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Lancaster copulas

Angelo Koudou, Yves Ngounou, Denys Pommeret

Abstract

This paper introduces a new class of copulas called Lancaster copula based on the orthogonal expansion of continuous Lancaster probability measures. We also define the associated density. We derive their estimators and investigate the asymptotic properties and performances. Monte Carlo simulation and real data are studied.

Keywords: Copula, Lancaster distributions, Maxiset theory, Minimax theory, Non-parametric estimation, Orthogonal polynomials

1. Introduction

Consider a d -random vector $\mathbf{X} = (X_1, \dots, X_d)^T$ with joint distribution function H and marginal distribution functions F_1, \dots, F_d , that we assumed to be continuous. According to Sklar's Theorem (Sklar (1959)), there exists a unique d -variate function C such that

$$H(x_1, \dots, x_d) = C(F_1(x_1), \dots, F_d(x_d)). \quad (1)$$

The function C is called the copula (function) associated to \mathbf{X} . The copula is a joint distribution function on $I^d = [0, 1]^d$ with uniform margins and satisfying $C(u_1, \dots, u_d) = H(F_1^{-1}(u_1), \dots, F_d^{-1}(u_d))$, where, for $j = 1, \dots, d$, $F_j^{-1}(u) = \inf\{x; F_j(x) \geq u\}$, is the quantile function of F_j . Assuming that for $j = 1, \dots, d$, F_j is differentiable, we can express the joint density h of \mathbf{X} as

$$h(x_1, \dots, x_d) = c(F_1(x_1), \dots, F_d(x_d)) \prod_{j=1}^d f_j(x_j), \quad (2)$$

where for $j = 1, \dots, d$, f_j is the marginal density of X_j and where

$$c = \frac{\partial^d C}{\partial x_1 \cdots \partial x_d}, \quad (3)$$

is called the copula density of \mathbf{X} . Copulas and density copulas have a large spectra of applications as described for instance in [Joe \(2014\)](#). For a general account on copulas, see, e.g. [Nelsen \(2007\)](#). Several copula families or classes are based on the Sklar's Theorem and have been derived from certain families of multivariate distribution. Various constructions have been proposed, such as, for instance, Archimedean copulas (see [Hofert et al. \(2013\)](#)), Vines (see [Aas et al. \(2009\)](#)), Elliptical copulas (see [Frahm et al. \(2003\)](#)) which are simply the distribution functions of componentwise transformed elliptically distributed random vectors, or the Extreme-value copulas (see [Gudendorf and Segers \(2010\)](#)) which relies to the copulas of random vectors distributed according to multivariate extreme value distributions.

In this paper we propose a novel copula class called *Lancaster copulas*. This class is based on the class of distributions satisfying an orthogonality condition involving orthogonal polynomials with respect to their marginal laws. That is the copula of random vectors whose joint distributions are in the class of Lancaster distributions. The reader is referred to Lancaster ([Lancaster \(1958, 1963, 1975\)](#)) the synthesis by [Koudou \(1995, 1996\)](#) and the characterization by [Pommeret \(2004\)](#); [Koudou and Pommeret \(2000\)](#) for more details. To recapitulate this class of Lancaster distributions we generally follow the standard notation in that area, as given by these authors.

If X and Y are two random variables with probability measures μ and ν , and if $(P_n)_{n \in \mathbb{N}}$ (respectively $(Q_n)_{n \in \mathbb{N}}$) is a μ (respectively ν) complete orthonormal basis of polynomials, the the joint density of (X, Y) , says

Consider bivariate distributions $\sigma(dx, dy)$ with margins μ and ν defined on spaces \mathcal{X} and \mathcal{Y} respectively. Also, we denote by (X, Y) a random vector with distribution σ . If there exists bi-orthonormal polynomials, $(P_n)_{n \in \mathbb{N}}$ and $(Q_n)_{n \in \mathbb{N}}$, associated to μ and ν , respectively, such that

$$\mathbb{E}(P_n(X)P_k(X)) = \delta_{n,k}; \mathbb{E}(Q_n(Y)Q_k(Y)) = \delta_{n,k} \text{ and } \mathbb{E}(P_n(X)P_k(Y)) = \rho_n \delta_{n,k},$$

for $n, k \geq 0$, then the class of bivariate distributions σ are known as Lancaster probabilities and the sequence $\rho_n = \mathbb{E}(P_n(X)P_k(Y))$ is known as the Lancaster sequence of σ .

These types of distributions are widely used in various areas of probability and statistics. For instance we find them in the literature concerning stationary Markov processes (see [Bussgang \(1952\)](#), [Barrett and Lampard \(1955\)](#), [Brown \(1958\)](#), [Wong and Thomas \(1962\)](#), [Wong \(1964\)](#), etc.) and also in canonical analysis (see [Dauxois and Pousse \(1975\)](#) and [Buja \(1990\)](#)).

They are also used for disjunctive kriging in geostatistics (see [Droesbeke et al. \(2002\)](#) and the reference given there). There are also a few more recent references: [Griffiths \(2009\)](#) on reversible Markov processes whose proper functions are orthogonal polynomials, [Diaconis et al. \(2008\)](#) where new techniques are proposed to calculate convergence speeds to stationary measurement in the context of bivariate Gibbs sampling and where it is shown that these techniques give good results if the target measurements are Lancaster distributions. In the area of discrete probabilities, [Diaconis and Griffiths \(2012\)](#) provide an interpretation of Lancaster probabilities of binomial margins using the generalized Ehrenfest ballot box model and ([Griffiths and Spanò, 2013](#)) for Dirichlet measurements.

The plan of the paper is as follows: Section 2 introduces the Lancaster copula and their densities. Some examples of these functions are presented in 3. Non-parametric estimators are proposed in Section 4. provides an extension to the multivariate case. The proofs and some mathematical and statistical properties are given in Appendix.

2. Lancaster copulas and their densities

2.1. Bivariate case

Let μ and ν be two probability measures on \mathcal{X} and \mathcal{Y} with density f_X and f_Y with respect to Lebesgue such that there exists an open set Θ satisfying $\int e^{\theta x} \mu(dx) < \infty$ and $\int e^{\theta y} \nu(dy) < \infty$, for all $\theta \in \Theta$, respectively. Assume that (X, Y) is a random vector with margins μ and ν and with joint distribution σ . We say that σ is a Lancaster probability with margins μ and ν if it satisfies the following decomposition:

$$\sigma(dx, dy) = \sum_{n \in \mathbb{N}} \rho_n P_n(x) Q_n(y) \mu(dx) \nu(dy),$$

where $(P_n)_n$ and $(Q_k)_k$ are orthonormal polynomials sequences with respect to μ and ν . Here we assume that μ , ν and σ are absolutely continuous and σ has density h . This yields the expansion,

$$h(x, y) = f_X(x) f_Y(y) \sum_{n \in \mathbb{N}} \rho_n P_n(x) Q_n(y) \tag{4}$$

Identifying this expression (12) with (2) from Sklar's Theorem ??, we obtain the Lancaster copula density given by

$$c(u, v) = \sum_{n \in \mathbb{N}} \rho_n P_n(F_X^-(u)) Q_n(F_Y^-(v)), \quad (5)$$

where F^- denotes the pseudo inverse of F . By integration, we get the Lancaster copula function as follows

$$C(u, v) = \sum_{n \in \mathbb{N}} \rho_n \int_0^{F_X^{-1}(u)} P_n(x) \mu(dx) \int_0^{F_Y^{-1}(v)} Q_n(y) \nu(dy) \quad (6)$$

Remark 1. *The Lancaster copula function (6) and the Lancaster copula density function (5) are completely specified by their marginal measures and the correlations ρ_n between $P_n(X)$ and $Q_n(X)$. In terms of canonical analysis, the variables $P_n(X)$ and $Q_n(Y)$ are the canonical variables. So, it is important to note that this Lancaster sequence $(\rho_n)_{n \in \mathbb{N}}$ characterizes the copula and contains the polynomial correlations between all the marginal random variables.*

3. Examples of Lancaster Copulas

3.1. Bivariate normal Lancaster copula

Let $(X, Y) \sim \mathcal{N}(0, \Sigma)$ follow a bivariate normal distribution with mean vector 0 and covariance $\Sigma = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$. For $n \in \mathbb{N}$, let

$$H_n(x) = \frac{(-1)^n}{\sqrt{n! \sqrt{2\pi}}} \exp \frac{1}{2} x^2 \left(\frac{d}{dx} \right)^n \exp -\frac{1}{2} x^2, \quad x \in \mathbb{R}$$

denote the n th Hermite polynomial. It is well-known that the polynomials $\{H_n : n \in \mathbb{N}\}$ are orthonormal with respect to the standard normal distribution μ and form a complete orthonormal basis for the Hilbert space $L^2(\mu)$. Sarmanov and Bratoeva (1967) consider series expansions of the form

$$\phi_{X,Y}(x, y) = \phi_X(x) \phi_Y(y) \sum_{n=0}^{\infty} \frac{\rho_n}{n!} H_n(x) H_n(y), \quad (x, y) \in \mathbb{R}^2 \quad (7)$$

as the Lancaster expansion of (X, Y) where the sequence of real numbers $(\rho_n)_{n \in \mathbb{N}}$ satisfies $\sum_{n=0}^{\infty} \rho_n^2 < \infty$; $\phi_{X,Y}$ denoted the density function of (X, Y) and ϕ_X, ϕ_Y their marginal densities. We also denote by Φ the probability distribution function and ϕ^{-1} the associated quantile function. [Sarmanov and Bratoeva \(1967\)](#) proved that for the expansion (7) to be nonnegative, and therefore to be a valid probability density function, it is necessary and sufficient that the sequence $(\rho_n)_{n \in \mathbb{N}}$ be the moment sequence of a random variable supported on the interval $[-1, 1]$ and set $\rho_n = \begin{cases} \rho^n, & \text{if } n \text{ even} \\ 0, & \text{otherwise} \end{cases}$. Thereby, the Lancaster copula density associated to (X, Y) is given by

$$c(u, v) = \sum_{n=0}^{\infty} \frac{\rho^n}{n!} H_n(\Phi_X^{-1}(u)) H_n(\Phi_Y^{-1}(v)), \quad (u, v) \in I^2$$

and its Lancaster copula function is as follows

$$C(u, v) = \sum_{n \in \mathbb{N}} \frac{\rho^n}{n!} \int_0^{\Phi_X^{-1}(u)} H_n(x) \mu(dx) \int_0^{\Phi_Y^{-1}(v)} H_n(y) \nu(dy)$$

3.2. Bivariate gamma Lancaster copula

The Lancaster expansion for a bivariate gamma distribution, which was derived by [Sarmanov \(1970a,b\)](#) can be stated as follows (see [Kotz et al. \(2004\)](#), page 437 – 438). Let $\alpha > -1$ and for $n \in \mathbb{N}$, the classical Laguerre polynomial is defined by

$$\begin{aligned} \tilde{\mathcal{L}}_n^{(\alpha)}(x) &= \frac{x^{-\alpha} e^{-x}}{n!} \frac{d^n}{dx^n} (e^{-x} x^{n+\alpha}) \\ &= \frac{(\alpha+1)_n}{n!} \sum_{k=0}^n \frac{(-n)_k}{(\alpha+1)_k} \frac{x^k}{k!}, \quad x > 0 \end{aligned}$$

where $(\alpha)_j = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} = \alpha(\alpha+1) \cdots (\alpha+j-1)$ denotes the rising factorial. By standardizing the classical Laguerre polynomial, we obtain the orthonormal version

$$\mathcal{L}_n^{(\alpha)}(x) = \tilde{\mathcal{L}}_n^{(\alpha)}(x) / q_k \quad \text{where } q_k^2 = \tilde{\mathcal{L}}_n^{(\alpha)2} = \frac{(\alpha+1)_n}{n!}$$

Let $\lambda \in (0, 1)$ and let α and β satisfy $\alpha \geq \beta > 0$. Sarmanov (1970a,b) derived for certain bivariate gamma random variables (X, Y) the joint probability density function,

$$f_{XY} = f_X(x)f_Y(y) \sum_{n=0}^{\infty} a_n \mathcal{L}_n^{(\alpha-1)}(x) \mathcal{L}_n^{(\beta-1)}(y), \quad x, y > 0 \text{ where } a_n = \left(\frac{(\beta)_n}{(\alpha)_n} \right)^{1/2} \lambda^n \quad (8)$$

The corresponding marginal density functions are

$$f_X(x) = \frac{1}{\Gamma(\alpha)} x^{\alpha-1} \exp(-x) \text{ and } f_Y(y) = \frac{1}{\Gamma(\beta)} y^{\beta} \exp(-y),$$

which we recognize as the density functions of one-dimensional gamma random variables with index parameters α and β respectively. Kotz et al. (2004) proved that if $\alpha = \beta$ then the density function (8) reduces to the Kibble-Moran bivariate gamma density function with $Corr(X, Y) = \lambda$ and (8) represents the Lancaster expansion for (X, Y) . Therefore, the Lancaster copula density associated to the bivariate gamma distribution (X, Y) as follows

$$c(u, v) = \sum_{n=0}^{\infty} \lambda^n \mathcal{L}_n^{(\alpha)}(F_X^{-1}(u)) \mathcal{L}_n^{(\alpha)}(F_Y^{-1}(v)), \quad (u, v) \in I^2, \quad \alpha > 1$$

and its Lancaster copula function is given by

$$C(u, v) = \sum_{n \in \mathbb{N}} \lambda^n \int_0^{F_X^{-1}(u)} \mathcal{L}_n^{(\alpha)}(x) \mu(dx) \int_0^{F_Y^{-1}(v)} \mathcal{L}_n^{(\alpha)}(y) (dy), \quad (u, v) \in I^2, \quad \alpha > 1$$

where μ is the gamma measure.

3.3. Bivariate beta Lancaster copulas

Let considering the triangular bivariate beta distribution (X, Y) with

$$\sigma(dx, dy) = \frac{a+b}{\mathbf{B}(a, b)} x^{a-1} y^{b-1} \mathbf{1}_{\mathcal{A}}(x, y) \text{ where } \mathcal{A} = \{(x, y) : x, y \geq 0, x+y \leq 1, 1 \leq a, b < \infty\}$$

and \mathbf{B} the beta function. This of joint distribution can be found in Buja (1990), page 1049.

The marginal distribution of X is $\beta_{a, b+1}$ i.e $\mu(dx) \propto \beta_{a, b+1}(dx)$ which has the Jacobi polynomials with paramter $p = a + b$ and $q = a$ on the interval

$(0, 1)$ as orthogonal polynomials where β is the univariate beta distribution. Similarly, $\nu(dy) \propto \beta_{a+1,b}(dy)$. Thus, the Lancaster sequence is given by

$$\rho_n = \frac{(-1)^n \sqrt{ab}}{\sqrt{(a+n)(b+n)}}$$

and so an explicit expression of (5) and (6) follows without difficulty. Bivariate quadratic natural exponential families

Lancaster proposed a construction of Lancaster distributions associated to natural exponential families.

4. An estimation procedure

For any positive integer N we can define the following N -th order approximations

$$c^{[N]}(u, v) = \sum_n^N \rho_n P_n(F_X^{-1}(u)) Q_n(F_Y^{-1}(v))$$

and

$$C^{[N]}(u, v) = \sum_n^N \rho_n \int_0^{F_X^{-1}(u)} P_n(x) \mu(dx) \int_0^{F_Y^{-1}(v)} Q_n(y) \nu(dy)$$

If we observe an m -sample $(X_1, Y_1), \dots, (X_m, Y_m)$, of iid random data, having joint distribution function F_{XY} and joint density f_{XY} of (X, Y) , we can estimate the ρ_n by

$$\hat{\rho}_n = \frac{1}{m} \sum_{i=1}^m P_n(X_i) Q_n(Y_i).$$

Therefore, a N -th order non-parametric estimator of the Lancaster copula density c is given by

$$\hat{c}^{[N]}(u, v) = \sum_n^N \hat{\rho}_n P_n(\hat{F}_X^{-1}(u)) Q_n(\hat{F}_Y^{-1}(v)) \quad (9)$$

where \widehat{F}_Z^{-1} is the empirical quantile estimation of the random variable Z . By integration, we get a non-parametric estimator of the Lancaster copula function as follows

$$\widehat{C}^{[N]}(u, v) = \sum_n^N \widehat{\rho}_n \int_0^{\widehat{F}_X^{-1}(u)} P_n(x) \mu(dx) \int_0^{\widehat{F}_Y^{-1}(v)} Q_n(y) \nu(dy) \quad (10)$$

Remark 1. • *Marginal quantiles can be estimate through kernel method in the following way*

$$\widehat{F}_Z^{-1}(t) = \inf\{z : \widehat{F}_Z(z) \geq t\} \text{ with } \widehat{F}_Z(z) = \frac{1}{m} \sum_{i=1}^m K\left(\frac{z - Z_i}{h_m}\right)$$

where k is a kernel on \mathbb{R} , that is, a symmetric density function, $K(z) = \int_{-\infty}^z k(s) ds$ and h_m is a sequence of strictly positive smoothing parameter which controls the smoothness of the estimator, with $\lim_{m \rightarrow \infty} h_m = 0$ and $\lim_{m \rightarrow \infty} mh_m = \infty$.

- *The estimator of C and c have the same simple form whether the margins are known or not.*

Properties of such estimators are summarized in Appendix. In particular, we have a truncation parameter selection, asymptotic properties and minimax results in the bi-orthogonal case.

5. Multivariate extensions

To extend the definition of the Lancaster copula in multivariate case, we introduce the following multivariate notations. Given $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ and $n = (n_1, \dots, n_d) \in \mathbb{N}^d$, we write $x^n = x_1^{n_1} \times \dots \times x_d^{n_d}$ and we call the integer $|n| = n_1 + \dots + n_d$ the order of n . A $|n|$ th degree polynomial, say P_n , has the following form

$$P_n(x) = \sum_{\substack{k \in \mathbb{N}^d \\ |k| \leq |n|}} \alpha_k x^k,$$

where at least one real α_k is non null for $|k| = |n|$. Let \mathbf{X} and \mathbf{Y} be two random variables on \mathbb{R}^d with probability measures μ and ν respectively. Let

$(P_n)_{n \in \mathbb{N}^d}$ and $(Q_n)_{n \in \mathbb{N}^d}$ be two basis of orthonormal polynomials with respect to the distributions of \mathbf{X} and \mathbf{Y} , respectively.

Here and subsequently, P_n and Q_n denote the polynomials of the $|n|$ th and $|k|$ th degrees.

The joint distribution σ is called a Lancaster probability if

$$\mathbb{E}(P_n(\mathbf{X})Q_k(\mathbf{Y})) \text{ vanishes for } n \neq k.$$

that is

$$\int P_n(x)Q_k(y)\sigma(dx, dy) = \rho_n\delta_{n,k}$$

where δ_{nk} is the Kronecker's delta and σ the joint distribution of (\mathbf{X}, \mathbf{Y}) . ρ_n is called a Lancaster sequence. It is well known that the joint probability measure σ has a polynomial expansion under the condition that σ belongs to the Hilbert space $L_2(\mu \otimes \nu)$. Then we can write

$$\sigma(dx, dy) = \sum_{n \in \mathbb{N}^d} \rho_n P_n(x)Q_n(y)\mu(dx)\nu(dy) \quad (11)$$

We assume that μ, ν and σ are absolutely continuous with respect to Lebesgue measure and we denote their corresponding probability density functions by $f_{\mathbf{X}}, f_{\mathbf{Y}}$ and $f_{\mathbf{XY}}$ respectively. This yields the expansion,

$$f_{\mathbf{XY}}(x, y) = f_{\mathbf{X}}(x)f_{\mathbf{Y}}(y) \sum_{n \in \mathbb{N}^d} \rho_n P_n(x)Q_n(y) \quad (12)$$

Identifying (2) and (12) we obtain the relation

$$c(\mathbf{u}, \mathbf{v}) = \sum_{n \in \mathbb{N}^d} \rho_n P_n(\mathbf{F}_{\mathbf{X}}^{-1}(\mathbf{u})) Q_n(\mathbf{F}_{\mathbf{Y}}^{-1}(\mathbf{v})), \quad (13)$$

where $\mathbf{u} = (u_1, \dots, u_d), \mathbf{v} = (v_1, \dots, v_d) \in I^d = [0, 1]^d$ and $\mathbf{F}_{\mathbf{Z}}^{-1}(\mathbf{z}) = (\mathbf{F}_{Z_1}^{-1}(z_1), \dots, \mathbf{F}_{Z_d}^{-1}(z_d))$

We deduce an expression of the copula C , as follows

$$C(\mathbf{u}, \mathbf{v}) = \sum_{n \in \mathbb{N}^d} \rho_n \left(\int_0^{F_{X_1}^{-1}(u_1)} \dots \int_0^{F_{X_d}^{-1}(u_d)} P_n(x)\mu(dx) \right) \left(\int_0^{F_{Y_1}^{-1}(v_1)} \dots \int_0^{F_{Y_d}^{-1}(v_d)} Q_n(y)\nu(dy) \right)$$

5.1. Estimation

Let assume $((X(1), Y(1)), \dots, ((X(m), Y(m)))$ are i.i.d with Lancaster distribution such that $((X(k), Y(k)))$ are distributed as (\mathbf{X}, \mathbf{Y}) . $X(k)$ denote the vector $(X_1(k), \dots, X_d(k))$ and $Y(k)$ denote the vector $(Y_1(k), \dots, Y_d(k))$.

The lancaster copula density can be approximated by the following truncated sum

$$c^{[N]}(\mathbf{u}, \mathbf{v}) = \sum_{|n| \leq N} \rho_n P_n(\mathbf{F}_{\mathbf{X}}^{-1}(\mathbf{u})) Q_n(F_{\mathbf{Y}}^{-1}(\mathbf{v})),$$

where N is an arbitrary integer. The square norm of the error of such approximation can be easily expressing using the orthogonality of the polynomials. We obtain

$$\int (c - c^{[N]})^2(dx, dy) = \sum_{|n| > N} \rho_n^2.$$

We can also the estimate the Lancaster sequence ρ_n by the following empirical correlations

$$\hat{\rho}_n = \frac{1}{m} \sum_{i=1}^m P_n(X(i)) Q_n(Y(i)).$$

Hence, we get the following unbiased estimation of c

$$\hat{c}^{[N]}(\mathbf{u}, \mathbf{v}) = \sum_{|n| \leq N} \hat{\rho}_n P_n(\mathbf{F}_{\mathbf{X}}^{-1}(\mathbf{u})) Q_n(F_{\mathbf{Y}}^{-1}(\mathbf{v})),$$

The same reasoning makes it possible to obtain an analog result for the Lancaster copula function. All properties in bivariate case can also be extended in the multivariate case.

6. Discussion

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