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Some classes of singular ODEs flows

Marc Briane*

February 21, 2024

Abstract

This paper is devoted to some singular ODEs flows $X(\cdot, x)$ for x in the d -dimensional torus \mathbb{T}^d , solutions to $\partial_t X(t, x) = b(X(t, x))$, with the initial condition $X(0, x) = x$. On the one hand, in dimension two assuming that the Herman rotation set for the flow X is parallel to some integer vector $k \neq 0_{\mathbb{R}^2}$, and that the component $b(x) \cdot k^\perp$ is dominated by a function only depending on $x \cdot k^\perp$ and changing sign, we prove that the flow X cannot have any invariant probability measure with positive Lebesgue density. As a by-product such a flow is also devoid of any first integral with periodic gradient. This result is illustrated by an explicit class of singular flows which contains in particular two-dimensional Euler flows. We also extend this class in dimension three restricting ourselves to the singular invariant measures. On the other hand, we consider a vector field b satisfying the non-negativity condition $b \cdot (\nabla u - \overline{\nabla u}) \geq 0$ in \mathbb{T}^d but not identically null, for some periodic gradient field ∇u . Then, the flow associated with b turns out to have only singular invariant probability measures in any dimension $d \geq 2$, and to be not integrable in dimension two. This class is illustrated by rather general examples.

Keywords: ODEs flow, rotation set, invariant measure, first integral

Mathematics Subject Classification: 34E05, 34E10, 37C10, 37C40

1 Introduction

In this paper we study some ODEs flows associated with a regular vector field b defined in the torus \mathbb{T}^d , solutions to

$$\begin{cases} \partial_t X(t, x) = b(X(t, x)), & t \in [0, \infty) \\ X(0, x) = x \in \mathbb{T}^d. \end{cases} \quad (1.1)$$

A flow X will be said *singular* if it has one or both of the following properties:

1. The flow X admits only invariant probability measures on \mathbb{T}^d ⁽¹⁾ which are singular with respect to Lebesgue's measure. For example, consider the Stepanoff flow [15] associated with the vector field defined by $b(x) := \rho(x) \xi$, where ρ is a non-negative function in $C_\#^1(\mathbb{T}^d)$ with a finite number of roots in \mathbb{T}^d and $\sigma := 1/\rho \in L_\#^1(\mathbb{T}^d)$ and where $\xi \in \mathbb{R}^d$ has incommensurable coordinates. By virtue of [4, Pro. 5.1]) $\sigma(x)/\bar{\sigma} dx$ is the unique invariant probability measure for this flow which is absolutely continuous with respect to Lebesgue's measure. In contrast, if the function ρ changes sign and has a (at most) countable zero set $\{\rho = 0\}$, then by virtue of [5, Pro. 2.1] any invariant probability measure on \mathbb{T}^d has support in $\{\rho = 0\}$, and is thus singular with respect to Lebesgue's measure.

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¹*i.e.* the probability measures μ on \mathbb{T}^d such that $\mu \circ X^{-1}(t, \cdot) = \mu \circ X(-t, \cdot) = \mu$ for any $t \in \mathbb{R}$.

2. The flow X has no *first integral* u ⁽²⁾ with \mathbb{Z}^d -periodic gradient, which is real-analytic. From now on, such a flow will be said *not integrable*. For example, the so-called ABC flow (Arnold-Beltrami-Childress) introduced by Arnold [1] is shown in [17] to be not integrable when $A = B$.

In view of the previous (apparently) simple examples the singular properties of some given flow are difficult to derive. So, the dynamical systems of classical mechanics (see, *e.g.* [8, §2.2]) have a smooth invariant probability measure (with a constant density for the Hamiltonian systems) by virtue of Liouville's theorem [8, Thm. 1, §2.2] (see also [3, Pro. 2.2]). In the present case, we need thus to prove that any invariant probability measure μ on \mathbb{T}^d for the flow (1.1) associated with some vector field b , *i.e.* solution to the PDE

$$\operatorname{div}(\mu b) = 0 \text{ in } \mathbb{T}^d \quad (1.2)$$

(see below for an equivalent variational formulation (1.10)), is actually singular with respect to the Lebesgue measure. However, such a solution μ is far to be explicit, which makes the problem quite intricate.

Our aim is to provide different classes of ODEs flows which have only singular invariant probability measures on \mathbb{T}^d for $d = 2, 3$, and which are possibly not integrable. First, in Section 2 we prove a general two-dimensional result for a flow (1.1) (see Theorem 2.1) the assumptions of which combine the structure of the vector field b and the one of the Herman rotation set C_b (see definition (1.14) below) induced by b . More precisely, we assume that the set C_b – which by virtue of the Franks-Misiurewicz theorem (see Theorem 1.1) is known to a closed line segment along a line passing through $0_{\mathbb{R}^2}$ – is contained in the line $\mathbb{R}k$ for some $k \in \mathbb{Z}^2 \setminus \{0_{\mathbb{R}^2}\}$. We also assume that the vector field b satisfies the representation

$$\begin{aligned} \forall x \in \mathbb{T}^2, \quad b(x) \cdot k^\perp &= \alpha(x \cdot k^\perp) + \beta(x), \\ \text{with } \max_{x \in \mathbb{T}^2} |\alpha(x \cdot k^\perp)| &> \max_{\mathbb{T}^2} |\beta|. \end{aligned} \quad (1.3)$$

In other words, (1.3) means that the component $b(x) \cdot k^\perp$ is dominated by a function which only depends on the variable $x \cdot k^\perp$ and which changes sign. Then, the flow associated with b has no regular invariant probability measure of the type $\sigma(x) dx$, where σ is a positive (in the broad sense of (2.3)) function in $L^1_{\sharp}(\mathbb{T}^2)$. If in addition the vector field does not vanish in \mathbb{T}^2 , then the flow has no real-analytic first integral u such that ∇u is \mathbb{Z}^2 -periodic. Such a class is illustrated by the flows of Proposition 2.4 (in which $x \cdot k^\perp = x_1$) which satisfy the two previous conditions. In particular, this class contains the two-dimensional parametrized Euler flows (see §2.4.3). Next, we extend the two-dimensional class of singular flows to dimension three through Proposition 2.7, except for the question of integrability which is more delicate in dimension three. Finally, the results of Section 2 are illustrated by various examples in §2.4.

Alternatively, in Section 3 we show (see Theorem 3.1) that if the vector field b in $C^1_{\sharp}(\mathbb{T}^d)^d$ satisfies the non-negativity condition

$$b \cdot (\nabla u - \overline{\nabla u}) \geq 0 \text{ in } \mathbb{T}^d, \quad \text{but not identically null,} \quad (1.4)$$

for some periodic gradient field $\nabla u \in C^1_{\sharp}(\mathbb{T}^d)^d$, then the flow (1.1) associated with b has only singular invariant probability measures in any dimension $d \geq 2$, and is not integrable in dimension two. This result is illustrated by the class of gradient flows of §3.2.1 in any dimension, and by the class of two-dimensional flows of §3.2.2 which is not restricted to gradient flows.

²*i.e.* a continuous function u satisfying $u(X(\cdot, x)) = u(x)$ for any $x \in \mathbb{T}^d$.

1.1 Notations

- (e_1, \dots, e_d) is the canonical basis of \mathbb{R}^d , and $0_{\mathbb{R}^d}$ is the null vector of \mathbb{R}^d .
- I_d is the unit matrix of $\mathbb{R}^{d \times d}$, and $R_{\perp} := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.
- A^T denotes the transposed matrix of $A \in \mathbb{R}^{d \times d}$, and $\mathbb{R}_S^{d \times d}$ denotes the subspace of the symmetric matrices in $\mathbb{R}^{d \times d}$.
- $\text{tr } A$ denotes the trace and $\det A$ denotes the determinant of the matrix $A \in \mathbb{R}^{d \times d}$.
- For $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$, we denote $\xi^{\perp} := R_{\perp} \xi = (-\xi_2, \xi_1)$.
- “ \cdot ” denotes the scalar product and $|\cdot|$ the euclidean norm in \mathbb{R}^d .
- \mathbb{T}^d denotes the d -dimensional torus $\mathbb{R}^d/\mathbb{Z}^d$ for $d \in \mathbb{N}$.
- dx denotes the Lebesgue measure on \mathbb{R}^d and on \mathbb{T}^d .
- $\mathcal{M}_{\sharp}(\mathbb{T}^d)$ denotes the set of Radon measures on the torus \mathbb{T}^d , and $\mathcal{M}_p(\mathbb{T}^d)$ denotes the set of probability measures on \mathbb{T}^d .
- $C_{\sharp}^k(\mathbb{T}^d)$, respectively $W_{\sharp}^{1,k}(\mathbb{T}^d)$, $k \in \mathbb{N} \cup \{\infty\}$, denotes the space of the real-valued functions $f \in C^k(\mathbb{R}^d)$, respectively $f \in W_{\sharp}^{1,k}(\mathbb{R}^d)$, which are \mathbb{Z}^d -periodic, *i.e.*

$$\forall k \in \mathbb{Z}^d, \forall x \in \mathbb{R}^d, \quad f(x+k) = f(x). \quad (1.5)$$

- For any function $f \in L_{\sharp}^1(\mathbb{T}^d)$, we denote

$$\begin{cases} \bar{f}^{x'}(x_1) := \int_{\mathbb{T}^{d-1}} f(x_1, x') dx', & x_1 \in \mathbb{T}^1 \\ \bar{f}^{x_1}(x') := \int_{\mathbb{T}^1} f(x_1, x') dx_1, & x' = (x_2, \dots, x_d) \in \mathbb{T}^{d-1}, \end{cases} \quad \text{where } \bar{f} := \int_{\mathbb{T}^d} f(x) dx. \quad (1.6)$$

For any Borel measure μ on \mathbb{T}^d and any function $f \in L^1(\mathbb{T}^d, \mu)$, we denote

$$\mu(f) := \int_{\mathbb{T}^d} f(x) \mu(dx) \quad (1.7)$$

which is also extended to vector-valued functions in $L^1(\mathbb{T}^d, \mu)^d$.

- For two functions f, g defined on a non-empty set E , “ $f \not\equiv g$ ” means that f and g are not identically equal in E .
- C denotes a positive constant which may vary from line to line.

1.2 Some recalls of ergodic theory

1.2.1 Invariant measures and rotation sets

Let b a vector field in $C_{\sharp}^1(\mathbb{T}^d)^d$. A probability measure μ in $\mathcal{M}_{\sharp}(\mathbb{T}^d)$ is said to be *invariant for the flow* X associated with the vector field b by (1.1), if one has

$$\forall t \in \mathbb{R}, \forall \psi \in C_{\sharp}^0(\mathbb{T}^d), \quad \int_{\mathbb{T}^2} \psi(X(t, y)) d\mu(y) = \int_{\mathbb{T}^d} \psi(y) d\mu(y). \quad (1.8)$$

By virtue of Liouville's theorem (see, *e.g.*, [8, Thm. 1, §2.2]) condition (1.8) means that

$$\operatorname{div}(b\mu) = 0 \quad \text{in } \mathbb{T}^d, \quad (1.9)$$

or equivalently, with the variational formulation (see, *e.g.*, [4, Pro. 2.2])

$$\forall \varphi \in C_{\sharp}^1(\mathbb{T}^d), \quad \int_{\mathbb{T}^d} b(x) \cdot \nabla \varphi(x) \mu(dx) = 0. \quad (1.10)$$

Alternatively, define the set

$$\mathcal{I}_b := \left\{ \mu \in \mathcal{M}_p(\mathbb{T}^2) : \mu \text{ invariant for the flow } X \right\}, \quad (1.11)$$

where $\mathcal{M}_p(\mathbb{T}^d)$ is the set of probability measures on \mathbb{T}^d . Also define the two following non-empty subsets of \mathbb{R}^d :

- The set of all the limit points of the sequences $(X(n, x)/n)_{n \geq 1}$ for $x \in \mathbb{T}^d$ (denoted by $\rho_p(b)$ in [13])

$$A_b := \bigcup_{x \in \mathbb{T}^2} \left[\bigcap_{n \geq 1} \overline{\left\{ \frac{X(k, x)}{k} : k \geq n \right\}} \right]. \quad (1.12)$$

When b does not vanish in \mathbb{T}^2 , by virtue of the Peirone result [14, Thm. 3.1] the former definition is reduced to

$$A_b = \left\{ \lim_{t \rightarrow \infty} \frac{X(t, x)}{t} : x \in \mathbb{T}^d \right\}. \quad (1.13)$$

- The so-called Herman [11] rotation set (recall notation (1.7))

$$C_b := \left\{ \mu(b) : \mu \in \mathcal{I}_b \right\} \quad (1.14)$$

which is a compact convex subset of \mathbb{R}^d .

An implicit consequence of the Misiurewicz-Ziemian results [13, Thm. 2.4, Rem. 2.5] and [13, Cor. 2.6] shows that

$$A_b \subset C_b = \operatorname{conv}(A_b).$$

1.2.2 Herman's rotation set in dimension two

In dimension two [13, Thm. 3.4 (c)] shows that actually the set A_b (1.12) is a convex set of \mathbb{R}^2 , which thus implies that

$$C_b = A_b. \quad (1.15)$$

If in addition b does not vanish in \mathbb{T}^2 , we thus deduce from (1.13) that

$$\mathbf{C}_b = \{\mu(b) : \mu \in \mathcal{A}_b\} = \left\{ \lim_{t \rightarrow \infty} \frac{X(t, x)}{t} : x \in \mathbb{T}^2 \right\} = \mathbf{A}_b. \quad (1.16)$$

This connects closely the invariant probability measures for the flow X (1.1) and the limit points of the flow. Moreover, Herman's rotation set satisfies the celebrated Franks-Misiurewicz ergodic theorem [9] (which actually extends to any continuous flow).

Theorem 1.1 (Franks-Misiurewicz, 1990). *Let b be a vector field in $C_{\sharp}^1(\mathbb{T}^2)^2$. Then, there exists a vector $\xi \in \mathbb{R}^2$ such that the Herman rotation set satisfies $\mathbf{C}_b \subset \mathbb{R}\xi$. More precisely, one of the three following mutually disjoint cases holds:*

- (i) \mathbf{C}_b is a unit set.
- (ii) \mathbf{C}_b is a closed line segment of a line passing through $0_{\mathbb{R}^2}$, and some another point of \mathbb{Q}^2 .
- (iii) \mathbf{C}_b is a closed line segment with one end at $0_{\mathbb{R}^2}$, and an irrational slope.

2 A class involving some change of sign

2.1 A general two-dimensional result

We have the following result which allows us to exhibit singular two-dimensional ODEs flows.

Theorem 2.1. *Let b be a vector field in $C_{\sharp}^1(\mathbb{T}^2)^2$ such that the Herman set \mathbf{C}_b reads as*

$$\mathbf{C}_b = I k, \quad \text{where } I \text{ is a closed segment of } (0, \infty) \text{ and } k \in \mathbb{Z}^2 \setminus \{0_{\mathbb{R}^2}\}. \quad (2.1)$$

Assume that there exist two functions $\alpha \in C_{\sharp}^0(\mathbb{T}^1)$ and $\beta \in C_{\sharp}^0(\mathbb{T}^2)$ such that

$$\begin{aligned} \forall x \in \mathbb{T}^2, \quad b(x) \cdot k^{\perp} &= \alpha(x \cdot k^{\perp}) + \beta(x), \\ \text{with } \max_{x \in \mathbb{T}^2} |\alpha(x \cdot k^{\perp})| &> \max_{\mathbb{T}^2} |\beta|. \end{aligned} \quad (2.2)$$

Then, for any function $\sigma \in L_{\sharp}^1(\mathbb{T}^2)$ the sections of which satisfy the positivity property

$$\text{for any closed line segment } S \subset \mathbb{R}^2, \quad \int_S \sigma > 0 \quad \text{a.e. in } T, \quad (2.3)$$

where T is any closed line segment orthogonal to S from some end point of S (so that $S \times T$ is a closed rectangle of \mathbb{R}^2), the probability measure $\sigma(x)/\bar{\sigma} dx$ cannot be invariant for the flow (1.1) associated with the vector field b .

Moreover, if the vector field b does not vanish in \mathbb{T}^2 , there does not exist any real-analytic first integral u for the flow with $\nabla u \in C_{\sharp}^1(\mathbb{T}^2)^2$.

Remark 2.2. In view of the Franks-Misiurewicz theorem, the assumption (2.1) of Theorem 2.1 corresponds in the statement of Theorem 1.1, either to the case (i) when \mathbf{C}_b is reduced to a commensurable vector in $\mathbb{R}^2 \setminus \{0_{\mathbb{R}^2}\}$, or to the case (ii).

Remark 2.3. It is not *a priori* evident to provide explicitly a vector field b satisfying the assumptions of Theorem 2.1. Indeed, the same integer vector k has to satisfy both condition (2.2) and the equality set $\mathbf{C}_b = I k$. The representation of the Herman rotation set \mathbf{C}_b (1.14) cannot be simply deduced from the data of the vector field b . However, in §2.2 we provide a class of suitable vector fields b illustrating the theoretical result of Theorem 2.1.

Proof of Theorem 2.1. Let b be a vector field in $C_{\sharp}^1(\mathbb{T}^2)^2$ satisfying the condition (2.1) with some $k \in \mathbb{Z}^2 \setminus \{0_{\mathbb{R}^2}\}$. Assume by contradiction that there exists an invariant probability measure for the flow (1.1), *i.e.* of the type $\sigma(x) dx$ where σ is in $L_{\sharp}^1(\mathbb{T}^2)$ and satisfies the positivity property (2.3). By Liouville's theorem combined with the representation of the divergence free vector fields in \mathbb{R}^2 , there exists a current field $\varphi \in W_{\text{loc}}^1(\mathbb{R}^2)$ with $\nabla\varphi \in L_{\sharp}^1(\mathbb{T}^2)^2$, such that

$$\sigma b = \nabla^{\perp}\varphi = (\nabla\varphi)^{\perp} \quad \text{in } \mathbb{T}^2.$$

Making the change of variables

$$x = Ky \quad \text{where} \quad K := \frac{1}{|k|} \begin{pmatrix} -k_2 & -k_1 \\ k_1 & -k_2 \end{pmatrix} = (K^T)^{-1}, \quad (2.4)$$

which implies that $K^T k = -|k| e_2$ and $K^T k^{\perp} = |k| e_1$. Then, using (2.2) we get that for a.e. y in the quotient group $\mathbb{R}^2/(K^{-1}\mathbb{Z}^2)$,

$$\sigma(Ky) b(Ky) \cdot k^{\perp} = \sigma(Ky) \alpha(|k| y_1) + \sigma(Ky) \beta(Ky) = \nabla^{\perp}\varphi(Ky) \cdot k^{\perp} = \nabla\varphi(Ky) \cdot k. \quad (2.5)$$

Moreover, by (2.1) we have $\overline{\sigma b} \cdot k^{\perp} = 0$. Taking the mean value of (2.5) this yields $\overline{\nabla\varphi(K\cdot)} \cdot k = 0$. Hence, defining the function $\varphi_{\sharp} : (x \mapsto \varphi(x) - \overline{\nabla\varphi} \cdot x)$ which belongs to $W_{\sharp}^1(\mathbb{T}^2)$, we deduce that

$$\nabla\varphi_{\sharp} \cdot k = \nabla\varphi \cdot k - \overline{\nabla\varphi} \cdot k = \nabla\varphi \cdot k.$$

Putting this in (2.5) and recalling $K^T k = -|k| e_2$, we get that

$$\begin{aligned} \alpha(|k| y_1) + \sigma(Ky) \beta(Ky) &= \nabla^{\perp}\varphi_{\sharp}(Ky) \cdot k^{\perp} = K \nabla(\varphi_{\sharp}(Ky)) \cdot k \\ &= -|k| \nabla(\varphi_{\sharp}(Ky)) \cdot e_2 = -|k| \partial_{y_2}(\varphi_{\sharp}(Ky)). \end{aligned} \quad (2.6)$$

Now, integrating (2.6) with respect to $y_2 \in \mathbb{R}/(k/|k| \cdot \mathbb{Z}^2)$ (due to (2.4)) and taking into account the periodicity of φ_{\sharp} , we obtain that

$$\overline{\sigma(K\cdot)}^{y_2}(y_1) \alpha(|k| y_1) + \overline{\sigma(K\cdot) \beta(K\cdot)}^{y_2}(y_1) = 0 \quad \text{for } y_1 \in \mathbb{R}/(k^{\perp}/|k| \cdot \mathbb{Z}^2),$$

which combined with the positivity property (2.3) of σ , implies that

$$\begin{aligned} \max_{x \in \mathbb{T}^2} \left| \alpha(x \cdot k^{\perp}) \right| &= \max_{y_1 \in \mathbb{R}/(k^{\perp}/|k| \cdot \mathbb{Z}^2)} \left| \alpha(|k| y_1) \right| \\ &\leq \max_{y_1 \in \mathbb{R}/(k^{\perp}/|k| \cdot \mathbb{Z}^2)} \frac{\left| \overline{\sigma(K\cdot) \beta(K\cdot)}^{y_2}(y_1) \right|}{\left| \overline{\sigma(K\cdot)}^{y_2}(y_1) \right|} \leq \max_{\mathbb{T}^2} |\beta|. \end{aligned}$$

This contradicts the strict inequality of (2.2) and concludes the first part of the proof.

Next, assume that there exists a real-analytic first integral u for the flow, such that the gradient field ∇u is \mathbb{Z}^2 -periodic and has a finite number of roots in \mathbb{T}^2 . Hence, taking the time derivative in $u(X(t, x)) = u(x)$, we get that

$$\forall x \in \mathbb{T}^2, \forall t \in [0, \infty), \quad \nabla u(X(t, x)) \cdot b(X(t, x)) = 0,$$

and thus for $t = 0$, $\nabla u \cdot b = 0$ in \mathbb{T}^2 . Now, if the vector field b does not vanish in \mathbb{T}^2 , there exists a function $\sigma \in C_{\sharp}^0(\mathbb{T}^2)$ such that $\nabla^{\perp}u = \sigma b$ in \mathbb{T}^2 . So, the continuous function σ does not vanish in $\mathbb{R}^2 \setminus \{\nabla u = 0\}$ which by assumption is a connected open set of \mathbb{R}^2 . It follows that σ has a constant sign, say positive, in $\mathbb{R}^2 \setminus \{\nabla u = 0\}$. This combined with the continuity of σ yields the positivity property (2.3), which thus contradicts the first part of Theorem 2.1. Therefore, the gradient field ∇u admits at least a cluster point in \mathbb{T}^2 , so that u cannot be real-analytic.

The proof of Theorem 2.1 is thus done. \square

2.2 A class of two-dimensional singular flows

We have the following result.

Proposition 2.4. *Let $b \in C_{\sharp}^1(\mathbb{T}^2)^2$ and let $\alpha, \beta \in C_{\sharp}^1(\mathbb{T}^2)$ be such that the first coordinate of b reads as*

$$b_1(x) = \alpha(x_1) + \beta(x) \text{ for } x \in \mathbb{T}^2, \quad (2.7)$$

with $B := \max_{\mathbb{T}^2} |\beta| < \min \left(-\min_{\mathbb{T}^1} \alpha, \max_{\mathbb{T}^1} \alpha \right)$.

Then, the two-dimensional flow (1.1) is singular in the sense of Theorem 2.1.

Remark 2.5. It is easy to check that condition (2.7) already prevents Lebesgue's measure from being invariant for the flow associated with the vector field b satisfying (2.7). Indeed, let $p_1, q_1 \in \mathbb{T}^1$ be such that

$$\min_{\mathbb{T}^1} \alpha = \alpha(p_1) < 0 \text{ and } \max_{\mathbb{T}^1} \alpha = \alpha(q_1) > 0.$$

Then, we deduce from the inequality of (2.7) that

$$\int_{p_1}^{q_1} \left(\int_{\mathbb{T}^1} (\operatorname{div} b)(x_1, x_2) dx_2 \right) dx_1 = \underbrace{(\alpha + \bar{\beta}^{x_2})(q_1)}_{>0} - \underbrace{(\alpha + \bar{\beta}^{x_2})(p_1)}_{<0} > 0.$$

Therefore, the vector field b of (2.7) cannot be divergence free in \mathbb{T}^2 .

Proof of Proposition 2.4. Let us start to prove that for any $x^0 = (x_1^0, x_2^0)$ in \mathbb{T}^2 , the first coordinate $x_1 := x_1(\cdot, x^0)$ of the flow $X := X(\cdot, x^0)$ (1.1) associated with b , is bounded in $[0, \infty)$. Due to the strict inequality of (2.7) there exists a point $p_1 \in \mathbb{T}^1$ such that $\alpha(x_1^0) \neq \alpha(p_1) < -B$. Hence, we have

$$x_1' - \alpha(x_1) = b_1(X) - \alpha(x_1) = \beta(X) \leq B < -\alpha(p_1) \neq -\alpha(x_1^0) \text{ in } [0, \infty).$$

Then, applying the first inequality (2.10) of Lemma 2.6 below to the function x_1 , with $f := -\alpha$ and $x_0 := p_1$, we get that

$$x_1 - x_1^0 = x_1 - x_1(0) < 1 \text{ in } [0, \infty).$$

Similarly, let q_1 be a point in \mathbb{T}^1 such that $\alpha(x_1^0) \neq \alpha(q_1) > B$. Then, we have

$$x_1' - \alpha(x_1) = b_1(X) - \alpha(x_1) = \beta(X) \geq -B > -\alpha(q_1) \neq -\alpha(x_1^0) \text{ in } [0, \infty).$$

Now, applying the second inequality (2.10) of Lemma 2.6 below to the function x_1 , with $f := -\alpha$ and $x_0 := q_1$, we get that

$$x_1 - x_1^0 = x_1 - x_1(0) > -1 \text{ in } [0, \infty).$$

Collecting the two inequalities satisfied by x_1 , we obtain the estimate

$$\left| x_1(\cdot, x^0) - x_1^0 \right| < 1 \text{ in } [0, \infty). \quad (2.8)$$

Next, let us prove that the Herman rotation set satisfies

$$\mathbf{C}_b \subset \mathbb{R}e_2. \quad (2.9)$$

Let $\mu(b) \in \mathbf{C}_b$ be associated with an invariant probability measure μ on \mathbb{T}^2 . Using successively the invariance property and Fubini's theorem we have for any $t > 0$,

$$\begin{aligned} \mu(b) \cdot e_1 &= \int_{\mathbb{T}^2} b_1(x) \mu(dx) = \frac{1}{t} \int_0^t \left(\int_{\mathbb{T}^2} b_1(X(s, x)) \mu(dx) \right) ds \\ &= \int_{\mathbb{T}^2} \left(\frac{1}{t} \int_0^t b_1(X(s, x)) ds \right) \mu(dx) = \int_{\mathbb{T}^2} \frac{x_1(t, x) - x_1}{t} \mu(dx) \end{aligned}$$

Moreover, taking into account the bound (2.8) Lebesgue's theorem implies that

$$\lim_{t \rightarrow \infty} \int_{\mathbb{T}^2} \frac{x_1(t, x) - x_1}{t} \mu(dx) = 0,$$

which proves the inclusion (2.9).

Finally, choosing the integer vector $k := -e_2$, or equivalently $k^\perp = e_1$, we get that the expression of b_1 in (2.7) agrees with the expression of $b \cdot k^\perp$ in (2.2). Moreover, it is clear that the inequality satisfied by B in (2.7) implies the inequality in (2.2). Therefore, the vector field b under the assumption (2.7) satisfies the conditions of Theorem 2.1.

We have just proved that the flow associated with any vector field b characterized by the condition (2.7) is singular in the sense of Theorem 2.1. \square

Lemma 2.6. *Let f be a function in $W_{\sharp}^{1,\infty}(\mathbb{R}/T\mathbb{Z})$, $T > 0$ (i.e. f is Lipschitz and periodic of period T in \mathbb{R}), let $x_0 \in \mathbb{R}$, and let x be a function in $C^1([0, \infty))$ such that $f(x(0)) \neq f(x_0)$. Then, we have the two following implications:*

$$\begin{cases} x' + f(x) < f(x_0) & \text{in } [0, \infty) \Rightarrow x - x(0) < T & \text{in } [0, \infty), \\ x' + f(x) > f(x_0) & \text{in } [0, \infty) \Rightarrow x - x(0) > -T & \text{in } [0, \infty), \end{cases} \quad (2.10)$$

where $f(x) := f \circ x$.

The proof of Lemma 2.6 is given in [2, Lem. 2.5].

2.3 An extension to dimension three

The two dimensional Proposition 2.4 can be extended partially in dimension three as follows.

Proposition 2.7. *Let $b \in C_{\sharp}^1(\mathbb{T}^3)^3$ and let $\alpha, \beta \in C_{\sharp}^1(\mathbb{T}^3)$ be such that the first coordinate of b reads as*

$$\begin{aligned} b_1(x) &= \alpha(x_1) + \beta(x) \quad \text{for } x \in \mathbb{T}^3, \\ \text{with } B &:= \max_{\mathbb{T}^3} |\beta| < \min \left(-\min_{\mathbb{T}^1} \alpha, \max_{\mathbb{T}^1} \alpha \right). \end{aligned} \quad (2.11)$$

Then, for any non-negative function σ in $L_{\sharp}^1(\mathbb{T}^3)$ satisfying the weak positivity property

$$\bar{\sigma}^{x'}(x_1) := \int_{\mathbb{T}^2} \sigma(x_1, x') dx' > 0 \quad \text{a.e. } x_1 \in \mathbb{T}^1, \quad (2.12)$$

the probability measure $\sigma(x)/\bar{\sigma} dx$ cannot be invariant for the flow (1.1) associated with b .

Proof of Proposition 2.7. First of all, using condition (2.11) and proceeding similarly to the proof Proposition 2.4 with Lemma 2.6, we get that for any $x^0 \in \mathbb{T}^3$, the first coordinate $x_1(\cdot, x^0)$ of the flow (1.1) is bounded in $[0, \infty)$. Hence, it follows immediately that

$$\forall x \in \mathbb{T}^3, \quad \lim_{t \rightarrow \infty} \frac{x_1(t, x)}{t} = 0, \quad (2.13)$$

which similarly to (2.9) yields

$$\mathbb{C}_b \subset (\mathbb{R} e_2 + \mathbb{R} e_3). \quad (2.14)$$

Now, assume by contradiction that there exists an invariant probability measure for the flow (1.1) of the type $\sigma(x) dx$ where σ is in $L_{\sharp}^1(\mathbb{T}^3)$ and satisfies the positivity property (2.3). By Liouville's

theorem combined with the representation of a divergence free field in dimension free (see, *e.g.*, [10, Thm. 3.4]) there exists a periodic current field $\varphi = (\varphi_1, \varphi_2, \varphi_3) \in W_{\#}^1(\mathbb{T}^3)^3$ such that

$$\sigma b = \overline{\sigma b} + \nabla \times \varphi \text{ in } \mathbb{T}^3.$$

Moreover, by (2.14) we have $\overline{\sigma b} \cdot e_1 = 0$. Then, the representation (2.11) of b_1 gives

$$\sigma(x) b_1(x) = \sigma(x) \alpha(x_1) + \sigma(x) \beta(x) = \partial_{x_2} \varphi_3 - \partial_{x_3} \varphi_2 \quad \text{for } x \in \mathbb{T}^3. \quad (2.15)$$

Then, integrating the former equality with respect to $x' = (x_2, x_3)$ and taking into account the periodicity of the functions φ_2 and φ_3 , we obtain that

$$\overline{\sigma}^{x'}(x_1) \alpha(x_1) + \overline{\sigma \beta}^{x'}(x_1) = 0 \quad \text{a.e. } x_1 \in \mathbb{T}^1.$$

This combined with the positivity property (2.12) of σ leads us to

$$\max_{\mathbb{T}^1} |\alpha| \leq \max_{\mathbb{T}^1} \left| \frac{\overline{\sigma \beta}^{x'}}{\overline{\sigma}^{x'}} \right| \leq \max_{\mathbb{T}^3} |\beta|,$$

which contradicts the strict inequality of (2.11), and concludes the proof of Proposition 2.7. \square

2.4 Examples

2.4.1 Gradient flows in any dimension

Consider the flow (1.1) associated with the gradient field $b = \nabla u \in C_{\#}^1(\mathbb{T}^d)^d$, where the function u is ‘‘affine plus periodic’’, *i.e.* the function $(x \mapsto u(x) - \overline{\nabla u} \cdot x)$ is in $C_{\#}^2(\mathbb{T}^d)^d$. Define the \mathbb{T}^{d-1} -mean value function $\bar{u}^{x'}$ of the function u by

$$\bar{u}^{x'}(x_1) := \int_{\mathbb{T}^{d-1}} u(x_1, x') dx' \quad \text{for } x_1 \in \mathbb{T}^1.$$

Then, assume that the function \bar{u} changes sign, and that the function u is sufficiently close to its mean value $\bar{u}^{x'}$ in the sense that

$$\max_{\mathbb{T}^d} \left| \partial_{x_1} (u - \bar{u}^{x'}) \right| < \frac{1}{2} \max_{\mathbb{T}^d} |\partial_{x_1} u|.$$

Therefore, due to the equality $b_1 = \partial_{x_1} u$, we easily deduce that the condition (2.11) is fulfilled by the functions $\alpha := \partial_{x_1} \bar{u}^{x'}$ and $\beta := \partial_{x_1} (u - \bar{u}^{x'})$.

The gradient fields of Section 3 below will be used in a completely different manner.

2.4.2 A modified nonlinear pendulum

In [12, §9.2, §9.4, §9.5] the dynamics of the ideal pendulum and of the one with constant forcing are investigated. Here, instead we consider the two-dimensional flow $X(\cdot, x) = (\omega, \theta)$ solution to the equation

$$\begin{cases} \partial_t \omega(t, x) = \varphi(\omega(t, x)) - \sin(\theta(t, x)) \\ \partial_t \theta(t, x) = \psi(\omega(t, x)), \end{cases} \quad \text{with } (\omega(0, x), \theta(0, x)) = x \in \mathbb{T}^2, \quad (2.16)$$

where θ denotes the angular position of the pendulum, $\psi(\omega)$ (rather than ω) denotes the angular velocity (which ensures the periodicity condition), and φ, ψ are two given functions in $C_{\#}^1(\mathbb{T}^1)$.

If the two following inequalities hold

$$\min_{\mathbb{T}^1} \varphi < -1 \quad \text{and} \quad \max_{\mathbb{T}^1} \varphi > 1,$$

then the flow of the modified pendulum (2.16) is singular in the sense of Proposition 2.4.

2.4.3 A two-dimensional Euler flow

We consider the two-dimensional Euler flow $X(\cdot, x)$ for x in the torus $\mathbb{R}^2/(2\pi\mathbb{Z}^2)$, solution to the ODEs system

$$\begin{cases} \partial_t x_1(t, x) = -A \cos(x_1(t, x)) - B \sin(x_2(t, x)) =: b_1(X(t, x)) \\ \partial_t x_2(t, x) = A \sin(x_1(t, x)) + B \cos(x_2(t, x)) =: b_2(X(t, x)), \end{cases} \quad (2.17)$$

where A, B are two fixed non-zero real parameters.

The vector field b defined by (2.17) represents the velocity solution to the steady Euler equation (see, e.g., [16])

$$\partial_t \omega + \psi_1 \omega_2 - \psi_2 \omega_1 = 0, \quad (2.18)$$

where the function ω denotes the (scalar) fluid vorticity, and ψ solution to $-\Delta\psi = \omega$, denotes the stream function. By definition (2.17) we have

$$\omega(x) = (\text{curl } b)(x) = (\partial_{x_1} b_2 - \partial_{x_2} b_1)(x) = A \cos x_1 + B \cos x_2 \quad \text{and} \quad \psi(x) = \omega(x).$$

According to [16, 7] the study of the stability of the flow (2.17) on the torus is actually relevant for the atmospheric flows.

We have the three following alternatives (see [2, Thm. 2.1] for further details):

- If $|A| > |B|$, then the functions

$$b_1, \quad (x \mapsto \alpha(x_1) := -A \cos x_1), \quad (x \mapsto \beta(x) := -B \sin x_2)$$

of the flow (2.17) clearly satisfy condition (2.7). Therefore, the flow (2.17) is singular in the sense of Theorem 2.1, and $C_b \subset \mathbb{R} e_2$. More precisely, since the vector field b does not vanish in $\mathbb{R}^2/(2\pi\mathbb{Z}^2)$, the commensurable case of [14, Thm. 3.1] combined with (1.16) implies that $C_b = I_2 e_2$ where I_2 is a closed interval of $(0, \infty)$.

- If $|A| < |B|$, then permuting the roles of x_1 and x_2 , the flow (1.1) is singular in the sense of Theorem 2.1, and $C_b = I_1 e_1$ where I_1 is a closed interval of $(0, \infty)$.
- If $|A| = |B|$, then the ODEs system (2.17) is integrable according to [2, Thm. 2.1].

3 A class involving some gradient fields

3.1 The main result

We have the following class of singular flows which is based on gradient fields.

Theorem 3.1. *Let b be a vector field in $C_{\sharp}^1(\mathbb{T}^d)^d$ such that there exists a gradient field ∇u in $C_{\sharp}^0(\mathbb{T}^d)^d$ satisfying the non-negativity condition*

$$\begin{cases} b \cdot (\nabla u - \overline{\nabla u}) \geq 0 \\ b \cdot (\nabla u - \overline{\nabla u}) \not\equiv 0 \end{cases} \quad \text{in } \mathbb{T}^d. \quad (3.1)$$

Then, for any non-negative function $\sigma \in L_{\sharp}^1(\mathbb{T}^d)$ which satisfies the weak positivity property

$$\text{for any non-empty open set } U \text{ of } \mathbb{T}^d, \quad \int_U \sigma(x) dx > 0, \quad (3.2)$$

the probability measure $\sigma(x)/\bar{\sigma} dx$ cannot be invariant for the flow (1.1) associated b .

Moreover, restricting ourselves to dimension two, if the vector field b does not vanish in \mathbb{T}^2 , there does not exist any real-analytic first integral v for the flow with $\nabla v \in C_{\sharp}^1(\mathbb{T}^2)^2$.

Remark 3.2. In dimension two the weak positivity on open sets (3.2) is finer than the weak positivity on closed line segments (2.3), and it is also finer than the almost-everywhere positivity on \mathbb{R}^2 . The last implication is clear. The first one is deduced by applying Fubini's theorem on any square U of \mathbb{R}^2 . Similarly, in dimension three the weak positivity on open sets (3.2) is finer than the weak positivity (2.12).

Remark 3.3. In Theorem 3.1 for a given vector field $b \in C_{\sharp}^1(\mathbb{T}^d)^d$, the non existence of a regular invariant probability measure on \mathbb{T}^d for the flow (1.1) having a Lebesgue density in $L_{\sharp}^1(\mathbb{T}^d)$, does not exclude the existence of a regular invariant measure $\sigma(x) dx$ with a positive density $\sigma \in L_{\text{loc}}^1(\mathbb{R}^d)$, *i.e.* solution to $\text{div}(\sigma b) = 0$ in \mathbb{R}^d by virtue of Liouville's theorem. So, by virtue of [6, Thm. 2.15] if b is a non vanishing gradient field in $C_{\sharp}^1(\mathbb{T}^d)^d$, such a positive density σ does exist but it is not periodic in general (see [6, Thm. 2.17] for a necessary and sufficient condition). For instance in [6, Exa. 2.12] the two-dimensional gradient field b defined by

$$b := \nabla u \quad \text{where} \quad u(x) := x_1 - \cos(2\pi x_2) \quad \text{for } x = (x_1, x_2) \in \mathbb{R}^2, \quad (3.3)$$

satisfies the condition (3.1) with the same gradient field, since

$$b \cdot (\nabla u - \overline{\nabla u}) = (e_1 + 2\pi \sin(2\pi x_2) e_2) \cdot 2\pi \sin(2\pi x_2) e_2 = 4\pi^2 \sin^2(2\pi x_2) \geq 0.$$

However, one gets the following explicit regular but non-periodic (with respect to the variable x_1) function

$$\sigma(x) := \begin{cases} \frac{1 + \tan^2(\pi x_2)}{e^{4\pi^2 x_1} + e^{-4\pi^2 x_1} \tan^2(\pi x_2)} & \text{if } x_2 \notin \frac{1}{2} + \mathbb{Z} \\ e^{4\pi^2 x_1} & \text{if } x_2 \in \frac{1}{2} + \mathbb{Z}. \end{cases} \quad (3.4)$$

Proof of Theorem 3.1. Let b be a vector field in $C_{\sharp}^1(\mathbb{T}^d)^d$ satisfying (3.1) with some gradient field $\nabla u \in C_{\sharp}^0(\mathbb{T}^d)^d$. Assume by contradiction that there exists a non-negative function $\sigma \in L_{\sharp}^1(\mathbb{T}^d)$ satisfying (3.2) and $\text{div}(\sigma b) = 0$ in \mathbb{T}^d . Since the function $u_{\sharp} : (x \mapsto u(x) - \overline{\nabla u} \cdot x)$ belongs to $C_{\sharp}^1(\mathbb{T}^d)$, the variational formulation (1.10) on \mathbb{T}^d yields

$$\int_{\mathbb{T}^d} \underbrace{b(x) \cdot (\nabla u(x) - \overline{\nabla u})}_{\geq 0} \sigma(x) dx = \int_{\mathbb{T}^d} b(x) \cdot \nabla u_{\sharp}(x) \sigma(x) dx = 0,$$

which implies that

$$b \cdot (\nabla u - \overline{\nabla u}) \sigma = 0 \quad \text{a.e. in } \mathbb{T}^d.$$

Now, since $b \cdot (\nabla u - \overline{\nabla u})$ is continuous and not identically null in \mathbb{T}^d , there exists a non-empty open ball B of \mathbb{T}^d on which it is bounded from below by a constant $c > 0$. Hence, recalling the weak positivity property (3.2) satisfied by σ , it follows that

$$0 = \int_B b(x) \cdot (\nabla u(x) - \overline{\nabla u}) \sigma(x) dx \geq c \int_B \sigma(x) dx > 0,$$

which leads us to a contradiction.

In dimension two, assuming that the vector field b does not vanish in \mathbb{T}^2 , the proof of the absence of any first integral with smooth periodic gradient is quite similar to the one of Theorem 2.1. This concludes the proof of Theorem 3.1. \square

3.2 Examples

The first example provides a class of singular gradient flows in any dimension.

3.2.1 Gradient flows in any dimension

Let u be the ‘‘affine plus periodic’’ function defined by

$$u(x) := \xi \cdot x + \sum_{k=1}^n u_k^\sharp(M_k x) \quad \text{for } x \in \mathbb{T}^d, \quad (3.5)$$

where $\xi \in \mathbb{R}^d$, $M_k \in \mathbb{Z}^{d \times d}$ and $u_k^\sharp \in C_\sharp^2(\mathbb{T}^d)$ for $k = 1, \dots, n$, $n \in \mathbb{N}$,

and assume that

$$\forall k = 1, \dots, n, \quad M_k \xi = 0_{\mathbb{R}^d}. \quad (3.6)$$

Then, by (3.5) and noting that $\overline{\nabla u} = \xi$, we get that for any $x \in \mathbb{T}^d$,

$$\begin{aligned} b(x) \cdot (\nabla u(x) - \overline{\nabla u}) &= \left(\xi + \sum_{k=1}^n M_k^T \nabla u_k^\sharp(M_k x) \right) \cdot \left(\sum_{k=1}^n M_k^T \nabla u_k^\sharp(M_k x) \right) \\ &= \left| \sum_{k=1}^n M_k^T \nabla u_k^\sharp(M_k x) \right|^2 + \sum_{k=1}^n \nabla u_k^\sharp(M_k x) \cdot M_k \xi \\ &= \left| \sum_{k=1}^n M_k^T \nabla u_k^\sharp(M_k x) \right|^2 = |\nabla u(x) - \overline{\nabla u}|^2 \geq 0. \end{aligned}$$

Therefore, if condition (3.6) is satisfied and the function u is not affine, the gradient field $b := \nabla u$ satisfies the non-negativity condition (3.1), so that the flow (1.1) associated with the gradient field b is singular in the sense of Theorem 3.1.

The next example provides a class of singular two-dimensional flows including gradient flows.

3.2.2 A two-dimensional class of singular flows

Let A be a symmetric positive definite matrix-valued function in $C_\sharp^1(\mathbb{T}^2, \mathbb{R}_S^{2 \times 2})$, let $\alpha \in C_\sharp^1(\mathbb{T}^2)$ and let B be the matrix-valued function defined by (recall §1.1)

$$B := R_\perp A - A R_\perp \in C_\sharp^1(\mathbb{T}^2, \mathbb{R}_S^{2 \times 2}) \quad \text{which satisfies} \quad \begin{cases} \operatorname{tr} B = 0 \\ \det B \leq 0 \end{cases} \quad \text{in } \mathbb{T}^2. \quad (3.7)$$

Assume that

$$\det(2A + \alpha B) = 4 \det A + \alpha^2 \det B > 0 \quad \text{in } \mathbb{T}^2. \quad (3.8)$$

Inequality (3.8) holds in particular for small enough $\max_{\mathbb{T}^2} |\alpha|$, since $\det A$ is bounded from below in \mathbb{T}^2 by a positive constant.

Let ∇u be a gradient field in $C_\sharp^1(\mathbb{T}^2)^2$. Then, we consider the vector field b defined by

$$b := A \nabla u + \alpha (A \nabla u)^\perp = A \nabla u + \alpha R_\perp A \nabla u \in C_\sharp^1(\mathbb{T}^2)^2. \quad (3.9)$$

Using the definition (3.7) of B and that R_\perp is skew-symmetric, we have

$$\begin{aligned} b \cdot (\nabla u - \overline{\nabla u}) &= (A \nabla u + \alpha R_\perp A \nabla u) \cdot (\nabla u - \overline{\nabla u}) \\ &= (A + \alpha R_\perp A) \nabla u \cdot \nabla u - (A + \alpha R_\perp A) \nabla u \cdot \overline{\nabla u} \\ &= \frac{1}{2} (2A + \alpha B) \nabla u \cdot \nabla u - (A - \alpha A R_\perp) \overline{\nabla u} \cdot \nabla u. \end{aligned} \quad (3.10)$$

Note that, since A is symmetric positive definite and inequality (3.8) holds with $\text{tr } B = 0$, the matrix $(2A + \alpha B)$ is also symmetric positive definite. Then, setting $X := \nabla u$ and $\xi := \overline{\nabla u}$, and fixing a symmetric positive definite matrix A and a real number α , by virtue of (3.10) the function $F_{\xi, A, \alpha}$ defined by

$$F_{\xi, A, \alpha}(X) := \frac{1}{2} (2A + \alpha B)X \cdot X - (A - \alpha AR_{\perp})\xi \cdot X \quad \text{for } X \in \mathbb{R}^2, \quad (3.11)$$

provides the representation

$$b \cdot (\nabla u - \overline{\nabla u}) = F_{\overline{\nabla u}, A, \alpha}(\nabla u) \quad \text{in } \mathbb{T}^2. \quad (3.12)$$

On the other hand, again using that $(2A + \alpha B)$ is symmetric positive definite, the function $F_{\xi, A, \alpha}$ satisfies

$$\min_{X \in \mathbb{R}^2} F_{\xi, A, \alpha}(X) = F_{\xi, A, \alpha}(X_{\xi, A, \alpha}) = -\frac{1}{2} (2A + \alpha B)^{-1} \eta \cdot \eta \leq 0, \quad (3.13)$$

$$\text{where } X_{\xi, A, \alpha} := (2A + \alpha B)^{-1} \eta \text{ and } \eta := (A - \alpha AR_{\perp})\xi,$$

which implies the equivalences

$$F_{\xi, A, \alpha} \geq 0 \text{ in } \mathbb{R}^2 \Leftrightarrow \eta = 0_{\mathbb{R}^2} \Leftrightarrow (A - \alpha AR_{\perp})\xi = 0_{\mathbb{R}^2} \Leftrightarrow \xi = 0_{\mathbb{R}^2}. \quad (3.14)$$

The last equivalence of (3.14) is an immediate consequence of

$$\det(A - \alpha AR_{\perp}) = (1 + \alpha^2) \det A > 0. \quad (3.15)$$

Then, we have the following alternative:

- Assume that $\overline{\nabla u} = 0_{\mathbb{R}^2}$.

By (3.12) and (3.13) we get the inequalities

$$\begin{cases} b \cdot \nabla u \geq 0 & \text{in } \mathbb{T}^2, \\ b \cdot \nabla u > 0 & \text{if } \nabla u \neq 0_{\mathbb{R}^2}. \end{cases} \quad (3.16)$$

Hence, it follows that if the function u is not a constant and condition (3.8) holds true, the vector field b defined by (3.9) satisfies the non-negativity condition (3.1). Therefore, the flow (1.1) associated with b is singular in the sense of Theorem 3.1.

- Assume that $\xi := \overline{\nabla u} \neq 0_{\mathbb{R}^2}$.

Let $K_{\xi, A, \alpha}$ be the subset of \mathbb{R}^2 defined by (recall (3.13))

$$\begin{aligned} K_{\xi, A, \alpha} &:= \{X \in \mathbb{R}^2 : F_{\xi, A, \alpha}(X) \leq 0\} \\ &= \{X \in \mathbb{R}^2 : (2A + \alpha B)X \cdot X - 2\eta \cdot X \leq 0\}. \end{aligned} \quad (3.17)$$

Since the matrix $(2A + \alpha B)$ is symmetric positive definite, $K_{\xi, A, \alpha}$ is a convex compact set of \mathbb{R}^2 , which due to (3.14) has a non-empty interior $\overset{\circ}{K}_{\xi, A, \alpha}$. Also note that the boundary $\partial K_{\xi, A, \alpha}$ is the ellipse of equation

$$(2A + \alpha B)(X - \lambda) \cdot (X - \lambda) = (2A + \alpha B)\lambda \cdot \lambda,$$

centered on the point

$$\lambda := (2A + \alpha B)^{-1} \eta = (2A + \alpha B)^{-1} (A - \alpha AR_{\perp})\xi \neq 0_{\mathbb{R}^2} \text{ by (3.15),}$$

and passing through $0_{\mathbb{R}^2}$. Then, collecting (3.12), (3.13) and (3.17) we get that

$$\begin{cases} b \cdot (\nabla u - \overline{\nabla u}) \geq 0 & \text{if } \nabla u \in \mathbb{R}^2 \setminus \overset{\circ}{K}_{\overline{\nabla u}, A, \alpha}, \\ b \cdot (\nabla u - \overline{\nabla u}) > 0 & \text{if } \nabla u \in \mathbb{R}^2 \setminus K_{\overline{\nabla u}, A, \alpha} \neq \emptyset, \end{cases} \quad (3.18)$$

Hence, under assumption (3.8) and

$$\begin{cases} \forall x \in \mathbb{T}^2, & \nabla u(x) \in \mathbb{R}^2 \setminus \overset{\circ}{K}_{\overline{\nabla u}, A(x), \alpha(x)} \\ \exists x \in \mathbb{T}^2, & \nabla u(x) \in \mathbb{R}^2 \setminus K_{\overline{\nabla u}, A(x), \alpha(x)}, \end{cases}$$

it follows from (3.18) that the vector field b defined by (3.9) satisfies the non-negativity condition (3.1). Therefore, the flow (1.1) associated with b is singular in the sense of Theorem 3.1.

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