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## CONVOLUTION SUM OF RAMANUJAN'S SUM

GYE HWAN JO, ABDELMEJID BAYAD AND DAEYEOUL KIM\*

ABSTRACT. This article is the result of calculating the convolution of Ramanujan's sum and natural number multiplied. Among these results, special values are expressed by Euler and Bernoulli functions.

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Key words and phrases : Ramanujan's sum, convolution sum.

### 1. Introduction

The theory of convolution sums of arithmetic functions is being studied a lot because it is helpful for the various number theory and special functions. Various results can be seen in the convolution sum for Ramanujan's sum. In fact,

Let  $n$  and  $r$  be positive integers. Define

$$c(n, r) := \sum_{\substack{x=1 \\ \gcd(x, r)=1}}^r e^{\frac{2\pi n x}{r}}$$

and

$$c(m, n, r) := \sum_{d|\gcd(m, n, r)} d^2 \mu(r/d), \quad (1)$$

where  $\mu$  is the Möbius function.  $c(n, r)$  is called Ramanujan's sum. In this article, we will use  $c(m, n, r) = c^{(2)}(\gcd(m, n), r)$  interchangeably. It is well known that  $c(n, r) = \sum_{d|\gcd(n, r)} \mu(r/d)d$  and  $c(0, r) = \phi(r)$ .

In 1918, Srinivasa Ramanujan considered Ramanujan's sums (see [6]). In fact, Srinivasa Ramanujan sum was usefully used when proving Vinogradov's theorem [5, Chapter 8]. In this paper we define the quasi-generalized Ramanujan's sum and give the relation to the Ramanujan's sum, Bernoulli polynomials and Euler

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polynomials. Bernoulli polynomials and Euler polynomials have been studied by many mathematicians (see [1, 2, 4, 7]).

For any rational numbers  $k_1, k_2, k_3$ , the quasi-generalized Ramanujan's sums are

$$C_{(k_1, k_2, k_3)}^+(l) := \sum_{n=1}^l \sum_{\substack{m=1 \\ r|n}}^{m \leq \frac{n}{k_1}} c(k_1 m, n - k_1 m, r) m^{k_2} n^{k_3}$$

and

$$C_{(k_1, k_2, k_3)}^-(l) := \sum_{n=1}^l \sum_{\substack{m=1 \\ r|n}}^{m \leq \frac{n}{k_1}} c(k_1 m, n - k_1 m, r) (-1)^n m^{k_2} n^{k_3},$$

where  $l$  is a positive integer.

In fact, we will prove the following result.

**Theorem 1.1.** *Let  $k_3$  be an integer and  $l$  be a positive integer. Then we have*

$$C_{(1,0,k_3)}^+(l) = \begin{cases} \frac{B_{k_3+3}(l+1) - B_{k_3+3}}{k_3+3} & \text{if } k_3 > -2, \\ l & \text{if } k_3 = -2, \\ H_l^{(-k_3-2)} & \text{if } k_3 < -2, \end{cases}$$

$$C_{(1,1,k_3)}^+(l) = \begin{cases} \frac{B_{k_3+4}(l+1) - B_{k_3+4}}{2k_3+8} & \text{if } k_3 > -3, \\ \frac{l}{2} & \text{if } k_3 = -3, \\ \frac{1}{2} H_l^{(-k_3-3)} & \text{if } k_3 < -3, \end{cases}$$

$$C_{(1,0,k_3)}^-(l) = \begin{cases} \frac{E_{k_3+2}(0) + (-1)^l E_{k_3+2}(l+1)}{2} & \text{if } k_3 > -2, \\ \chi(l) & \text{if } k_3 = -2, \\ S_l^{(-k_3-2)} & \text{if } k_3 < -2 \end{cases}$$

and

$$C_{(1,1,k_3)}^-(l) = \begin{cases} \frac{E_{k_3+3}(0) + (-1)^l E_{k_3+3}(l+1)}{4} & \text{if } k_3 > -3, \\ \frac{\chi(l)}{2} & \text{if } k_3 = -3, \\ S_l^{(-k_3-3)} & \text{if } k_3 < -3. \end{cases}$$

Here,  $H_l^{(t)} = 1 + \frac{1}{2^t} + \dots + \frac{1}{l^t}$ ,  $S_l^{(t)} = -1 + \frac{1}{2^t} - \dots + (-1)^l \frac{1}{l^t}$ ,  $B_n(z)$  (resp.,  $E_n(z)$ ) is the  $n$ th Bernoulli polynomial (resp., Euler polynomial) and

$$\chi(l) = \begin{cases} 0, & \text{if } l \text{ is even,} \\ -1, & \text{otherwise.} \end{cases}$$

**Example 1.2.** Let  $l$  be a positive integer. Then

$$C_+(l) := C_{(1,0,0)}^+(l) = \sum_{n=1}^l \sum_{\substack{m=1 \\ r|n}}^n c(m, n - m, r) = \frac{B_3(l+1)}{3} = \frac{l(l+1)(2l+1)}{6}$$

and

$$C_-(l) := C_{(1,0,0)}^-(l) = \sum_{n=1}^l \sum_{\substack{m=1 \\ r|n}}^n (-1)^n c(m, n-m, r) = \frac{(-1)^l E_2(l+1)}{2} = \frac{(-1)^l l(l+1)}{2}.$$

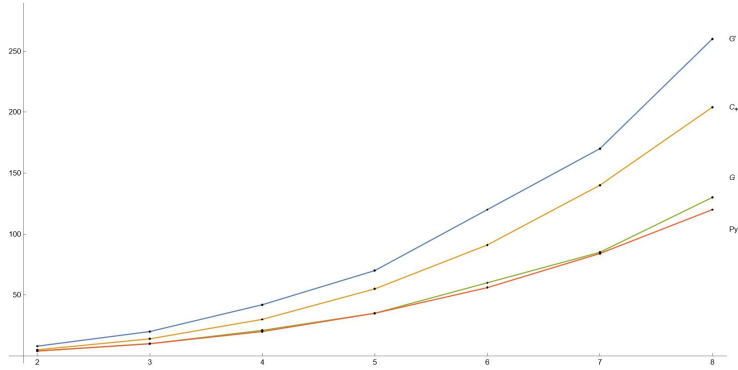


FIGURE 1. Values of  $G'(l)$ ,  $C_+(l)$ ,  $G(l)$ ,  $Pyr_3(l)$  ( $2 \leq l \leq 8$ )

**Remark 1.1.** Let  $G(l) := \sum_{(a+b+c)x=l+2} a$  and  $G'(l) := \sum_{(a+b+c)x=l+2} 2a$  with  $a, b, c, l, x \in \mathbb{N}$ . The result of studying the values of functions such as  $G(l)$  is in [8].

Using [3, Theorem 1.1 and Remark 2.2] and Theorem 1.1, we obtain

$$G'(l) > C_+(l) > G(l) \geq Pyr_3(l)$$

with  $l \geq 1$ . Here,  $Pyr_a(z) = \frac{1}{6}(z(z+1)((a-2)z+5-a))$  be the  $a$ th order pyramid number. In Figure 1, we plot the graphs for the values of  $G'(l)$ ,  $C_+(l)$ ,  $G(l)$  and  $Pyr_3(l)$  when  $l = 2, 3, 4, 5, 6, 7, 8$ .

It is easily checked that if  $l = 2$  or  $l + 2$  is a prime number for  $l \geq 1$ , then we obtain

$$G(l) = Pyr_3(l) = \frac{l(l+1)(l+2)}{6}.$$

## 2. Proof of Theorem 1.1

We need several lemmas to prove Theorem 1.1, so we introduce them first.

**Lemma 2.1.** *Let  $k$ ,  $n$  and  $r$  be positive integers. Then*

$$\sum_{m=1}^n c(m, n-m, r)^k = (\varphi * c^{(2)}(\cdot, r)^k)(n) \quad (2)$$

and

$$\sum_{m=1}^{n-1} c(m, n-m, r)^k = (\varphi * c^{(2)}(\cdot, r)^k)(n) - c^{(2)}(n, r)^k.$$

Here,  $*$  is a Dirichlet convolution, that is,  $(f * g)(n) = \sum_{d|n} f(d)g(\frac{n}{d})$  and  $c^{(2)}(\cdot, r)$  means that one of the two variables,  $r$ , is given as a fixed number, and the other is regarded as a one variable function.

*Proof.* Let

$$A := \sum_{m=1}^n c(m, n-m, r)^k = \sum_{d|n} \sum_{\substack{x=1 \\ \gcd(x, n/d)=1}}^{n/d} c(xd, n-xd, r)^k.$$

Since  $\gcd(x, n/d) = 1$  implies  $\gcd(x, n/d - x) = 1$ ,  $\gcd(xd, n-xd) = d$  and  $c(xd, n-xd, r) = c^{(2)}(d, r)$  by (1). Thus, we deduce that

$$A = \sum_{d|n} c^{(2)}(d, r)^k \sum_{\substack{x=1 \\ \gcd(x, n/d)=1}}^{n/d} 1 = \sum_{d|n} \varphi(n/d) c^{(2)}(d, r)^k.$$

From (2), we obtain that

$$\sum_{m=1}^{n-1} c(m, n-m, r)^k = (\varphi * c^{(2)}(\cdot, r)^k)(n) - c^{(2)}(n, r)^k.$$

This completes the proof of Lemma 2.1.  $\square$

**Lemma 2.2.** *Let  $n$  and  $r$  be positive integers. Then*

$$(\varphi * c^{(2)}(\cdot, r))(n) = nc(n, r). \quad (3)$$

*Proof.* To prove Lemma 2.2, let us define  $g(n, r)$  as follows

$$g(n, r) := \begin{cases} r & \text{if } r|n \\ 0 & \text{otherwise.} \end{cases}$$

From the left hand side of (3), we get the following process by rearranging terms. That is,

$$\begin{aligned} (\varphi * c^{(2)}(\cdot, r))(n) &= \sum_{d|n} \varphi(n/d) \sum_{e|\gcd(d, r)} e^2 \mu(r/e) = \sum_{d|n} \varphi(n/d) \sum_{d|r} e \mu(r/e) g(d, e) \\ &= \sum_{e|r} e^2 \mu(r/e) \sum_{\substack{d|n \\ e|d}} \varphi(n/d) = \sum_{e|\gcd(n, r)} e^2 \mu(r/e) \sum_{\substack{D|n/e \\ d=De}} \varphi(n/De) \\ &= \sum_{e|\gcd(n, r)} e^2 \mu(r/e) \frac{n}{e} = n \sum_{e|\gcd(n, r)} e \mu(r/e) = nc(n, r). \end{aligned}$$

$\square$

By Lemma 2.2, we have Corollary 2.3.

**Corollary 2.3.** *Let  $n$  and  $r$  be positive integers. Then*

$$\sum_{m=1}^{n-1} c(m, n-m, r) = nc(n, r) - c^{(2)}(n, r) = \sum_{d|\gcd(n, r)} d(n-d)\mu(r/d).$$

Furthermore, by using the Möbius inversion formula, we have

$$c^{(2)}(n, r) = (\zeta_1 \mu * \zeta_0 * \zeta_1 c(\cdot, r))(n).$$

Here,  $\zeta_i(t) = t^i$  and  $fg(t) = f(t)g(t)$ , where  $i$  is a non-negative integer and  $t$  is a positive integer.

**Lemma 2.4.** *Let  $n$  and  $r$  be positive integers. If  $r|n$ , then*

$$\sum_{m=1}^n c(m, r)c(n-m, r) = nc(n, r). \quad (4)$$

*Proof.* Let  $B$  be the left hand side of (4) and let  $r|n$ . By the definition of the Ramanujan' sum, we derive that

$$\begin{aligned} B &= \sum_{m=1}^n \sum_{\substack{x=1 \\ \gcd(x, r)=1}}^r \sum_{\substack{y=1 \\ \gcd(y, r)=1}}^r e^{2\pi m x/r} \cdot e^{2\pi(n-m)y/r} \\ &= \sum_{\substack{y=1 \\ \gcd(y, r)=1}}^r e^{2\pi n y/r} \sum_{\substack{x=1 \\ \gcd(x, r)=1}}^r \left[ \sum_{m=1}^n e^{2\pi m(x-y)/r} \right]. \end{aligned}$$

Since  $1 \leq x, y \leq r$  and  $r|(x-y)$  implies  $x=y$ , the sum in brackets is equal to  $n$  if  $x=y$  and 0 otherwise. Therefore,

$$B = n \sum_{\substack{x=1 \\ \gcd(x, r)=1}}^r e^{2\pi n x/r} = nc(n, r).$$

□

By Corollary 2.3 and Lemma 2.4, we have that

**Corollary 2.5.** *Let  $n$  and  $r$  be positive integers. If  $r|n$ , then*

$$\sum_{m=1}^n c(m, n-m, r) = \sum_{m=1}^n c(m, r)c(n-m, r) = nc(n, r) = n\varphi(r).$$

**Proof of Theorem 1.1**

It is well-known that

$$\sum_{r|n} \varphi(r) = n \quad (5)$$

by Lemma 2.5 and (5), we obtain that

$$\sum_{r|n} \sum_{m=1}^n c(m, n-m, r) = \sum_{r|n} n\varphi(r) = n^2. \quad (6)$$

It is easily checked that

$$\sum_{m=1}^n c(m, n-m, r)m^2 = \sum_{m=1}^n c(m, n-m, r)(n-m)^2$$

and

$$\sum_{m=1}^n c(m, n-m, r)m = \sum_{m=1}^n c(m, n-m, r)\frac{n}{2}. \quad (7)$$

Using (6), (7) and two results ((2.2), (2.3)) in Sun's article (see [7]), the proof of Theorem 1.1 is completed.  $\square$

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