On universality of general Dirichlet series

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ON UNIVERSALITY OF GENERAL DIRICHLET SERIES

FRÉDÉRIC BAYART AND ATHANASIOS KOUROUPIS

ABSTRACT. In the present work we establish sufficient conditions for a Dirichlet series induced by general frequencies to be universal with respect to vertical translations. Applying our methodology we give examples of universal Dirichlet series such as the alternating prime zeta function $\sum_{n\geq 1}(-1)^n p_n^{-s}$.

1. INTRODUCTION

The study of the universal properties of Dirichet series goes back to 1975 with the seminal work of Voronin on the Riemann zeta function [19]. Voronin's theorem says:

> Let K be a compact subset of $\{1/2 < \Re e(s) < 1\}$ with connected complement, let f be a nonvanishing function continuous on K and holomorphic in the interior of K. Then
>
> $$\underline{\mathrm{dens}}\left\{\tau \geq 0 : \sup_{s\in K} |\zeta(s+i\tau) - f(s)| < \varepsilon\right\} > 0$$
>
> where $\underline{\mathrm{dens}}(A)$ denotes the lower density of $A \subset \mathbb{R}_+$, that is
>
> $$\underline{\mathrm{dens}}(A) = \liminf_{T\to\infty} \int_0^T \mathbf{1}_A(t)\, dt.$$

Let us introduce the following definitions: let $\Omega_1 \subset \Omega \subset \mathbb{C}$ be two domains such that $\Omega_1 + i\tau \subset \Omega_1$ for all $\tau > 0$, and, for all compact sets $K \subset \Omega$, there exists $\tau > 0$ with $K + i\tau \subset \Omega_1$. Let $D : \Omega_1 \to \mathbb{C}$ be holomorphic. We say that D is universal in Ω if, for all compact subsets K of Ω with connected complement, for all nonvanishing functions $f : K \to \Omega$, continuous on K and holomorphic in the interior of K,

$$\underline{\mathrm{dens}}\left\{\tau \geq 0 : \sup_{s\in K} |D(s+i\tau) - f(s)| < \varepsilon\right\} > 0.$$

We say that f is strongly universal if the restriction that it is non-vanishing can be eased.

Since Voronin's work, the area of universality gained popularity. Many authors studied aspects of (strong) universality for various classes of Dirichlet series $\sum_n a_n e^{-\lambda_n s}$, where $(a_n) \subset \mathbb{C}^{\mathbb{N}}$ and (λ_n) is an increasing sequence of nonnegative real numbers tending to $+\infty$. The case $(\lambda_n) = (\log n)$ corresponds to ordinary Dirichlet series. The survey paper [12] provides a thorough examination of the subject up to 2015.

The first author in [4] improving the work of [11] on strong universality of general Dirichel series obtained the following result:

> Let $P \in \mathbb{R}_d[X]$ with $d \geq 1$ and $\lim_{+\infty} P = +\infty$, let $Q \in \mathbb{R}_{d-1}[X]$, let $\omega \in \mathbb{R}\backslash 2\pi\mathbb{Z}$ and let $\kappa \in \mathbb{R}$. Assume moreover that the sequence $(\log(P(n))_{n\geq 1}$ is $\mathbb{Q}$-linearly independent. Then the Dirichlet series $D(s) = \sum_{n\geq 1} Q(n)(\log n)^\kappa e^{i\omega n}(P(n))^{-s}$ is strongly universal in $\{(2d-1)/2d < \Re e(s) < 1\}$.

This generalizes the case of the Lerch zeta function (see [10]) when $Q(n) = 1$, $\kappa = 0$ and $P(n) = n + \alpha$ with α transcendental.

One of the objectives of this article is to study potential cases of universal Dirichlet series for which the methods of [4] are not applicable, see for example [4, Question 6.8]. The first one concerns the frequencies (λ_n). The simplest example of a frequency (λ_n) such that $(\log(\lambda_n))$ is $\mathbb{Q}$-linearly independent is probably the sequence (p_n) of prime numbers. However, this sequence is not regular enough to be handled by the methods of [4]. The main barrier for this problem is that the sequence of primes is not regular enough to estimate its partial sums using the classical techniques from harmonic analysis like the method of non-stationary phase/ Van der Corput type lemmas 2.5 or decoupling [6]. We are able to prove strong universality for Dirichlet series with type of frequencies and with similar (and even slightly relaxed) conditions on the coefficients. However, the universal property is only shown on a smaller strip.

Theorem 1.1. *Let $D(s) = \sum_{n\geq 1} a_n e^{-\lambda_n s}$ be a Dirichlet series and let $d \in \mathbb{N}$. Assume that*

- *$\lambda_n = \log\big(P(p_n)\big)$ where $P \in \mathbb{R}_d[X]$ and $\lim_{+\infty} P = +\infty$.*
- *there exist $C, \kappa > 0$ such that for $n \geq 2$,*

$$\frac{n^{d-1}}{C(\log n)^\kappa} \leq |a_n| \leq Cn^{d-1}(\log(n))^\kappa.$$

- *$\sigma_c(D) = 0$.*
- *The sequence (λ_n) is $\mathbb{Q}$-linearly independent.*

Then D is strongly universal in $\{(3d-1)/d < \Re e(s) < 1\}$.

Corollary 1.2. *The Dirichlet series $\sum_n (-1)^n p_n^{-s}$ is strongly universal in the strip $\{\frac{2}{3} < \Re e(s) < 1\}$.*

Observe that for the examples coming from [4] or from Theorem 1.1, the Dirichlet series itself converges in its strip of universality. This does not cover the case of the Riemann zeta function or that of the Hurwitz zeta functions $\sum_n (n+\alpha)^{-s}$, α transcendental, which have a pole at 1 and are known to be universal in $\{\frac{1}{2} < \Re e(s) < 1\}$. We extend those results to a large class of general Dirichlet series, even in the case 1 is a branching point and not a pole. In what follows we denote by $\mathbb{C}_\sigma$ the half-plane $\{\Re e(s) > \sigma\}$ and by $\mathbb{C}_\sigma^+$ its restriction to the complex numbers of positive imaginary part, $\{s \in \mathbb{C} :\ \Re e(s) > \sigma,\ \Im m(s) > 0\}$.

Theorem 1.3. *Let $P \in \mathbb{R}_d[X]$ with $\lim_{+\infty} P = +\infty$, let $Q \in \mathbb{R}_{d-1}[X]$ and let $\kappa \in \mathbb{R}$. Assume moreover that the sequence $(\log(P(n)))$ is $\mathbb{Q}$-linearly independent. Then the Dirichlet series $D(s) = \sum_n Q(n)(\log(n))^\kappa [P(n)]^{-s}$ admits a holomorphic continuation to $\mathbb{C}^+_{1-\frac{1}{d}} \cup \mathbb{C}_1$ and even to $\mathbb{C}_{1-\frac{1}{d}} \backslash \{1\}$ if $\kappa \in \mathbb{N}_0$. Moreover, it is strongly universal in the strip $\{(2d-1)/d < \Re e(s) < 1\}$.*

Notation. Throughout the paper, if $f, g : E \to \mathbb{R}$ are two functions defined on the same set E, the notation $f \lesssim g$ will mean that there exists some constant $C > 0$ such that $f \leq Cg$ on E.

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2. Preliminaries

2.1. **Abscissas of convergence.** To a Dirichlet series $D = \sum_{n=1}^{+\infty} a_n e^{-\lambda_n s}$ we will associate three abscissas, its abscissa of convergence,

$$\sigma_c(D) := \inf\left\{\Re e(s) :\ \sum_n a_n e^{-\lambda_n s} \text{ converges}\right\},$$

its abscissa of absolute convergence

$$\sigma_a(D) := \inf\left\{\sigma\in\mathbb{R}:\ \sum_n |a_n|e^{-\lambda_n\sigma}\ \text{converges}\right\},$$

and also

$$\sigma_2(D) := \inf\left\{\sigma\in\mathbb{R}:\ \sum_n |a_n|^2e^{-2\lambda_n\sigma}\ \text{converges}\right\}.$$

It is well-known that $D=\sum_n a_n e^{-\lambda_n s}$ converges in the half-plane $\mathbb{C}_{\sigma_c(D)}$ and that it defines a holomorphic function there.

2.2. **How to prove universality.** Let us introduce two definitions from [4].

Definition 2.1. *Let $\sigma_0\in\mathbb{R}$. We say that a Dirichlet series $D(s)=\sum_n a_n e^{-\lambda_n s}$ with finite abscissa of convergence belongs to $\mathcal{D}_{\text{w.a.}}(\sigma_0)$ provided*

(1) it extends holomorphically to $\mathbb{C}^+_{\sigma_0}\cup\mathbb{C}_{\sigma_c(D)}$;
(2) $\sigma_2(D)\le\sigma_0$;
(3) for all $\sigma_1>\sigma_0$, there exist $t_0,\ B>0$ such that, for all $s=\sigma+it$ with $\sigma\ge\sigma_1$ and $t\ge t_0$, $|D(\sigma+it)|\le t^B$;
(4) for all $\sigma_2>\sigma_1>\sigma_0$,

$$\sup_{\sigma\in[\sigma_1,\sigma_2]}\sup_{T>0}\frac{1}{T}\int_1^T |D(\sigma+it)|^2dt<+\infty;$$

(5) the sequence (λ_n) is $\mathbb{Q}$-linearly independent.

Definition 2.2. *We say that a Dirichlet series $D=\sum_n a_n e^{-\lambda_n s}$ belongs to $\mathcal{D}_{\text{dens}}$ provided for all $\alpha,\beta>0$, there exist $C>0$ and $x_0\ge 1$ such that, for all $x\ge x_0$,*

$$\sum_{\lambda_n\in\left[x,x+\frac{\alpha}{x^\beta}\right]}|a_n|\ge Ce^{(\sigma_a(D)-\beta)x}.$$

The main interest of introducing these definitions is the following theorem (see [4]).

Theorem 2.3. *Let D be a Dirichlet series and let $\sigma_0>\sigma_2(D)$. Assume that $D\in\mathcal{D}_{\text{w.a.}}(\sigma_0)\cap\mathcal{D}_{\text{dens}}$. Then D is strongly universal in the strip $\{\sigma_0<\Re e(s)<\sigma_a(D)\}$*

It should be pointed out that Definition 2.1 in [4] mentions the whole half-plane $\mathbb{C}_{\sigma_0}$ and not the quarter half-plane as here. However, this does not change anything for the proofs. The key points are that the half vertical lines $\sigma+it,\ t>0,\ \sigma>\sigma_0$, are contained in $\mathbb{C}^+_{\sigma_0}$ and that for any compact set K included in the strip $\{\sigma_0<\Re e(s)<\sigma_a(D)\}$, there exists $\tau>0$ such that $K+i\tau\subset\mathbb{C}^+_{\sigma_0}$.

2.3. **Two lemmas to estimate exponential sums.** We shall need two inequalities which have been widely used in this context. The first one deals with exponential sums and is due to Montgomery and Vaughan (see [13]).

Lemma 2.4. *Let (a_n) be a sequence of complex numbers such that $\sum_n|a_n|^2<+\infty$. Let (λ_n) be a sequence of real numbers and set $\theta_n:=\inf_{m\ne n}|\lambda_n-\lambda_m|>0$ for every n. Then*

$$\int_0^T\left|\sum_n a_n e^{i\lambda_n t}\right|^2dt=T\sum_n|a_n|^2+O\left(\sum_n\frac{|a_n|^2}{\theta_n}\right)$$

where the O-constant is absolute.

We also need the following classical inequality for exponential sums, which goes back to J. G. van der Corput (see [5, Lemma 11.5]).

Lemma 2.5. *Let $a < b$ and let $f, g : [a, b] \to \mathbb{R}$ be two functions of class $\mathcal{C}^2$. Assume that*

- *f' is monotonic with $|f'| < 1/2$;*
- *g is positive, non-increasing and convex.*

Then

$$\sum_{n=a}^{b} g(n)e^{2\pi i f(n)} = \int_a^b g(u)e^{2\pi i f(u)}du + O(g(a) + |g'(a)|).$$

2.4. **The incomplete Gamma function/ Prym's function.** We will make a short presentation and we refer the interested reader to [1, 16, 17]. For $\Re e(a) > 0$ and $\Re e(z) > 0$, we define the incomplete Gamma function $\Gamma(a, z)$ by

$$\Gamma(a, z) = \int_z^{+\infty} t^{a-1}e^{-t}\, dt.$$

For fixed z, as in the classical case it has a meromorphic extension in $\mathbb{C}$ with simple poles at the nonpositive integers. This can be easily obtained from the recurrence relation:

$$\Gamma(a + 1, z) = a\Gamma(a, z) + z^a e^{-z}.$$

For a fixed value of a, Γ admits a holomorphic extension (its principal branch) to $\mathbb{C}\backslash\mathbb{R}_-$ and even to $\mathbb{C}$ when a is a positive integer. When a is not a nonpositive integer, this follows for instance from the relation

$$\Gamma(a, z) = \Gamma(a)(1 - z^{a-1}\gamma^*(a, z)),$$

where the function γ^* is entire in both a and z. When a is a nonpositive integer, this follows from the corresponding statement for $a = 0$ (in that case, the incomplete Gamma function is also called the exponential integral).

For this principal branch (and for a fixed a), we have the estimation

$$\Gamma(a, z) = e^{-z}\int_0^{+\infty} e^{-u}\,(z + u)^{a-1}\, du = O(z^{a-1}e^{-z}), \tag{1}$$

as $|z| \to +\infty$.

2.5. **A remark on** [4]**.** In [4, Theorem 1.6], the theorem on rearrangement universality of Dirichlet series states that a Dirichlet series $\sum_n a_n e^{-\lambda_n s}$ is rearrangement universal if for any $f \in H(\Omega)$, where Ω is the strip $\sigma_c(D) < \Re e(s) < \sigma_a(D)$, there exists a permutation σ of $\mathbb{N}$ such that $\sum_n a_{\sigma(n)}e^{-\lambda_{\sigma(n)}s}$ converges to f in $H(\Omega)$). This theorem is false. Indeed it would imply that $\sum_n(-1)^n n^{-s}$ is rearrangement universal. This cannot hold: any rearrangement of $\sum_n(-1)^n n^{-s}$ will take values in $\mathbb{R}$ for real values of the parameter s. The mistake which is made in [4] lies on the fact that a lemma due to Banaszczyk is only true for some real Fréchet spaces and was applied to the complex Fréchet space $H(\Omega)$.

3. Proof of Theorem 1.1

The main difficulty in order to apply Theorem 2.3 is to estimate the square moments of D on vertical lines. We follow a method introduced in [2] where the authors estimate the square moments of the logarithm of the zeta function.

Lemma 3.1. *Let $D = \sum_{n\geq 1} a_n e^{-\lambda_n s}$ be a Dirichlet series with $\sigma_a(D) \leq 1$. Assume that D extends continuously to $\overline{\mathbb{C}_\alpha}$, $0 \leq \alpha < 1$, analytically in $\mathbb{C}_\alpha$, and that D has order B in $\overline{\mathbb{C}_\alpha}$.*

Let $\sigma_0 \in (\alpha, 1)$ with $\sigma_0 > \sigma_2(D)$ and let us set $A = B/(\sigma_0 - \alpha)$. Then for all $T \geq 1$, for all $\sigma > \sigma_1 > \sigma_0$,

$$\int_0^T |D(\sigma + it)|^2 dt \lesssim T + \sum_n \frac{|a_n|^2 \exp(-2\lambda_n \sigma) \exp\left(-2\frac{e^{\lambda_n}}{T^A}\right)}{\min(\lambda_n - \lambda_{n-1}, \lambda_{n+1} - \lambda_n)}.$$

Proof. The inverse Mellin transform (see [14, Appendix 3]) applied to the Γ function says that, for all $x > 0$,

$$e^{-x} = \frac{1}{2\pi i} \int_{2-\sigma_0 - i\infty}^{2-\sigma_0 + i\infty} x^{-w} \Gamma(w) dw. \tag{2}$$

Let $T \geq 2$ and set $X = T^A$. We apply (2) for $x = \frac{e^{\lambda_n}}{X}$, yielding to

$$\exp\left(-\frac{e^{\lambda_n}}{X}\right) = \frac{1}{2\pi i} \int_{2-\sigma_0 - i\infty}^{2-\sigma_0 + i\infty} \exp(-\lambda_n w) X^w \Gamma(w) dw.$$

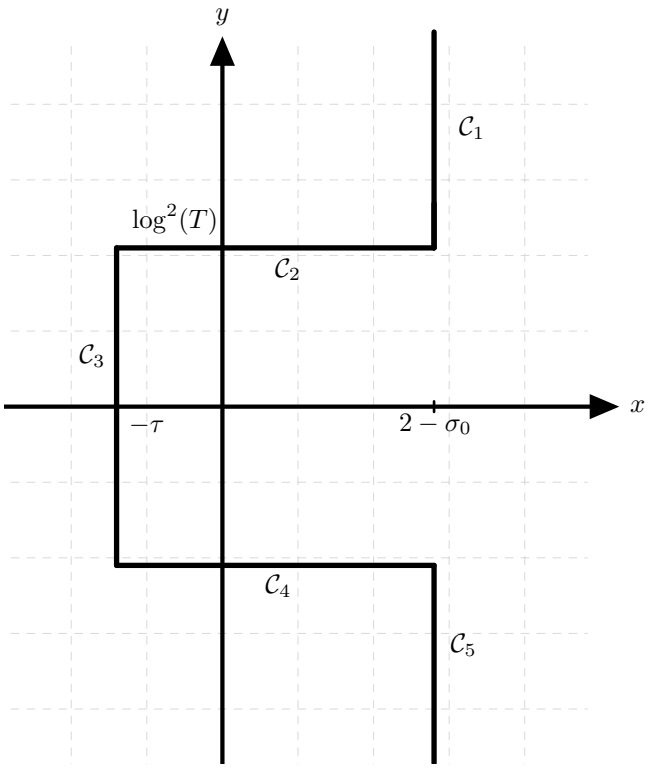
Therefore, for any $\sigma > \sigma_0$ and any $t \in [0, T]$, setting $s = \sigma + it$, for any $n \geq 1$,

$$a_n \exp(-\lambda_n s) \exp\left(-\frac{e^{\lambda_n}}{X}\right) = \frac{1}{2\pi i} \int_{2-\sigma_0 - i\infty}^{2-\sigma_0 + i\infty} a_n \exp(-\lambda_n (s+w)) X^w \Gamma(w) dw.$$

Since $\Re e(s + w) > 1$ provided $\Re e(w) = 2 - \sigma_0$, we can sum these equalities and interchange summation and integral to get

$$\sum_{n \geq 1} a_n \exp(-\lambda_n s) \exp\left(\frac{e^{-\lambda_n}}{X}\right) = \frac{1}{2\pi i} \int_{2-\sigma_0 - i\infty}^{2-\sigma_0 + i\infty} D(s+w) X^w \Gamma(w)\, dw. \tag{3}$$

We set $\tau = \sigma_0 - \alpha$ and we introduce the following contour $\mathcal{C}$, defined as the union of five segments or half-lines.



On $\mathcal{C}_1 \cup \mathcal{C}_5$, $|D(s+w)| \lesssim 1$. Moreover, writing $w = u + iv$ with $u = 2 - \sigma_0$, Stirling's formula for the Γ-function (see again [14, Appendix 3]) says that

$$|\Gamma(u+iv)| \lesssim e^{-C|v|}$$

for some $C > 0$ (independent of $w \in \mathcal{C}_1 \cup \mathcal{C}_5$). Hence,

$$\begin{aligned}\int_{\mathcal{C}_1 \cup \mathcal{C}_5} |D(s+w)X^w\Gamma(w)|dw &\lesssim X^{2-\sigma_0} \int_{\log^2(T)}^{+\infty} e^{-Cv}dv \\ &\lesssim 1.\end{aligned}$$

Pick now $w = u + iv \in \mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{C}_4$. Then

$$\left\{ \begin{array}{rcl} |\Im m(s+w)| & \leq & \log^2(T) + T \lesssim T \\ \Re e(s+w) & \geq & \sigma_0 - \tau = \alpha. \end{array} \right.$$

Therefore, $|D(s+w)| \lesssim T^B$. This implies

$$\begin{aligned}\int_{\mathcal{C}_2 \cup \mathcal{C}_4} |D(s+w)X^w\Gamma(w)|dw &\lesssim T^B T^{A(2-\sigma_0)} e^{-C\log^2(T)} \\ &\lesssim 1.\end{aligned}$$

Finally,

$$\begin{aligned}\int_{\mathcal{C}_3} |D(s+w)X^w\Gamma(w)|dw &\lesssim T^{-A\tau}T^B \int_{\mathbb{R}} |\Gamma(-\tau+it)|dt \\ &\lesssim 1\end{aligned}$$

by our choice of A and τ. Summing the estimates we obtain

$$\int_{\mathcal{C}} |D(s+w)X^w\Gamma(w)|dw \lesssim 1. \tag{4}$$

Let now $\mathcal{R}$ be the rectangle $\mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{C}_4 \cup \mathcal{C}_6$ with $\mathcal{C}_6 = [2-\sigma_0 - i\log^2 T, 2-\sigma_0 + i\log^2 T]$ so that

$$\int_{\mathcal{C}} = \int_{2-\sigma_0-i\infty}^{2-\sigma_0+i\infty} + \int_{\mathcal{R}}. \tag{5}$$

The function $w \mapsto D(s+w)X^w\Gamma(w)$ has a single pole 0 inside $\mathcal{R}$, with residue $D(s)$. Hence, by (3), (4) and (5),

$$|D(s)| \lesssim 1 + \left| \sum_{n\geq 1} a_n \exp(-\lambda_n s) \exp\left(\frac{-e^{\lambda_n}}{X}\right)\right|.$$

We set $b_n = a_n \exp(-\lambda_n\sigma)\exp\left(-\frac{e^{\lambda_n}}{X}\right)$. Taking the square and integrating over $[0,T]$, we find

$$\int_0^T |D(\sigma+it)|^2 dt \lesssim T + \int_0^T \left|\sum_{n\geq 1} b_n \exp(-i\lambda_n t)\right|^2 dt.$$

We apply the Montgomery-Vaughan inequality, yielding to the following estimate:

$$\int_0^T |D(\sigma+it)|^2 dt \lesssim T + T\sum_{n\geq 1} |a_n|^2 \exp(-2\lambda_n\sigma) + \sum_{n\geq 1} \frac{|b_n|^2}{\min(\lambda_n - \lambda_{n-1}, \lambda_{n+1}-\lambda_n)}.$$

Since $\sigma_0 > \sigma_2(D)$, we have proven Lemma 3.1. □

To estimate the sum appearing in the last lemma, we shall use the following result.

Lemma 3.2. *Let (λ_n) be a sequence of frequencies, let (a_n) be a sequence of complex numbers. Assume that there exist $a \in \mathbb{R}$, b, c, $\beta > 0$ such that*

$$|a_n| \lesssim n^a, \quad \lambda_n \geq b\log n, \quad \min(\lambda_n - \lambda_{n-1}, \lambda_{n+1}-\lambda_n) \geq c\exp(-\beta\lambda_n).$$

Then for all $X > 0$ and all $\sigma > \beta/2$,

$$\sum_{n\geq 1} \frac{|a_n|^2 \exp(-2\lambda_n\sigma)\exp\left(-2\frac{e^{\lambda_n}}{X}\right)}{\min(\lambda_n - \lambda_{n-1}, \lambda_{n+1}-\lambda_n)} \lesssim \max\left(1, X^{\frac{2a+(\beta-2\sigma)b+1}{b}}\right).$$

In particular, if $e^{\lambda_{n+1}} - e^{\lambda_n} \geq 1$ for all n, we may choose $\beta = 1$ in the previous lemma. Indeed, the inequality $\lambda_{n+1} - \lambda_n \geq \exp(-\lambda_n)$ is clear if $\lambda_{n+1} - \lambda_n \geq 1$. Otherwise,

$$\lambda_{n+1} - \lambda_n \gtrsim e^{\lambda_{n+1}-\lambda_n} - 1 = (e^{\lambda_{n+1}} - e^{\lambda_n})e^{-\lambda_n}.$$

Proof. Denote by S the sum. Then

$$\begin{aligned} S &\lesssim \sum_n n^{2a} \exp\left((\beta-2\sigma)\lambda_n\right)\exp\left(-2\frac{e^{\lambda_n}}{X}\right) \\ &\lesssim \sum_n n^{2a+(\beta-2\sigma)b}\exp\left(-2\frac{n^b}{X}\right). \end{aligned}$$

We split the sum into two parts. We first sum up to $X^{1/b}$ and we denote by S_1 this sum. Then

$$S_1 \lesssim \sum_{n\leq X^{1/b}} n^{2a+(\beta-2\sigma)b} \lesssim \max\left(1, X^{\frac{2a+(\beta-2\sigma)b+1}{b}}\right).$$

Regarding the second sum, say S_2, we write

$$\begin{aligned} S_2 &= \sum_{n\geq X^{1/b}} n^{2a+(\beta-2\sigma)b}\exp\left(-\frac{2n^b}{X}\right)\\ &\lesssim \int_{X^{1/b}}^{+\infty} t^{2a+(\beta-2\sigma)b}\exp\left(-\frac{2t^b}{X}\right)dt \end{aligned}$$

We do the change of variables $u=t^b/X$ which yields

$$\begin{aligned} S_2 &\lesssim X^{\frac{2a+(\beta-2\sigma)b+1}{b}}\int_1^{+\infty} u^{\frac{2a+(\beta-2\sigma)b+1-b}{b}}\exp(-2u)du\\ &\lesssim X^{\frac{2a+(\beta-2\sigma)b+1}{b}}. \end{aligned}$$

□

Using what has been done above, we may complete the first part of the proof of Theorem 1.1, that is:

Corollary 3.3. *Let $D(s)=\sum_{n\geq 1}a_n e^{-\lambda_n s}$ be a Dirichlet series and let $d\in\mathbb{N}$. Assume that*

- *$\lambda_n=\log\big(P(p_n)\big)$ where $P\in\mathbb{R}_d[X]$ and $\lim_{+\infty}P=+\infty$.*
- *there exist $C,\kappa>0$ such that for $n\geq 2$,*

$$\frac{n^{d-1}}{C(\log n)^\kappa}\leq |a_n|\leq Cn^{d-1}(\log(n))^\kappa.$$

- *$\sigma_c(D)=0$.*
- *the sequence (λ_n) is $\mathbb{Q}$-linearly independent.*

Then D belongs to $\mathcal{D}_{\text{w.a.}}(\sigma_0)$ where $\sigma_0=1-\frac{1}{3d}$.

Proof. Under the above conditions, $\sigma_a(D)=1$, $\sigma_c(D)=0$ and $\sigma_2(D)=1-\frac{1}{2d}$. Therefore, conditions (1) to (3) of Definition 2.1 are satisfied. Let us prove (4). Let $\sigma_1>\sigma_0$ and let $\varepsilon<\sigma_1-\sigma_0$. We set $\alpha=\varepsilon$, and observe that D has order 1 in $\overline{\mathbb{C}_\alpha}$ (see [8, Theorem 12]). We apply Lemma 3.1 and then Lemma 3.2 with $a=(d-1)+\varepsilon$, $b=d-\varepsilon$ and $\beta=1$ to obtain that, for $\sigma\geq\sigma_1$, for $T\geq 2$,

$$\int_0^T |D(\sigma+it)|^2dt\lesssim T+T^{B(\varepsilon)}$$

with

$$B(\varepsilon)=\frac{2\big((d-1)+\varepsilon\big)+(1-2\sigma_0)(d-\varepsilon)+1}{(d-\varepsilon)(\sigma_1-\varepsilon)}.$$

Now,

$$\lim_{\varepsilon\to 0}B(\varepsilon)=\frac{3d-2d\sigma_0-1}{d\sigma_1}=\frac{\sigma_0}{\sigma_1}<1.$$

Hence, choosing $\varepsilon>0$ small enough, we have shown that $\int_0^T|D(\sigma+it)|^2dt\lesssim T$. □

Let us proceed with the second half. First let us state the following lemma proved in [4, Lemma 6.1]:

Lemma 3.4. *Let $P(X)=\sum_{k=0}^d b_kX^k$ be a polynomial of degree d, with $b_d>0$. Then, there exist x_0, $y_0>0$ such that P induces a bijection from $[x_0,+\infty]$ to $[y_0,+\infty]$, and*

$$P^{-1}(x)=\frac{x^{1/d}}{(b_d^{1/d})}-\frac{b_{d-1}}{b_d^{(d-1)/d}}+o(1),$$

as $x\to+\infty$.

Proposition 3.5. *Let $D(s) = \sum_{n\geq 1} a_n e^{-\lambda_n s}$ be a Dirichlet series and let $d \in \mathbb{N}$. Assume that*

- *$\lambda_n = \log\left(P(p_n)\right)$ where $P \in \mathbb{R}_d[X]$ and $\lim_{+\infty} P = +\infty$.*
- *there exist $C, \kappa > 0$ such that for $n \geq 2$,*

$$\frac{n^{d-1}}{C(\log n)^\kappa} \leq |a_n| \leq Cn^{d-1}(\log(n))^\kappa.$$

Then D belongs to $\mathcal{D}_{\text{dens}}$.

Proof. Let $\alpha, \beta > 0$. Without loss of generality, we may assume that P is one-to-one on $[\log(p_1), +\infty)$. Then

$$\lambda_n \in \left[x, x + \frac{\alpha}{x^2}\right] \qquad \text{if and only if} \qquad p_n \in \left[P^{-1}(e^x), P^{-1}(e^{x+\frac{\alpha}{x^2}})\right].$$

Using Lemma 3.4, there exist $c_1 > 0$, $c_2 \in \mathbb{R}$ such that for small $\varepsilon > 0$ and for every x sufficiently large:

$$p_n \in \left[c_1 e^{\frac{x}{d}} + c_2 + \varepsilon, c_1 e^{\frac{x}{d} + \frac{\alpha}{dx^2}} + c_2 - \varepsilon\right] \qquad \text{implies that} \qquad \lambda_n \in \left[x, x + \frac{\alpha}{x^2}\right].$$

By Hadamard - De la Vallée Poussin estimate, we also know that

$$\begin{aligned}\Pi(u) &:= \text{card}\{n : \ p_n \leq u\} \\ &= \int_2^u \frac{dt}{\log(t)} + O\left(ue^{-c\sqrt{\log u}}\right).\end{aligned}$$

If $p_n \gtrsim e^{\frac{x}{d}}$, then $n \gtrsim \frac{e^{\frac{x}{d}}}{x}$. Therefore

$$\begin{aligned}\sum_{\lambda_n \in \left[x, x+\frac{\alpha}{x^2}\right]} |a_n| &\gtrsim \frac{e^{\frac{x(d-1)}{d}}}{x^{d-1+\kappa}} \text{card}\left\{n : \ p_n \in \left[c_1 e^{\frac{x}{d}} + c_2 + \varepsilon, c_1 e^{\frac{x}{d} + \frac{\alpha}{dx^2}} + c_2 - \varepsilon\right]\right\} \\ &\gtrsim \frac{e^{\frac{x(d-1)}{d}}}{x^{d-1+\kappa}} \left(\int_{c_1 e^{\frac{x}{d}} - c_2 + \varepsilon}^{c_1 e^{\frac{x}{d} + \frac{\alpha}{dx^2}} - c_2 - \varepsilon} \frac{dt}{\log(t)} + O\left(e^{\frac{x}{d} - c\sqrt{x}}\right)\right) \\ &\gtrsim \frac{e^{\frac{x(d-1)}{d}}}{x^{d-1+\kappa}} \left(e^{\frac{x}{d}} \frac{e^{\frac{\alpha}{dx^2}} - 1}{x} + O\left(e^{\frac{x}{d} - c'\sqrt{x}}\right)\right) \\ &\gtrsim \frac{e^x}{x^{d+\kappa+2}} \\ &\gtrsim e^{(1-\beta)x}.\end{aligned}$$

□

Proof of Theorem 1.1. It follows immediately from Theorem 2.3, Corollary 3.3 and Proposition 3.5. □

Remark. To prove Theorem 1.1, we use rather weak properties of prime numbers to prove Corollary 3.3. Hence we could easily replace $(\log(P(p_n)))$ by another sequence of frequencies satisfying some conditions in the statement of this corollary. Nevertheless, in Proposition 3.5, we use the much more profound Hadamard - De la Vallée Poussin estimate.

4. Proof of Theorem 1.3

As above the main difficulty is to estimate the square moments of D. The situation is not as clear as in the previous case since D now will be defined via an analytic continuation. We need to understand how to define this analytic continuation and how close it is to the partial sums of D.

Lemma 4.1. *Let $P \in \mathbb{R}_d[X]$ with $\lim_{+\infty} P = +\infty$, let $Q \in \mathbb{R}_{d-1}[X]$ and let $\kappa \in \mathbb{R}$. Then the Dirichlet series $D(s) = \sum_n Q(n)(\log n)^\kappa (P(n))^{-s}$ admits a holomorphic continuation to $\mathbb{C}^+_{1-\frac{1}{d}} \cup \mathbb{C}_1$ and even to $\mathbb{C}_{1-\frac{1}{d}} \backslash \{1\}$ provided $\kappa \in \mathbb{N}_0$. Moreover, let $\sigma_1 > 1 - \frac{1}{d}$ and $\sigma_2 > 1$.*

(a) There exist $t_0,\ B > 0$ such that, for all $s = \sigma + it$ with $\sigma \geq \sigma_1$ and $t \geq t_0$,

$$|D(s)| \leq t^B.$$

(b) There exist $\delta, \varepsilon > 0$ such that, for all $x > 0$, for all $s = \sigma + it$ with $\sigma \in [\sigma_1, \sigma_2]$ and $1 \leq t \leq \delta x$,

$$D(s) = \sum_{n=2}^{x} Q(n)(\log n)^\kappa (P(n))^{-s} + O(x^{-\varepsilon}) + O\left(\frac{(\log P(x))^\kappa}{(s-1)P(x)^{s-1}}\right)$$

(here, the O-constants do not depend neither on s nor on x).

Proof. As in the classical case of the Riemann zeta function, see for example [5], our plan is to use the regularity and smoothness of the coefficients and the frequencies of our Dirichlet series D to estimate its order and how close the partial sums approximate D. We will rely again on the principle of non-stationary phase, that is Lemma 2.5. But first we need to deal with some technical difficulties that arise from the "unknown" polynomials P and Q. We start with $s = \sigma + it$, $\sigma > 1$ and let $N \geq 1$. We write

$$D(s) = \sum_{n=2}^{N-1} Q(n)(\log n)^\kappa (P(n))^{-s} + \sum_{n=N}^{+\infty} Q(n)(\log n)^\kappa (P(n))^{-s}$$

and we apply Euler's summation formula (see [5, (11.3)]). Setting

$$\phi(u) = Q(u)(\log u)^\kappa (P(u))^{-s} \text{ and } \rho(u) = u - \lfloor u \rfloor - \frac{1}{2},$$

we get

$$D(s) = \sum_{n=2}^{N-1} Q(n)(\log n)^\kappa (P(n))^{-s} + \int_N^{+\infty} \phi(u)du + \int_N^{+\infty} \rho(u)\phi'(u)du + \frac{1}{2}\phi(N). \tag{6}$$

These integrals are convergent when $s \in \mathbb{C}_1$. Moreover it is easy to check that there exists $\varepsilon > 0$ such that, provided $s = \sigma + it$ with $\sigma \geq \sigma_1 > 1 - \frac{1}{d}$, for any $u > 2$,

$$|\phi(u)| \lesssim u^{-\varepsilon} \text{ and } |\phi'(u)| \lesssim |s| u^{-1-\varepsilon}.$$

In particular, the last integral in (6) defines a holomorphic function in $\mathbb{C}_{1-\frac{1}{d}}$ which is $O(|s|N^{-\varepsilon})$ in $\mathbb{C}_{\sigma_1}$. Let us now see how to control the first integral. Up to multiply Q by some constant, we may write it $Q(u) = P'(u) + Q_1(u)$ with $\deg(Q_1) \leq d-2$. As above, the integral $\int_N^{+\infty} Q_1(u)(\log u)^\kappa (P(u))^{-s} du$ defines an analytic function in $\mathbb{C}_{1-\frac{1}{d}}$ which is $O(N^{-\varepsilon})$. Therefore we have obtained so far that D may be written in $\mathbb{C}_1$

$$D(s) = \sum_{n=2}^{N-1} Q(n)(\log n)^\kappa (P(n))^{-s} + \int_N^{+\infty} \frac{P'(u)(\log u)^\kappa}{(P(u))^s} du + R_N(s)$$

where R_N is analytic in $\mathbb{C}_{1-\frac{1}{d}}$ and $|R_N(s)| \lesssim |s| N^{-\varepsilon}$ uniformly for $\sigma \geq \sigma_1$.

We choose N sufficiently large such that P is one-to-one on $[N,+\infty)$. By change of variables we obtain:

$$\int_N^{+\infty} \frac{P'(u)(\log u)^\kappa}{(P(u))^s}du = \int_{P(N)}^{+\infty} \frac{(\log P^{-1}(u))^\kappa}{u^s}du.$$

By Lemma 3.4 we have the following formula:

$$P^{-1}(u) = a_d u^{1/d}(1+\varepsilon_1(u)) \text{ with } |\varepsilon_1(u)| \lesssim u^{-1/d},$$

where $a_d > 0$. Therefore,

$$(\log P^{-1}(u))^\kappa = \log^\kappa(a_d u^{1/d}) + \varepsilon_2(u)$$

with

$$|\varepsilon_2(u)| \lesssim u^{-1/d}\log^{\kappa-1}(u).$$

As before, the integral $\int_{P(N)}^{+\infty} \varepsilon_2(u)u^{-s}du$ defines an analytic function in $\mathbb{C}_{1-\frac{1}{d}}$ which is $O(P(N)^{-\varepsilon})$ in $\mathbb{C}_{\sigma_1}$. On the other hand, setting $b_d = a_d^d$ and restricting ourselves to $s \in \mathbb{C}_1$, we may write

$$\begin{aligned}
\int_{P(N)}^{+\infty} \frac{\log^\kappa(a_d u^{1/d})}{u^s}du &= \int_{P(N)}^{+\infty} \frac{1}{d^\kappa}\frac{\log^\kappa(b_d u)}{u^s}du \\
&= \frac{b_d^{s-1}}{d^\kappa}\int_{b_d P(N)}^{+\infty} \frac{\log^\kappa(v)}{v^s}dv && (v = b_d u) \\
&= \frac{b_d^{s-1}}{d^\kappa}\int_{\log(b_d P(N))}^{+\infty} y^\kappa e^{(1-s)y}dy && (y = \log v) \\
&= \frac{b_d^{s-1}}{d^\kappa(s-1)^{\kappa+1}}\Gamma(\kappa+1,(s-1)\log(b_d P(N))).
\end{aligned}$$

Hence we have shown that for $s \in \mathbb{C}_1$, we may write

$$D(s) = \sum_{n=2}^{N-1} Q(n)(\log n)^\kappa (P(n))^{-s} + \widetilde{R_N}(s) + \frac{b_d^{s-1}}{d^\kappa(s-1)^{\kappa+1}}\Gamma(\kappa+1,(s-1)\log(b_d P(N)))$$

where $\widetilde{R_N}(s)$ is holomorphic in $\mathbb{C}_{1-\frac{1}{d}}$ and is $O(|s|N^{-\varepsilon}) + O(P(N)^{-\varepsilon})$ in $\mathbb{C}_{\sigma_1}$. Since we know that $\Gamma(\kappa+1,\cdot)$ admits an analytic continuation to $\mathbb{C}\backslash\mathbb{R}_-$ we can conclude to the analytic continuation of D to $\mathbb{C}^+_{1-\frac{1}{d}} \cup \mathbb{C}_1$. When $\kappa \in \mathbb{N}$, the analytic continuation even holds on $\mathbb{C}_{1-\frac{1}{d}}\backslash\{1\}$. The estimation (a) (which is trivial for $\sigma \geq \sigma_2 > 1$) follows easily for $\sigma \in [\sigma_1,\sigma_2]$ by what we already know on $\widetilde{R_N}$ and by (1).

Let us turn to the proof of (b). Choosing $N \geq x$, we may write

$$D(s) = \sum_{n=2}^{x} Q(n)(\log n)^\kappa (P(n))^{-s} + \sum_{n=x+1}^{N} Q(n)(\log n)^\kappa (P(n))^{-s} + \frac{b_d^{s-1}}{d^\kappa(s-1)^{\kappa+1}}\Gamma(\kappa+1,(s-1)\log(b_d P(N))) + O(|s|N^{-\varepsilon}) + O(P(N)^{-\varepsilon}).$$

We apply Lemma 2.5 to the second sum with

$$g(u) = Q(u)\log^\kappa(u)(P(u))^{-\sigma},\ f(u) = \frac{-t\log(P(u))}{2\pi}.$$

Observe that, for $\sigma \in [\sigma_1, \sigma_2]$ and $u \in [x, N]$, provided $t \leq \delta x$ with δ small enough,

$$|g(u)| \lesssim x^{-\varepsilon},\ |g'(u)| \lesssim \sigma x^{-1-\varepsilon} \leq x^{-\varepsilon},\ |f'(u)| \leq \frac{1}{2}.$$

Hence,

$$D(s) = \sum_{n=2}^{x} Q(n)(\log n)^{\kappa}(P(n))^{-s} + \int_x^N \frac{Q(u)\log^{\kappa}(u)}{(P(u))^s}du + \frac{b_d^{s-1}}{d^{\kappa}(s-1)^{\kappa+1}}\Gamma(\kappa+1,(s-1)\log(b_dP(N))) + O(x^{-\varepsilon} + |s|N^{-\varepsilon} + P(N)^{-\varepsilon}).$$

We let $N \to +\infty$, yielding to:

$$D(s) = \sum_{n=2}^{x} Q(n)(\log n)^{\kappa}(P(n))^{-s} + \int_x^{+\infty} \frac{Q(u)\log^{\kappa}(u)}{(P(u))^s}du + O(x^{-\varepsilon}), \qquad s \in \mathbb{C}_1.$$

Now writing $Q(u) = P'(u) + Q_1(u)$ and repeating the argument in the first part of this proof one can obtain the following identity

$$D(s) = \sum_{n=2}^{x} Q(n)(\log n)^{\kappa}(P(n))^{-s} + \frac{b_d^{s-1}}{d^{\kappa}(s-1)^{\kappa+1}}\Gamma(\kappa+1,(s-1)\log(b_dP(x))) + \widetilde{R}(s),$$

where $\widetilde{R}(s)$ is holomorphic in $\mathbb{C}_{1-\frac{1}{d}}$ and is $O(x^{-\varepsilon})$ in $\mathbb{C}_{\sigma_1}$. Using one last time (1), we obtain (b) of Lemma 4.1. □

From this and Lemma 2.4, we may deduce the first half of Theorem 1.3.

Proposition 4.2. *Let $P \in \mathbb{R}_d[X]$ with $\lim_{+\infty} P = +\infty$, let $Q \in \mathbb{R}_{d-1}[X]$ and let $\kappa \in \mathbb{R}$. Assume moreover that $(\log(P(n)))$ is $\mathbb{Q}$-linearly independent. Then the Dirichlet series $D(s) = \sum_n Q(n)(\log n)^{\kappa}(P(n))^{-s}$ belongs to $\mathcal{D}_{\text{w.a.}}(\sigma_0)$ with $\sigma_0 = 1 - \frac{1}{2d}$.*

Proof. It is clear that $\sigma_2(D) = (2d-1)/2d$ and thus it just remains to prove (4). We fix $T \geq 1$ and we first estimate $\int_{T/2}^{T} |D(\sigma+it)|^2 dt$ where $1 - \frac{1}{2d} < \sigma_1 \leq \sigma \leq \sigma_2$. We apply the estimate given by Lemma 4.1 with $x = T/\delta$ so that $O(x^{-\varepsilon}) = O(T^{-\varepsilon})$ and

$$\left|\frac{\log^{\kappa}(P(x))}{(s-1)P(x)^{s-1}}\right| \lesssim \frac{\log^{\kappa} T}{TT^{d(\sigma-1)}} \lesssim T^{-\varepsilon}.$$

Hence, applying Lemma 2.4

$$\begin{aligned}\int_{T/2}^{T} |D(\sigma+it)|^2 dt &\lesssim \int_{T/2}^{T} \left|\sum_{n=2}^{T/\delta} |Q(n)(\log n)^{\kappa}(P(n))^{-s}\right|^2 dt + T^{1-2\varepsilon} \\ &\lesssim T\sum_{n=2}^{T/\delta} |Q(n)|^2(\log n)^{2\kappa}|P(n)|^{-2\sigma} + \\ &\quad \sum_{n=2}^{T/\delta} \frac{|Q(n)|^2(\log n)^{2\kappa}|P(n)|^{-2\sigma}}{\log(P(n+1)) - \log(P(n))} + T^{1-2\varepsilon}.\end{aligned}$$

The first sum is dominated by some constant since $\sigma \geq \sigma_1 > \sigma_2(D)$. Regarding the second sum, for $n \in [2, T/\delta]$,

$$\frac{|Q(n)|^2(\log n)^{2\kappa}|P(n)|^{-2\sigma}}{\log(P(n+1)) - \log(P(n))} \lesssim Tn^{2(d-1)-2d\sigma}(\log n)^{2\kappa} \lesssim Tn^{2d(1-\sigma_1)-2}(\log n)^{2\kappa},$$

and we get the estimate

$$\sum_{n=2}^{T/\delta} \frac{|Q(n)|^2 (\log n)^{2\kappa} |P(n)|^{-2\sigma}}{\log(P(n+1)) - \log(P(n))} \lesssim T,$$

since $2d(1-\sigma_1) < 1$. Hence, we have obtained

$$\int_{T/2}^{T} |D(\sigma + it)|^2 dt \lesssim T,$$

for all $T \geq 1$ and all $\sigma \in [\sigma_1, \sigma_2]$, where the involved constant does not depend neither on σ nor on T. Taking $T2^{-j}$ instead of T in the latter formula and summing over j, we get the proposition. □

The second half of the proof of Theorem 1.3 has been proven in [4, Proposition 6.2], for the sake of completeness we repeat the argument below.

Proposition 4.3. *Let $P \in \mathbb{R}_d[X]$ with $\lim_{+\infty} P = +\infty$, let $Q \in \mathbb{R}_{d-1}[X]$ and let $\kappa \in \mathbb{R}$. Then the Dirichlet series $D(s) = \sum_n Q(n)(\log n)^\kappa (P(n))^{-s}$ belongs to $\mathcal{D}_{\text{dens}}$.*

Proof. Let $\alpha, \beta > 0$. There exists $x_0 > 1$ such that for every $x \geq x_0$ the polynomial P is positive and increasing and Q behaves like its leading term. By Lemma 3.4 there exist constants $c_1 > 0$, $c_2 \in \mathbb{R}$ such that

$$n \in \left[c_1 e^{\frac{x}{d}} - c_2 + \varepsilon, c_1 e^{\frac{x}{d} + \frac{\alpha}{dx^2}} - c_2 - \varepsilon\right] \qquad \text{implies that} \qquad \lambda_n \in \left[x, x + \frac{\alpha}{x^2}\right].$$

Thus

$$\sum_{\lambda_n \in \left[x, x + \frac{\alpha}{x^2}\right]} |Q(n)(\log n)^\kappa| \gtrsim \frac{e^{\frac{x}{d}}}{x^2} e^{\frac{d-1}{d} x(1-\frac{\beta}{2})} \gtrsim e^{(1-\beta)x}.$$

□

Theorem 1.3 now follows from Proposition 4.2, Proposition 4.3 and Theorem 2.3.

5. Random models and further discussion

One of the motivations behind our work is to give concrete examples of convergent universal objects like the alternating prime zeta function $P_-(s) = \sum_{n\geq 1} (-1)^n p_n^{-s}$. As we have already proved P_- is strongly universal in $\{\frac{2}{3} < \Re e < 1\}$. But, the universality of this object in the whole critical strip remains an open question.

Question 5.1. Is P_- strongly universal in $\{\frac{1}{2} < \Re e \leq \frac{2}{3}\}$?

It is worth mentioning that Theorem 1.1 implies that every series of the form

$$P_\chi(s) = \sum_{n\geq 1} \chi_n p_n^{-s}, \qquad |\chi_n| = 1,$$

with $\sigma_c(P_\chi) \leq 0$, is strongly universal in $\{\frac{2}{3} < \Re e < 1\}$.

We expect such P_χ to be strongly universal in the whole critical strip even when we only have $\sigma_c(P_\chi) \leq 1/2$. To justify our claim let us randomize our series. Let $\{X_n\}$ be a sequence of unimodular independent identically distributed Steinhaus or Rademacher (coin tossing) random variables and $P_X(s) = \sum_{n\geq 1} X_n p_n^{-s}$. Kolmogorov's three-series theorem [15, Chapter 5] implies that P_X converges almost surely in $\mathbb{C}_{\frac{1}{2}}$. To obtain that such series are strongly universal almost surely, we need to obtain more information about their order in the critical strip.

Proposition 5.2. *Let $P_X(s) = \sum_{n\geq 1} X_n p_n^{-s}$, where $\{X_n\}$ is as above. Then, P_X is of sub-logarithmic order in the critical strip and as a consequence is strongly universal, almost surely.*

Proof. We consider the corresponding randomized zeta functions

$$\zeta_X(s) = \prod_{n\geq 1} \frac{1}{1 - X_n p_n^{-s}}. \tag{7}$$

It is easy to see that ζ_X converges absolutely for $\Re es > 1 + \varepsilon$, $\varepsilon > 0$. It is also known that ζ_X and the reciprocal $1/\zeta_X$ converge in $\mathbb{C}_{\frac{1}{2}}$, almost surely. For Steinhaus random variables $(X_1, X_2, \dots) \in \mathbb{T} \times \mathbb{T} \times \dots$ this can be obtained from the work of Helson [9] or as an application of Menchoff's theorem [3]. In the case of Rademacher random variables $X_n = r_n(t) = \operatorname{sign}(\sin(2\pi 2^n t))$, $0 < t < 1$ this has been done by Carlson and Wintner [7, 20].

Starting from (7) one can prove that there exists an absolutely convergent Dirichlet series $F(s, X)$ in $\mathbb{C}_{\frac{1}{2}}$ such that:

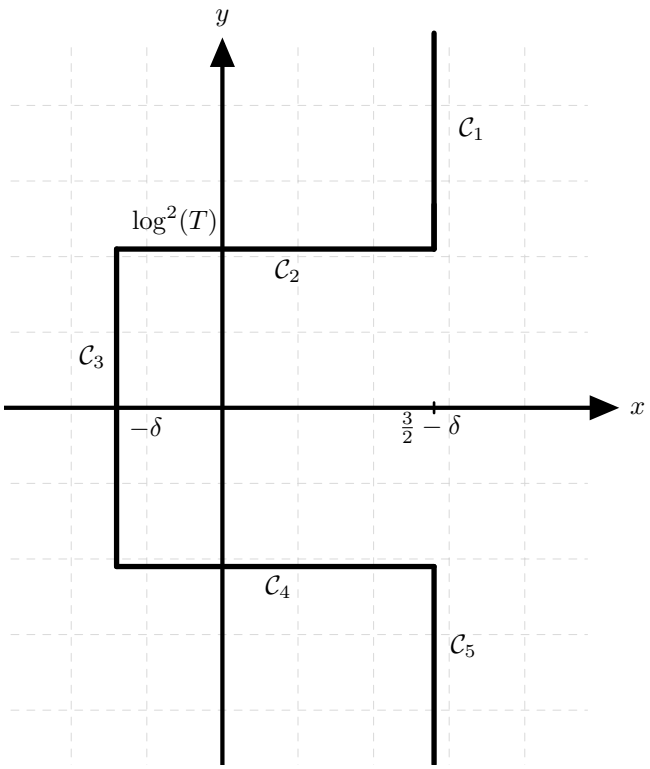
$$\log \zeta_X - P_X = F, \qquad \text{in} \qquad \mathbb{C}_{\frac{1}{2}}.$$

Using the Borel–Carathéodory theorem in a similar manner as in the proof of the implication the Riemann hypothesis implies the Lindelöf hypothesis, [18, Chapter XIV], we obtain that for all $\varepsilon > 0$

$$|P_X(\sigma + it)| = O\left((\log t)^{2-2\sigma+\varepsilon}\right), \qquad t \to \infty,$$

uniformly for $\sigma \geq \sigma_0 > \frac{1}{2}$.

Fix such X_0, the fact that $P_{X_0} \in \mathcal{D}_{\text{dens}}$ follows immediately from Proposition 3.5. It remains to show that $P_{X_0} \in \mathcal{D}_{\text{w.a.}}(\frac{1}{2})$. We will work as in Lemma 3.1. Let $T > 2$, we set $X = T^{\varepsilon}$, $\tau = \delta$ and $\sigma \geq \sigma_0 > \frac{1}{2} + 2\delta$, where ε, $\delta > 0$ are sufficiently small. We consider the contour $\mathcal{C} = \cup_{i=1}^{5} \mathcal{C}_i$, where



We observe that

$$\sum_n X_n p_n^{-s} e^{-\frac{p_n}{X}} = \frac{1}{2\pi i}\int_{\frac{3}{2}-\delta-i\infty}^{\frac{3}{2}-\delta+i\infty} P_{X_0}(s+w)X^w\Gamma(w)\,dw,$$

$$\int_{\mathcal{C}_i} |P_{X_0}(s+w)X^w\Gamma(w)|\,dw \lesssim 1, \qquad i=1,2,4,5.$$

To prove that the integral over the line segment $\mathcal{C}_3$ is bounded note that our function P_{X_0} is of zero order uniformly in $\mathbb{C}_{\frac{1}{2}+\delta}$, thus

$$\int_{\mathcal{C}_3} |P_{X_0}(s+w)X^w\Gamma(w)|\,dw \lesssim T^{\frac{\varepsilon}{2}-\varepsilon} \lesssim 1.$$

Applying Cauchy's theorem and the Montgomery-Vaughan inequality as in Lemma 3.1, we obtain

$$\frac{1}{T}\int_0^T |P_{X_0}(\sigma+it)|^2 dt \lesssim 1+\sum_n p_n^{1-2\sigma}e^{-2\frac{p_n}{X}} \lesssim \sum_n n^{1-2\sigma}e^{-2\frac{n}{X}} \lesssim 1+T^{-1+\varepsilon(2-(1+\delta))} \lesssim 1.$$

Therefore $P_{X_0} \in \mathcal{D}_{\text{w.a.}}(\frac{1}{2})$. □

Question 5.3. Is it true that if the series P_X converges, then it will be strongly universal in $\{\frac{1}{2} < \Re e < 1\}$?

In view of the proof of Proposition 5.2, this question is clearly linked to the order of the Dirichlet series in $\mathbb{C}_{1/2}$.

Question 5.4. Let $\alpha > 0$. What is the order of $\sum_{n\geq 1}(-1)^n p_n^{-s}$ in $\mathbb{C}_\alpha$?

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