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## ON EXTENDED EULERIAN NUMBERS

ABDELMEJID BAYAD, MOHAND OUAMAR HERNANE, ALAIN TOGBÉ

**Abstract:** In this paper, we will study some properties of the extended Eulerian numbers  $H(n, \lambda)$ , with a parameter  $\lambda$ . In fact, for any integer  $n$ , we investigate the asymptotic behavior, find lower and upper bounds for  $H(n, \lambda)$ . As application, for a champion number  $N$ , we obtain asymptotic formulas, lower and upper bounds of the arithmetic functions  $\omega(N)$  and  $\Omega(N)$ .

**Keywords:** Kalmár's function, extended Eulerian numbers, Champion numbers, asymptotic formula, Ikehara-Wiener theorem.

### 1. Introduction and preliminaries

Let  $\lambda$  be a complex number. An extended Eulerian number  $H(n, \lambda)$  is defined by means of its Dirichlet series

$$\frac{\lambda - 1}{\lambda - \zeta(s)} = \sum_{n \geq 1} \frac{H(n, \lambda)}{n^s}, \quad (1.1)$$

where  $\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}$  is the Riemann Zeta function defined for  $\Re(s) > 1$ . A champion number  $N$  for the function  $H$  is a number that satisfies

$$n < N \implies H(n, \lambda) < H(N, \lambda).$$

In this paper, we will study several properties of extended Eulerian numbers  $H(n, \lambda)$ , with a real parameter  $\lambda > 1$ . We will extend and improve some results obtained by Kalmár [17], Hille [14], Erdős [8], Evans [9], Klazar and Luca [18], Deleglise, Hernane, and Nicolas [13], concerning the maximal order of extended Eulerian numbers. As an application of our results, we investigate the  $H$ -champion numbers.

We are motivated by the following important particular cases:

- 1) If  $\lambda = 0$ , then  $H(n, 0) = \mu(n)$  is a Möbius number.

- 2) If  $n = p_1 p_2 \cdots p_r$  is square-free, then  $H(n, \lambda) = H_r(\lambda)$ , where  $H_r(\lambda)$  is an Eulerian number. In addition, if  $\lambda = -1$ , thus

$$H(n, -1) = H_r(-1) = E_r$$

is the so-called Euler number. One can notice that an Euler number  $E_n$  corresponds to  $2^{-n} C_n$  in [21, p.28]. One can also see [3, p.688] and [1, p.354, formula (1.1.5)].

- 3) If  $\lambda = 2$ , then  $H(n, 2) = K(n)$ , where  $K$  is the Kalmár arithmetic function which counts the number of ordered factorizations of a positive integer  $n$  in factors bigger than 1. Various properties of this function were studied by many mathematicians. In fact, Kalmár found the average order of  $K(n)$ , for  $x \rightarrow \infty$

$$\sum_{n \leq x} K(n) = -\frac{x^\rho}{\rho \zeta'(\rho)} \{1 + o(1)\}, \tag{1.2}$$

where  $\rho = 1.72864 \dots$  is the positive real solution to  $\zeta(s) = 2$ . On the other hand, this result was improved by Hwang [15]. Moreover, bounds on the maximal order of  $K(n)$  were studied by Erdős [8]), Chor, Lemke and Mador [5], Coppersmith and Lewenstein [6], and Hille [14]. Recently, Klazar and Luca [18], Deleglise, Hernane, and Nicolas [13] improved the bounds for the maximal order of  $K(n)$ .

It is well-known that the extended Eulerian numbers  $H(n, \lambda)$  satisfy the following properties:

- 1) The recurrence formulas are given by

$$\begin{cases} \lambda H(n, \lambda) = \sum_{d|n} H(d, \lambda), & n \geq 2, \lambda \neq 1, \\ H(1, \lambda) = 1. \end{cases} \tag{1.3}$$

See formula (1.15) in [3].

- 2) The expression  $(\lambda - 1)^{\Omega(n)} H(n, \lambda)$  is a polynomial in  $\lambda$  of degree less than  $\Omega(n)$ , where  $\Omega(n) = \sum_{k=1}^r k_i$ , if  $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$ . In the special case where  $n = p_1 p_2 \cdots p_r$ , we have

$$(\lambda - 1)^{\Omega(n)} H(n, \lambda) = (\lambda - 1)^r H_r(\lambda) = \sum_{t=1}^r A_{r,t} \lambda^{t-1}, \quad r \geq 1$$

with  $A_{r,t} = \sum_{j=0}^t (-1)^j \binom{r+1}{j} (t-j)^r$ . See formulas (5.12), (5.13) in [3].

Writing

$$\frac{\lambda - 1}{\lambda - \zeta(s)} = \frac{\lambda - 1}{\lambda} \frac{1}{1 - \frac{\zeta(s)}{\lambda}} \tag{1.4}$$

and expanding it, we obtain the explicit formula

$$H(n, \lambda) = \frac{\lambda - 1}{\lambda} \sum_{k=1}^{\Omega(n)} \frac{d_k(n)}{\lambda^k}, \quad \text{where } d_k(n) = \sum_{\substack{n_1 n_2 \cdots n_k = n \\ n_1, n_2, \dots, n_k \geq 1}} 1. \tag{1.5}$$

If we rewrite  $\frac{\lambda-1}{\lambda-\zeta(s)} = \frac{1}{1-\left(\frac{\zeta(s)-1}{\lambda-1}\right)}$  and expand it, we obtain another explicit formula proved in [10]

$$H(n, \lambda) = \sum_{k=1}^{\Omega(n)} \frac{d'_k(n)}{(\lambda-1)^k}, \quad \text{where } d'_k(n) = \sum_{\substack{n_1 n_2 \cdots n_k = n \\ n_1, n_2, \dots, n_k \geq 2}} 1. \tag{1.6}$$

For  $\lambda > 1$ , we consider the Mellin inverse integral

$$\frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\lambda-1}{\lambda-\zeta(s)} x^s \frac{ds}{s} = \sum_{n \geq 1} H(n, \lambda) \int_{\sigma-i\infty}^{\sigma+i\infty} \left(\frac{x}{n}\right)^s \frac{ds}{s} \tag{1.7}$$

by Perron’s formula, we get

$$\frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\lambda-1}{\lambda-\zeta(s)} x^s \frac{ds}{s} = \sum_{n \leq x}^* H(n, \lambda). \tag{1.8}$$

Here, the symbol \* on the summation indicates that the last term of the sum must be multiplied by 1/2 when  $x$  is an integer. Now, using Ikehara-Wiener theorem, we obtain the average formula for extended Eulerian numbers

$$\sum_{n \leq x}^* H(n, \lambda) \sim C(\lambda)x^{\rho(\lambda)}, \tag{1.9}$$

where  $\rho(\lambda)$  is the positive real number solution to the equation  $\zeta(s) = \lambda$  and

$$C(\lambda) = \frac{1-\lambda}{\rho(\lambda)\zeta'(\rho(\lambda))}.$$

Formula (1.5) was proved by Evans [9]. Formulas (1.6), (1.8) and (1.9) were also obtained by Grosswald [10].

In this paper, we will study the behavior of the functions  $\lambda \mapsto C(\lambda)$  and  $\lambda \mapsto \rho(\lambda)$ . As an application of this study, we will investigate  $H$ -champions numbers, specially their asymptotic, lower, and upper bounds. The size of the exponents of their prime factors will be estimated.

### 2. Statement of the main results

Let  $k$  be a positive integer,  $\mathcal{N}_k$  a multiplicative system (including 1) associated to the set of primes numbers,  $\mathcal{P}_k = \{p_1, p_2, \dots, p_k\}$ , with  $p_1 = 2 < p_2 = 3 < \dots < p_k$ . Define

$$\zeta_k(s) = \prod_{p \in \mathcal{P}_k} (1 - p^{-s})^{-1}, \tag{2.1}$$

which is equivalent to

$$\zeta_k(s) = \sum_{n \in \mathcal{N}_k} \frac{1}{n^s}. \tag{2.2}$$

Let  $\rho_k(\lambda)$  be the positive real solution to  $\zeta_k(s) = \lambda$  and  $a_k(\lambda) = -\frac{\lambda}{\zeta'_k(\rho_k(\lambda))}$ . For  $k = \infty$ , we get the Riemann zeta function  $\zeta(s)$  and  $\rho(\lambda) = \rho_\infty(\lambda)$ ,  $\zeta(\rho(\lambda)) = \lambda$  and  $a(\lambda) = -\frac{\lambda}{\zeta'(\rho(\lambda))}$ . We state our first main result.

**Theorem 1.** *Let  $\lambda > 1$ . Then*

1. *The function  $\lambda \mapsto \rho(\lambda)$  is strictly increasing and  $\rho(\lambda) > 1$ , for any  $\lambda > 1$ .*
2. *The sequence  $(\rho_k(\lambda))_{k \geq 1}$  is strictly increasing and bounded. Moreover, for  $\Re(s) \geq \sigma > 1$ , the function  $\zeta_k(s)$  uniformly converges to  $\zeta(s)$  as  $k$  tends to  $\infty$ .*
3. *We have*

$$\rho(\lambda) - \rho_k(\lambda) = \frac{\lambda}{(\rho(\lambda) - 1)|\zeta'(\rho(\lambda))|k^{\rho(\lambda)-1}(\log k)^{\rho(\lambda)}} \left\{ 1 + O\left(\frac{\log \log k}{\log k}\right) \right\}. \tag{2.3}$$

4. *The sequence  $(a_k(\lambda))$  is decreasing and converges to  $a(\lambda)$ , as  $k$  tends to  $\infty$ . Moreover, there exists a positive constant  $\gamma(\lambda)$ , depending only on  $\lambda$ , such that for any prime  $p$ , we have the inequality*

$$\left| \frac{a_k(\lambda)}{p^{\rho_k(\lambda)} - 1} - \frac{a(\lambda)}{p^{\rho(\lambda)} - 1} \right| \leq \gamma(\lambda) \frac{\log p}{p^{\rho_2(\lambda)}(k \log k)^{\rho_k(\lambda)-1}}. \tag{2.4}$$

To state our remaining main results, we need some notations and definitions. So let  $\underline{x} = (x_j)_{1 \leq j \leq \omega}$  be a finite or infinite sequence of positive real numbers such that

$$0 < \Omega(\underline{x}) = \sum_{j=1}^{\omega} x_j < \infty.$$

There exists a unique  $C = C(\underline{x}) > 0$  such that

$$\prod_{j=1}^{\omega} \left( 1 + \frac{x_j}{C(\underline{x})} \right) = \lambda. \tag{2.5}$$

Put

$$T(\underline{x}) = \sum_{i=1}^{\omega} \frac{x_i}{x_i + C(\underline{x})}, \tag{2.6}$$

$$A(\underline{x}) = \frac{\lambda - 1}{\lambda \sqrt{\lambda}} \exp(-\Omega(\underline{x})) \prod_{i=1}^{\omega} \frac{(x_i + C(\underline{x}))^{x_i}}{\Gamma(x_i + 1)}, \tag{2.7}$$

and

$$B(\underline{x}) = \sqrt{\frac{2C(\underline{x})}{T(\underline{x})}}. \tag{2.8}$$

If  $n = q_1^{\alpha_1} q_2^{\alpha_2} \dots q_k^{\alpha_k}$ ,  $\underline{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_k)$ ,  $\alpha_i > 0$ ,  $\Omega = \Omega(\underline{\alpha})$ , R. Evans [9] proved the following important result

$$H(n, \lambda) = \sqrt{\pi} A(\underline{\alpha})B(\underline{\alpha}) \{ 1 + O(\Omega^{-F}) \}, \tag{2.9}$$

when  $\Omega(\underline{\alpha})$  tends to  $\infty$ , for any  $F$  such that  $0 < F < \frac{1}{2}$ . Next, we state main important inequalities needed in our study.

**Theorem 2.** For  $\lambda > 1$ , we have

$$\frac{\Omega(\underline{x})}{\lambda - 1} < C(\underline{x}) \leq \frac{\Omega(\underline{x})}{\log \lambda}, \tag{2.10}$$

$$\frac{\lambda - 1}{\lambda} \leq T(\underline{x}) \leq \lambda - 1, \tag{2.11}$$

$$\sqrt{\frac{2}{\lambda - 1} C(\underline{x})} \leq B(\underline{x}) \leq \sqrt{\frac{2\lambda}{\lambda - 1} C(\underline{x})}. \tag{2.12}$$

If  $n = 2^{\alpha_1} 3^{\alpha_2} \dots p_k^{\alpha_k}$ ,  $\alpha_1, \alpha_2, \dots, \alpha_k \geq 1$ , then there exist absolute positive constants  $C_1$  and  $C_2$  such that we have

$$C_1 \frac{\exp(-k)}{\lambda \sqrt{\lambda}} \frac{\exp(F(\underline{\alpha}))}{\sqrt{\alpha_1 \alpha_2 \dots \alpha_k}} \leq H(n, \lambda) \leq C_2 \frac{1}{\lambda} \sqrt{\frac{\lambda - 1}{\log \lambda}} \frac{n^{\rho_k(\lambda)}}{\pi^{k/2}}, \tag{2.13}$$

where

$$F(\underline{x}) = \sum_{j=1}^k x_j \log \left( 1 + \frac{C(\underline{x})}{x_j} \right). \tag{2.14}$$

Put

$$f(\lambda) = C_2 \frac{1}{\lambda} \sqrt{\frac{\lambda - 1}{\log \lambda}}, C_2 > 0 \quad \text{and} \quad g(\lambda) = C_1 \frac{1}{\lambda \sqrt{\lambda}}.$$

**Theorem 3.** Let  $n$  be a positive integer and  $N$  an  $H$ -champion number.

1) For  $n$  large, we have

$$\log H(n, \lambda) \leq U(n, \lambda), \tag{2.15}$$

where

$$U(n, \lambda) = \rho(\lambda) \log n - \gamma_1(\lambda) \frac{(\log n)^{1/\rho(\lambda)}}{\log \log n}$$

and

$$\gamma_1(\lambda) = \min \left( \frac{1}{2}, \frac{\lambda}{(\rho(\lambda) - 1) |\zeta'(\rho(\lambda))|} \right).$$

2) For a large  $H$ -champion number  $N$ , there exist a positive constant  $\gamma_2(\lambda)$  depending only on  $\lambda$  such that

$$L(N, \lambda) \leq \log H(N, \lambda) \leq U(N, \lambda), \tag{2.16}$$

where

$$L(N, \lambda) = \rho(\lambda) \log N - \gamma_2(\lambda) \frac{(\log N)^{1/\rho(\lambda)}}{\log \log N} + \log g(\lambda) - \rho(\lambda) \log 2.$$

3) For a large  $H$ -champion number  $N$ , there exist  $\gamma_3(\lambda), \gamma_4(\lambda) > 0$  such that

$$\gamma_3(\lambda) \frac{(\log N)^{1/\rho(\lambda)}}{\log \log N} \leq \omega(N) \leq \gamma_4(\lambda) \frac{(\log N)^{1/\rho(\lambda)}}{\log \log N}.$$

Here is the statement of our last result.

**Theorem 4.** *Let  $N$  be a  $H$ -champion number and  $\delta(\lambda) = \left(1 + \frac{1}{\rho(\lambda)}\right) / 2$ . Then, for*

$$N = 2^{\alpha_1} 3^{\alpha_2} \dots p_k^{\alpha_k}$$

such that  $N$  tends to  $\infty$ , we have

1.  $\Omega(N) = \sum_{i=1}^k \alpha_i = a(\lambda) \log N \sum_{i=1}^k \frac{1}{p_i^{\rho(\lambda)-1}} + O((\log N)^{\delta(\lambda)})$  and  $\alpha_i = \frac{a(\lambda)}{p_i^{\rho(\lambda)-1}} \log N + O\left(\frac{(\log N)^{\delta(\lambda)}}{\log p_i}\right)$ ;
2.  $C(\underline{\alpha}) = a(\lambda) \log N + O((\log N)^{\delta(\lambda)})$ ;
3.  $T(\underline{\alpha}) = \zeta_{\mathcal{P}}(\rho(\lambda)) + O((\log N)^{\delta(\lambda)-1})$ ;
4.  $B(\underline{\alpha}) = \sqrt{\frac{2a(\lambda)}{\zeta_{\mathcal{P}}(\rho(\lambda))} \log N \{1 + O((\log N)^{\delta(\lambda)-1})\}}$ ;
5.  $H(N, \lambda) = \sqrt{\frac{2a(\lambda)}{\zeta_{\mathcal{P}}(\rho(\lambda))} \pi \log N A(\underline{\alpha}) \{1 + O((\log N)^{\delta(\lambda)-1})\}}$ .

The next sections will be devoted to the proofs of the above theorems.

### 3. Proof of Theorem 1

Let  $k \geq 2$ . There exist unique positive real numbers  $\rho_k(\lambda), \rho(\lambda)$  such that  $\zeta_k(\rho_k(\lambda)) = \zeta(\rho(\lambda)) = \lambda$ . Then, we have

$$\prod_{p \in \mathcal{P}_k} \left(1 - p^{-\rho_k(\lambda)}\right)^{-1} = \prod_{p \in \mathcal{P}} \left(1 - p^{-\rho(\lambda)}\right)^{-1} = \lambda. \tag{3.1}$$

Thus, we obtain

$$\lambda = \prod_{p \in \mathcal{P}_k} \left(1 - p^{-\rho_k(\lambda)}\right)^{-1} = \prod_{p \in \mathcal{P}_k} \left(1 - p^{-\rho_k(\lambda)}\right)^{-1} \prod_{p > p_k} \left(1 - p^{-\rho(\lambda)}\right)^{-1}. \tag{3.2}$$

Taking the logarithm of the above equation, we get

$$\log \zeta_k(\rho_k(\lambda)) - \log \zeta_k(\rho(\lambda)) = - \sum_{p > p_k} \log \left(1 - p^{-\rho(\lambda)}\right). \tag{3.3}$$

For fixed  $\lambda > 1$ , there exists a positive integer  $m(\lambda) \geq 2$  such that for all  $k \geq m(\lambda)$  we have  $\rho_k(\lambda) > 1$ . By Lagrange's mean-value, we obtain

$$\log \zeta_k(\rho_k(\lambda)) - \log \zeta_k(\rho(\lambda)) = (\rho(\lambda) - \rho_k(\lambda)) \sum_{p \in \mathcal{P}_k} \frac{\log p}{p^{\sigma_p(\lambda)} - 1}, \tag{3.4}$$

for some  $\sigma_p(\lambda) \in (\rho_k(\lambda), \rho(\lambda)) \subset (1, 2)$ . We have

$$- \sum_{p > p_k} \log \left(1 - p^{-\rho(\lambda)}\right) = \sum_{p > p_k} \frac{1}{p^{\rho(\lambda)}} + O\left(\sum_{p > p_k} \frac{1}{p^{2\rho(\lambda)}}\right), \tag{3.5}$$

and by standard estimates we get

$$\sum_{p > p_k} \frac{1}{p^{\rho(\lambda)}} = \frac{(\rho(\lambda) - 1)^{-1}}{p_k^{\rho(\lambda)-1} \log p_k} + O\left(\frac{1}{p_k^{\rho(\lambda)-1} (\log p_k)^2}\right), \tag{3.6}$$

since  $p_k = k \log k + k \log \log k + O(k)$ . Therefore, we obtain

$$- \sum_{p > p_k} \log(1 - p^{-\rho(\lambda)}) = \frac{(\rho(\lambda) - 1)^{-1}}{k^{\rho(\lambda)-1} (\log k)^{\rho(\lambda)}} \left\{ 1 + O\left(\frac{\log \log k}{\log k}\right) \right\}. \tag{3.7}$$

On the other hand, we have

$$\frac{\log p}{p^{\sigma_p(\lambda)} - 1} = \frac{\log p}{p^{\sigma_p(\lambda)} - 1} \left\{ 1 + \frac{p^{\sigma_p(\lambda)}}{p^{\sigma_p(\lambda)} - 1} (p^{\rho(\lambda) - \sigma_p(\lambda)} - 1) \right\}. \tag{3.8}$$

Note that

$$1 \leq \frac{p^{\sigma_p(\lambda)}}{p^{\sigma_p(\lambda)} - 1} \leq 2 \tag{3.9}$$

and for  $2 \leq p \leq p_k$ , we have

$$\begin{aligned} p^{\rho(\lambda) - \sigma_p(\lambda)} - 1 &\leq p_k^{\rho(\lambda) - \rho_k(\lambda)} - 1 = \exp((\rho(\lambda) - \rho_k(\lambda)) \log p_k) - 1 \\ &\ll (\rho(\lambda) - \rho_k(\lambda)) \log p_k \\ &\ll \frac{1}{k^{\rho(\lambda)-1} (\log k)^{\rho(\lambda)-1}}. \end{aligned} \tag{3.10}$$

For these inequalities, we used  $\exp(x) \ll x$ , when  $x$  tends to 0 and

$$\rho(\lambda) - \rho_k(\lambda) \ll \frac{1}{k^{\rho(\lambda)-1} (\log k)^{\rho(\lambda)}}, \tag{3.11}$$

since  $p_k = k \log k + k \log \log k + O(k)$ . We remark that

$$\begin{aligned} &\log \zeta_k(\rho(\lambda)) - \log \zeta_k(\rho_k(\lambda)) \\ &= (\rho(\lambda) - \rho_k(\lambda)) \sum_{p \in \mathcal{P}_k} \frac{\log p}{p^{\sigma_p(\lambda)} - 1} \\ &= (\rho(\lambda) - \rho_k(\lambda)) \left( \frac{\log 2}{2^{\sigma_2(\lambda)} - 1} + \frac{\log 3}{3^{\sigma_3(\lambda)} - 1} + \dots + \frac{\log p_k}{p_k^{\sigma_{p_k}(\lambda)} - 1} \right) \\ &> (\rho(\lambda) - \rho_k(\lambda)) \frac{\log 2}{2^{\sigma_2(\lambda)} - 1}, \quad \text{with } \sigma_2(\lambda) > 1. \end{aligned} \tag{3.12}$$



From (3.3), (3.8), (3.10), and (3.12), we obtain

$$\begin{aligned}
 (\rho(\lambda) - \rho_k(\lambda)) \sum_{p \in \mathcal{P}_k} \frac{\log p}{p^{\sigma_p(\lambda)} - 1} &= (\rho(\lambda) - \rho_k(\lambda)) \sum_{p \in \mathcal{P}_k} \frac{\log p}{p^{\rho(\lambda)} - 1} \\
 &\quad \times \left\{ 1 + O\left(k^{1-\rho(\lambda)}(\log k)^{1-\rho(\lambda)}\right) \right\} \\
 &= (\rho(\lambda) - \rho_k(\lambda)) \sum_{p \in \mathcal{P}_k} \frac{\log p}{p^{\rho(\lambda)} - 1} \left\{ 1 + O\left(k^{1-\rho(\lambda)}\right) \right\}.
 \end{aligned} \tag{3.13}$$

Therefore, from (3.3) and (3.7), we get

$$(\rho(\lambda) - \rho_k(\lambda)) \sum_{p \in \mathcal{P}_k} \frac{\log p}{p^{\sigma_p(\lambda)} - 1} = \frac{(\rho(\lambda) - 1)^{-1}}{k^{\rho(\lambda)-1}(\log k)^{\rho(\lambda)}} \left\{ 1 + O\left(\frac{\log \log k}{\log k}\right) \right\}. \tag{3.14}$$

On the other hand, we have

$$\frac{|\zeta'(\rho(\lambda))|}{\zeta(\rho(\lambda))} = \sum_{p \in \mathcal{P}} \frac{\log p}{p^{\rho(\lambda)} - 1} = \sum_{p \in \mathcal{P}_k} \frac{\log p}{p^{\rho(\lambda)} - 1} + \sum_{p > p_k} \frac{\log p}{p^{\rho(\lambda)} - 1}. \tag{3.15}$$

From (3.6), we deduce

$$\frac{|\zeta'(\rho(\lambda))|}{\lambda} = \sum_{p \in \mathcal{P}_k} \frac{\log p}{p^{\rho(\lambda)} - 1} + O\left(k^{1+\varepsilon-\rho(\lambda)}\right), \tag{3.16}$$

for any small  $\varepsilon > 0$  and for any integer  $k$  such that  $k > m(\lambda)$ . Thus, we get

$$\begin{aligned}
 (\rho(\lambda) - \rho_k(\lambda)) \left[ \frac{|\zeta'(\rho(\lambda))|}{\lambda} + O\left(k^{1+\varepsilon-\rho(\lambda)}\right) \right] \\
 = \frac{(\rho(\lambda) - 1)^{-1}}{k^{\rho(\lambda)-1}(\log k)^{\rho(\lambda)}} \left\{ 1 + O\left(\frac{\log \log k}{\log k}\right) \right\},
 \end{aligned} \tag{3.17}$$

which implies that

$$\rho(\lambda) - \rho_k(\lambda) = \frac{\lambda}{(\rho(\lambda) - 1)|\zeta'(\rho(\lambda))|k^{\rho(\lambda)-1}(\log k)^{\rho(\lambda)}} \left\{ 1 + O\left(\frac{\log \log k}{\log k}\right) \right\}, \tag{3.18}$$

$\forall k > m(\lambda)$ .

Now, we will prove relation (2.4). We apply the mean-value theorem to the function  $x \mapsto \frac{1}{p^x-1}$  to obtain

$$\begin{aligned} \frac{a_k(\lambda)}{p^{\rho_k(\lambda)}-1} - \frac{a(\lambda)}{p^{\rho(\lambda)}-1} &= \left( \frac{1}{p^{\rho_k(\lambda)}-1} - \frac{1}{p^{\rho(\lambda)}-1} \right) a_k(\lambda) + \frac{a_k(\lambda) - a(\lambda)}{p^{\rho(\lambda)}-1} \\ &= (\rho(\lambda) - \rho_k(\lambda))a_k(\lambda) \frac{p^{x_0} \log p}{(p^{x_0}-1)^2} + \frac{a_k(\lambda) - a(\lambda)}{p^{\rho(\lambda)}-1}, \end{aligned} \tag{3.19}$$

with  $\rho_k(\lambda) \leq x_0 \leq \rho(\lambda)$ . Moreover, we get

$$\frac{p^{x_0}}{(p^{x_0}-1)^2} \leq \frac{p^{\rho_2(\lambda)}}{(p^{\rho_2(\lambda)}-1)^2} \leq \frac{C(\lambda)^2}{p^{\rho_2(\lambda)}}, \tag{3.20}$$

$$\frac{1}{p^{x_0}-1} \leq \frac{1}{p^{\rho_2(\lambda)}-1} \leq \frac{1}{p^{\rho_2(\lambda)}} \cdot C(\lambda) \leq \frac{3\rho_3(\lambda)C(\lambda) \log p}{2 \log 2} \cdot \frac{1}{p^{\rho_2(\lambda)}}, \tag{3.21}$$

where

$$C(\lambda) = \frac{2^{\rho_2(\lambda)}}{2^{\rho_2(\lambda)}-1}.$$

On the other hand,  $(a_k(\lambda))_k$  is a decreasing sequence such that

$$\frac{2^{\rho_2(\lambda)}-1}{\log 3} \leq a_2(\lambda) \leq \frac{3^{\rho_3(\lambda)}}{2 \log 2} \leq \frac{3^{\rho_3(\lambda)}}{2} \tag{3.22}$$

and

$$\begin{aligned} \left| \frac{a_k(\lambda)}{p^{\rho_k(\lambda)}-1} - \frac{a(\lambda)}{p^{\rho(\lambda)}-1} \right| \\ \leq \frac{3^{\rho_3(\lambda)} \log p}{2p^{\rho_2(\lambda)}} [C(\lambda)^2(\rho(\lambda) - \rho_k(\lambda)) + C(\lambda)(a_k(\lambda) - a(\lambda))]. \end{aligned} \tag{3.23}$$

We have

$$a_k(\lambda) - a(\lambda) = a(\lambda)a_k(\lambda) \left[ \frac{1}{a(\lambda)} - \frac{1}{a_k(\lambda)} \right] \sim a(\lambda)^2 \left( \frac{1}{a(\lambda)} - \frac{1}{a_k(\lambda)} \right), \tag{3.24}$$

and

$$\frac{1}{a(\lambda)} - \frac{1}{a_k(\lambda)} = L'(\rho(\lambda)) - L'_k(\rho(\lambda)) + L'_k(\rho(\lambda)) - L'_k(\rho_k(\lambda)), \tag{3.25}$$

where

$$L'(\rho(\lambda)) - L'_k(\rho(\lambda)) = \sum_{i=k+1}^{\infty} \frac{\log p_i}{p_i^{\rho(\lambda)}-1} \sim \left( \frac{1}{\rho(\lambda)-1} \right) \frac{1}{(k \log k)^{\rho(\lambda)-1}}, \tag{3.26}$$

$$L'_k(\rho(\lambda)) - L'_k(\rho_k(\lambda)) = (\rho(\lambda) - \rho_k(\lambda))L''(\rho(\lambda)). \tag{3.27}$$

Using Theorem 1, 3), we obtain

$$L'_k(\rho(\lambda)) - L'_k(\rho_k(\lambda)) \sim \frac{\lambda L''(\rho(\lambda))}{|\zeta'(\rho(\lambda))|(\rho(\lambda) - 1)k^{\rho(\lambda)-1}(\log k)^{\rho(\lambda)}}. \tag{3.28}$$

Therefore, we have

$$\frac{1}{a(\lambda)} - \frac{1}{a_k(\lambda)} \sim \frac{1}{(\rho(\lambda) - 1)(k \log k)^{\rho(\lambda)-1}} \tag{3.29}$$

and

$$a_k(\lambda) - a(\lambda) \sim \frac{a^2(\lambda)}{\rho(\lambda) - 1} \cdot \frac{1}{(k \log k)^{\rho(\lambda)-1}}. \tag{3.30}$$

From relations (2.3), (3.23), and (3.30), there exists a constant  $\gamma(\lambda)$  depending on  $\lambda$  such that

$$\left| \frac{a_k(\lambda)}{p^{\rho_k(\lambda)} - 1} - \frac{a(\lambda)}{p^{\rho(\lambda)} - 1} \right| \leq \gamma(\lambda) \frac{\log p}{(\rho(\lambda) - 1)(k \log k)^{\rho_k(\lambda)-1}}. \tag{3.31}$$

Therefore, the proof of Theorem 1 is completed.

#### 4. Proof of Theorem 2

We consider the function

$$t \mapsto H(\underline{x}, t) = \sum_{j=1}^{\omega} \log \left( 1 + \frac{x_j}{t} \right). \tag{4.1}$$

We have

$$H(\underline{x}, t) \leq \sum_{j=1}^{\omega} \log \frac{x_j}{t}, \quad \text{for any } x_j > 0. \tag{4.2}$$

The function  $t \mapsto H(\underline{x}, t)$  is decreasing from  $\infty$  to 0, when  $t \in [0, \infty)$ . Thus, there exists a unique  $C(\underline{x}) > 0$  such that

$$H(\underline{x}, C(\underline{x})) = \log \lambda, \quad \lambda > 1.$$

Then, for  $t = \frac{\Omega(\underline{x})}{\lambda - 1}$ , we obtain

$$H(\underline{x}, t) = \log \left( \prod_{j=1}^{\omega} \left( 1 + (\lambda - 1) \frac{x_j}{\Omega(\underline{x})} \right) \right) \geq \log \left( 1 + (\lambda - 1) \sum_{j=1}^{\omega} \frac{x_j}{\Omega(\underline{x})} \right) = \log \lambda. \tag{4.3}$$

This implies that

$$C(\underline{x}) \geq \frac{\Omega(\underline{x})}{\lambda - 1}.$$

For the upper bound, we write

$$\log \lambda = \sum_{j=1}^{\omega} \log \left( 1 + \frac{x_j}{C(\underline{x})} \right) \leq \sum_{j=1}^{\omega} \frac{x_j}{C(\underline{x})} = \frac{\Omega(\underline{x})}{C(\underline{x})}. \tag{4.4}$$

Therefore, we get

$$\frac{\Omega(\underline{x})}{\lambda - 1} \leq C(\underline{x}) \leq \frac{\Omega(\underline{x})}{\log \lambda}. \tag{4.5}$$

Now, we will prove relation (2.11). From equation (2.10), we obtain

$$T(\underline{x}) = \sum_{j=1}^{\omega} \frac{x_j}{x_j + C(\underline{x})} \leq \sum_{j=1}^{\omega} \frac{x_j}{C(\underline{x})} = \frac{\Omega(\underline{x})}{C(\underline{x})} \leq \lambda - 1. \tag{4.6}$$

We set  $\gamma_j = \frac{x_j}{x_j + C(\underline{x})}$ . One can see that  $0 \leq \gamma_j < 1$  and

$$T(\underline{x}) = \sum_{j=1}^{\omega} \gamma_j \geq \prod_{j=1}^{\omega} (1 - \gamma_j) = 1 - \prod_{j=1}^{\omega} \frac{x_j}{x_j + C(\underline{x})} = 1 - \frac{1}{\lambda} = \frac{\lambda - 1}{\lambda}. \tag{4.7}$$

Hence, we have

$$\frac{\lambda - 1}{\lambda} \leq T(\underline{x}) \leq \lambda - 1. \tag{4.8}$$

Relation (2.12) is immediate from (2.11).

To finish the proof of Theorem 2, first we will use formula (2.9). We take  $F = \frac{1}{4}$ . So there exist positive constants  $C_1$  and  $C_2$  such that

$$C_1 \sqrt{\pi} A(\underline{\alpha}) B(\underline{\alpha}) \leq H(n, \lambda) \leq C_2 \sqrt{\pi} A(\underline{\alpha}) B(\underline{\alpha}). \tag{4.9}$$

Second, from the expression of  $A(\underline{\alpha})$ , we have

$$C_1 \sqrt{\pi} \sqrt{\frac{2}{\lambda - 1} C(\underline{\alpha})} \frac{\lambda - 1}{\lambda \sqrt{\lambda}} \exp(F(\underline{\alpha})) \leq H(n, \lambda) \tag{4.10}$$

and

$$H(n, \lambda) \leq C_2 \sqrt{\pi} \sqrt{\frac{2\lambda}{\lambda - 1} C(\underline{\alpha})} \frac{\lambda - 1}{\lambda \sqrt{\lambda}} \exp(F(\underline{\alpha})) \frac{1}{(2\pi)^{k/2} \sqrt{\alpha_1 \alpha_2 \cdots \alpha_k}}. \tag{4.11}$$

By the Stirling's formula  $\Gamma(x + 1) = x^x e^{-x} s(x)$ , where  $\sqrt{2\pi x} \leq s(x) \leq e\sqrt{x}$ ,  $x \geq 1$ , the relation  $F(\underline{\alpha}) \leq \rho_k(\lambda) \log n$  and equations (2.10), (2.11) imply equation (2.13). Notice that the inequality  $F(\underline{\alpha}) \leq \rho_k(\lambda) \log n$  can be obtained by the use of the result given in [13, Lemma 3.5], with  $A = \log n$ ,  $\rho_k = \rho_k(\lambda)$ ,  $a_k = a_k(\lambda)$ . This completes the proof of Theorem 2.

**5. Proof of Theorem 3**

We start by proving inequality (2.15). Put

$$\theta_1(\lambda) = \frac{\lambda}{(\rho(\lambda) - 1)|\zeta'(\rho(\lambda))|}. \tag{5.1}$$

From Theorem 2 and inequality (2.13), we have

$$\log H(n, \lambda) \leq \log \left( \frac{C_2}{\lambda} \sqrt{\frac{\lambda - 1}{\log \lambda}} \right) - \frac{k}{2} \log \pi + \rho_k(\lambda) \log n. \tag{5.2}$$

Write

$$\rho_k(\lambda) \log n = \rho(\lambda) \log n - (\rho(\lambda) - \rho_k(\lambda)) \log n.$$

By Theorem 1, there exists a positive constant  $\theta_1(\lambda)$  such that

$$\rho(\lambda) - \rho_k(\lambda) \geq \frac{\theta_1(\lambda)}{k^{\rho(\lambda)-1}(\log k)^{\rho(\lambda)}}. \tag{5.3}$$

For  $n \geq 16$ , we have  $\log \log n > 1$ . Thus, for  $2 \leq k \leq \frac{(\log n)^{1/\rho(\lambda)}}{\log \log n} < \log n$ , we obtain

$$\rho(\lambda) - \rho_k(\lambda) \geq \frac{\theta_1(\lambda)}{k^{\rho(\lambda)-1}(\log k)^{\rho(\lambda)}} \geq \theta_1(\lambda) \frac{(\log n)^{\frac{1}{\rho(\lambda)}-1}}{\log \log n} \tag{5.4}$$

and for  $k > \frac{(\log n)^{1/\rho(\lambda)}}{\log \log n}$ , we get

$$\frac{k}{2} \log \pi > \frac{1}{2} \frac{(\log n)^{1/\rho(\lambda)}}{\log \log n}. \tag{5.5}$$

Therefore, we deduce that

$$\begin{aligned} (\rho(\lambda) - \rho_k(\lambda)) \log n + \frac{k}{2} \log \pi &\geq \min \left( \frac{1}{2}, \theta_1(\lambda) \right) \frac{(\log n)^{1/\rho(\lambda)}}{\log \log n} \\ &= \gamma_1(\lambda) \frac{(\log n)^{1/\rho(\lambda)}}{\log \log n}. \end{aligned} \tag{5.6}$$

For a large positive integer  $n$ , we deduce inequality (2.15).

Now, we will prove inequalities (2.16). For this, we take a large  $H$ -champion number  $N$ . We apply [13, Lemme 3.5], with  $A = \log n$ ,  $k = \left\lceil \alpha \frac{(\log n)^{1/\rho(\lambda)}}{\log \log n} \right\rceil$ , and  $0 < \alpha < \rho(\lambda) a(\lambda)^{1/\rho(\lambda)}$ . So, for a large  $H$ -champion number  $n = 2^{\alpha_1} \cdot 3^{\alpha_2} \dots p_k^{\alpha_k}$ , we follow the procedure used in [13] (see pages 1694-1697) to obtain formula (2.16).

By Theorem 2 and inequality (2.13), we have

$$\log H(n, \lambda) \geq F(\underline{\alpha}) - k - \frac{1}{2} \sum_{i=1}^k \log \alpha_i + \log g(\lambda).$$

On the other hand, we know that

$$F(\underline{\alpha}) \geq F(\underline{x}^*) - \rho(\lambda) \log 2 - k,$$

where  $\underline{x}^*$  satisfies

$$F(\underline{x}^*) = \rho(\lambda) \log n,$$

see [13, Lemme 3.5]. Therefore, we get

$$\log H(n, \lambda) \geq \rho(\lambda) \log n - (\rho(\lambda) - \rho_k(\lambda)) \log n - 2k - \frac{1}{2} \sum_{i=1}^k \log \alpha_i + \log g(\lambda) - \rho(\lambda) \log 2.$$

Using equation (2.3) of Theorem 1, we obtain

$$(\rho(\lambda) - \rho_k(\lambda)) \log n = O\left(\frac{(\log n)^{\frac{1}{\rho(\lambda)}}}{\log \log n}\right) = O(k)$$

and

$$\frac{1}{2} \sum_{i=1}^k \log \alpha_i = O(k).$$

See [13, Lemme 4.2]. Hence, we deduce that

$$\log H(n, \lambda) \geq \rho(\lambda) \log n - \gamma_2(\lambda) \frac{(\log n)^{\frac{1}{\rho(\lambda)}}}{\log \log n} + \log g(\lambda) - \rho(\lambda) \log 2,$$

for some positive constant  $\gamma_2$ . Therefore, we complete the proof of the second part of Theorem 3.

It remains to prove the third part of Theorem 3. So let  $k = \omega(N)$ . From equation (2.13), we have

$$\log H(N, \lambda) \leq \rho_k(\lambda) \log N - \frac{k}{2} \log \pi + \log f(\lambda) \tag{5.7}$$

and

$$\log H(N, \lambda) \leq \rho(\lambda) \log N - (\rho(\lambda) - \rho_k(\lambda)) \log N - \frac{k \log \pi}{2} + \log f(\lambda). \tag{5.8}$$

Thus, we get

$$(\rho(\lambda) - \rho_k(\lambda)) \log N \leq \rho(\lambda) \log N - \log H(N, \lambda) - k \frac{\log \pi}{2} + \log f(\lambda). \tag{5.9}$$

From Theorem 3, 1), we obtain

$$(\rho(\lambda) - \rho_k(\lambda)) \log N \leq \gamma_1(\lambda) \frac{(\log N)^{1/\rho(\lambda)}}{\log \log N} + \log \left( \frac{f(\lambda)}{g(\lambda)} \right). \tag{5.10}$$

One can see that there exists  $C > 0$  such that

$$k^{\rho(\lambda)-1} (\log k)^{\rho(\lambda)} \geq C (\log N)^{1-1/\rho(\lambda)} \log \log N. \tag{5.11}$$

Note that the function  $g : t \mapsto t^{\rho(\lambda)-1}(\log t)^{\rho(\lambda)}$  tends to infinity when  $t \rightarrow \infty$  and is increasing, for  $t \geq 1$ . Its inverse function satisfies  $g^{-1}(y) \sim \left(\frac{y}{\log(y)^\rho}\right)^{\frac{1}{\rho-1}}$  as  $y \rightarrow \infty$ . Then, we get

$$k \geq g^{-1}\left(C(\lambda)\frac{(\log N)^{1-1/\rho(\lambda)}}{\log \log N}\right) \sim C'(\lambda)\frac{(\log N)^{1/\rho(\lambda)}}{\log \log N}, \quad C'(\lambda) > 0. \quad (5.12)$$

Therefore, we obtain the lower bound for  $k = \omega(N)$

$$\omega(N) = k \geq \gamma_3(\lambda)\frac{(\log N)^{1/\rho(\lambda)}}{\log \log N}. \quad (5.13)$$

Similarly, we obtain the second inequality. This completes the proof of Theorem 3.

### 6. Proof of Theorem 4

Let  $N = 2^{\alpha_1} \cdot 3^{\alpha_2} \cdots p_k^{\alpha_k}$ ,  $\alpha_1, \alpha_2, \dots, \alpha_k \geq 1$ , where  $p_k$  is the  $k^{\text{th}}$  prime number. We have  $\log N = \sum_{i=1}^k \alpha_i \log p_i$ . We consider the maximization problem for the function

$$F(\underline{x}) = \sum_{j=1}^k x_j \log\left(1 + \frac{C(\underline{x})}{x_j}\right), \quad \underline{x} = (x_1, x_2, \dots, x_k) \quad (6.1)$$

on the set

$$\mathcal{D}_N = \{\underline{x} \in \mathbb{R}_+^{*k} : \sum_{j=1}^k x_j \log p_j \leq \log N\}. \quad (6.2)$$

This problem is equivalent to the following one

$$\begin{cases} \sum_{j=1}^k x_j \log p_j = \log N \\ \max F(\underline{x}). \end{cases} \quad (6.3)$$

Referring to [9, Lemma 6] and [13], there exists a unique  $\underline{x}^* = (x_1^*, x_2^*, \dots, x_k^*) \in \mathcal{D}_N$  such that

$$\log N = \sum_{i=1}^k x_i \log p_i, \quad C(\underline{x}^*) = a_k(\lambda) \log N, \quad (6.4)$$

and

$$x_i^* = \frac{a_k(\lambda) \log N}{p_i^{\rho_k(\lambda)} - 1}, \quad F(\underline{x}^*) = \rho_k(\lambda) \log N. \quad (6.5)$$

Moreover, for  $\underline{\beta} = (\beta_1, \beta_2, \dots, \beta_k) \in \mathcal{D}_N$ , we have

$$F(\underline{x}^*) \geq F(\underline{\beta}) + \frac{1}{4 \log N \log p_k} \sum_{i=1}^k (\beta_i - x_i^*)^2 (\log p_i)^2, \quad \forall k \geq 2. \quad (6.6)$$

If  $N$  is a  $H$ -champion number, then we have

$$\log p_k \sim \frac{1}{\rho(\lambda)} \log N \log N, \quad \text{and} \quad \sum_{i=1}^k (\beta_i - x_i^*) (\log p_i) = O(\log N)^{\delta(\lambda)}. \quad (6.7)$$

Now, we can start the proof of Theorem 4. We set

$$\beta_i = \frac{a(\lambda)}{p_i^{\rho(\lambda)} - 1} \quad \text{and} \quad k = \omega(N). \quad (6.8)$$

Then, we have

$$\left| \sum_{i=1}^k (\alpha_i - \beta_i \log N) \right| \leq \sum_{i=1}^k |\alpha_i - \beta_i \log N| \leq \sum_{i=1}^k |\alpha_i - x_i^*| + \sum_{i=1}^k |x_i^* - \beta_i \log N|, \quad (6.9)$$

where

$$\sum_{i=1}^k |\alpha_i - x_i^*| \leq O(\log N)^{\delta(\lambda)} \quad (6.10)$$

and

$$\begin{aligned} \sum_{i=1}^k |x_i^* - \beta_i \log N| &= (\log N) \sum_{i=1}^k \left| \frac{a_k(\lambda)}{p_i^{\rho_k(\lambda)} - 1} - \frac{a(\lambda)}{p_i^{\rho(\lambda)} - 1} \right| \\ &\leq \frac{\gamma(\lambda)(\log N)}{(k \log k)^{\rho(\lambda)-1}} \sum_{i=1}^k \frac{\log p_i}{p_i^{\rho(\lambda)}} = O\left(\frac{\log N}{k^{\rho(\lambda)-1}}\right). \end{aligned} \quad (6.11)$$

By Theorem 3. 2), we have

$$k \geq \gamma_3(\lambda) \frac{(\log N)^{1/\rho(\lambda)}}{\log N \log N}. \quad (6.12)$$

Thus, we get

$$\frac{\log N}{k^{\rho(\lambda)-1}} = O\left((\log N)^{1/\rho(\lambda)} (\log \log N)^{\rho(\lambda)-1}\right) = O\left((\log N)^{\delta(\lambda)}\right), \quad (6.13)$$

where  $\rho(\lambda) > 1$ . One can see that

$$\sum_{i=k+1}^{\infty} \beta_i = \sum_{i=k+1}^{\infty} \frac{a(\lambda)}{p_i^{\rho(\lambda)} - 1} \leq 2a(\lambda) \sum_{i=k+1}^{\infty} \frac{1}{p_i^{\rho(\lambda)}} \leq \frac{2a(\lambda)}{(\rho(\lambda) - 1)k^{\rho(\lambda)-1}}. \quad (6.14)$$

Hence, we have

$$\left| \Omega(N) - \sum_{i=1}^{\infty} \beta_i \log N \right| \leq O(\log N)^{\delta(\lambda)}. \quad (6.15)$$



For  $1 \leq i \leq k$ , we obtain

$$|\alpha_i - x_i^*| \ll \frac{(\log N)^{\delta(\lambda)}}{p_i} \tag{6.16}$$

and

$$|x_i^* - \beta_i \log N| \leq \gamma(N) \frac{\log p_i}{p_i^{\rho(\lambda)}} \frac{\log N}{(k \log k)^{\rho(\lambda)-1}} = \frac{1}{\log p_i} O\left(\frac{\log N}{k^{\rho(\lambda)-1}}\right), \quad \forall i. \tag{6.17}$$

We deduce that

$$|x_i^* - \beta_i \log N| \ll \frac{(\log N)^{1/\delta(\lambda)}}{\log p_i} \tag{6.18}$$

and

$$|x_i^* - \beta_i \log N| \leq O\left(\frac{(\log N)^{\delta(\lambda)}}{\log p_i}\right), \quad \forall 1 \leq i \leq k. \tag{6.19}$$

This shows the first part of Theorem 4.

One can see that

$$|C(\underline{\alpha}) - C(\underline{\beta}) \log N| \leq \sum_{i=1}^k |\alpha_i - \beta_i \log N| + \sum_{i=1}^{\infty} \beta_i \log N = O(\log N)^{\delta(\lambda)}, \tag{6.20}$$

$$\lambda = \zeta(\rho(\lambda)) = \prod_{i=1}^{\infty} (1 - p_i^{-\rho(\lambda)})^{-1} = \prod_{i=1}^{\infty} \frac{p_i^{\rho(\lambda)}}{p_i^{\rho(\lambda)} - 1} = \prod_{i=1}^{\infty} \left(1 + \frac{1}{p_i^{\rho(\lambda)} - 1}\right). \tag{6.21}$$

Therefore, equations

$$C\left(\frac{1}{2^{\rho(\lambda)} - 1}, \dots\right) = 1 \quad \text{and} \quad C\left(\frac{a(\lambda)}{2^{\rho(\lambda)} - 1}, \frac{a(\lambda)}{3^{\rho(\lambda)} - 1}, \dots\right) = a(\lambda) \tag{6.22}$$

imply that

$$C(\underline{\beta}) = a(\lambda). \tag{6.23}$$

Thus, the second part of Theorem 4 is proved.

From Theorem 4 1) and relation (2.6), we obtain

$$T(\underline{\alpha}) = \zeta_{\mathcal{P}}(\rho(\lambda)) + O((\log N)^{\delta(\lambda)-1}). \tag{6.24}$$

Using Theorem 4. 2), 3) and equation (2.8), we complete the proof of Theorem 4.

### 7. Comments on numerical computations

In this section, we do some numerical computations to study the behavior of some constants. Let  $\lambda$  be a complex number,  $k$  a positive integer, and  $\mathcal{P}_k$  the set of

primes numbers defined by  $\mathcal{P}_k = \{p_1, p_2, \dots, p_k\}$ , with  $p_1 = 2 < p_2 = 3 < \dots < p_k$ . We know that

$$\zeta_k(s) = \prod_{p \in \mathcal{P}_k} (1 - p^{-s})^{-1},$$

(see (2.1)),  $\rho_k(\lambda)$  is a positive real solution to  $\zeta_k(s) = \lambda$  and

$$a_k(\lambda) = -\frac{\lambda}{\zeta'_k(\rho_k(\lambda))}.$$

Moreover, we consider

$$p_k(\lambda) = \frac{\lambda}{(\rho_k(\lambda) - 1)\zeta'_k(\rho_k(\lambda))}$$

the truncated part of the coefficient in equation (2.3) and

$$\delta(\lambda) = \frac{1}{2} \left( 1 + \frac{1}{\rho(\lambda)} \right),$$

(see Theorem 4). We wrote a program in Maple that we run under a MacBook Air. We restricted the computations to  $3 \leq \lambda \leq 7$  and  $3 \leq k \leq 20$ , because the result is known for  $\lambda = 2$  and the computations become extremely slow for higher values of  $\lambda$  and  $k$ . Here are some comments on these computations, which confirm the properties that we proved:

1. As we consider  $\rho_k(\lambda) > 1$ , we notice that the first value of  $\rho_k(\lambda)$  is obtained for higher values of  $k$  when  $\lambda$  is higher.
2. The values of  $\rho_k(\lambda)$  increase when  $k$  increases. They are smaller when  $\lambda$  increases. But in all cases, we have  $\rho_k(\lambda) < 2$ .
3. When  $k$  increases, the values of  $a_k(\lambda), p_k(\lambda), \delta_k(\rho_k)$  decrease. We notice that as  $\lambda$  increases, the behavior is different. In all cases, we see that

$$a_k(\lambda) < 0.772, \quad p_k(\lambda) < 125.406, \quad \delta_k(\rho_k) < 1.$$

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**Addresses:** Abdelmejid Bayad: Département de mathématiques, Université d’Evry Val d’Essonne, 23 Bd. De France, 91037 Evry Cedex, France;

Mohand Ouamar Hernane: Université des Sciences et de la Technologie, Faculté de Mathématiques, BP 32, USTHB, 16123 Bab-Ezzouar Alger, Algérie;

Alain Togbé: Mathematics Department, Purdue University North Central, 1401 S, U.S. 421, Westville IN 46391, USA.

**E-mail:** abayad@maths.univ-evry.fr, mhernane@usthb.dz, atogbe@pnw.edu

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