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UNRESTRICTED VIRTUAL BRAIDS AND CRYSTALLOGRAPHIC BRAID GROUPS

PAOLO BELLINGERI, JOHN GUASCHI, AND STAVROULA MAKRI

ABSTRACT. We show that the crystallographic braid group $B_n/[P_n, P_n]$ embeds naturally in the group of unrestricted virtual braids UVB_n , we give new proofs of known results about the torsion elements of $B_n/[P_n, P_n]$, and we characterise the torsion elements of UVB_n .

1. INTRODUCTION

Let $n \geq 1$. The group of unrestricted virtual braids, denoted throughout this paper by UVB_n , was introduced by Kauffman and Lambropoulou in [22] as the analogue of fused links in the setting of braids. The classification of fused links is now well known. Such links are distinguished by their *virtual linking number*, see for instance [25], where they are considered as the closure of unrestricted virtual braids, as well as [1] for their classification in terms of Gauss diagrams. Unrestricted virtual braid groups occur as natural quotients of virtual and welded braid groups. They appear for instance in [20] in the study of local representations of welded braid groups, where they are called *symmetric loop braid groups*, and they may be decomposed as a semi-direct product of a right-angled Artin group, which is in fact the pure subgroup UVP_n of UVB_n , by the symmetric group S_n [3]. The main aim of this paper is to characterise the torsion elements of UVB_n using this decomposition, namely to show that any element of finite order is a conjugate of an element of S_n by an element of UVP_n .

The structure of this paper is as follows. In Section 2, we give presentations of the virtual braid groups VB_n , of the welded braid groups WB_n , and of UVB_n . In this way, UVB_n may be viewed as a quotient of both VB_n and WB_n . We also recall two important results of [3] that describe the structure of the pure unrestricted virtual braid group UVP_n as a direct sum of copies of the free group F_2 on two generators, which as mentioned above, allows us to decompose UVB_n as a semi-direct product of UVP_n and S_n in a natural way. A similar decomposition holds for VB_n and WB_n , but the canonical homomorphism $\eta: B_n \rightarrow UVB_n$, where B_n is the Artin braid group, is not injective, which is in contrast with the nature of the corresponding homomorphisms for VB_n and WB_n . In Section 3, we study the image of η , and in Proposition 3.1, we prove that it is isomorphic to the quotient $B_n/[P_n, P_n]$, where P_n is the pure Artin braid group, and $[P_n, P_n]$ is its commutator subgroup. This quotient has been the subject of recent study, one of the reasons being that it is a crystallographic group [16–18]. The results of [16] have been generalised to quasi-Abelianised quotients of complex reflection groups [4, 24], and to surface braid groups [19]. This enables us to give an alternative proof in Proposition 3.5 and Remark 3.6 of the fact that $B_n/[P_n, P_n]$ embeds in the semi-direct product $P_n/[P_n, P_n] \rtimes S_n$. In the final section of the paper, Section 4, we study the torsion elements of UVB_n . We apply the embedding of Proposition 3.1 to give a new combinatorial proof in Theorem 4.1 of the fact that there are no elements of even order in $B_n/[P_n, P_n]$, and in Theorem 4.3, we characterise the torsion elements of UVB_n by showing that every such element is conjugate to an element of S_n by an element of UVP_n . To our knowledge, it is not known whether an analogous result holds for VB_n and WB_n .

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2. UNRESTRICTED VIRTUAL BRAIDS

In order to define unrestricted virtual braid groups, in this section we recall first the notions of virtual and welded braid groups by exhibiting their usual group presentations.

Definition 2.1. The *virtual braid group* VB_n is the group defined by the group presentation:

Generators: $\{\sigma_1, \dots, \sigma_{n-1}, \rho_1, \dots, \rho_{n-1}\}$

Relations:

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad \text{for } i = 1, \dots, n-2 \quad (\text{BR1})$$

$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{for } |i-j| \geq 2 \quad (\text{BR2})$$

$$\rho_i \rho_{i+1} \rho_i = \rho_{i+1} \rho_i \rho_{i+1} \quad \text{for } i = 1, \dots, n-2 \quad (\text{SR1})$$

$$\rho_i \rho_j = \rho_j \rho_i \quad \text{for } |i-j| \geq 2 \quad (\text{SR2})$$

$$\rho_i^2 = 1 \quad \text{for } i = 1, \dots, n-1 \quad (\text{SR3})$$

$$\sigma_i \rho_j = \rho_j \sigma_i \quad \text{for } |i-j| \geq 2 \quad (\text{MR1})$$

$$\rho_i \rho_{i+1} \sigma_i = \sigma_{i+1} \rho_i \rho_{i+1} \quad \text{for } i = 1, \dots, n-2. \quad (\text{MR2})$$

A diagrammatic description of generators and relations of VB_n may be found for instance in [2, 5, 21]. For a topological interpretation, we refer the reader to [10], and for an algebraic one (in terms of actions on root systems) to [6]. Note that the relations (BR1)–(BR2) (resp. (SR1)–(SR3)) correspond to the usual relations of the Artin braid group B_n (resp. the symmetric group S_n) for the set $\{\sigma_1, \dots, \sigma_{n-1}\}$ (resp. for the set $\{\rho_1, \dots, \rho_{n-1}\}$), and the remaining relations (MR1)–(MR2) are ‘mixed’ in the sense that they involve generators of both of these sets.

Recall that the pure braid group P_n is the kernel of the homomorphism $\pi: B_n \rightarrow S_n$ defined on the generators $\sigma_1, \dots, \sigma_{n-1}$ of B_n by $\pi(\sigma_i) = s_i$ for all $i = 1, \dots, n-1$, where s_i is the transposition $(i, i+1)$. Analogously, we define the *virtual pure braid group*, denoted by VP_n , to be the kernel of the homomorphism $\pi_{VP}: VB_n \rightarrow S_n$ that for all $i = 1, 2, \dots, n-1$, maps the generators σ_i and ρ_i to s_i . A group presentation for VP_n is given in [2]. Let $\iota: S_n \rightarrow VB_n$ be the homomorphism defined by $\iota(s_i) = \rho_i$ for $i = 1, \dots, n-1$. Since ι is a section for π_{VP} , it follows that ι is injective and that VB_n is a semi-direct product of VP_n by S_n . The canonical homomorphism $\eta: B_n \rightarrow VB_n$ defined by $\eta(\sigma_i) = \sigma_i$ for $i = 1, \dots, n-1$ is injective [5, 15, 21].

The welded braid group WB_n may be defined as a quotient of VB_n by adding the following family of relations to the presentation given in Definition 2.1:

$$\rho_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \rho_{i+1}, \quad \text{for } i = 1, \dots, n-2, \quad (\text{OC})$$

where OC stands for ‘Over Commuting’. Welded braid groups have several different equivalent definitions [11]. In particular, they may be defined as *basis conjugating automorphisms* of free groups [14], from which it follows that the homomorphism $\eta: B_n \rightarrow WB_n$ defined by $\eta(\sigma_i) = \sigma_i$ for $i = 1, \dots, n-1$ is injective [21]. As in the case of virtual braid groups, the *welded pure braid group*, denoted by WP_n , is defined to be the kernel of the homomorphism $\pi_{WP}: WB_n \rightarrow S_n$ given by sending the generators σ_i and ρ_i of WB_n to s_i for all $i = 1, 2, \dots, n-1$. A group presentation for WP_n may be found in [8, 11, 26]. Abusing notation, we define $\iota: S_n \rightarrow WB_n$ to be also the homomorphism defined by $\iota(s_i) = \rho_i$ for $i = 1, \dots, n-1$. As in the virtual case, ι is a section for π_{WP} , and therefore ι is injective and WB_n is a semi-direct product of WP_n by S_n . Note that the symmetrical relations:

$$\rho_{i+1} \sigma_i \sigma_{i+1} = \sigma_i \sigma_{i+1} \rho_i, \quad \text{for } i = 1, \dots, n-2, \quad (\text{UC})$$

where UC stands for ‘Under Commuting’, do not hold in WB_n (see for instance [7, 21]). Consequently, the following group, which was first defined by Kauffman and Lambropoulou in [22], is a proper quotient of WB_n .

Definition 2.2. The *unrestricted virtual braid group* UVB_n is the group defined by the following group presentation:

Generators: $\{\sigma_1, \dots, \sigma_{n-1}, \rho_1, \dots, \rho_{n-1}\}$

Relations: the seven relations (BR1)–(BR2), (SR1)–(SR3) and (MR1)–(MR2) of Definition 2.1, plus the relations of types (OC) and (UC).

As in the case of VB_n , let $\pi_{UVP}: UVB_n \rightarrow S_n$ be the homomorphism defined by $\pi_{UVP}(\sigma_i) = \pi_{UVP}(\rho_i) = s_i$ for $i = 1, \dots, n-1$. The kernel of π_{UVP} is the *unrestricted virtual pure braid group*, denoted by UVP_n . For $1 \leq i, j \leq n, i \neq j$, we define the elements $\lambda_{i,j}$ of UVP_n as follows:

$$\begin{cases} \lambda_{i,i+1} = \rho_i \sigma_i^{-1} & \text{for } i = 1, \dots, n-1 \\ \lambda_{i+1,i} = \sigma_i^{-1} \rho_i & \text{for } i = 1, \dots, n-1 \\ \lambda_{i,j} = \rho_{j-1} \rho_{j-2} \cdots \rho_{i+1} \lambda_{i,i+1} \rho_{i+1} \cdots \rho_{j-2} \rho_{j-1} & \text{for } 1 \leq i < j-1 \leq n-1 \\ \lambda_{j,i} = \rho_{j-1} \rho_{j-2} \cdots \rho_{i+1} \lambda_{i+1,i} \rho_{i+1} \cdots \rho_{j-2} \rho_{j-1} & \text{for } 1 \leq i < j-1 \leq n-1. \end{cases} \quad (1)$$

Theorem 2.3 (Bardakov–Bellingeri–Damiani [3]). *The group UVP_n admits the following presentation: Generators: $\lambda_{i,j}$, where $1 \leq i, j \leq n, i \neq j$.*

Relations: The generators commute pairwise except for the pairs $\lambda_{i,j}$ and $\lambda_{j,i}$ for all $1 \leq i, j \leq n, i \neq j$.

It follows from Theorem 2.3 that the group UVP_n is a right-angled Artin group that is isomorphic to the direct product of the $n(n-1)/2$ free groups $F_{i,j}$ of rank 2, where $1 \leq i < j \leq n$, and where $\{\lambda_{i,j}, \lambda_{j,i}\}$ is a basis of $F_{i,j}$. By convention, if $1 \leq i, j \leq n$ then we set $F_{j,i} = F_{i,j}$. As for VB_n , let $\iota: S_n \rightarrow UVB_n$ be the homomorphism defined by $\iota(s_i) = \rho_i$ for $i = 1, \dots, n-1$. Then ι is injective, and is a section for π_{UVP} . This leads to the following natural decomposition of UVB_n .

Theorem 2.4 (Bardakov–Bellingeri–Damiani [3]). *The group UVB_n is isomorphic to the semi-direct product $UVP_n \rtimes S_n$, where S_n acts on UVP_n by permuting the indices of the elements of the generating set of UVP_n given in Theorem 2.3. More precisely, for all $\lambda_{i,j} \in UVP_n$, where $1 \leq i, j \leq n, i \neq j$, and for all $s \in S_n$ we have:*

$$\iota(s) \lambda_{i,j} \iota(s)^{-1} = \lambda_{s(i),s(j)}. \quad (2)$$

3. CRYSTALLOGRAPHIC GROUPS

As we mentioned in Section 2, the Artin braid group B_n embeds naturally in VB_n and WB_n via the injective homomorphism η defined in each of these two cases. However this is no longer the case for the canonical homomorphism $\eta: B_n \rightarrow UVB_n$ defined by $\eta(\sigma_i) = \sigma_i$ for $i = 1, \dots, n-1$. In this section, we show that the image of B_n under the homomorphism η is isomorphic to the quotient of B_n by the commutator subgroup $[P_n, P_n]$. This quotient was introduced by Tits in [27] as *groupes de Coxeter étendus*, and has been studied more recently by various authors (see for example [4, 13, 16–18, 24]).

Proposition 3.1. *Let $\eta: B_n \rightarrow UVB_n$ be the canonical homomorphism defined by $\eta(\sigma_i) = \sigma_i$ for $1 \leq i \leq n-1$. Then $\eta(B_n)$ is isomorphic to the group $B_n/[P_n, P_n]$.*

Proof. Consider the homomorphism $\eta: B_n \rightarrow UVB_n$. Since $P_n = \text{Ker}(\pi)$ (resp. $UVP_n = \text{Ker}(\pi_{UVP})$), where $\pi: B_n \rightarrow S_n$ (resp. $\pi_{UVP}: UVB_n \rightarrow S_n$) is defined by $\pi(\sigma_i) = s_i$ (resp. $\pi_{UVP}(\sigma_i) = \pi_{UVP}(\rho_i) = s_i$) for all $1 \leq i \leq n-1$, it follows from the definition of $\eta: B_n \rightarrow UVB_n$ that $\pi = \pi_{UVP} \circ \eta$, and thus $\pi = \pi_{UVP} \Big|_{\eta(B_n)} \circ \eta$. Since π is surjective, the homomorphism $\pi_{UVP} \Big|_{\eta(B_n)}: \eta(B_n) \rightarrow S_n$ is too. From the equality $\pi = \pi_{UVP} \circ \eta$, we obtain the following commutative diagram of short exact sequences:

$$\begin{array}{ccccccc} 1 & \longrightarrow & P_n & \longrightarrow & B_n & \xrightarrow{\pi} & S_n \longrightarrow 1 \\ & & \downarrow \eta|_{P_n} & & \downarrow \eta & & \parallel \\ 1 & \longrightarrow & UVP_n & \longrightarrow & UVB_n & \xrightarrow{\pi_{UVP}} & S_n \longrightarrow 1. \end{array} \quad (3)$$

We claim that $\eta(P_n) = \text{Ker}(\pi_{UVP} \Big|_{\eta(B_n)})$. To prove the claim, the exactness of (3) implies that $\eta(P_n) \subset \text{Ker}(\pi_{UVP} \Big|_{\eta(B_n)})$. Conversely, let $x \in \text{Ker}(\pi_{UVP} \Big|_{\eta(B_n)})$. Then $x \in \eta(B_n)$, and there exists $y \in B_n$ such that $\eta(y) = x$. The commutativity of (3) implies that $y \in P_n$, and so $x \in \eta(P_n)$, so $\text{Ker}(\pi_{UVP} \Big|_{\eta(B_n)}) \subset \eta(P_n)$, and the claim follows.

Considering η and its restriction to P_n , we obtain the following commutative diagram of short exact sequences:

$$\begin{array}{ccccccc}
& & & 1 & & 1 & \\
& & & \downarrow & & \downarrow & \\
1 & \longrightarrow & \text{Ker}(\eta|_{P_n}) & \longrightarrow & P_n & \xrightarrow{\eta|_{P_n}} & \eta(P_n) \longrightarrow 1 \\
& & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & \text{Ker}(\eta) & \longrightarrow & B_n & \xrightarrow{\eta} & \eta(B_n) \longrightarrow 1. \\
& & & & \downarrow \pi & & \downarrow \pi_{UVB_n|_{\eta(B_n)}} \\
& & & & S_n & \xlongequal{\quad} & S_n \\
& & & & \downarrow & & \downarrow \\
& & & & 1 & & 1
\end{array} \tag{4}$$

Note that the exactness of the rightmost column of (4) follows from the claim of the previous paragraph. By a standard diagram-chasing argument, we conclude from (4) that $\text{Ker}(\eta) = \text{Ker}(\eta|_{P_n})$. Applying the isomorphism $UVB_n \cong UVP_n \times S_n$ of Theorem 2.4 and using equation (1), we see that $\eta(\sigma_i) = \lambda_{i,i+1}^{-1} \rho_i$. Recall that the pure braid group P_n is generated by the set $\{a_{ij} \mid 1 \leq i < j \leq n\}$, where:

$$\begin{aligned}
a_{i,i+1} &= \sigma_i^2, \\
a_{i,j} &= \sigma_{j-1} \sigma_{j-2} \cdots \sigma_{i+1} \sigma_i^2 \sigma_{i+1}^{-1} \cdots \sigma_{j-2}^{-1} \sigma_{j-1}^{-1}, \quad \text{for } i+1 < j \leq n.
\end{aligned}$$

In [3, Corollary 2.8] it was shown that $\eta(a_{i,j}) = \lambda_{i,j}^{-1} \lambda_{j,i}^{-1}$ for $1 \leq i+1 < j \leq n$. From the construction, we deduce that $[P_n, P_n]$ is contained in $\text{Ker}(\eta|_{P_n})$. Further, $\eta(P_n)$ is isomorphic to $P_n/[P_n, P_n] = \mathbb{Z}^{n(n-1)/2}$ (see [3, Corollary 2.8]). It follows that $[P_n, P_n]$ coincides with $\text{Ker}(\eta|_{P_n})$. The statement follows by combining this with the fact that $\text{Ker}(\eta) = \text{Ker}(\eta|_{P_n})$ and using the exactness of (4). \square

Remark 3.2. We can rephrase Proposition 3.1 by saying that the canonical homomorphism $\eta: B_n \rightarrow UVB_n$ factors through an embedding of $B_n/[P_n, P_n]$ into UVB_n . On the other hand, by [9, Theorem 2.5], the canonical homomorphism from B_n to VB_n induces an embedding of $B_n/[P_n, P_n]$ in $VB_n/[VP_n, VP_n]$. Since $VB_n/[VP_n, VP_n]$ is isomorphic to $UVB_n/[UVP_n, UVP_n]$ [9, Theorem 4.1], the canonical homomorphism B_n to UVB_n gives rise to an embedding of the quotient $B_n/[P_n, P_n]$ in $UVB_n/[UVP_n, UVP_n]$. We therefore conclude that the composition of the embedding of $B_n/[P_n, P_n]$ in UVB_n defined in the proof of Proposition 3.1 and the canonical projection $q_n: UVB_n \rightarrow UVB_n/[UVP_n, UVP_n]$ is also injective.

We recall that an abstract group Γ is said to be *crystallographic* if it can be realised as an extension of a free Abelian subgroup of Γ of maximal rank by a finite group [12, Theorem 2.1.4]. Using the following result, in [16], Gonçalves–Guaschi–Ocampo proved that the group $B_n/[P_n, P_n]$ is a crystallographic group.

Proposition 3.3 (Gonçalves–Guaschi–Ocampo [16]). *Let $n \geq 2$. Then the following sequence:*

$$1 \longrightarrow \mathbb{Z}^{n(n-1)/2} \longrightarrow B_n/[P_n, P_n] \xrightarrow{\hat{\pi}} S_n \longrightarrow 1, \tag{5}$$

is short exact, where $\hat{\pi}$ is the homomorphism induced by $\pi: B_n \rightarrow S_n$.

Note that the sequence (5) does not split [13, 16]. It follows from Propositions 3.1 and 3.3 that the group $\eta(B_n)$ is a crystallographic subgroup of UVB_n . We remark also that $\text{Im}(\eta)$ is not normal in UVB_n , and that the normal closure of $\text{Im}(\eta)$ is of index 2 in UVB_n . This follows from the fact that in the quotient of UVB_n by $\text{Im}(\eta)$, the generators ρ_i are identified to a single element due to relations (OC) and (UC) of Definition 2.2. The group UVB_n contains other crystallographic groups of rank $n(n-1)/2$ that are not isomorphic to $\eta(B_n)$. One such example is given in the following proposition that is a straightforward consequence of Theorems 2.3 and 2.4

Proposition 3.4. *Let C_n be the subgroup of UVB_n generated by $\{\lambda_{i,j}\lambda_{j,i}^{-1}\}$ for $1 \leq i < j \leq n$ and by ρ_i for $i = 1, \dots, n$, and let H_n be the subgroup of C_n generated by $\{\lambda_{i,j}\lambda_{j,i}^{-1}\}$ for $1 \leq i < j \leq n$. The restriction of π_{UVP} to C_n defines the following split exact sequence:*

$$1 \longrightarrow H_n \longrightarrow C_n \xrightarrow{\pi_{UVP}|_{C_n}} S_n \longrightarrow 1,$$

In particular C_n is crystallographic.

Proposition 3.5. *Let H_n be the subgroup of UVB_n defined in the statement of Proposition 3.4, let $\langle\langle H_n \rangle\rangle$ its normal closure in UVB_n , and let $\{x_{i,j}\}_{1 \leq i < j \leq n}$ be a set of generators of $\mathbb{Z}^{n(n-1)/2}$.*

(a) *The quotient $UVB_n/\langle\langle H_n \rangle\rangle$ is isomorphic to the semi-direct product $\mathbb{Z}^{n(n-1)/2} \rtimes S_n$, where for any $s \in S_n$, $s \cdot x_{i,j} = x_{s(i),s(j)}$, where we take $x_{i,j} = x_{j,i}$.*

(b) *Let $\theta: B_n \rightarrow UVB_n/\langle\langle H_n \rangle\rangle$ be the composition of $\eta: B_n \rightarrow UVB_n$ and the projection $\pi_H: UVB_n \rightarrow UVB_n/\langle\langle H_n \rangle\rangle$. The group $\theta(B_n)$ is isomorphic to $B_n/[P_n, P_n]$.*

Proof. Part (a) follows from Theorems 2.3 and 2.4 that imply that $UVB_n/\langle\langle H_n \rangle\rangle = UVP_n/\langle\langle H_n \rangle\rangle \rtimes S_n$. The identification $\lambda_{i,j} = \lambda_{j,i}$ for $i \neq j$ implies that $UVP_n/H_n = \mathbb{Z}^{n(n-1)/2}$, and it also defines the induced action of S_n on this quotient. The proof of Proposition 3.1 can be adapted to prove part (b) (the image of P_n by θ is a free Abelian group of rank $n(n-1)/2$, and is thus isomorphic to $P_n/[P_n, P_n]$). \square

Remark 3.6. The quotient $UVB_n/\langle\langle H_n \rangle\rangle$ coincides with the semi-direct product considered in [27, Section 2.7] and in the Remark following [23, Proposition 5.12]. Thus Proposition 3.5 yields a new proof of the fact that the crystallographic braid group $B_n/[P_n, P_n]$ embeds in the semi-direct product $\mathbb{Z}^{n(n-1)/2} \rtimes S_n$.

4. TORSION ELEMENTS OF UVB_n

Let $n \geq 3$. In [16], it was shown that $B_n/[P_n, P_n]$ has torsion, but that it possesses no elements of even order. Viewing $B_n/[P_n, P_n]$ as the subgroup $\text{Im}(\eta)$ of UVB_n via Proposition 3.1, we may give an alternative proof of this latter fact. We do this by first characterising the elements of UVB_n of order 2, and then showing that these elements do not belong to $\text{Im}(\eta)$.

Theorem 4.1. *Let $n \geq 3$.*

(a) *An element $v \in UVB_n$ is of order 2 if and only if there exist $\rho \in \text{Im}(\iota)$ of order 2 and $g \in UVP_n$ such that $v = g\rho g^{-1}$.*

(b) *The elements of order 2 of UVB_n do not belong to $\text{Im}(\eta)$. In particular, $B_n/[P_n, P_n]$ has no elements of even order.*

Proof. Let $n \geq 3$.

(a) First observe that the given condition is clearly sufficient. Conversely, suppose that $v \in UVB_n$ is of order 2. Identifying UVB_n with the internal semi-direct product $UVP_n \rtimes \text{Im}(\iota)$ using Theorem 2.4, there exist unique elements $w \in UVP_n$ and $\rho \in \text{Im}(\iota)$ such that $v = w\rho$, and the pair (w, ρ) is non trivial. Further, the fact that UVP_n is torsion free implies that $\rho \neq 1$. Since v is of order 2, we have $1 = v^2 = (w\rho)^2 = w \cdot \rho w \rho^{-1} \cdot \rho^2$, where $w \cdot \rho w \rho^{-1} \in UVP_n$ and $\rho^2 \in \text{Im}(\iota)$, from which we deduce using Theorem 2.4 that ρ is also of order 2. Hence $\pi_{UVP}(\rho) \in S_n$ is also of order 2, and thus may be written as a non-trivial product of disjoint transpositions. So there exists $\tau \in \text{Im}(\iota)$ such that $\pi_{UVP}(\tau\rho\tau^{-1}) = \sigma$, where $\sigma = (1, 2)(3, 4) \cdots (m, m+1)$, and $1 \leq m \leq n-1$ is odd. Setting $\tilde{v} = \tau v \tau^{-1}$, $\tilde{w} = \tau w \tau^{-1}$ and $\tilde{\rho} = \tau \rho \tau^{-1}$, it follows that $\tilde{v} = \tilde{w}\tilde{\rho}$, where $\tilde{v} \in UVB_n$ is of order 2, $\tilde{w} \in UVP_n$, $\tilde{\rho} \in \text{Im}(\iota)$ is of order 2, $\pi_{UVP}(\tilde{\rho}) = \sigma$, and $\tilde{w} \cdot \tilde{\rho} \tilde{w} \tilde{\rho}^{-1} = 1$. For $1 \leq i < j \leq n$, let $\pi_{i,j}: UVP_n \rightarrow F_{i,j}$ denote the projection of UVP_n onto $F_{i,j}$ (see Theorem 2.3 and the comments that follow it). Applying (2), we observe that conjugation by an element of UVB_n permutes the $\{F_{i,j}\}_{1 \leq i < j \leq n}$. Let T denote the subset of $n(n-1)/2$ transpositions of S_n , and let $T_\sigma = \{(i, i+1) \in T \mid i \in \{1, 3, \dots, m\}\}$. Then the subgroup $\langle \sigma \rangle$ of S_n of order 2 acts on T by conjugation, and if $1 \leq i < j \leq n$, the orbit $\mathcal{O}(i, j)$ of (i, j) is equal to $\{(i, j), (\sigma(i), \sigma(j))\}$, which is equal to $\{(i, j)\}$ if and only if either $(i, j) \in T_\sigma$ or if $m+2 \leq i < j \leq n$, and contains two elements otherwise. Further, for all $1 \leq i < j \leq n$, $\sigma(i) < \sigma(j)$

if and only if $(i, j) \notin T_\sigma$. Let \mathcal{T} be a transversal for this action of $\langle \sigma \rangle$ on T . Using Theorem 2.3, we obtain:

$$\tilde{w} = \prod_{(i,j) \in \mathcal{T}} \prod_{(k,l) \in \mathcal{O}(i,j)} w_{k,l}(\lambda_{k,l}, \lambda_{l,k}), \quad (6)$$

where $w_{k,l} = w_{k,l}(\lambda_{k,l}, \lambda_{l,k}) \in F_{k,l}$, and this decomposition is unique up to permutation of the factors. So:

$$\begin{aligned} 1 &= \tilde{w} \cdot \tilde{\rho} \tilde{w} \tilde{\rho}^{-1} = \left(\prod_{(i,j) \in \mathcal{T}} \prod_{(k,l) \in \mathcal{O}(i,j)} w_{k,l}(\lambda_{k,l}, \lambda_{l,k}) \right) \tilde{\rho} \left(\prod_{(i,j) \in \mathcal{T}} \prod_{(k,l) \in \mathcal{O}(i,j)} w_{k,l}(\lambda_{k,l}, \lambda_{l,k}) \right) \tilde{\rho}^{-1} \\ &= \left(\prod_{(i,j) \in \mathcal{T}} \prod_{(k,l) \in \mathcal{O}(i,j)} w_{k,l}(\lambda_{k,l}, \lambda_{l,k}) \right) \left(\prod_{(i,j) \in \mathcal{T}} \prod_{(k,l) \in \mathcal{O}(i,j)} w_{k,l}(\lambda_{\sigma(k), \sigma(l)}, \lambda_{\sigma(l), \sigma(k)}) \right) \\ &= \prod_{(i,j) \in \mathcal{T}} \prod_{(k,l) \in \mathcal{O}(i,j)} w_{k,l}(\lambda_{k,l}, \lambda_{l,k}) \cdot w_{\sigma(k), \sigma(l)}(\lambda_{k,l}, \lambda_{l,k}), \end{aligned} \quad (7)$$

where we have used the fact that $\sigma^2 = 1$. Note also that if $(i, j) \in T_\sigma$ then $j = \sigma(i) > \sigma(j) = i$, and in this case, any term of the form $w_{\sigma(i), \sigma(j)}(\lambda_{i,j}, \lambda_{j,i})$ in (7) should be interpreted as $w_{i,j}(\lambda_{j,i}, \lambda_{i,j})$. The expression (7) is written with respect to the direct product structure of $UV P_n$. It follows from Theorem 2.3 that $w_{i,j}(\lambda_{i,j}, \lambda_{j,i}) \cdot w_{\sigma(i), \sigma(j)}(\lambda_{i,j}, \lambda_{j,i}) = 1$ for all $1 \leq i < j \leq n$, and hence:

$$w_{\sigma(i), \sigma(j)}(\lambda_{\sigma(i), \sigma(j)}, \lambda_{\sigma(j), \sigma(i)}) = w_{i,j}^{-1}(\lambda_{\sigma(i), \sigma(j)}, \lambda_{\sigma(j), \sigma(i)}). \quad (8)$$

Suppose that $(i, j) \in T_\sigma$. Then $w_{i,j}(\lambda_{i,j}, \lambda_{j,i}) \cdot w_{i,j}(\lambda_{j,i}, \lambda_{i,j}) = 1$. Writing $a = \lambda_{i,j}$, $b = \lambda_{j,i}$ and $w_{i,j}(a, b) = a^{k_1} b^{l_1} \dots a^{k_m} b^{l_m}$, where $m \geq 0$, $k_1, l_1, \dots, k_m, l_m \in \mathbb{Z}$ and $l_1, \dots, k_m \neq 0$, it follows from the relation $w_{i,j}(a, b) \cdot w_{i,j}(b, a) = 1$ that $j_q = -k_{m+1-q}$ for all $q = 1, \dots, m$. Taking $y_{i,j}(a, b) = a^{k_1} b^{-k_m} \dots a^{k_{m/2}} b^{-k_{(m+2)/2}} a^{k_{(m+1)/2}}$ if m is even (resp. odd), we see that $w_{i,j}(a, b) = y_{i,j}(a, b) \cdot y_{i,j}^{-1}(b, a)$. Thus $w_{i,j}(\lambda_{i,j}, \lambda_{j,i}) = y_{i,j}(\lambda_{i,j}, \lambda_{j,i}) \cdot y_{i,j}^{-1}(\lambda_{j,i}, \lambda_{i,j})$. Setting $z_{i,j} = w_{i,j}$ if $(i, j) \notin T_\sigma$ and $z_{i,j} = y_{i,j}$ if $(i, j) \in T_\sigma$, it follows from (6) and (8) that:

$$\tilde{v} = \tilde{w} \tilde{\rho} = \left(\prod_{(i,j) \in \mathcal{T}} z_{i,j}(\lambda_{i,j}, \lambda_{j,i}) z_{i,j}^{-1}(\lambda_{j,i}, \lambda_{i,j}) \right) \tilde{\rho}. \quad (9)$$

Now the terms $z_{i,j}(\lambda_{i,j}, \lambda_{j,i})$ (resp. $z_{i,j}^{-1}(\lambda_{j,i}, \lambda_{i,j})$) appearing in (9) commute pairwise, and setting $\tilde{g} = \prod_{(i,j) \in \mathcal{T}} z_{i,j}(\lambda_{i,j}, \lambda_{j,i}) \in UV P_n$, we obtain:

$$\tilde{v} = \left(\prod_{(i,j) \in \mathcal{T}} z_{i,j}(\lambda_{i,j}, \lambda_{j,i}) \right) \left(\prod_{(i,j) \in \mathcal{T}} z_{i,j}^{-1}(\lambda_{j,i}, \lambda_{i,j}) \right) \tilde{\rho} = \tilde{g} \tilde{\rho} \tilde{g}^{-1}.$$

Hence $v = \tau^{-1} \tilde{v} \tau = g \rho g^{-1}$, where $\rho \in \text{Im}(\iota)$ is of order 2 and $g = \tau^{-1} \tilde{g} \tau \in UV P_n$, which proves that the condition of part (a) is also necessary.

(b) We start by characterising the elements of $\text{Im}(\eta)$. Let $v \in UV B_n$. Then $\pi_{UV P}(v) \in S_n$, and so $\pi_{UV P}(v) = s_{i_1} \dots s_{i_r}$, where $r \geq 0$, and for all $j = 1, \dots, r$, $s_{i_j} \in \{(k, k+1) \mid k = 1, \dots, n-1\}$. Hence $v = w \rho$, where $w \in UV P_n$ and $\rho = \iota(\pi_{UV P}(v)) = \rho_{i_1} \dots \rho_{i_r}$. Let $\beta = \sigma_{i_1} \dots \sigma_{i_r} \in B_n$. Then by (1), we have $\eta(\beta) = \sigma_{i_1} \dots \sigma_{i_r} = \lambda_{i_1, i_1+1}^{-1} \rho_{i_1} \dots \lambda_{i_r, i_r+1}^{-1} \rho_{i_r} = y \rho_{i_1} \dots \rho_{i_r} = y \rho \in \text{Im}(\eta)$, where $y \in UV P_n$. It follows that:

$$v \in \text{Im}(\eta) \iff (\eta(\beta))^{-1} v \in \text{Im}(\eta) \iff \rho^{-1} y^{-1} w \rho \in \text{Im}(\eta).$$

Now $y^{-1} w \in UV P_n$, so $\rho^{-1} y^{-1} w \rho \in UV P_n$, and it follows that $v \in \text{Im}(\eta)$ if and only if $\rho^{-1} y^{-1} w \rho \in \eta(P_n)$. From Section 3, P_n is generated by the set $\{a_{ij} \mid 1 \leq i < j \leq n\}$, and since $\eta(a_{ij}) = \lambda_{i,j}^{-1} \lambda_{j,i}^{-1}$ for all $1 \leq i < j \leq n$, we see that $v \in \text{Im}(\eta)$ if and only if $\rho^{-1} y^{-1} w \rho$ belongs to the free Abelian subgroup Γ of $UV P_n$ of rank $n(n-1)/2$ generated by the set $\{\lambda_{i,j}^{-1} \lambda_{j,i}^{-1} \mid 1 \leq i < j \leq n\}$. In particular, if $1 \leq i < j \leq n$ and $\varepsilon_{i,j}: UV P_n \rightarrow \mathbb{Z}$ denotes the evaluation homomorphism defined by $\varepsilon_{i,j}(\lambda_{k,l}) = 1$ if $\{k, l\} = \{i, j\}$ and $\varepsilon_{i,j}(\lambda_{k,l}) = 0$ otherwise, where $1 \leq k, l \leq n$, $k \neq l$, then $\varepsilon_{i,j}(\Gamma) = 2\mathbb{Z}$ for all $1 \leq i < j \leq n$.

Now suppose on the contrary that $\text{Im}(\eta)$ possesses an element v of order 2. From part (a), there exist $g \in UV P_n$ and $\rho \in \text{Im}(\iota)$ of order 2 such that $v = g\rho g^{-1}$. Then $\pi_{UV P}(\rho)$ is also of order 2, and so may be written as a non-trivial product of transpositions whose supports are pairwise disjoint. So there exist $1 \leq r \leq n/2$ and distinct elements $i_1, j_1, \dots, i_r, j_r$ of $\{1, \dots, n\}$ such that $i_k < j_k$ for all $1 \leq k \leq r$ and $\rho = \iota((i_1, j_1) \cdots (i_r, j_r))$. Let $\tau \in \text{Im}(\iota)$ be such that $\pi_{UV P}(\tau\rho\tau^{-1}) = (1, 2) \cdots (2r-1, 2r)$, let $\tilde{\rho} = \tau\rho\tau^{-1} \in \text{Im}(\iota)$, let $\hat{\tau} \in B_n$ be such that $\pi(\hat{\tau}) = \pi_{UV P}(\tau)$, let $\beta = \sigma_1\sigma_3 \cdots \sigma_{2r-1}$, and let $\hat{\beta} = \hat{\tau}^{-1}\beta\hat{\tau}$. Then $\tilde{\rho} = \rho_1\rho_3 \cdots \rho_{2r-1}$ and $\eta(\beta) = \sigma_1\sigma_3 \cdots \sigma_{2r-1} = \lambda_{1,2}^{-1}\lambda_{3,4}^{-1} \cdots \lambda_{2r-1,2r}^{-1}\tilde{\rho}$ by (1) and Theorem 2.3. Since $v \in \text{Im}(\eta)$, it follows that $(\eta(\hat{\beta}))^{-1} \cdot v \in \text{Im}(\eta)$. Hence:

$$(\eta(\hat{\beta}))^{-1} \cdot v = \eta(\hat{\tau}^{-1})\tilde{\rho}^{-1}\eta(\hat{\tau}) \cdot \eta(\hat{\tau}^{-1})\lambda_{2r-1,2r} \cdots \lambda_{3,4}\lambda_{1,2}\eta(\hat{\tau}) \cdot g\rho g^{-1}. \quad (10)$$

Now using (3), we have $\pi_{UV P}(\eta(\hat{\tau})) = \pi(\hat{\tau}) = \pi_{UV P}(\tau)$, so:

$$\eta(\hat{\tau}^{-1})\lambda_{2r-1,2r} \cdots \lambda_{3,4}\lambda_{1,2}\eta(\hat{\tau}) = \lambda_{i_r, j_r} \cdots \lambda_{i_2, j_2}\lambda_{i_1, j_1}. \quad (11)$$

It follows from (10) and (11) that:

$$\begin{aligned} (\eta(\hat{\beta}))^{-1} \cdot v &= \eta(\hat{\tau}^{-1})\tilde{\rho}^{-1}\eta(\hat{\tau})\rho \cdot \rho^{-1}\lambda_{i_r, j_r} \cdots \lambda_{i_2, j_2}\lambda_{i_1, j_1}\rho \cdot \rho^{-1}g\rho g^{-1} \\ &= \eta(\hat{\tau}^{-1})\tilde{\rho}^{-1}\eta(\hat{\tau})\rho\lambda_{j_r, i_r} \cdots \lambda_{j_2, i_2}\lambda_{j_1, i_1}\rho^{-1}g\rho \cdot g^{-1} \end{aligned} \quad (12)$$

Since $\pi_{UV P}(\eta(\hat{\tau})) = \pi(\hat{\tau}) = \pi_{UV P}(\tau)$, there exists $z \in UV P_n$ such that $\eta(\hat{\tau}) = \tau z$, and hence:

$$\begin{aligned} (\eta(\hat{\beta}))^{-1} \cdot v &= z^{-1}\tau^{-1}\tilde{\rho}^{-1}\tau z\rho \cdot \lambda_{j_r, i_r} \cdots \lambda_{j_2, i_2}\lambda_{j_1, i_1} \cdot \rho^{-1}g\rho \cdot g^{-1} \\ &= z^{-1} \cdot \rho^{-1}z\rho \cdot \lambda_{j_r, i_r} \cdots \lambda_{j_2, i_2}\lambda_{j_1, i_1} \cdot \rho^{-1}g\rho \cdot g^{-1}. \end{aligned} \quad (13)$$

The expression on the right-hand side of (13) is written as a product of elements of $UV P_n$, which we will now project onto F_{i_1, j_1} . Let $\alpha(\lambda_{i_1, j_1}, \lambda_{j_1, i_1}) = \pi_{i_1, j_1}(z)$ et $\beta(\lambda_{i_1, j_1}, \lambda_{j_1, i_1}) = \pi_{i_1, j_1}(g)$. Using (2) and the fact that $\pi_{UV P}(\rho) = (i_1, j_1) \cdots (i_r, j_r)$, by (13), we have:

$$\pi_{i_1, j_1}((\eta(\hat{\beta}))^{-1} \cdot v) = (\alpha(\lambda_{i_1, j_1}, \lambda_{j_1, i_1}))^{-1} \cdot \alpha(\lambda_{j_1, i_1}, \lambda_{i_1, j_1}) \cdot \lambda_{j_1, i_1} \cdot \beta(\lambda_{j_1, i_1}, \lambda_{i_1, j_1}) \cdot (\beta(\lambda_{i_1, j_1}, \lambda_{j_1, i_1}))^{-1},$$

from which it follows that $\varepsilon_{i_1, j_1}(\eta(\hat{\beta}))^{-1} \cdot v = 1$. We conclude from the previous paragraph that $(\eta(\hat{\beta}))^{-1} \cdot v \notin \Gamma$, and thus $v \notin \text{Im}(\eta)$, which yields a contradiction. Therefore $\text{Im}(\eta)$ contains no elements of order 2, and we deduce from Proposition 3.1 that $B_n/[P_n, P_n]$ has no elements of even order. \square

Proposition 4.2. *If $n \geq 3$ then any torsion element of $UV B_n$ belongs to the normal closure of $\text{Im}(\iota)$ in $UV B_n$.*

Proof. From Definition 2.2, one may check that the quotient of $UV B_n$ by the normal closure of $\text{Im}(\iota)$ in $UV B_n$ is isomorphic to \mathbb{Z} . If g is a torsion element of $UV B_n$, its image in this quotient is thus trivial, in other words g belongs to the normal closure of $\text{Im}(\iota)$ in $UV B_n$. \square

One may show in the same manner that the statement of Proposition 4.2 also holds for VB_n and WB_n . In the case of $UV B_n$, we may strengthen the results of Theorem 4.1(a) and Proposition 4.2.

Theorem 4.3. *Let $n \geq 2$, and let w be a torsion element of $UV B_n$ of order r . Then there exists $s_w \in S_n$ of order r such that w is conjugate to $\iota(s_w)$ by an element of $UV P_n$.*

Before proving Theorem 4.3, we define some notation, and we make study the action of the symmetric group S_n on the group $UV P_n$ that will used in the proof. If $\tau \in S_n$, let $\text{Supp}(\tau)$ denote its support. Let $s \in S_n$, let $o(s)$ denote the order of s and let G_s denote the cyclic subgroup of S_n of order $o(s)$ generated by the element s . By Theorem 2.4, the permutation s acts on $UV P_n = \langle \lambda_{i,j} \mid 1 \leq i \neq j \leq n \rangle$ by permuting the indices of the elements $\lambda_{i,j}$. To simplify the notation, if $g \in UV P_n$ and $s \in S_n$, we set $s(g) = \iota(s)g\iota(s)^{-1}$, and we shall identify s with its image $\iota(s)$ in $UV B_n$. Let $s = s_1 \cdots s_{m(s)}$ be the cycle decomposition of s , where $1 \leq m(s) \leq n$, and the subsets $\text{Supp}(s_1), \dots, \text{Supp}(s_{m(s)})$ form a partition of the set $\{1, \dots, n\}$ (in particular, some of the $\text{Supp}(s_j)$, where $1 \leq j \leq m(s)$, may be singletons).

Let $I = \{(i, j) \mid 1 \leq i \neq j \leq n\}$. The action of G_s on I gives rise to a partition of I in $n(s)$ disjoint orbits, where $n(s) \in \mathbb{N}$, and the cardinality of each such orbit is either the order of one cycle or is the

least common multiple of the orders of two cycles of s . If $(i, j) \in I$, let $\mathcal{O}(i, j)$ denote its orbit, and let $|\mathcal{O}(i, j)|$ denote the cardinality of $\mathcal{O}(i, j)$. Clearly, $|\mathcal{O}(j, i)| = |\mathcal{O}(i, j)|$ for all $(i, j) \in I$. Further, if $\mathcal{O}(j, i) = \mathcal{O}(i, j)$ then there exist $1 \leq q \leq m(s)$ and $p \in \mathbb{N}$ such that $i, j \in \text{Supp}(s_q)$, $s_q^p(i) = j$ and $s_q^p(j) = i$, from which it follows that $|\mathcal{O}(i, j)| = o(s_q)$ is even, and if $(k, l) \in I$ then $(k, l) \in \mathcal{O}(i, j)$ if and only if $(l, k) \in \mathcal{O}(i, j)$. This gives rise to a partition $\sqcup_{k=1}^{n(s)} \mathcal{O}_k$ of I , where $n(s) \in \mathbb{N}$, that dominates the partition given by the action of G_s on I , and for $k = 1, \dots, n(s)$, \mathcal{O}_k is defined by the property that if $(i, j) \in I$, then $(i, j) \in \mathcal{O}_k$ if and only if $\mathcal{O}_k = \mathcal{O}(i, j) \cup \mathcal{O}(j, i)$. Note that $n(s)$ is the number of orbits of the action of G_s on the set $\{\{i, j\} \mid 1 \leq i \neq j \leq n\}$ of *unordered* pairs of elements of $\{1, \dots, n\}$. Since $|\mathcal{O}(j, i)| = |\mathcal{O}(i, j)|$, and $\mathcal{O}(i, j)$ is even if $\mathcal{O}(j, i) = \mathcal{O}(i, j)$, it follows that for all $1 \leq k \leq n(s)$, the cardinality $|\mathcal{O}_k|$ of \mathcal{O}_k is even, so $|\mathcal{O}_k| = 2n_k$ for some $n_k \in \mathbb{N}$. Let $F_{\mathcal{O}_k} = \bigoplus_{(i,j) \in \mathcal{O}_k} F_{i,j}$, where as in Section 2, we identify $F_{i,j}$ with $F_{j,i}$. Then:

$$UV P_n = \bigoplus_{1 \leq k \leq n(s)} F_{\mathcal{O}_k}. \quad (14)$$

The following lemma is folklore.

Lemma 4.4. *Let $F_2 = F_2(x, y)$ be the free group of rank 2 freely generated by $\{x, y\}$, and let $\alpha: F_2 \rightarrow F_2$ be the involutive automorphism defined by $\alpha(x) = y$ and $\alpha(y) = x$. Let $w \in F_2$ be such that $w\alpha(w) = 1$. Then there exists $u \in F_2$ such that $w = u\alpha(u^{-1})$.*

Proof. Let $w \in F_2$ be such that $w\alpha(w) = 1$. If $w = 1$ then we may take $u = 1$. So suppose that $w \neq 1$. Since $w\alpha(w) = 1$, w cannot be written in reduced form as a word starting and ending with a non-zero power of the same generator. By replacing w by $\alpha(w)$ if necessary, we may thus suppose that $w = x^{\varepsilon_1} y^{\varepsilon_2} \dots y^{\varepsilon_{2t-1}} y^{\varepsilon_{2t}} \neq 1$, where $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{2t} \in \mathbb{Z} \setminus \{0\}$, and thus:

$$1 = w\alpha(w) = \underbrace{(x^{\varepsilon_1} y^{\varepsilon_2} \dots x^{\varepsilon_{2t-1}} y^{\varepsilon_{2t}})}_{\neq 1} \cdot \underbrace{(y^{\varepsilon_1} x^{\varepsilon_2} \dots y^{\varepsilon_{2t-1}} x^{\varepsilon_{2t}})}_{\neq 1} \in F_2.$$

It follows that $\varepsilon_{2t-j} = -\varepsilon_{j+1}$ for $j = 0, \dots, 2t-1$, from which we conclude that $w = u\alpha(u^{-1})$, where $u = x^{\varepsilon_1} y^{\varepsilon_2} \dots y^{\varepsilon_t}$. \square

In what follows, we shall identify S_n with its image in $UV B_n$ by ι . In particular, if $s \in S_n$, then $\iota(s)$ shall be denoted simply by s .

Lemma 4.5. *Let $n \geq 2$, let $s \in S_n$, let $\gamma_1 \in F_{\mathcal{O}_1}$, where $1 \leq l \leq n(s)$, and let $w_1 = \gamma_1 s \in UV B_n$. If w_1 is of finite order then it is conjugate to s by an element of $F_{\mathcal{O}_1}$.*

Proof. Let $s \in S_n$, $\gamma_1 \in F_{\mathcal{O}_1}$, where $1 \leq l \leq n(s)$, and $w_1 \in UV B_n$ be as in the statement. By reordering the orbits $\mathcal{O}_1, \dots, \mathcal{O}_{n(s)}$ if necessary, we may suppose that $l = 1$, so $\gamma_1 \in F_{\mathcal{O}_1}$. The result is clear if $o(w_1) = 1$. If $o(w_1) = 2$, then by Theorem 4.1(a), there exists $g \in UV P_n$ such that $w_1 = gsg^{-1}$. With respect to the decomposition (14), let $g = \delta_1 \dots \delta_{n(s)}$, where for $k = 1, \dots, n(s)$, $\delta_k \in F_{\mathcal{O}_k}$. Since $gsg^{-1} = \gamma_1 s$, we have:

$$\gamma_1 = \delta_1 \dots \delta_{n(s)} \cdot s (\delta_1 \dots \delta_{n(s)})^{-1} s^{-1} = \delta_1 \dots \delta_{n(s)} \cdot s \delta_{n(s)}^{-1} s^{-1} \dots s \delta_1^{-1} s^{-1} \in F_{\mathcal{O}_1}. \quad (15)$$

Now for all $k = 1, \dots, n(s)$, $F_{\mathcal{O}_k}$ is invariant under conjugation by s , and identifying the components of (15) with respect to (14), we see that $\delta_k \cdot s \delta_k^{-1} s^{-1} = 1$ for all $2 \leq k \leq n(s)$, or in other words, δ_k and s commute. It follows from (15) that $\gamma_1 = \delta_1 \cdot s \delta_1^{-1} s^{-1}$, and thus $w_1 = \gamma_1 s = \delta_1 s \delta_1^{-1}$, where $\delta_1 \in F_{\mathcal{O}_1}$, as required.

So suppose that $o(w_1) \geq 3$. For all $m \in \mathbb{N}$, we have:

$$w_1^m = (\gamma_1 s)^m = \gamma_1 s \gamma_1 s^{-1} s^2 \gamma_1 s^{-2} \dots s^{m-1} \gamma_1 s^{-(m-1)} s^m = \gamma_1 s (\gamma_1) \dots s^{m-1} (\gamma_1) s^m. \quad (16)$$

Since $UV P_n$ is torsion free, it follows from (16) that $o(w_1) = o(s)$. As we mentioned above, $F_{\mathcal{O}_1}$ is invariant under conjugation by s , so for all $j = 0, 1, \dots, m-1$, the term $s^j (\gamma_1) = s^j \gamma_1 s^{-j}$ of (16) belongs to $F_{\mathcal{O}_1}$. Let $(i_0, j_0) \in \mathcal{O}_1$, where $1 \leq i_0 < j_0 \leq n$. If $\mathcal{O}(i_0, j_0) \neq \mathcal{O}(j_0, i_0)$ (resp. $\mathcal{O}(i_0, j_0) = \mathcal{O}(j_0, i_0)$), let $\varepsilon = 1$ (resp. $\varepsilon = 2$). Then $|\mathcal{O}(i_0, j_0)| = \varepsilon n_1$, and $|\mathcal{O}_1| = 2n_1$, $F_{\mathcal{O}_1} = \bigoplus_{(i,j) \in \mathcal{O}_1} F_{i,j} = \bigoplus_{q=0}^{n_1-1} F_{s^q(i_0), s^q(j_0)}$,

and for $j = 0, \dots, n_1 - 1$, there exists $u_j \in F_{s^j(i_0), s^j(j_0)}$ such that $\gamma_1 = u_0 \cdots u_{n_1-1}$. Hence u_0, \dots, u_{n_1-1} belong to distinct free factors of $UV P_n$, so commute pairwise. Further:

$$s^k(u_j) \in F_{s^{j+k}(i_0), s^{j+k}(j_0)} \text{ for all } k \in \mathbb{Z} \text{ and } j = 0, \dots, n_1 - 1, \quad (17)$$

where the superscript $j+k$ is taken modulo n_1 . In particular, if $v \in F_{s^q(i_0), s^q(j_0)}$ for some $q = 0, \dots, n_1 - 1$, then $s^{n_1}(v) = v$ (resp. $s^{n_1}(v) = \alpha(v)$, where $\alpha: F_{s^q(i_0), s^q(j_0)} \rightarrow F_{s^q(i_0), s^q(j_0)}$ is the automorphism that exchanges $\lambda_{s^q(i_0), s^q(j_0)}$ and $\lambda_{s^q(j_0), s^q(i_0)}$, as in Lemma 4.4), and thus $s^{\varepsilon n_1}(v) = v$. It follows from this, the fact that $o(w_1) = o(s)$ and (16) that:

$$\begin{aligned} 1 &= w_1^{\varepsilon n_1 o(s)} = (w_1^{\varepsilon n_1})^{o(s)} = (\gamma_1 s(\gamma_1) \cdots s^{\varepsilon n_1-1}(\gamma_1) s^{\varepsilon n_1})^{o(s)} = (\gamma_1 s(\gamma_1) \cdots s^{\varepsilon n_1-1}(\gamma_1))^{o(s)} s^{\varepsilon n_1 o(s)} \\ &= (\gamma_1 s(\gamma_1) \cdots s^{\varepsilon n_1-1}(\gamma_1))^{o(s)}. \end{aligned}$$

Since $UV P_n$ is torsion free, we conclude that $\gamma_1 s(\gamma_1) \cdots s^{\varepsilon n_1-1}(\gamma_1) = 1$. Hence:

$$u_0 \cdots u_{n_1-1} s(u_0 \cdots u_{n_1-1}) \cdots s^{\varepsilon n_1-1}(u_0 \cdots u_{n_1-1}) = 1. \quad (18)$$

Applying (17), and projecting (18) into $F_{1,2}$, we see that:

$$1 = \prod_{i=0}^{\varepsilon-1} s^{i n_1}(u_0) s^{i n_1+1}(u_{n_1-1}) \cdots s^{(i+1)n_1-1}(u_1) = \prod_{i=0}^{\varepsilon-1} s^{i n_1}(u_0 s(u_{n_1-1}) \cdots s^{n_1-1}(u_1)). \quad (19)$$

We deduce from (19) (resp. from (19) and Lemma 4.4) that $u_0 = \prod_{j=1}^{n_1-1} s^{n_1-j}(u_j^{-1})$ (resp. that there exists $\tilde{u} \in F_{1,2}$ such that $u_0 = \tilde{u} s^{n_1}(\tilde{u}^{-1})(\prod_{j=1}^{n_1-1} s^{n_1-j}(u_j^{-1}))$). Using the fact that u_1, \dots, u_{n_1-1} commute pairwise, it follows that:

$$\gamma_1 = u_0 u_1 \cdots u_{n_1-1} = \beta s^{n_1-1}(u_1^{-1}) \cdots s(u_{n_1-1}^{-1}) \cdot u_{n_1-1} \cdots u_1, \quad (20)$$

where $\beta = 1$ (resp. $\beta = \tilde{u} s^{n_1}(\tilde{u}^{-1})$). In what follows, if $a, b \in UV B_n$, we shall write $a \sim b$ if a and b are conjugate by an element of $UV P_n$. Let us show by induction that:

$$\gamma_1 \cdot s \sim \beta s^{n_1-1}(u_1^{-1}) \cdots s^{t+1}(u_{n_1-(t+1)}^{-1}) s^t(u_{n_1-t}^{-1}) \cdot u_{n_1-t} \cdots u_1 \cdot s \quad (21)$$

for all $t = 1, \dots, n_1$. If $t = 1$ then the result follows directly from (20). So suppose that (21) holds for some $1 \leq t \leq n_1 - 1$. Let $M_0 = \prod_{c=1}^{n_1-1} s^c(\tilde{u})$, and for $1 \leq t \leq n_1 - 1$, let $M_t = \prod_{c=1}^{t-1} s^{t-c}(u_{n_1-t})$. Note that $M_1 = 1$, and that $M_0 = 1$ if $\mathcal{O}(i_0, j_0) \neq \mathcal{O}(j_0, i_0)$. By (17), for $c = 1, \dots, t-1$, $s^{t-c}(u_{n_1-t}) \in F_{s^{n_1-c}(i_0), s^{n_1-c}(j_0)}$, so for $1 \leq t \leq n_1 - 1$, the factors of M_t commute pairwise, and $M_t \in \bigoplus_{c=n_1-t+1}^{n_1-1} F_{s^c(i_0), s^c(j_0)}$. Thus:

$$M_t^{-1} s^t(u_{n_1-t}^{-1}) = \left(\prod_{c=1}^{t-1} s^{t-c}(u_{n_1-t}) \right)^{-1} s^t(u_{n_1-t}^{-1}) = \left(\prod_{c=1}^{t-1} s^c(u_{n_1-t}^{-1}) \right) s^t(u_{n_1-t}^{-1}) = \prod_{c=1}^t s^c(u_{n_1-t}^{-1}). \quad (22)$$

and since $s^t(u_{n_1-t}^{-1}) \in F_{i_0, j_0}$ and $u_{n_1-t} \cdots u_1 \in \bigoplus_{i=1}^{n_1-t} F_{s^i(i_0), s^i(j_0)}$, we see that $M_t^{-1} s^t(u_{n_1-t}^{-1})$ commutes with $u_{n_1-t} \cdots u_1$. So by (21) and (22), we have:

$$\begin{aligned} \gamma_1 \cdot s &\sim \beta s^{n_1-1}(u_1^{-1}) \cdots s^{t+1}(u_{n_1-(t+1)}^{-1}) M_t M_t^{-1} s^t(u_{n_1-t}^{-1}) \cdot u_{n_1-t} \cdots u_1 \cdot s \\ &= \beta s^{n_1-1}(u_1^{-1}) \cdots s^{t+1}(u_{n_1-(t+1)}^{-1}) M_t \cdot u_{n_1-t} \cdots u_1 \left(\prod_{c=1}^t s^c(u_{n_1-t}^{-1}) \right) \cdot s \\ &= \beta s^{n_1-1}(u_1^{-1}) \cdots s^{t+1}(u_{n_1-(t+1)}^{-1}) \left(\prod_{c=1}^t s^{t-c}(u_{n_1-t}) \right) u_{n_1-(t+1)} \cdots u_1 \cdot s \cdot \prod_{c=0}^{t-1} s^c(u_{n_1-t}^{-1}), \quad (23) \end{aligned}$$

where we have used the fact that $M_t \cdot u_{n_1-t} = \prod_{c=1}^t s^{t-c}(u_{n_1-t})$. Now by (17), $\prod_{c=1}^t s^{t-c}(u_{n_1-t}) \in \bigoplus_{c=n_1-t}^{n_1-1} F_{s^c(i_0), s^c(j_0)}$, and since $\beta s^{n_1-1}(u_1^{-1}) \cdots s^{t+1}(u_{n_1-(t+1)}^{-1}) \in F_{i_0, j_0}$, we see that these two terms commute, and it follows from (23) that:

$$\gamma_1 \cdot s \sim \left(\prod_{c=1}^t s^{t-c}(u_{n_1-t}) \right) \beta s^{n_1-1}(u_1^{-1}) \cdots s^{t+1}(u_{n_1-(t+1)}^{-1}) u_{n_1-(t+1)} \cdots u_1 \cdot s \cdot \prod_{c=0}^{t-1} s^c(u_{n_1-t}^{-1}). \quad (24)$$

One may check that $\prod_{c=1}^t s^{t-c}(u_{n_1-t})$ is the inverse of $\prod_{c=0}^{t-1} s^c(u_{n_1-t}^{-1})$, and using (24), we conclude that (21) holds for $t+1$. Taking $t = n_1$ in (21), we obtain $\gamma_1 \cdot s \sim \beta \cdot s$. If $\mathcal{O}(i_0, j_0) \neq \mathcal{O}(j_0, i_0)$ then $\beta = 1$, and the statement of the lemma holds in this case. So suppose that $\mathcal{O}(i_0, j_0) = \mathcal{O}(j_0, i_0)$. Then $\tilde{u} \in F_{i_0, j_0}$, and for $c = 1, \dots, n_1 - 1$, $s^c(\tilde{u}) \in F_{s^c(i_0), s^c(j_0)}$. Thus the factors of $\tilde{u}M_0 = \prod_{c=0}^{n_1-1} s^c(\tilde{u})$ commute pairwise, and hence $M_0^{-1}s^{n_1}(\tilde{u}^{-1}) = \prod_{c=1}^{n_1} s^c(\tilde{u}^{-1})$. It follows that:

$$\begin{aligned} \gamma_1 \cdot s \sim \beta \cdot s &= \tilde{u}s^{n_1}(\tilde{u}^{-1}) \cdot s = \tilde{u}M_0M_0^{-1}s^{n_1}(\tilde{u}^{-1}) \cdot s = \left(\prod_{c=0}^{n_1-1} s^c(\tilde{u}) \right) \left(\prod_{c=1}^{n_1} s^c(\tilde{u}^{-1}) \right) \cdot s \\ &= \left(\prod_{c=0}^{n_1-1} s^c(\tilde{u}) \right) \cdot s \cdot \left(\prod_{c=0}^{n_1-1} s^c(\tilde{u}^{-1}) \right). \end{aligned}$$

Now $\prod_{c=0}^{n_1-1} s^c(\tilde{u})$ may be seen to be the inverse of $\prod_{c=0}^{n_1-1} s^c(\tilde{u}^{-1})$, and therefore $\gamma_1 \cdot s \sim s$ also in this case, which completes the proof of the lemma. \square

This enables us to prove Theorem 4.3.

Proof of Theorem 4.3. Let $w \in UVB_n$ be a torsion element of order r . By Theorems 2.3 and 2.4 there exist (unique) $u \in UVP_n$ and $s \in S_n$ such that $w = us$. As in (16), we see that $u \cdot s(u) \cdots s^{r-1}(u) = 1$ in UVP_n and $s^r = 1$. Further, from (14), $u = \gamma_1 \cdots \gamma_{n(s)}$, where for $l = 1, \dots, n(s)$, $\gamma_l \in F_{\mathcal{O}_l}$. Let $l \in \{1, \dots, n(s)\}$. Since $F_{\mathcal{O}_l}$ is invariant under conjugation by s , it follows from (14) that $\gamma_l s(\gamma_l) \cdots s^{r-1}(\gamma_l) = 1$ in $F_{\mathcal{O}_l}$. Since $\gamma_l s(\gamma_l) \cdots s^{r-1}(\gamma_l)$ may also be written as $(\gamma_l s)^r$ using (16), we conclude from Lemma 4.5 that $\gamma_l s$ is conjugate to s by an element of $F_{\mathcal{O}_l}$, or in other words, there exists $\Lambda_l \in F_{\mathcal{O}_l}$ such that $\gamma_l s = \Lambda_l \cdot s \cdot \Lambda_l^{-1}$. Let us prove by reverse induction that for all $m = 1, \dots, n(s) + 1$:

$$w = \Lambda_{n(s)} \Lambda_{n(s)-1} \cdots \Lambda_m \gamma_1 \cdots \gamma_{m-1} \cdot s \cdot \Lambda_m^{-1} \cdots \Lambda_{n(s)-1}^{-1} \Lambda_{n(s)}^{-1}. \quad (25)$$

If $m = n(s) + 1$ then (25) follows directly from the fact that $w = us = \gamma_1 \cdots \gamma_{n(s)} \cdot s$. So suppose that (25) holds for some $m \in \{2, \dots, n(s) + 1\}$. Then using (14) and the facts that $\gamma_1 \cdots \gamma_{m-2} \in \bigoplus_{i=1}^{m-2} F_{\mathcal{O}_i}$ and $\gamma_{m-1} \in F_{\mathcal{O}_{m-1}}$, we obtain:

$$\begin{aligned} w &= \Lambda_{n(s)} \Lambda_{n(s)-1} \cdots \Lambda_m \gamma_1 \cdots \gamma_{m-2} (\gamma_{m-1} \cdot s) \cdot \Lambda_m^{-1} \cdots \Lambda_{n(s)-1}^{-1} \Lambda_{n(s)}^{-1} \\ &= \Lambda_{n(s)} \Lambda_{n(s)-1} \cdots \Lambda_m \gamma_1 \cdots \gamma_{m-2} \Lambda_{m-1} \cdot s \cdot \Lambda_{m-1}^{-1} \Lambda_m^{-1} \cdots \Lambda_{n(s)-1}^{-1} \Lambda_{n(s)}^{-1} \\ &= \Lambda_{n(s)} \Lambda_{n(s)-1} \cdots \Lambda_m \Lambda_{m-1} \gamma_1 \cdots \gamma_{m-2} \cdot s \cdot \Lambda_{m-1}^{-1} \Lambda_m^{-1} \cdots \Lambda_{n(s)-1}^{-1} \Lambda_{n(s)}^{-1}, \end{aligned}$$

which proves (25) for $m-1$. Taking $m = 1$, we see that $w = \Lambda \cdot s \cdot \Lambda^{-1}$, where $\Lambda = \Lambda_{n(s)} \cdots \Lambda_1 \in UVP_n$, and this completes the proof of the theorem. \square

Remark 4.6. It is an open question whether a result similar to that of Theorem 4.3 holds for VB_n and for WB_n .

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