



HAL
open science

CONVOLUTION SUM OF RAMANUJAN'S SUM

Gye Hwan Jo, Abdelmejid Bayad, Daeyeoul Kim

► **To cite this version:**

Gye Hwan Jo, Abdelmejid Bayad, Daeyeoul Kim. CONVOLUTION SUM OF RAMANUJAN'S SUM. Journal of Applied and Pure Mathematics, 2022, 4 (1-2), pp.63-69. <10.23091/japm.2022.063>. <hal-04466081>

HAL Id: hal-04466081

<https://hal.science/hal-04466081v1>

Submitted on 19 Feb 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



HAL Authorization

CONVOLUTION SUM OF RAMANUJAN'S SUM

GYE HWAN JO, ABDELMEJID BAYAD AND DAEYEOUL KIM*

ABSTRACT. This article is the result of calculating the convolution of Ramanujan's sum and natural number multiplied. Among these results, special values are expressed by Euler and Bernoulli functions.

AMS Mathematics Subject Classification : 11A25, 11B68.

Key words and phrases : Ramanujan's sum, convolution sum.

1. Introduction

The theory of convolution sums of arithmetic functions is being studied a lot because it is helpful for the various number theory and special functions. Various results can be seen in the convolution sum for Ramanujan's sum. In fact,

Let n and r be positive integers. Define

$$c(n, r) := \sum_{\substack{x=1 \\ \gcd(x, r)=1}}^r e^{\frac{2\pi n x}{r}}$$

and

$$c(m, n, r) := \sum_{d|\gcd(m, n, r)} d^2 \mu(r/d), \quad (1)$$

where μ is the Möbius function. $c(n, r)$ is called Ramanujan's sum. In this article, we will use $c(m, n, r) = c^{(2)}(\gcd(m, n), r)$ interchangeably. It is well known that $c(n, r) = \sum_{d|\gcd(n, r)} \mu(r/d)d$ and $c(0, r) = \phi(r)$.

In 1918, Srinivasa Ramanujan considered Ramanujan's sums (see [6]). In fact, Srinivasa Ramanujan sum was usefully used when proving Vinogradov's theorem [5, Chapter 8]. In this paper we define the quasi-generalized Ramanujan's sum and give the relation to the Ramanujan's sum, Bernoulli polynomials and Euler

Received January 24, 2022. Revised February 23, 2022. Accepted February 25, 2022.

*Corresponding author.

The corresponding author is supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MSIT). (No. 2021R1F1A1051093)

© 2022 KSCAM.

polynomials. Bernoulli polynomials and Euler polynomials have been studied by many mathematicians (see [1, 2, 4, 7]).

For any rational numbers k_1, k_2, k_3 , the quasi-generalized Ramanujan's sums are

$$C_{(k_1, k_2, k_3)}^+(l) := \sum_{n=1}^l \sum_{\substack{m=1 \\ r|n}}^{m \leq \frac{n}{k_1}} c(k_1 m, n - k_1 m, r) m^{k_2} n^{k_3}$$

and

$$C_{(k_1, k_2, k_3)}^-(l) := \sum_{n=1}^l \sum_{\substack{m=1 \\ r|n}}^{m \leq \frac{n}{k_1}} c(k_1 m, n - k_1 m, r) (-1)^n m^{k_2} n^{k_3},$$

where l is a positive integer.

In fact, we will prove the following result.

Theorem 1.1. *Let k_3 be an integer and l be a positive integer. Then we have*

$$C_{(1,0,k_3)}^+(l) = \begin{cases} \frac{B_{k_3+3}(l+1) - B_{k_3+3}}{k_3+3} & \text{if } k_3 > -2, \\ l & \text{if } k_3 = -2, \\ H_l^{(-k_3-2)} & \text{if } k_3 < -2, \end{cases}$$

$$C_{(1,1,k_3)}^+(l) = \begin{cases} \frac{B_{k_3+4}(l+1) - B_{k_3+4}}{2k_3+8} & \text{if } k_3 > -3, \\ \frac{l}{2} & \text{if } k_3 = -3, \\ \frac{1}{2} H_l^{(-k_3-3)} & \text{if } k_3 < -3, \end{cases}$$

$$C_{(1,0,k_3)}^-(l) = \begin{cases} \frac{E_{k_3+2}(0) + (-1)^l E_{k_3+2}(l+1)}{2} & \text{if } k_3 > -2, \\ \chi(l) & \text{if } k_3 = -2, \\ S_l^{(-k_3-2)} & \text{if } k_3 < -2 \end{cases}$$

and

$$C_{(1,1,k_3)}^-(l) = \begin{cases} \frac{E_{k_3+3}(0) + (-1)^l E_{k_3+3}(l+1)}{4} & \text{if } k_3 > -3, \\ \frac{\chi(l)}{2} & \text{if } k_3 = -3, \\ S_l^{(-k_3-3)} & \text{if } k_3 < -3. \end{cases}$$

Here, $H_l^{(t)} = 1 + \frac{1}{2^t} + \dots + \frac{1}{l^t}$, $S_l^{(t)} = -1 + \frac{1}{2^t} - \dots + (-1)^l \frac{1}{l^t}$, $B_n(z)$ (resp., $E_n(z)$) is the n th Bernoulli polynomial (resp., Euler polynomial) and

$$\chi(l) = \begin{cases} 0, & \text{if } l \text{ is even,} \\ -1, & \text{otherwise.} \end{cases}$$

Example 1.2. Let l be a positive integer. Then

$$C_+(l) := C_{(1,0,0)}^+(l) = \sum_{n=1}^l \sum_{\substack{m=1 \\ r|n}}^n c(m, n - m, r) = \frac{B_3(l+1)}{3} = \frac{l(l+1)(2l+1)}{6}$$

and

$$C_-(l) := C_{(1,0,0)}^-(l) = \sum_{n=1}^l \sum_{\substack{m=1 \\ r|n}}^n (-1)^n c(m, n-m, r) = \frac{(-1)^l E_2(l+1)}{2} = \frac{(-1)^l l(l+1)}{2}.$$

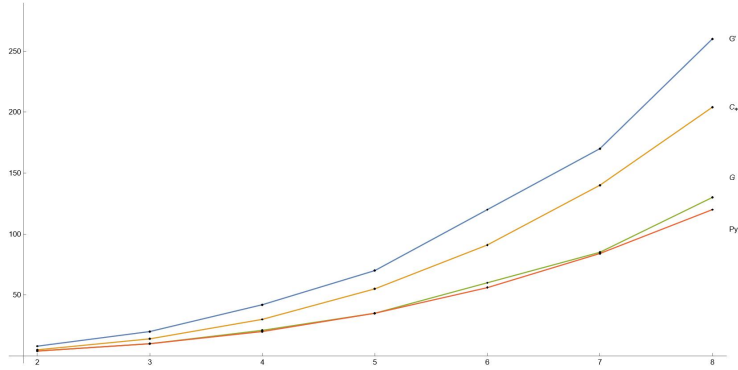


FIGURE 1. Values of $G'(l)$, $C_+(l)$, $G(l)$, $Pyr_3(l)$ ($2 \leq l \leq 8$)

Remark 1.1. Let $G(l) := \sum_{(a+b+c)x=l+2} a$ and $G'(l) := \sum_{(a+b+c)x=l+2} 2a$ with $a, b, c, l, x \in \mathbb{N}$. The result of studying the values of functions such as $G(l)$ is in [8].

Using [3, Theorem 1.1 and Remark 2.2] and Theorem 1.1, we obtain

$$G'(l) > C_+(l) > G(l) \geq Pyr_3(l)$$

with $l \geq 1$. Here, $Pyr_a(z) = \frac{1}{6}(z(z+1)((a-2)z+5-a))$ be the a th order pyramid number. In Figure 1, we plot the graphs for the values of $G'(l)$, $C_+(l)$, $G(l)$ and $Pyr_3(l)$ when $l = 2, 3, 4, 5, 6, 7, 8$.

It is easily checked that if $l = 2$ or $l + 2$ is a prime number for $l \geq 1$, then we obtain

$$G(l) = Pyr_3(l) = \frac{l(l+1)(l+2)}{6}.$$

2. Proof of Theorem 1.1

We need several lemmas to prove Theorem 1.1, so we introduce them first.

Lemma 2.1. *Let k , n and r be positive integers. Then*

$$\sum_{m=1}^n c(m, n-m, r)^k = (\varphi * c^{(2)}(\cdot, r)^k)(n) \quad (2)$$

and

$$\sum_{m=1}^{n-1} c(m, n-m, r)^k = (\varphi * c^{(2)}(\cdot, r)^k)(n) - c^{(2)}(n, r)^k.$$

Here, $*$ is a Dirichlet convolution, that is, $(f * g)(n) = \sum_{d|n} f(d)g(\frac{n}{d})$ and $c^{(2)}(\cdot, r)$ means that one of the two variables, r , is given as a fixed number, and the other is regarded as a one variable function.

Proof. Let

$$A := \sum_{m=1}^n c(m, n-m, r)^k = \sum_{d|n} \sum_{\substack{x=1 \\ \gcd(x, n/d)=1}}^{n/d} c(xd, n-xd, r)^k.$$

Since $\gcd(x, n/d) = 1$ implies $\gcd(x, n/d - x) = 1$, $\gcd(xd, n-xd) = d$ and $c(xd, n-xd, r) = c^{(2)}(d, r)$ by (1). Thus, we deduce that

$$A = \sum_{d|n} c^{(2)}(d, r)^k \sum_{\substack{x=1 \\ \gcd(x, n/d)=1}}^{n/d} 1 = \sum_{d|n} \varphi(n/d) c^{(2)}(d, r)^k.$$

From (2), we obtain that

$$\sum_{m=1}^{n-1} c(m, n-m, r)^k = (\varphi * c^{(2)}(\cdot, r)^k)(n) - c^{(2)}(n, r)^k.$$

This completes the proof of Lemma 2.1. \square

Lemma 2.2. *Let n and r be positive integers. Then*

$$(\varphi * c^{(2)}(\cdot, r))(n) = nc(n, r). \quad (3)$$

Proof. To prove Lemma 2.2, let us define $g(n, r)$ as follows

$$g(n, r) := \begin{cases} r & \text{if } r|n \\ 0 & \text{otherwise.} \end{cases}$$

From the left hand side of (3), we get the following process by rearranging terms. That is,

$$\begin{aligned} (\varphi * c^{(2)}(\cdot, r))(n) &= \sum_{d|n} \varphi(n/d) \sum_{e|\gcd(d, r)} e^2 \mu(r/e) = \sum_{d|n} \varphi(n/d) \sum_{d|r} e \mu(r/e) g(d, e) \\ &= \sum_{e|r} e^2 \mu(r/e) \sum_{\substack{d|n \\ e|d}} \varphi(n/d) = \sum_{e|\gcd(n, r)} e^2 \mu(r/e) \sum_{\substack{D|n/e \\ d=De}} \varphi(n/De) \\ &= \sum_{e|\gcd(n, r)} e^2 \mu(r/e) \frac{n}{e} = n \sum_{e|\gcd(n, r)} e \mu(r/e) = nc(n, r). \end{aligned}$$

\square

By Lemma 2.2, we have Corollary 2.3.

Corollary 2.3. *Let n and r be positive integers. Then*

$$\sum_{m=1}^{n-1} c(m, n-m, r) = nc(n, r) - c^{(2)}(n, r) = \sum_{d|\gcd(n, r)} d(n-d)\mu(r/d).$$

Furthermore, by using the Möbius inversion formula, we have

$$c^{(2)}(n, r) = (\zeta_1\mu * \zeta_0 * \zeta_1 c(\cdot, r))(n).$$

Here, $\zeta_i(t) = t^i$ and $fg(t) = f(t)g(t)$, where i is a non-negative integer and t is a positive integer.

Lemma 2.4. *Let n and r be positive integers. If $r|n$, then*

$$\sum_{m=1}^n c(m, r)c(n-m, r) = nc(n, r). \quad (4)$$

Proof. Let B be the left hand side of (4) and let $r|n$. By the definition of the Ramanujan' sum, we derive that

$$\begin{aligned} B &= \sum_{m=1}^n \sum_{\substack{x=1 \\ \gcd(x, r)=1}}^r \sum_{\substack{y=1 \\ \gcd(y, r)=1}}^r e^{2\pi mx/r} \cdot e^{2\pi(n-m)y/r} \\ &= \sum_{\substack{y=1 \\ \gcd(y, r)=1}}^r e^{2\pi ny/r} \sum_{\substack{x=1 \\ \gcd(x, r)=1}}^r \left[\sum_{m=1}^n e^{2\pi m(x-y)/r} \right]. \end{aligned}$$

Since $1 \leq x, y \leq r$ and $r|(x-y)$ implies $x = y$, the sum in brackets is equal to n if $x = y$ and 0 otherwise. Therefore,

$$B = n \sum_{\substack{x=1 \\ \gcd(x, r)=1}}^r e^{2\pi nx/r} = nc(n, r).$$

□

By Corollary 2.3 and Lemma 2.4, we have that

Corollary 2.5. *Let n and r be positive integers. If $r|n$, then*

$$\sum_{m=1}^n c(m, n-m, r) = \sum_{m=1}^n c(m, r)c(n-m, r) = nc(n, r) = n\varphi(r).$$

Proof of Theorem 1.1

It is well-known that

$$\sum_{r|n} \varphi(r) = n \quad (5)$$

by Lemma 2.5 and (5), we obtain that

$$\sum_{r|n} \sum_{m=1}^n c(m, n-m, r) = \sum_{r|n} n\varphi(r) = n^2. \quad (6)$$

It is easily checked that

$$\sum_{m=1}^n c(m, n-m, r)m^2 = \sum_{m=1}^n c(m, n-m, r)(n-m)^2$$

and

$$\sum_{m=1}^n c(m, n-m, r)m = \sum_{m=1}^n c(m, n-m, r)\frac{n}{2}. \quad (7)$$

Using (6), (7) and two results ((2.2), (2.3)) in Sun's article (see [7]), the proof of Theorem 1.1 is completed. \square

REFERENCES

1. S.D. Kim and M-S. Kim, *New identities involving Bernoulli, Euler and Genocchi numbers*, Adv. Difference Equ. **2013:74** (2013), 15 pp.
2. N.S. Jung and C.S. Ryoo, *A reserach on a new approach to Euler polynomials and Bernstein polynomials with variable $[x]_q$* , J. Appl. Math. & Informatics **35** (2017), 205-215.
3. D. Kim, A. Kim and N.Y. Ikikardes, *Bernoulli numbers and certain convolution sums with divisor functions*, Adv. Difference Equ. **2013:277** (2013), 20pp.
4. M-S. Kim, *Generalized Euler power series*, J. Appl. Math. & Informatics **38** (2020), 591-600.
5. M.B. Nathanson, *Additive Number Theory*, The Classical Bases, Graduate Texts in Mathematics 164, Springer, 1996.
6. S. Ramanujan, *On Certain Trigonometric Sums and their Applications in the Theory of Numbers*, Transactions of the Cambridge Philosophical Society XXII **13** (1918), 259-276.
7. Z. Sun, *Introduction to Bernoulli and Euler polynomials*, <http://maths.nju.edu.cn/~zwsun/BerE.pdf>.
8. K.S. Williams, *Number Theory in the Spirit of Liouville*, Cambridge University Press, New York, 2011.

Gye Hwan Jo is currently a M.S. student at Jeonbuk National University. His research intersets focus on number theory.

Department of Mathematics and Institute of Pure and Applied Mathematics, Jeonbuk National University, 567 Baekje-daero, Deokjin-gu, Jeonju-si, Jeollabuk-do, 54896, Republic of Korea.

e-mail: johh1986@hanmail.net

Abdelmejid Bayad received Ph.D. degree from Université Bordeaux 1. His research intersets focus on number theory.

Université d'Evry Val d'Essonne, Université Paris-Saclay, CNRS (UMR 8071), Laboratoire de Mathématiques et Modélisation d'Évry, I.B.G.B.I., 3ème étage, 23 Boulevard de France, 91037 Evry cedex, France.

e-mail: abdelmejid.bayad@univ-evry.fr, abayad@maths.univ-evry.fr

Daeyeoul Kim received Ph.D. degree from Jeonbuk National University. His research intersets focus on number theory.

Department of Mathematics and Institute of Pure and Applied Mathematics, Jeonbuk National University, 567 Baekje-daero, Deokjin-gu, Jeonju-si, Jeollabuk-do, 54896, Republic of Korea.

e-mail: kdaeyeoul@jbnu.ac.kr