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► **To cite this version:**

Abdelmejid Bayad, Takao Komatsu. Zeta functions interpolating the convolution of the Bernoulli polynomials. *Integral Transforms and Special Functions*, 2018, 29 (8), pp.611-622. 10.1080/10652469.2018.1478822 . hal-04463438

**HAL Id: hal-04463438**

**<https://hal.science/hal-04463438v1>**

Submitted on 4 Apr 2024

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# Zeta functions interpolating the convolution of the Bernoulli polynomials

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## Abstract

We prove nonlinear relation on multiple Hurwitz-Riemann zeta functions. Using analytic continuation of these multiple Hurwitz-Riemann zeta function, we quote at negative integers Euler's nonlinear relation for generalized Bernoulli polynomials and numbers.

## 1 Introduction and Preliminaries

Let

$$f(x) = \frac{1}{e^x - 1}$$

and

$$b(x) = xf(x) = \frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} \quad (|x| < 1),$$

where  $B_n$  is the  $n$ -th Bernoulli number. Then we have the formulae:

**Lemma 1** For  $n \geq 0$

$$f^{(n)}(x) = (-1)^n \sum_{k=1}^{n+1} (k-1)! \left\{ \begin{matrix} n+1 \\ k \end{matrix} \right\} f(x)^k \quad (1)$$

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2010 *Mathematics subject classification.* 11B68,11M35,11F67,

*Key words and phrases.* Bernoulli numbers, Bernoulli polynomials, Zeta functions, Special values of zeta functions, Convolution identities.

and

$$f(x)^n = \frac{(-1)^{n-1}}{(n-1)!} \sum_{k=0}^{n-1} \begin{bmatrix} n \\ k+1 \end{bmatrix} f^{(k)}(x), \quad (2)$$

where  $\begin{bmatrix} n \\ k \end{bmatrix}$  denotes the (unsigned) Stirling number of the first kind and  $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$  denotes the Stirling number of the second kind.

*Proof.* The formulae (1) and (2) are equivalent via the orthogonal relation satisfied by Stirling numbers

$$\sum_{k=0}^{\max\{n,m\}} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} \left\{ \begin{smallmatrix} k \\ m \end{smallmatrix} \right\} = (-1)^n \delta_{n,m}. \quad (3)$$

It is easy to see that both formulae are valid for  $n = 1$ . Let  $n \geq 2$ . If the formula (1) is true, then

$$\begin{aligned} & \frac{(-1)^{n-1}}{(n-1)!} \sum_{k=0}^{n-1} \begin{bmatrix} n \\ k+1 \end{bmatrix} f^{(k)}(x) \\ &= \frac{(-1)^{n-1}}{(n-1)!} \sum_{k=0}^{n-1} \begin{bmatrix} n \\ k+1 \end{bmatrix} (-1)^k \sum_{i=1}^{k+1} (i-1)! \left\{ \begin{smallmatrix} n+1 \\ i \end{smallmatrix} \right\} f(x)^i \\ &= \frac{(-1)^{n-1}}{(n-1)!} \sum_{i=1}^n (i-1)! f(x)^i \sum_{k=i-1}^{n-1} (-1)^k \begin{bmatrix} n \\ k+1 \end{bmatrix} \left\{ \begin{smallmatrix} n+1 \\ i \end{smallmatrix} \right\} \\ &= \frac{(-1)^{n-1}}{(n-1)!} \sum_{i=1}^n (i-1)! f(x)^i (-1)^{n-1} \delta_{n,i} \\ &= f(x)^n. \end{aligned}$$

On the other hand, the relation (1) can be proven by induction on  $n$  by using the recurrence formula

$$\left\{ \begin{smallmatrix} n \\ m \end{smallmatrix} \right\} = m \left\{ \begin{smallmatrix} n-1 \\ m \end{smallmatrix} \right\} + \left\{ \begin{smallmatrix} n-1 \\ m-1 \end{smallmatrix} \right\} \quad (n, m \geq 1).$$

with

$$\left\{ \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right\} = 1, \quad \left\{ \begin{smallmatrix} n \\ 0 \end{smallmatrix} \right\} = \left\{ \begin{smallmatrix} 0 \\ n \end{smallmatrix} \right\} = 0 \quad (n \geq 1).$$

■

Applying the Leibniz derivative formula to  $b(x) = xf(x)$ , we obtain

$$b^{(n)}(x) = xf^{(n)}(x) + nf^{(n-1)}(x).$$

Then we have

$$b^{(n)}(x) = (-1)^n \sum_{k=1}^{n+1} (k-1)! \left( \left\{ \begin{smallmatrix} n+1 \\ k \end{smallmatrix} \right\} x - n \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} \right) f(x)^k. \quad (4)$$

Thus, for  $n \geq 0$ , we have

**Proposition 1**

$$b^{(n)}(x) = (-1)^n \sum_{k=1}^{n+1} \frac{(k-1)!}{x^k} \left( \left\{ \begin{matrix} n+1 \\ k \end{matrix} \right\} x - n \left\{ \begin{matrix} n \\ k \end{matrix} \right\} \right) b(x)^k. \quad (5)$$

## 2 Zeta function

Consider the zeta function defined by Mellin integral

$$Z_m(s, x) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} F^{(m)}(t) e^{-xt} dt, \quad (6)$$

where

$$F(t) = -f(-t) = \frac{1}{1 - e^{-t}},$$

which is defined for  $\Re(s) > m + 1$  and  $\Re(x) > 0$ . It is not difficult, by use of the classical Riemann's methods, to prove its analytic continuation to whole complex plane  $\mathbb{C}$  except simple poles at  $s = 1, 2, \dots, m + 1$ . We omit it.

On the other hand, we have the following results.

**Theorem 1** *Let  $m$  be a nonnegative integer. Then we have*

$$(-1)^m \sum_{k=1}^{m+1} (k-1)! \left\{ \begin{matrix} m+1 \\ k \end{matrix} \right\} \zeta_k(s, x) = \sum_{k=0}^m (-1)^k \binom{m}{k} x^{m-k} \zeta(s-k, x),$$

where

$$\zeta(s, x) = \sum_{n=0}^{\infty} \frac{1}{(x+n)^s}$$

denotes the Hurwitz zeta function and

$$\zeta_n(s, x) = \sum_{k_1, \dots, k_n=0}^{\infty} \frac{1}{(x+k_1+\dots+k_n)^s}$$

denotes the multiple Hurwitz zeta function of order  $n$ .

Using the orthogonality property (3), from Theorem 1 we obtain

**Corollary 1** *For any positive integer  $n$ , we have*

$$\begin{aligned} \zeta_n(s, x) &= \frac{(-1)^n}{(n-1)!} \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} B_{n-k-1}^{(n)}(x) \zeta(s-k, x) \\ &= \frac{(-1)^n}{(n-1)!} \sum_{k=0}^{n-1} (-1)^k \sum_{m=0}^{n-k-1} \binom{m+k}{m} \left[ \begin{matrix} n \\ m+k+1 \end{matrix} \right] x^m \zeta(s-k, x), \end{aligned}$$

where  $B_j^{(n)}(x)$  is the  $j$ -th Bernoulli polynomial of order  $n$ .

*Proof of Theorem 1.* By integration by parts, we have

$$Z_m(s, x) = \frac{1}{\Gamma(s)} \int_0^\infty (t^{s-1} e^{-xt})^{(m)} F(t) dt,$$

where  $F(t) = 1/(1 - e^{-t})$ . Using the general Leibniz rule, we obtain

$$Z_m(s, x) = \sum_{k=0}^m \binom{m}{k} (-1)^{m-k} x^{m-k} \zeta(s-k, x). \quad (7)$$

On the other hand, from the relation (1), we have

$$\begin{aligned} Z_m(s, x) &= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} F^{(m)}(t) e^{-xt} dt \\ &= \sum_{k=1}^{m+1} (k-1)! \left\{ \begin{matrix} m+1 \\ k \end{matrix} \right\} \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \frac{e^{-xt}}{(1-e^{-t})^k} dt. \end{aligned}$$

Therefore,

$$Z_m(s, x) = \sum_{k=1}^{m+1} (k-1)! \left\{ \begin{matrix} m+1 \\ k \end{matrix} \right\} \zeta_k(s, x). \quad (8)$$

The result follows from (7) and (8). ■

*Remark.* From the analytic continuations of  $\zeta_n(s; x)$  and  $\zeta(s; x)$  to the whole complex plane, the generalized  $m$ -th Bernoulli polynomial  $B_m^{(n)}(x)$  of order  $n$  and the  $m$ -th Bernoulli polynomial  $B_m(x)$  are related to  $\zeta_n(-m; x)$  and  $\zeta(-m; x)$  by the formulas

$$\zeta_n(-m; x) = (-1)^n \frac{m!}{(m+n)!} B_{m+n}^{(n)}(x), \quad (9)$$

$$\zeta(-m; x) = -\frac{B_{m+1}(x)}{m+1}, \quad (10)$$

for any non-negative integer  $n$ . In special, if  $s$  is a negative integer with  $s = -m$  in Corollary 1, then we get the well-known formula [3, Corollary 1.8]:

$$B_{m+n}^{(n)}(x) = (m+n) \binom{m+n-1}{n-1} \sum_{k=0}^{n-1} (-1)^k \binom{m-1}{k} B_{n-k-1}^{(n)}(x) \frac{B_{m+k+1}(x)}{m+k+1}, \quad (11)$$

which recovers once more Euler's identity and Dilcher's results in [5]; it is also reminiscent of the convolution identities in [10]. For more details, on various generalizations on Euler's identity for Bernoulli numbers, see [1, 2, 4, 5, 7, 8, 9] and references therein.

Next, we study the zeta function associated with Mellin integral given by

$$\widehat{Z}_m(s, x) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} G^{(m)}(t) e^{-xt} dt, \quad (12)$$

where

$$G(t) = b(-t) = \frac{t}{1 - e^{-t}}.$$

We have the following.

**Theorem 2**

$$\begin{aligned} (-1)^m \sum_{k=1}^{m+1} (k-1)! \left( s \begin{Bmatrix} m+1 \\ k \end{Bmatrix} \zeta_k(s+1, x) - m \begin{Bmatrix} m \\ k \end{Bmatrix} \zeta_k(s, x) \right) \\ = \sum_{k=0}^m \binom{m}{k} (-1)^k x^{m-k} (s-k) \zeta(s-k+1, m). \end{aligned}$$

**Corollary 2** *If  $s$  is a negative integer with  $s = -n$ , we obtain*

$$\begin{aligned} (-1)^m \sum_{k=0}^m \binom{m}{k} (-1)^k x^{m-k} B_{n+k}(x) \\ = \sum_{k=1}^{m+1} (-1)^{k-1} \frac{(k-1)!n!}{(n+k)!} \left( \begin{Bmatrix} m+1 \\ k \end{Bmatrix} (n+k) B_{n+k-1}^{(k)}(x) + m \begin{Bmatrix} m \\ k \end{Bmatrix} B_{n+k}^{(k)}(x) \right). \end{aligned}$$

In addition, if  $x = 0$ , we obtain

$$B_{n+m} = \sum_{k=1}^{m+1} (-1)^{k-1} \frac{(k-1)!n!}{(n+k)!} \left( \begin{Bmatrix} m+1 \\ k \end{Bmatrix} (n+k) B_{n+k-1}^{(k)} + m \begin{Bmatrix} m \\ k \end{Bmatrix} B_{n+k}^{(k)} \right). \quad (13)$$

### 3 Zeta function of higher order

Let  $m_1, \dots, m_d$  be nonnegative integers. We study the zeta function  $Z_{m_1, \dots, m_d}(s, x)$  defined by

$$Z_{m_1, \dots, m_d}(s, x) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} G^{(m_1)}(t) \cdots G^{(m_d)}(t) e^{-xt} dt$$

( $\Re(s) > m_1 + \cdots + m_d + d$ ,  $\Re(x) > 0$ ).

**Theorem 3** *The function  $Z_{m_1, \dots, m_d}(s, x)$  can be analytically continued to the whole  $\mathbb{C}$  plane except simple poles at  $s = 1, 2, \dots, m_1 + \cdots + m_d + d$ , and its values at non-positive integers are given as*

$$\begin{aligned} Z_{m_1, \dots, m_d}(-n, x) &= (B_{m_1}(x) + \cdots + B_{m_d}(x))^n \\ &:= \sum_{\substack{k_1 + \cdots + k_d = n \\ k_1, \dots, k_d \geq 0}} \binom{n}{k_1, \dots, k_d} B_{m_1+k_1}(x) \cdots B_{m_d+k_d}(x). \end{aligned} \quad (14)$$

*Proof.* Since

$$\begin{aligned} G^{(m)}(t)e^{-xt} &= (-1)^m b^{(m)}(-t)e^{-xt} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n B_{m+n}(x)}{n!} t^n, \end{aligned}$$

we have

$$\begin{aligned} G^{(m_1)}(t) \cdots G^{(m_d)}(t)e^{-xt} &= \sum_{n=0}^{\infty} \left( \sum_{k_1+\cdots+k_d=n} (-1)^n \frac{B_{m_1+k_1}(x) \cdots B_{m_d+k_d}(x)}{k_1! \cdots k_d!} \right) t^n \\ &= \sum_{n=0}^{\infty} (-1)^n \left( \sum_{k_1+\cdots+k_d=n} \binom{n}{k_1, \dots, k_d} B_{m_1+k_1}(x) \cdots B_{m_d+k_d}(x) \right) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} (-1)^n (B_{m_1}(x) + \cdots + B_{m_d}(x))^n \frac{t^n}{n!}. \end{aligned}$$

Now,

$$\begin{aligned} &\int_0^{\infty} t^{s-1} G^{(m_1)}(t) \cdots G^{(m_d)}(t)e^{-xt} dt \\ &= \sum_{k=0}^{\infty} (-1)^k \frac{(B_{m_1}(x) + \cdots + B_{m_d}(x))^k}{k!} \int_0^1 t^{s+k-1} e^{-xt} dt \\ &\quad + \sum_{k=0}^{\infty} (-1)^k \frac{(B_{m_1}(x) + \cdots + B_{m_d}(x))^k}{k!} \int_1^{\infty} t^{s+k-1} e^{-xt} dt. \end{aligned}$$

The first integral converges if  $\Re(s) > m_1 + \cdots + m_d + d$  and the second integral converges absolutely for any  $s \in \mathbb{C}$  and represents an entire function of  $s$ . Hence, the right-hand side defines holomorphic function for  $\Re(s) > m_1 + \cdots + m_d + d$ . Since

$$\int_0^{\infty} t^{s+k-1} e^{-xt} dt = \frac{\Gamma(s+k)}{x^{s+k}}$$

for  $\Re(x) > 0$ , we obtain

$$\begin{aligned} Z_{m_1, \dots, m_d}(-n, x) &= \lim_{s \rightarrow -n} \left( \frac{1}{\Gamma(s)} \sum_{k=0}^{\infty} (-1)^k \frac{(B_{m_1}(x) + \cdots + B_{m_d}(x))^k}{k!} \frac{\Gamma(s+k)}{x^{s+k}} \right) \\ &= (B_{m_1}(x) + \cdots + B_{m_d}(x))^n \lim_{s \rightarrow -n} \frac{(-1)^n \Gamma(s+n)}{n! \Gamma(s)} \frac{1}{x^{s+n}} \\ &= (B_{m_1}(x) + \cdots + B_{m_d}(x))^n. \end{aligned}$$

■

**Theorem 4** For nonnegative integers  $k_1, \dots, k_d$  and  $m_1, \dots, m_d$ , we have

$$\begin{aligned} \sum_{m_1+\dots+m_d=m} \binom{m}{m_1, \dots, m_d} Z_{m_1+k_1, \dots, m_d+k_d}(s, x) \\ = \sum_{k=0}^m \binom{m}{k} (-1)^k x^{m-k} Z_{k_1, \dots, k_d}(s-k, x). \end{aligned} \quad (15)$$

*Proof.* By the general Leibniz rule, the left-hand side of (15) is equal to

$$\begin{aligned} \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \left( \sum_{m_1+\dots+m_d=m} \binom{m}{m_1, \dots, m_d} G^{(m_1+k_1)}(t) \dots G^{(m_d+k_d)}(t) \right) e^{-xt} dt \\ = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} (G^{(k_1)}(t) \dots G^{(k_d)}(t))^{(m)} e^{-xt} dt. \end{aligned}$$

By integration by parts,

$$\int_0^\infty (t^{s-1} e^{-xt})^{(m)} G^{(k_1)}(t) \dots G^{(k_d)}(t) dt = (-1)^m \int_0^\infty t^{s-1} e^{-xt} (G^{(k_1)}(t) \dots G^{(k_d)}(t))^{(m)} dt.$$

On the other hand, we have

$$\begin{aligned} \int_0^\infty (t^{s-1} e^{-xt})^{(m)} G^{(k_1)}(t) \dots G^{(k_d)}(t) dt \\ = \sum_{k=0}^m \binom{m}{k} (-x)^{m-k} (s-1)_k \int_0^\infty t^{s-k-1} G^{(k_1)}(t) \dots G^{(k_d)}(t) e^{-xt} dt. \end{aligned}$$

Hence, we get

$$\begin{aligned} \sum_{m_1+\dots+m_d=m} \binom{m}{m_1, \dots, m_d} Z_{m_1+k_1, \dots, m_d+k_d}(s, x) \\ = \sum_{k=0}^m \binom{m}{k} (-1)^k x^{m-k} \frac{(s-1)_k}{\Gamma(s)} \int_0^\infty t^{s-k-1} e^{-xt} dt \\ = \sum_{k=0}^m \binom{m}{k} (-1)^k x^{m-k} Z_{k_1, \dots, k_d}(s-k, x), \end{aligned}$$

which is (15). ■

**Corollary 3** We have

$$\sum_{m_1+\dots+m_d=m} \binom{m}{m_1, \dots, m_d} Z_{m_1, \dots, m_d}(s, x) = \sum_{k=0}^m \binom{m}{k} (-1)^k x^{m-k} \zeta_d(s-k, x). \quad (16)$$



Then by putting  $s = -n$ , we obtain the following.

**Corollary 4**

$$\begin{aligned} \sum_{m_1+\dots+m_d=m} \binom{m}{m_1, \dots, m_d} (B_{m_1}(x) + \dots + B_{m_d}(x))^n \\ = \sum_{k=0}^m \binom{m}{k} (-1)^{k+d} x^{m-k} \frac{(n+k)!}{(n+k+d)!} B_{n+m+d}^{(d)}(x). \end{aligned}$$

In special, for  $x = 0$ , we have

$$\sum_{m_1+\dots+m_d=m} \binom{m}{m_1, \dots, m_d} (B_{m_1} + \dots + B_{m_d})^n = \frac{(-1)^{m+d} (n+m)! B_{n+m+d}^{(d)}}{(n+m+d)!}.$$

## 4 Another method and further formulas

In this section, we study the zeta function

$$Z_{m_1, \dots, m_d}(s, x) := \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} G^{(m_1)}(t) \dots G^{(m_d)}(t) e^{-xt} dt$$

by another method. First, we consider the function  $Z_{m_1, m_2}(s, x)$  when  $d = 2$ .

**Lemma 2**

$$Z_{m_1, m_2}(s, x) = (-1)^{m_2} \sum_{0 \leq k_1 \leq k_2 \leq m_2} \binom{m_2}{m_2 - k_2, k, k_2 - k} (-x)^{k_2 - k} Z_{m_1 + m_2 - k_2, 0}(s - k, x). \quad (17)$$

*Proof.* By integration by parts and the general Leibniz rule, we have

$$\begin{aligned} Z_{m_1, m_2}(s, x) &= \frac{(-1)^{m_2}}{\Gamma(s)} \int_0^\infty (t^{s-1} e^{-xt} G^{(m_1)})^{(m_2)} G(t) dt \\ &= (-1)^{m_2} \sum_{k_2=0}^{m_2} \binom{m_2}{k_2} \frac{1}{\Gamma(s)} \int_0^\infty (t^{s-1} e^{-xt})^{(k_2)} G^{(m_1 + m_2 - k_2)}(t) G(t) dt. \end{aligned}$$

Hence,

$$\begin{aligned} Z_{m_1, m_2}(s, x) \\ &= (-1)^{m_2} \sum_{0 \leq k \leq k_2 \leq m_2} \binom{m_2}{k_2} \binom{k_2}{k} (-x)^{k_2 - k} \frac{1}{\Gamma(s - k)} \int_0^\infty t^{s-k-1} e^{-xt} G^{(m_1 + m_2 - k_2)}(t) G(t) dt \\ &= (-1)^{m_2} \sum_{0 \leq k_1 \leq k_2 \leq m_2} \binom{m_2}{m_2 - k_2, k, k_2 - k} (-x)^{k_2 - k} Z_{m_1 + m_2 - k_2, 0}(s - k, x). \end{aligned}$$

■

By Proposition 1 and Lemma 2, we obtain the following theorem.

**Theorem 5** *We have*

$$\begin{aligned}
Z_{m_1, m_2}(s, x) &= \sum_{\substack{0 \leq k \leq k_2 \leq m_2 \\ 0 \leq j \leq m_1 + m_2 - k_2}} (-1)^{j+m_2-1} (j-1)! \binom{m_2}{m_2 - k_2, k_2 - k, k} (-x)^{k_2 - k} \\
&\quad \times \left( \binom{m_1 + m_2 - k_2 + 1, j}{j} (s-k)(s-k+1) \zeta_{j+1}(s-k+2, x) \right. \\
&\quad \left. + (m_1 + m_2 - k_2) \binom{m_1 + m_2 - k_2}{j} (s-k) \zeta_{j+1}(s-k+1, x) \right).
\end{aligned}$$

*Proof.* By Proposition 1, we have

$$\begin{aligned}
G^{(m_1+m_2-k_2)}(t) &= (-1)^{m_1+m_2-k_2} b^{(m_1+m_2-k_2)}(-t) \\
&= \sum_{j=1}^{m_1+m_2-k_2+1} \frac{(j-1)!(-1)^{j-1}}{t^j} \left( \binom{m_1 + m_2 - k_2 + 1}{j} t \right. \\
&\quad \left. + (m_1 + m_2 - k_2) \binom{m_1 + m_2 - k_2}{j} \right) B(-t)^j \\
&= \sum_{j=1}^{m_1+m_2-k_2+1} (j-1)!(-1)^{j-1} \left( \binom{m_1 + m_2 - k_2 + 1}{j} t \right. \\
&\quad \left. + (m_1 + m_2 - k_2) \binom{m_1 + m_2 - k_2}{j} \right) F(t)^j.
\end{aligned}$$

Note that

$$G(t) = \frac{t}{1 - e^{-t}} = tF(t).$$

Therefore,

$$\begin{aligned}
&Z_{m_1+m_2-k_2, 0}(s-k, x) \\
&= \frac{1}{\Gamma(s-k)} \int_0^\infty t^{s-k-1} e^{-xt} G^{(m_1+m_2-k_2)}(t) G(t) dt \\
&= \sum_{j=1}^{m_1+m_2-k_2+1} (j-1)!(-1)^{j-1} \left( \binom{m_1 + m_2 - k_2 + 1}{j} \frac{1}{\Gamma(s-k)} \int_0^\infty t^{s-k+1} e^{-xt} F(t)^{j+1} dt \right. \\
&\quad \left. + (m_1 + m_2 - k_2) \binom{m_1 + m_2 - k_2}{j} \frac{1}{\Gamma(s-k)} \int_0^\infty t^{s-k} e^{-xt} F(t)^{j+1} dt \right) \\
&= \sum_{j=1}^{m_1+m_2-k_2+1} (j-1)!(-1)^{j-1} \left( \binom{m_1 + m_2 - k_2 + 1}{j} (s-k+1)(s-k) \zeta_{j+1}(s-k+2, x) \right. \\
&\quad \left. + (m_1 + m_2 - k_2) \binom{m_1 + m_2 - k_2}{j} (s-k) \zeta_{j+1}(s-k+1, x) \right).
\end{aligned}$$

Together with Lemma 2, we get the desired result. ■

Putting  $s = -n$  in Theorem 5, we obtain the following.

**Corollary 5** *We have*

$$\begin{aligned} & (B_{m_1} + B_{m_2})^n \\ &= \sum_{\substack{0 \leq k \leq k_2 \leq m_2 \\ 0 \leq j \leq m_1 + m_2 - k_2}} (-1)^{m_2} \binom{m_2}{k_2} (j-1)! \left( \left\{ \begin{matrix} m_1 + m_2 - k_2 + 1, j \\ j \end{matrix} \right\} \frac{(n+k_2)! B_{n+k_2+j-1}^{(j+1)}}{(n+k_2+j-1)!} \right. \\ & \quad \left. - (m_1 + m_2 - k_2) \left\{ \begin{matrix} m_1 + m_2 - k_2 \\ j \end{matrix} \right\} \frac{(n+k_2)! B_{n+k_2+j}^{(j+1)}}{(n+k_2+j)!} \right). \end{aligned}$$

We find a very intriguing formula in the above corollary. Similarly, we can do the same for any  $d \geq 3$  and obtain a generalization of the previous Theorem. However, the formulas obtained for  $(B_{m_1} + \cdots + B_{m_d})^n$  are not as pretty as in the case  $d = 2$ . For this reason, we do not wish to state them here.

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