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CONVOLUTION SUM OF RAMANUJAN'S SUM

GYE HWAN JO, ABDELMEJID BAYAD AND DAEYEOUL KIM*

ABSTRACT. This article is the result of calculating the convolution of Ramanujan's sum and natural number multiplied. Among these results, special values are expressed by Euler and Bernoulli functions.

AMS Mathematics Subject Classification : 11A25, 11B68.

Key words and phrases : Ramanujan's sum, convolution sum.

1. Introduction

The theory of convolution sums of arithmetic functions is being studied a lot because it is helpful for the various number theory and special functions. Various results can be seen in the convolution sum for Ramanujan's sum. In fact,

Let n and r be positive integers. Define

$$c(n, r) := \sum_{\substack{x=1 \\ \gcd(x, r)=1}}^r e^{\frac{2\pi n x}{r}}$$

and

$$c(m, n, r) := \sum_{d|\gcd(m, n, r)} d^2 \mu(r/d), \quad (1)$$

where μ is the Möbius function. $c(n, r)$ is called Ramanujan's sum. In this article, we will use $c(m, n, r) = c^{(2)}(\gcd(m, n), r)$ interchangeably. It is well known that $c(n, r) = \sum_{d|\gcd(n, r)} \mu(r/d)d$ and $c(0, r) = \phi(r)$.

In 1918, Srinivasa Ramanujan considered Ramanujan's sums (see [6]). In fact, Srinivasa Ramanujan sum was usefully used when proving Vinogradov's theorem [5, Chapter 8]. In this paper we define the quasi-generalized Ramanujan's sum and give the relation to the Ramanujan's sum, Bernoulli polynomials and Euler

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polynomials. Bernoulli polynomials and Euler polynomials have been studied by many mathematicians (see [1, 2, 4, 7]).

For any rational numbers k_1, k_2, k_3 , the quasi-generalized Ramanujan's sums are

$$C_{(k_1, k_2, k_3)}^+(l) := \sum_{n=1}^l \sum_{\substack{m=1 \\ r|n}}^{m \leq \frac{n}{k_1}} c(k_1 m, n - k_1 m, r) m^{k_2} n^{k_3}$$

and

$$C_{(k_1, k_2, k_3)}^-(l) := \sum_{n=1}^l \sum_{\substack{m=1 \\ r|n}}^{m \leq \frac{n}{k_1}} c(k_1 m, n - k_1 m, r) (-1)^n m^{k_2} n^{k_3},$$

where l is a positive integer.

In fact, we will prove the following result.

Theorem 1.1. *Let k_3 be an integer and l be a positive integer. Then we have*

$$C_{(1,0,k_3)}^+(l) = \begin{cases} \frac{B_{k_3+3}(l+1) - B_{k_3+3}}{k_3+3} & \text{if } k_3 > -2, \\ l & \text{if } k_3 = -2, \\ H_l^{(-k_3-2)} & \text{if } k_3 < -2, \end{cases}$$

$$C_{(1,1,k_3)}^+(l) = \begin{cases} \frac{B_{k_3+4}(l+1) - B_{k_3+4}}{2k_3+8} & \text{if } k_3 > -3, \\ \frac{l}{2} & \text{if } k_3 = -3, \\ \frac{1}{2} H_l^{(-k_3-3)} & \text{if } k_3 < -3, \end{cases}$$

$$C_{(1,0,k_3)}^-(l) = \begin{cases} \frac{E_{k_3+2}(0) + (-1)^l E_{k_3+2}(l+1)}{2} & \text{if } k_3 > -2, \\ \chi(l) & \text{if } k_3 = -2, \\ S_l^{(-k_3-2)} & \text{if } k_3 < -2 \end{cases}$$

and

$$C_{(1,1,k_3)}^-(l) = \begin{cases} \frac{E_{k_3+3}(0) + (-1)^l E_{k_3+3}(l+1)}{4} & \text{if } k_3 > -3, \\ \frac{\chi(l)}{2} & \text{if } k_3 = -3, \\ S_l^{(-k_3-3)} & \text{if } k_3 < -3. \end{cases}$$

Here, $H_l^{(t)} = 1 + \frac{1}{2^t} + \dots + \frac{1}{l^t}$, $S_l^{(t)} = -1 + \frac{1}{2^t} - \dots + (-1)^l \frac{1}{l^t}$, $B_n(z)$ (resp., $E_n(z)$) is the n th Bernoulli polynomial (resp., Euler polynomial) and

$$\chi(l) = \begin{cases} 0, & \text{if } l \text{ is even,} \\ -1, & \text{otherwise.} \end{cases}$$

Example 1.2. Let l be a positive integer. Then

$$C_+(l) := C_{(1,0,0)}^+(l) = \sum_{n=1}^l \sum_{\substack{m=1 \\ r|n}}^n c(m, n - m, r) = \frac{B_3(l+1)}{3} = \frac{l(l+1)(2l+1)}{6}$$

and

$$C_-(l) := C_{(1,0,0)}^-(l) = \sum_{n=1}^l \sum_{\substack{m=1 \\ r|n}}^n (-1)^n c(m, n-m, r) = \frac{(-1)^l E_2(l+1)}{2} = \frac{(-1)^l l(l+1)}{2}.$$

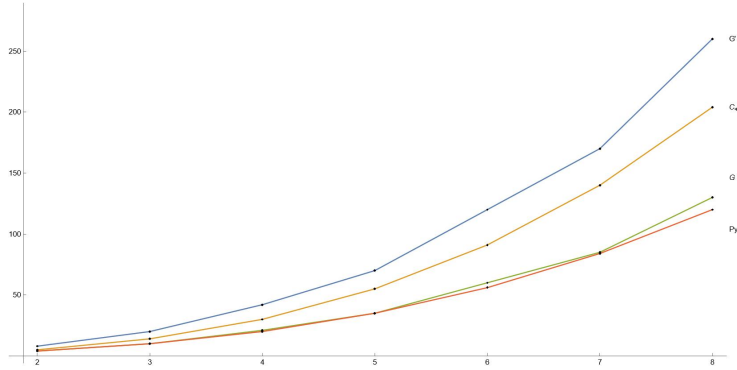


FIGURE 1. Values of $G'(l), C_+(l), G(l), Pyr_3(l)$ ($2 \leq l \leq 8$)

Remark 1.1. Let $G(l) := \sum_{(a+b+c)x=l+2} a$ and $G'(l) := \sum_{(a+b+c)x=l+2} 2a$ with $a, b, c, l, x \in \mathbb{N}$. The result of studying the values of functions such as $G(l)$ is in [8].

Using [3, Theorem 1.1 and Remark 2.2] and Theorem 1.1, we obtain

$$G'(l) > C_+(l) > G(l) \geq Pyr_3(l)$$

with $l \geq 1$. Here, $Pyr_a(z) = \frac{1}{6}(z(z+1)((a-2)z+5-a))$ be the a th order pyramid number. In Figure 1, we plot the graphs for the values of $G'(l), C_+(l), G(l)$ and $Pyr_3(l)$ when $l = 2, 3, 4, 5, 6, 7, 8$.

It is easily checked that if $l = 2$ or $l + 2$ is a prime number for $l \geq 1$, then we obtain

$$G(l) = Pyr_3(l) = \frac{l(l+1)(l+2)}{6}.$$

2. Proof of Theorem 1.1

We need several lemmas to prove Theorem 1.1, so we introduce them first.

Lemma 2.1. *Let k, n and r be positive integers. Then*

$$\sum_{m=1}^n c(m, n-m, r)^k = (\varphi * c^{(2)}(\cdot, r)^k)(n) \tag{2}$$

and

$$\sum_{m=1}^{n-1} c(m, n-m, r)^k = (\varphi * c^{(2)}(\cdot, r)^k)(n) - c^{(2)}(n, r)^k.$$

Here, $*$ is a Dirichlet convolution, that is, $(f * g)(n) = \sum_{d|n} f(d)g(\frac{n}{d})$ and $c^{(2)}(\cdot, r)$ means that one of the two variables, r , is given as a fixed number, and the other is regarded as a one variable function.

Proof. Let

$$A := \sum_{m=1}^n c(m, n-m, r)^k = \sum_{d|n} \sum_{\substack{x=1 \\ \gcd(x, n/d)=1}}^{n/d} c(xd, n-xd, r)^k.$$

Since $\gcd(x, n/d) = 1$ implies $\gcd(x, n/d-x) = 1$, $\gcd(xd, n-xd) = d$ and $c(xd, n-xd, r) = c^{(2)}(d, r)$ by (1). Thus, we deduce that

$$A = \sum_{d|n} c^{(2)}(d, r)^k \sum_{\substack{x=1 \\ \gcd(x, n/d)=1}}^{n/d} 1 = \sum_{d|n} \varphi(n/d) c^{(2)}(d, r)^k.$$

From (2), we obtain that

$$\sum_{m=1}^{n-1} c(m, n-m, r)^k = (\varphi * c^{(2)}(\cdot, r)^k)(n) - c^{(2)}(n, r)^k.$$

This completes the proof of Lemma 2.1. \square

Lemma 2.2. *Let n and r be positive integers. Then*

$$(\varphi * c^{(2)}(\cdot, r))(n) = nc(n, r). \quad (3)$$

Proof. To prove Lemma 2.2, let us define $g(n, r)$ as follows

$$g(n, r) := \begin{cases} r & \text{if } r|n \\ 0 & \text{otherwise.} \end{cases}$$

From the left hand side of (3), we get the following process by rearranging terms. That is,

$$\begin{aligned} (\varphi * c^{(2)}(\cdot, r))(n) &= \sum_{d|n} \varphi(n/d) \sum_{e|\gcd(d, r)} e^2 \mu(r/e) = \sum_{d|n} \varphi(n/d) \sum_{d|r} e \mu(r/e) g(d, e) \\ &= \sum_{e|r} e^2 \mu(r/e) \sum_{\substack{d|n \\ e|d}} \varphi(n/d) = \sum_{e|\gcd(n, r)} e^2 \mu(r/e) \sum_{\substack{D|n/e \\ d=De}} \varphi(n/De) \\ &= \sum_{e|\gcd(n, r)} e^2 \mu(r/e) \frac{n}{e} = n \sum_{e|\gcd(n, r)} e \mu(r/e) = nc(n, r). \end{aligned}$$

\square

By Lemma 2.2, we have Corollary 2.3.

Corollary 2.3. *Let n and r be positive integers. Then*

$$\sum_{m=1}^{n-1} c(m, n-m, r) = nc(n, r) - c^{(2)}(n, r) = \sum_{d|\gcd(n, r)} d(n-d)\mu(r/d).$$

Furthermore, by using the Möbius inversion formula, we have

$$c^{(2)}(n, r) = (\zeta_1 \mu * \zeta_0 * \zeta_1 c(\cdot, r))(n).$$

Here, $\zeta_i(t) = t^i$ and $fg(t) = f(t)g(t)$, where i is a non-negative integer and t is a positive integer.

Lemma 2.4. *Let n and r be positive integers. If $r|n$, then*

$$\sum_{m=1}^n c(m, r)c(n-m, r) = nc(n, r). \quad (4)$$

Proof. Let B be the left hand side of (4) and let $r|n$. By the definition of the Ramanujan' sum, we derive that

$$\begin{aligned} B &= \sum_{m=1}^n \sum_{\substack{x=1 \\ \gcd(x, r)=1}}^r \sum_{\substack{y=1 \\ \gcd(y, r)=1}}^r e^{2\pi m x/r} \cdot e^{2\pi(n-m)y/r} \\ &= \sum_{\substack{y=1 \\ \gcd(y, r)=1}}^r e^{2\pi n y/r} \sum_{\substack{x=1 \\ \gcd(x, r)=1}}^r \left[\sum_{m=1}^n e^{2\pi m(x-y)/r} \right]. \end{aligned}$$

Since $1 \leq x, y \leq r$ and $r|(x-y)$ implies $x=y$, the sum in brackets is equal to n if $x=y$ and 0 otherwise. Therefore,

$$B = n \sum_{\substack{x=1 \\ \gcd(x, r)=1}}^r e^{2\pi n x/r} = nc(n, r).$$

□

By Corollary 2.3 and Lemma 2.4, we have that

Corollary 2.5. *Let n and r be positive integers. If $r|n$, then*

$$\sum_{m=1}^n c(m, n-m, r) = \sum_{m=1}^n c(m, r)c(n-m, r) = nc(n, r) = n\varphi(r).$$

Proof of Theorem 1.1

It is well-known that

$$\sum_{r|n} \varphi(r) = n \quad (5)$$

by Lemma 2.5 and (5), we obtain that

$$\sum_{r|n} \sum_{m=1}^n c(m, n-m, r) = \sum_{r|n} n\varphi(r) = n^2. \quad (6)$$

It is easily checked that

$$\sum_{m=1}^n c(m, n-m, r)m^2 = \sum_{m=1}^n c(m, n-m, r)(n-m)^2$$

and

$$\sum_{m=1}^n c(m, n-m, r)m = \sum_{m=1}^n c(m, n-m, r)\frac{n}{2}. \quad (7)$$

Using (6), (7) and two results ((2.2), (2.3)) in Sun's article (see [7]), the proof of Theorem 1.1 is completed. \square

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