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ALGEBRAIC GROMOV'S ELLIPTICITY OF CUBIC HYPERSURFACES

SHULIM KALIMAN AND MIKHAIL ZAIDENBERG

ABSTRACT. We show that every smooth cubic hypersurface X in \mathbb{P}^{n+1} , $n \geq 2$ is algebraically elliptic in Gromov's sense. This gives the first examples of non-rational projective manifolds elliptic in Gromov's sense. We also deduce that the punctured affine cone over X is elliptic.

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1. INTRODUCTION

Gromov's ellipticity appeared (and was extremely useful) in complex analysis within the Oka-Grauert theory, see [Gro89], [For17a] and [For23]. In the same paper [Gro89] Gromov developed an algebraic version of this notion. We deal below exceptionally with algebraic ellipticity. Thus, all varieties and vector bundles in this paper are algebraic and "ellipticity" means the algebraic ellipticity.

Let \mathbb{K} be an algebraically closed field of characteristic zero and \mathbb{A}^n resp. \mathbb{P}^n be the affine resp. projective n -space over \mathbb{K} .

Given a smooth algebraic variety X , a *spray of rank r* over X is a triple (E, p, s) where $p: E \rightarrow X$ is a vector bundle of rank r with zero section Z and $s: E \rightarrow X$ is a morphism such that $s|_Z = p|_Z$. A spray (E, p, s) is *dominating at $x \in X$* if the restriction $s|_{E_x}: E_x \rightarrow X$ to the fiber $E_x = p^{-1}(x)$ is dominant at the origin $0_x \in E_x \cap Z$ of the vector space E_x i.e. $ds(T_0 E_x) = T_x X$.

Key words and phrases. spray, Gromov's ellipticity, unirationality, stable rationality, cubic threefold, cubic hypersurface, affine cone.

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One says that the variety X is *elliptic* if it admits a spray (E, p, s) which is dominating at each point $x \in X$, see [Gro89, 3.5A]. It is immediate from the definition that every elliptic manifold is unirational. An algebraic variety X is said to be *uniformly rational* (or *regular* in the terminology of [Gro89, 3.5D]) if every point $x \in X$ has a neighborhood isomorphic to an open subset of \mathbb{A}^n . It is called *stably uniformly rational* if $X \times \mathbb{A}^k$ is uniformly rational for some $k \geq 0$. A complete stably uniformly rational variety is elliptic, see [AKZ23, Theorem 1.3 and Corollary 3.7].

It is shown in [BB14, Example 2.4] that a rational smooth cubic hypersurface X in \mathbb{P}^{n+1} , $n \geq 2$, is uniformly rational (cf. also [Gro89, Remark 3.5E''']). Hence X is elliptic. There are examples of such hypersurfaces X of any even dimension $n = 2k \geq 2$. Another class of uniformly rational (and hence elliptic) projective varieties is provided by smooth complete intersections of two quadrics in \mathbb{P}^{n+2} , $n \geq 3$, see [BB14, Example 2.5]; cf. also [AKZ23, Examples 4.8].

Gromov asked in [Gro89, 3.5B''] whether any rational smooth projective variety is elliptic; the answer is still unknown. Gromov also discussed in [Gro89, 3.5B''] a conjectural equivalence between ellipticity and unirationality. In the opposite direction, the question arises whether the ellipticity implies the (stable, uniform) rationality.

The answer to the latter question is negative. Indeed, for a certain natural number n there exists a finite subgroup F of $\mathrm{SL}(n, \mathbb{C})$ such that the quotient $Y = \mathrm{SL}(n, \mathbb{C})/F$ is stably irrational, see e.g. [Sal84, Theorem 3.6], [Bog87], [Pop11, Example 1.22] and [Pop13]. Being an affine homogeneous manifold of a semisimple group, Y is flexible, that is, the subgroup $\mathrm{SAut}(Y) \subseteq \mathrm{Aut}(Y)$ generated by all one-parameter unipotent subgroups of $\mathrm{Aut}(Y)$ acts highly transitively on Y , see [AFKKZ13, Proposition 5.4]. The latter implies that Y is elliptic, see e.g. [AFKKZ13, Proposition A.3]. Thus, Y is an example of a stably irrational elliptic affine manifold.

Our aim is to find an irrational smooth projective variety X which is Gromov's elliptic. It is not clear if one could explore to this end the above example. Indeed, given a smooth elliptic affine variety Y , we are lacking the construction of a smooth elliptic completion X of Y .

Nevertheless, we will establish the ellipticity of every smooth cubic threefold $X \subseteq \mathbb{P}^4$, see Theorem 1.1. By the celebrated Clemens-Griffiths theorem [CG72] such a threefold defined over \mathbb{C} is irrational. According to Murre [Mur73] the same holds for every smooth cubic threefold defined over an algebraically closed field of characteristic different from 2.

Our main result is the following theorem.

Theorem 1.1. *A smooth cubic hypersurface $X \subseteq \mathbb{P}^{n+1}$ of dimension $n \geq 2$ is elliptic in the sense of Gromov.*

Remarks 1.2.

1. It is not known whether there exist a stably (uniformly) rational smooth cubic threefold, see e.g. [CT19]. Also, no example of an irrational smooth cubic hypersurface of dimension ≥ 4 is known. Recall that every smooth cubic hypersurface of dimension ≥ 2 is unirational, see [Kol02] or [Huy23, Remark 5.13].

2. A smooth cubic threefold X is far from being homogeneous. Indeed, due to Matsumura–Monsky's theorem (see [MM63]) $\text{Aut}(X)$ is a finite group.

3. Notice that any rational smooth projective surface X has a covering by open subsets isomorphic to \mathbb{A}^2 . Hence it is uniformly rational and also elliptic. In the sequel we deal with cubic hypersurfaces of dimension ≥ 3 .

Let us recall the following open question, cf. [Gro89, Remark 3.5E''']:

Question. *Does the Gromov ellipticity of a smooth complete variety survive the birational maps generated by blowups and contractions with smooth centers?*

There are some partial results concerning this question. The uniform rationality is preserved under blowups with smooth centers, see [BB14, Proposition 2.6] (cf. [Gro89, 3.5E-E'']). Hence, also the ellipticity of a complete uniformly rational variety is preserved, see [AKZ23, Corollary 1.5]. For not necessary complete or uniformly rational elliptic varieties, the ellipticity is preserved under blowups with smooth centers under some additional assumptions, see [Gro89, Corollary 3.5.D''], [LT17, Corollary 2], [KKT18] and [KZ23b, Theorem 0.1]. The preservation of ellipticity under blowdowns with smooth centers is unknown.

Forstnerič [For17b] proved that every elliptic projective manifold X of dimension n admits a surjective morphism from \mathbb{C}^n , whose restriction to an open subset of \mathbb{C}^n is smooth and also surjective. Due to Kusakabe [Kus22] this remains true for not necessary complete elliptic algebraic manifold X over \mathbb{K} , provided one replaces \mathbb{C}^n by $\mathbb{A}_{\mathbb{K}}^{n+1}$. Exploring these facts and also [KZ23b, Corollary 3.8 and Proposition 6.1] and Theorem 1.1, we deduce the following immediate corollary.

Corollary 1.3. *Let $X \subseteq \mathbb{P}^{n+1}$, $n \geq 2$ be a smooth cubic hypersurface and Y be a punctured generalized affine cone over X defined by an ample polarization of X . Then the following hold.*

- Y is elliptic in Gromov's sense.

- *There exist surjective morphisms $\mathbb{A}^{n+1} \rightarrow X$ resp. $\mathbb{A}^{n+2} \rightarrow Y$ which are smooth and surjective on appropriate open subsets $U \subseteq \mathbb{A}^{n+1}$ resp. $V \subseteq \mathbb{A}^{n+2}$.*
- *If $\mathbb{K} = \mathbb{C}$, then there exists a surjective morphism $\mathbb{A}^n \rightarrow X$ which is smooth and surjective on an appropriate open subset $U \subseteq \mathbb{A}^n$.*
- *The monoid of endomorphisms $\text{End}(Y)$ acts m -transitively on Y for any natural number m .*

Recall that a *generalized affine cone* \hat{Y} over a smooth projective variety X defined by an ample \mathbb{Q} -divisor D on X is the affine variety

$$\hat{Y} = \text{Spec} \left(\bigoplus_{k=0}^{\infty} H^0(X, \mathcal{O}_X([kD])) \right),$$

see, e.g., [KPZ13, Sec. 1.15]. In the case where D is a hyperplane section of $X \subseteq \mathbb{P}^n$ the cone \hat{Y} is the usual affine cone over X . The corresponding punctured cone Y over X is obtained from \hat{Y} by removing the vertex of \hat{Y} ; so Y is a smooth quasiprojective variety.

2. CRITERIA OF ELLIPTICITY

For the proofs of the following proposition see [Gro89, 3.5B], [For17a, Propositions 6.4.1 and 6.4.2], [LT17, Remark 3] and [KZ23b, Appendix B].

Proposition 2.1 (Gromov's extension lemma). *Let X be a smooth complete variety and (E, p, s) be a spray on an open subset $U \subseteq X$ with values in X , that is, $p: E \rightarrow U$ is a vector bundle on U with zero section Z and $s: E \rightarrow X$ is a morphism such that $s|_Z = p|_Z$. Then for each $x \in U$ there is a smaller neighborhood $V \subseteq U$ of x and a spray (E', p', s') on X such that $(E', p', s')|_V = (E, p, s)|_V$.*

Recall that a variety X is said to be *locally elliptic* if for any $x \in X$ there is a local spray (E_x, p_x, s_x) defined in a neighborhood U of x and dominating at x such that $s_x: E \rightarrow X$ takes values in X . The variety X is called *subelliptic* if there is a family of sprays $\{(E_i, p_i, s_i)\}_{i \in I}$ on X which is dominating at each point $x \in X$, that is,

$$T_x X = \text{span}(ds_i(T_{0_i, x} E_{i, x}) \mid i \in I) \quad \forall x \in X.$$

The following corollary on the equivalence of local and global ellipticity is indicated in [Gro89, 3.5B']; see also [KZ23a, Theorem 1.1 and Corollary 2.4].

Corollary 2.2 (Gromov's Localization Lemma). *The ellipticity of a smooth variety is equivalent to its local ellipticity.*

Sketch of the proof. Choose a finite covering of X equipped with sprays with values in X dominating on the elements of this covering. We may suppose that the vector bundles of these sprays are trivial, and so can be decomposed in a direct sum of linear bundles. By Proposition 2.1 one can extend the resulting rank 1 sprays to sprays defined on the whole X , and then compose them in order to get a spray dominating on X . \square

Likewise, we have the following fact.

Proposition 2.3 ([KZ23a, Theorem 1.1]). *The ellipticity of a smooth variety is equivalent to its subellipticity.*

Given a spray (E, p, s) with values in X and a point $x \in X$, the constructible subset $O_x := s(E_x) \subseteq X$ is called the s -orbit of x . The proof of Proposition 2.3 leads to the following lemma, cf. [KZ23a].

Lemma 2.4. *A smooth variety X of dimension n is elliptic if for every $x \in X$ there exist local rank 1 sprays (E_i, p_i, s_i) , $i = 1, \dots, n$ with values in X defined on respective neighborhoods U_i of x such that the s_i -orbits $O_{i,x}$ of x are curves with local parameterizations $s: (E_{i,x}, 0) \rightarrow (O_{i,x}, x)$ étale at x whose tangent vectors at x span $T_x X$.*

Sketch of the proof. Shrinking U_i we may suppose that (E_i, p_i, s_i) is defined on the whole X for $i = 1, \dots, n$, see Proposition 2.1. Since the tangent lines $ds(T_0 E_{i,x})$ at x of the s_i -orbits $O_{i,x}$, $i = 1, \dots, n$ span $T_x X$, the composition of these extended sprays is a spray of rank n on X dominating at x . This domination spreads to a neighborhood of x in X . It follows that X is locally elliptic (and also subelliptic). By Gromov's Localization Lemma, see Corollary 2.2 (or alternatively by Proposition 2.3) X is elliptic. \square

In our proof of Theorem 1.1 in the next section we construct on any smooth cubic hypersurface X of dimension $n \geq 3$ such an n -tuple of independent rank 1 sprays (E_i, p_i, s_i) dominating in the orbit directions.

3. ELLIPTICITY OF CUBIC HYPERSURFACES

We use the following notation.

Notation 3.1. For a subset $M \subseteq \mathbb{P}^n$ we let $\langle M \rangle$ be the smallest projective subspace which contains M . Given a smooth hypersurface $X \subseteq \mathbb{P}^{n+1}$ and a point $x \in X$ we let $S_x = \mathbb{T}_x X \cap X$, where $\mathbb{T}_x X$ stands for the projective tangent space of X at x . Let \mathcal{C}_x stand for the union of lines on X through the point $x \in X$. Clearly, $\mathcal{C}_x \subseteq S_x$. Recall that for $n \geq 3$ every smooth cubic hypersurface $X \subseteq \mathbb{P}^{n+1}$ is covered by

projective lines (this follows, for instance, from [CG72, Corollary 8.2]). Therefore, \mathcal{C}_x has positive dimension for every $x \in X$.

Given a cubic threefold $X \subseteq \mathbb{P}^4$, the equality $\mathcal{C}_x = S_x$ holds if and only if x is an Eckardt point of X . Recall that a point $x \in X$ is called an *Eckardt point* if there is an infinite number of lines on X passing through x . In the latter case \mathcal{C}_x is the cone with vertex x over a plane elliptic cubic curve. Notice that a general cubic threefold has no Eckardt points, a cubic threefold can contain at most 30 Eckardt points (see [CG72, p. 315]), and the maximal number of 30 Eckardt points is attained only for the Fermat cubic threefold, see [CG72, p. 315], [Rou09] and [Huy23, Chapter 5, Remark 1.7].

Lemma 3.2. *Given a smooth cubic hypersurface $X \subseteq \mathbb{P}^{n+1}$ where $n \geq 3$, the tangent space $T_x X$ at a general point $x \in X$ is spanned by some n tangent lines to projective lines on X passing through x .*

Proof. Let us start with the case $n = 3$, that is, let X be a smooth cubic threefold. Through a general point $x \in X$ pass exactly 6 lines, see e.g. [AK77, Proposition (1.7)] or [Huy23, Chapter 5, Exercise 1.4]. More precisely, the lines of the second type on X sweep a proper closed subset, say $Y \subseteq X$, and through every point $x \in X \setminus Y$ pass exactly 6 distinct lines on X , see [Mur72, Lemma (1.19)]. These 6 lines form the cone \mathcal{C}_x . No four of them are coplanar, hence the tangent space $T_x X$ is spanned by the tangent lines to some three of these projective lines through x .

Let us show now that the latter property holds as well in higher dimensions. Given a smooth cubic hypersurface $X \subseteq \mathbb{P}^{n+1}$, $n \geq 4$ and a point $x \in X$, consider the projective variety

$$\mathbb{P}\mathcal{C}_x := \{T_x l \mid x \in l \subseteq \mathcal{C}_x\} \subseteq \mathbb{P}T_x X$$

where l stands for a line on X . Clearly, $T_x X$ is spanned by the tangent lines $T_x l$ for $l \subseteq \mathcal{C}_x$ if and only if $\mathbb{P}\mathcal{C}_x$ is linearly nondegenerate in $\mathbb{P}T_x X \cong \mathbb{P}^{n-1}$. There exists an open dense subset $U \subseteq X$ such that for every $x \in U$ the projective cone \mathcal{C}_x has codimension 2 in $T_x X$. The dimension $\dim \langle \mathbb{P}\mathcal{C}_x \rangle$ is lower semicontinuous on U and attains its maximal value, say m on an open dense subset $U_0 \subseteq U$. Let $\mathcal{V} \subseteq \mathbb{P}T_x X|_{U_0}$ be the subvariety swept out by the $\langle \mathbb{P}\mathcal{C}_x \rangle$ for $x \in U_0$.

Suppose that $m < n - 1$ (contrary to the assertion of the lemma), so that \mathcal{V} is a proper closed subset of $\mathbb{P}T_x X|_{U_0}$. Choose a general linear section Y of X by a 4-plane $\mathbb{P}^4 \subseteq \mathbb{P}^{n+1}$. Then Y is a smooth cubic threefold. Furthermore, for a general $x \in Y \cap U_0$ the proper subspaces $\mathbb{P}T_x Y \cong \mathbb{P}^2$ and $\langle \mathbb{P}\mathcal{C}_x \rangle \cong \mathbb{P}^m$ of $\mathbb{P}T_x X \cong \mathbb{P}^{n-1}$ are transversal. On the other hand, $T_x Y$ is spanned by the tangent lines to the lines on Y

passing through x . Thus, we have $\mathbb{P}T_x Y \subseteq \langle \mathbb{P}\mathcal{C}_x \rangle$. This contradiction ends the proof. \square

Notation 3.3. For a general point $u \in X$ we let S_u^* be the set of points $x \in X$ such that $u \in \mathbb{T}_x X$. Let $x \in S_u^*$ be a general point and $u^* = \mathbb{T}_u X$ resp. $x^* = \mathbb{T}_x X$ be the corresponding points of the dual hypersurface $X^* \subseteq (\mathbb{P}^{n+1})^\vee$. Then $u \in \mathbb{T}_x X$ if and only if $x^* \in \mathbb{T}_{u^*} X^*$, that is, $x \in S_u^*$ if and only if $x^* \in S_{u^*}$. It follows that S_u^* is a hypersurface in X passing through u . It is easily seen that $\mathcal{C}_u \subseteq S_u^*$, and so $\mathcal{C}_u \subseteq S_u \cap S_u^*$.

Lemma 3.4. *For a smooth cubic hypersurface $X \subseteq \mathbb{P}^{n+1}$, $n \geq 1$ one has $S_u \cap S_u^* = \mathcal{C}_u$ for every $u \in X$.*

Proof. If $x \in S_u \cap S_u^*$ is different from u then $l := \langle x, u \rangle \subseteq \mathbb{T}_x X \cap \mathbb{T}_u X$. By the Bézout theorem, the bitangent line l of X is contained in X . It follows that $x \in \mathcal{C}_u$. Hence we have $S_u \cap S_u^* \subseteq \mathcal{C}_u$. Since also $\mathcal{C}_u \subseteq S_u \cap S_u^*$ one has $S_u \cap S_u^* = \mathcal{C}_u$. \square

Let again X be a smooth cubic hypersurface in \mathbb{P}^{n+1} , $n \geq 1$. Following [BB14, Example 2.4] and [BKK13] let us consider the birational self-map $\tau_u: X \dashrightarrow X$ which sends a general point $x \in X$ to the third point $y = \tau(x)$ of intersection of the line $\langle x, u \rangle$ with X . Thus, X cuts out on the line $\langle x, u \rangle$ the divisor $x + u + y$. In fact, τ_u is well defined unless $x = u$ or the line $\langle x, u \rangle$ is contained in X . Thus, $\tau_u: X \dashrightarrow X$ is regular on $X \setminus \mathcal{C}_u$.

Lemma 3.5. *Let $X \subseteq \mathbb{P}^{n+1}$, $n \geq 1$ be a smooth cubic hypersurface and $x, u \in X$ be such that $x \in X \setminus \mathcal{C}_u$. Then $\tau_u(x) = \tau_u(x')$ for a point $x' \in X$ if and only if either $u \in \mathbb{T}_x X$ (i.e. $x \in S_u^*$) and $x' = x$, or $x \in \mathbb{T}_u X$ (i.e. $x \in S_u$) and $x' = u$.*

Proof. Notice that by our assumption $x \neq u$ and $\langle x, u \rangle \not\subseteq X$. Suppose that $\tau_u(x) = \tau_u(x')$ holds. Then we have $\langle x, u \rangle = \langle x', u \rangle$. So the points x, x', u are aligned. If $x' \notin \{x, u\}$, then $\tau_u(x) = x'$ and $\tau_u(x') = x$. Applying τ_u once again we obtain $x' = \tau_u(x) = \tau_u(x') = x$, a contradiction. Thus, $x' \in \{x, u\}$, and so the divisor cut out by X on the line $\langle x, u \rangle$ is either $2x + u$, or $2u + x$. We have $x' = x$ in the former case and $x' = u$ in the latter case. Now the assertion follows. \square

The following corollary is immediate.

Corollary 3.6.

- (a) *The morphism $\tau_u: X \setminus \mathcal{C}_u \rightarrow X$ contracts the hyperplane section $S_u \setminus \mathcal{C}_u$ to the point u and fixes pointwise the variety $S_u^* \setminus S_u = S_u^* \setminus \mathcal{C}_u$ (see Lemma 3.4).*
- (b) *The indeterminacy points of τ_u are contained in \mathcal{C}_u .*

- (c) *The affine threefold $X \setminus S_u \subseteq X \setminus \mathcal{C}_u$ is invariant under τ_u and $\tau_u|_{X \setminus S_u}$ is a biregular involution with mirror $S_u^* \setminus \mathcal{C}_u$ which acts freely on $X \setminus (S_u \cup S_u^*)$.*

In the proof of Theorem 1.1 we use the following construction of a rank 1 spray on a smooth cubic hypersurface.

Proposition 3.7. *Let $X \subseteq \mathbb{P}^{n+1}$, $n \geq 3$ be a smooth cubic hypersurface, $y \in X$ be an arbitrary point and $x \in X$ be a general point. Let $u \in X \cap \langle x, y \rangle$ be a point different from x and y . Choose a line $l \subseteq \mathcal{C}_x$ on X through x , and let $C^* = \tau_u(l \setminus S_u) \cong \mathbb{A}^1$. Then there exists a rank 1 spray (E, p, s) on X such that the s -orbit O_y of y coincides with C^* and $ds(T_0 E_y)$ is the tangent line to $O_y = C^*$ at y .*

Proof. Since X cuts out on the line $\langle x, y \rangle$ the reduced divisor $x + u + y$ we have

$$x, y \in U_u := X \setminus (S_u \cup S_u^*) \quad \text{and} \quad u, y \in U_x := X \setminus (S_x \cup S_x^*),$$

see Lemma 3.5. In particular, $x \notin \mathbb{T}_u X$. Hence the line $l \subseteq X$ through x meets $\mathbb{T}_u X$ in a single point, say $z \in S_u = \mathbb{T}_u X \cap X$.

Fix an isomorphism $f: \mathbb{P}^1 \xrightarrow{\cong} l$ sending 0 to x and ∞ to z . Then f embeds $\mathbb{A}^1 = \mathbb{P}^1 \setminus \{\infty\}$ onto $l \setminus \{z\} \subseteq X \setminus S_u$. Since $\tau_u|_{X \setminus S_u} \in \text{Aut}(X \setminus S_u)$, see Corollary 3.6(c), the map $\varphi_u := \tau_u \circ f|_{\mathbb{A}^1}: \mathbb{A}^1 \rightarrow C^*$ is an isomorphism.

By Corollary 3.6(c) $\tau_x|_{U_x}$ is a biregular involution acting freely on U_x and interchanging u and y . Since the projective line $\langle x, u \rangle$ is not tangent to X , we can choose an open neighborhood $V_u \subseteq U_x$ of u such that also the line $\langle x, u' \rangle$ is not tangent to X for each $u' \in V_u$. Letting $y' = \tau_{u'}(x) \in \langle x, u' \rangle \cap X$ we have $\tau_x(u') = y'$. Letting $V_y = \tau_x(V_u) \subseteq U_x$ the restriction $\tau_x|_{V_u}: V_u \xrightarrow{\cong} V_y$ is biregular, and so V_y is a neighborhood of $y = \tau_u(x)$ in X .

Shrinking V_u if necessary we may assume that for any $u' \in V_u$ the tangent hyperplane $\mathbb{T}_{u'} X$ does not pass through x , so that $z(u') := l \cap \mathbb{T}_{u'} X \neq x$.

Consider the trivial \mathbb{P}^1 -bundle $\pi: V_u \times l \rightarrow V_u$ over V_u along with the constant section $Z_0 = V_u \times \{x\}$ of π . The subset $\{z(u') | u' \in V_u\} \subseteq V_u \times l$ defines a section, say Z' of π disjoint with Z_0 . The restriction $\pi: (V_u \times l) \setminus Z' \rightarrow V_u$ is a smooth \mathbb{A}^1 -fibration with irreducible fibers.

There is an automorphism of the \mathbb{P}^1 -bundle $\pi: V_u \times l \rightarrow V_u$ identical on the base V_u that preserves Z_0 and sends Z' to a constant section, say, Z_∞ disjoint with Z_0 . This yields a trivialization

$$F: V_u \times \mathbb{A}^1 \xrightarrow{\cong|_{V_u}} (V_u \times l) \setminus Z'$$

that extends f and sends the zero section of $V_u \times \mathbb{A}^1 \rightarrow V_u$ to Z_0 .

Consider the morphism

$$\varphi: V_u \times \mathbb{A}^1 \rightarrow X, \quad (u', t) \mapsto \varphi_{u'}(t) := \tau_{u'}(F(t)).$$

Let

$$s = \varphi \circ (\tau_x \times \text{id}_{\mathbb{A}^1}): V_y \times \mathbb{A}^1 \rightarrow X, \quad (y', t) \mapsto \varphi_{u'}(t) \text{ where } u' = \tau_x(y') \in V_u.$$

Consider also the trivial line bundle $p: E = V_y \times \mathbb{A}^1 \rightarrow V_y$, where p is the first projection, with zero section $Z = V_y \times \{0\}$. We have $s(y', 0) = y'$, that is $s|_Z = p|_Z$. Thus, the triplet (E, p, s) is a spray of rank 1 on V_y with values in X . The s -orbit O_y of y coincides with $\varphi_u(\mathbb{A}^1) = C^*$ and the map $ds|_{T_0 E_y}: T_0 E_y \rightarrow T_y O_y$ is onto. Due to Proposition 2.1 (E, p, s) can be extended to a spray on X . \square

Remark 3.8. We claim that the general s -orbits $O_{y'}$ in X are smooth affine conics whose closures are fibers of the conic bundle $\hat{X} \rightarrow \mathbb{P}^2$ resulting from the blowup of $\hat{X} \rightarrow X$ with center l . Indeed, let $y' \in V_y$ be a general point. Then $\langle x, y' \rangle$ being a general projective line in \mathbb{P}^4 through y' , X cuts out the plane $L' := \langle l, y' \rangle$ along the line l passing through x and the residual smooth conic $C_{y'}$ passing through u' and y' . So, the plane L' is $\tau_{u'}$ -invariant and $\tau_{u'}$ interchanges l and $C_{y'}$ fixing their intersection points that are points of $l \cap S_{u'}^*$, see Corollary 3.6(c). Furthermore, $C_{y'}$ is the closure of the s -orbit $O_{y'} = \tau_{u'}(l \setminus \{z(u')\})$.

We are ready now to prove Theorem 1.1.

Proof of Theorem 1.1. Since every complete smooth rational surface is elliptic, see [KZ23b, Theorem 1.1], we may suppose that $n \geq 3$. Due to Corollary 2.2 it suffices to show that X is locally elliptic. Fixing an arbitrary $y \in X$, choose a general point $x \in X$ and n projective lines l_1, \dots, l_n on X through x such that their tangent lines $T_x l_i$ span $T_x X$, see Lemma 3.2. Applying Proposition 3.7 to each of the lines l_i yields rank 1 sprays (E_i, p_i, s_i) , $i = 1, \dots, n$, on X dominating at y along their respective orbits $O_{i,y}$. We claim that the collection of sprays $\{(E_i, p_i, s_i)\}_{i=1, \dots, n}$ is dominating at y . The latter implies that X is subelliptic, and so elliptic, see Corollary 2.2 (cf. also Lemma 2.4). The above domination is equivalent to the fact that the tangent lines $T_y O_{i,y}$ span the tangent space $T_y X$. By the construction of Proposition 3.7 we have

$$T_y O_{i,y} = d\varphi_{i,u}(T_x l_i) \text{ where } \varphi_i = \tau_u \circ f_i \text{ with } f_i: (\mathbb{A}^1, 0) \xrightarrow{\cong} (l_i \setminus \{z_i\}, x).$$

Therefore,

$$\text{span}(T_y O_{1,y}, \dots, T_y O_{n,y}) = d\tau_u(\text{span}(T_x l_1, \dots, T_x l_n)) = d\tau_u(T_x X) = T_y X,$$

which proves our claim. The domination at y spreads to a neighborhood of y in X , which gives the local ellipticity (and the subellipticity), hence also the ellipticity. \square

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