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## Surveys in Geometry II. Introduction

Athanase Papadopoulos

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**SURVEYS IN GEOMETRY, II**  
**CHAPTER 1:**  
**INTRODUCTION**

ATHANASE PAPADOPOULOS

**ABSTRACT.** This chapter contains a description of the various topics and results covered in the book.

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**Keywords:** Conformal geometry, metric geometry, Teichmüller spaces, surfaces immersed in 3-manifolds, symplectic geometry, Grassmann spaces, finite homogeneous metric spaces, polytopes, the Gauss–Bonnet formula, isoperimetry, Coxeter groups.

This is the second volume of a collection of multi-authored surveys in geometry. The project of publishing these volumes arose from the conviction that the mathematical community is in need of good surveys of topics that are at the heart of current research. Thus I have asked some colleagues and friends to write an expository article on a subject they have been working on, which I find particularly important or interesting. I hope the result will be of use to mathematicians, both beginners and experienced.

The topics surveyed in the present volume include the conformal and the metric geometry of surfaces, Teichmüller spaces, surfaces immersed with prescribed extrinsic curvature in 3-dimensional manifolds, symplectic geometry, the metric theory of Grassmann spaces, finite homogeneous metric spaces, regular and semi-regular polytopes, the Gauss–Bonnet formula and its higher-dimensional versions, isoperimetry in finitely generated groups, and Coxeter groups. Let me review now each chapter in some detail.

Chapter 2, by Norbert A'Campo and me, is titled *Geometry on surfaces, a source for mathematical developments*. It is an overview of the theory of surfaces equipped with various structures: volume forms, almost complex structures, Riemannian metrics of constant curvature, quasiconformal structures, and others. Several topics discussed originate in the Riemann mapping theorem and its generalization, the uniformization theorem. The theory of Riemann surfaces is intertwined with topology and combinatorics. Higher-dimensional structures are also discussed. We also review several applications of a colored graph associated with a branched covering of the sphere, which we call a *Speiser graph*. Nevanlinna, Teichmüller and others obtained criteria, using this graph, for the determination of the type of a simply connected surface, that is, to know whether the surface is conformally equivalent to the Euclidean plane or to the unit disc. The same graph appears in the theory of dessins d'enfants as well as in a realization theorem of Thurston concerning the characterization of some branched coverings of the sphere. More precisely, Thurston's

theorem gives a necessary and sufficient condition for a 4-valent graph on the 2-sphere to be obtained as a pullback by a branched cover of the sphere of a closed curve passing through all the singular values. Combinatorial characterizations of polynomials and rational maps among a class of branched covers of the sphere are discussed. We also survey the notion of rooted colored trees associated with slalom polynomials, and certain graphs that are used in the stratification of the space of monic polynomials. New models of the Riemann sphere and of hyperbolic 2- and 3-spaces appear at several places of the survey, some of them in an expected manner, using algebra (the ring of polynomials). By analogy, some constructions based on the field of complex numbers are extended to arbitrary fields.

Chapter 3, by Ken'ichi Ohshika, is titled *Teichmüller spaces and their various metrics*. This chapter is a survey of three different Finsler metrics on Teichmüller space: the Teichmüller metric, Thurston's asymmetric metric and the earthquake metric. In particular, the author presents some recent results he obtained with Y. Huang, H. Pan and A. Papadopoulos on the so-called earthquake metric, also introduced by Thurston. He reviews a duality established in that paper between the tangent space at an arbitrary point of Teichmüller space equipped with the earthquake norm and the cotangent space at the same point equipped with Thurston's co-norm. He also reports on a result, also obtained in that paper, saying that the earthquake metric is not complete, and he provides a description of its completion.

Chapter 4 by Marc Troyanov is titled *Double forms, curvature integrals and the Gauss–Bonnet formula*. We recall that the Gauss–Bonnet formula for surfaces is a major achievement of nineteenth century differential geometry, and is one of the very good examples of how topology is closely related to geometry. In its simplest form, the formula says that for a closed surface  $S$  equipped with a Riemannian metric, the integral of the Gaussian curvature is equal (up to a universal constant) to the Euler characteristic of the surface. There are much more evolved forms of the formula. Troyanov, in Chapter 4, recalls that the Gauss–Bonnet formula is due to the effort of several mathematicians, namely, K. F. Gauss, P. Bonnet, J. Binet, and W. von Dyck, and that extensions of this formula to higher-dimensional Riemannian manifolds were obtained in the twentieth century by H. Hopf, W. Fenchel, C. B. Allendoerfer, A. Weil and S.S. Chern. The extended formula establishes relations between the Euler characteristic of a smooth, compact Riemannian manifold with (possibly empty) boundary and a curvature integral over the manifold plus a boundary term that involves a combination of curvature and the second fundamental form over the boundary. In this chapter, the author revisits the higher-dimensional formula using the formalism of double forms, a tool introduced by G. de Rham and further developed by R. Kulkarni, J. Thorpe and A. Gray in the 1960s and 1970s. In particular, he surveys the history and the techniques around Chern's version of the formula, which uses É. Cartan's moving frame formalism. He explores the geometric nature of the boundary term and he provides examples and applications. This chapter is also an occasion for Troyanov to make a detailed account of the various tools that were used by various authors in the proof of the higher-dimensional Gauss–Bonnet formula, starting with H. Hopf's problem on the *Curvatura Intega*, including the theories of double forms with their geometric applications, the Pfaffian, the moving frame, the Gauss–Kronecker curvature of hypersurfaces, the Lipschitz–Killing curvature, and several other notions.

Chapter 5, by Graham Smith, is titled *Quaternions, Monge–Ampère structures and  $\kappa$ -surfaces*. This chapter combines Riemannian geometry, conformal geometry, symplectic geometry and quaternionic geometry (an analogue of complex geometry where the field of complex numbers is replaced by that of quaternions). The study is based on a work of F. Labourie. The latter developed, in his paper *Problèmes de Monge–Ampère, courbes pseudo-holomorphes et laminations*, published in 1997, a theory of surfaces immersed in three-manifolds with prescribed extrinsic curvature. This theory has found applications in hyperbolic geometry, general relativity, Teichmüller theory and other domains. Labourie’s key insight is that if the second fundamental form of an immersed surface of prescribed extrinsic curvature is positive definite (the surface is then called infinitesimally strictly convex), then its Gauss lift is a pseudo-holomorphic curve for some suitable almost complex structure. This allows the application of Gromov’s theory developed in his paper *Pseudo-holomorphic curves in symplectic manifolds* (1987), and in particular, his compactness results for families of immersed surfaces of prescribed extrinsic curvature in 3-dimensional Riemannian manifolds. Smith, in this chapter, builds on Labourie’s work, of which he gives a quaternionic reformulation. This leads him to simpler proofs of Labourie’s results, and at the same time, to generalisations to higher-dimensions. Two theorems of Labourie are in the background: a compactness result for sequences of quasicomplete pointed infinitesimally strictly convex immersed surfaces in a complete, oriented, 3-dimensional Riemannian manifold with prescribed curvature, and a description of the accumulation points of such a sequence. A key result of this chapter establishes a relation between the solutions of the 2-dimensional Monge–Ampère equation and pseudo-holomorphic curves.

Chapter 6, by Peter Kristel and Eric Schippers, is titled *Lagrangian Grassmannians of polarizations*. This chapter is an introduction to polarization theory in symplectic and orthogonal geometries. In this setting, one starts with a triple of structures on a real vector space, namely, an inner product, a symplectic form and a complex structure, the triple satisfying a compatibility condition that ensures that when we are provided with two out of these three structures, we can reconstruct the third one, if it exists (which is not always the case). The specification of such a compatible triple is equivalent to a decomposition of the complexified ambient vector space into the eigenspaces of the complex structure. A familiar example of such a triple of structures is that of a Kähler manifold, where the manifold is equipped with three compatible structures: a Riemannian metric, an integrable complex structure and a symplectic form. Kristel and Schippers adopt the natural point of view of fixing *one* of these structures, and studying the space of structures of a second type allowing the reconstruction of a compatible structure of the third type. For example, we can fix a symplectic manifold and study the space of integrable complex structures on the manifold such that this manifold admits a compatible Riemannian metric. In the case where either the symplectic form or the inner product is fixed, we get a Grassmannian of polarizations. Polarizations appear in several contexts of algebraic and complex geometry: in the study of moduli spaces, in the theory of metaplectic and spin representations, and in conformal field theory. In Chapter 6, Kristel and Schippers survey this circle of ideas, in which the underlying vector spaces are allowed to be infinite-dimensional, emphasizing the symmetry of the symplectic and orthogonal settings. The authors consider two particular situations: the Riemannian one, in which the polarization

is an orthogonal decomposition, and the symplectic one, in which the polarization is a symplectic decomposition. The potential fields of applications of this theory of polarizations include representation theory, loop groups, complex geometry, moduli spaces, quantization, and conformal field theory.

The next three chapters are concerned with metric geometry.

Chapter 7, by Árpád Kurusa, is titled *Metric characterizations of projective-metric spaces*. The author starts by recalling Hilbert's Problem IV of the list of problems he proposed in 1900 at the Paris International Congress of Mathematicians. The problem asks for the construction of all the projective metrics on an open convex subset of projective space, that is, the metrics whose geodesics are the intersections of this open subset with the projective lines of the ambient space. Kurusa addresses the general question of characterizing such spaces under the effect of adding some additional conditions on the metric. On the same occasion, he surveys notions like Hilbert and Minkowski metrics, projective center, Ptolemaic metric, Erdős ratio, conics in metric spaces, general Finsler projective metrics, the Ceva and Menelaus properties, and other properties of triangles in a projective-metric setting. Notions such as plane and line perpendicularity, equidistance of lines, bisectors, medians, bounded curvature and others, that hold in a general metric space, are used. These properties were introduced in such a general setting by Herbert Busemann. The questions of characterizing Minkowski planes, of Hilbert geometries and of the three classical geometries, which were extensively investigated by Busemann, are addressed by Kurusa, who also mentions several open problems on this topic.

Chapter 8, by the same author, titled *Supplement to "Metric Characterization of projective-metric spaces"*, is a supplement to the previous chapter whose goal is to provide proofs of two theorems. The first one, due to B. B. Phadke, says that a projective-metric space is a Minkowski plane if all equidistants to geodesics are geodesics. The proof that the author provides is different from Phadke's original proof. The second theorem, due to the author, says that a projective-metric plane has the Ceva property (a generalization of the classical Ceva property) if and only if it is either a Minkowski plane or a model of hyperbolic or elliptic geometry. The proofs are long, which is the reason why these theorems are included in a separate chapter.

Chapter 9, by Boumediene Et-Taoui, is titled *Metric problems in projective and Grassmann spaces*. In this chapter, the author studies metric problems in real, complex and quaternionic spaces concerning equiangular lines and equi-isoclinic  $n$ -subspaces. Let us recall the setting. We take  $\mathbb{F}$  to be the real, complex or quaternionic field. Let  $p \geq 3$  and  $r \geq 2$  be two integers. A collection of lines in  $\mathbb{F}^r$  is said to be *equiangular* if any two lines in this collection make the same nonzero angle. A set of  $p$   $n$ -subspaces in  $\mathbb{F}^r$  is said to be *equi-isoclinic* if this set spans  $\mathbb{F}^r$  and if any two lines in this set make the same non-zero angle. As the author notes, such structures appear, under various names, in fields such as discrete geometry, combinatorics, harmonic analysis, frame theory, coding theory and quantum information theory. The author addresses two natural questions, namely, (i) How many equiangular lines can be placed in  $\mathbb{F}^r$ ? (ii) How many equiangular equi-isoclinic  $n$ -subspaces can be placed in  $\mathbb{F}^r$ ? The chapter is a survey of the work done and the developments due to various authors in the last 70 years on these and related questions, interpreted in the setting of the metric geometry

of projective and Grassmann spaces. This metric setting involves classical and fundamental works of Menger, Blumenthal, Lemmens, Seidel and others, as well as works of the author himself with co-authors.

Chapter 10, by Valerii Berestovskii and Yuriï Nikonorov, is titled *On the geometry of finite homogeneous subsets of Euclidean spaces*. The authors review recent results on finite homogeneous metric spaces, that is, spaces on which their isometry group acts transitively. They are especially interested in finite homogeneous metric subspaces of a Euclidean space that represent vertex sets (assumed to lie on a sphere) of compact convex polytopes whose isometry groups are transitive on the vertex set. In particular, the authors are led to the classification of regular and semiregular polytopes in Euclidean spaces, according to whether or not they satisfy the normal homogeneity property or the Clifford–Wolf homogeneity property on their vertex sets. The metric spaces that satisfy these two properties constitute a remarkable subclass of the class of homogeneous metric spaces. These properties are stronger than the usual homogeneity properties used for homogeneous metric spaces. The definitions of normal and Clifford–Wolf homogeneity involve a property satisfied by the isometry taking one point to the other, in the definition of homogeneity. The fact that such a study is closely related to the theory of convex polytopes in Euclidean spaces makes it natural to first check the presence of these properties for the vertex sets of regular and semiregular polytopes. Berestovskii and Nikonorov are then led to the study of the  $m$ -point homogeneity property and to the notion of point homogeneity degree for finite metric spaces. They discuss several recent results, in particular, the classification of polyhedra with all edges of equal length and with 2-point homogeneous vertex sets, and they present results on the point homogeneity degree for some important classes of polytopes. While discussing these classification results, the authors explain in detail the main tools used for the study of the relevant objects, and they discuss prospects for future results, presenting several open problems.

Chapter 11 by Gue-Seon Lee and Ludovic Marquis, is titled *Discrete Coxeter groups*. It is an introduction to Coxeter groups, with a focus on how these groups can be used for the construction of discrete subgroups of Lie groups. Coxeter groups are geometrically defined groups. They were introduced in 1934 by H. S. M. Coxeter. They are generated by reflections in some spaces. They generalize the Euclidean reflection groups and the symmetry groups of regular polyhedra. They appear in several areas of mathematics, in particular in the theory of representations of discrete groups in Lie groups. The topics discussed in this chapter include, besides the general Coxeter groups, the theories of reflection groups in hyperbolic space, convex cocompact projective reflection groups, projective reflection groups, divisible and quasi-divisible domains in Hilbert geometry, and Anosov representations. The latter constitute a generalization of the class of discrete convex cocompact representations of hyperbolic groups into rank one Lie groups to the setting of representations of hyperbolic groups into semi-simple Lie groups. Anosov representations were introduced by Labourie in his paper *Anosov flows, surface groups and curves in projective space* (2006), in which he studies Hitchin representations.

Chapter 12, by Bruno Luiz Santos et Marc Troyanov, is titled *Isoperimetry in finitely generated groups*. The setting is that of infinite finitely generated groups equipped with word metrics associated with finite symmetric sets of generators. An isoperimetric inequality is an inequality between the size of an arbitrary finite set

and the size of its boundary (for an appropriate definition of boundary). In this chapter, the authors revisit the work done in the 1980s-1990s by N. Varopoulos, T. Coulhon and L. Saloff-Coste on isoperimetric inequalities in finitely generated groups, adding new results and establishing relations with other topics. In particular, they obtain lower bounds on the isoperimetric quotient that appears in the isoperimetric inequality in terms of the  $\mathcal{U}$ -transform, which is a variant of the classical Legendre transform of a function, or its generalization, the Legendre–Fenchel transform, which is used in physics. The chapter also includes a review of some basic elements from geometric group theory (growth functions, amenability, the Cheeger constant, etc.) as well as some basic elements from the theory of  $\mathcal{U}$ -transform, including some computational techniques, and the relation between the  $\mathcal{U}$ -transform and the Legendre transform.

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ATHANASE PAPADOPOULOS, UNIVERSITÉ DE STRASBOURG AND CNRS, 7 RUE RENÉ DESCARTES,  
67084 STRASBOURG CEDEX, FRANCE

*Email address:* `papadop@math.unistra.fr`