal{T}, b)$ and $(M, \mathcal{T}, c)$ are not anomalies.*

*Then neither $\mathcal{Q}(\mathcal{T}, b) \trianglelefteq \mathcal{Q}(\mathcal{T}, c)$ nor $\mathcal{Q}(\mathcal{T}, c) \trianglelefteq \mathcal{Q}(\mathcal{T}, b)$.*

*Proof.* See [Ste10, Theorem 6.12]. □

If $\mathcal{T}$ is a normal tree on $M$ of limit length and if the next branch to be picked has the Q-structure, we might try to guess that structure before actually knowing the branch. We now work towards introducing this structure and finally succeed in doing so in Definition 2.9.

countably iterable

**Definition 2.6.** Suppose that $M$ is a premouse. Then $M$ is countably iterable iff for all countable premice $\bar{M}$ and all elementary $i : \bar{M} \to M$, it holds that $\bar{M}$ is $(\omega_1 + 1)$-iterable. □

**Lemma 2.7.** *Suppose that $P$ is a premouse. Then there exists at most one premouse $Q$ satisfying that*

*a. $P \trianglelefteq Q$,*

*b. $o(P)$ is a strong cutpoint in $Q$,*

*c. $Q$ is the Q-structure of $Q$ at $o(P)$,*[5]

*d. $Q$ is sound above $o(P)$,*

*e. $Q$ is countably iterable.*

*Proof.* Let us assume otherwise and let $Q_0 \neq Q_1$ be two witnesses. Since we can always take a countable hull of some $H_\theta$, for a large enough $\theta$, we may assume w.l.o.g. that $P$, $Q_0$, and $Q_2$ are countable. By the argument of [Ste10, Corollary 3.12], we have that either $Q_0 \trianglelefteq Q_1$ or $Q_1 \trianglelefteq Q_0$. However, the minimality which is a part of the definition of a Q-structure would then imply $Q_0 = Q_1$, which is a contradiction. □

$\mathcal{Q}(P)$

**Definition 2.8.** Suppose that $P$ is a premouse. The Q-structure above $P$, denoted by $\mathcal{Q}(P)$, is the unique premouse $Q$ satisfying:

a. $P \trianglelefteq Q$,

b. $o(P)$ is a strong cutpoint in $Q$,

c. $Q$ is the Q-structure of $Q$ at $o(P)$,

d. $Q$ is sound above $o(P)$,

e. $Q$ is countably iterable. □

**Definition 2.9.** Suppose that

1. $M$ is a premouse,

2. $\mathcal{T}$ is a normal tree on $M$ of limit length.

$\mathcal{Q}(\mathcal{T})$

Then $\mathcal{Q}(\mathcal{T})$ exists iff $\mathcal{Q}(\mathcal{M}(\mathcal{T}))$ exists. In that case, we define

$$\mathcal{Q}(\mathcal{T}) := \mathcal{Q}(\mathcal{M}(\mathcal{T})).$$

□

[5] Or in other words, $Q = \mathcal{Q}(Q, o(P))$.

If mice do not have extenders overlapping local Woodins, they are said to be tame. Tame mice are simple enough so that Q-structures can be used to identify the right branches through trees on them, i.e. the branches that must be pick by any sufficiently strong strategy for those mice.

tame

**Definition 2.10.** Suppose that $M$ is a premouse. Then $M$ is tame iff for all $\eta < \hat{o}(M)$, if $E^M_\eta \neq \emptyset$, then for all $\delta \in [\mathsf{crit}(E), \eta)$, $M\|\eta \models$"$\delta$ is not Woodin". □

**Proposition 2.11.** *Suppose that*

1. *$M$ is a tame premouse,*
2. *$\mathcal{T}$ is a normal tree on $M$ of limit length.*

*Then there exists at most one cofinal wellfounded branch $b$ through $\mathcal{T}$ such that*

a. *$(M, \mathcal{T}, b)$ is not an anomaly,*
b. *$\mathcal{Q}(\mathcal{T}, b)$ exists,*
c. *$\mathcal{Q}(\mathcal{T}, b)$ is countably iterable.*

*Proof.* Let us assume otherwise and let $b_0 \neq b_1$ be two such branches. Since $M$ is tame, $\delta(\mathcal{T})$ is a strong cutpoint in both $\mathcal{Q}(\mathcal{T}, b_0)$ and $\mathcal{Q}(\mathcal{T}, b_1)$. By the uniqueness of Q-structures (cf. Lemma 2.7), we have that $\mathcal{Q}(\mathcal{T}, b_0) = \mathcal{Q}(\mathcal{T}, b_1)$, which contradicts Proposition 2.5. □

**Proposition 2.12.** *Suppose that*

1. *$M$ is a tame premouse,*
2. *$\Sigma_M$ is an $(\omega_1 + 1)$-iteration strategy for $M$,*
3. *$\mathcal{T}$ is a normal tree on $M$ of limit length according to $\Sigma_M$,*
4. *$b := \Sigma_M(\mathcal{T})$,*
5. *$\mathcal{Q}(\mathcal{T}, b)$ exists.*

*Then $\mathcal{Q}(\mathcal{T})$ exists and is equal to $\mathcal{Q}(\mathcal{T}, b)$.*

*Proof.* The Q-structure $\mathcal{Q}(\mathcal{T}, b)$ is countably iterable since $\mathcal{M}^{\mathcal{T}}_b$ is countably iterable. The ordinal $\delta(\mathcal{T})$ is a strong cutpoint in $\mathcal{Q}(T, b)$ because $M$ is tame. This suffices for the conclusion. □

The ordinal $(\boldsymbol{\delta}^2_1)^{L(\mathbb{R})}$ will figure prominently in the present work. There are many different characterizations of this ordinal, but we choose the one most useful for our purposes (cf. [Ste83, Lemma 1.12]). Since we will not compute this ordinal in any other model except $L(\mathbb{R})$, we omit the superscript.

**Definition 2.13.** The ordinal $\boldsymbol{\delta}^2_1$ is the least ordinal $\delta$ satisfying that $\Sigma_1$-formulas with parameters in $\mathbb{R} \cup \{\mathbb{R}\}$ are absolute between $L_\delta(\mathbb{R})$ and $L(\mathbb{R})$. □

This reflection implies that all countable mice that have $\omega_1$-iteration strategies in $L(\mathbb{R})$, have such strategies in $L_{\boldsymbol{\delta}^2_1}(\mathbb{R})$. The following proposition is an example of how this fact can be used.

**Proposition 2.14.** *Suppose that*

1. *$M$ is a countable $\omega$-small*[6] *premouse,*
2. *for all $\delta < o(M)$, if $M \models$"$\delta$ is Woodin", then $\mathcal{Q}(M, \delta)$ exists,*
3. *$M$ is $(\omega_1 + 1)$-iterable.*

*Then $L(\mathbb{R}) \models$"$M$ is $(\omega_1 + 1)$-iterable". Moreover, there exists a set of reals in $L_{\boldsymbol{\delta}^2_1}(\mathbb{R})$ which canonically codes an $\omega_1$-iteration strategy for $M$.*

*Proof.* Since $M$ is $(\omega_1 + 1)$-iterable, we have that it is weakly $\omega$-iterable[7]. By the proof of [Ste10, Theorem 7.10], we have that $M$ has an $(\omega_1 + 1)$-iteration strategy in $L(\mathbb{R})$. The moreover part follows from Definition 2.13. □

# 3 Suitable Premice

We are really interested in super-suitable premice. They are the appropriate initial segments of iterates of $\mathcal{M}_\omega$ and they are going to be introduced in Section 4. However, super-suitable premice cannot be defined internally in $L(\mathbb{R})$, so we need to work with an approximate notion, that of a suitable premouse. Of course, it will be a theorem that super-suitable premice are suitable.

**Definition 3.1.** Suppose that

1. $a$ is countable and transitive,
2. $M$ is an $a$-premouse.

Lp-good

Then $M$ is Lp-good iff all of the following conditions are met:

a. $k(M) = \omega$,

b. $\rho_\omega(M) = 1$[8],

c. $M$ has an $\omega_1$-iteration strategy in $L(\mathbb{R})$. □

Lp-good $a$-premice extend each other and there is no longest one among them. We will be interested in their supremum.

**Lemma 3.2.** *Suppose that*

1. *$a$ is countable and transitive,*

[6] See [Ste10, Definition 7.1]
[7] See [Ste10, Definition 7.7]
[8] See Definition 1.4 and the comment after it.

2. *$M, N$ are $\mathsf{Lp}$-good.*

*Then either $M \trianglelefteq N$ or $N \trianglelefteq M$.*

*Proof.* This is a straightforward generalization of [Ste10, Corollary 3.12]. □

**Lemma 3.3.** *Suppose that $a$ is countable and transitive. Then for all $\mathsf{Lp}$-good $a$-premice $M$, there exists an $\mathsf{Lp}$-good $a$-premouse $N$ such that $M \triangleleft N$.*

*Proof.* Since $M$ projects to 1 and is sound, there exists a surjection $f : J_1(a) \twoheadrightarrow M$ which is definable over $M$. Now, if look at $J_1(M)$, organized as an $a$-premouse, we see that it is obtained as the rudimentary closer of $M \cup \{M\}$. However, the rudimentary functions can be listed recursively, so the facts that $V_\omega \cup \{f\} \subseteq J_1(a)$ allow us to define a surjection

$$g : J_1(a) \twoheadrightarrow J_1(M)$$

over $J_1(M)$. This means that $J_1(M)$ projects to 1 and is consequently $\mathsf{Lp}$-good. □

Supremum of all $\mathsf{Lp}$-good $a$-mice is denoted by $\mathsf{Lp}(a)$. This object is itself an $a$-mouse and it looks like the power set of $a$. Since sets appearing in $\mathsf{Lp}(a)$ are not too complicated, i.e. they come from mice that have strategies in $L(\mathbb{R})$, this object can be understood as a *lower part* of the full powerset of $a$ (hence the abbreviation $\mathsf{Lp}$).

$\mathsf{Lp}(a)$ **Definition 3.4.** Suppose that $a$ is countable and transitive. Then $\mathsf{Lp}(a)$ is the unique $a$-premouse satisfying:

a. for all $\mathsf{Lp}$-good $a$-premice $M$, we have that $M \trianglelefteq \mathsf{Lp}(a)$;

b. no proper initial segment of $\mathsf{Lp}(a)$ satisfies the previous close. □

**Lemma 3.5.** *Suppose that $a$ is countable and transitive. Then the following holds:*

*a.* $k(\mathsf{Lp}(a)) = 0$,

*b.* $\mathsf{Lp}(a)$ *is countable.*

*Proof.*

1° If it were the case that $k(\mathsf{Lp}(a)) > 0$, then the $a$-premouse obtained from $\mathsf{Lp}(a)$ by decreasing $k(\mathsf{Lp}(a))$ by 1 would also satisfy condition a of Definition 3.4, while being a strict initial segment of $\mathsf{Lp}(a)$. This shows that a must hold.

2° To establish b, let us assume otherwise. Then the set $C$ of all $\alpha \in (\mathsf{rank}(a), \omega_1)$ such that there exists an $\mathsf{Lp}$-good $a$-premouse $M$ satisfying $\hat{o}(M) = \alpha$ is cofinal in $\omega_1$.

3° For all $\alpha \in C$, an $M$ witnessing this fact is unique and we denote it by $M_\alpha$.

4° Let $(f_\alpha : \alpha \in C), g$ be as follows:

1. for all $\alpha \in C$, $f_\alpha : a \cup \{a\} \twoheadrightarrow M_\alpha$ is given by the soundness,
2. $g : \omega \twoheadrightarrow a \cup \{a\}$ is an arbitrary enumeration,
3. for all $\alpha \in C$, $e_\alpha := \{(m, n) : f_\alpha(g(m)) \in f_\alpha(g(n))\} \subseteq \omega^2$.

5° The sequence $(e_\alpha : \alpha \in C)$ is injective and it belongs to $L(\mathbb{R})$. This contradicts $L(\mathbb{R}) \models \mathsf{AD}$.

□

We said that $\mathsf{Lp}(a)$ should be understood as a lower part of the powerset of $a$. It turns out that if $a$ is countable, transitive, and self-wellorderable[9], it is in fact the case that

$$\mathsf{Lp}(a) = H(|a|^+)^{\mathcal{M}_\omega(a)} = H(|a|^+)^{\mathsf{HOD}^{L(\mathbb{R})}(a \cup \{a\})}.$$

This is the content of [SW16, Theorem 6.4], which we reproduce here.

**Theorem 3.6.** *Suppose that*

1. *$a$ is countable and transitive,*
2. *$b \subseteq a$.*

*Then the following are equivalent.*

a. *$b$ is definable over $(L(\mathbb{R}), \in)$ from parameters in $\mathsf{Ord} \cup a \cup \{a\}$.*

b. *$b \in \mathsf{Lp}(a)$.*

c. *$b \in \mathcal{M}_\omega(a)$.* □

**Corollary 3.7.** *Suppose that $a$ is countable, transitive, and self-wellorderable. Then it holds that*

a. *$\mathsf{Lp}(a) = H(|a|^+)^{\mathcal{M}_\omega}$,*

b. *$\mathsf{Lp}(a) \models \mathsf{ZFC}^-$,*

c. *$\mathsf{Lp}(a)$ is $\omega$-sound.* □

We are ready to introduce the notion of a suitable premouse. Here and later, we will need reorganize premice into premice over their initial segments (when this is possible).

[9] A transitive set $a$ is *self-wellorderable* iff $J_1(a) \models$"there exists a wellordering on $a$".

$M/\delta$

**Notation 3.8.** Suppose that $M$ is a premouse and $\delta$ is a strong cutpoint of $M$. Then we denote by $M/\delta$ the canonical reorganization of $M$ into an $(M\|\delta)$-premouse. □

suitable

**Definition 3.9.** Suppose that $M$ is a premouse. Then $M$ is suitable iff there exists $\delta < \omega_1$ such that

a. $M \models \mathsf{ZFC}^- +$"$\delta$ is the largest cardinal"+"$\delta$ is Woodin",

b. $\delta$ is a strong cutpoint of $M$,

c. for all $\eta \leq \delta$, $\mathsf{Lp}(M\|\eta) \subseteq M$,

d. for all $\eta \leq \delta$, if $\eta$ is a strong cutpoint of $M$, then $\mathsf{Lp}(M\|\eta) = (M\|\eta^{+M})/\eta$,

e. for all $\eta < \delta$, $\mathsf{Lp}(M\|\eta) \models$"$\eta$ is not Woodin". □

We observe that the part of the definition is for $\delta$ to be countable. This has as an immediate consequence the fact that all suitable premice are countable.

**Lemma 3.10.** *Suppose that $M$ is suitable. Then $M$ is countable.*

*Proof.* This follows from Lemma 3.5. and the fact that there exists $\delta < \omega_1$ such that $M/\delta = \mathsf{Lp}(M\|\delta)$. □

# 4 Super-suitable Premice

A super-suitable premouse is obtained from an iterate of $\mathcal{M}_\omega$ by cutting it at the successor of its least Woodin. We make precise in next few definitions. Here and later, we shall introduce several variations on the notion of an iterate. What we call here simply "an iterate" is elsewhere called (more cumbersomely) "a nondropping iterate". Since we will not have the need to talk about dropping iterates, we omit this additional qualifier. Similarly, since we will not have the need to talk about uncountable iterates, we omit the qualifier "countable" as well and incorporate the countability in the definitions.

**Definition 4.1.** Suppose that

1. $M, N$ are countable premice,

2. $\Sigma_M$ is an $(\omega_1 + 1)$-iteration strategy for $M$.

normal $\Sigma_M$-iterate

Then $N$ is a normal $\Sigma_M$-iterate of $M$ iff there exists a countable normal tree $\mathcal{T}$ on $M$ according to $\Sigma_M$ whose last model is $N$ and whose main branch does not drop. □

**Definition 4.2.** Suppose that

1. $M, N$ are countable premice,

2. $\Sigma_M$ is an $(\omega_1, \omega_1 + 1)$-iteration strategy for $M$.

$\Sigma_M$-iterate

Then $N$ is a $\Sigma_M$-iterate of $M$ iff there exists a countable stack of countable normal trees on $M$ according to $\Sigma_M$ whose last model is $N$ and whose main branch does not drop. □

**Definition 4.3.** Suppose that

1. $\tau := \delta(\mathcal{M}_\omega)^{+\mathcal{M}_\omega}$,
2. $\mathcal{M} := \mathcal{M}_\omega \| \tau$,
3. $\Sigma_{\mathcal{M}}$ is the iteration strategy for $\mathcal{M}$ obtained from the unique $(\omega_1, \omega_1 + 1)$-iteration strategy for $\mathcal{M}_\omega$,
4. $M, N$ are countable premice.

Then we define the following.

super-suitable

a. $M$ is super-suitable iff $M$ is a $\Sigma_{\mathcal{M}}$-iterate of $\mathcal{M}$.

$\Sigma_M$

b. If $M$ is super-suitable, then $\Sigma_M$ is the $(\omega_1, \omega_1 + 1)$-iteration strategy for $M$ induced by $\Sigma_{\mathcal{M}}$. □

Since the supremum of Woodin cardinals of $\mathcal{M}_\omega$ is countable and since all iterates are countable by our choice of the definition, we have that all super-suitable mice are countable. Furthermore, since all super-suitable mice are elementarily equivalent to the mouse $\mathcal{M}$ of the previous definition, they are all tame. We highlight this in the following lemma.

**Lemma 4.4.** *All super-suitable premice are countable and tame.* □

When we consider super-suitable mice, we will only consider them together with their canonical strategies. Thus, we can simplify the terminology of iterates a bit.

**Notation 4.5.** Suppose that $M$ is super-suitable and that $N$ is a countable premouse. Then we define the following.

normal iterate

a. $N$ is a normal iterate of $M$ iff $N$ is a normal $\Sigma_M$-iterate of $M$.

iterate

b. $N$ is an iterate of $M$ iff $N$ is a $\Sigma_M$-iterate of $M$. □

As we have already pointed it out, the suitability is an approximation to the super-suitability, so the following proposition is to be expected.

**Proposition 4.6.** *Suppose that $M$ is a super-suitable premouse. Then $M$ is suitable.*

*Proof.* See the paragraph immediately below Definition 6.8 of [SW16]. □

If $M$ is super-suitable, then $M$ is in particular tame. This has for a consequence that $\delta(\mathcal{T})$ is a strong cutpoint in $\mathcal{Q}(\mathcal{T}, b)$ whenever $\mathcal{T}$ is a normal tree on $M$, $b$ a cofinal wellfounded branch through $\mathcal{T}$, and $\mathcal{Q}(\mathcal{T}, b)$ exists. In particular, the $\mathcal{M}(\mathcal{T})$-premouse $\mathcal{Q}(\mathcal{T}, b)/\delta(\mathcal{T})$ is defined. If $\mathcal{Q}(\mathcal{T}, b)$ is iterable, then teh canonical strategy $\Sigma_M$ must pick the branch $b$ for the tree $\mathcal{T}$, which is the content of the following proposition.

**Proposition 4.7.** *Suppose that*

1. *$M$ is a super-suitable premouse,*
2. *$\mathcal{T}$ is a countable normal tree on $M$ of limit length according to $\Sigma_M$,*
3. *$b$ is a cofinal wellfounded branch through $\mathcal{T}$,*
4. *$\mathcal{Q}(\mathcal{T}, b)$ exists and is $(\omega_1 + 1)$-iterable.*

*Then $\mathcal{Q}(\mathcal{T}, b)/\delta(\mathcal{T}) \trianglelefteq \mathsf{Lp}(\mathcal{M}(\mathcal{T}))$ and $\Sigma_M(\mathcal{T}) = b$.*

*Proof.*

1° By Proposition 2.14, we have that $\mathcal{Q}(\mathcal{T}, b)$ has an $(\omega_1 + 1)$-iteration strategy in $L(\mathbb{R})$. In particular, $\mathcal{Q}(\mathcal{T}, b)$ is $\mathsf{Lp}$-good and consequently,

$$\mathcal{Q}(\mathcal{T}, b)/\delta(\mathcal{T}) \trianglelefteq \mathsf{Lp}(\mathcal{M}(\mathcal{T})).$$

2° Let us now verify that $\Sigma_M(\mathcal{T}) = b$. We assume towards contradiction that the branch $c := \Sigma_M(\mathcal{T})$ is distinct from $b$.

3° The Q-structure $\mathcal{Q}(\mathcal{T}, c)$ does not exist, for otherwise it would be $(\omega_1 + 1)$-iterable, while there can be only one branch with an iterable Q-structure. Hence, we get that $c$ does not drop and $\mathcal{M}_c^{\mathcal{T}} \models$"$\delta(\mathcal{T})$ is Woodin".

4° Since $c$ does not drop, we have that $\mathcal{M}_c^{\mathcal{T}}$ is super-suitable.

5° Putting together 3° and 4°, we get that $\mathcal{M}_c^{\mathcal{T}}$ is suitable and that $\delta(\mathcal{T})$ is its unique Woodin. This means that

$$\mathcal{M}_c^{\mathcal{T}}/\delta(\mathcal{T}) = \mathsf{Lp}(\mathcal{M}_c^{\mathcal{T}} \| \delta(\mathcal{T})) = \mathsf{Lp}(\mathcal{M}(\mathcal{T})).$$

6° Adding to this the conclusion of 1°, we have that

$$\mathcal{Q}(\mathcal{T}, b) \trianglelefteq \mathsf{Lp}(\mathcal{M}(\mathcal{T})) = \mathcal{M}_c^{\mathcal{T}}.$$

7° Since $\delta(\mathcal{T})$ is Woodin in $\mathcal{M}_c^{\mathcal{T}}$, it follows that $\mathcal{Q}(\mathcal{T}, b) = \mathcal{M}_c^{\mathcal{T}}$.

8° Now, this would mean that $\mathcal{Q}(\mathcal{T}, c)$ does exist (and is equal to $\mathcal{Q}(\mathcal{T}, b)$), which is in contradiction with 3°.

□

# 5 Short tree Iterability

The model $L(\mathbb{R})$ does not contain an $\omega_1$-itertion strategy for $\mathcal{M}_\omega$. The reason for this is that otherwise one could easily construct, using the genericity iterations, $\mathbb{R}^\sharp$ inside $L(\mathbb{R})$. A consequence of this fact is that $L(\mathbb{R})$ cannot check if a given countable normal tree on the suitable initial segment of $\mathcal{M}_\omega$ is according to the strategy for $\mathcal{M}_\omega$. This also means that one cannot expect to be able to verify inside $L(\mathbb{R})$ whether a premouse is super-suitable and for this reason, we introduced the notion of suitability. Now, given a super-suitable premouse $M$, we can verify inside $L(\mathbb{R})$ that it is suitable, but we cannot know the strategy $\Sigma_M$. What we do have is a partial strategy, one which knows how to continue the tree as long as the tree is *short* and breaks down once the tree stops being short (such trees are called *maximal*). This partial strategy is called *the short tree strategy* and denoted by $\Sigma_M^{\mathrm{st}}$.

**Definition 5.1.** Suppose that

1. $M$ is suitable,
2. $\mathcal{T}$ is a countable normal tree on $M$ of limit length.

short, maximal, $\Sigma_M^{\mathrm{st}}$

The predicates "$\mathcal{T}$ is short", "$\mathcal{T}$ is maximal", "$\mathcal{T} \in \mathsf{dom}(\Sigma_M^{\mathrm{st}})$" and the value $\Sigma_M^{\mathrm{st}}(\mathcal{T})$ are defined by recursion on $\mathsf{lh}(\mathcal{T})$, as follows. If $\mathcal{T}$ is according to $\Sigma_M^{\mathrm{st}}$, then

a. $\mathcal{T}$ is short iff $\mathcal{Q}(\mathcal{T})$ exists,

b. $\mathcal{T}$ is maximal iff it is not short,

c. $\mathcal{T} \in \mathsf{dom}(\Sigma_M^{\mathrm{st}})$ iff $\mathcal{T}$ is short and there exists a cofinal wellfounded branch $b$ through $\mathcal{T}$ such that $\mathcal{Q}(\mathcal{T}) \trianglelefteq \mathcal{M}_b^{\mathcal{T}}$,

d. if $\mathcal{T} \in \mathsf{dom}(\Sigma_M^{\mathrm{st}})$, the $\Sigma_M^{\mathrm{st}}(\mathcal{T})$ is the unique cofinal wellfounded branch $b$ through $\mathcal{T}$ satisfying $\mathcal{Q}(\mathcal{T}) \trianglelefteq \mathcal{M}_b^{\mathcal{T}}$. □

To explain the previous definition, at successor stages, the tree is built by picking and applying an extender, so the strategy is concerned by limit stages only. If the tree is built according to the short tree strategy and if we reach a limit stage, several things can occur. One possibility is that the tree is maximal, in which case the strategy breaks down. In particular, no extension of such a tree will be according to the short tree strategy. The other possibility is that the tree is short. In this case, we would like to say that $\Sigma_M^{\mathrm{st}}(\mathcal{T})$ is defined, but the fact that $\mathcal{Q}(\mathcal{T})$ exists does not guarantee that there is a branch whose Q-structure is exactly $\mathcal{Q}(\mathcal{T})$. This last scenario is a pathology and we would like to exclude it. Assuming that this pathology does not occur, we then know that the branch $b$ whose Q-structure is equal to $\mathcal{Q}(\mathcal{T})$ is unique, so we can set $\Sigma_M^{\mathrm{st}}(\mathcal{T}) := b$. The pathology just mentioned is dealt with by introducing the notion of short tree iterability. This notion also excludes another pathology, that of the possibility that an ultrapower in the tree may produce an illfounded model, and it additionally guarantees that the iterates of $M$ stay suitable (which is sometimes called *being fullness preserving*).

short tree iterable

**Definition 5.2.** Suppose that $M$ is suitable. Then $M$ is short tree iterable iff for all countable trees $\mathcal{T}$ on $M$ according to $\Sigma_M^{\mathrm{st}}$, we have that

a. if $\mathcal{T}$ has a last model $N$, then

   i. $\mathcal{T}$ can be normally extended by any extender on the $N$-sequence without producing an illfounded model,

   ii. if the branch $M$-to-$N$ of $\mathcal{T}$ does not drop, then $N$ is suitable,

b. if $\mathcal{T}$ is short, then $\mathcal{T} \in \mathsf{dom}(\Sigma_M^{\mathrm{st}})$,

c. if $\mathcal{T}$ is maximal, then there exists a nondropping cofinal wellfounded branch $b$ through $\mathcal{T}$ such that $\mathcal{M}_b^{\mathcal{T}}$ is suitable. □

We have already seen that a super-suitable premouse is suitable. In this sense, the suitability is an approximation to the super-suitability. We want to see next that $\Sigma_M^{\mathrm{st}}$ is an approximation to $\Sigma_M$ whenever $M$ is super-suitable.

**Proposition 5.3.** *Suppose that $M$ is super-suitable. Then $\Sigma_M^{\mathrm{st}} \subseteq \Sigma_M$.*

*Proof.*

1° By induction on $\mathsf{lh}(\mathcal{T})$, we show that for all $\mathcal{T} \in \mathsf{dom}(\Sigma_M^{\mathrm{st}})$, if

$$\forall \eta < \mathsf{lh}(\mathcal{T}) \text{ limit}, \ \mathcal{T}\restriction\eta \in \mathsf{dom}(\Sigma_M),$$

then $\mathcal{T} \in \mathsf{dom}(\Sigma_M)$ and $\Sigma_M(\mathcal{T}) = \Sigma_M^{\mathrm{st}}(\mathcal{T})$.

2° Since $\mathcal{T}$ is according to $\Sigma_M$, we immediately have that $\mathcal{T} \in \mathsf{dom}(\Sigma_M)$.

3° Let us denote by $b$ the branch $\Sigma_M^{\mathrm{st}}(\mathcal{T})$. By the definition of $\Sigma_M^{\mathrm{st}}$, we have that $\mathcal{Q}(\mathcal{T})$ exists and satisfies $\mathcal{Q}(\mathcal{T}) \trianglelefteq \mathcal{M}_b^{\mathcal{T}}$. Thus, $\mathcal{Q}(\mathcal{T}, b) = \mathcal{Q}(\mathcal{T})$ exists and is $(\omega_1 + 1)$-iterable.

4° By Proposition 4.7, we must have that $\Sigma_M(\mathcal{T}) = b$, as required.

□

Super-suitable mice are not only suitable, but also iterable. Since we are interested in $L(\mathbb{R})$-approximations, we should verify the short tree iterability of these mice.

**Proposition 5.4.** *Suppose that $M$ is super-suitable. Then $M$ is short tree iterable.*

*Proof.*

1° Let $\mathcal{T}$ be a countable normal tree on $M$ according to $\Sigma_M^{\mathrm{st}}$. By Proposition 5.3, $\mathcal{T}$ is also according to $\Sigma_M$.

2° What needs to be verified depends on the type of the tree $\mathcal{T}$, as can be seen from Definition 5.2. We have three cases.

3° **Case.** $\mathcal{T}$ is of successor length.

*Proof.* First, we need to verify that when $\mathcal{T}$ is extended by one extender, we do not get an illfounded model. However, this is obvious since $\mathcal{T}$ is according to $\Sigma_M$ and $\Sigma_M$ is an $(\omega_1, \omega_1 + 1)$-iteration strategy for $M$. The second thing to verify is that if the main branch of $\mathcal{T}$ does not drop, then the last model of $\mathcal{T}$ is suitable. For this, observe that if the main branch of $\mathcal{T}$ does not drop, then the last model of $\mathcal{T}$ is an iterate of a super-suitable premouse $M$, so it must itself be super-suitable. □

4° **Case.** $\mathcal{T}$ is short.

*Proof.*

1′ Being short means that $\mathcal{Q}(\mathcal{T})$ exists, so we only need to verify that there exists a cofinal wellfounded branch $b$ through $\mathcal{T}$ such that $\mathcal{Q}(\mathcal{T}) \trianglelefteq \mathcal{M}_b^{\mathcal{T}}$ works. We claim that $b := \Sigma_M(\mathcal{T})$ works.

2′ Let us assume towards contradiction that $b$ does not drop. Then $\mathcal{M}_b^{\mathcal{T}}$ is suitable and $\delta(\mathcal{T})$ is its Woodin.

3′ By definition, the premouse $\mathcal{Q}(\mathcal{T})$ is $(\omega_1 + 1)$-iterable and $\delta(\mathcal{T})$ is its strong cutpoint. This means that $\mathcal{Q}(\mathcal{T})/\delta(\mathcal{T})$ is $\mathsf{Lp}$-good and consequently,

$$\mathcal{Q}(\mathcal{T})/\delta(\mathcal{T}) \trianglelefteq \mathsf{Lp}(\mathcal{M}(\mathcal{T})).$$

4′ However, we have by suitability that

$$\mathcal{M}_b^{\mathcal{T}}/\delta(\mathcal{T}) = \mathsf{Lp}(\mathcal{M}_b^{\mathcal{T}} \| \delta(\mathcal{T})) = \mathsf{Lp}(\mathcal{M}(\mathcal{T})),$$

so we conclude that $\mathcal{Q}(\mathcal{T}) \trianglelefteq \mathcal{M}_b^{\mathcal{T}}$.

5′ If it were the case that $o(\mathcal{Q}(\mathcal{T})) < o(\mathcal{M}_b^{\mathcal{T}})$, it would hold that $\delta(\mathcal{T})$ is not Woodin in $\mathcal{M}_b^{\mathcal{T}}$. This being absurd, we conclude that the J-structures underlying premice $\mathcal{Q}(\mathcal{T})$ and $\mathcal{M}_b^{\mathcal{T}}$ are equal.

6′ However, this is a contradiction since $\rho(\mathcal{Q}(\mathcal{T})) \leq \delta(\mathcal{T})$ while

$$\rho_\omega(\mathcal{M}_b^{\mathcal{T}}) = \hat{o}(\mathcal{M}_b^{\mathcal{T}}).$$

7′ The conclusion is then that $b$ drops, which means that $\mathcal{Q}(\mathcal{T}, b)$ exists.

8′ By the uniqueness of the Q-structure above $\mathcal{M}(\mathcal{T})$ (cf. Proposition 2.7), it follows that $\mathcal{Q}(\mathcal{T}, b) = \mathcal{Q}(\mathcal{T})$ and consequently, $\mathcal{Q}(\mathcal{T}) \trianglelefteq \mathcal{M}_b^{\mathcal{T}}$, as required.

□

5° **Case.** $\mathcal{T}$ is maximal.

*Proof.*

1′ In this case, we need to verify that there exists a nondropping cofinal wellfounded branch $b$ through $\mathcal{T}$ such that $\mathcal{M}_b^{\mathcal{T}}$ is suitable. We claim that $b := \Sigma_M(\mathcal{T})$ works.

2′ If $b$ dropped, then $\mathcal{Q}(\mathcal{T}, b)$ would exist and be $(\omega_1 + 1)$-iterable. In other words, $\mathcal{Q}(\mathcal{T})$ would exist and be equal to $\mathcal{Q}(\mathcal{T}, b)$, which is not the case.

3′ Thus, $b$ dos not drop.

4′ It follows that $\mathcal{M}_b^{\mathcal{T}}$ is an iterate of a super-suitable premouse $M$, as witnessed by $\mathcal{T}^\frown b$, so $\mathcal{M}_b^{\mathcal{T}}$ is super-suitable.

5′ In particular, $\mathcal{M}_b^{\mathcal{T}}$ is suitable, as required.

□

6° The three cases above cover all possibilities, so the verification is concluded.

□

Hence, the right $L(\mathbb{R})$-approximations for super-suitable mice are suitable, short tree iterable premice. The notion of a normal iterate for them is not in general defined, so we introduce a more general notion, which we shall call a *normal pseudo-iterate*.

$M \longrightarrow^{\mathrm{st}}_{\mathcal{T}} N$

**Definition 5.5.** The relation $M \longrightarrow^{\mathrm{st}}_{\mathcal{T}} N$ holds iff

a. $M$ is suitable and short tree iterable,

b. $\mathcal{T}$ is a tree on $M$ according to $\Sigma^{\mathrm{st}}_M$,

c. if $\mathcal{T}$ has a last model, then $N$ is the last model of $\mathcal{T}$ and the $M$-to-$N$ branch of $\mathcal{T}$ does not drop,

d. if $\mathcal{T}$ does not have a last model, then $\mathcal{T}$ is maximal and there exists a nondropping cofinal wellfounded branch $b$ through $\mathcal{T}$ such that $\mathcal{M}_b^{\mathcal{T}}$ is suitable and $\mathcal{M}_b^{\mathcal{T}} = N$. □

This relation is absolute between $L(\mathbb{R})$ and $V$. In the part c, one has the behavior that one would expect from a normal iterate. It is the part d that is unusual insomuch that we do not know what is the branch leading to the final model. Let us look closer to this case.

**Lemma 5.6.** *Suppose that $M \longrightarrow^{\mathrm{st}}_{\mathcal{T}} N$. Then $N$ is suitable and if $\mathcal{T}$ is maximal, then $\delta(\mathcal{T})$ is the Woodin of $N$ and $N\|\delta(\mathcal{T}) = \mathcal{M}(\mathcal{T})$.*

*Proof.* This is immediate from the definition since it subsumes the short tree iterability of $M$. □

What could happen is that there could be many branches that are "good enough", so we do not force ourselves to choose, but we know that they all lead to the same model. In other words, by Lemma 5.6, we have that

$$N/\delta(\mathcal{T}) = \mathsf{Lp}(\mathcal{M}(\mathcal{T})),$$

which means that $N$ depends only on $\mathcal{T}$, but not on $b$. That there are indeed such branches follows from the definition of the short tree iterability, but (assuming that $M$ is super-suitable) we do not know which one of them is chosen by $\Sigma_M$. We observe just below that in the case that $M$ is super-suitable, the relation

$$M \longrightarrow^{\mathrm{st}}_{\mathcal{T}} N$$

simply means that $N$ is a normal iterate of $M$ via $\mathcal{T}$.

**Lemma 5.7.** *Suppose that*

1. *$M$ is super-suitable,*
2. *$N$ is a countable premouse,*
3. *$\mathcal{T}$ is a countable normal tree on $M$,*
4. *for all limit $\gamma < \mathsf{lh}(\mathcal{T})$, $\mathcal{Q}(\mathcal{T}\restriction\gamma)$ exists.*

*Then $M \longrightarrow^{\mathrm{st}}_{\mathcal{T}} N$ holds if and only if the following holds:*

a. *$\mathcal{T}$ is according to $\Sigma_M$,*

b. *if $\mathcal{T}$ has a last model, then $\mathcal{T}\restriction(\mathsf{lh}(\mathcal{T})-1)$ is not maximal*[10]*, $N$ is the last model of $\mathcal{T}$, and the $M$-to-$N$ branch of $\mathcal{T}$ does not drop,*

c. *if $\mathcal{T}$ does not have a last model, then for $b := \Sigma_M(\mathcal{T})$, we have that $b$ does not drop and $N = \mathcal{M}_b^{\mathcal{T}}$.*

*Proof.*

**Implication ($\Rightarrow$)**

1° The part a follows from Proposition 5.3, while the part b follows from the part a.

2° We want to verify the part c. In this case, the tree $\mathcal{T}$ is maximal.

3° If $b$ dropped, then $\mathcal{Q}(\mathcal{T}, b)$ would exist and be $(\omega_1+1)$-iterable, which would mean that $\mathcal{Q}(\mathcal{T})$ exists and is equal to $\mathcal{Q}(\mathcal{T}, b)$. This being contradictory, we conclude that $b$ does not drop.

4° It follows that $\mathcal{M}_b^{\mathcal{T}}$ is super-suitable and that $\delta(\mathcal{T})$ is its Woodin. In particular,

$$\mathcal{M}_b^{\mathcal{T}}/\delta(\mathcal{T}) = \mathsf{Lp}(\mathcal{M}(\mathcal{T})) = N/\delta(\mathcal{T}),$$

where the second equality follows from Lemma 5.6.

[10] We do not say "short" here because we $\mathcal{T}\restriction(\mathsf{lh}(\mathcal{T})-1)$ might not be of limit length.

5° This suffices for the conclusion that $N = \mathcal{M}_b^{\mathcal{T}}$.

**Implication ($\Leftarrow$)**

6° The assumptions yield that if $\mathcal{T}$ has a last model, then $\mathcal{T}\restriction(\mathsf{lh}(\mathcal{T}) - 1)$ is not maximal.

7° We also know that there cannot exist $\xi$ such that $\xi + 1 < \mathsf{lh}(\mathcal{T})$ and $\mathcal{T}\restriction\xi$ is maximal. The reason for this is that $\delta(\mathcal{T}) = \delta(\mathcal{M}_\xi^{\mathcal{T}})$ is a strong cutpoint of $\mathcal{M}_\xi^{\mathcal{T}}$, so any cofinal branch through $\mathcal{T}$ must contain $\xi$ and must drop at the successor of $\xi$.

8° These two fact together with Proposition 5.3 imply that $\mathcal{T}$ is according to $\Sigma_M^{\mathrm{st}}$. The rest is now a routine verification.

□

normal pseudo-iterate

**Definition 5.8.** Suppose that $M$ is suitable and $N$ is a countable premouse. Then $N$ is a normal pseudo-iterate of $M$ iff there exists $\mathcal{T}$ such that $M \longrightarrow_{\mathcal{T}}^{\mathrm{st}} N$. □

**Proposition 5.9.** *Suppose that*

1. *$M$ is super-suitable,*
2. *$N$ is a countable premouse.*

*Then the following are equivalent.*

a. *$N$ is a normal pseudo-iterate of $M$.*
b. *$N$ is a normal iterate of $M$.*

*Proof.* This follows from Lemma 5.7. □

It turns out that the short tree iterability suffices for the comparison.

**Proposition 5.10.** *Suppose that $M, N$ are suitable and short tree iterable. Then there exists $P$ which is a normal pseudo-iterate of both $M$ and $N$.*

*Proof.*

1° Let $T$ be the tree of a $(\Sigma_1^2)^{L(\mathbb{R})}$-scale on a universal $(\Sigma_1^2)^{L(\mathbb{R})}$-set and let

$$\mu := \max\{\delta(M), \delta(N)\}^{+L[T,M,N]}.$$

We want to show that $\mu < \omega_1$, for once we do this, the conclusion follows from [SW16, Theorem 6.14].

2° Let $x$ be a real coding $M$ and $N$. It suffices to show that $\omega_1^{L[T,x]} < \omega_1$.

3° The theorem at the bottom of the page 77 of [Ste95] implies that

$$L[T]\|\boldsymbol{\delta}_1^2 = \mathsf{HOD}\|\boldsymbol{\delta}_1^2,$$

while [Lar22, Corollary 10.3.7] implies that $\mathsf{HOD}_x = \mathsf{HOD}[x]$.

4° We can now compute that

$$L[T,x]\|\boldsymbol{\delta}_1^2 = (L[T]\|\boldsymbol{\delta}_1^2)[x] = (\mathsf{HOD}\|\boldsymbol{\delta}_1^2)[x] = \mathsf{HOD}[x]\|\boldsymbol{\delta}_1^2 = \mathsf{HOD}_x\|\boldsymbol{\delta}_1^2.$$

5° By [SW16, Lemma 6.34], $\omega_1$ is the least measurable of $\mathsf{HOD}$. Exactly the same arguments yields the same conclusion for $\mathsf{HOD}_x$.

6° Thus, it follows that

$$\omega_1^{\mathsf{HOD}_x} < (\text{“the first measurable”})^{\mathsf{HOD}_x} = \omega_1,$$

as required.

□

# 6 HOD as a Direct Limit of Mice

In this section, we describe $\mathsf{HOD}\|\Theta$ as the direct limit of a certain directed family of mice. This is just a slight reformulation of the well-known results of [SW16].

**Definition 6.1.** Suppose that $M, N$ are super-suitable. Then we define the following.

$\mathcal{I}(M)$ a. $\mathcal{I}(M)$ is the set of all normal iterates of $M$.

$M \leq_{\mathcal{I}} N$ b. $M \leq_{\mathcal{I}} N$ iff $N$ is a normal iterate of $M$. □

We note that if the main branch of a normal tree on a super-suitable mouse does not drop, then the tree is based below the Woodin. The reason why extenders above the Woodin cannot be used is that once such an extender is used, all later extenders need to be above the Woodin, while any such extender necessarily leads to a drop. Instead of restricting ourselves to normal iterates, we could have looked at all iterates (via stacks of normal trees). However, by results of [Sch21], stacks of trees can be normalized, so nothing would be gained from this extension and we are still the scenario of [Ste10, Section 8] and [SW16, Section 6]. In particular, the structure $(\mathcal{I}(M), \leq_{\mathcal{I}})$ is a countably directed partial order.

**Lemma 6.2.** *Suppose that $M$ is super-suitable. Then $(\mathcal{I}(M), \leq_{\mathcal{I}})$ is a countably directed partial order.* □

This partial order will index our directed family. To every pair $(M, N)$ satisfying $M \leq_{\mathcal{I}} N$, we need to assign an arrow $M \to N$, which is provided by the next lemma.

**Lemma 6.3.** *Suppose that*

1. *$M$ is super-suitable,*
2. *$N$ is a normal iterate of $M$.*

*Then there exists a unique normal tree $\mathcal{T}$ on $M$ according to $\Sigma_M$ with the last model $N$. This tree is countable and its main branch does not drop.*

*Proof.* Such a tree $\mathcal{T}$ exists by the fact that $N$ is a normal iterate of $M$. The uniqueness follows from the fact that any such tree must be the first coordinate of the coiteration of $(M, \Sigma_M)$ and $(N, \Sigma_N)$. □

**Definition 6.4.** Suppose that

1. $M$ is super-suitable,
2. $N$ is a normal iterate of $M$.

$\pi_{M,N}$ Then the mapping $\pi_{M,N}$ is defined to be the mapping

$$\pi^{\mathcal{T}} : M \to N$$

where $\mathcal{T}$ is the normal tree on $M$ according to $\Sigma_M$ whose last model is $N$. □

We are ready to introduce the directed system.

$\vec{\mathcal{I}}(M)$ **Definition 6.5.** Suppose that $M$ is super-suitable. Then we define $\vec{\mathcal{I}}(M)$ to be the system

$$(P, \pi_{P,Q} : P, Q \in \mathcal{I}(M),\ P \leq_{\mathcal{I}} Q).$$

□

**Proposition 6.6.** *Suppose that $M$ is super-suitable. Then $\vec{\mathcal{I}}(M)$ is a directed system with a wellfounded direct limit.*

*Proof.* The commutativity follows from the normalization (cf. [Sch21, Theorem 1.1]), while wellfoundedness follows from the fact that $(\mathcal{I}(M), \leq_{\mathcal{I}})$ is countably directed. □

By the results of Section 5, and in particular Proposition 5.10, if $M$ and $N$ are two super-suitable mice, then there exists a super-suitable $P$ which is a normal iterate of both. We get that

$$\vec{\mathcal{I}}(M) \restriction P = \vec{\mathcal{I}}(N) \restriction P = \vec{\mathcal{I}}(P).$$

This leads to the following consequence.

**Proposition 6.7.** *Suppose that $M, N$ are super-suitable. Then the direct limits of the systems $\vec{\mathcal{I}}(M)$ and $\vec{\mathcal{I}}(N)$ are equal and for all $P \in \mathcal{I}(M) \cap \mathcal{I}(N)$,*

$$\pi^{\vec{\mathcal{I}}(M)}_{P,\infty} = \pi^{\vec{\mathcal{I}}(N)}_{P,\infty}.$$

□

$\mathcal{M}_\infty$, $\pi_{P,\infty}$

**Definition 6.8.** Let us denote $\mathcal{M} := \mathcal{M}_\omega \parallel \delta(\mathcal{M}_\omega)^{+\mathcal{M}_\omega}$. We define objects $\mathcal{M}_\infty$, $\pi_{P,\infty}$ (for all $P \in \mathcal{I}(\mathcal{M})$) as follows:

$$(\mathcal{M}_\infty, \pi_{P,\infty} : P \in \mathcal{I}(\mathcal{M}))$$

is the direct limit of the system $\vec{\mathcal{I}}(\mathcal{M})$. □

We can now state the characterization of $\mathsf{HOD}\|\Theta$ in the style of [SW16] that we shall use.

**Theorem 6.9.** *It holds that $\delta(\mathcal{M}_\infty) = \Theta$ and $\mathcal{M}_\infty\|\Theta = \mathsf{HOD}\|\Theta$.* □

# 7 HOD as a Normal Iterate

In order to get from a super-suitable mouse to the mouse $\mathsf{HOD}\|\Theta$, we did the direct limit construction. In that limit construct, one moves from one mouse to another via a normal tree, so we see that, after collapsing everything relevant to be countable, we get a stack of normal tree leading from a super-suitable mouse to $\mathcal{M}_\infty$. This stack again can be normalized, leading to a single normal tree on the super-suitable mouse, ending with $\mathcal{M}_\infty$. This tree will not be countable, but it will be according to the canonical extension of the canonical strategy for the super-suitable mouse. We will dissect this normal tree in order to come up with a desired local definition of $\mathsf{HOD}\|\Theta$.

**Proposition 7.1.** *Suppose that $M$ is super-suitable. Then there exists a unique normal tree $\mathcal{T}$ on $M$ according*[11] *to $\Sigma_M$ whose last model is $\mathcal{M}_\infty$. Moreover, the main branch of $\mathcal{T}$ does not drop, $\pi^{\mathcal{T}} = \pi_{M,\infty}$, and $\mathsf{lh}(\mathcal{T}) = \Theta + 1$.*

*Proof.* The existence of $\mathcal{T}$ follows from [Sch21, Corollary 1.2]. The uniqueness follows from the fact that, when comparing $M$ to $\mathcal{M}_\infty$ using $\Sigma_M$, all disagreements are on the $M$ side and the comparison tree is exactly $\mathcal{T}$. This characterization also implies that $\mathsf{lh}(\mathcal{T}) \leq \Theta + 1$, while the regularity[12] of $\Theta$ in $L(\mathbb{R})$ implies the equality. The rest of the proposition follows from [Sch21, Theorem 1.1]. □

$\mathcal{T}_M$
$b_M$

**Notation 7.2.** Suppose that $M$ is super-suitable. Then $\mathcal{T}_M$ denotes the unique normal tree on $M$ according to $\Sigma_M$ with the last model $\mathcal{M}_\infty$ and $b_M$ denotes the main branch of $\mathcal{T}_M$. □

For initial segments of $\mathcal{T}_M$, the Q-structures exist, so $L(\mathbb{R})$ knows how to pick the right branches of those trees. This means that the proper initial segments of $\mathcal{T}_M$ are in $L(\mathbb{R})$. On the other hand, once we reach $\mathcal{T}_M \restriction \Theta$, we reach the Woodin cardinal and we need the strategy to tell us which branch to pick, the strategy

[11] A priori, $\mathsf{lh}(\mathcal{T}) \leq \mathfrak{c}^+ + 1$, so $\Gamma := \Sigma_M$ needs to be extended. This is done in the standard way: for $\mathcal{U} \subseteq H(\mathfrak{c}^+)$, pick any $g$ which is $V$-generic for $\mathsf{Col}(\omega, \mathscr{P}(\mathbb{R}))$ and define $\Gamma(\mathcal{U}) := \Gamma_g(\mathcal{U})$.

[12] Cf. [KW10, Lemma 2.19]

which is not in $L(\mathbb{R})$. In fact, the final branch is not in $L(\mathbb{R})$, for otherwise we would have the embedding

$$\pi^{\mathcal{T}_M} : M \to \mathcal{M}_\infty$$

which is continuous at the Woodin and whose image would thus witness that

$$\mathsf{cof}^{L(\mathbb{R})}(\Theta) = \omega$$

(whereas $\Theta$ is regular in $L(\mathbb{R})$).

**Proposition 7.3.** *Suppose that $M$ is super-suitable. Then every proper initial segment of $\mathcal{T}_M$ belongs to $L(\mathbb{R})$.*

*Proof.*

1° We will show that extenders and branches of $\mathcal{T}_M$, except possibly the last branch, are picked definably over $L(\mathbb{R})$, which suffices for the conclusion.

2° For all $\xi$ satisfying $\xi + 1 < \mathsf{lh}(\mathcal{T}_M)$, the extender $E_\xi^{\mathcal{T}_M}$ is exactly the first extender on the $\mathcal{M}_\xi^{\mathcal{T}_M}$-sequence which is not on the $\mathsf{HOD}\|\Theta$-sequence, hence chosen definably.

3° Let $\xi$ be a limit ordinal satisfying $\xi + 1 < \mathsf{lh}(\mathcal{T}_M)$. We want to show that the branch $[0, \xi)_{\mathcal{T}_M}$ is picked definably over $L(\mathbb{R})$.

4° **Claim.** $\mathcal{Q}(\mathcal{T}_M \restriction \xi, [0, \xi)_{\mathcal{T}_M})$ exists and it is an initial segment of $\mathsf{HOD}\|\Theta$.

*Proof.*

1′ Let $\delta := \delta(\mathcal{T}_M \restriction \xi) = \bigcup_{i<\xi} \mathsf{lh}(E_i^{\mathcal{T}_M})$. Since $\xi + 1 < \mathsf{lh}(\mathcal{T}_M)$, we must have that $\delta < \Theta$.

2′ If $[0, \xi)_{\mathcal{T}_M}$ drops, then $\mathcal{Q}(\mathcal{T}_M \restriction \xi, [0, \xi)_{\mathcal{T}_M})$ exists.

3′ If $[0, \xi)_{\mathcal{T}_M}$ does not drop, then $\delta < \delta(\mathcal{M}_\xi^{\mathcal{T}_M})$, for otherwise $\mathcal{T}_M \restriction \xi$ could not have a strict normal extension whose main branch does not drop. In particular, $\mathcal{Q}(\mathcal{T}_M \restriction \xi, [0, \xi)_{\mathcal{T}_M})$ exists in this case as well.

4′ Let $Q := \mathcal{Q}(\mathcal{T}_M \restriction \xi, [0, \xi)_{\mathcal{T}_M}) \trianglelefteq \mathcal{M}_\xi^{\mathcal{T}_M}$. We want to show that $Q \trianglelefteq \mathsf{HOD}\|\Theta$.

5′ Since $\mathcal{T}_M$ is normal, $\mathsf{lh}(E_\xi^{\mathcal{T}_M}) \geq \delta$.

6′ Recall that for all $i < \xi$, $\mathsf{lh}(E_i^{\mathcal{T}_M})$ is a cardinal of $\mathcal{M}_{i+1}^{\mathcal{T}_M}$, which means that $\delta$ is a limit cardinal of $\mathcal{M}_\xi^{\mathcal{T}_M}$.

7′ Consequently, there are no extenders in $\mathcal{M}_\xi^{\mathcal{T}_M}$ indexed at $\delta$ and it must be the case that $\mathsf{lh}(E_\xi^{\mathcal{T}_M}) > \delta$.

8′ Thus, $\mathscr{P}(\delta) \cap \mathcal{M}_\xi^{\mathcal{T}_M} = \mathscr{P}(\delta) \cap \mathsf{HOD}$.

9′ Since $\delta$ is not Woodin[13] in $\mathsf{HOD}$, there exists $\zeta < \delta^{+\mathsf{HOD}}$ such that

$$\mathsf{HOD}\|(\zeta+1) \models \text{“}\delta \text{ is not Woodin”}.$$

10′ Note now that $\mathsf{HOD}\|(\zeta+1) \lhd \mathcal{M}_\xi^{\mathcal{T}_M}$, so $Q = \mathsf{HOD}\|\zeta$.

□

5° For all cofinal wellfounded branches $b$ through $\mathcal{T}_M\restriction\xi$, $(M, \mathcal{T}_M\restriction\xi, b)$ is not an anomaly. This simply follows from the fact that $M$ is sound.

6° If $b$ is a cofinal wellfounded branch through $\mathcal{T}\restriction\xi$ different from $[0,\xi)_{\mathcal{T}_M}$ which has a Q-structure, then that Q-structure is not equal to

$$\mathcal{Q}(\mathcal{T}_M\restriction\xi, [0,\xi)_{\mathcal{T}_M}).$$

In particular, $\mathcal{Q}(\mathcal{T}_M\restriction\xi, b) \not\trianglelefteq \mathsf{HOD}\|\Theta$.

7° We conclude that $[0,\xi)_{\mathcal{T}_M}$ is exactly the unique cofinal wellfounded branch through $\mathcal{T}_M\restriction\xi$ which has a Q-structure which is an initial segment of $\mathsf{HOD}\|\Theta$. This shows that $[0,\xi)_{\mathcal{T}_M}$ is picked definably over $L(\mathbb{R})$.

□

We now start working towards establishing local definability of $\mathsf{HOD}$. We will work at some fixed $\mathsf{HOD}$-cardinal $\eta$.

$\eta$ **Declaration 7.4.** We fix a $\mathsf{HOD}$-cardinal $\eta$ satisfying that $\boldsymbol{\delta}_1^2 \leq \eta < \Theta$. □

We want to show that $\mathsf{HOD}\|\eta^{+\mathsf{HOD}}$ is definable over some local structure at $\eta$. We would like to say that it is definable over $H(\eta^+)^{L(\mathbb{R})}$, but this is not good enough since Choice fails. We propose two alternatives, the first one being the structure $\mathcal{A}_\eta$ defined just below and the second one being $\mathcal{B}_\eta$ defined in Notation 9.5. The structure $\mathcal{B}_\eta$ is arguably more canonical, but establishing the definability over it is considerably harder.

**Notation 7.5.** We fix the following notation.

$\mathcal{H}_\eta$ a. $\mathcal{H}_\eta$ is the premouse $\mathsf{HOD}\|\eta$,

$\mathcal{Q}_\eta$ b. $\mathcal{Q}_\eta$ is the Q-structure $\mathcal{Q}(\mathsf{HOD}\|\Theta, \eta)$,

$\kappa_\eta$ c. $\kappa_\eta$ is the least $\kappa > o(\mathcal{Q}_\eta)$ such that $L_\kappa(\mathbb{R}, \mathcal{Q}_\eta) \models \mathsf{ZF}^-$,

$\mathcal{A}_\eta$ d. $\mathcal{A}_\eta$ is the structure $(L_{\kappa_\eta}(\mathbb{R}, \mathcal{Q}_\eta), \in, \mathbb{R}, \mathcal{Q}_\eta)$, where $\mathbb{R}$ and $\mathcal{Q}_\eta$ are treated as constants. □

**Lemma 7.6.** *The ordinal $\kappa_\eta$ is well defined and is strictly less than $\Theta$.*

[13] This is because $\delta < \Theta$, the models $\mathsf{HOD}$ and $\mathcal{M}_\infty$ agree below $\Theta$, and $\Theta$ is the only Woodin of $\mathcal{M}_\infty$.

*Proof.*

1° We work in $L(\mathbb{R})$ and we denote by $\theta$ the cardinal $\Theta^{+10}$.

2° Let us first verify that $L_\theta(\mathbb{R})$ satisfies $\mathsf{ZF}^-$. What is nontrivial is to see that this model satisfies Collection, so we concentrate on that.

3° Let $x \in L_\theta(\mathbb{R})$ and $R \subseteq L_\theta(\mathbb{R})^2$ be arbitrary. We want to find $y \in L_\theta(\mathbb{R})$ such that for all $u \in x \cap \mathsf{dom}(R)$, $R[u] \cap y \neq \emptyset$.

4° For all $u \in x$, let $g(u)$ be the least $\eta < \theta$ such that $R[u] \cap L_\eta(\mathbb{R}) \neq \emptyset$. We have defined a function $g : x \to \theta$.

5° **Claim.** $g$ is not cofinal.

*Proof.*

1′ Let us assume otherwise.

2′ There exist $\xi < \theta$ and a surjection $f : \xi \times \mathbb{R} \twoheadrightarrow x$.

3′ Let $h : \omega \twoheadrightarrow \mathbb{R}$ code a generic for $\mathsf{Col}(\omega, \mathbb{R})$ and let us work in $L(\mathbb{R})[h] = L[h]$. We have that $\theta$ remains a successor.

4′ Since Choice holds, we get that $\theta$ regular.

5′ The mapping $g \circ f \circ (\mathsf{id}_\xi \times h) : \xi \times \omega \to \theta$ is cofinal. This contradicts the regularity of $\theta$.

□

6° Let $\zeta := \sup(\mathsf{ran}(g)) < \theta$ and let $y := L_\zeta(\mathbb{R})$. It is immediate that $y$ is as required in 2°.

7° We have established that $L_\theta(\mathbb{R}) \models \mathsf{ZF}^-$. Since $\mathcal{Q}_\eta$ is coded by a set of reals and since $\theta > \Theta$, we get that

$$L_\theta(\mathbb{R}, \mathcal{Q}_\eta) = L_\theta(\mathbb{R}) \models \mathsf{ZF}^-.$$

8° Since there exists a surjection $\theta \times \mathbb{R} \twoheadrightarrow L_\theta(\mathbb{R})$, there exists $X \prec L_\theta(\mathbb{R}, \mathcal{Q}_\eta)$ satisfying that $\mathbb{R} \cup \mathcal{Q}_\eta \cup \{\mathbb{R}, \mathcal{Q}_\eta\} \subseteq X$ and satisfying that there exists a surjection $\mathbb{R} \twoheadrightarrow X$.

9° The transitive collapse of $X$ is of the form $L_\kappa(\mathbb{R}, \mathcal{Q}_\eta)$ with $\kappa < \Theta$. This suffices for the conclusion.

□

Showing that $\mathsf{HOD}\|\eta^{+\mathsf{HOD}}$ is definable over $\mathcal{A}_\eta$ amounts to showing that $\mathscr{P}(\eta) \cap \mathsf{HOD}$ is a subset of and definable over $\mathcal{A}_\eta$. We now work towards giving a characterization of this set which we will be able to approximate in $\mathcal{A}_\eta$.

**Definition 7.7.** Suppose that $M$ is super-suitable. Then we define the following.

$\gamma_M$ a. $\gamma_M$ is the least $i$ such that $\mathcal{M}_i^{\mathcal{T}_M}\|\eta = \mathsf{HOD}\|\eta$.

$\mathcal{U}_M$ b. $\mathcal{U}_M$ is the tree $\mathcal{T}_M\restriction(\gamma_M+1)$.

$\mathcal{P}_M$ c. $\mathcal{P}_M$ is the premouse $\mathcal{M}_{\gamma_M}^{\mathcal{T}_M}$. □

**Lemma 7.8.** *Suppose that $M$ is super-suitable. Then it holds that*

*a.* $\mathsf{lh}(\mathcal{U}_M) \leq \eta + 1$,

*b.* $\mathcal{U}_M \in L(\mathbb{R})$,

*c.* $|\mathcal{P}_M|^{L(\mathbb{R})} \leq |\eta|^{L(\mathbb{R})} < \Theta$.

*Proof.* Part a follows from the fact that $\mathcal{U}_M$ the tree arising from comparing $(M, \Sigma_M)$ to $\mathsf{HOD}\|\eta$, part b follows from Proposition 7.3, while part c follows from the previous two. □

**Definition 7.9.** Suppose that $M$ is super-suitable. Then

$\eta$-exact a. $M$ is $\eta$-exact iff $\eta \in \mathsf{ran}(\pi_{M,\infty})$,

$\eta_M$ b. if $M$ is $\eta$-exact, then $\eta_M$ denotes the preimage of $\eta$ by $\pi_{M,\infty}$. □

The characterization of $\mathscr{P}(\eta) \cap \mathsf{HOD}$ that we announced is given below. Namely, this powerset is the same as the powerset $\mathscr{P}(\eta) \cap \mathcal{P}_M$ where $M$ is any $\eta$-exact super-suitable mouse.

**Proposition 7.10.** *Suppose that $M$ is super-suitable and $\eta$-exact. Then it holds that*

*a.* $\gamma_M \in b_M$,

*b.* $\mathsf{crit}(\pi^{\mathcal{T}_M}_{\gamma_M,\mathsf{lh}(\mathcal{T}_M)-1}) > \eta$,

*c.* $\eta \in \mathsf{ran}(\pi^{\mathcal{T}_M}_{0,\gamma_M})$,

*d.* $\mathscr{P}(\eta) \cap \mathcal{P}_M = \mathscr{P}(\eta) \cap \mathsf{HOD}$.

*Proof.*

1° Let us assume towards contradiction that a fails. Then there exist $\xi, \zeta \in b_M$ such that $\gamma_M \in (\xi, \zeta)$ and $\xi$ is a $\mathcal{T}_M$-predecessor of $\zeta$.

2° Note that $\zeta = \bar{\zeta} + 1$ for some $\bar{\zeta} \geq \gamma_M$ and

$$\mathcal{M}_\zeta^{\mathcal{T}_M} = \mathsf{Ult}(\mathcal{M}_\xi^{\mathcal{T}_M}, E_{\bar{\zeta}}^{\mathcal{T}_M}).$$

3° We have that
$$\mathsf{crit}(E^{\mathcal{T}_M}_{\bar{\zeta}}) < \mathsf{lh}(E^{\mathcal{T}_M}_{\xi}) \leq \eta \leq \mathsf{lh}(E^{\mathcal{T}_M}_{\bar{\zeta}}),$$
where the first inequality follows from the fact that $\mathcal{T}_M$ is normal, the second one follows by definition of $\gamma_M$, and the third one follows from the fact $\mathcal{M}^{\mathcal{T}_M}_{\bar{\zeta}}\|\eta = \mathcal{M}^{\mathcal{T}_M}_{\gamma_M}\|\eta = \mathcal{M}_\infty\|\eta$.

4° The previous two points then imply that $\eta \notin \mathsf{ran}(\pi^{\mathcal{T}_M}_{\xi,\zeta})$, which contradicts the fact that $\eta \in \mathsf{ran}(\pi^{\mathcal{T}_M})$.

5° Let us now verify part b. Let $E$ be the extender used at $\gamma_M$ along $b_M$. We have that $\mathsf{lh}(E) \geq \eta$.

6° The case $\mathsf{crit}(E) \leq \eta$ would imply $\eta \notin \mathsf{ran}(\pi^{\mathcal{T}_M}_{\gamma_M,\mathsf{lh}(\mathcal{T}_M)-1})$, which is not possible since $\eta \in \mathsf{ran}(\pi^{\mathcal{T}_M})$.

7° Thus, $\mathsf{crit}(\pi^{\mathcal{T}^M}_{\gamma_M,\mathsf{lh}(\mathcal{T}_M)-1}) = \mathsf{crit}(E) > \eta$, which establishes part b.

8° Parts c and d now easily follow.

□

# 8 Local Definition of HOD I

We will now establish the definability of $\mathsf{HOD}\|\eta^{+\mathsf{HOD}}$ over $\mathcal{A}_\eta$. This is done by approximating the characterization of Proposition 7.10 inside $\mathcal{A}_\eta$. The first step is to expand the notions of the previous section to a larger domain of premice, in order to make them definable inside the structure that we are interested in.

**Definition 8.1.** Suppose that $M$ is a countable premouse. Then we define the following.

$\mathcal{U}_M$

a. The tree $\mathcal{U}_M$ on $M$ is the unique normal tree $\mathcal{U}$ on $M$ of the maximal possible length satisfying that

   i. for all $i$ such that $i+1 < \mathsf{lh}(\mathcal{U})$, the least disagreement between $\mathcal{M}^{\mathcal{U}}_i$ and $\mathcal{H}_\eta$ is on the $\mathcal{M}^{\mathcal{U}}_i$-side and $E^{\mathcal{U}}_i$ is that disagreement,

   ii. for all limit $i < \mathsf{lh}(\mathcal{U})$, we have that the Q-structure $\mathcal{Q}(\mathcal{U}\restriction i, [0,i)_{\mathcal{U}})$ exists and is an initial segment of $\mathcal{Q}_\eta$.

$\mathcal{U}_M$ converges

b. We say that $\mathcal{U}_M$ converges iff there exists $N$ such that

   i. $N$ is the last model of $\mathcal{U}_M$,

   ii. the $M$-to-$N$ branch of $\mathcal{U}_M$ does not drop,

   iii. either $N \trianglelefteq \mathcal{H}_\eta$ or $\mathcal{H}_\eta \trianglelefteq N$.

$\mathcal{P}_M$, $\sigma_M$

c. If $\mathcal{U}_M$ converges, we denote by $\mathcal{P}_M$ the last model of $\mathcal{U}_M$ and we denote by $\sigma_M$ the mapping
$$\pi^{\mathcal{U}_M} : M \to \mathcal{P}_M.$$

$\eta$-exact

d. The premouse $M$ is $\eta$-exact iff $\mathcal{U}_M$ converges, $\mathcal{P}_M \rhd \mathcal{H}_\eta$, and $\eta \in \mathsf{ran}(\sigma_M)$.

$\eta_M$, $\gamma_M$

e. If $M$ is $\eta$-exact, we denote $\eta_M := \sigma_M^{-1}(\eta)$ and $\gamma_M := \mathsf{lh}(\mathcal{U}_M) - 1$. □

The tree $\mathcal{U}_M$ is built by comparing $M$ to $\mathcal{H}_\eta$, using the strategy given by Q-structures at limit stages. We want that $\mathcal{H}_\eta$ does not move in this comparison, so we set up the definition in such a way that the construction breaks down if $\mathcal{H}_\eta$ has to move (or if the strategy is not strong enough). The notion of convergence is simply ensuring that breakdown does not happen. If $M$ is super-suitable and $\eta$-exact in the sense of Definition 7.9, then it is $\eta$-exact in this new sense (cf. Proposition 7.10) and the objects $\eta_M$, $\mathcal{U}_M$, $\gamma_M$, and $\mathcal{P}_M$ correspond to those introduced in Definition 7.9 and Definition 7.7. In that case, the embedding $\sigma_M$ is exactly the embedding

$$\pi^{\mathcal{T}_M}_{0,\gamma_M} = \pi^{\mathcal{U}_M} : M \to \mathcal{P}_M.$$

Recall that we want to locally approximate the notion of an $\eta$-exact super-suitable mouse. We do so as follows.

$\eta$-good

**Definition 8.2.** Suppose that $M$ is a countable premouse. Then $M$ is $\eta$-good iff

a. $M$ is suitable,

b. $M$ is short tree iterable,

c. all normal pseudo-iterates $N$ of $M$ are $\eta$-exact and satisfy that

$$\mathscr{P}(\eta) \cap \mathcal{P}_N = \mathscr{P}(\eta) \cap \mathcal{P}_M.$$

□

**Proposition 8.3.** *Suppose that $M$ is super-suitable and $\eta$-exact. Then $M$ is $\eta$-good.*

*Proof.*

1° We already know that super-suitable premice are suitable and short tree iterable, so it remains to verify condition c of Definition 8.2.

2° Let $N$ be a normal pseudo-iterate of $M$. We want to show that $N$ is $\eta$-exact and that $\mathscr{P}(\eta) \cap \mathcal{P}_N = \mathscr{P}(\eta) \cap \mathcal{P}_M$.

3° Let $\mathcal{T}$ be the tree satisfying $M \longrightarrow^{\mathrm{st}}_{\mathcal{T}} N$. By Lemma 5.7, we have that $\mathcal{T}$ is according to $\Sigma_M$ and $N$ is either the last model of $\mathcal{T}$ or $N = \mathcal{M}^{\mathcal{T}}_{\Sigma_M(\mathcal{T})}$. In addition, the main branch of $\mathcal{T}$ or the branch $\Sigma_M(\mathcal{T})$ does not drop.

4° This implies that $N$ is super-suitable. Since $\eta \in \mathsf{ran}(\pi_{M,\infty})$ and $\pi_{M,\infty} = \pi_{N,\infty} \circ \pi_{M,N}$, we get that $\eta \in \mathsf{ran}(\pi_{N,\infty})$, which is another way of saying that $N$ is $\eta$-exact.

5° By Proposition 7.10, we have that

$$\mathscr{P}(\eta) \cap \mathcal{P}_N = \mathscr{P}(\eta) \cap \mathsf{HOD} = \mathscr{P}(\eta) \cap \mathcal{P}_M,$$

which concludes the verification.

□

We are finally ready to state our local characterization of $\mathscr{P}(\eta) \cap \mathsf{HOD}$.

**Proposition 8.4.** *Suppose that $A \subseteq \eta$. Then $A \in \mathsf{HOD}$ if and only if there exists an $\eta$-good $M$ such that $A \in \mathcal{P}_M$.*

*Proof.*

1° For the implication ($\Rightarrow$), we can pick a super-suitable $M$ satisfying $\eta \in \mathsf{ran}(\pi_{M,\infty})$ and reach the conclusion by referencing Propositions 8.3, 5.4, and 7.10. We now focus on the implication ($\Leftarrow$).

2° Let $M$ be $\eta$-good such that $A \in \mathcal{P}_M$. We want to show that $A \in \mathsf{HOD}$.

3° There exists a super-suitable $N$ such that $\eta \in \mathsf{ran}(\pi_{N,\infty})$.

4° By Proposition 5.10, there exists a suitable $P$ which is a normal pseudo-iterate of both $M$ and $N$.

5° Since $M$ is $\eta$-good, we have that

$$\mathscr{P}(\eta) \cap \mathcal{P}_M = \mathscr{P}(\eta) \cap \mathcal{P}_P.$$

6° Since $N$ is super-suitable and $\eta \in \mathsf{ran}(\pi_{N,\infty})$, we have that $P$ is super-suitable as well and $\eta \in \mathsf{ran}(\pi_{P,\infty})$.

7° By Proposition 7.10, we have that

$$\mathscr{P}(\eta) \cap \mathcal{P}_P = \mathscr{P}(\eta) \cap \mathsf{HOD}.$$

8° By 5° and 7°, we have that $A \in \mathsf{HOD}$.

□

To see that this characterization definable in $\mathcal{A}_\eta$, the main ingredient is to observe that the function $\mathsf{Lp}$ is definable over $\mathcal{A}_\eta$. Recall that $\mathsf{Lp}$ is defined on countable transitive sets and returns such sets as values. Since these sets are coded by reals, the function $\mathsf{Lp}$ is indeed a (partial) function in the structure $\mathcal{A}_\eta$. In other words, it really makes sense to ask whether now this function is definable over that structure.

**Lemma 8.5.** *The function $\mathsf{Lp}$ is lightface definable over $\mathcal{A}_\eta$.*

*Proof.* Definition 2.13 implies that $\Sigma_1$-formulas with parameters in $\mathbb{R} \cup \{\mathbb{R}\}$ are absolute between $\mathcal{A}_\eta$ and $L(\mathbb{R})$, from which it easily follows that the definition of $\mathsf{Lp}$ is absolute between $\mathcal{A}_\eta$ and $L(\mathbb{R})$. □

Once we have $\mathsf{Lp}$ in $\mathcal{A}_\eta$, we can run the standard definitions of the suitability, short tree iterability, and $\eta$-goodness inside this structure and obtain their definability as well.

**Corollary 8.6.** *The following sets are contained in $\mathcal{A}_\eta$ and definable over it:*

*a.* $\{M : M$ *is suitable*$\}$

*b.* $\{M : M$ *is short tree iterable*$\}$,

*c.* $\{M : M$ *is $\eta$-good*$\}$. □

Putting this corollary together with Proposition 8.4, we get the conclusion.

**Corollary 8.7.** *The premouse $\mathsf{HOD}\|\eta^{+\mathsf{HOD}}$ is contained in $L_{\kappa_\eta}(\mathbb{R}, \mathcal{Q}_\eta)$ and definable over it from the parameter $\mathcal{Q}_\eta$.* □

As an immediate consequence, we get an upper bound on $\eta^{+\mathsf{HOD}}$.

**Corollary 8.8.** $\eta^{+\mathsf{HOD}} < \kappa_\eta$. □

# 9 Local Definition of HOD II

In the previous section, we established local definability of $\mathsf{HOD}\|\eta^{+\mathsf{HOD}}$ from $\mathcal{Q}_\eta$. We now want to obtain a more canonical structure to define $\mathsf{HOD}\|\eta^{+\mathsf{HOD}}$ over it, a structure that does not refer to $\mathcal{Q}_\eta$. Hence, the work done in this section has for its goal locally defining $\mathcal{Q}_\eta$ from some other objects (cf. the structure $\mathcal{B}_\eta$ of Notation 9.5). Most of this work will be done internally in $L(\mathbb{R})$, so we position ourselves accordingly.

**Declaration 9.1.** In this section, we work in $L(\mathbb{R})$, unless otherwise stated. □

We remind the reader that a supercompactness measure on $[X]^\omega$ is just a normal fine ultrafilter on that set. Since $\mathsf{AD}$ holds (in $L(\mathbb{R})$), the sets that can be wellordered in the ordertype $< \Theta$ come with unique supercompactness measures.

**Theorem 9.2.** *For all $X$ satisfying $|X| < \Theta$, there exists a unique supercompactness measure on $[X]^\omega$.*

*Proof.* See the first theorem on page 32 of [Nee07]. □

$\mu_X$ **Definition 9.3.** Suppose that $|X| < \Theta$. Then $\mu_X$ denotes the unique supercompactness measure on $[X]^\omega$. □

Since these measures are unique, they must project to each other.

**Lemma 9.4.** *Suppose that*

1. $|X|, |Y| < \Theta$,
2. $X \subseteq Y$,
3. $D \in \mu_Y$.

*Then* $\{\sigma \cap X : \sigma \in D\} \in \mu_X$.

*Proof.* We have that

$$\nu := \{\{\sigma \cap X : \sigma \in E\} : E \in \mu_Y\}$$

is a supercompactness measure on $[X]^\omega$. By the uniqueness of supercompactness measures (Theorem 9.2), we get that $\nu = \mu_X$. This now means that

$$\{\sigma \cap X : \sigma \in D\} \in \nu = \mu_X.$$

□

We are now able to define the new structure over which we will define $\mathsf{HOD}\|\eta^{+\mathsf{HOD}}$. This structure has the measure $\mu_\eta$ as one of its predicates.

**Notation 9.5.** We use the following notation.

$\chi_\eta$

1. $\chi_\eta$ denotes the second of ordinals $\chi > \eta$ satisfying that

$$L_\chi(\mathcal{H}_\eta^\omega)[\mu_\eta] \models \mathsf{ZF}^- + \text{“}\mathscr{P}([\eta]^\omega) \text{ exists”}.$$

$\mathcal{B}_\eta$

2. $\mathcal{B}_\eta$ denotes the structure

$$(L_{\chi_\eta}(\mathcal{H}_\eta^\omega)[\mu_\eta], \in, \mu_\eta, \mathcal{H}_\eta),$$

where (the restriction of) $\in$ is a binary predicate, (the restriction of) $\mu_\eta$ is a unary predicate, and $\mathcal{H}_\eta$ is a constant. □

**Lemma 9.6.** *The ordinal* $\chi_\eta$ *is well defined and strictly less than* $\Theta$.

*Proof.* This is verified similarly to Lemma 7.6. □

The goal is to define $\mathcal{Q}_\eta$ over $\mathcal{B}_\eta$. Let $M$ be an arbitrary $\eta$-exact super-suitable mouse. This mouse belongs to $\mathcal{B}_\eta$ and it is not hard to see that the tree $\mathcal{U}_M \restriction \gamma_M$ also belongs to $\mathcal{B}_\eta$. What will require some work is to see that the branch $b_M$ of this tree belongs to $\mathcal{B}_\eta$ as well. Once we know this, we will get that $\mathcal{Q}_\eta$ is definable over $\mathcal{B}_\eta$ as the Q-structure corresponding the branch $b_M$ of the tree $\mathcal{U}_M \restriction \gamma_M$. Our approach to establishing that $b_M$ belongs to $\mathcal{B}_\eta$ is to represent it as a certain ultraproduct and then argue that that ultraproduct can be computed inside $\mathcal{B}_\eta$. Now, in the absence of Choice, we do not have full Łoś's Theorem, but we will be able to salvage a part of it which will suffice for our purposes.

**Notation 9.7.** Suppose that

1. $W \ni \eta$ is a rudimentarily closed transitive set,
2. $\mathcal{W} := (W, \in, \dots)$ is a first-order structure in a countable language with a lightface definable wellordering,
3. $\sigma \subseteq W$.

Then we denote by

$\mathsf{Hull}^{\mathcal{W}}(\sigma)$

a. $\mathsf{Hull}^{\mathcal{W}}(\sigma)$ the Skolem hull of $\sigma$ inside $\mathcal{W}$, understood both as a set and as a substructure,

$\mathsf{cHull}^{\mathcal{W}}(\sigma)$

b. $\mathsf{cHull}^{\mathcal{W}}(\sigma)$ the transitive collapse of $\mathsf{Hull}^{\mathcal{W}}(\sigma)$,

$\pi_\sigma^{\mathcal{W}}$

c. $\pi_\sigma^{\mathcal{W}}$ the anticollapse associated to $\mathsf{Hull}^{\mathcal{W}}(\sigma)$, where the superscript is omitted whenever it is clear from the context. □

Theses hulls are well defined because we assumed that $\mathcal{W}$ defines a wellordering. We can now get the following form of Łoś's Theorem.

**Lemma 9.8.** *Suppose that*

1. *$W \ni \eta$ is a rudimentarily closed transitive set,*
2. *$\mathcal{W} := (W, \in, \dots)$ is a first-order structure in a countable language with a lightface definable wellordering,*
3. *$\mathcal{W}^* := \prod_{\sigma \in [\eta]^\omega} \mathsf{cHull}^{\mathcal{W}}(\sigma)/\mu_\eta$,*
4. *$\phi(x)$ is a formula in the language of the structure $\mathcal{W}$,*
5. *$f \in \prod_{\sigma \in [\eta]^\omega} \mathsf{cHull}^{\mathcal{W}}(\sigma)$.*

*Then*

$$\mathcal{W}^* \models \phi[[f]] \iff \{\sigma \in [\eta]^\omega : \mathsf{cHull}^{\mathcal{W}}(\sigma) \models \phi[f(\sigma)]\} \in \mu_\eta.$$

*Proof.*

1° This is shown by induction on the complexity of $\phi(\overline{x})$, as in the usual proof of Łoś's Theorem. We verify here the nontrivial step.

2° Suppose that

$$\{\sigma \in [\eta]^\omega : \mathsf{cHull}^{\mathcal{W}}(\sigma) \models \exists y, \phi(f(\sigma), y)\} \in \mu_\eta.$$

We want to show that $\mathcal{W}^* \models \exists y, \phi([f], y)$.

3° Let

$$C := \{\sigma \in [\eta]^\omega : \mathsf{cHull}^{\mathcal{W}}(\sigma) \models \exists y, \phi(f(\sigma), y)\}.$$

There exists a mapping $C \ni \sigma \mapsto (t_\sigma, \alpha_\sigma)$ such that for all $\sigma \in C$, $t_\sigma$ is a Skolem term, $\alpha_\sigma \in \sigma$, and

$$\mathsf{cHull}^{\mathcal{W}}(\sigma) \models \phi(f(\sigma), t_\sigma(\alpha_\sigma)).$$

4° Since $\mu_\eta$ is countably closed, we may assume w.l.o.g. that there exists a Skolem term $t$ such that for all $\sigma \in C$, $t_\sigma = t$.

5° By normality of $\mu_\eta$, we may assume w.l.o.g. that there exists $\alpha < \eta$ such that for all $\sigma \in C$, $\alpha_\sigma = \alpha$. In particular, for all $\sigma \in C$,

$$\mathsf{cHull}^{\mathcal{W}}(\sigma) \models \phi[f(\sigma), t(\alpha)].$$

6° Let $g \in \prod_{\sigma \in [\eta]^\omega} \mathsf{cHull}^{\mathcal{W}}(\sigma)$ be defined by

$$\forall \sigma \in [\eta]^\omega,\ g(\sigma) := \pi_\sigma^{-1}(t(\alpha)).$$

By the inductive hypothesis, we have that $\mathcal{W}^* \models \phi[[f], [g]]$.

□

We can use the above lemma to show that in fact $\mathcal{W}^*$ embeds canonically into $\mathcal{W}$. This in particular shows that the ultraproduct $\mathcal{W}^*$ is wellfounded.

**Lemma 9.9.** *Suppose that*

1. *$W \ni \eta$ is a rudimentarily closed transitive set,*
2. *$\mathcal{W} := (W, \in, \dots)$ is a first-order structure in a countable language with a lightface definable wellordering,*
3. *$\mathcal{W}^* := \prod_{\sigma \in [\eta]^\omega} \mathsf{cHull}^{\mathcal{W}}(\sigma)/\mu_\eta$.*

*Then $\mathcal{W}^*$ is wellfounded and there exists a unique embedding*

$$j : \mathcal{W}^* \xrightarrow[\Sigma_\omega]{} \mathcal{W}$$

*satisfying that for all $\alpha < \eta$,*

$$j\left([(\mathsf{otp}(\alpha \cap \sigma) : \sigma \in [\eta]^\omega)]\right) = \alpha.$$

*Moreover,* $\mathsf{ran}(j) = \mathsf{Hull}^{\mathcal{W}}(\eta)$.

*Proof.*

1° The wellfoundedness follows from the existence of $j$.

2° Let $f \in \prod_{\sigma \in [\eta]^\omega} \mathsf{cHull}^{\mathcal{W}}(\sigma)$ be arbitrary. By applying the countable completeness and the normality of $\mu_\eta$ as in the proof of Lemma 9.8, we see that there exist $C \in \mu_\eta$, a Skolem term $t$, and an ordinal $\alpha < \eta$ such that for all $\sigma \in C$, $f(\sigma) = \pi_\sigma^{-1}(t(\alpha))$. We denote by $(t_f, \alpha_f)$ the minimal such $(t, \alpha)$.

3° For all $f \in \prod_{\sigma \in [\eta]^\omega} \mathsf{cHull}^{\mathcal{W}}(\sigma)$, we define $j([f]) := t_f(\alpha_f)$. By Lemma 9.8, it follows that $j$ is well defined and

$$j\left([(\mathsf{otp}(\alpha \cap \sigma) : \sigma \in [\eta]^\omega)]\right) = \alpha.$$

4° The "moreover" part is now easily verified.

□

We are now ready to do the main work, which involves showing that $\mathcal{U}_M$ belongs to $\mathcal{B}_\eta$, where $M$ is $\eta$-exact and super-suitable. This breaks down into first observing that $\mathcal{U}_M \restriction \gamma_M$ belongs to $\mathcal{B}_\eta$ and then proving that $b_M = [0, \gamma_M)_{\mathcal{U}_M}$ also belongs to $\mathcal{B}_\eta$.

**Lemma 9.10.** *Suppose that $M$ is an $\eta$-exact super-suitable premouse. Then $\mathcal{U}_M \restriction \gamma_M$ belongs to $\mathcal{B}_\eta$.*

*Proof.* The tree $\mathcal{U}_M \restriction \gamma_M$ is obtained by comparing $M$ to $\mathcal{H}_\eta$, where at limit stages, we pick the branches whose Q-structures are initial segments of $\mathcal{H}_\eta$. In this comparison, $\mathcal{H}_\eta$ does not move. This description relativizes correctly to $\mathcal{B}_\eta$, yielding the conclusion. □

The previous proof does not go through when we need to pick the final branch of $\mathcal{U}_M$. The reason for this is that this branch has for its Q-structure the structure $\mathcal{Q}_\eta$, which we are actually trying to obtain. We thus need a more complicated argument, which we give below.

**Proposition 9.11.** *Suppose that $M$ is an $\eta$-exact super-suitable premouse. Then the branch $b_M$ belongs to $\mathcal{B}_\eta$.*

*Proof.*

1° Let $x_M$ be a real coding $M$ and let $\theta > \eta$ be the least ordinal satisfying that $W := L_\theta(x_M, \mathcal{H}_\eta, \mathcal{U}_M \restriction \gamma_M) \models \mathsf{ZFC}^-$. We have that $\theta < \chi_\eta$ and that $W$ belongs to $\mathcal{B}_\eta$.

2° We use the following notation:

1. $U := \mathcal{U}_M \restriction \gamma_M$,
2. $\mathcal{W} := (W, \in, x_M, \mathcal{H}_\eta, U)$,
3. for all $\sigma \in [\eta]^\omega$, $U_\sigma := (\pi_\sigma^{\mathcal{W}})^{-1}(U)$.

3° Let $B$ consist of all pairs $(\sigma, B_\sigma)$ where

a. $\sigma \in [\eta]^\omega$,
b. $U_\sigma$ is a short tree on $M$,
c. $B_\sigma = \Sigma_M^{\mathrm{st}}(U_\sigma)$.

We have that $B$ is a function and belongs to $\mathcal{B}_\eta$ (cf. Proposition 2.14).

4° Let

1. $\lambda \in (\theta, \Theta)$ be such that $W^* := L_\lambda(\mathcal{W}, \mathcal{Q}_\eta) \models \mathsf{ZFC}^-$,
2. $\mathcal{W}^* := (W^*, \in, \mathcal{W}, \mathcal{Q}_\eta)$,

3. for all $\sigma \in [\eta]^\omega$, $\mathsf{Hull}^{\mathcal{W}}_+(\sigma) := \mathsf{Hull}^{\mathcal{W}^*}(\sigma) \cap W$,
4. for all $\sigma \in [\eta]^\omega$, $\mathsf{cHull}^{\mathcal{W}}_+(\sigma)$ be the transitive collapse of $\mathsf{Hull}^{\mathcal{W}}_+(\sigma)$,
5. for all $\sigma \in [\eta]^\omega$, $k_\sigma : \mathsf{cHull}^{\mathcal{W}}(\sigma) \underset{\Sigma_\omega}{\longrightarrow} \mathsf{cHull}^{\mathcal{W}}_+(\sigma)$ be such that the diagram

$$\begin{array}{ccc} \mathsf{Hull}^{\mathcal{W}}(\sigma) & \xrightarrow{\subseteq} & \mathsf{Hull}^{\mathcal{W}}_+(\sigma) \\ {\scriptstyle \pi^{\mathcal{W}}_\sigma}\uparrow & & {\scriptstyle \pi^{\mathcal{W}^*}_\sigma}\uparrow \\ \mathsf{cHull}^{\mathcal{W}}(\sigma) & \xrightarrow{k_\sigma} & \mathsf{cHull}^{\mathcal{W}}_+(\sigma) \end{array}$$

commutes.

Note that $b_M \in W^*$ and it is definable over $\mathcal{W}^*$ (with no additional parameters) as the unique cofinal wellfounded branch through $U$ having $\mathcal{Q}_\eta$ for its Q-structure.

5° **Claim.** There exists $S \in \mu_\eta$ such that for all $\sigma \in S$,

a. $\sigma \in \mathsf{dom}(B)$,
b. $\mathsf{crit}(k_\sigma) > (\pi^{\mathcal{W}}_\sigma)^{-1}(\eta)$,
c. $B_\sigma \in \mathsf{cHull}^{\mathcal{W}^*}(\sigma)$.

*Proof.*

1′ Let $C^* := \{X^* \in [W^*]^\omega : X^* \prec \mathcal{W}^*\} \in \mu_{W^*}$. By Proposition 9.4, we have that
$$S := \{X^* \cap \eta : X^* \in C^*\} \in \mu_\eta.$$

2′ We want to show that $S$ is as required. To that end, fix an arbitrary $\sigma \in S$.

3′ Let $X^* \in C^*$ be such that $\sigma = X^* \cap \eta$. We have that
$$\mathsf{Hull}^{\mathcal{W}}(\sigma) \cap \eta = \mathsf{Hull}^{\mathcal{W}^*}(\sigma) \cap \eta = \sigma,$$
and in particular, $k_\sigma \restriction (\pi^{\mathcal{W}}_\sigma)^{-1}(\eta)$ is the identity.

4′ In addition, $\eta$ is definable over both $\mathcal{W}$ and $\mathcal{W}^*$. This suffices for b.

5′ Note that $U \in \mathsf{Hull}^{\mathcal{W}^*}(\sigma)$ and that $U$ is coded by a subset of $\eta$. Since $\mathsf{crit}(k_\sigma) > (\pi^{\mathcal{W}}_\sigma)^{-1}(\eta)$, we conclude that $(\pi^{\mathcal{W}^*}_\sigma)^{-1}(U) = U_\sigma$.

6′ We have that $\mathcal{Q}_\eta \in \mathsf{Hull}^{\mathcal{W}^*}(\sigma)$, so we can denote $Q := (\pi^{\mathcal{W}^*}_\sigma)^{-1}(\mathcal{Q}_\eta)$. By elementarity, $\mathsf{cHull}^{\mathcal{W}^*}(\sigma) \models$"$Q$ is the Q-structure of the tree $U_\sigma$".

7′ Being the case that $Q$ embeds into $\mathcal{Q}_\eta$, it is $(\omega_1 + 1)$-iterable (in $V$ and in $L(\mathbb{R})$; cf. Proposition 2.14). This means that $Q$ is the true Q-structure of the tree $U_\sigma$.

8′ In particular, $U_\sigma$ is short and $\sigma \in \mathsf{dom}(B)$. This establishes a.

9′ Now, $B_\sigma$ is the unique cofinal wellfounded branch through $U_\sigma$ which has $Q$ as its Q-structure, which means that

$$B_\sigma = (\pi_\sigma^{\mathcal{W}^*})^{-1}([0, \gamma_M)_{\mathcal{U}_M}) \in \mathsf{cHull}^{\mathcal{W}^*}(\sigma),$$

as required in c.

□

6° Let

$$b := \{\alpha < \eta : \forall^*_{\mu_\eta} \sigma,\ \mathsf{otp}(\alpha \cap \sigma) \in B_\sigma\} \subseteq \eta.$$

By 3°, $b$ belongs to $\mathcal{B}_\eta$.

7° **Claim.** $b = b_M$

*Proof.*

1′ Let

$$j : \prod_{\sigma \in [\eta]^\omega} \mathsf{cHull}^{\mathcal{W}^*}(\sigma)/\mu_\eta \xrightarrow[\Sigma_\omega]{} \mathcal{W}^*$$

be the unique embedding satisfying $j \restriction \eta = \mathsf{id}_\eta$ (cf. Lemma 9.9). Since $\eta$ is definable over $\mathcal{W}^*$, we also have that $j(\eta) = \eta$.

2′ By 4° and 1′, we have that

a. $j^{-1}(\mathcal{Q}_\eta) = \mathcal{Q}_\eta$,

b. $b_M$ is definable over $\mathcal{W}^*$, $b_M \in \mathsf{ran}(j)$, and $j^{-1}(b_M) = b_M$,

c. $\prod_{\sigma \in [\eta]^\omega} \mathsf{cHull}^{\mathcal{W}^*}(\sigma)/\mu_\eta \models$"$b_M$ is the unique cofinal wellfounded branch through $U$ having $\mathcal{Q}_\eta$ for its Q-structure".

3′ By Lemma 9.8, if we fix some $f$ satisfying $[f] = b_M$, we get that for $\mu_\eta$-almost all $\sigma$, it holds that $\mathsf{cHull}^{\mathcal{W}^*}(\sigma) \models$"$f(\sigma)$ is the unique cofinal wellfounded branch through $(\pi_\sigma^{\mathcal{W}^*})^{-1}(U)$ having $(\pi_\sigma^{\mathcal{W}^*})^{-1}(\mathcal{Q}_\eta)$ for its Q-structure".

4′ By Claim 5° and the step 3′, there exists $S \in \mu_\eta$ such that for all $\sigma \in S$,

a. $\sigma \in \mathsf{dom}(B)$,

b. $\mathsf{crit}(k_\sigma) > (\pi_\sigma^{\mathcal{W}})^{-1}(\eta)$,

c. $B_\sigma \in \mathsf{cHull}^{\mathcal{W}^*}(\sigma)$,

d. $f(\sigma)$ is the unique cofinal wellfounded branch through

$$(\pi_\sigma^{\mathcal{W}^*})^{-1}(U)$$

having $(\pi_\sigma^{\mathcal{W}^*})^{-1}(\mathcal{Q}_\eta)$ for its Q-structure.

5′ Thus, for all $\sigma \in S$, we have that

a. $U_\sigma$ is short,

b. $(\pi^{\mathcal{W}^*}_\sigma)^{-1}(U) = U_\sigma$,

c. $(\pi^{\mathcal{W}^*}_\sigma)^{-1}(\mathcal{Q}_\eta)$ is $(\omega_1+1)$-iterable (for this, we reason as in the step 5°7′).

6′ Putting 4′ and 5′ together, we get that for all $\sigma \in S$,

a. $U_\sigma$ is short,

b. $(\pi^{\mathcal{W}^*}_\sigma)^{-1}(\mathcal{Q}_\eta)$ is the Q-structure of the tree $U_\sigma$,

c. $f(\sigma)$ is the unique cofinal wellfounded branch through $U_\sigma$ having $(\pi^{\mathcal{W}^*}_\sigma)^{-1}(\mathcal{Q}_\eta)$ for its Q-structure.

Hence, for all $\sigma \in S$, $f(\sigma) = B_\sigma$.

7′ Now, we can compute as follows:

$$\begin{aligned} b_M &= \{\alpha < \eta : \alpha \in [f]\} && (1)\\ &= \{\alpha < \eta : \forall^*_{\mu_\eta}\sigma,\ \mathsf{otp}(\alpha \cap \sigma) \in f(\sigma)\} && (2)\\ &= \{\alpha < \eta : \forall^*_{\mu_\eta}\sigma,\ \mathsf{otp}(\alpha \cap \sigma) \in B_\sigma\} && (3)\\ &= b. && (4) \end{aligned}$$

□

8° By the point 6° and Claim 7°, we conclude that $b_M$ belongs to $\mathcal{B}_\eta$.

□

Thanks to the previous proposition, we can obtain $\mathcal{Q}_\eta$ as a definable element of $\mathcal{B}_\eta$, which then implies that the structure $\mathcal{A}_\eta$ is a definable element of the structure $\mathcal{B}_\eta$.

**Corollary 9.12.** *$\mathcal{A}_\eta$ belongs to and is definable over $\mathcal{B}_\eta$ with no additional parameters.*

*Proof.* Let $M$ be an arbitrary $\eta$-exact super-suitable premouse. Since $\mathcal{U}_M$ belongs to $\mathcal{B}_\eta$, so does $\mathcal{Q}_\eta = \mathcal{Q}(\mathcal{U}_M \restriction \gamma_M, b_M)$. It remains to show that $\mathcal{Q}_\eta$ is lightface definable over $\mathcal{B}_\eta$. To that end, notice that $\mathcal{Q}_\eta$ is the Q-structure above $\mathcal{H}_\eta$ (in the sense of Definition 2.8). Since $\mathcal{B}_\eta$ has the strategies for the hulls of this structure (cf. Proposition 2.14), this characterization relativizes correctly to $\mathcal{B}_\eta$. □

We now get the main local definability result.

**Corollary 9.13.** *The premouse $\mathsf{HOD}\|\eta^{+\mathsf{HOD}}$ is lightface definable over the structure*[14] $L_{\chi_\eta}(\mathcal{H}^\omega_\eta)[\mu_\eta]$. □

We of course get one more upper bound for $\eta^{+\mathsf{HOD}}$.

**Corollary 9.14.** $\eta^{+\mathsf{HOD}} < \chi_\eta$. □

[14] The natural structure that we have in mind here has $\mathcal{H}^\omega_\eta$ as a constant and restrictions of $\in$ and $\mu_\eta$ as unary predicates.

## References


[KW10] Peter Koellner and W. Hugh Woodin. Large cardinals from determinacy. In *Handbook of set theory. In 3 volumes*, pages 1951–2119. Dordrecht: Springer, 2010.

[Lar22] Paul B. Larson. *Extensions of the Axiom of Determinacy.* Book manuscript from December 20, 2022.

[MS21] Sandra Müller and Grigor Sargsyan. HOD in inner models with Woodin cardinals. *J. Symb. Log.*, 86(3):871–896, 2021.

[Nee07] Itay Neeman. Inner models and ultrafilters in $L(\mathbb{R})$. *Bull. Symb. Log.*, 13(1):31–53, 2007.

[Nee10] Itay Neeman. Determinacy in $L(\mathbb{R})$. In *Handbook of set theory. In 3 volumes*, pages 1877–1950. Dordrecht: Springer, 2010.

[Sch21] Farmer Schlutzenberg. Full normalization for transfinite stacks. *arXiv preprint arXiv:2102.03359*, 2021.

[Ste83] John R. Steel. Scales in L(R). Cabal Semin. 79-81, Proc. Caltech-UCLA Logic. Semin. 1979-81, Lect. Notes Math. 1019, 107-156 (1983)., 1983.

[Ste95] John R. Steel. $\mathrm{HOD}^{L(\mathbb{R})}$ is a core model below $\Theta$. *Bull. Symb. Log.*, 1(1):75–84, 1995.

[Ste96] John R. Steel. *The core model iterability problem*, volume 8 of *Lect. Notes Log.* Berlin: Springer, 1996.

[Ste09] J. R. Steel. The derived model theorem. In *Logic colloquium 2006. Proceedings of the annual European summer meeting of the Association for Symbolic Logic (ASL), Nijmegen, Netherlands, July 27–August 2, 2006*, pages 280–327. Cambridge: Cambridge University Press; Urbana, IL: Association for Symbolic Logic (ASL), 2009.

[Ste10] John R. Steel. An outline of inner model theory. In *Handbook of set theory. In 3 volumes*, pages 1595–1684. Dordrecht: Springer, 2010.

[Ste23] John R. Steel. *A comparison process for mouse pairs*, volume 51 of *Lect. Notes Log.* Cambridge: Cambridge University Press; Ithaca, NY: Association for Symbolic Logic (ASL), 2023.

[SW16] John R. Steel and W. Hugh Woodin. HOD as a core model. In *Ordinal definability and recursion theory. The Cabal Seminar, Vol. III. Reprints of papers and new material based on the Los Angeles Caltech-UCLA Logic Cabal Seminar 1976–1985*, pages 257–345. Cambridge: Cambridge University Press; Ithaca, NY: Association of Symbolic Logic (ASL), 2016.