FURTHER EXPANSION FORMULAS FOR A CLASS OF GENERALIZED HURWITZ-LERCH ZETA FUNCTIONS OBTAINED BY MEANS OF A NEW LEIBNIZ RULE FOR FRACTIONAL DERIVATIVES

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FURTHER EXPANSION FORMULAS FOR A CLASS OF GENERALIZED HURWITZ-LERCH ZETA FUNCTIONS OBTAINED BY MEANS OF A NEW LEIBNIZ RULE FOR FRACTIONAL DERIVATIVES

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Abstract. The aim of this paper is to make use of a new generalized Leibniz rule for fractional derivatives obtained recently by Tremblay et al. [Tremblay, Gaboury and Fugère, A new Leibniz rule and its integral analogue for fractional derivatives, Integral Transforms Spec. Funct. 24 (2013), 111-128] by means of a representation based on the Pochhammer's contour of integration for fractional derivatives in order to derive new expansion formulas for several families of the Hurwitz-Lerch zeta function. Special cases are also given.

2010 Mathematics Subject Classification. 11M35, 26A33, 33C05, 11B68.

Keywords and phrases. Fractional derivatives, expansion formula, Lerch's functional equation, Hurwitz-Lerch zeta functions, Leibniz rule, Riemann zeta function, Generalized Bernoulli polynomials.

1. Introduction

It is well known that the generalized Hurwitz-Lerch zeta function as well as its extended version have many applications in various areas of mathematics and physics. In number theory, the Riemann and Hurwitz zeta functions are closely related to Dedekind zeta functions and Artin L-functions, which play a central role in the discipline. In addition, the generalized Hurwitz-Lerch zeta functions, evaluated at negative integers, are closely related to the Apostol-Bernoulli polynomials, the Apostol-Euler polynomials and the Frobenius-Euler polynomials [1, 5, 21]. These functions are also connected to the generalized Fermi-Dirac functions and the generalized Bose-Einstein functions [1]. The generalized Fermi-Dirac and Bose-Einstein functions, which appear in quantum statistics, quantum interference and in the theory of quantum entanglement, have been introduced recently by Srivastava et al. [22]. Moreover, the generalized Hurwitz-Lerch zeta functions have interesting applications in geometric function theory [19] and finally, Gupta et al. [7] investigated the generalized Hurwitz-Lerch zeta distribution and applied this new distribution in reliability.

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The generalized Hurwitz zeta function $\zeta(s,a)$ is defined by [23, p. 88 et seq.]

(1)

$$\zeta(s,a) := \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} \qquad (\mathrm{Re}(s) > 1; a \in \mathbb{C} \setminus \mathbb{Z}_0^-; \mathbb{Z}_0^- := \{0, -1, -2, ...\})$$

where

$$\zeta(s,1) = \zeta(s) = \frac{1}{2^s - 1} \zeta\left(s, \frac{1}{2}\right) \tag{2}$$

yields the celebrated Riemann zeta function $\zeta(s)$. The Riemann zeta function is continued meromorphically to the whole complex s-plane except for a simple pole at $s = 1$ with residue 1.

The Hurwitz-Lerch zeta function $\Phi(z,s,a)$ is defined, as in [23, p. 121 et seq.], by

$$\Phi(z,s,a) := \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s} \tag{3}$$

$$(a \in \mathbb{C} \setminus \mathbb{Z}_0^-; s \in \mathbb{C} \text{ when } |z| < 1; \mathrm{Re}(s) > 1 \text{ when } |z| = 1).$$

Clearly, we have the following relations

$$\Phi(1,s,a) = \zeta(s,a) \quad \text{and} \quad \Phi(1,s,1) = \zeta(s). \tag{4}$$

The Hurwitz-Lerch zeta function has the well known integral representation

$$\Phi(z,s,a) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1} \mathrm{e}^{-at}}{1 - z\mathrm{e}^{-\mathrm{t}}} dt \tag{5}$$

$$(\mathrm{Re}(a) > 0; \mathrm{Re}(s) > 0 \text{ when } |z| \leq 1 \ (z \neq 1); \mathrm{Re}(s) > 1 \text{ when } z = 1).$$

Recently, Lin and Srivastava [10] investigated a more general family of Hurwitz-Lerch zeta functions. They introduced the function $\Phi_{\mu,\nu}^{(\rho,\sigma)}(z,s,a)$ defined by

$$\Phi_{\mu,\nu}^{(\rho,\sigma)}(z,s,a) := \sum_{n=0}^{\infty} \frac{(\mu)_{\rho n}}{(\nu)_{\sigma n}} \frac{z^n}{(a+n)^s} \tag{6}$$

$$\left(\mu \in \mathbb{C};\ a, \nu \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ \rho, \sigma \in \mathbb{R}^+;\ \rho < \sigma \text{ when } s, z \in \mathbb{C};\right.$$

$$\left.\rho = \sigma \text{ and } s \in \mathbb{C} \text{ when } |z| < 1;\ \rho = \sigma \text{ and } \mathrm{Re}(s - \mu + \nu) > 1 \text{ when } |z| = 1\right),$$

where $(\lambda)_\kappa$ denotes the Pochhammer symbol defined, in terms of the Gamma function, by

$$(\lambda)_\kappa := \frac{\Gamma(\lambda + \kappa)}{\Gamma(\lambda)} = \begin{cases} \lambda(\lambda+1)\cdots(\lambda+n-1) & (\kappa = n \in \mathbb{N}; \lambda \in \mathbb{C}) \\ 1 & (\kappa = 0; \lambda \in \mathbb{C} \setminus \{0\}). \end{cases} \tag{7}$$

It is easily seen that

$$\Phi_{\nu,\nu}^{(\sigma,\sigma)}(z,s,a) = \Phi_{\mu,\nu}^{(0,0)}(z,s,a) = \Phi(z,s,a) \tag{8}$$

and

$$\Phi_{\mu,1}^{(1,1)}(z,s,a) = \Phi_{\mu}^{*}(z,s,a) := \sum_{n=0}^{\infty} \frac{(\mu)_n}{n!} \frac{z^n}{(n+a)^s}. \tag{9}$$

The function $\Phi_{\mu}^{*}(z,s,a)$ is, in fact, a generalized Hurwitz-Lerch zeta function investigated by Goyal and Laddha [6, p. 100, Equation (1.5)].

Let us recall some other important special cases of the Hurwitz-Lerch zeta function $\Phi(z,s,a)$. The Lerch zeta function defined by

$$l_s(\xi) := \sum_{n=1}^{\infty} \frac{\mathrm{e}^{2n\pi i\xi}}{n^s} \quad (\xi \in \mathbb{R};\ \mathrm{Re}(s) > 1) \tag{10}$$

is related to the Hurwitz-Lerch zeta function by the following relation

$$l_s(\xi) = \mathrm{e}^{2\pi i\xi} \Phi\left(\mathrm{e}^{2\pi i\xi}, s, 1\right). \tag{11}$$

Also, we note, as a special case of the Hurwitz-Lerch zeta functions, the Lipschitz-Lerch zeta function [23, p. 122, Equation 2.5 (11)]:

$$\phi(\xi,a,s) := \sum_{n=0}^{\infty} \frac{\mathrm{e}^{2n\pi i\xi}}{(n+a)^s} = \Phi\left(\mathrm{e}^{2\pi i\xi}, s, a\right) \tag{12}$$

$$(a \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ \mathrm{Re}(s) > 0 \text{ when } \xi \in \mathbb{R} \setminus \mathbb{Z};\ \mathrm{Re}(s) > 1 \text{ when } \xi \in \mathbb{Z}).$$

Setting $z = \exp\left(\dfrac{2\pi ip}{q}\right)$ with $p \in \mathbb{Z}$ and $q \in \mathbb{N}$, and using the next series identity

$$\sum_{n=0}^{\infty} f(n) = \sum_{j=0}^{k-1} \sum_{n=0}^{\infty} f(kn+j), \tag{13}$$

we obtain the following summation formula for the Lipschitz-Lerch zeta function $\phi(\xi,a,s)$:

$$\phi\left(\frac{p}{q}, a, s\right) = q^{-s} \sum_{j=1}^{q} \zeta\left(s, \frac{a+j-1}{q}\right) \exp\left(\frac{2(j-1)p\pi i}{q}\right) \tag{14}$$

in terms of the Hurwitz zeta function $\zeta(s,a)$ defined by equation (1) (see also [21, p. 81, Equation (3.9)]).

A very important result for the sequel is the Lerch's functional equation [3, p. 29, Equation 1.11 (7)]:

$$\begin{aligned} \Phi(z,s,a) = i\, z^{-a}(2\pi)^{s-1}\Gamma(1-s)\Bigg\{ & \exp\left(-\frac{1}{2}s\pi i\right) \Phi\left(\mathrm{e}^{-2\pi ia}, 1-s, \frac{\log z}{2\pi i}\right) \\ & - \exp\left[\left(2a+\frac{1}{2}s\right)\pi i\right] \Phi\left(\mathrm{e}^{2\pi ia}, 1-s, 1-\frac{\log z}{2\pi i}\right)\Bigg\} \end{aligned} \tag{15}$$

$$\left(\mathrm{Re}(s) < 0; |\arg(-\log(z)\ mod\ 2\pi i)| \le \pi; a \in \mathbb{C} \setminus \mathbb{Z}_0^-\right).$$

Recently, Lin et al. [11] obtained some expansion formulas for a generalized Hurwitz-Lerch zeta function $\Phi_{\mu}^{*}(z,s,a)$ defined by (9) in terms of the

Hurwitz-Lerch zeta function (3) by making use of the generalized Leibniz rule for fractional derivative [14, p. 95, Equation (4.3)]:

$$\mathcal{D}_z^{\mu} f(z)g(z) = \sum_{j=0}^{\infty} \binom{\mu}{j} \mathcal{D}_z^{\mu-j} f(z) D_z^j g(z) \tag{16}$$

where $\mathcal{D}_z^{\mu}$ denotes the Riemann-Liouville fractional calculus operator [4, p. 181 et seq.] defined by:

$$\mathcal{D}_z^{\mu} f(z) := \begin{cases} \dfrac{1}{\Gamma(-\mu)} \displaystyle\int_0^z f(t)(z-t)^{-\mu-1} \mathrm{d}t & \text{if } \mathrm{Re}(\mu) < 0, \\ \dfrac{d^m}{dz^m} \mathcal{D}_z^{\mu-m} f(z) & \text{if } m-1 \leq \mathrm{Re}(\mu) < m. \end{cases} \tag{17}$$

Explicitly, they proved the following expansion formula for the generalized Hurwitz-Lerch zeta function $\Phi_{\mu}^{*}(z,s,a)$:

$$\begin{aligned} \Phi_{\mu}^{*}(z,s,a) = i\, z^{-a}\Gamma(1-s) \sum_{j=0}^{\infty} \frac{(j-a+1)_{\mu-j-1}}{\Gamma(\mu-j)j!} \sum_{k=0}^{j} \binom{j-1}{k-1} (1-s)_k B_{j-k}^{(j)} \\ \cdot (2\pi)^{s-k-1} \left[\exp\left(-\frac{1}{2}(s-k)\pi i \right) \Phi\left(\mathrm{e}^{-2\pi i a}, 1-s+k, \frac{\log z}{2\pi i} \right) \right. \\ \left. - \exp\left[\left(2a + \frac{1}{2}(s-k) \right) \pi i \right] \Phi\left(\mathrm{e}^{2\pi i a}, 1-s+k, 1-\frac{\log z}{2\pi i} \right) \right] \end{aligned} \tag{18}$$

with $\mathrm{Re}(\mu) > 0$, $s \in \mathbb{C}$, $|\arg(-\log(z)\ mod\ 2\pi i)| \leq \pi$, $a \in \mathbb{C} \setminus \mathbb{Z}_0^-$, where $B_n^{(\alpha)} := B_n^{(\alpha)}(0)$ are the generalized Bernoulli numbers [13] given by the following generating function:

$$\left(\frac{z}{\mathrm{e}^z - 1} \right)^{\alpha} \mathrm{e}^{xz} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x) \frac{z^n}{n!} \qquad (|z| < 2\pi;\ 1^{\alpha} := 1). \tag{19}$$

The aim of this paper is to make use of a new generalized Leibniz rule for fractional derivatives obtained recently by Tremblay et al. [25] by means of a representation based on the Pochhammer's contour of integration for fractional derivatives [8] in order to derive a new expansion formula for the generalized Hurwitz-Lerch zeta function $\Phi_{\mu}^{*}(z,s,a)$ in terms of the Hurwitz-Lerch zeta function $\Phi(z,s,a)$. Moreover, an expansion theorem is obtained for the generalized family of Hurwitz-Lerch zeta functions $\Phi_{\mu,\nu}^{(\rho,\sigma)}(z,s,a)$. Special cases are also given. The paper is arranged as follows. Section 2 is devoted to the representation of the fractional derivatives based on the Pochhammer's contour of integration and the new generalized Leibniz rule. In section 3, we prove the expansions formulas and finally, section 4 aims to provide some special cases.

2. Pochhammer contour integral representation for fractional derivative and a new generalized Leibniz rule

The use of contour of integration in the complex plane provides a very powerful tool in both classical and fractional calculus. The most familiar

representation for fractional derivative of order α of $z^p f(z)$ is the Riemann-Liouville integral [2, 12, 20] that is

$$\mathcal{D}_z^\alpha z^p f(z) = \frac{1}{\Gamma(-\alpha)} \int_0^z f(\xi)\xi^p (\xi - z)^{-\alpha-1} d\xi, \tag{20}$$

which is valid for $\mathrm{Re}(\alpha) < 0$, $\mathrm{Re}(p) > 1$. Here the integration is done along a straight line from 0 to z in the ξ-plane. By integrating by part m times, we obtain

$$\mathcal{D}_z^\alpha z^p f(z) = \frac{d^m}{dz^m} \mathcal{D}_z^{\alpha-m} z^p f(z). \tag{21}$$

This allows to relax the restriction $\mathrm{Re}(\alpha) < 0$ by $\mathrm{Re}(\alpha) < m$ [20]. Another used representation for the fractional derivative is the one based on the Cauchy integral formula widely used by Osler [15, 16, 17, 18]. These two representations have been used in many interesting research papers. It appears that the less restrictive representation of fractional derivative according to parameters is the Pochhammer's contour definition introduced in [9, 24].

Definition 2.1. Let $f(z)$ be analytic in a simply connected region $\mathcal{R}$. Let $g(z)$be regular and univalent on $\mathcal{R}$ and let $g^{-1}(0)$ be an interior point of $\mathcal{R}$. If α is not a negative integer, p is not an integer, and z is in $\mathcal{R} - \{g^{-1}(0)\}$, we define the fractional derivative of order α of $g(z)^p f(z)$ with respect to $g(z)$ by

$$D_{g(z)}^\alpha g(z)^p f(z) =$$

$$\frac{e^{-i\pi p}\Gamma(1+\alpha)}{4\pi \sin(\pi p)} \int_{C(z+,g^{-1}(0)+,z-,g^{-1}(0)-;F(a),F(a))} \frac{f(\xi)g(\xi)^p g'(\xi)}{(g(\xi) - g(z))^{\alpha+1}} d\xi \tag{22}$$

For non-integer α and p, the functions $g(\xi)^p$ and $(g(\xi) - g(z))^{-\alpha-1}$ in the integrand have two branch lines which begin respectively at $\xi = z$ and $\xi = g^{-1}(0)$, and both pass through the point $\xi = a$ without crossing the Pochhammer contour $P(a) = \{C_1 \cup C_2 \cup C_3 \cup C_4\}$ at any other point as shown in Figure 1. $F(a)$ denotes the principal value of the integrand in (22) at the beginning and ending point of the Pochhammer contour $P(a)$ which is closed on Riemann surface of the multiple-valued function $F(\xi)$.

Remark 2.2. In the Definition 2.1, the function $f(z)$ must be analytic at $\xi = g^{-1}(0)$. However it is interesting to note here that we could also allow $f(z)$ to have an essential singularity at $\xi = g^{-1}(0)$, and the equation (22) would still be valid.

Remark 2.3. The Pochhammer contour never crosses the singularities at $\xi = g^{-1}(0)$ and $\xi = z$ in (22). Then we know that the integral is analytic for all p and α and for z in $\mathcal{R} - \{g^{-1}(0)\}$. Indeed, the only possible singularities of $D_{g(z)}^\alpha g(z)^p f(z)$ are $\alpha = -1, -2, \ldots$, and $p = 0, \pm 1, \pm 2, \ldots$ which can directly be identified from the coefficient of the integral (22). However, integrating by parts N times the integral in (22) by two different ways, we can show that $\alpha = -1, -2, \ldots$, and $p = 0, 1, 2, \ldots$ are removable singularities (see [9]) .

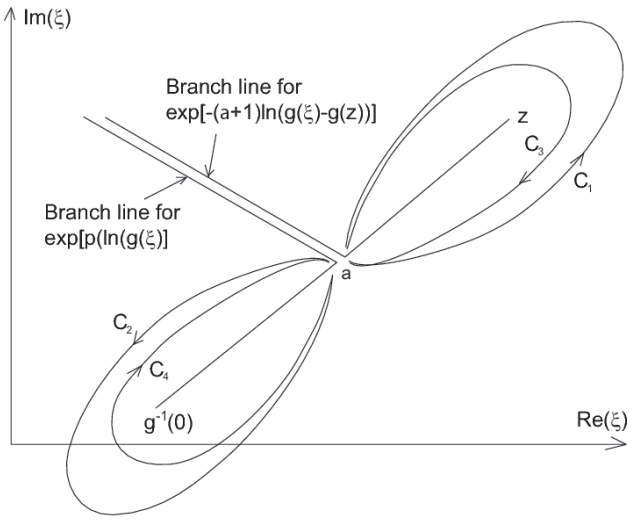

FIGURE 1. Pochhammer's contour

It is well known that [14, p. 83, Equation (2.4)]

$$D_z^{\alpha} z^p = \frac{\Gamma(1+p)}{\Gamma(1+p-\alpha)} z^{p-\alpha} \qquad (\mathrm{Re}(p) > -1), \tag{23}$$

but by adopting the Pochhammer based representation for the fractional derivative this last restriction becomes p not a negative integer. In view of definition 6, the fractional derivative formula for the generalized Hurwitz-Lerch zeta function $\Phi_{\mu,\nu}^{(\rho,\sigma)}(z,s,a)$ with $\rho = \sigma$ [10, p. 730, Equation (24)] is

$$D_z^{\mu-\nu} z^{\mu-1} \Phi(z^{\sigma}, s, a) = \frac{\Gamma(\mu)}{\Gamma(\nu)} z^{\nu-1} \Phi_{\mu,\nu}^{(\sigma,\sigma)}(z^{\sigma}, s, a) \tag{24}$$

with $\mu - 1$ not a negative integer.

A very interesting special case is obtained when setting $\nu = \sigma = 1$, and equation (24) reduces to the following form:

$$\Phi_{\mu}^{*}(z,s,a) = \frac{1}{\Gamma(\mu)} D_z^{\mu-1} z^{\mu-1} \Phi(z,s,a) \tag{25}$$

with $\mu - 1$ not a negative integer.

As remarked by Lin and Srivastava [10], the function $\Phi_{\mu}^{*}(z,s,a)$ is essentially a fractional derivative of the classical Hurwitz-Lerch function $\Phi(z,s,a)$. Many other interesting explicit representations for $\Phi_{\mu}^{*}(z,s,a)$ have been proven by Lin and Srivastava [10].

Recently, one of the authors obtained a new generalized Leibniz rule for fractional derivatives by making use of the properties of this representation [25, 26]. Explicitly, we proved the following theorem:

Theorem 2.4. *(i) Let $\mathcal{R}$ be a simply connected region containing the origin. (ii) Let $u(z)$ and $v(z)$ satisfy the conditions of* Definition 2.1 *for the existence*

of the fractional derivative. (iii) Let $\mathcal{U} \subset \mathcal{R}$ be the region of analyticity of the function $u(z)$ and $\mathcal{V} \subset \mathcal{R}$ the one for the function $v(z)$. Then for $z \neq 0$, $z \in \mathcal{U} \cap \mathcal{V}$, $\mathrm{Re}(1-\beta) > 0$ the following product rule holds

(26)
$$D_z^{\alpha} z^{\alpha+\beta-1} u(z)v(z) = \frac{z \sin(\beta\pi)\Gamma(1+\alpha)\sin(\mu\pi)\sin((\alpha+\beta-\mu)\pi)}{\sin((\alpha+\beta)\pi)\sin((\beta-\mu-\nu)\pi)\sin((\mu+\nu)\pi)} \cdot \sum_{n=-\infty}^{\infty} \frac{D_z^{\alpha+\nu+1-n} z^{\alpha+\beta-\mu-1-n} u(z) D_z^{-1-\nu+n} z^{\mu-1+n} v(z)}{\Gamma(2+\alpha+\nu-n)\Gamma(-\nu+n)}.$$

3. Expansion formulas

In this section, we present the two expansion formulas obtained by means of fractional calculus techniques for the functions $\Phi_{\mu,\nu}^{(\rho,\sigma)}(z,s,a)$ and $\Phi_{\mu}^{*}(z,s,a)$. The first expansion formula is a direct applications of the generalized Leibniz rule (26). The second expansion formula is analogous to Lerch's functional equation. We first state the first one in the form of the theorem below:

Theorem 3.1. *The following expansion formula holds true*

$$\Phi_{\mu,\nu}^{(\sigma,\sigma)}\left(z^{\sigma},s,a\right)$$

$$= \frac{\Gamma(\nu)\Gamma(\mu-\nu+1)\sin\beta\pi\ \sin(\mu-\nu+\beta-\lambda)\pi}{\Gamma(\mu)\Gamma(\nu-\gamma-\lambda-1)\Gamma(1+\gamma+\lambda)\sin(\mu-\nu+\beta)\pi\ \sin(\beta-\lambda-\gamma)\pi}$$

(27)
$$\cdot \frac{\sin\lambda\pi}{\sin(\lambda+\gamma)\pi} \sum_{n=-\infty}^{\infty} \frac{\Gamma(\mu-\lambda-n)\Gamma(\lambda+n)}{\Gamma(2+\mu-\nu+\gamma-n)\Gamma(-\gamma+n)} \Phi_{\lambda+n,1+\gamma+\lambda}^{(\sigma,\sigma)}\left(z^{\sigma},s,a\right)$$

$$\Bigg(z \neq 0;\ \mathrm{Re}(1-\beta) > 0;\ (\lambda+n) \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ (1+\lambda+\gamma) \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ a,\nu,\mu \in \mathbb{C}\backslash\mathbb{Z}_0^-;$$

$$\sigma \in \mathbb{R}^+;\ s \in \mathbb{C}\ \textit{when}\ |z^{\sigma}| < 1\Bigg).$$

Proof. First, making the substitutions $\mu \mapsto \lambda$, $\nu \mapsto \gamma$ in Theorem 2.4 and afterwards setting $\alpha = \mu - \nu$, $u(z) = z^{\nu-\beta}$ and $v(z) = \Phi\left(z^{\sigma},s,a\right)$ (both $u(z)$ and $v(z)$ satisfied the conditions of Theorem 2.4), we have

$$D_z^{\mu-\nu} z^{\mu-1}\ \Phi\left(z^{\sigma},s,a\right)$$

$$= \frac{z\,\Gamma(\mu-\nu+1)\sin\beta\pi\ \sin\lambda\pi\ \sin(\mu-\nu+\beta-\lambda)\pi}{\sin(\mu-\nu+\beta)\pi\ \sin(\beta-\lambda-\gamma)\pi\ \sin(\lambda+\gamma)\pi}$$

(28)
$$\cdot \sum_{n=-\infty}^{\infty} \frac{D_z^{\mu-\nu+\gamma+1-n} z^{\mu-\lambda-1-n}\ D_z^{-1-\gamma+n} z^{\lambda-1+n}\ \Phi\left(z^{\sigma},s,a\right)}{\Gamma(2+\mu-\nu+\gamma-n)\Gamma(-\gamma+n)}.$$

Using (23) and (24), we find that

(29)
$$D_z^{\mu-\nu+\gamma+1-n} z^{\mu-\lambda-1-n} = \frac{\Gamma(\mu-\lambda-n)}{\Gamma(\nu-\gamma-\lambda-1)} z^{\nu-\gamma-\lambda-2}$$

and

(30)
$$D_z^{-1-\gamma+n} z^{\lambda-1+n}\ \Phi\left(z^\sigma, s, a\right) = \frac{\Gamma(\lambda+n)}{\Gamma(\gamma+\lambda+1)} z^{\gamma+\lambda}\ \Phi^{(\sigma,\sigma)}_{\lambda+n,\gamma+\lambda+1}\left(z^\sigma, s, a\right).$$

It is important to mention that in the case where $\rho = \sigma$ and $|z^\sigma| = 1$, the function $\Phi^{(\rho,\sigma)}_{\mu,\nu}(z,s,a)$ exists for $\mathrm{Re}(s-\mu+\nu) > 1$. This implies that for $|z^\sigma| = 1$, the function $\Phi^{(\sigma,\sigma)}_{\lambda+n,\gamma+\lambda+1}\left(z^\sigma, s, a\right)$ in (30) must satisfy $\mathrm{Re}(s+\gamma-n) > 0$ for all $n \in \mathbb{Z}$ which case is impossible. Thus, the expansion holds true only for $|z^\sigma| < 1$.

Finally, the results follow by combining (29), (30) and (28). □

Now let us shift our focus on the second expansion formula involving the generalizerd Hurwitz-Lerch zeta function $\Phi^*_\mu(z,s,a)$ which is stated in the next theorem:

Theorem 3.2. *The following expansion formula holds true for* $\Phi^*_\mu(z,s,a)$*:*

$$\begin{aligned}
\Phi^*_\mu(z,s,a) = {} & \frac{i\,\Gamma(1-s) z^{-a} \sin\beta\pi\ \sin(\beta+\mu-\omega)\pi}{\Gamma(\mu)\Gamma(1-\omega-a)\sin(\mu+\beta)\pi\ \sin(\beta-\omega)\pi} \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{j=0}^{k} \binom{n-1}{k-1} \\
& \cdot \binom{k}{j} \frac{\Gamma(\mu-a-\omega-n)}{\Gamma(\mu-n)n!} B^{(n)}_{n-k}\ (2\pi)^{s-1-j} (1-s)_j (\omega-1+n)_{k-j} \\
& \cdot \left[\exp\left(-\frac{(s-j)\pi i}{2}\right) \Phi\left(\mathrm{e}^{-2\pi i a}, 1-s+j, \frac{\log z}{2\pi i}\right) \right. \\
& \left. - \exp\left[\left(2a+\frac{s-j}{2}\right)\pi i\right] \Phi\left(\mathrm{e}^{2\pi i a}, 1-s+j, 1-\frac{\log z}{2\pi i}\right) \right] \\
& \left(\mu \in \mathbb{C}\setminus\mathbb{Z}_0^-;\ \omega\in\mathbb{C};\ \mathrm{Re}(1-\beta)>0;\ s\in\mathbb{C}; \right. \\
& \left. |\arg(-\log(z)\ mod\ 2\pi i)| \le \pi;\ a \in \mathbb{C}\setminus\mathbb{Z}_0^- \right).
\end{aligned} \tag{31}$$

Proof. Multiplying each member of equation (15) by $z^{\mu-1}$ and operating both sides of the resulting equation by $D_z^{\mu-1}$, we get

$$\begin{aligned}
& D_z^{\mu-1} z^{\mu-1}\ \Phi(z,s,a) \\
& = i\,(2\pi)^{s-1}\Gamma(1-s)\left[\exp\left(-\frac{i\pi s}{2}\right) D_z^{\mu-1} z^{\mu-1-a}\ \Phi\left(\mathrm{e}^{-2\pi i a}, 1-s, \frac{\log z}{2\pi i}\right) \right. \\
& \left. - \exp\left[\left(2a+\frac{s}{2}\right) i\pi\right] D_z^{\mu-1} z^{\mu-1-a}\ \Phi\left(\mathrm{e}^{2\pi i a}, 1-s, 1-\frac{\log z}{2\pi i}\right) \right].
\end{aligned} \tag{32}$$

Using (25), we have

$$\begin{aligned}
& \Phi^*_\mu(z,s,a) \\
& = \frac{i\,(2\pi)^{s-1}\Gamma(1-s)}{\Gamma(\mu)} \left[\exp\left(-\frac{i\pi s}{2}\right) D_z^{\mu-1} z^{\mu-1-a}\ \Phi\left(\mathrm{e}^{-2\pi i a}, 1-s, \frac{\log z}{2\pi i}\right) \right.
\end{aligned}$$

(33)
$$-\exp\left[\left(2a+\frac{s}{2}\right)i\pi\right] D_z^{\mu-1} z^{\mu-1-a}\ \Phi\left(\mathrm{e}^{2\pi i a}, 1-s, 1-\frac{\log z}{2\pi i}\right)\Bigg].$$

We now have to evaluate the two fractional derivatives on the right hand side of (33). By applying the Leibniz rule (26) when $\nu = -1$ on each member of the right hand side of (32) with $u(z) = z^{1-\beta-a}$ and $v(z) = \Phi\left(\mathrm{e}^{-2\pi i a}, 1-s, \frac{\log z}{2\pi i}\right)$ in the first member, and with $u(z) = z^{1-\beta-a}$ and $v(z) = \Phi\left(\mathrm{e}^{2\pi i a}, 1-s, 1-\frac{\log z}{2\pi i}\right)$ in the second member (we see that the functions $u(z)$ and $v(z)$ involved satisfied the conditions of Theorem 2.4), we obtain respectively:

$$D_z^{\mu-1} z^{\mu-1-a}\ \Phi\left(\mathrm{e}^{-2\pi i a}, 1-s, \frac{\log z}{2\pi i}\right) = \frac{z\,\Gamma(\mu)\sin\beta\pi\ \sin(\mu+\beta-\omega)\pi}{\sin(\mu+\beta)\pi\ \sin(\beta-\omega)\pi}$$

(34)
$$\cdot \sum_{n=0}^{\infty} \frac{D_z^{\mu-1-n} z^{\mu-\omega-a-1-n}\ D_z^n z^{\omega-1+n}\ \Phi\left(\mathrm{e}^{-2\pi i a}, 1-s, \frac{\log z}{2\pi i}\right)}{\Gamma(\mu-n)n!}$$

and

$$D_z^{\mu-1} z^{\mu-1-a}\ \Phi\left(\mathrm{e}^{2\pi i a}, 1-s, 1-\frac{\log z}{2\pi i}\right) = \frac{z\,\Gamma(\mu)\sin\beta\pi\ \sin(\mu+\beta-\omega)\pi}{\sin(\mu+\beta)\pi\ \sin(\beta-\omega)\pi}$$

(35)
$$\cdot \sum_{n=0}^{\infty} \frac{D_z^{\mu-1-n} z^{\mu-\omega-a-1-n}\ D_z^n z^{\omega-1+n}\ \Phi\left(\mathrm{e}^{2\pi i a}, 1-s, 1-\frac{\log z}{2\pi i}\right)}{\Gamma(\mu-n)n!}.$$

Equations (34) and (35) reduce after simple calculations to:

$$D_z^{\mu-1} z^{\mu-1-a}\ \Phi\left(\mathrm{e}^{-2\pi i a}, 1-s, \frac{\log z}{2\pi i}\right)$$

$$= \frac{z^{1-\omega-a}\,\Gamma(\mu)\sin\beta\pi\ \sin(\mu+\beta-\omega)\pi}{\Gamma(1-\omega-a)\sin(\mu+\beta)\pi\ \sin(\beta-\omega)\pi} \sum_{n=0}^{\infty} \frac{\Gamma(\mu-\omega-a-n)z^{-n}}{\Gamma(\mu-n)\ n!}$$

(36)
$$\cdot \left(z^n D_z^n\right) z^{\omega-1+n}\ \Phi\left(\mathrm{e}^{-2\pi i a}, 1-s, \frac{\log z}{2\pi i}\right)$$

and

$$D_z^{\mu-1} z^{\mu-1-a}\ \Phi\left(\mathrm{e}^{2\pi i a}, 1-s, 1-\frac{\log z}{2\pi i}\right)$$

$$= \frac{z^{1-\omega-a}\,\Gamma(\mu)\sin\beta\pi\ \sin(\mu+\beta-\omega)\pi}{\Gamma(1-\omega-a)\sin(\mu+\beta)\pi\ \sin(\beta-\omega)\pi} \sum_{n=0}^{\infty} \frac{\Gamma(\mu-\omega-a-n)z^{-n}}{\Gamma(\mu-n)\ n!}$$

(37)
$$\cdot \left(z^n D_z^n\right) z^{\omega-1+n}\ \Phi\left(\mathrm{e}^{2\pi i a}, 1-s, 1-\frac{\log z}{2\pi i}\right).$$

We know from [13, p. 24, Equations (5) and (6)] that

$$z^n D_z^n = \sum_{k=0}^{n} \binom{n-1}{k-1} B_{n-k}^{(n)} (zD_z)^k. \tag{38}$$

Therefore upon setting

$$\frac{\log z}{2\pi i} = x \quad \text{and} \quad zD_z = \frac{1}{2\pi i} D_x \tag{39}$$

and

$$1 - \frac{\log z}{2\pi i} = y \quad \text{and} \quad zD_z = \frac{-1}{2\pi i} D_y \tag{40}$$

equations (36) and (37) become respectively

$$D_z^{\mu-1} z^{\mu-1-a}\ \Phi\left(\mathrm{e}^{-2\pi i a}, 1-s, \frac{\log z}{2\pi i}\right) =$$

$$\frac{z^{1-\omega-a}\ \Gamma(\mu) \sin\beta\pi\ \sin(\mu+\beta-\omega)\pi}{\Gamma(1-\omega-a) \sin(\mu+\beta)\pi\ \sin(\beta-\omega)\pi} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n-1}{k-1} \frac{\Gamma(\mu-\omega-a-n) z^{-n}}{\Gamma(\mu-n)\ n!}$$

$$\cdot B_{n-k}^{(n)} \left(\frac{D_x}{2\pi i}\right)^k\ \mathrm{e}^{2\pi i x(\omega-1+n)}\ \Phi\left(\mathrm{e}^{-2\pi i a}, 1-s, x\right) \tag{41}$$

and

$$D_z^{\mu-1} z^{\mu-1-a}\ \Phi\left(\mathrm{e}^{2\pi i a}, 1-s, 1-\frac{\log z}{2\pi i}\right) =$$

$$\frac{z^{1-\omega-a}\ \Gamma(\mu) \sin\beta\pi\ \sin(\mu+\beta-\omega)\pi}{\Gamma(1-\omega-a) \sin(\mu+\beta)\pi\ \sin(\beta-\omega)\pi} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n-1}{k-1} \frac{\Gamma(\mu-\omega-a-n) z^{-n}}{\Gamma(\mu-n)\ n!}$$

$$\cdot\ B_{n-k}^{(n)} \left(\frac{-D_y}{2\pi i}\right)^k\ \mathrm{e}^{-2\pi i(y-1)(\omega-1+n)}\ \Phi\left(\mathrm{e}^{2\pi i a}, 1-s, y\right). \tag{42}$$

With the help of the classical Leibniz rule and the fact that

$$\frac{\partial^k}{\partial a^k} \Phi(z,s,a) = (-1)^k (s)_k \Phi(z, s+k, a), \tag{43}$$

the result follows after some simple calculations. □

4. Special cases of Theorems 3.1 and 3.2

In this section, we first give two special cases of Theorem 3.1 involving the Hurwitz-Lerch zeta function $\Phi_\mu^*(z,s,a)$. Next, we shift our focus on some interesting consequences of Theorem 3.2.

If we put $\sigma = 1$, $\nu = 1$ and $\gamma = -1$ in equation (27) and use equation (9), we find

$$\begin{aligned} \Phi_\mu^*(z,s,a) =& \frac{\sin\beta\pi\ \sin(\mu+\beta-\lambda)\pi}{\Gamma(1-\lambda)\Gamma(\lambda) \sin(\mu+\beta)\pi\ \sin(\beta-\lambda)\pi} \\ & \cdot \sum_{n=0}^{\infty} \frac{\Gamma(\mu-\lambda-n)\Gamma(\lambda+n)}{\Gamma(\mu-n)\ n!} \Phi_{\lambda+n,\lambda}^{(1,1)}(z,s,a) \end{aligned} \tag{44}$$

$$\left(z \neq 0;\ \mathrm{Re}(1-\beta) > 0;\ (\lambda+n) \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ a, \mu \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ s \in \mathbb{C} \text{ when } |z| < 1\right).$$

Setting $\sigma = 1$, $\mu = \nu$ in equation (27) and with the help of equation (8), we obtain the following expansion formula:

$$\begin{aligned}\Phi(z,s,a) =& \frac{\sin\lambda\pi\ \sin(\beta-\lambda)\pi}{\Gamma(\nu-\gamma-\lambda-1)\Gamma(1+\gamma+\lambda)\ \sin(\beta-\lambda-\gamma)\pi\ \sin(\lambda+\gamma)\pi} \\ &\cdot \sum_{n=-\infty}^{\infty} \frac{\Gamma(\nu-\lambda-n)\Gamma(\lambda+n)}{\Gamma(2+\gamma-n)\Gamma(-\gamma+n)} \Phi^{(1,1)}_{\lambda+n,1+\gamma+\lambda}(z,s,a) \qquad (45)\end{aligned}$$

$$\left(z \neq 0;\ \mathrm{Re}(1-\beta) > 0;\ (\lambda+n) \in \mathbb{C}\setminus\mathbb{Z}_0^-;\ (1+\lambda+\gamma) \in \mathbb{C}\setminus\mathbb{Z}_0^-;\ a, \nu \in \mathbb{C}\setminus\mathbb{Z}_0^-;\right.$$
$$\left. s \in \mathbb{C} \text{ when } |z| < 1\right).$$

Let us now shift our focus on some interesting consequences of Theorem 3.2. Let

$$z = \mathrm{e}^{2\pi \mathrm{i}\xi} \quad \text{and} \quad a = \frac{p}{q}\ \ (p \in \mathbb{Z};\ q \in \mathbb{N};\ \xi \in \mathbb{R})$$

and replace s by $\mu - s$, and then apply definition (12) and the series identity (14), we obtain from our theorem:

$$\Phi^*_\mu\left(\mathrm{e}^{2\pi\mathrm{i}\xi}, \mu-s, \frac{p}{q}\right)$$

$$\begin{aligned}&= \frac{i\,\Gamma(1-\mu+s)\sin\beta\pi\ \sin(\beta+\mu-\omega)\pi}{\Gamma(\mu)\Gamma\left(1-\omega-\frac{p}{q}\right)\sin(\mu+\beta)\pi\ \sin(\beta-\omega)\pi} \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{j=0}^{k}\binom{n-1}{k-1}\binom{k}{j} \\ &\cdot \frac{\Gamma\left(\mu-\frac{p}{q}-\omega-n\right)}{\Gamma(\mu-n)n!} B^{(n)}_{n-k}\,(2\pi q)^{\mu-s-1-j}(1-\mu+s)_j(\omega-1+n)_{k-j} \\ &\cdot \left[\sum_{r=1}^{q}\zeta\left(1-\mu+s+j, \frac{\xi+r-1}{q}\right)\exp\left[-i\pi\left(\frac{\mu-s-j}{2}+\frac{2(r-1+\xi)p}{q}\right)\right]\right. \\ &\left. -\sum_{r=1}^{q}\zeta\left(1-\mu+s+j, \frac{r-\xi}{q}\right)\exp\left[i\pi\left(\frac{\mu-s-j}{2}+\frac{2(r-\xi)p}{q}\right)\right]\right] \qquad (46)\end{aligned}$$

$$\left(\mu \in \mathbb{C}\setminus\mathbb{Z}_0^-;\ \omega \in \mathbb{C};\ \mathrm{Re}(1-\beta) > 0;\ s \in \mathbb{C};\ 0 < \xi < 1;\ a \in \mathbb{C}\setminus\mathbb{Z}_0^-\right).$$

Moreover, if we set $\mu = m$ $(m \in \mathbb{N})$ in equations (31) and (46), we obtain respectively

$$\begin{aligned}\Phi^*_m(z,s,a) =& \frac{i\,\Gamma(1-s)z^{-a}}{\Gamma(m)\Gamma(1-\omega-a)} \sum_{n=0}^{m-1}\sum_{k=0}^{n}\sum_{j=0}^{k}\binom{n-1}{k-1}\binom{k}{j} \\ &\cdot \frac{\Gamma(m-a-\omega-n)}{\Gamma(m-n)n!} B^{(n)}_{n-k}\,(2\pi)^{s-1-j}(1-s)_j\,(\omega-1+n)_{k-j}\end{aligned}$$

$$\cdot\left[\exp\left(-\frac{(s-j)\pi i}{2}\right)\Phi\left(\mathrm{e}^{-2\pi ia},1-s+j,\frac{\log z}{2\pi i}\right)\right.$$
$$\left.-\exp\left[\left(2a+\frac{s-j}{2}\right)\pi i\right]\Phi\left(\mathrm{e}^{2\pi ia},1-s+j,1-\frac{\log z}{2\pi i}\right)\right] \tag{47}$$

and

$$\Phi_m^*\left(\mathrm{e}^{2\pi i\xi},m-s,\frac{p}{q}\right)=\frac{i\,\Gamma(1-m+s)}{\Gamma(m)\Gamma\left(1-\omega-\frac{p}{q}\right)}\sum_{n=0}^{m-1}\sum_{k=0}^{n}\sum_{j=0}^{k}\binom{n-1}{k-1}\binom{k}{j}$$
$$\cdot\frac{\Gamma\left(m-\frac{p}{q}-\omega-n\right)}{\Gamma(m-n)n!}B_{n-k}^{(n)}\,(2\pi q)^{m-s-1-j}(1-m+s)_j(\omega-1+n)_{k-j}$$
$$\cdot\left[\sum_{r=1}^{q}\zeta\left(1-m+s+j,\frac{\xi+r-1}{q}\right)\exp\left[-i\pi\left(\frac{m-s-j}{2}+\frac{2(r-1+\xi)p}{q}\right)\right]\right.$$
$$\left.-\sum_{r=1}^{q}\zeta\left(1-m+s+j,\frac{r-\xi}{q}\right)\exp\left[i\pi\left(\frac{m-s-j}{2}+\frac{2(r-\xi)p}{q}\right)\right]\right] \tag{48}$$
$$\left(\omega\in\mathbb{C};\ \mathrm{Re}(1-\beta)>0;\ s\in\mathbb{C};\ 0<\xi<1;\ a\in\mathbb{C}\setminus\mathbb{Z}_0^-\right).$$

Setting $m=1$ in (47) yields the Lerch's functional equation (15).

For the sake of completeness, we choose to recall the expansion formulas analogue to equation (47) obtained recently by Lin et al. [11, p. 825, Equation (40)] and Garg et al. [5, Theorem 2]. Respectively, they obtained:

$$\Phi_m^*(z,s,a)=i\,\Gamma(1-s)z^{-a}\sum_{j=0}^{m-1}\frac{(j-a+1)_{m-j-1}}{j!\,(m-j-1)!}\sum_{k=0}^{j}\binom{j-1}{k-1}(1-s)_k\,B_{j-k}^{(j)}$$
$$\cdot(2\pi)^{s-1-k}\left[\exp\left(-\frac{(s-k)\pi i}{2}\right)\Phi\left(\mathrm{e}^{-2\pi ia},1-s+k,\frac{\log z}{2\pi i}\right)\right.$$
$$\left.-\exp\left[\left(2a+\frac{s-k}{2}\right)\pi i\right]\Phi\left(\mathrm{e}^{2\pi ia},1-s+k,1-\frac{\log z}{2\pi i}\right)\right] \tag{49}$$

and

$$\Phi_m^*(z,s,a)=\frac{i\,\Gamma(1-s)z^{-a}}{(m-1)!}\sum_{k=0}^{m-1}\binom{m-1}{k}B_{m-k-1}^{(m)}\sum_{j=0}^{k}(-1)^{m-k+j-1}\binom{k}{j}\,j!$$
$$\cdot\binom{s-1}{j}(-a)^{k-j}(2\pi)^{s-j-1}\left[\exp\left(-\frac{(s-j)\pi i}{2}\right)\Phi\left(\mathrm{e}^{-2\pi ia},1-s+j,\frac{\log z}{2\pi i}\right)\right.$$
$$\left.-\exp\left[\left(2a+\frac{s-j}{2}\right)\pi i\right]\Phi\left(\mathrm{e}^{2\pi ia},1-s+j,1-\frac{\log z}{2\pi i}\right)\right] \tag{50}$$

with $s\in\mathbb{C}$, $|\arg(-\log(z)\ mod\ 2\pi i)|\leq\pi$, $a\in\mathbb{C}\setminus\mathbb{Z}_0^-$.

References

1. A. Bayad and J. Chikhi, *Reduction and duality of the generalized Hurwitz-Lerch zetas*, Fixed Point Theory and Applications 2013 **2013:82**.
2. A. Erdelyi, *An integral equation involving Legendre polynomials*, SIAM J. Appl. Math. **12** (1964), 15–30.
3. A. Erdelyi, W. Magnus, F. Oberhettinger, and F. Tricomi, *Higher transcendental functions, vols.1-3*, New York, Toronto and London, McGraw-Hill Book Company, 1953.
4. ———, *Tables of integral transforms, vols.1-2,*, New York, Toronto and London, McGraw-Hill Book Company, 1953.
5. M. Garg, K. Jain, and H.M. Srivastava, *Some relationships between the generalized Apostol-Bernoulli polynomials and Hurwitz-Lerch zeta functions*, Integral Transform Spec. Funct. **17** (2006), no. 11, 803–815.
6. S. P. Goyal and R. K. Laddha, *On the generalized Rieman Zeta function and the generalized Lambert transform*, Ganita Sandesh **11** (1997), 99–108.
7. P.L. Gupta, R.C. Gupta, S.-H. Ong, and H.M. Srivastava, *A class of Hurwitz-Lerch zeta distributions and their applications in reliability*, Appl. Math. Comput. **196** (2008), 521–531.
8. J.-L. Lavoie, T.J. Osler, and R. Tremblay, *Fundamental properties of fractional derivatives via Pochhammer integrals*, Lecture Notes in Mathematics, 1976.
9. ———, *Fundamental properties of fractional derivatives via Pochhammer integrals*, Lecture Notes in Mathematics, Springer-Verlag (1976).
10. S.-D. Lin and H.M. Srivastava, *Some families of the Hurwitz-Lerch zeta functions and associated fractional derivative and other integral representations*, Appl. Math. Comput **154** (2004), 725–733.
11. S.-D. Lin, H.M. Srivastava, and P.-Y. Wang, *Some expansion formulas for a class of generalized Hurwitz-Lerch zeta functions*, Integral Transform Spec. Funct. **17** (2006), 817–827.
12. J. Liouville, *Mémoire sur le calcul des différentielles à indices quelconques*, J. de l'École Polytechnique **13** (1832), 71–162.
13. Y. Luke, *The special functions and their approximations, vols. 1-2*, Mathematics in Science and Engineering, New York and London: Academic Press, 1969.
14. K.S. Miller and B. Ross, *An introduction of the fractional calculus and fractional differential equations*, New York, Chichester, Brisbane, Toronto and Singapore, John Wiley and Sons, 1993.
15. T.J. Osler, *Fractional derivatives of a composite function*, SIAM J.Math.Anal. **1** (1970), 288–293.
16. ———, *Leibniz rule for the fractional derivatives and an application to infinite series*, SIAM J. Appl. Math. **18** (1970), 658–674.
17. ———, *Leibniz rule, the chain rule and Taylor's theorem for fractional derivatives*, Ph.D. thesis, New York University, 1970.
18. ———, *Fractional derivatives Leibniz rule*, Amer.Math.Monthly **78** (1971), 645–649.
19. D. Raducanu and H.M. Srivastava, *A new class of analytic functions defined by means of a convolution operator involving the Hurwitz-Lerch zeta function*, Integral Transforms Spec. Funct. **18** (2007), 933–943.
20. M. Riesz, *L'intégrale de Riemann-Liouville et le problème de Cauchy*, Acta. Math. **81** (1949), 1–233.
21. H.M. Srivastava, *Some formulas for the Bernoulli and Euler polynomials at rational arguments*, Math. Proc. Cambridge Philos. Soc. **129** (2000), 77–84.
22. H.M. Srivastava, M.A. Chaudhry, A. Qadir, and A. Tassaddiq, *Some extensions of the Fermi-Dirac and Bose-Einstein functions with applications to the family of the zeta and related functions*, Russ. J. Math. Phys. **18 (1)** (2011), 107–121.
23. H.M. Srivastava and J. Choi, *Series associated with zeta and related functions*, Kluwer Academin Publishers, Dordrecht, Boston and London, 2001.
24. R. Tremblay, *Une contribution à la théorie de la dérivée fractionnaire*, Ph.D. thesis, Laval University, Canada, 1974.

25. R. Tremblay, S. Gaboury, and B.-J. Fugère, *A new Leibniz rule and its integral analogue for fractional derivatives*, Integral Transforms Spec. Funct. **24 issue 2** (2013), 111–128.
26. ———, *Taylor-like expansion in terms of a rational function obtained by means of fractional derivatives*, Integral Transforms Spec. Funct. **24 issue 1** (2013), 50–64.

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