Spectrum of hyperbolic and parabolic composition operators

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SPECTRUM OF HYPERBOLIC AND PARABOLIC WEIGHTED COMPOSITION OPERATORS

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Abstract. We compute the spectrum of a weighted composition operator induced by a hyperbolic or a parabolic symbol when it acts on the Fréchet space of holomorphic functions on the unit disc. We discuss situations where these results extend and do not extend to the unit ball.

Contents

1. Introduction

This paper is a contribution to a series of works which intend to compute the spectrum of weighted composition operators acting on spaces of holomorphic functions. Let Ω be a domain in $\mathbb{C}^d$, and $\mathrm{Hol}(\Omega)$ the set of all complex-valued holomorphic functions on Ω.

For $\varphi : \Omega \to \Omega$ holomorphic and $m \in \mathrm{Hol}(\Omega)$, the weighted composition operator with symbol φ and multiplier m is defined as

$$W_{m,\varphi}(f) = m(f \circ \varphi), \quad f \in \mathrm{Hol}(\Omega).$$

We shall only consider the cases where $\Omega = \mathbb{D}$ is the unit disc or $\Omega = \mathbb{B}_d$ is the (euclidean) unit ball in $\mathbb{C}^d$. There is a huge literature on this subject (see e.g. [12, 17] for a general study, or the recent articles [7, 8]), in particular when they are considered as acting on Banach spaces of holomorphic functions like Hardy spaces or Bergman spaces.

In this paper, we focus on the Fréchet space $\mathrm{Hol}(\Omega)$ endowed with the compact-open topology. When $\Omega = \mathbb{D}$, the spectrum of $W_{m,\varphi}$ has already been thoroughly studied in [3, 2, 1]. Our aim in this paper is to complete the picture by filling the cases left open in these papers and by studying how the results can be extended to the ball.

The results obtained so far depend on the nature of the Denjoy-Wolff point of the symbol and on the behaviour of the multiplier near this attractive fixed point. Recall that for a holomorphic self-map φ of $\mathbb{D}$, one of the following two properties hold.

- The map φ has a fixed point ξ inside $\mathbb{D}$. In that case, φ is said to be *elliptic*. If φ is not an automorphism, then the sequence of iterates $(\varphi^{[n]})$ of φ converges to ξ in $\mathrm{Hol}(\mathbb{D})$.
- The map φ has no fixed point in $\mathbb{D}$. In that case, there exists $\xi \in \mathbb{T}$ such that $(\varphi^{[n]})$ converges to ξ in $\mathrm{Hol}(\mathbb{D})$. Moreover, φ admits an angular derivative $\varphi'(\xi) \in (0, 1]$ at ξ. The symbol φ is called *parabolic* if $\varphi'(\xi) = 1$ and *hyperbolic* if $\varphi'(\xi) < 1$.

Elliptic automorphisms behave very differently and we refer to [1] for results concerning them. Regarding the other cases, assuming some regularity of m and φ at the Denjoy-Wolff point ξ if it belongs to $\mathbb{T}$, the following table summarizes the results of [2] on the point spectrum of $W_{m,\varphi}$.

	Elliptic, $\varphi'(\xi) \neq 0$	Elliptic, $\varphi'(\xi) = 0,\ \varphi \neq \xi$	Hyperbolic	Parabolic
$m = 1$	$\{\varphi'(\xi)^n : n \in \mathbb{N}\}$	$\{1\}$	$\mathbb{C}^*$	$\mathbb{C}^*$
$m(\xi) \neq 0$	$\{m(\xi)\varphi'(\xi)^n : n \in \mathbb{N}\}$	$\{m(\xi)\}$	$\mathbb{C}^*$	?
$m(\xi) = 0$	$\varnothing$	$\varnothing$	?	?

TABLE 1. Results in the disc

Moreover, if φ is not an automorphism, then $\sigma(W_{m,\varphi}) = \sigma_p(W_{m,\varphi}) \cup \{0\}$. Otherwise, $\sigma(W_{m,\varphi}) = \sigma_p(W_{m,\varphi})$.

The cases in grey are left open in [2], even if some interesting examples are given. Our first main result will be that, for these three cases, under natural regularity assumptions on φ and m (precise statements will be given later on), $\sigma_p(W_{m,\varphi}) = \mathbb{C}^*$. This property keeps true if we assume that ξ is a pole for m, again assuming some regularity assumptions. As in [2], the proof will be based on a careful study on how $\varphi^{[n]}(z)$ goes to ξ and on a renormalization of $m(\varphi^{[n]}(z))$. Nevertheless, we will have to change the normalization for parabolic maps and we will need much more precise estimates, leading to replace the results of Valiron [18], Pommerenke [15] and Baker-Pommerenke [4] by those of Bourdon-Shapiro [9].

Regarding the several variables situation, we will mainly concentrate on the two-dimensional case. A similar classification occurs and we will be able, when $m(\xi) \neq 0$, to prove similar statements, using now [5] and [11]. When $m(\xi) = 0$, the situation will become much more involved. Indeed, a new difficulty arises : in dimension two or more, we cannot easily factorize a holomorphic function vanishing at some point.

We will obtain some partial results, and also exhibit a very simple example of a weight m and symbol φ, regular at the Denjoy-Wolff point, with $m(\xi) = 0$, such that $\sigma_p(W_{m,\varphi}) = \varnothing$. Hence the results dramatically break down on the ball.

We end up this introduction with the following lemma, that we will use repeatedly.

Lemma 1.1. *Let $m_1, m_2 \in \mathrm{Hol}(\Omega)$ and $\varphi : \Omega \to \Omega$ holomorphic. Then*

$$\sigma_p(W_{m_1,\varphi}) \cdot \sigma_p(W_{m_2,\varphi}) \subset \sigma_p(W_{m_1 m_2,\varphi}).$$

Proof. Let $\lambda_1 \in \sigma_p(W_{m_1,\varphi})$ and $\lambda_2 \in \sigma_p(W_{m_2,\varphi})$. By definition, there exist two nonzero maps f_1, f_2 of $\mathrm{Hol}(\Omega)$ such that

$$m_1(f_1 \circ \varphi) = \lambda_1 f_1, \qquad m_2(f_2 \circ \varphi) = \lambda_2 f_2.$$

Hence, considering $f = f_1 f_2$ and $m = m_1 m_2$, we get

$$m(f \circ \varphi) = m_1 m_2 (f_1 \circ \varphi)(f_2 \circ \varphi) = m_1 (f_1 \circ \varphi) m_2 (f_2 \circ \varphi) = \lambda_1 f_1 \lambda_2 f_2 = \lambda_1 \lambda_2 f.$$

Finally, we obtain $\lambda_1 \lambda_2 \in \sigma_p(W_{m,\varphi})$, the desired result. □

Notation. Throughout the paper, if $f, g : X \to \mathbb{R}$ are two functions defined on the same set X, the notation $f \lesssim g$ will mean that there exists some constant $C > 0$ such that $f \leq Cg$ on X. The notation $f \asymp g$ will simply mean $f \lesssim g$ and $g \lesssim f$.

2. On the disc

We begin this section by recalling some definitions.

Definition 2.1. The *Stolz angle* with vertex $\xi \in \mathbb{T}$ and radius $R > 0$ is

$$S(\xi, R) := \left\{ z \in \mathbb{D} : \frac{|z - \xi|}{1 - |z|} < R \right\}.$$

The *horodisc* of vertex $\xi \in \mathbb{T}$ and radius $\alpha > 0$ is

$$\mathcal{H}(\xi, \alpha) := \{ z \in \mathbb{D} : |z - \xi|^2 < \alpha(1 - |z|^2) \} = D\left(\frac{\xi}{1+\alpha}, \frac{\alpha}{1+\alpha} \right)$$

where $D(w, r)$ is the (euclidean) disc of center w and radius $r > 0$. The *right half-plane* of $\mathbb{C}$ is $\mathbb{H} := \{ z \in \mathbb{C} : \Re(z) > 0 \}$.

In what follows, as it is often the case, we will transfer the problem on the right half-plane $\mathbb{H}$ through the Cayley transform

$$\tau(z) = \frac{1+z}{1-z}.$$

For $\varphi : \mathbb{D} \to \mathbb{D}$ holomorphic with Denjoy-Wolff point at 1, we will set $\Phi = \tau \circ \varphi \circ \tau^{-1}$, which is a self-map of $\mathbb{H}$, with ∞ as attractive fixed point.

Setting $w = \tau(z)$, we will repeatedly use

$$1 - z = \frac{2}{1+w} \quad \text{and} \quad 1 - \varphi(z) = \frac{2}{1 + \Phi(w)}.$$

The analogues of the Stolz angles (with vertex 1) on the right half-plane are the *angular sectors* $\{w \in \mathbb{H} : \arg(w) < \alpha\}$, for $\alpha \in (0, \pi/2)$. We will still call them Stolz angles at ∞.

As mentioned in the introduction, we will sometimes need that φ has some regularity properties at its Denjoy-Wolff point. Here are the relevant definitions.

Definition 2.2. Let $\xi \in \mathbb{T}$, and $\varphi : \mathbb{D} \cup \{\xi\} \to \mathbb{D} \cup \{\xi\}$ a holomorphic map on $\mathbb{D}$. For $n \in \mathbb{N}$ and $0 \leq \varepsilon < 1$, we say that $\varphi \in \mathcal{C}^{n+\varepsilon}(\xi)$ if φ has a n^{th} derivative at ξ, and if

$$\varphi(z) = \sum_{k=0}^{n} \frac{\varphi^{(k)}(\xi)}{k!}(z-\xi)^k + o(|z-\xi|^{n+\varepsilon}), \qquad z \in \mathbb{D}.$$

Definition 2.3. A holomorphic map $\varphi : \mathbb{D} \to \mathbb{D}$ is *regular* if φ is continuous on $\overline{\mathbb{D}}$, and if $\varphi(\overline{\mathbb{D}}) \subset \mathbb{D} \cup \{\xi\}$, where ξ is the Denjoy-Wolff point of φ.

We assume now, and for the rest of the paper, that $\xi = 1$ and $\varphi(1) = 1$.

If we translate Definition 2.2 onto the map Φ, it means that we may write

$$\Phi(w) = a_0 w + a_1 + \sum_{j=2}^{n-2} \frac{a_j}{(w+1)^j} + o\left(\frac{1}{|w+1|^{n-2+\varepsilon}}\right),$$

with $a_0, \cdots, a_{n-2} \in \mathbb{C}$.

2.1. **Hyperbolic symbol with nonvanishing weight at the Denjoy-Wolff point.** We begin by a well-known result about the iterates of hyperbolic selfmaps of $\mathbb{D}$ within a Stolz angle.

Lemma 2.4 ([12, Lemma 2.66])**.** *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be a hyperbolic holomorphic map, with Denjoy-Wolff point $\xi \in \mathbb{T}$. Then, for all compact subsets K of $\mathbb{D}$, there exists $R > 0$ such that for all $n \in \mathbb{N}_0$,*

$$\varphi^{[n]}(K) \subset S(\xi, R).$$

The computation of $\sigma_p(W_{m,\varphi})$ when φ is hyperbolic and m does not vanish at the Denjoy-Wolff point ξ of φ has already been done in [2], where the following theorem is obtained. Recall that ℓ is the *angular limit* of $f \in \mathrm{Hol}(\mathbb{D})$ at $\xi \in \mathbb{T}$ if there exists a Stolz angle S such that $f(z) \to \ell$ when $z \to \xi$ and $z \in S$.

Theorem 2.5 ([2, Theorem 6.3])**.** *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be a hyperbolic holomorphic map, with Denjoy-Wolff point ξ, and $m \in \mathrm{Hol}(\mathbb{D})$ with a non-zero angular limit $m(\xi)$, and a finite angular derivative at ξ. Then, $\sigma_p(W_{m,\varphi}) = \mathbb{C}^*$.*

We shall improve it by slightly relaxing the assumptions. This will be important to handle the case $m(\xi) = 0$.

Theorem 2.6. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be a hyperbolic holomorphic map, with Denjoy-Wolff point ξ, and $m \in \mathrm{Hol}(\mathbb{D})$ with non-zero angular limit $m(\xi)$ at ξ. Assume that*

$$m(z) = m(\xi) + \Lambda(z), \tag{1}$$

and that for all Stolz angles S with vertex ξ, there exist $\beta \in (0,1)$ and a nondecreasing map $g : \mathbb{R} \to \mathbb{R}_+$ such that

$$\forall x \in (0,1),\ \sum_{n\geq 0} g(x^n) < +\infty \quad \textit{and} \quad z \in S,\ |z| \geq 1-\beta \implies |\Lambda(z)| \leq g(|z-\xi|).$$

Then, $\sigma_p(W_{m,\varphi}) = \mathbb{C}^*$.

The assumptions of Theorem 2.5 mean that (1) is true with $\Lambda(z) = m'(\xi)(z-\xi) + o(z-\xi)$. Therefore, we may choose $g(x) = Mx$ for a suitable $M > 0$. It is nevertheless natural (think at Definition 2.2) to be able to use $g(x) = Mx^\varepsilon$ for some $\varepsilon > 0$, or even $g(x) = \frac{M}{|\log(x)|^{1+\eta}}$, $\eta > 0$.

Proof. Since φ and m are non-constant maps, if $m(f \circ \varphi) = 0$, then $f = 0$, so we have $0 \notin \sigma_p(W_{m,\varphi})$. For $n \in \mathbb{N}$ and $z \in \mathbb{D}$, we define

$$f_n(z) = \frac{1}{m(\xi)^n} \prod_{k=0}^{n-1} m(\varphi^{[k]}(z)).$$

Hence,

$$\begin{aligned} W_{m,\varphi}(f_n)(z) &= m(z)\frac{1}{m(\xi)^n} \prod_{k=0}^{n-1} m(\varphi^{[k+1]}(z)) \\ &= m(\xi)\frac{1}{m(\xi)^{n+1}} \prod_{k=0}^{n} m(\varphi^{[k]}(z)) = m(\xi) f_{n+1}(z). \end{aligned}$$

We show that $(f_n)_{n\geq 0}$ converges uniformly on all compact subsets of $\mathbb{D}$. Let K be such a compact. By Lemma 2.4, there exists a Stolz angle S with vertex ξ such that for all $n \in \mathbb{N}_0$, $\varphi^{[n]}(K) \subset S$. For all $z \in K$,

$$\left| \frac{m(\varphi^{[n]}(z))}{m(\xi)} - 1 \right| = \frac{\left| m(\varphi^{[n]}(z)) - m(\xi) \right|}{|m(\xi)|} \leq \frac{1}{|m(\xi)|} g\left(\left| \varphi^{[n]}(z) - \xi \right| \right).$$

Let us now consider $\alpha > 0$ such that $K \subset \mathcal{H}(\xi, \alpha)$. Using Julia's lemma, and Julia-Wolff-Caratheodory theorem ([10], [17]), for all $n \in \mathbb{N}$,

$$\varphi^{[n]}(K) \subset \mathcal{H}(\xi, \alpha\varphi'(\xi)^n).$$

Therefore, for all $n \in \mathbb{N}$ and $z \in K$,

$$\left| \varphi^{[n]}(z) - \xi \right|^2 < 2\alpha\varphi'(\xi)^n.$$

Hence, if n is sufficiently large and $x \in (\varphi'(\xi)^{1/2}, 1)$,

$$g(\left| \varphi^{[n]}(z) - \xi \right|) \leq g(x^n).$$

Finally, $(f_n)_{n\geq 0}$ converges uniformly on K to a nonzero function f, which satisfies $W_{m,\varphi}(f) = m(\xi)f$, so that $m(\xi) \in \sigma_p(W_{m,\varphi})$. Since $\sigma_p(C_\varphi) = \mathbb{C}^*$ (see [2, Theorem 6.1]), by Lemma 1.1, we obtain $\mathbb{C}^* \subset \sigma_p(W_{m,\varphi})$. □

2.2. **Hyperbolic symbol with vanishing weight at the Denjoy-Wolff point.** In this section, we intend to prove the following result.

Theorem 2.7. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be a hyperbolic regular holomorphic map, with Denjoy-Wolff point ξ, and $m \in \mathrm{Hol}(\mathbb{D})$. Assume that there exist $\varepsilon \in (0,1)$, $a \in \mathbb{C}^*$, $\alpha \in \mathbb{R}^*$ and $\Lambda \in \mathrm{Hol}(\mathbb{D})$ such that $\varphi \in \mathcal{C}^{1+\varepsilon}(\xi)$,*

$$m(z) = a(z-\xi)^\alpha(1+\Lambda(z)),$$

and for all Stolz angles S with vertex ξ, there exist $\beta \in (0,1)$ and a nondecreasing map $g : \mathbb{R} \to \mathbb{R}_+$ such that

$$\forall x \in (0,1),\ \sum_{n\geq 0} g(x^n) < +\infty \quad \textit{and} \quad z \in S,\ |z| \geq 1-\beta \implies |\Lambda(z)| \leq g(|z-\xi|).$$

Then $\sigma_p(W_{m,\varphi}) = \mathbb{C}^*$.

In particular, we may apply this theorem when m is meromorphic at ξ.

Proof. We shall prove the theorem on $\mathbb{H}$, by assuming that $\xi = 1$. Let $\Phi : \mathbb{H} \to \mathbb{H}$ be holomorphic, which extends to a continuous function on $\overline{\mathbb{H}} \cup \{\infty\}$ and maps $\Phi(\overline{\mathbb{H}})$ into $\mathbb{H}$. We shall also assume that Φ may be written

$$\Phi(w) = \lambda w + \Theta(w),$$

with $\lambda > 1$ and $|\Theta(w)| \lesssim |w+1|^{1-\varepsilon}$. Let also $m : \mathbb{H} \to \mathbb{C}$ be such that

$$m(w) = \frac{a}{(w+1)^\alpha}\big(1+\Lambda(w)\big).$$

In this context, for all Stolz angles S with vertex ∞, there exist $C > 0$ and a non-decreasing map $g : \mathbb{R} \to \mathbb{R}_+$ such that

$$\forall x \in (0,1),\ \sum_{n\geq 0} g(x^n) < +\infty \quad \text{and} \quad w \in S,\ |w| \geq C \implies |\Lambda(w)| \leq g(1/|w|).$$

By the linear fractional model (see [9, Theorem 4.9]), there exist a holomorphic self-map $H : \mathbb{H} \to \mathbb{H}$ and $c > 1$ such that $H \circ \Phi = cH$ and $H(w) = (w+1)(1+G(w))$ with $|G(w)| \lesssim |w+1|^{-\varepsilon}$ for $|w|$ large enough.

Our strategy for the proof is the following. We will first consider the weighted composition operator $W_{H^{-\alpha},\Phi}$ and show that it admits a nonzero eigenvalue by exhibiting an eigenvector. Next, we will set $m_1 = mH^\alpha$ and we will apply Theorem 2.6 to $W_{m_1,\Phi}$. We will finally conclude by Lemma 1.1.

Step 1 : We show that $\sigma_p(W_{H^{-\alpha},\Phi}) \cap \mathbb{C}^* \neq \varnothing$. Set $\lambda = \alpha/2\log(c)$, and

$$F = \exp[\lambda \log^2(H)].$$

Then,

$$\begin{aligned} W_{H^{-\alpha},\Phi}(F) &= \frac{\exp[\lambda \log^2(H\circ\Phi)]}{H^\alpha} = \frac{\exp[\lambda \log^2(cH)]}{H^\alpha} \\ &= \frac{\exp[\lambda \log^2(c) + 2\lambda \log(c)\log(H) + \lambda \log^2(H)]}{H^\alpha} \\ &= e^{\lambda \log^2(c)} H^{2\lambda \log(c)-\alpha} F = e^{\lambda \log^2(c)} F. \end{aligned}$$

Since $e^{\lambda \log^2(c)} \neq 0$, we obtain the desired result.

Step 2 : We show that $\sigma_p(W_{m_1,\Phi}) = \mathbb{C}^*$. Indeed, using the estimates on m and H, we know that $m_1 = mH^\alpha$ can be written

$$m_1(w) = a(1+\Lambda(w))(1+G(w))^\alpha.$$

In addition, $(1+G(w))^\alpha = 1 + \alpha G(w) + \eta(w)$, with $|\eta(w)| \lesssim |w+1|^{-2\varepsilon}$. Finally, we may write

$$m_1(w) = a + \tilde{\Lambda}(w),$$

with $\tilde{\Lambda} = a(\Lambda + \alpha G + \alpha G\Lambda + \eta + \eta\Lambda)$. By sums and products, for all Stolz angles S with vertex ∞, there exist $\tilde{C} > 0$ and a nondecreasing map $\tilde{g}$ such that for all $x \in (0,1)$, $\sum \tilde{g}(x^n) < +\infty$, and

$$w \in S,\ |w| \geq \tilde{C} \quad \Longrightarrow \quad |\tilde{\Lambda}(w)| \leq \tilde{g}(1/|w|).$$

We obtain the result by applying Theorem 2.6.

Conclusion : Using Lemma 1.1 together with the results of Step 1 and Step 2, we get $\sigma_p(W_{m,\Phi}) = \mathbb{C}^*$. □

2.3. **General discussion on parabolic symbols.** We turn to the case where the symbol φ is a parabolic self-map of $\mathbb{D}$, and we will always assume that 1 is its Denjoy-Wolff point. The strategy to prove that, under some regularity assumptions, $\sigma_p(W_{m,\varphi}) = \mathbb{C}^*$ is the same as in the hyperbolic case.

We will first prove that this holds true if m does not vanish at 1 by renormalizing the sequence $m(\varphi^{[n]}(z))$. Then, we will handle the case where m vanishes at 1 by exhibiting an eigenvector when m has a specific form related to the symbol, and we will conclude in the general case by factorizing the multiplier.

Nevertheless, every step becomes harder. The key point is that $\varphi^{[n]}(z)$ converges much more slowly to 1 when φ is parabolic, and the convergence of the sequences which come into play will be more difficult to prove.

The linear fractional model gives us now the existence of a map $H : \mathbb{H} \to \mathbb{C}$ such that $H \circ \Phi = H + a$. As in the proof of Theorem 2.7, we will need estimates on the behaviour of H near infinity, and even more precise estimates due to the parabolic nature of the symbol.

Our starting point is again [9], now Theorem 4.12. Bourdon and Shapiro give there a linear fractional model with estimates for φ a parabolic self-map of $\mathbb{D}$ belonging to

$\mathcal{C}^{3+\varepsilon}(1)$, with $\varphi''(1) \neq 0$ and φ regular. Indeed, in such case, we get $\Re(\varphi''(1)) \geq 0$, and we can write

$$\varphi(z) = 1 + (z-1) + \varphi''(1)\frac{(z-1)^2}{2} + \varphi'''(1)\frac{(z-1)^3}{6} + \gamma(z),$$

with $\gamma(z) = o(|z-1|^{3+\varepsilon})$. Setting again $\Phi = \tau \circ \varphi \circ \tau^{-1}$, then for all $w \in \mathbb{H}$,

$$\Phi(w) = w + a + \frac{b}{w+1} + \Theta(w), \qquad \Theta(w) = o\left(\frac{1}{|w+1|^{1+\varepsilon}}\right),$$

with $a = \varphi''(1)$ and $b = \varphi''(1)^2 - 2\varphi'''(1)/3$. If $\Re(\varphi''(1)) = 0$, we may and shall assume that $\Im(\varphi''(1)) > 0$. Bourdon-Shapiro's theorem reads as follows.

Theorem 2.8 ([9], Theorem 4.12)**.**
Let $\varphi : \mathbb{D} \to \mathbb{D}$ be a parabolic holomorphic map, with Denjoy-Wolff point 1. Assume that φ is regular, $\mathcal{C}^{3+\varepsilon}(1)$, and that $\varphi''(1) \neq 0$. Then, there exists a function $H : \mathbb{H} \to \mathbb{C}$ such that $H \circ \Phi = H + a$, and

$$H(w) = w - \frac{b}{a}\log(1+w) + B(w), \qquad w \in \overline{\Omega}, \tag{2}$$

with B holomorphic, bounded on Ω and continuous on $\overline{\Omega}$, where

- *if $\Re(a) > 0$, then $\Omega = \mathbb{H}$,*
- *if $\Re(a) = 0$ and $\Im(a) > 0$, then $\Omega = \mathbb{H}^+ = \{z \in \mathbb{C} : \Re(z) > 0, \Im(z) > 0\}$.*

It turns out that we will need the following enhancement of this theorem, giving a slightly more precise information on the behaviour of H.

Proposition 2.9. *Under the assumptions of Theorem 2.8, there exist $R > 0$, a domain $\Omega \subset \mathbb{H}$ and a map $H : \mathbb{H} \to \mathbb{C}$ such that $H \circ \Phi = H + a$ and*

$$H(w) = w + c - \frac{b}{a}\log(1+w) + \theta(w), \qquad w \in \overline{\Omega}, \tag{3}$$

with $c \in \mathbb{C}$, $\theta : \mathbb{H} \to \mathbb{C}$ holomorphic, satisfying $|\theta(w)| \lesssim |w|^{-\varepsilon/2}$ for $|w|$ sufficiently large and

- *if $\Re(a) > 0$, then $\Omega = \mathbb{H}$,*
- *if $\Re(a) = 0$ and $\Im(a) > 0$, then $\Omega = \mathbb{H}_R = \{z \in \mathbb{C} : \Re(z) > 0, \Im(z) > R\}$.*

Proof. We follow the method of [9] but we have to go a step further in the expansion. We start from the last formula p.74 and [9, Proposition 4.19] : there exists some $R > 0$ such that H is the uniform limit on compact subsets of $\overline{\Omega}$ of the maps H_n defined by

$$\boxed{H_n(w) = w - w_0 - bw\sum_{j=0}^{n-1}\frac{1}{w(j)w_0(j)} + bS_{2,n}(w) + S_{3,n}(w),}$$

with $w_0 \in \Omega$, $w(n) = \Phi^{[n]}(w) + 1$, and

$$S_{2,n}(w) = \sum_{j=0}^{n-1}\frac{w_0 - (H_j(w) - H_0(w))}{w(j)w_0(j)}, \quad S_{3,n}(w) = \sum_{j=0}^{n-1}[\Theta(w(j)) - \Theta(w_0(j))].$$

To achieve the proof, we need to study all the terms of H_n. First, [9, Proposition 4.18] gives us the very useful inequality

$$|w(n)| \gtrsim |w| + n, \quad w \in \Omega, \quad n \geq 0.$$

- Let us start with $S_{3,n}$. For $j \geq 1$ and $w \in \Omega$, if we set $\gamma = \max(|w|, j)$, then

$$|w|^{\varepsilon/2} |\Theta(w(j))| \lesssim \frac{|w|^{\varepsilon/2}}{|w(j)|^{1+\varepsilon}} \lesssim \frac{|w|^{\varepsilon/2}}{(|w|+j)^{1+\varepsilon}} \leq \frac{\gamma^{\varepsilon/2}}{\gamma^{1+\varepsilon}} \leq \frac{1}{\gamma^{1+\varepsilon/2}} \leq \frac{1}{j^{1+\varepsilon/2}}.$$

  Moreover, the sum $\sum \Theta(w_0(j))$ converges, so

$$S_3(w) = \lim_{n\to\infty} S_{3,n}(w) = \sum_{j\geq 0} \Theta(w(j)) - \sum_{j\geq 0} \Theta(w_0(j)) = \theta_3(w) + c_3,$$

  with $|\theta_3(w)| \lesssim |w|^{-\varepsilon/2}$ for $|w|$ large enough.

- We now focus on $S_{2,n}$. For $j \geq 1$ and $w \in \Omega$,

$$\frac{|w|^{\varepsilon}}{|w(j)w_0(j)|} \lesssim \frac{|w|^{\varepsilon}}{(|w|+j)(|w_0|+j)} \leq \frac{\gamma^{\varepsilon}}{\gamma j} \leq \frac{1}{\gamma^{1-\varepsilon} j} \leq \frac{1}{j^{2-\varepsilon}}.$$

  Since $|H_j(w) - H_0(w)| \lesssim \log(j+1)$ (cf. [9, p. 76]), we also get

$$|w|^{\varepsilon} \left| \frac{H_j(w) - H_0(w)}{w(j)w_0(j)} \right| \lesssim \frac{|w|^{\varepsilon} \log(j+1)}{(|w|+j)(|w_0|+j)} \leq \frac{\log(j+1)}{\gamma^{1-\varepsilon} j} \leq \frac{\log(j+1)}{j^{2-\varepsilon}}.$$

  Therefore, we have shown that

$$S_2(w) = \lim_{n\to\infty} S_{2,n}(w) = \theta_2(w) + c_2,$$

  with $|\theta_2(w)| \lesssim |w|^{-\varepsilon}$ for $|w|$ large enough.

For the moment, setting $d_1 = c_2 + c_3 - w_0$ and $\Delta_1 = \theta_2 + \theta_3$, we get

$$\boxed{H(w) = w - bw \sum_{j\geq 0} \frac{1}{w(j)w_0(j)} + d_1 + \Delta_1(w), \qquad |\Delta_1(w)| \lesssim |w|^{-\varepsilon/2}.}$$

Now, note that using Lemma 4.20 of [9], for each $w \in \Omega$ and each $j \geq 0$,

$$|w + ja| \gtrsim (|w| + j), \qquad |w(j) - w - ja| \lesssim \log(j+1).$$

Hence, let us write

$$bw \sum_{j\geq 0} \frac{1}{w(j)w_0(j)} = I_1(w) + I_2(w) + I_3(w) + b \sum_{j\geq 1} \frac{w}{(w+ja)(w_0+ja)},$$

with $I_1(w) = bw/(w+1)(w_0+1)$, and

$$I_2(w) = b \sum_{j\geq 1} \frac{w}{w+ja} \left[\frac{1}{w_0(j)} - \frac{1}{w_0+ja} \right], \quad I_3(w) = b \sum_{j\geq 1} \frac{w}{w_0(j)} \left[\frac{1}{w(j)} - \frac{1}{w+ja} \right].$$

Then,

$$I_1(w) = \frac{b}{w_0+1} - \frac{b}{(w_0+1)(w+1)} = c_4 - \theta_4(w),$$

with $|\theta_4(w)| \lesssim |w|^{-1}$ for $|w|$ large. In addition,

$$I_2(w) = \underbrace{b\sum_{j\geq 1}\left[\frac{1}{w_0(j)} - \frac{1}{w_0+ja}\right]}_{=c_5} - \underbrace{b\sum_{j\geq 1}\frac{ja}{w+ja}\left[\frac{1}{w_0(j)} - \frac{1}{w_0+ja}\right]}_{=\theta_5(w)},$$

with

$$\left|\frac{ja}{w+ja}\left[\frac{1}{w_0(j)} - \frac{1}{w_0+ja}\right]\right| \lesssim \frac{j\log(j+1)}{(|w|+j)(|w_0|+j)^2} \leq \frac{\log(j+1)}{|w|^{1/2}\, j^{3/2}},$$

so $|\theta_5(w)| \lesssim |w|^{-1/2}$. Finally, to study $I_3(w)$, we only have to see that

$$\left|\frac{w}{w_0(j)}\left[\frac{1}{w(j)} - \frac{1}{w+ja}\right]\right| \lesssim \frac{|w|\log(j+1)}{(|w_0|+j)(|w|+j)^2} \leq \frac{\log(j+1)}{|w|^{1/2}\, j^{3/2}}.$$

Thus, $|I_3(w)| \lesssim |w|^{-1/2}$. Setting $d_2 = d_1 - c_4 - c_5$ and $\Delta_2 = \Delta_1 + \theta_4 + \theta_5 - I_3$, we get

$$\boxed{H(w) = w - b\sum_{j\geq 1}\frac{w}{(w+ja)(w_0+ja)} + d_2 + \Delta_2(w), \qquad |\Delta_2(w)| \lesssim |w|^{-\varepsilon/2}.}$$

Next, note that

$$\sum_{j\geq 1}\frac{w}{(w+ja)(w_0+ja)} = \underbrace{\sum_{j\geq 1}\left[\frac{1}{w_0+ja} - \frac{1}{w+ja}\right]}_{=J_1(w)} + \underbrace{w_0\sum_{j\geq 1}\frac{1}{(w+ja)(w_0+ja)}}_{=J_2(w)}.$$

For the second term, we have

$$|J_2(w)| \leq |w_0|\sum_{j\geq 1}\left|\frac{1}{(w+ja)(w_0+ja)}\right| \leq \frac{|w_0|}{|w|^{1/2}}\sum_{j\geq 1}\frac{1}{j^{3/2}} \lesssim \frac{1}{|w|^{1/2}},$$

and for the first one,

$$\begin{aligned} &J_1(w) - \int_1^\infty \left[\frac{1}{w_0+ta} - \frac{1}{w+ta}\right]\,\mathrm{d}t \\ &= \underbrace{\sum_{j\geq 1}\int_j^{j+1}\frac{a(t-j)}{(w_0+ja)(w_0+ta)}\,\mathrm{d}t}_{=c_6} - \underbrace{\sum_{j\geq 1}\int_j^{j+1}\frac{a(t-j)}{(w+ja)(w+ta)}\,\mathrm{d}t}_{=\theta_6(w)}, \end{aligned}$$

with, for $j \leq t \leq j+1$,

$$\left|\frac{a(t-j)}{(w_0+ja)(w_0+ta)}\right| \lesssim \frac{1}{(|w_0|+j)^2} \leq \frac{1}{j^2}$$

$$\text{and} \quad \left|\frac{a(t-j)}{(w+ja)(w+ta)}\right| \lesssim \frac{1}{(|w|+j)^2} \leq \frac{1}{|w|^{1/2}\, j^{3/2}}.$$

Finally,

$$J_1(w) - \int_1^\infty \left[\frac{1}{w_0+ta} - \frac{1}{w+ta}\right] \mathrm{d}t = c_6 + \theta_6(w), \qquad |\theta_6(w)| \lesssim \frac{1}{|w|^{1/2}}.$$

Therefore, setting $d_3 = d_2 - c_6$ and $\Delta_3 = \Delta_2 + \theta_6 - J_2$, we obtain

$$\boxed{H(w) = w - b\int_1^\infty \left[\frac{1}{w_0+ta} - \frac{1}{w+ta}\right] \mathrm{d}t + d_3 + \Delta_3(w), \qquad |\Delta_3(w)| \lesssim |w|^{-\varepsilon/2}.}$$

But it is well-known that

$$\int_1^\infty \left[\frac{1}{w_0+ta} - \frac{1}{w+ta}\right] \mathrm{d}t = \frac{1}{a}\log\left(\frac{w+a}{w_0+a}\right),$$

so

$$\boxed{H(w) = w - \frac{b}{a}\log\left(\frac{w+a}{w_0+a}\right) + d_3 + \Delta_3(w), \qquad |\Delta_3(w)| \lesssim |w|^{-\varepsilon/2}.}$$

We conclude, using the identity

$$\log\left(\frac{w+a}{w_0+a}\right) - \log(w+1) = \underbrace{\log\left(1 + \frac{a-1}{w+1}\right)}_{=\theta_7(w)} - \underbrace{\log(w_0+a)}_{=c_7},$$

with $|\theta_7(w)| \lesssim |w|^{-1}$ for $|w|$ large. Finally, if $c = d_3 + bc_7/a$ and $\theta = \Delta_3 - b\theta_7/a$, then

$$\boxed{H(w) = w - \frac{b}{a}\log(w+1) + c + \theta(w), \qquad |\theta(w)| \lesssim |w|^{-\varepsilon/2}.} \qquad \square$$

Regarding the behaviour of the sequences $(\varphi^{[n]}(z))$, we will need the following lemma.

Lemma 2.10 ([6, Lemma 5.8])**.** *Under the assumptions of Theorem 2.8, for all $w \in \mathbb{H}$,*

$$\Phi^{[n]}(w) = an + \frac{b}{a}\log(n) + B_n(w),$$

with B_n bounded (independently of n) on all compact subsets of $\mathbb{H}$.

2.4. **Parabolic symbols with nonvanishing weight at the Denjoy-Wolff point.** The preliminary results of the previous section allow us to get the following theorem. Here again, we set $a = \varphi''(1)$, $b = \varphi''(1)^2 - 2\varphi'''(1)/3$.

Theorem 2.11. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be a parabolic holomorphic map, with Denjoy-Wolff point 1. Assume that φ is regular, $\mathcal{C}^{3+\varepsilon}(1)$, and that $\varphi''(1) \neq 0$. Let $m \in \mathrm{Hol}(\mathbb{D})$. Assume also that there exist a neighborhood U of 1, $r > 0$, $\delta > 0$ and $\Lambda : \mathbb{D} \to \mathbb{C}$ such that setting $\mathbb{D}_r = \tau^{-1}(\mathbb{H}_r)$ and*

$$V(r) = \begin{cases} U \cap \mathbb{D} & \text{if } \Re(\varphi''(1)) > 0, \\ U \cap \mathbb{D}_r & \text{if } \Re(\varphi''(1)) = 0 \text{ and } \Im(\varphi''(1)) > 0, \end{cases}$$

the map m satisfies

$$\forall z \in \mathbb{D}, \quad m(z) = m(1) + \Lambda(z),$$

with $m(1) \neq 0$*, and*

$$\forall z, z' \in V(r), \qquad |\Lambda(z) - \Lambda(z')| \lesssim |z-1|^{1+\delta} + |z'-1|^{1+\delta} + |z-z'|^{\frac{1}{2}+\delta}$$

Then, $\sigma_p(W_{m,\varphi}) = \mathbb{C}^*$.

In particular, our assumptions imply that m admits a nonzero limit at 1 along $\mathbb{D}_r$. They are satisfied if $m \in \mathcal{C}^{1+\delta}(1)$ and $m(1) \neq 0$.

Proof. We may assume that $r \geq R$, where R is given by Proposition 2.9. We move on $\mathbb{H}$ and consider $m \in \mathrm{Hol}(\mathbb{H})$ which can be written $m(w) = m(\infty) + \Lambda(w)$, with $m(\infty) \neq 0$ and for all $w \in \Omega$,

$$|\Lambda(w) - \Lambda(w')| \lesssim \frac{1}{|w+1|^{1+\delta}} + \frac{1}{|w'+1|^{1+\delta}} + \left|\frac{w-w'}{(w+1)(w'+1)}\right|^{\frac{1}{2}+\delta}, \tag{4}$$

where $\Omega = \mathbb{H} \cap \{|w| > M\}$ if $\Re(\varphi''(1)) > 0$, and $\Omega = \mathbb{H}_r \cap \{|w| > M\}$ otherwise, with $M > 0$ sufficiently large.

Arguing as in the proof of Theorem 2.6, it is sufficient to show that $m(\infty)$ is an eigenvalue of $W_{m,\Phi}$. Let K be a compact subset of $\mathbb{H}$ containing 1. We claim that $\Phi^{[n]}(K) \subset \Omega$ provided n is large enough. This is a consequence of the Denjoy-Wolff theorem if $\Re(\varphi''(1)) > 0$. If $\Re(\varphi''(1)) = 0$, we set $w_0 = 1 + i(r+1)$ and consider $\rho > 0$ so that $K \subset \Delta(w_0, \rho)$, where $\Delta(\zeta, r)$ is the pseudo-hyperbolic disc with center ζ and radius r. Using Schwarz-Pick's lemma, for all $n \in \mathbb{N}$,

$$\Phi^{[n]}(K) \subset \Delta(\Phi^{[n]}(w_0), \rho) = \Re(\Phi^{[n]}(w_0))\Delta(1, \rho) + \Im(\Phi^{[n]}(w_0)).$$

However, the sequence $(\Re(\Phi^{[n]}(w_0)))_{n\geq 0}$ is bounded (see [9, Proposition 4.25]), and $\Im(\Phi^{[n]}(w_0)) \xrightarrow[n\to\infty]{} +\infty$. Thus, for n large, $\Phi^{[n]}(K) \subset \Omega$.

In particular, there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$, $m(\Phi^{[k]}(1)) \neq 0$. Hence, for all $w \in \mathbb{H}$ and $n \geq n_0$, we may set

$$F_n(w) = \prod_{k=n_0}^{n} \frac{m(\Phi^{[k]}(w))}{m(\Phi^{[k]}(1))}.$$

We intend to show that (F_n) converges uniformly on K. For $w \in K$ and n large,

$$\begin{aligned}\left|\frac{m(\Phi^{[n]}(w))}{m(\Phi^{[n]}(1))} - 1\right| &\lesssim \left|m(\Phi^{[n]}(w)) - m(\Phi^{[n]}(1))\right| \\ &\leq \left|\frac{1}{\Phi^{[n]}(w)+1}\right|^{1+\delta} + \left|\frac{1}{\Phi^{[n]}(1)+1}\right|^{1+\delta} + \left|\frac{B_n(w) - B_n(1)}{(\Phi^{[n]}(w)+1)(\Phi^{[n]}(1)+1)}\right|^{\frac{1}{2}+\delta}.\end{aligned}$$

Since $|\Phi^{[n]}(w)+1|, |\Phi^{[n]}(1)+1| \geq C(K)n$, an application of Lemma 2.10 yields the desired convergence. We conclude since

$$W_{m,\Phi}(F)(w) = m(w)F(\Phi(w)) = m(w)\lim_{n\to+\infty}\prod_{k=n_0}^{n}\frac{m(\Phi^{[k+1]}(w))}{m(\Phi^{[k]}(1))}$$
$$= \lim_{n\to+\infty} m(\Phi^{[n+1]}(1))\prod_{k=n_0}^{n+1}\frac{m(\Phi^{[k]}(w))}{m(\Phi^{[k]}(1))} = m(\infty)F(w).\square$$

When φ is a parabolic automorphism (hence, excluding the identity), the problem is much simpler to solve. Indeed, in such case, there exists $a \in \mathbb{R}^*$ such that for all $w \in \mathbb{H}$, $\Phi(w) = w + ia$. Hence, for all compact subsets K of $\mathbb{H}$ containing 1, and for all $w \in K$,

$$|w| \leq C \quad \Longrightarrow \quad \left|\Phi^{[n]}(w)\right| = |w + ina| \geq n\,|a| - |w| \geq n\,|a| - C.$$

For $n_0 \geq (M + C)/|a|$ ($M > 0$ being in the definition of Ω) and $n \geq n_0$, we obtain $m(\Phi^{[n]}(w)) \neq 0$, and

$$\left|\frac{m(\Phi^{[n]}(w))}{m(\Phi^{[n]}(1))} - 1\right| \lesssim \left|m(\Phi^{[n]}(w)) - m(\Phi^{[n]}(1))\right|.$$

Under the same hypotheses as in Theorem 2.11, (F_n) converges uniformly on K, and we get the result.

2.5. **Parabolic symbols with vanishing weight at the Denjoy-Wolff point.** If the weight m does vanish, or admits a pole at the Denjoy-Wolff point of the symbol, we get a similar result.

Theorem 2.12. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be a parabolic holomorphic map, with Denjoy-Wolff point 1. Assume that φ is regular, $\mathcal{C}^{3+\varepsilon}(1)$, and that $\varphi''(1) \neq 0$. Let $m \in \mathrm{Hol}(\mathbb{D})$. Assume also that there exist a neighborhood U of 1, $r > 0$, $\delta > 0$ and $\Lambda : \mathbb{D} \to \mathbb{C}$ such that setting $\mathbb{D}_r = \tau^{-1}(\mathbb{H}_r)$ and*

$$V(r) = \begin{cases} U \cap \mathbb{D} & \text{if } \Re(\varphi''(1)) > 0, \\ U \cap \mathbb{D}_r & \text{if } \Re(\varphi''(1)) = 0 \text{ and } \Im(\varphi''(1)) > 0, \end{cases}$$

the map m satisfies

$$\forall z \in \mathbb{D}, \quad m(z) = (z-1)^{\alpha}(\beta + \Lambda(z)),$$

with $\alpha \in \mathbb{R}^*$, $\beta \in \mathbb{C}^*$, and

$$\forall z, z' \in V(r), \qquad |\Lambda(z) - \Lambda(z')| \lesssim |z-1|^{1+\delta} + |z'-1|^{1+\delta} + |z-z'|^{\frac{1}{2}+\delta}.$$

Then, $\sigma_p(W_{m,\varphi}) = \mathbb{C}^*$.

Proof. Again, we shall prove the theorem on $\mathbb{H}$ and we follow a strategy similar to that of Theorem 2.7. Therefore, we fix $m : \mathbb{H} \to \mathbb{C}$, which may be written

$$m(w) = (w+1)^{-\alpha}(\beta + \Lambda(w)),$$

where Λ satisfies (4). By Theorem 2.8 and Proposition 2.9, we shall find a map $H : \mathbb{H} \to \mathbb{C}$ such that $H \circ \Phi = H + a$, and

$$H(w) = w - \frac{b}{a}\log(1+w) + c + \theta(w), \qquad w \in \overline{\Omega}, \tag{5}$$

with $\Omega = \mathbb{H}$ or $\mathbb{H}^+$, and $|\theta(w)| \lesssim |w+1|^{-\varepsilon}$. First, we show that we may always assume that $H(\mathbb{H}) \cap \{ya : y \leq 0\} = \emptyset$.

- If $\Re(a) = 0$, then $H(\mathbb{H}) \subset \mathbb{H}$ (cf. [9, p.64]), which gives the result.
- Otherwise, let $x \geq 0$ be such that if $B(w) = c + \theta(w)$, then $|B(w)| \leq x/2$ on $\mathbb{H}$. We first show that for $|w|$ large enough, say $|w| > \rho$ and $y \leq -x/\Re(a)$, then one cannot have $H(w) = ya$. Otherwise

$$z := w - \frac{b}{a}\log(1+w) = ya - B(w)$$

would belong to $D(w, |w|/2)$ and to $B(ya, |y|\Re(a)/2)$, but these two discs have empty intersection. Moreover, H is bounded on the compact set $\overline{\mathbb{H}} \cap \{|w| \leq \rho\}$, so we get the existence of $s > 0$ such that

$$H(\mathbb{H}) \cap \{ya : y \leq 0\} \subset \{ya : \ -s \leq y \leq 0\}.$$

We conclude by considering $H + sa$ instead of H.

Step 1 : We show that $\sigma_p(W_{H^{-\alpha},\Phi}) \cap \mathbb{C}^* \neq \varnothing$. Indeed, set $F(w) = \Gamma^\alpha(H(w)/a)$, where Γ is the gamma function. Then, F is well-defined since $H(w)/a \notin \mathbb{R}^-$, and

$$H^{-\alpha}(w)(F \circ \Phi)(w) = H^{-\alpha}(w)\Gamma^\alpha\left(\frac{H(w)}{a} + 1\right) = H^{-\alpha}(w)\frac{H^\alpha(w)}{a^\alpha}F(w) = a^{-\alpha}F(w).$$

Therefore, $a^{-\alpha} \in \sigma_p(W_{H^{-\alpha},\Phi})$.

Step 2 : We show that $\sigma_p(W_{m_1,\Phi}) = \mathbb{C}^*$, where $m_1 = mH^\alpha$. This will be a consequence of Theorem 2.11. Indeed, we may write, for $w \in \Omega$,

$$\begin{aligned} m_1(w) &= (w+1)^{-\alpha}\big(\beta + \Lambda(w)\big)\left(w + 1 - \frac{b}{a}\log(1+w) + (c-1) + \theta(w)\right)^\alpha \\ &= \big(\beta + \Lambda(w)\big)\left(1 - \frac{b}{a}\frac{\log(w+1)}{w+1} + \frac{c-1}{w+1} + \frac{\theta(w)}{w+1}\right)^\alpha \\ &= \big(\beta + \Lambda(w)\big)\left(1 - \frac{\alpha b}{a}\frac{\log(w+1)}{w+1} + \frac{\alpha(c-1)}{w+1} + \frac{\alpha\theta(w)}{w+1} + \Lambda_2(w)\right), \end{aligned}$$

with $|\theta(w)| \lesssim |w|^{-\varepsilon/2}$ and $|\Lambda_2(w)| \lesssim |w|^{-3/2}$, for $|w|$ large.

We expand the product and verify that we can write $m_1(w) = m_1(\infty) + \tilde{\Lambda}(w)$, with $m_1(\infty) = \beta \neq 0$ and $\tilde{\Lambda}(w)$ satisfying (4). The last point follows from the easy fact that the product of two functions which are bounded on Ω and satisfy (4) still verifies (4). Thus we just have to check that each term satisfies this condition.

- The map $p_1 : w \mapsto \frac{\log(w+1)}{w+1}$ satisfies, for $w, w' \in \Omega$,

$$|p_1(w) - p_1(w')| \lesssim \left| \frac{w - w'}{(w+1)(w'+1)} \right|^{3/4}.$$

- The map $p_2 : w \mapsto (w+1)^{-1}$ satisfies, for $w, w' \in \Omega$,

$$|p_2(w) - p_2(w')| \leq \left| \frac{w - w'}{(w+1)(w'+1)} \right|.$$

- The map $p_3 : w \mapsto \frac{\theta(w)}{w+1}$ satisfies, for $w, w' \in \Omega$,

$$|p_3(w) - p_3(w')| \lesssim \left| \frac{1}{w+1} \right|^{1+\varepsilon} + \left| \frac{1}{w'+1} \right|^{1+\varepsilon}.$$

- The map Λ_2 satisfies, for $w, w' \in \Omega$,

$$|\Lambda_2(w) - \Lambda_2(w')| \lesssim \left| \frac{1}{w+1} \right|^{3/2} + \left| \frac{1}{w'+1} \right|^{3/2}.$$

We conclude by applying Lemma 1.1. □

3. Multivariable case

We now study the spectrum of weighted composition operators defined on the euclidean unit ball $\mathbb{B}_d$ of $\mathbb{C}^d$. We also denote by $\mathbb{S}^{d-1}$ the unit sphere.

As proved by MacCluer in [13], the Denjoy-Wolff theorem keeps true for holomorphic self-maps of $\mathbb{B}_d$. Precisely, let $\varphi : \mathbb{B}_d \to \mathbb{B}_d$ be such a map without any fixed point in $\mathbb{B}_d$. Then there exists $\xi \in \mathbb{S}^{d-1}$ such that $\varphi^{[n]}$ converges uniformly on all compact subsets of $\mathbb{B}_d$ to ξ. Moreover, the boundary dilation coefficient $\lambda(\varphi)$ of φ at ξ, defined by

$$\lambda(\varphi) = \liminf_{z \to \xi} \frac{1 - \|\varphi(z)\|}{1 - \|z\|},$$

belongs to $(0, 1]$. We will say that φ is parabolic if $\lambda(\varphi) = 1$ and hyperbolic otherwise.

When $m(\xi) \neq 0$ and φ is hyperbolic, the one-dimensional proof can be easily modified to cover $\mathbb{B}_d$. The parabolic case will be substantially more difficult and we will use the results of [5]. We shall restrict ourselves to the case $d = 2$. Since the point spectrum is stable under conjugation by automorphisms, we are allowed to reduce the matrix of $d\varphi(\xi)$ and we will distinguish two cases.

- the parabolic diagonalizable case, with $d\varphi(\xi) = \begin{pmatrix} 1 & 0 \\ 0 & \mu \end{pmatrix}$,
- the parabolic nondiagonalizable case, with $d\varphi(\xi) = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}$ and $\alpha > 0$,

(see [5] for details). The orbits $(\varphi^{[n]}(0))$ behave very differently for these two kinds of maps and we will need to handle them separately.

3.1. **Hyperbolic symbols with nonvanishing weight at the Denjoy-Wolff point.** We begin this paragraph by recalling some definitions.

Definition 3.1. The *horodisc* with vertex $\xi \in \mathbb{S}^{d-1}$ and radius $r > 0$ is defined by

$$\mathcal{H}(\xi, r) = \left\{ z \in \mathbb{B}_d : \frac{|1 - \langle z, \xi \rangle|^2}{1 - \|z\|^2} < r \right\}.$$

We obtain the following theorem, very close to the one-variable one.

Theorem 3.2. *Let $\varphi : \mathbb{B}_d \to \mathbb{B}_d$ be a hyperbolic holomorphic map, with Denjoy-Wolff point $\xi \in \mathbb{S}^{d-1}$, and $m \in \mathrm{Hol}(\mathbb{B}_d)$ having a non-zero K-limit at ξ. Assume that*

$$m(z) = m(\xi) + \Lambda(z),$$

and that for all horodisc $\mathcal{H}$ with vertex ξ, there exist $\delta \in (0,1)$ and a nondecreasing map $g : \mathbb{R} \to \mathbb{R}_+$ such that

$$\forall x \in (0,1),\ \sum_{n \geq 0} g(x^n) < +\infty \quad \textit{and} \quad z \in \mathcal{H},\ \|z\| \geq 1 - \delta \implies |\Lambda(z)| \leq g(\|z - \xi\|).$$

Then, $\sigma_p(W_{m,\varphi}) = \mathbb{C}^*$.

Proof. By [16, Theorem 8.5.3] (see also [5, Theorem 2.3]), setting $\lambda \in (0,1)$ the boundary dilation coefficient of φ, for all $r > 0$, $\varphi(\mathcal{H}(\xi, r)) \subset \mathcal{H}(\xi, \lambda r)$. By induction, we obtain $\varphi^{[n]}(\mathcal{H}(\xi, r)) \subset \mathcal{H}(\xi, \lambda^n r)$, $n \in \mathbb{N}$.

Consider once again the sequence (f_n) defined by

$$f_n(z) = \frac{1}{m(\xi)^n} \prod_{k=0}^{n-1} m(\varphi^{[k]}(z)) = \prod_{k=0}^{n-1} \frac{m(\varphi^{[k]}(z))}{m(\xi)}.$$

Then, as in the proof of Theorem 2.6,

$$\left| \frac{m(\varphi^{[n]}(z))}{m(\xi)} - 1 \right| \leq \frac{1}{|m(\xi)|} g\left(\left\| \varphi^{[n]}(z) - \xi \right\| \right).$$

Let K be a compact subset of $\mathbb{B}_d$. Consider $R > 0$ such that $K \subset \mathcal{H}(\xi, R)$. Hence, for all $z \in K$,

$$\begin{aligned}
\left\| \varphi^{[n]}(z) - \xi \right\|^2 &= \left\| \varphi^{[n]}(z) \right\|^2 + 1 - 2\Re \left\langle \varphi^{[n]}(z), \xi \right\rangle \\
&\leq 2[1 - \Re \left\langle \varphi^{[n]}(z), \xi \right\rangle] + \left\| \varphi^{[n]}(z) \right\|^2 - 1 \\
&\leq 2\Re(1 - \left\langle \varphi^{[n]}(z), \xi \right\rangle) \\
&\leq 2 \left| 1 - \left\langle \varphi^{[n]}(z), \xi \right\rangle \right| \leq 2\sqrt{R}\lambda^{n/2} \sqrt{1 - \|\varphi^{[n]}(z)\|^2} \leq 2\sqrt{R}\lambda^{n/2}.
\end{aligned}$$

Finally, for n large enough, if $x \in (\lambda^{1/4}, 1)$, we get $\left\| \varphi^{[n]}(z) - \xi \right\| \leq x^n$ for n large enough. Thus,

$$\left| \frac{m(\varphi^{[n]}(z))}{m(\xi)} - 1 \right| \leq \frac{1}{|m(\xi)|} g\left(x^n \right),$$

so that (f_n) converges uniformly on K. We conclude as in Theorem 2.6, using the fact that $\sigma_p(C_\varphi) = \sigma_p(W_{1,\varphi}) = \mathbb{C}^*$ (see [14, Proposition 3.3]). □

3.2. **Parabolic non-diagonalizable symbol.** In the following subsections, we will focus on two-variables symbols, and we will always assume that φ is a holomorphic self-map of $\mathbb{B}_2$ with Denjoy-Wolff point at $e_1 = (1, 0)$.

An element of $\mathbb{C}^2$ will be denoted (w, z). To apply the results of [5], we need a notion of regularity at e_1 which emphasizes the dissymmetry between $w - 1$ and z as (w, z) goes to e_1. Indeed, one always has $|z|^2 \leq 1 - |w|^2 \leq 2(1 - |w|)$. This leads up to the following definition.

Definition 3.3. A holomorphic map $\varphi : \mathbb{B}_2 \to \mathbb{B}_2$ is in $\mathcal{D}^m(e_1)$, $m \geq 2$, if we can write it as

$$\varphi(w,z) = \left(\begin{array}{l} 1 + \lambda(w-1) + \displaystyle\sum_{j+k\geq 2,\ j/2+k\leq m/2} a_{j,k} z^j (w-1)^k + \varepsilon_1(w,z), \\ \alpha(w-1) + \mu z + \displaystyle\sum_{j+k\geq 2,\ j/2+k\leq m/2} b_{j,k} z^j (w-1)^k + \varepsilon_2(w,z) \end{array} \right),$$

with $\varepsilon_i(w, z) = o(|w - 1|^{m/2})$. Similarly, we say that φ is in $\mathcal{D}^{m+\varepsilon}(e_1)$, $m \geq 2$, $\varepsilon \in (0, 1)$, if $\varepsilon_i(w, z) = O(|w - 1|^{(m+\varepsilon)/2})$.

As in the one-variable case, we will move from $\mathbb{B}_2$ to another domain where the computations will be easier. The relevant domain here is the *Siegel half-space*, defined by

$$\mathbb{H}_2 := \{(w, z) \in \mathbb{C}^2 : \Re(w) > |z|^2\}.$$

Let τ be the Cayley map from $\mathbb{B}_2$ onto $\mathbb{H}_2$ defined by

$$\tau(w,z) = \left(\frac{w+1}{1-w}, \frac{z}{1-w} \right), \quad \tau^{-1}(w,z) = \left(\frac{w-1}{w+1}, \frac{2z}{w+1} \right).$$

We will set $\Phi = \tau \circ \varphi \circ \tau^{-1}$. Regarding distances between points, it is easy to check that if $(w_0, z_0) = \tau(w, z)$ and $(w'_0, z'_0) = \tau(w', z')$ then

$$\|(w,z) - (w',z')\|^2 \asymp \frac{|w_0 - w'_0|^2}{|w_0+1|^2 |w'_0+1|^2} + \left| \frac{z_0}{w_0+1} - \frac{z'_0}{w'_0+1} \right|^2.$$

In this paragraph, we will assume that φ is parabolic and that

$$d\varphi(e_1) = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix},$$

with $\alpha > 0$. We call it a parabolic non-diagonalizable map.

As for the one variable case, the construction of eigenvectors will follow from a careful analysis of the behaviour of $\varphi^{[n]}(w, z)$. Moving to $\mathbb{H}_2$, for $(w, z) \in \mathbb{H}_2$, we will denote $(w(n), z(n)) = \Phi^{[n]}(w, z)$. To study how $(w(n), z(n))$ goes to ∞, it is convenient to introduce the following sets. For $A, M > 0$, let

$$K_{M,A} := \{(w, z) \in \mathbb{H}_2 : \Re(z) \geq A, \Re(w) \geq M, 4\Im(z) \leq \Re(z), 2\,|\Im(w)| \leq \Re(w)\}.$$

From the results of [5], we will extract the following lemma.

Lemma 3.4. *Let $\varepsilon \in (0,1)$ and assume that $\varphi \in \mathcal{D}^{5+\varepsilon}(e_1)$ is parabolic non-diagonalizable. Let $A, M > 0$. Then, for all compact sets $K \subset \mathbb{H}_2$, there exists $n_0 \in \mathbb{N}$ such that $\Phi^{[n]}(K) \subset K_{M,A}$ for $n \geq n_0$. Morever there exists $a > 0$ such that, for all $\eta \in (0,1)$, for all $(w,z) \in K$,*

$$\left|w(n) - n(n-1)a^2\right| \lesssim n^{1+\eta}, \qquad |z(n) - na| \lesssim n^{\eta}, \qquad \left|\frac{z(n)}{w(n)} - \frac{1}{an}\right| \lesssim \frac{1}{n^{2-\eta}}.$$

Proof. The first assertion is [5, Lemma 3.3]. The estimates are direct consequences of [5, Proposition 3.8 and Lemma 3.10]. □

We deduce the following result.

Theorem 3.5. *Let $\varepsilon > 0$. Assume that $\varphi \in \mathcal{D}^{5+\varepsilon}(e_1)$ is parabolic nondiagonalizable. Let $m \in \mathrm{Hol}(\mathbb{D})$. Assume also that there exist $M, A > 0$ and $\delta \in (0,1)$ such that, setting $V_{M,A} = \tau^{-1}(K_{M,A})$, the map m satisfies*

$$m(z) = m(e_1) + \Lambda(z),$$

with $m(e_1) \neq 0$ and for all $(w,z), (w',z') \in V_{M,A}$,

$$|\Lambda(w,z) - \Lambda(w',z')| \lesssim \|(w,z) - e_1\|^{1+\delta} + \|(w',z') - e_1\|^{1+\delta} + \|(w,z) - (w',z')\|^{\frac{1}{2}+\delta}.$$

Then, $\sigma_p(W_{m,\varphi}) = \mathbb{C}^*$.

Proof. We move on $\mathbb{H}_2$ and consider $m \in \mathrm{Hol}(\mathbb{H}_2)$, which can be written

$$m(w,z) = m(\infty) + \Lambda(w,z),$$

with $m(\infty) \neq 0$ and for all $(w,z), (w',z') \in K_{M,A}$,

$$|\Lambda(w,z) - \Lambda(w',z')|^2 \lesssim \left(\frac{1+|z|^2}{|w+1|^2}\right)^{1+\delta} + \left(\frac{1+|z'|^2}{|w'+1|^2}\right)^{1+\delta} + \left(\frac{|w-w'|}{|w+1|\cdot|w'+1|}\right)^{1+2\delta} + \left|\frac{z}{w+1} - \frac{z'}{w'+1}\right|^{1+2\delta}.$$

As in the one-dimensional case we consider

$$F_n(w,z) = \prod_{k=n_0}^{n} \frac{m(\Phi^{[k]}(w,z))}{m(\Phi^{[k]}(1,0))},$$

and it is sufficient to prove that (F_n) converges uniformly on all compact subsets of $\mathbb{H}_2$ to conclude that $m(\infty) \in \sigma_p(W_{m,\Phi})$. Setting $(w_0,z_0) = (1,0)$, we get

$$\begin{aligned}
&\left|m(\Phi^{[n]}(w,z)) - m(\Phi^{[n]}(1,0))\right|^2 \\
\leq\ &|\Lambda(w(n),z(n)) - \Lambda(w_0(n),z_0(n))|^2 \\
\leq\ &\left(\frac{1+|z(n)|^2}{|w(n)+1|^2}\right)^{1+\delta} + \left(\frac{1+|z_0(n)|^2}{|w_0(n)+1|^2}\right)^{1+\delta} \\
&\quad + \left(\frac{|w(n)-w_0(n)|}{|w(n)+1|\cdot|w_0(n)+1|}\right)^{1+2\delta} + \left|\frac{z(n)}{w(n)+1} - \frac{z_0(n)}{w_0(n)+1}\right|^{1+2\delta}.
\end{aligned}$$

Now Lemma 3.4 allows us to conclude that

$$\left|m(\Phi^{[n]}(w,z)) - m(\Phi^{[n]}(1,0))\right| \lesssim \frac{1}{n^\beta},$$

for some $\beta > 1$. Indeed,

$$\left(\frac{1+|z(n)|^2}{|w(n)+1|^2}\right)^{1+\delta} \lesssim \left(\frac{n^2}{n^4}\right)^{1+\delta} \leq \frac{1}{n^{2+2\delta}},$$

whereas

$$\left(\frac{|w(n)-w_0(n)|}{|w(n)+1|\cdot|w_0(n)+1|}\right)^{1+2\delta} \lesssim \left(\frac{n^{1+\eta}}{n^4}\right)^{1+2\delta} \lesssim \frac{1}{n^{2+2\delta}},$$

provided $\eta > 0$ is small enough. Finally,

$$\begin{aligned}\left|\frac{z(n)}{1+w(n)} - \frac{z_0(n)}{1+w_0(n)}\right| &\lesssim \left|\frac{z(n)-z_0(n)}{w(n)w_0(n)}\right| + \left|\frac{z(n)}{w(n)} - \frac{z_0(n)}{w_0(n)}\right| \\ &\lesssim \frac{1}{n^{4-\eta}} + \frac{1}{n^{2-\eta}}.\end{aligned}$$

By choosing $\eta > 0$ sufficiently small,

$$\left|\frac{z(n)}{1+w(n)} - \frac{z_0(n)}{1+w_0(n)}\right|^{\frac{1}{2}+\delta} \lesssim \frac{1}{n^{1+\gamma}}$$

for some $\gamma > 0$.

Thus, $m(\infty) \in \sigma_p(W_{m,\Phi})$. To conclude, we need to observe that $\sigma_p(C_\Phi) = \mathbb{C}^*$. This follows from [5, Theorem 3.1], which gives the existence of $G \in \mathrm{Hol}(\mathbb{H}_2)$ and $a \in \mathbb{C}^*$ such that $G\circ\Phi = G+a$. For all $\lambda \in \mathbb{C}$, the function $H = \exp(\lambda G)$ is then an eigenvector of C_Φ corresponding to the eigenvalue $\exp(\lambda a)$. □

3.3. **Parabolic diagonalizable symbol.** We firstly come back to general considerations for φ a holomorphic self-map of $\mathbb{B}_2$ with e_1 as Denjoy-Wolff point. Assume that $\varphi \in \mathcal{D}^2(e_1)$ and (by reduction)

$$d\varphi(e_1) = \begin{pmatrix}\lambda_1 & 0\\ \alpha_1 & \mu_1\end{pmatrix}.$$

Then, φ may be written

$$\varphi(w,z) = (1+\lambda_1(w-1)+\gamma_1 z^2 + o(|w-1|), \alpha_1(w-1)+\mu_1 z + \kappa_1 z^2 + o(|w-1|))$$

To ensure that φ maps $\mathbb{B}_2$ into itself, it is mandatory that

$$\lambda_1 \geq |\mu_1|^2 + 2|\gamma_1| \tag{6}$$

(see [5, Equation (2.1)]) and the $\gamma_1 z^2$ term is the source of many troubles.

If we translate this on Φ, we find that

$$\begin{aligned}\Phi_1(w,z) &= \frac{\lambda_\infty w}{1-\gamma_\infty \frac{z^2}{w+1}} + o(w),\\ \Phi_2(w,z) &= \frac{\mu_\infty z}{1-\gamma_\infty \frac{z^2}{w+1}} - \frac{\alpha_\infty}{1-\gamma_\infty \frac{z^2}{w+1}} + F_\infty\left(\frac{z^2}{w+1}\right) + o\left(\frac{z^2}{w+1}\right),\end{aligned}$$

with $\lambda_\infty = 1/\lambda_1$, $\mu_\infty = \mu_1/\lambda_1$, $\gamma_\infty = 2\gamma_1/\lambda_1$, $\alpha_\infty = \alpha_1/\lambda_1$ and F_∞ holomorphic in $\mathbb{D}$. In particular, (6) becomes

$$\lambda_\infty \geq |\mu_\infty|^2 + \lambda_\infty |\gamma_\infty|.$$

As for the one-dimensional case, the study of the orbits of φ done in [5] require some regularity assumptions on φ. Recall that Wolff's lemma says that, for any $(w,z) \in \mathbb{H}_2$,

$$\Re(\Phi_1(w,z)) - |\Phi_2(w,z)|^2 \geq \Re(w) - |z|^2.$$

Definition 3.6. We say that φ satisfies the *strong Wolff's condition* if

$$\lambda_\infty > |\mu_\infty|^2 + \lambda_\infty |\gamma_\infty|$$

and there exists $\delta > 0$ such that for any $(w,z) \in \mathbb{H}_2$,

$$\Re(\Phi_1(w,z)) - |\Phi_2(w,z)|^2 \geq \Re(w) - |z|^2 + \delta.$$

To study the spectrum of $W_{m,\Phi}$, we will need the following estimates on $(w(n), z(n))$.

Lemma 3.7. *Let $\varepsilon \in (0,1)$, and assume that $\varphi \in \mathcal{D}^{6+\varepsilon}(e_1)$ is parabolic diagonalizable and that φ satisfies the strong Wolff condition. There exist $a \in \mathbb{C}^*$ and a sequence of complex numbers (a_n) such that for any compact $K \subset \mathbb{H}_2$, there exist $C(K)$ satisfying, for all $(w,z) \in K$, for all $n \geq 1$,*

$$|w(n) - na| \leq C(K)\log(n+1)$$
$$|z(n)| \leq \frac{C(K)}{n}$$
$$|z(n) - a_n| \leq \frac{C(K)\log(n+1)}{n^2}.$$

Proof. The two first inequalities already appear in [5] (see Lemma 5.4 and the discussion after Lemma 5.3). For the third one, we shall need that our assumptions on φ ensure that on $\mathbb{H}_2$, we can write

$$\Phi_2(w,z) = \mu_\infty z + \frac{b}{w+1} + \sum_{\substack{p+q=2 \\ q>0}} \frac{z^p}{(w+1)^q} F_{p,q}(w,z) + \delta(w,z),$$

with $b, b_{p,q} \in \mathbb{C}$, $|F_{p,q}(w,z)| \lesssim 1$, $|\delta(w,z)| \lesssim \frac{1}{|w|^{2+\varepsilon}}$ and $|\mu_\infty| < 1$ (see [5, p. 811]).

If we iterate this, then we find

$$z(n) = \mu_\infty^n \left(z + \sum_{k=0}^{n-1} \frac{1}{\mu_\infty^{k+1}} \left(\frac{b}{w(k)+1} + \sum_{\substack{p+q=2 \\ q>0}} \frac{z^p(k)}{(w(k)+1)^q} F_{p,q}(w(k),z(k)) + \delta(w(k),z(k)) \right) \right).$$

Now the known estimate on $w(k)$ implies that

$$\left| \frac{1}{w(k)+1} - \frac{1}{ka+1} \right| \leq \frac{C(K)\log(k+1)}{k^2}.$$

Taking into account the estimate on $|z(k)|$ we get for $p+q=2$,

$$\left|\frac{z^p(k)}{(w(k)+1)^q}\right| \leq \frac{C(K)}{k^2}.$$

Setting

$$a_n = b\mu_\infty^n \sum_{k=0}^{n-1} \frac{1}{\mu_\infty^{k+1}(ka+1)},$$

we infer the result from the estimate

$$\sum_{k=0}^{n-1} \frac{1}{|\mu_\infty|^{k+1}(k+1)^2} \lesssim \frac{1}{|\mu_\infty|^n n^2}. \qquad \square$$

We are now ready to state the theorem regarding the spectrum of $W_{m,\Phi}$.

Theorem 3.8. *Let $\varepsilon \in (0,1)$, and assume that $\varphi \in \mathcal{D}^{6+\varepsilon}(e_1)$ is parabolic diagonalizable, and that φ satisfies the strong Wolff condition. Let $m \in \mathrm{Hol}(\mathbb{B}_2)$ be such that*

$$m(w,z) = m(e_1) + \Lambda(w,z),$$

with $m(e_1) \neq 0$ and, for any $(w,z),(w',z') \in U \cap \mathbb{B}_2$,

$$|\Lambda(w,z) - \Lambda(w',z')| \lesssim \|(w,z)-e_1\|^{1+\delta} + \|(w',z')-e_1\|^{1+\delta} + \|(w,z)-(w',z')\|^{\frac{1}{2}+\delta},$$

with $\delta > 0$ and U a neighbourhood of e_1. Then $\sigma_p(W_{m,\Phi}) = \mathbb{C}^*$.

Proof. We follow eactly the proof of Theorem 3.5, using now the estimates coming from Lemma 3.4 which yield

$$\begin{aligned}
\frac{1+|z(n)|^2}{|w(n)+1|^2} &\lesssim \frac{1}{n^4}, \\
\frac{|w(n)-w_0(n)|}{|w(n)+1|\cdot|w_0(n)+1|} &\lesssim \frac{\log(n)}{n^2}, \\
\left|\frac{z(n)}{w(n)+1} - \frac{z_0(n)}{w_0(n)+1}\right| &\lesssim \frac{\log(n)}{n^2}.
\end{aligned}$$

We conclude the proof thanks to Lemma 1.1 and [5, Theorem 5.1] which asserts that $\sigma_p(C_\Phi) = \mathbb{C}^*$. $\square$

3.4. **Hyperbolic symbols and vanishing weights.** We now consider the case where the weight vanishes at the Denjoy-Wolff point. It turns out that the situation becomes much more complicated than in the disc and that, even for very simple symbols and multipliers, the point spectrum can be empty. We provide an example on $\mathbb{H}_2$.

Example 3.9. Let $\Phi(w,z) = (2w,z)$ and $m(w,z) = \frac{z}{w+1}$. Then $\sigma_p(W_{m,\Phi}) = \varnothing$.

Proof. Suppose that f is a nonzero eigenvector of $W_{m,\Phi}$ corresponding to $\lambda \neq 0$. For a fixed w with $\Re(w) > 0$, $z \mapsto f(w,z)$ is analytic in $|z|^2 < \Re(w)$. Therefore we can write it $f(w,z) = \sum_{n\geq 0} a_n(w) z^n$.

Evaluating $mf \circ \Phi = \lambda f$ at $(w,0)$, we find $f(w,0) = 0$ thus $a_0(w) = 0$. Now the relation $mf \circ \Phi = \lambda f$ also implies that for all $n \geq 0$,

$$\lambda a_{n+1}(w) = \frac{a_n(2w)}{w+1},$$

which in turn yields $a_n(w) = 0$ for all $n \in \mathbb{N}$ and all $w \in \mathbb{C}$ with $\Re(w) > 0$. Therefore $f = 0$, a contradiction. □

Observe that the previous weight has a very simple expression on $\mathbb{B}_2$: $m(w,z) = z$. This is one of the easiest example of a weight vanishing at e_1 !

Nevertheless, we will be able to get positive results. We did not look for the most general statement, and we will only handle the case where m is a polynomial vanishing at e_1 with no linear term in z, and with a nonzero linear term in $w-1$.

Theorem 3.10. *Let $\varphi \in \mathcal{D}^3(e_1)$ be a hyperbolic map with a Denjoy-Wolff point at e_1. Let $m : \mathbb{B}_2 \to \mathbb{C}$ be a polynomial writing*

$$m(w,z) = w - 1 + \sum_{p+q \geq 2} a_{p,q}(w-1)^p z^q.$$

Then, $\sigma_p(W_{m,\varphi}) = \mathbb{C}^*$.

Proof. We move on the Siegel half-space $\mathbb{H}_2$ and show that, for

$$m(w,z) = \frac{1}{w+1} + \sum_{p+q \geq 2} a_{p,q} \frac{z^q}{(w+1)^{p+q}},$$

then $\sigma_p(W_{m,\Phi}) = \mathbb{C}^*$. First, we focus on the case $\lambda_\infty > |\mu_\infty|^2$. We will use some results of [5], in particular Lemma 4.2, Lemma 4.4 and Proposition 4.5. Let us review this material. For $M > 0$, we set

$$L_M = \left\{ (w,z) \in \mathbb{H}_2 : |w| > M \text{ and } \frac{|z|}{|w+1|^{1/2}} < \frac{1}{\log|w+1|} \right\}.$$

There exist $\kappa \in (0,1)$ and $M > 0$ such that

- L_M is Φ-stable,
- For each compact set $K \subset \mathbb{H}_2$, for each $(w,z) \in K$,

$$\frac{|z(n)|^2}{|w(n)+1|} \lesssim \kappa^n, \tag{7}$$

- For each compact set $K \subset \mathbb{H}_2$, there exists $n_0 \in \mathbb{N}$ such that $\Phi^{[n_0]}(K) \subset L_M$,
- For each compact set $K \subset \mathbb{H}_2$, for all $(w,z) \in K$, $|w(n)| \gtrsim \lambda_\infty^n$, as proved in [5, Proposition 4.5] or [11, Lemma 2.4]),
- There exists $E : \mathbb{H}_2 \to \mathbb{H}$ holomorphic such that on L_M,

$$E \circ \Phi = \lambda_\infty E \quad \text{and} \quad |E(w,z) - w| \lesssim |z|^2 + |w+1|^{1/2}.$$

That E maps $\mathbb{H}_2$ into $\mathbb{H}$ is not written explicitly in [5], but this follows from the definition of E as the limit of $w(n)/\lambda_\infty^n$.

We set $m_1 = mE$, and first show that $\sigma_p(W_{m_1,\Phi}) = \mathbb{C}^*$. Arguing as in the proof of Theorem 3.2, it is enough to prove that $\sum |m_1(w(n), z(n)) - 1|$ converges uniformly on each compact set $K \subset \mathbb{H}_2$. We may even assume that $K \subset L_M$. We write

$$|m_1(w(n), z(n)) - 1| \leq \left| \frac{E(w(n), z(n))}{w(n) + 1} - 1 \right| + \sum_{p+q \geq 2} |a_{p,q}| \left| \frac{E(w(n), z(n)) z(n)^q}{(w(n) + 1)^{p+q}} \right|.$$

On the one hand, using (7),

$$\left| \frac{E(w(n), z(n)) z(n)^q}{(w(n) + 1)^{p+q}} \right| \lesssim \frac{\kappa^{nq}}{|w(n) + 1|^{p+\frac{q}{2}-1}}.$$

For $p + q \geq 2$, we deduce $p + \frac{q}{2} - 1 \geq 0$ and this gives rise to a convergent series. On the other hand,

$$\begin{aligned} \left| \frac{E(w(n), z(n))}{w(n) + 1} - 1 \right| &\lesssim \left| \frac{E(w(n), z(n)) - w(n)}{w(n) + 1} \right| + \frac{1}{\lambda_\infty^n} \\ &\lesssim \left(\frac{|z^2(n)| + |w(n) + 1|^{1/2}}{|w(n) + 1|} + \frac{1}{\lambda_\infty^n} \right) \\ &\lesssim \left(\kappa^{2n} + \frac{1}{\lambda_\infty^{n/2}} \right) \end{aligned}$$

and we also obtain uniform convergence of the induced series.

Finally, we show that $\sigma_p(W_{E^{-1},\Phi}) \cap \mathbb{C}^* \neq \varnothing$. But as in the one-dimensional case, it is easy to check that $f = \exp(\mu \log^2 E)$ is an eigenvector of $W_{E^{-1},\Phi}$ for $\mu = -1/2 \log(\lambda_\infty)$.

We turn to the case $|\mu_\infty|^2 = \lambda_\infty$, which is simpler (since it implies that $\gamma_\infty = 0$). From [11], in particular Proposition 2.1 and Lemma 2.4, we know that

- There exists $E : \mathbb{H}_2 \to \mathbb{H}$ such that
$$E \circ \Phi = \lambda_\infty E, \qquad |E(w, z) - w| \lesssim |w|^{1/2} \quad \text{for all } w \in \mathbb{H}_2.$$
- For all compact sets $K \subset \mathbb{H}_2$, for all $(w, z) \in K$,
$$|z(n)| \lesssim |\mu_\infty|^n = \lambda_\infty^{n/2} \quad \text{and} \quad |w(n)| \gtrsim \lambda_\infty^n.$$
This comes from the fact that $w(n)/\lambda_\infty^n$ and $z(n)/\mu_\infty^n$ converge uniformly on compact subsets of $\mathbb{H}_2$ to a holomorphic function and that E, which is the limit of $w(n)/\lambda_\infty^n$, does not vanish.

Hence, if $m_1 = mE$, we obtain

$$\begin{aligned} |m_1(w(n), z(n)) - 1| &\leq \left| \frac{E(w(n), z(n))}{w(n) + 1} - 1 \right| + \sum_{p+q \geq 2} |a_{p,q}| \left| \frac{E(w(n), z(n)) z(n)^q}{(w(n) + 1)^{p+q}} \right| \\ &\lesssim \frac{1}{\lambda_\infty^{n/2}} + \frac{1}{\lambda_\infty^{n(p-1+q/2)}}, \end{aligned}$$

and we conclude as above. □

Open question. What happens if the symbol φ is parabolic and the weight m does vanish at the Denjoy-Wolff point ?

Acknowledgments : This research is partly supported by the Bézout Labex, funded by ANR, reference ANR-10-LABX-58.

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