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On the boundary controlled wave equation

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Abstract

The HUM (Hilbert Uniqueness Method) is the method to find the boundary control satisfying the exact controllability of a partial differential equation such as heat equation, wave equation and Navier-Stoke equation. We study discretization of the one dimensional boundary controlled wave equation with finite difference method. Under appropriate assumption, we prove some theorems relating to the controllability of the discrete system as well as the observability inequality with some cure methods.

1 Introduction

The HUM (Hilbert Uniqueness Method) is firstly introduced by J.L.Lions [6] and explained rigorously in [7] and in his book [11] with R.Glowinski and J.He. R. Glowinski, C-H-Li and J.L.Lions [2] gave theoretical and numerical implementation of this method for two dimensional wave equation. Enrique Zuazua [9] make a summary of optimal control results with one dimension wave system. The research about optimal control of wave equation and other equation has became active with some remarkable papers of M.Asch and G.Lebeau [17] in 1998, J.A.Infante and E.zuazua [5] in 1999, L.León and E.Zuazua [3] in 2002, M.Negreanu and E.Zuazua [16] in 2004, P.Loreti and M.Mehrenberger [18] in 2008 and S.Ervedoza and E.Zuazua [14] in 2011. In this paper, we study the boundary controlled wave equation together with fully discrete scheme and finite difference method. We also prove some theorems related to this scheme compared to continuous controlled wave equation. Furthermore, we indicate the efficiency of some cure methods namely Tikhonov regularization theoretically and numerically.

2 Model description and HUM

Similarly in [9], we consider the one dimensional controlled wave equation. For any $(w^0, w^1) \in L^2(0, 1) \times H^{-1}(0, 1)$, find control $v(t) \in L^2(0, T)$ such that

$$\begin{cases} w_{tt} - w_{xx} = 0, & 0 < x < 1, 0 < t < T, \\ w(0, t) = 0, w(1, t) = v(t), & 0 < t < T, \\ w(x, 0) = w^0(x), w_t(x, 0) = w^1(x), & 0 < x < 1, \end{cases} \quad (1)$$

satisfies null-controllability $w(x, T) = w_t(x, T) = 0$.
Wave equation with zero boundary condition.

$$\begin{cases} u_{tt} - u_{xx} = 0 & 0 < x < 1, 0 < t < T \\ u(0, t) = u(1, t) = 0 & 0 < t < T \\ u(x, 0) = u^0(x), u_t(x, 0) = u^1(x) & 0 < x < 1, \end{cases} \quad (2)$$

where $(u^0(x), u^1(x)) \in H_0^1(0, 1) \times L^2(0, 1)$.
The total energy is defined as

$$E(t) = \frac{1}{2} \int_0^1 [|u_x(x, t)|^2 + |u_t(x, t)|^2] dx = E(0). \quad (3)$$

The null-controllability of (1) is equivalent to the observability of (2).

Definition 2.1. *System (2) is said to be observable if there exists a constant $C(T)$ such that*

$$E(0) \leq C(T) \int_0^T |u_x(1, t)|^2 dt. \quad (4)$$

There is a fundamental result indicating that we have the observability as well as null-controllability when $T \geq 2$.

Proposition 2.1 ([14]). *For an arbitrary $T \geq 2$, system (2) is observable. Otherwise, when $T < 2$, we have*

$$\sup_u \left[\frac{E(0)}{\int_0^T |u_x(1, t)|^2 dt} \right] = \infty, \quad (5)$$

where sup takes over all solutions u of (2).

The Hilbert Uniqueness Method is described in detail in [2].

3 Fully discrete scheme and some related results

Given two integers M and N , we define mesh point $\{x_i\}_{i=0}^{N+1}$ and $\{t_j\}_{j=0}^{M+1}$. Set $\Delta x = \frac{1}{N+1}$ and $\Delta t = \frac{T}{M+1}$. We obtain the fully discrete system of (1) and (2) (change the notation of initial data (1) to $e^0(x), e^1(x)$ and replace the initial data of (2) by zero final data).

$$\begin{cases} \frac{u_j^{k+1} - 2u_j^k + u_j^{k-1}}{|\Delta t|^2} - \frac{u_{j+1}^k - 2u_j^k + u_{j-1}^k}{|\Delta x|^2} = 0, & k = \overline{1, M}, j = \overline{1, N}, \\ u_0^k = 0, u_{N+1}^k = 0, & k = \overline{0, M+1}, \\ u_j^0 = e_j^0, \frac{u_j^1 - u_j^0}{\Delta t} = e_j^1, & j = \overline{0, N+1}, \end{cases} \quad (6)$$

where $(e_j^0, e_j^1) \in H_{0,\Delta x}^1(0, 1) \times L_{\Delta x}^2(0, 1)$,
and the backward equation

$$\begin{cases} \frac{w_j^{k+1} - 2w_j^k + w_j^{k-1}}{|\Delta t|^2} - \frac{w_{j+1}^k - 2w_j^k + w_{j-1}^k}{|\Delta x|^2} = 0, & k = \overline{1, M}, j = \overline{1, N}, \\ w_0^k = 0, w_{N+1}^k = -\frac{u_N^k}{\Delta x}, & k = \overline{0, M+1}, \\ w_j^{M+1} = 0, \frac{w_j^{M+1} - w_j^M}{\Delta t} = 0, & j = \overline{0, N+1}. \end{cases} \quad (7)$$

The energy and norm of the control are defined as

$$E_{\Delta x, \Delta t}^n = \frac{\Delta x}{2} \sum_{j=1}^N \left| \frac{u_j^{n+1} - u_j^n}{\Delta t} \right|^2 + \frac{\Delta x}{2} \sum_{j=1}^{N+1} \frac{u_j^{n+1} - u_{j-1}^{n+1}}{|\Delta x|} \frac{u_j^n - u_{j-1}^n}{|\Delta x|}, \quad (8)$$

and

$$\| |u_N / \Delta x \|_{l^2(0, T)}^2 = \sum_{k=0}^{M+1} \left| \frac{u_N^k}{\Delta x} \right|^2 \cdot |\Delta t|. \quad (9)$$

3.1 Discrete nonuniform observability

Proposition 3.1. *The energy defined as (8) is preserved in time, i.e. $E_{\Delta x, \Delta t}^{n+1} = E_{\Delta x, \Delta t}^n \forall n \in \mathbb{N}$.*

The energy in (8) is non-negative, which is crucial for proving nonuniform observability with fully discrete scheme.

Lemma 3.1 ([15]). *The energy (8) satisfies inequality*

$$E_{\Delta x, \Delta t}^n \geq \frac{\Delta x}{2} \left(1 - \left| \frac{\Delta t}{\Delta x} \right|^2 \right) \sum_{j=1}^N \left| \frac{u_j^{n+1} - u_j^n}{\Delta t} \right|^2. \quad (10)$$

Lemma 3.2 ([5]). *For any $\Delta x > 0$ and any eigenvalue λ of discrete laplacian operator and eigenvector w , we derive the formula*

$$(\Delta x) \sum_{j=0}^N \left| \frac{w_{j+1} - w_j}{\Delta x} \right|^2 = \frac{2}{4 - \lambda(\Delta x)^2} \left| \frac{w_N}{\Delta x} \right|^2. \quad (11)$$

For continuous case, we know that we have the observability when $T \geq 2$ (proposition (1)). However, with fully discrete scheme, we obtain the nonuniform observability for all T .

Theorem 3.1. *Given a number $0 < \alpha < 1$, under assumption $\Delta t = \alpha \Delta x$, we derive for any $T > 0$, when $\Delta x \rightarrow 0$*

$$\sup_u \frac{E_{\Delta x, \Delta t}^n}{\| |u_N / \Delta x \|_{l^2(0, T)}^2} \rightarrow \infty, \quad (12)$$

where u is solution of (6).

Proof. We shall construct a solution to the scheme (6) using the separation of variables. More specifically, we look for a solution u_j^k of the form $u_j^k = a^k w_{N,j}^{\Delta x}$ where $w_{N,j}^{\Delta x}$ is defined in

$$\frac{u_j^{k+1} - 2u_j^k + u_j^{k-1}}{(\Delta t)^2} + \lambda_N^{\Delta x} u_j^k = 0. \quad (13)$$

Then, we see that a_k satisfies the following recurrence equation

$$a^{k+1} + (\Delta t^2 \lambda_N^{\Delta x} - 2)a^k + a^{k-1} = 0, \quad (14)$$

whose associated characteristic equation is

$$y^2 + (\Delta t^2 \lambda_N^{\Delta x} - 2)y + 1 = 0. \quad (15)$$

We compute the delta for this quadratic equation

$$\begin{aligned} \Delta &= \left(4\alpha^2 \sin^2 \left(\frac{N\pi\Delta x}{2} \right) - 2 \right)^2 - 4, \\ &= 4^2 \alpha^2 \sin^2 \left(\frac{N\pi\Delta x}{2} \right) \left[\alpha^2 \sin^2 \left(\frac{N\pi\Delta x}{2} \right) - 1 \right] \end{aligned}$$

We can see that the $\Delta < 0$. Hence, we derive two roots as follow.

$$\begin{cases} y_1 = \frac{2 - \Delta t^2 \lambda_N^{\Delta x} + i\sqrt{-\Delta}}{2}, \\ y_2 = \frac{2 - \Delta t^2 \lambda_N^{\Delta x} - i\sqrt{-\Delta}}{2} \end{cases} \quad (16)$$

From that, we can obtain the formula a^k as follow.

$$a^k = e^{(\frac{2 - \Delta t^2 \lambda_N^{\Delta x}}{2})k\Delta t} \left(C_1 \cos \left(\frac{\sqrt{-\Delta}}{2} k\Delta t \right) + C_2 \sin \left(\frac{\sqrt{-\Delta}}{2} k\Delta t \right) \right), \quad (17)$$

where C_1 and C_2 are real numbers (independence on k).

In this proof, we choose $C_1 = 1$ and $C_2 = 0$. We get

$$a^k = e^{(\frac{2 - \Delta t^2 \lambda_N^{\Delta x}}{2})k\Delta t} \cos \left(\frac{\sqrt{-\Delta}}{2} k\Delta t \right). \quad (18)$$

We consider the $u^k = a^k w_N^{\Delta x}$. From this formula, we tend to compute the energy $E_{\Delta x, \Delta t}^n$ and $\|u_N/\Delta x\|^2$. From (10), we have

$$E_{\Delta x, \Delta t}^n \geq \frac{\Delta x}{2} \left(1 - \left| \frac{\Delta t}{\Delta x} \right|^2 \right) \sum_{j=1}^N \left| \frac{a^1 - a^0}{\Delta t} \right|^2 |w_{N,j}|^2 \quad (19)$$

and

$$\|u_N(t)/\Delta x\|^2 = \sum_{k=0}^{M+1} \left| \frac{a^k}{\Delta x} \right|^2 \cdot |\Delta t| \cdot |w_{N,N}^{\Delta x}|^2. \quad (20)$$

Remind that, we have identity

$$-\frac{w_{N,j+1}^{\Delta x} - 2w_{N,j}^{\Delta x} + w_{N,j-1}^{\Delta x}}{(\Delta x)^2} = \lambda_N^{\Delta x} w_{N,j}^{\Delta x}, \quad \forall j = 1, N. \quad (21)$$

Multiplying both side by $w_{N,j}^{\Delta x}$ and summing from $j = \overline{1, N}$, we derive

$$-\sum_{j=1}^N \frac{w_{N,j+1}^{\Delta x} - w_{N,j}^{\Delta x}}{(\Delta x)^2} w_{N,j}^{\Delta x} + \sum_{j=1}^N \frac{w_{N,j}^{\Delta x} - w_{N,j-1}^{\Delta x}}{(\Delta x)^2} w_{N,j}^{\Delta x} = \lambda_N^{\Delta x} \sum_{j=1}^N |w_{N,j}^{\Delta x}|^2. \quad (22)$$

Changing index of the left side, we indicate that

$$\sum_{j=0}^N \left| \frac{w_{N,j+1}^{\Delta x} - w_{N,j}^{\Delta x}}{\Delta x} \right|^2 = \lambda_N^{\Delta x} \sum_{j=1}^N |w_{N,j}^{\Delta x}|^2. \quad (23)$$

From the identity (24) and lemma 3.1, we derive

$$(\Delta x) \sum_{j=1}^N |w_{N,j}^{\Delta x}|^2 = \frac{2}{(\Delta x)^2 \lambda_N^{\Delta x} (4 - \lambda_N^{\Delta x} (\Delta x)^2)} |w_{N,N}^{\Delta x}|^2. \quad (24)$$

From (19) and (24), we have

$$E_{\Delta x, \Delta t}^n \geq \left(1 - \left| \frac{\Delta t}{\Delta x} \right|^2\right) \left| \frac{a^1 - a^0}{\Delta t} \right|^2 \frac{1}{\lambda_N^{\Delta x} (\Delta x)^2 (4 - \lambda_N^{\Delta x} (\Delta x)^2)} |w_{N,N}^{\Delta x}|^2. \quad (25)$$

We imply

$$\frac{E_{\Delta x, \Delta t}^n}{\|u_N(t)/\Delta x\|^2} \geq \left(1 - \left| \frac{\Delta t}{\Delta x} \right|^2\right) \left| \frac{a^1 - a^0}{\Delta t} \right|^2 \frac{1}{|\Delta x|^2 \lambda_N^{\Delta x} (4 - \lambda_N^{\Delta x} (\Delta x)^2) |\Delta t| \sum_{k=0}^{M+1} \left| \frac{a^k}{\Delta x} \right|^2} \quad (26)$$

or

$$\frac{E_{\Delta x, \Delta t}^n}{\|u_N(t)/\Delta x\|^2} \geq \frac{1}{\alpha^2} (1 - \alpha^2) (a^1 - a^0)^2 \frac{1}{|\Delta x|^2 \lambda_N^{\Delta x} (4 - \lambda_N^{\Delta x} (\Delta x)^2) |\Delta t| \sum_{k=0}^{M+1} (a^k)^2} \quad (27)$$

From here, we need to indicate that

$$\lim_{\Delta t \rightarrow 0} |\Delta t| \sum_{k=0}^{M+1} (a^k)^2 < +\infty. \quad (28)$$

Indeed, for any $k \geq 0$ and $k \leq M + 1$, we have

$$|a^k| = \left| e^{\left(\frac{2-\Delta t^2 \lambda_N^{\Delta x}}{2}\right) k \Delta t} \cos\left(\frac{\sqrt{-\Delta}}{2} k \Delta t\right) \right| \leq e, \quad (29)$$

which implies

$$|\Delta t| \sum_{k=0}^{M+1} (a^k)^2 \leq \frac{M+2}{M+1} e. \quad (30)$$

Hence, left side of (27) tends to $+\infty$ when N tends to $+\infty$ ($|\Delta x|^2 \lambda_N^{\Delta x} \rightarrow 4^-$). \square

Remark 3.1. *We might not prove the nonuniform observability when $\Delta t = \Delta x$. In practice, when $\Delta t = \Delta x$, the algorithm is stable.*

3.2 Property of operator Λ in discrete Hilbert Uniqueness Method

With fully discrete scheme, we consider the corresponding HUM. We have fully discrete operator $\Lambda_{\Delta t, \Delta x} : H_{0, \Delta x}^1(0, 1) \times L_{\Delta x}^2(0, 1) \rightarrow L_{\Delta x}^2(0, 1) \times H_{\Delta x}^{-1}(0, 1)$ as follow.

$$\Lambda_{\Delta t, \Delta x}(e_j^0, e_j^1) = \left(\frac{w_j^1 - w_j^0}{\Delta t}, -w_j^0 \right) \quad (31)$$

Lemma 3.3. $\Lambda_{\Delta t, \Delta x}$ defined in (31) is positive semidefinite and symmetric.

Proof. For two data (e_j^0, e_j^1) and $(\bar{e}_j^0, \bar{e}_j^1)$ corresponding to solution (u, w) and (\bar{u}, \bar{w}) .

$$\langle \Lambda_{\Delta t, \Delta x}(e^0, e^1), (\bar{e}^0, \bar{e}^1) \rangle = \sum_{j=0}^{N+1} \frac{w_j^1 - w_j^0}{\Delta t} \bar{u}_j^0 - w_j^0 \frac{\bar{u}_j^1 - \bar{u}_j^0}{\Delta t}. \quad (32)$$

We introduce the following identity

$$\begin{aligned} \frac{w^{i+1} - w^i}{\Delta t} u^{i+1} - w^{i+1} \frac{u^{i+1} - u^i}{\Delta t} &= \frac{w^{j+1} - w^j}{\Delta t} u^j - w^j \frac{u^{j+1} - u^j}{\Delta t}, \\ &+ \Delta t \sum_{k=i+1}^j \left(\frac{u^{k+1} + u^{k-1} - 2u^k}{|\Delta t|^2} w^k - \frac{w^{k+1} + w^{k-1} - 2w^k}{|\Delta t|^2} u^k \right). \end{aligned}$$

We have

$$\langle \Lambda_{\Delta t, \Delta x}(e^0, e^1), (\bar{e}^0, \bar{e}^1) \rangle = \sum_{j=0}^{N+1} \frac{w_j^1 - w_j^0}{\Delta t} \bar{u}_j^0 - w_j^0 \frac{\bar{u}_j^1 - \bar{u}_j^0}{\Delta t}. \quad (33)$$

Applying the identity, we derive

$$\begin{aligned} \langle \Lambda_{\Delta t, \Delta x}(e^0, e^1), (\bar{e}^0, \bar{e}^1) \rangle &= \Delta t \sum_{j=1}^N \sum_{k=1}^M \left(\frac{\bar{u}_j^{k+1} + \bar{u}_j^{k-1} - 2\bar{u}_j^k}{|\Delta t|^2} w_j^k - \frac{w_j^{k+1} + w_j^{k-1} - 2w_j^k}{|\Delta t|^2} \bar{u}_j^k \right), \\ &= \Delta t \sum_{k=1}^M \left(\sum_{j=1}^M \frac{\bar{u}_{j+1}^k - \bar{u}_j^k}{|\Delta x|^2} w_j^k - \frac{w_{j+1}^k - w_j^k}{|\Delta x|^2} \bar{u}_j^k \right) \\ &\quad - \left(\sum_{j=0}^{M-1} \frac{\bar{u}_{j+1}^k - \bar{u}_j^k}{|\Delta x|^2} w_{j+1}^k - \frac{w_{j+1}^k - w_j^k}{|\Delta x|^2} \bar{u}_{j+1}^k \right), \\ &= \Delta t \sum_{k=1}^M \frac{\bar{u}_{N+1}^k - \bar{u}_N^k}{|\Delta x|^2} w_N^k - \frac{w_{N+1}^k - w_N^k}{|\Delta x|^2} \bar{u}_N^k - \frac{\bar{u}_1^k - \bar{u}_0^k}{|\Delta x|^2} w_1^k + \frac{w_1^k - w_0^k}{|\Delta x|^2} \bar{u}_1^k \end{aligned}$$

Using the changing index technique, we obtain

$$\langle \Lambda_{\Delta t, \Delta x}(e^0, e^1), (\bar{e}^0, \bar{e}^1) \rangle = \frac{\Delta t}{\Delta x} \sum_{k=1}^M w_{N+1}^k \cdot \bar{w}_{N+1}^k \quad (34)$$

From that, we imply the positive semidefinite

$$\langle \Lambda_{\Delta t, \Delta x}(e^0, e^1), (e^0, e^1) \rangle = \frac{\Delta t}{\Delta x} \sum_{k=1}^M (w_{N+1}^k)^2. \quad (35)$$

□

Remark 3.2. *With this property of discrete $\Lambda_{\Delta t, \Delta x}$, the conjugate gradient method might be applied to compute the discrete control (see [2]).*

3.3 Observability inequality with Tikhonov regularization

There are some possibilities to eliminate the oscillations of discrete control. We mention here two methods which is discribed in [2].

Theorem 3.2. *For $T > 4 + 2\Delta t$, there exists a $C(T, \Delta t) > 0$ such that*

$$\begin{aligned} E_{\Delta x, \Delta t}^0 \leq & C(T, \Delta t) \left((\Delta t) \sum_{k=1}^M \left(\frac{u_N^k}{\Delta x} \right)^2 \right. \\ & + (\Delta x)^3 (\Delta t) \sum_{k=0}^M \sum_{j=0}^N \left(\frac{u_{j+1}^{k+1} - u_{j+1}^k - u_j^{k+1} + u_j^k}{(\Delta x)(\Delta t)} \right)^2 + \frac{(\Delta x)(\Delta t)^3}{4} \sum_{k=0}^M \sum_{j=0}^N \left(\frac{u_{j+1}^{k+1} - u_j^{k+1} - u_{j+1}^k + u_j^k}{(\Delta x)(\Delta t)} \right. \\ & \left. \left. + (\Delta x)(\Delta t)^2 \sum_{k=0}^M \sum_{j=0}^N \left(\frac{u_{j+1}^{k+1} - u_{j+1}^k - u_j^{k+1} + u_j^k}{(\Delta x)(\Delta t)} \right)^2 \right) \right). \quad (36) \end{aligned}$$

Furthermore, $\lim_{\Delta t \rightarrow 0} C(T, \Delta t) = c(T)$.

Proof. We have the following identity

$$\begin{aligned} |\Delta t| \sum_{k=1}^M \sum_{j=1}^N j \left(\frac{u_j^{k+1} - 2u_j^k + u_j^{k-1}}{|\Delta t|^2} \right) \left(\frac{u_{j+1}^k - u_{j-1}^k}{2} \right) \\ = |\Delta t| \sum_{k=1}^M \sum_{j=1}^N j \left(\frac{u_{j+1}^k - 2u_j^k + u_{j-1}^k}{|\Delta x|^2} \right) \left(\frac{u_{j+1}^k - u_{j-1}^k}{2} \right) \quad (37) \end{aligned}$$

We have

$$\begin{aligned}
& \sum_{k=1}^M \sum_{j=1}^N j \left(\frac{u_j^{k+1} - 2u_j^k + u_j^{k-1}}{|\Delta t|^2} \right) \left(\frac{u_{j+1}^k - u_{j-1}^k}{2} \right) \\
&= \sum_{k=1}^M \sum_{j=1}^N j \left(\frac{u_j^{k+1} - u_j^k}{|\Delta t|^2} - \frac{u_j^k - u_j^{k-1}}{|\Delta t|^2} \right) \left(\frac{u_{j+1}^k - u_{j-1}^k}{2} \right), \\
&= \sum_{j=1}^N j \left(\sum_{k=1}^M \left(\frac{u_j^{k+1} - u_j^k}{|\Delta t|^2} \right) \left(\frac{u_{j+1}^k - u_{j-1}^k}{2} \right) - \sum_{k=0}^{M-1} \left(\frac{u_j^{k+1} - u_j^k}{|\Delta t|^2} \right) \left(\frac{u_{j+1}^{k+1} - u_{j-1}^{k+1}}{2} \right) \right), \\
&= \sum_{j=1}^N j \left(\sum_{k=1}^M \left(\frac{u_j^{k+1} - u_j^k}{|\Delta t|^2} \right) \left(\frac{u_{j+1}^k - u_{j-1}^k}{2} \right) - \sum_{k=0}^{M-1} \left(\frac{u_j^{k+1} - u_j^k}{|\Delta t|^2} \right) \left(\frac{u_{j+1}^{k+1} - u_{j-1}^{k+1}}{2} \right) \right), \\
&= \sum_{j=1}^N j \left(\sum_{k=0}^M \left(\frac{u_j^{k+1} - u_j^k}{|\Delta t|^2} \right) \left(\frac{u_{j+1}^k - u_{j-1}^k}{2} - \frac{u_{j+1}^{k+1} - u_{j-1}^{k+1}}{2} \right) \right) + X_{\Delta x}, \\
&= \sum_{k=0}^M \sum_{j=1}^N \left(\frac{u_j^{k+1} - u_j^k}{|\Delta t|^2} \right) \left(\frac{u_{j+1}^k - u_{j-1}^k}{2} \right) + X_{\Delta x},
\end{aligned}$$

where $X_{\Delta x} = \sum_{j=1}^N j \left(\left(\frac{u_j^{M+1} - u_j^M}{|\Delta t|} \right) \left(\frac{u_{j+1}^{M+1} - u_{j-1}^{M+1}}{2|\Delta t|} \right) - \left(\frac{u_j^1 - u_j^0}{|\Delta t|} \right) \left(\frac{u_{j+1}^0 - u_{j-1}^0}{2|\Delta t|} \right) \right)$.

Moreover, we derive

$$\begin{aligned}
& \sum_{k=1}^M \sum_{j=1}^N j \left(\frac{u_{j+1}^k - 2u_j^k + u_{j-1}^k}{|\Delta x|^2} \right) \left(\frac{u_{j+1}^k - u_{j-1}^k}{2} \right) \\
&= \sum_{k=1}^M \sum_{j=1}^N j \left(\frac{(u_{j+1}^k)^2 - 2u_j^k u_{j+1}^k + 2u_j^k u_{j-1}^k - (u_{j-1}^k)^2}{|\Delta x|^2} \right), \\
&= \sum_{k=1}^M \sum_{j=1}^N j \left(\frac{(u_{j+1}^k - u_j^k)^2 - (u_j^k - u_{j-1}^k)^2}{2|\Delta x|^2} \right), \\
&= -\frac{1}{2} \sum_{k=1}^M \sum_{j=0}^N \left| \frac{u_{j+1}^k - u_j^k}{\Delta x} \right|^2 + \frac{1}{2|\Delta x|} \sum_{k=1}^M \left(\frac{u_N^k}{\Delta x} \right)^2.
\end{aligned}$$

Hence, we derive the following identity

$$\begin{aligned}
& |\Delta x| |\Delta t| \sum_{k=0}^M \sum_{j=1}^N \left(\frac{u_j^{k+1} - u_j^k}{|\Delta t|^2} \right) \left(\frac{u_{j+1}^k - u_{j-1}^k}{2} \right) + \frac{|\Delta x| |\Delta t|}{2} \sum_{k=1}^M \sum_{j=0}^N \left| \frac{u_{j+1}^k - u_j^k}{\Delta x} \right|^2 + |\Delta x| |\Delta t| X_{\Delta x}, \\
&= \frac{|\Delta t|}{2} \sum_{k=1}^M \left| \frac{u_N^k}{\Delta x} \right|^2.
\end{aligned}$$

We imply

$$\begin{aligned}
TE_{\Delta x, \Delta t}^0 &+ \frac{|\Delta x||\Delta t|}{2} \sum_{k=0}^M \sum_{j=0}^N (u_{j+1}^{k+1} - u_{j+1}^k - u_j^{k+1} + u_j^k) \left(\frac{u_j^{k+1} - u_j^k}{|\Delta t|^2} - \frac{u_{j+1}^k - u_j^k}{|\Delta x|^2} \right) + |\Delta x||\Delta t|X_{\Delta x} \\
&= \frac{|\Delta t|}{2} \sum_{k=1}^M \left| \frac{u_N^k}{\Delta x} \right|^2 + \frac{|\Delta x||\Delta t|}{2} \sum_{j=0}^N \left| \frac{u_{j+1}^0 - u_j^0}{\Delta x} \right|^2.
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
TE_{\Delta x, \Delta t}^0 &+ \frac{|\Delta x||\Delta t|}{2} \sum_{k=0}^M \sum_{j=0}^N (u_{j+1}^{k+1} - u_{j+1}^k - u_j^{k+1} + u_j^k) \left(\frac{u_{j+1}^{k+1} - u_{j+1}^k}{|\Delta t|^2} - \frac{u_{j+1}^k - u_j^k}{|\Delta x|^2} \right) + |\Delta x||\Delta t|X_{\Delta x} \\
&= \frac{|\Delta t|}{2} \sum_{k=1}^M \left| \frac{u_N^k}{\Delta x} \right|^2 + \frac{|\Delta x||\Delta t|}{2} \sum_{j=0}^N \left| \frac{u_{j+1}^0 - u_j^0}{\Delta x} \right|^2 + \frac{|\Delta x||\Delta t|}{2} \sum_{k=0}^M \sum_{j=0}^N \left(\frac{u_{j+1}^{k+1} - u_{j+1}^k - u_j^{k+1} + u_j^k}{|\Delta t|} \right)^2.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\frac{|\Delta x||\Delta t|}{2} \sum_{k=0}^M \sum_{j=0}^N (u_{j+1}^{k+1} - u_{j+1}^k - u_j^{k+1} + u_j^k) \left(\frac{u_{j+1}^{k+1} - u_{j+1}^k}{|\Delta t|^2} - \frac{u_{j+1}^k - u_j^k}{|\Delta x|^2} \right) \\
&= \frac{|\Delta x||\Delta t|}{2} \sum_{k=0}^M \sum_{j=0}^N \left(\frac{u_{j+1}^{k+1} - u_{j+1}^k}{|\Delta t|} \right)^2 - \frac{|\Delta x||\Delta t|}{2} \sum_{k=0}^M \sum_{j=0}^N \left(\frac{u_j^{k+1} - u_j^k}{|\Delta t|} \right) \left(\frac{u_{j+1}^{k+1} - u_{j+1}^k}{|\Delta t|} \right) \\
&\quad - \frac{|\Delta x||\Delta t|}{2} \sum_{k=0}^M \sum_{j=0}^N \left(\frac{u_{j+1}^{k+1} - u_j^{k+1}}{|\Delta x|} \right) \left(\frac{u_{j+1}^k - u_j^k}{|\Delta x|} \right) + \frac{|\Delta x||\Delta t|}{2} \sum_{k=0}^M \sum_{j=0}^N \left(\frac{u_{j+1}^k - u_j^k}{|\Delta x|} \right)^2
\end{aligned}$$

Applying Cauchy-Schwarz inequality, we have

$$\frac{|\Delta x||\Delta t|}{2} \sum_{k=0}^M \sum_{j=0}^N \left(\frac{u_{j+1}^{k+1} - u_{j+1}^k}{|\Delta t|} \right)^2 - \frac{|\Delta x||\Delta t|}{2} \sum_{k=0}^M \sum_{j=0}^N \left(\frac{u_j^{k+1} - u_j^k}{|\Delta t|} \right) \left(\frac{u_{j+1}^{k+1} - u_{j+1}^k}{|\Delta t|} \right) \geq 0,$$

and

$$\begin{aligned}
&- \frac{|\Delta x||\Delta t|}{2} \sum_{k=0}^M \sum_{j=0}^N \left(\frac{u_{j+1}^{k+1} - u_j^{k+1}}{|\Delta x|} \right) \left(\frac{u_{j+1}^k - u_j^k}{|\Delta x|} \right) + \frac{|\Delta x||\Delta t|}{2} \sum_{k=0}^M \sum_{j=0}^N \left(\frac{u_{j+1}^k - u_j^k}{|\Delta x|} \right)^2 \\
&\geq \frac{|\Delta x||\Delta t|}{4} \sum_{j=0}^N \left(\frac{u_{j+1}^0 - u_j^0}{\Delta x} \right)^2 - \frac{|\Delta x||\Delta t|}{4} \sum_{j=0}^N \left(\frac{u_{j+1}^{M+1} - u_j^{M+1}}{\Delta x} \right)^2
\end{aligned}$$

Hence, we get the following inequality

$$\begin{aligned}
TE_{\Delta x, \Delta t}^0 &+ |\Delta x||\Delta t|X_{\Delta x} - \frac{|\Delta x||\Delta t|}{4} \sum_{j=0}^N \left(\frac{u_{j+1}^0 - u_j^0}{\Delta x} \right)^2 + \frac{|\Delta x||\Delta t|}{4} \sum_{j=0}^N \left(\frac{u_{j+1}^{M+1} - u_j^{M+1}}{\Delta x} \right)^2 \\
&\leq \frac{|\Delta t|}{2} \sum_{k=1}^M \left| \frac{u_N^k}{\Delta x} \right|^2 + \frac{(\Delta x)^3(\Delta t)}{2} \sum_{k=0}^M \sum_{j=0}^N \left(\frac{u_{j+1}^{k+1} - u_{j+1}^k - u_j^{k+1} + u_j^k}{(\Delta x)(\Delta t)} \right)^2 \quad (38)
\end{aligned}$$

We have

$$(\Delta x)(\Delta t)|X_{\Delta x}| \leq 4E_{\Delta x, \Delta t}^n + \frac{(\Delta x)(\Delta t)^2}{2} \sum_{k=0}^M \sum_{j=0}^N \left(\frac{u_{j+1}^{k+1} - u_{j+1}^k - u_j^{k+1} + u_j^k}{(\Delta x)(\Delta t)} \right)^2 \quad (39)$$

Furthermore,

$$\begin{aligned} \frac{(\Delta x)(\Delta t)}{4} \left(\frac{u_{j+1}^0 - u_j^0}{\Delta x} \right)^2 &= \frac{(\Delta x)(\Delta t)^3}{4} \left(\frac{u_{j+1}^0 - u_j^0 - u_{j+1}^1 + u_j^1}{(\Delta x)(\Delta t)} \right)^2 \\ &\quad + \frac{(\Delta x)(\Delta t)}{2} \frac{(u_{j+1}^0 - u_j^0)}{\Delta x} \frac{(u_{j+1}^1 - u_j^1)}{\Delta x} - \left(\frac{u_j^1 - u_{j+1}^1}{\Delta x} \right)^2, \\ &\leq \frac{(\Delta x)(\Delta t)^3}{4} \sum_{j=0}^N \left(\frac{u_{j+1}^0 - u_j^0 - u_{j+1}^1 + u_j^1}{\Delta x} \right)^2 + (\Delta t)E_{\Delta t, \Delta x}^0. \end{aligned} \quad (40)$$

Similarly,

$$\frac{(\Delta x)(\Delta t)}{4} \sum_{j=0}^N \left(\frac{u_{j+1}^{M+1} - u_j^{M+1}}{(\Delta x)} \right)^2 \leq \frac{(\Delta x)(\Delta t)^3}{4} \sum_{j=0}^N \left(\frac{u_{j+1}^{M+1} - u_j^{M+1} - u_{j+1}^M + u_j^M}{(\Delta x)(\Delta t)} \right)^2 + (\Delta t)E_{\Delta t, \Delta x}^M. \quad (41)$$

From (40) and (41),

$$\begin{aligned} &\left| -\frac{(\Delta x)(\Delta t)}{4} \sum_{j=0}^N \left(\frac{u_{j+1}^0 - u_j^0}{\Delta x} \right)^2 + \frac{(\Delta x)(\Delta t)}{4} \sum_{j=0}^N \left(\frac{u_{j+1}^{M+1} - u_j^{M+1}}{\Delta x} \right)^2 \right| \\ &\leq \sum_{k=0}^M \sum_{j=0}^N \frac{(\Delta x)(\Delta t)^3}{4} \left(\frac{u_{j+1}^{k+1} - u_j^{k+1} - u_{j+1}^k + u_j^k}{\Delta x} \right)^2 + 2(\Delta t)E_{\Delta t, \Delta x}^0 \end{aligned} \quad (42)$$

From (38), (39) and (42), we have

$$\begin{aligned} (T-4-2\Delta t)E_{\Delta x, \Delta t}^0 &\leq \frac{|\Delta t|}{2} \sum_{k=1}^M \left| \frac{u_N^k}{\Delta x} \right|^2 + \frac{(\Delta x)^3(\Delta t)}{2} \sum_{k=0}^M \sum_{j=0}^N \left(\frac{u_{j+1}^{k+1} - u_{j+1}^k - u_j^{k+1} + u_j^k}{(\Delta x)(\Delta t)} \right)^2 \\ &\quad + \frac{(\Delta x)(\Delta t)^3}{4} \sum_{k=0}^M \sum_{j=0}^N \left(\frac{u_{j+1}^{k+1} - u_j^{k+1} - u_{j+1}^k + u_j^k}{(\Delta x)(\Delta t)} \right)^2 \\ &\quad + \frac{(\Delta x)(\Delta t)^2}{2} \sum_{k=0}^M \sum_{j=0}^N \left(\frac{u_{j+1}^{k+1} - u_{j+1}^k - u_j^{k+1} + u_j^k}{(\Delta x)(\Delta t)} \right)^2 \end{aligned} \quad (43)$$

From (43), we obtain $C(T, \Delta t) = \frac{1}{2(T-4-2\Delta t)}$. When $\Delta t \rightarrow 0$, $C(T, \Delta t) \rightarrow \frac{1}{2(T-4)}$. \square

With bigrid algorithm, we predict that the observability inequality satisfies when $T > 2\sqrt{2}$ (see [18]).

Conjecture 1. For N is an odd number and $T > 2\sqrt{2}$, there exists a constant $C(T)$ such that

$$E_{\Delta x, \Delta t}^n \leq C(T) \|u_N / \Delta x\|_{L^2(0, T)}^2. \quad (44)$$

for all solution u obtained by bigrid algorithm with system (6) and (7).

4 Numerical experiment

We consider two numerical examples.

Numerical experiment 1: We take y^0 is a piecewise constant function and y^1 is a zero function. With the mesh size in space $N = 41, 81, 101, 121$, we have

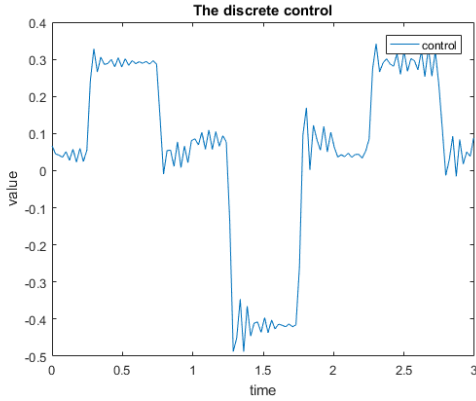


Figure 1: $N = 41$

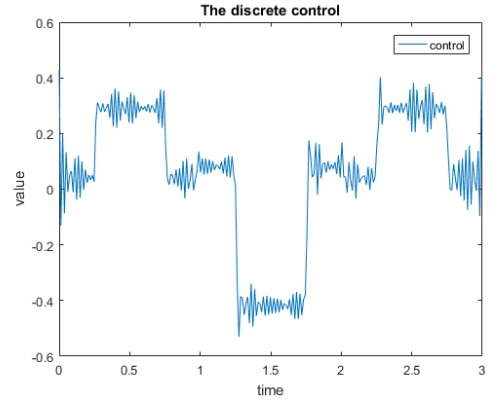


Figure 2: $N = 81$

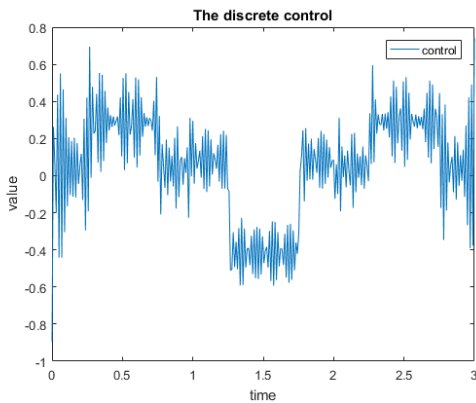


Figure 3: $N = 101$

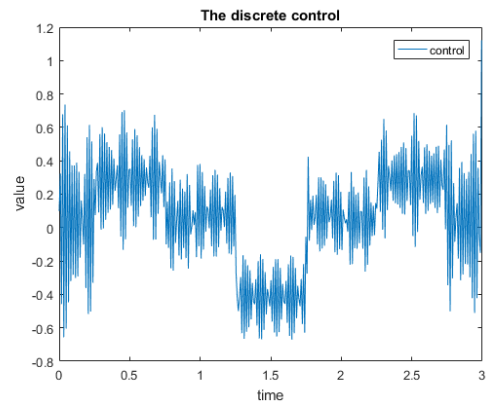


Figure 4: $N = 121$

Remark 4.1. We observe that when N is increased, there are more and more oscillations with discrete control and it seems that they will blow up.

With Tikhonov regularization, the discrete control converges to the exact control.

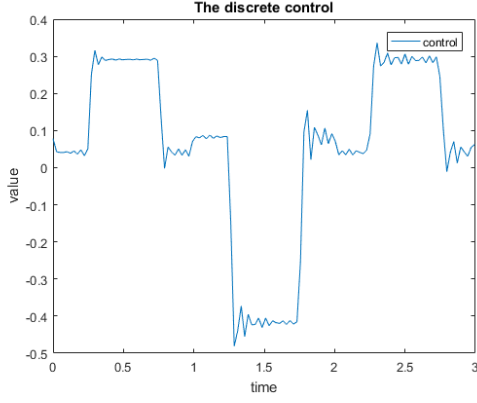


Figure 5: $N = 41$

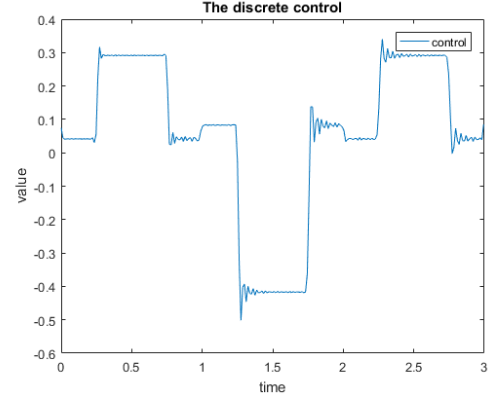


Figure 6: $N = 81$

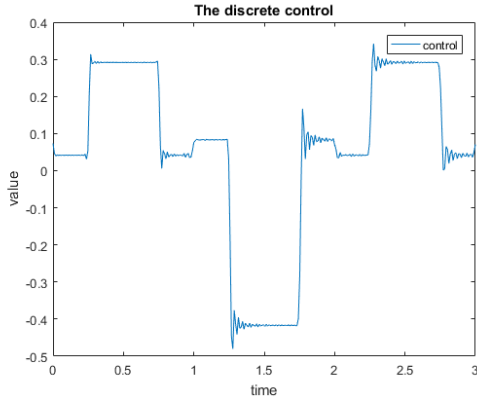


Figure 7: $N = 101$

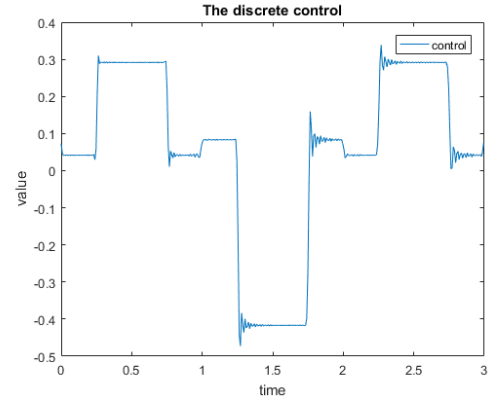


Figure 8: $N = 121$

Remark 4.2. Theoretically, we are able to prove Tikhonov works when $T > 4$ (theorem (3.2)). However, the numerics shows that Tikhonov might be implemented for $T < 4$.

Numerical experiment 2: We consider continuous functions $u^0(x) = \sin(\pi x)$ and $u^1(x) = \pi \sin(\pi x)$.

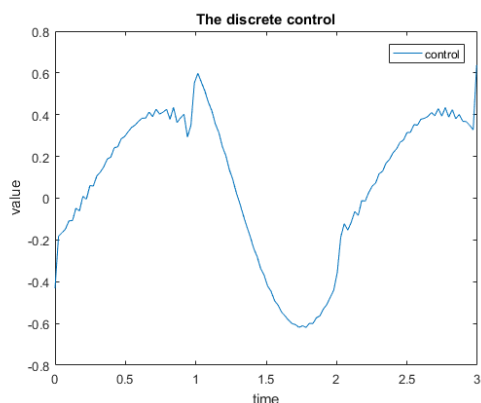


Figure 9: $N = 41$

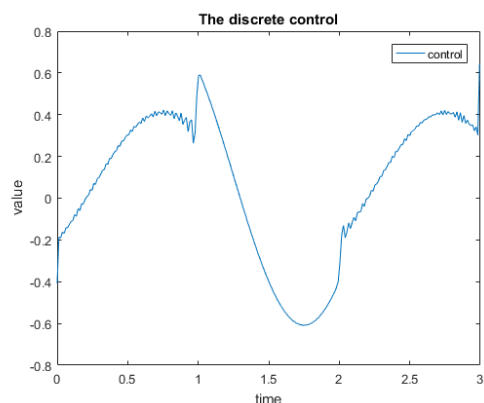


Figure 10: $N = 81$

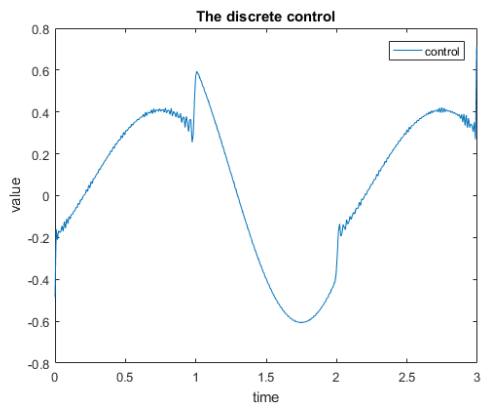


Figure 11: $N = 101$

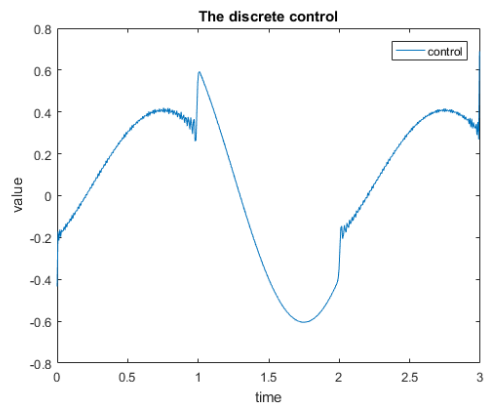


Figure 12: $N = 121$

Remark 4.3. *From this example, we predict that with continuous data $(u^0(x), u^1(x))$ or $(y^0(x), y^1(x))$, there are not many oscillations and they do not blow up.*

5 Conclusion

We use the finite difference scheme to discretize wave equation and compute the discrete control. Theoretically, we prove that the discrete observability inequality does not hold for any value $T > 0$, which is similar to [9]. Furthermore, we also have the symmetry and positive defined properties of discrete operator $\Lambda_{\Delta x, \Delta t}$ which is claimed by R.Glowinski, C-H.Li and J.L.Lions in [2]. With conjugate gradient method, we compute the discrete control with blowup oscillations when we have nonsmooth data. This phenomenon is not occurred with smooth data. Furthermore, we implement the Tikhonov regularization to eliminate the oscillations when $T = 3$. However, theoretical results show that this cure method works efficiently when $T > 4$.

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