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## ON THE SYMMETRY PROPERTIES OF THE GENERALIZED HIGHER-ORDER EULER POLYNOMIALS<sup>†</sup>

A. BAYAD, T. KIM\*, J. CHOI, Y. H. KIM, AND B. LEE

**ABSTRACT.** In this paper we prove a generalized symmetry relation between the generalized Euler polynomials and the generalized higher-order (attached to Dirichlet character) Euler polynomials. Indeed, we prove a relation between the power sum polynomials and the generalized higher-order Euler polynomials..

AMS Mathematics Subject Classification : 11B68, 11S80

*Key words and phrases* : Euler polynomials, Euler numbers, symmetry

### 1. Introduction

Let  $d$  be a fixed positive integer and let  $\chi$  be the Dirichlet's character with conductor  $d$ . Then the generalized Euler numbers and polynomials attached to  $\chi$  are defined as

$$\begin{aligned} \frac{2 \sum_{a=0}^{d-1} (-1)^a \chi(a) e^{at}}{e^{dt} + 1} &= \sum_{n=0}^{\infty} E_{n,\chi} \frac{t^n}{n!}, \\ \frac{2 \sum_{a=0}^{d-1} (-1)^a \chi(a) e^{at}}{e^{dt} + 1} e^{xt} &= \sum_{n=0}^{\infty} E_{n,\chi}(x) \frac{t^n}{n!}, \text{ for } |t| < \frac{\pi}{d}, \text{ (see [1-4])}. \end{aligned} \quad (1)$$

For a real or complex parameter  $\alpha$ , we define the generalized higher-order Euler numbers and polynomials, of order  $\alpha$ , attached to  $\chi$  as follows:

$$\begin{aligned} \left( \frac{2 \sum_{a=0}^{d-1} (-1)^a \chi(a) e^{at}}{e^{dt} + 1} \right)^{\alpha} &= \sum_{n=0}^{\infty} E_{n,\chi}^{(\alpha)} \frac{t^n}{n!}, \text{ and} \\ \left( \frac{2 \sum_{a=0}^{d-1} (-1)^a \chi(a) e^{at}}{e^{dt} + 1} \right)^{\alpha} e^{xt} &= \sum_{n=0}^{\infty} E_{n,\chi}^{(\alpha)}(x) \frac{t^n}{n!}, \text{ where } |t| < \frac{\pi}{d}. \end{aligned} \quad (2)$$

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The main purpose of this paper is to prove an identity of symmetry for the generalized higher-order Euler polynomials. It turn out that the recurrence relation and multiplication theorem for the generalized Euler polynomials attached to  $\chi$ .

For the basic definitions and properties of the Euler polynomials, see [1-6].

## 2. Symmetry Identities related to the generalized higher-order Euler polynomials

**Theorem 1.** *Let  $\chi$  be the Dirichlet's character with conductor  $d \in \mathbb{N}^*$ , and  $a, b$  be the natural numbers with the same parity. For  $n, m \in \mathbb{N}$ , we have*

$$\begin{aligned} & \sum_{k=0}^n \binom{n}{k} a^k b^{n-k} E_{n-k, \chi}^{(m-1)}(aY) \sum_{j=0}^{d-1} \sum_{i=0}^{a-1} \chi(j) (-1)^{i+j} E_{k, \chi}^{(m)} \left( bX + \frac{bd}{a}i + \frac{b}{a}j \right) \\ &= \sum_{k=0}^n \binom{n}{k} b^k a^{n-k} E_{n-k, \chi}^{(m-1)}(bY) \sum_{j=0}^{d-1} \sum_{i=0}^{b-1} \chi(j) (-1)^{i+j} E_{k, \chi}^{(m)} \left( aX + \frac{ad}{b}i + \frac{a}{b}j \right). \end{aligned} \quad (3)$$

*Proof.* The generating function of the left hand side of (3) is

$$\begin{aligned} & \sum_{n \geq 0} \sum_{k=0}^n \binom{n}{k} a^k b^{n-k} E_{n-k, \chi}^{(m-1)}(aY) \sum_{j=0}^{d-1} \sum_{i=0}^{a-1} \chi(j) (-1)^{i+j} E_{k, \chi}^{(m)} \left( bX + \frac{bd}{a}i + \frac{b}{a}j \right) \frac{t^n}{n!} \\ &= \sum_{j=0}^{d-1} \sum_{i=0}^{a-1} \chi(j) (-1)^{i+j} \sum_{n \geq 0} \sum_{k=0}^n E_{n-k, \chi}^{(m-1)}(aY) \frac{(bt)^{n-k}}{(n-k)!} E_{k, \chi}^{(m)} \left( bX + \frac{bd}{a}i + \frac{b}{a}j \right) \frac{(at)^k}{k!} \\ &= \sum_{j=0}^{d-1} \sum_{i=0}^{a-1} \chi(j) (-1)^{i+j} \left( \sum_{k \geq 0} E_{k, \chi}^{(m-1)}(aY) \frac{(bt)^k}{(k!)} \right) \\ & \quad \times \left( \sum_{k \geq 0} E_{k, \chi}^{(m)} \left( bX + \frac{bd}{a}i + \frac{b}{a}j \right) \frac{(at)^k}{(k!)} \right) \\ &= \sum_{j=0}^{d-1} \sum_{i=0}^{a-1} \chi(j) (-1)^{i+j} \left( \frac{2 \sum_{j=0}^{d-1} (-1)^j \chi(j) e^{jbt}}{e^{dbt} + 1} \right)^{m-1} e^{abYt} \\ & \quad \times \left( \frac{2 \sum_{j=0}^{d-1} (-1)^j \chi(j) e^{jat}}{e^{dat} + 1} \right)^m e^{(bX + \frac{bd}{a}i + \frac{b}{a}j)at}. \end{aligned}$$

Finally, we obtain

$$\begin{aligned} & \sum_{n \geq 0} \sum_{k=0}^n \binom{n}{k} a^k b^{n-k} E_{n-k, \chi}^{(m-1)}(aY) \sum_{j=0}^{d-1} \sum_{i=0}^{a-1} \chi(j) (-1)^{i+j} E_{k, \chi}^{(m)} \left( bX + \frac{bd}{a}i + \frac{b}{a}j \right) \frac{t^n}{n!} \\ &= \frac{1}{2} \left( \frac{2 \sum_{j=0}^{d-1} (-1)^j \chi(j) e^{jat}}{e^{dat} + 1} \right)^m e^{abXt} \left( 1 - (-1)^a e^{dabt} \right) \end{aligned}$$

$$\times \left( \frac{2 \sum_{j=0}^{d-1} (-1)^j \chi(j) e^{jbt}}{e^{dbt} + 1} \right)^m e^{abYt},$$

and in the same way we obtain the generating function of the right hand side of (3) is

$$\begin{aligned} & \sum_{n \geq 0} \sum_{k=0}^n \binom{n}{k} b^k a^{n-k} E_{n-k, \chi}^{(m-1)}(bY) \sum_{j=0}^{d-1} \sum_{i=0}^{b-1} \chi(j) (-1)^{i+j} E_{k, \chi}^{(m)} \left( aX + \frac{ad}{b}i + \frac{a}{b}j \right) \frac{t^n}{n!} \\ &= \frac{1}{2} \left( \frac{2 \sum_{j=0}^{d-1} (-1)^j \chi(j) e^{jat}}{e^{dat} + 1} \right)^m e^{abXt} \left( 1 - (-1)^b e^{dabt} \right) \\ & \quad \times \left( \frac{2 \sum_{j=0}^{d-1} (-1)^j \chi(j) e^{jbt}}{e^{dbt} + 1} \right)^m e^{abYt}. \end{aligned}$$

Since  $a \equiv b \pmod{2}$  then we have the same generating functions. Thus, the coefficients of these generating functions are the same. Hence, we obtain our desired Theorem 1.  $\square$

### Remark

1) Our theorem 1 is valid for arbitrary conductor  $d$  of the Dirichlet character  $\chi$ .  $a$  and  $b$  must only have the same parity.

2) Let  $m = 1$  in (3). Then we have

$$\begin{aligned} & a^n \sum_{j=0}^{d-1} \sum_{i=0}^{a-1} \chi(j) (-1)^{i+j} E_{n, \chi} \left( bX + \frac{bd}{a}i + \frac{b}{a}j \right) \\ &= b^n \sum_{j=0}^{d-1} \sum_{i=0}^{b-1} \chi(j) (-1)^{i+j} E_{n, \chi} \left( aX + \frac{ad}{b}i + \frac{a}{b}j \right). \end{aligned}$$

3) Let  $m = 1$  in (3) and  $\chi = 1$ . Then we have

$$a^n \sum_{i=0}^{a-1} (-1)^{i+j} E_n \left( bX + \frac{b}{a}i \right) = b^n \sum_{i=0}^{b-1} (-1)^i E_n \left( aX + \frac{a}{b}i \right),$$

for all  $a$  and  $b$  with same parity. For instance,

$$\sum_{i=0}^{a-1} (-1)^i E_n \left( X + \frac{i}{a} \right) = a^{-n} E_n(aX), \text{ if } a \text{ is odd, (here we take } b = 1).$$

And

$$\sum_{i=0}^{a-1} (-1)^i E_n \left( X + \frac{i}{a} \right) = a^{-n} 2^n \left( E_n(aX) - E_n \left( aX + \frac{a}{2} \right) \right), \text{ if } a \text{ is even,}$$

(here we take  $b = 2$ ).

Now, for the rest of this paper, we set

$$\tilde{T}_{k,\chi}(n) = \sum_{i=0}^{d-1} \sum_{j=0}^{n-1} (-1)^{i+j} \chi(j) (i+jd)^k.$$

By (1), we can also easily see that

$$(-1)^{n-1} E_{k,\chi}(nd) + E_{k,\chi} = 2\tilde{T}_{k,\chi}(n).$$

Note that if  $\chi_0$  is the trivial Dirichlet's character, then

$$\tilde{T}_k(n) := \tilde{T}_{k,\chi_0}(n) = \sum_{j=1}^n (-1)^{i-1} i^k = \frac{1}{2} \left( (-1)^{n-1} E_k(n) + E_k \right).$$

**Theorem 2.** *Let  $\chi$  be the Dirichlet's character with an arbitrary conductor  $d \in \mathbb{N}^*$ . For  $a, b \in \mathbb{N}$  with same parity we have*

$$\begin{aligned} & \sum_{j=0}^n \binom{n}{j} b^j a^{n-j} E_{n-j,\chi}^{(m)}(bX) \sum_{k=0}^j \binom{j}{k} \tilde{T}_{k,\chi}(a) E_{j-k,\chi}^{(m-1)}(aY) \\ &= \sum_{j=0}^n \binom{n}{j} a^j b^{n-j} E_{n-j,\chi}^{(m)}(aX) \sum_{k=0}^j \binom{j}{k} \tilde{T}_{k,\chi}(b) E_{j-k,\chi}^{(m-1)}(bY). \end{aligned} \quad (4)$$

*Proof.* For  $a, b \in \mathbb{N}$  with  $a \equiv b \pmod{2}$ , let us consider the following functional equation:

$$\begin{aligned} F_{d,m}(a, b; X, Y) &= \frac{1}{2} \left( \frac{2 \sum_{j=0}^{d-1} (-1)^j \chi(j) e^{jat}}{e^{dat} + 1} \right)^m e^{abXt} \left( 1 - (-1)^a e^{dabt} \right) \\ &\quad \times \left( \frac{2 \sum_{j=0}^{d-1} (-1)^j \chi(j) e^{jbt}}{e^{dbt} + 1} \right)^m e^{abYt} \\ &= \left( \frac{2 \sum_{j=0}^{d-1} (-1)^j \chi(j) e^{jat}}{e^{dat} + 1} \right)^m e^{abXt} \left( \frac{1 - (-1)^a e^{dabt}}{e^{dbt} + 1} \right) \\ &\quad \times \left( \sum_{j=0}^{d-1} \chi(j) (-1)^j e^{jbt} \right) \times \left( \frac{2 \sum_{j=0}^{d-1} (-1)^j \chi(j) e^{jbt}}{e^{dbt} + 1} \right)^{m-1} e^{abYt}. \end{aligned} \quad (5)$$

It is easy to see that

$$\begin{aligned} & \left( \frac{1 - (-1)^a e^{dabt}}{e^{dbt} + 1} \right) \left( \sum_{i=0}^{d-1} \chi(i) e^{ibt} (-1)^i \right) \\ &= \left( \sum_{l=0}^{a-1} e^{ldbt} (-1)^l \right) \left( \sum_{i=0}^{d-1} \chi(i) (-1)^i e^{ibt} \right) \\ &= \sum_{j=0}^{d-1} \sum_{i=0}^{a-1} \chi(j) (-1)^{i+j} e^{(id+j)bt} \end{aligned}$$

$$\begin{aligned}
 &= \sum_{k=0}^{\infty} \left( \sum_{j=0}^{d-1} \sum_{i=0}^{a-1} \chi(j) (-1)^{i+j} (id+j)^k \right) \frac{b^k t^k}{k!} \\
 &= \sum_{k=0}^{\infty} \tilde{T}_{k,\chi}(a) \frac{b^k t^k}{k!}.
 \end{aligned} \tag{6}$$

By (5) and (6), we have

$$\begin{aligned}
 &F_{d,m}(a, b; X, Y) \\
 &= \left( \sum_{i=0}^{\infty} E_{i,\chi}^{(m)}(bX) \frac{a^i t^i}{i!} \right) \left( \sum_{l=0}^{\infty} \tilde{T}_{l,\chi}(a) \frac{b^l t^l}{l!} \right) \left( \sum_{k=0}^{\infty} E_{k,\chi}^{(m-1)}(aY) \frac{b^k t^k}{k!} \right) \\
 &= \sum_{n=0}^{\infty} \left( \sum_{j=0}^n \binom{n}{j} b^j a^{n-j} E_{n-j,\chi}^{(m)}(bX) \sum_{k=0}^j \tilde{T}_{k,\chi}(a) \binom{j}{k} E_{j-k,\chi}^{(m-1)}(aY) \right) \frac{t^n}{n!}.
 \end{aligned} \tag{7}$$

By the symmetry of  $F_{d,m}(a, b; X, Y)$  in  $a$  and  $b$  (because  $a \equiv b \pmod{2}$ ), we also see that

$$F_{d,m}(a, b; X, Y) = \sum_{n=0}^{\infty} \left( \sum_{j=0}^n \binom{n}{j} a^j b^{n-j} E_{n-j,\chi}^{(m)}(aX) \sum_{k=0}^j \binom{j}{k} \tilde{T}_{k,\chi}(b) E_{j-k,\chi}^{(m-1)}(bY) \right) \frac{t^n}{n!}. \tag{8}$$

By comparing the coefficients on the both sides of (7) and (8), we obtain the Theorem 2.  $\square$

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