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The Large-Period Limit for Equations of Discrete Turbulence

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Abstract. We consider the damped/driven cubic NLS equation on the 5 torus of a large period L with a small nonlinearity of size λ , a properly 6 scaled random forcing and dissipation. We examine its solutions under 7 the subsequent limit when first $\lambda \to 0$ and then $L \to \infty$. The first limit, 8 called the limit of discrete turbulence, is known to exist, and in this 9 work we study the second limit $L \to \infty$ for solutions to the equations 10 of discrete turbulence. Namely, we decompose the solutions to formal 11 series in amplitude and study the second-order truncation of this series. 12 We prove that the energy spectrum of the truncated solutions becomes 13 close to solutions of a damped/driven nonlinear wave kinetic equation. 14 Kinetic nonlinearity of the latter is similar to that which usually appears 15 in works on wave turbulence, but is different from it (in particular, it is 16 non-autonomous). Apart from tools from analysis and stochastic analysis, 17 our work uses two powerful results from the number theory. 18

19 1. Introduction

20 1.1. The Setting

3

In this paper we continue the study of the Zakharov-L'vov stochastic model for wave turbulence (WT), initiated in [7,8]; see also a survey [9]. We start by recalling the classical and the Zakharov-L'vov stochastic settings of WT. See the introduction to [7] for more detailed discussions of the two models.

²⁵ Classical setting. Let $\mathbb{T}_{L}^{d} = \mathbb{R}^{d}/(L\mathbb{Z}^{d})$ be the *d*-dimensional torus, $d \geq 2$, of ²⁶ period $L \geq 2$. We denote by ||u|| the normalized L_2 -norm of a complex function ²⁷ u on \mathbb{T}_{L}^{d} , $||u||^{2} = L^{-d} \int_{\mathbb{T}_{L}^{d}} |u(x)|^{2} dx$, and write the Fourier series of u in the ²⁸ form

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$$u(x) = L^{-d/2} \sum_{s \in \mathbb{Z}_L^d} v_s e^{2\pi i s \cdot x}, \qquad \mathbb{Z}_L^d = L^{-1} \mathbb{Z}^d.$$
(1.1)

Here the vector of Fourier coefficients $v = (v_s)_{s \in \mathbb{Z}_L^d}$ is given by the Fourier transform of u(x),

$$v = \mathcal{F}(u), \quad v_s = L^{-d/2} \int_{\mathbb{T}_L^d} u(x) e^{-2\pi i s \cdot x} \, \mathrm{d}x \quad \text{for } s \in \mathbb{Z}_L^d,$$

so the Parseval identity takes form $||u||^2 = L^{-d} \sum_{s \in \mathbb{Z}_L^d} |v_s|^2$. We will study solutions u(t, x) whose norms satisfy $||u(t, \cdot)|| \sim 1$ as $L \to \infty$. This makes the chosen in (1.1) scaling of Fourier series convenient for our purposes.

36 We consider the cubic NLS equation with modified nonlinearity

37
$$\frac{\partial}{\partial t}u + i\Delta u - i\lambda \left(|u|^2 - 2||u||^2\right)u = 0, \qquad x \in \mathbb{T}_L^d, \tag{1.2}$$

where u = u(t, x), $\Delta = (2\pi)^{-2} \sum_{j=1}^{d} (\partial^2 / \partial x_j^2)$ and $\lambda \in (0, 1]$ is a small parameter. The modification of the nonlinearity by the term $2i\lambda ||u||^2 u$ keeps the main features of the standard cubic NLS equation, reducing some non-crucial technicalities; see the introduction to [7].

The objective of WT is to study solutions of (1.2) under the limit $L \to \infty$ 42 and $\lambda \to 0$ on long time intervals. There are plenty of physical works containing 43 some different (but consistent) approaches to the limit: many references may be 44 found in [25, 26, 30]. Despite the strong interest in physical and mathematical 45 communities to the addressed questions, significant progress in the rigorous 46 47 16,23]. See, for example, the introductions to [3,6,7] for discussions of the 48 obtained results. 49

Zakharov-L'vov setting. When studying Eq. (1.2), members of the WT community talk about "pumping energy to low modes and dissipating it in high modes". To make this rigorous, following Zakharov-L'vov [29], in the present paper as well as in [7,8] we consider the NLS equation (1.2) damped by a (hyper) viscosity and driven by a random force:

 $\frac{\partial}{\partial t}u + i\Delta u - i\lambda \left(|u|^2 - 2||u||^2\right)u = -\nu\mathfrak{A}(u) + \sqrt{\nu}\frac{\partial}{\partial t}\eta^{\omega}(t,x).$ (1.3)

Here $\nu \in (0, 1/2]$ is another small parameter, which should be properly agreed with λ and L. The dissipative linear operator \mathfrak{A} is defined as

$$\mathfrak{A}(u(x)) = L^{-d/2} \sum_{s \in \mathbb{Z}_L^d} \gamma_s v_s e^{2\pi i s \cdot x}, \quad v = \mathcal{F}(u), \ \gamma_s = \gamma^0(|s|^2), \tag{1.4}$$

where |s| stands for the Euclidean norm of a vector s and $\gamma^0(y)$ is a smooth real increasing function of y > 0, satisfying¹

58

$$\gamma^0 \ge 1$$
 and $c(1+y)^{r_*} \le \gamma^0(y) \le C(1+y)^{r_*} \quad \forall y > 0.$ (1.5)

¹ For example, if $\gamma_s = (1+|s|^2)^{r*}$, then $\mathfrak{A} = (1-\Delta)^{r*}$. In particular, we can take $\mathfrak{A} = 1-\Delta$.

⁶² The exponents $r_* > 0$ and c, C are positive constants. We also assume that

- all derivatives of γ^0 have at most polynomial growths at infinity.
- ⁶⁴ The random noise η^{ω} is given by a Fourier series

$$\eta^{\omega}(t,x) = L^{-d/2} \sum_{s \in \mathbb{Z}_L^d} b(s) \beta_s^{\omega}(t) e^{2\pi i s \cdot x},$$

where $\{\beta_s(t), s \in \mathbb{Z}_L^d\}$ are standard independent complex Wiener processes² and b(s) is a Schwartz function on $\mathbb{R}^d \supset \mathbb{Z}_L^d$.³

Solutions $u(\tau)$ of (1.3) are random processes in the space $H = L_2(\mathbb{T}_L^d, \mathbb{C})$, equipped with the norm $\|\cdot\|$. If r_* is sufficiently big in terms of d, Eq. 1.3 is known to be well posed, see Theorem 1.1 below and a discussion after its formulation. Moreover, Ito's formula shows that $\mathbb{E}\|u(\tau)\|^2$ is bounded uniformly in τ and L, ν, λ , once $\mathbb{E}\|u(0)\|^2$ is bounded uniformly in these parameters, see in [7].

We will study the equation on time intervals of order ν^{-1} . So, it is convenient to pass from t to the slow time $\tau = \nu t$ and write Eq. (1.3) as

$$\begin{split} \dot{u} + i\nu^{-1}\Delta u - i\rho \left(|u|^2 - 2||u||^2 \right) u &= -\mathfrak{A}(u) + \dot{\eta}^{\omega}(\tau, x), \\ \eta^{\omega}(\tau, x) &= L^{-d/2} \sum_{s \in \mathbb{Z}_{\tau}^d} b(s)\beta_s^{\omega}(\tau)e^{2\pi i s \cdot x}. \end{split}$$

Here $\rho = \lambda \nu^{-1}$, the upper dot stands for $d/d\tau$ and $\{\beta_s(\tau), s \in \mathbb{Z}_L^d\}$ is another set of standard independent complex Wiener processes. Below we use ρ , ν and L as parameters of the equation.

In the context of Eq. (1.6), the objective of WT is to study its solutions $u(\tau)$ when

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$$L \to \infty \quad \text{and} \quad \nu \to 0, \tag{1.7}$$

(1.6)

while $\rho = \rho(\nu, L)$ is scaled appropriately, mostly paying attention to their energy spectra

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92

$$N_s(\tau) := \mathbb{E}|v_s(\tau)|^2, \quad \text{where} \quad v(\tau) = \mathcal{F}(u(\tau)). \tag{1.8}$$

Exact meaning of the limit (1.7) is unclear since no relation between the parameters ν and L is postulated by the theory.

Motivated by physical works, in the present paper, as in [7,8], we study formal decompositions in ρ of solutions to Eq. (1.6) and of their energy spectra N_s under the limit (1.7). See the introduction to [7] for a discussion of our motivation, and see below Sect. 4. In [7,8] we understand the limit (1.7) as

first $L \to \infty$ and then $\nu \to 0$, or $L \gg \nu^{-2}$ while $\nu \to 0$. (1.9)

⁹³ There we have shown that principal terms of the decomposition of N_s in ρ ⁹⁴ have a non-trivial limiting behaviour, provided that ρ is scaled as $\rho \sim \nu^{-1/2}$,

² i.e. $\beta_s = \beta_s^1 + i\beta_s^2$, where $\{\beta_s^j, s \in \mathbb{Z}_L^d, j = 1, 2\}$ are standard independent real Wiener processes.

³Often it is assumed that the intensity b(s) of the noise η^{ω} is non-negative, but we do not impose this condition. Note that if $b(s) \equiv 0$, then our results become trivial since below we will provide (1.3) with the zero initial conditions.

governed by a nonlinear wave kinetic equation (WKE) with added dissipation
and a constant forcing. The WKE coincides with that, arising in physical
works, so this result agrees well with the predictions of the WT.

⁹⁸ In the present paper we are interested in the opposite order of limits,⁹⁹ which rarely appears in physical works:

firstly
$$\nu \to 0$$
 and then $L \to \infty$. (1.10)

Roughly speaking, our main result is that under the double limit (1.10) the behaviour of principal terms of the decomposition in ρ for the energy spectrum N_s is governed by a *modified* WKE. The latter is similar to the WKE arising in [7,8] and physical papers, but is different from them. The scaling of ρ now is $\rho \sim L \chi_d(L)$, where

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$$\chi_d(L) \equiv 1$$
 if $d \ge 3$ and $\chi_d(L) = (\ln L)^{-1/2}$ if $d = 2.$ (1.11)

To the best of our knowledge, this WKE did not appear in the literature be-107 fore. For the proof we start with the result obtained in [17, 21], where the 108 limiting as $\nu \to 0$ behaviour of Eq. (1.6) is examined (while L and ρ are kept 109 fixed). Then we pass to the limit as $L \to \infty$, following the approach of [7,8] 110 and using the developed there tools, such as a specific Feynman diagram pre-111 sentation. Another key ingredient of the proof is an obtained in [10] refinement 112 of the Heath–Brown circle method for quadratic forms [14], and certain upper 113 bounds for the number of integer points on intersections of quadrics. In the 114 next subsections we describe our results and methods in more detail. 115

In [20] a similar result concerning the iterated limit (1.10) was found heuristically; however, there ρ was scaled as $\rho \sim \sqrt{L}$. The present paper shows that the correct scaling is different: $\rho \sim L \chi_d(L)$.

Similar regimes, when $L \to \infty$ slowly while $\nu > 0$ fast decays to zero, were studied in [1,16]. However the elegant description of the limit, obtained there, is far from the prediction of WT. The works [1,16] should rather be regarded as a kind of averaging (similar to that of Krylov–Bogolyubov) since the considered there time scale is much shorter than the characteristic time scale of WT. Note that in [1] a similar to ours [10] refinement of the Heath– Brown method also is crucially used.

126 **1.2. The Limit of Discrete Turbulence**

127 We first consider the limit

128
$$\nu \to 0$$
 while L and ρ stay fixed. (1.12)

It is known as the limit of discrete turbulence (see [25, Section 10]) and has been
successfully studied in [17,21]. To explain the result, let us take the Fourier
transform of Eq. (1.6):

132
$$\dot{v}_{s} - i\nu^{-1}|s|^{2}v_{s} + \gamma_{s}v_{s} = i\rho L^{-d} \left(\sum_{1,2,3} \delta_{3s}^{\prime 12} v_{1}v_{2}\bar{v}_{3} - |v_{s}|^{2}v_{s} \right) + b(s)\dot{\beta}_{s}$$
133 (1.13)

for $s \in \mathbb{Z}_L^d$. Here, as it is common in WT, v_j abbreviates v_{s_j} , $\sum_{1,2,3}$ stands for $\sum_{s_1,s_2,s_3 \in \mathbb{Z}_L^d}$, and

137
$$\delta_{3s}^{\prime 12} = \delta_{s_3s}^{\prime s_1 s_2} := \begin{cases} 1, & \text{if } s_1 + s_2 = s_3 + s \text{ and } \{s_1, s_2\} \neq \{s_3, s\}, \\ 0, & \text{otherwise.} \end{cases}$$
(1.14)

138 Note that

if
$$\delta_{3s}^{\prime 12} = 1$$
, then $\{s_1, s_2\} \cap \{s_3, s\} = \emptyset$. (1.15)

140 We pass to the *interaction representation*,

$$\mathfrak{a}_s(\tau) = v_s(\tau)e^{-i\nu^{-1}\tau|s|^2}, \quad s \in \mathbb{Z}_L^d,$$
(1.16)

142 and denote

143
$$\omega_{3s}^{12} = \omega_{s_{3s}}^{s_{1}s_{2}} := |s_{1}|^{2} + |s_{2}|^{2} - |s_{3}|^{2} - |s|^{2} = -2(s_{1} - s) \cdot (s_{2} - s), \quad (1.17)$$

where the last equality holds if $\delta_{3s}^{\prime 12} = 1$ since then $s_3 = s_1 + s_2 - s$. Then Eq. (1.13) takes the form

141

$$\dot{\mathfrak{a}}_{s} + \gamma_{s}\mathfrak{a}_{s} = i\rho Y_{s}(\mathfrak{a},\nu^{-1}\tau) + b(s)\dot{\beta}_{s}, \qquad s \in \mathbb{Z}_{L}^{d},$$

$$Y_{s}(\mathfrak{a},t) = L^{-d}\left(\sum_{1,2,3} \delta_{3s}^{\prime 12}\mathfrak{a}_{1}\mathfrak{a}_{2}\bar{\mathfrak{a}}_{3}e^{it\omega_{3s}^{12}} - |\mathfrak{a}_{s}|^{2}\mathfrak{a}_{s}\right), \qquad (1.18)$$

where $\{\beta_s\}$ is yet another set of standard independent complex Wiener processes (and, again \mathfrak{a}_j stands for \mathfrak{a}_{s_j}). Note that the energy spectra of solutions to Eqs. (1.13) and (1.18) coincide:

$$N_s(\tau) = \mathbb{E}|v_s(\tau)|^2 = \mathbb{E}|\mathfrak{a}_s(\tau)|^2.$$
(1.19)

Sometimes we will write N_s and \mathfrak{a}_s as $N_s(\tau; \nu, L)$ and $\mathfrak{a}_s(\tau; \nu, L)$. The limiting dynamics in Eq. (1.18) under the limit (1.12) is governed by the *effective equation* of discrete turbulence. The latter has the form (1.18) with the modified nonlinearity Y^{res} , in which the sum is taken only over resonant vectors s_1, s_2, s_3 :

$$\dot{\mathfrak{a}}_{s} + \gamma_{s}\mathfrak{a}_{s} = i\rho Y_{s}^{res}(\mathfrak{a}) + b(s)\dot{\beta}_{s}, \quad s \in \mathbb{Z}_{L}^{d},$$

$$Y_{s}^{res}(\mathfrak{a}) = L^{-d}\left(\sum_{1,2,3} \delta_{3s}^{\prime 12} \delta(\omega_{3s}^{12})\mathfrak{a}_{1}\mathfrak{a}_{2}\bar{\mathfrak{a}}_{3} - |\mathfrak{a}_{s}|^{2}\mathfrak{a}_{s}\right).$$
(1.20)

Here $\delta(\omega_{3s}^{12}) = 1$ if $\omega_{3s}^{12} = 0$ and $\delta(\omega_{3s}^{12}) = 0$ otherwise. The following result is proven in [17,21] (concerning the well posedness of (1.18) also see [28]).

Theorem 1.1. If $\mathfrak{A}(u) = -\Delta u + u$ and $d \leq 3$, then Eqs. (1.18) and (1.20) are well posed. Under the limit (1.12), on time intervals of order 1,

(i) A solution $\mathfrak{a}^{\nu}(\tau)$ of (1.18) converges in distribution to a solution $\mathfrak{a}^{0}(\tau)$ of (1.20) once they have the same deterministic initial data at $\tau = 0$;

(ii) The energy spectrum $\mathbb{E}|\mathfrak{a}_{s}^{\nu}(\tau)|^{2} = N_{s}(\tau;\nu,L)$ converges to the energy spectrum $\mathbb{E}|\mathfrak{a}_{s}^{0}(\tau)|^{2}$.

It is very likely that the assertions of the theorem stays true with a similar 165 proof for any d if \mathfrak{A} is the operator (1.4, 1.5) with a sufficiently large r_* , and 166 in [21] this indeed is proved, provided that Eq. (1.18) is known to be well 167 posed. It is also shown in [17, 21] that if under the assumptions of Theorem 1.1 168 Eqs. (1.18) and (1.20) are mixing, then the stationary measure for the former 169 converges to that for the latter as $\nu \to 0$. The mixing property for Eq. (1.18) 170 is established in [28] for the case when all numbers b(s) are nonzero and d = 1. 171 It is plausible that a variation of the argument in [28] allows to establish the 172 mixing property for both Eqs. (1.18) and (1.20) if in (1.5) r_* is big in terms of 173 d and $b(s) \neq 0$ for all s. 174

175 1.3. The Main Result

In view of Theorem 1.1, to understand behaviour of the energy spectrum (1.19) of Eq. (1.18) under the limit (1.10), it remains to study that of the energy spectrum $\mathbb{E}|\mathfrak{a}_s(\tau;L)|^2$ of the effective Eq. (1.20) under the limit $L \to \infty$. Instead, following the logic of [7], we study the energy spectrum corresponding to a principal part of a decomposition in ρ for the solutions $\mathfrak{a}_s(\tau;L)$ of Eq. (1.20).

Quasisolutions and their energy spectra. To simplify presentation we assume that initially the system was at rest, i.e. supplement Eq. (1.20) with the zero initial condition

$$\mathfrak{a}_s(0) = 0 \qquad \forall s \in \mathbb{Z}_L^d.$$
(1.21)

We formally decompose the corresponding solution of (1.20) in ρ ,

187
$$\mathfrak{a}(\tau) = \mathfrak{a}^{(0)}(\tau) + \rho \mathfrak{a}^{(1)}(\tau) + \rho^2 \mathfrak{a}^{(2)}(\tau) + \dots, \qquad \mathfrak{a}^{(k)}(0) = 0, \quad (1.22)$$

188 $\mathfrak{a}^{(k)}(\tau) = \mathfrak{a}^{(k)}(\tau; L)$. The process $\mathfrak{a}^{(0)}(\tau)$ satisfies the linear equation

$$\dot{\mathfrak{a}}_s^{(0)} + \gamma_s \mathfrak{a}_s^{(0)} = b(s)\dot{\beta}_s, \quad s \in \mathbb{Z}_L^d.$$

190 So it is Gaussian,

$$\mathfrak{a}_s^{(0)}(\tau) = b(s) \int_0^\tau e^{-\gamma_s(\tau-l)} d\beta_s(l), \qquad (1.23)$$

and its components $\{\mathfrak{a}_s^{(0)}\}$ are independent. The process $\mathfrak{a}^{(1)}$ satisfies

193
$$\dot{\mathfrak{a}}_s^{(1)} + \gamma_s \mathfrak{a}_s^{(1)} = i Y_s^{res}(\mathfrak{a}^{(0)})$$

194 so that

195

196

$$\mathfrak{a}_{s}^{(1)}(\tau) = iL^{-d} \int_{0}^{\tau} e^{-\gamma_{s}(\tau-l)} \left(\sum_{1,2,3} \delta_{3s}^{\prime 12} \delta(\omega_{3s}^{12})(\mathfrak{a}_{1}^{(0)}\mathfrak{a}_{2}^{(0)}\bar{\mathfrak{a}}_{3}^{(0)}) - |\mathfrak{a}_{s}^{(0)}|^{2}\mathfrak{a}_{s}^{(0)} \right)(l) \mathrm{d}l \qquad (1.24)$$

is a Wiener chaos of third order (see [18]). Similar for $n \ge 1$,

$$\mathfrak{a}_{s}^{(n)}(\tau) = iL^{-d} \sum_{n_{1}+n_{2}+n_{3}=n-1} \int_{0}^{\tau} e^{-\gamma_{s}(\tau-l)} \times \left(\sum_{1,2,3} \delta_{3s}^{\prime 12} \delta(\omega_{3s}^{12}) \big(\mathfrak{a}_{1}^{(n_{1})} \mathfrak{a}_{2}^{(n_{2})} \bar{\mathfrak{a}}_{3}^{(n_{3})} \big) - \mathfrak{a}_{s}^{(n_{1})} \mathfrak{a}_{s}^{(n_{2})} \bar{\mathfrak{a}}_{s}^{(n_{3})} \right) (l) \, \mathrm{d}l,$$

$$(1.25)$$

199

198

is a Wiener chaos of order 2n + 1.

Next we consider the quadratic truncation of the series (1.22),

202
$$\mathcal{A}_{s}(\tau;L) = \mathcal{A}_{s}(\tau) = \mathfrak{a}_{s}^{(0)}(\tau) + \rho \mathfrak{a}_{s}^{(1)}(\tau) + \rho^{2} \mathfrak{a}_{s}^{(2)}(\tau), \qquad (1.26)$$

which we call the quasisolution⁴ of the effective Eqs. (1.20), 1.21). It is traditional in WT to analyse the quasisolution instead of the solution itself, postulating that the former well approximates the latter; see introduction to [7] for a discussion. The goal of the present paper is to study the behaviour of the energy spectrum of $\mathcal{A}(\tau)$,

$$\mathfrak{n}_{s,L}(\tau) = \mathbb{E}|\mathcal{A}_s(\tau;L)|^2, \quad s \in \mathbb{Z}_L^d, \tag{1.27}$$

as $L \to \infty$. Our results, formulated below, show that under this limit the energy spectrum $\mathbf{n}_{s,L}(\tau)$ has a non-trivial behaviour (i.e. stays finite and behaves differently from $\mathbb{E}|\mathbf{a}_s^{(0)}|^2$) only if $\rho \sim L \chi_d(L)$, where χ_d is defined in (1.11). Accordingly, from now on we assume that

$$\rho = \varepsilon L \chi_d(L), \tag{1.28}$$

where $0 < \varepsilon \leq 1/2$ is a small but fixed constant (see a few lines below for its discussion). Then the energy spectrum $\mathbf{n}_{s,L}$ expands as

216
$$\mathfrak{n}_{s,L}(\tau) = \mathfrak{n}_{s,L}^{(0)}(\tau) + \varepsilon \,\mathfrak{n}_{s,L}^{(1)}(\tau) + \varepsilon^2 \mathfrak{n}_{s,L}^{(2)}(\tau) + \varepsilon^3 \mathfrak{n}_{s,L}^{(3)}(\tau) + \varepsilon^4 \mathfrak{n}_{s,L}^{(4)}(\tau),$$
217 (1.29)

218 $s \in \mathbb{Z}_L^d$, where

$$\mathfrak{n}_{s,L}^{(k)}(\tau) = \left(L\chi_d(L)\right)^k \sum_{\substack{k_1+k_2=k\\0\le k_1,k_2\le 2}} \mathbb{E}\mathfrak{a}_s^{(k_1)}(\tau)\bar{\mathfrak{a}}_s^{(k_2)}(\tau).$$
(1.30)

 $_{220}$ In particular, by (1.23)

221
$$\mathfrak{n}_{s,L}^{(0)}(\tau) = \mathbb{E}|\mathfrak{a}_s^{(0)}(\tau;L)|^2 = \frac{b(s)^2}{\gamma_s} \left(1 - e^{-2\gamma_s \tau}\right), \qquad s \in \mathbb{Z}_L^d, \tag{1.31}$$

and a simple computation shows that $\mathfrak{n}_{s,L}^{(1)}(\tau) \equiv 0$. For higher-order terms, we prove that

224
$$\mathfrak{n}_{s,L}^{(2)} \sim 1$$
 and $|\mathfrak{n}_{s,L}^{(3)}|, |\mathfrak{n}_{s,L}^{(4)}| \lesssim 1$ as $L \to \infty$ uniformly in $\tau \ge 0;$
225 (1.32)

⁴By analogy with the *quasimodes* in the spectral theory of the Shrödinger operator.

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see a discussion in the next subsection. Thus, the parameter ε measures the properly scaled amplitude of the solutions, and indeed it should be small for the methodology of WT to apply. Then, the term $\varepsilon^2 \mathfrak{n}_{s,L}^{(2)}$ is the crucial non-trivial component of the energy spectrum $\mathfrak{n}_{s,L}$, while the terms $\varepsilon^3 \mathfrak{n}_{s,L}^{(3)}$, $\varepsilon^4 \mathfrak{n}_{s,L}^{(4)}$ are perturbative. This well agrees with the prediction of physical works concerning various models of WT.

Wave kinetic equation. In view of (1.32), to study the limiting as $L \to \infty$ behaviour of the energy spectrum $\mathfrak{n}_{s,L}(\tau)$ up to an error of size ε^3 it remains to investigate the behaviour of its principal component $\mathfrak{n}_{s,L}^{(0)}(\tau) + \varepsilon^2 \mathfrak{n}_{s,L}^{(2)}(\tau)$. We show that the latter is governed by a WKE. To state the result, let us consider the resonant quadric

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 $\Sigma_s = \left\{ (s_1, s_2) \in \mathbb{R}^{2d} : (s_1 - s) \cdot (s_2 - s) = 0 \right\},$ (1.33)

238 cf. (1.17), and a measure μ^{Σ_s} on it, given by

$$\mu^{\Sigma_s}(\mathrm{d}s_1\mathrm{d}s_2) = \left(|s_1 - s|^2 + |s_2 - s|^2\right)^{-1/2} \mathrm{d}s_1\mathrm{d}s_2|_{\Sigma_s},\tag{1.34}$$

where $ds_1 ds_2 |_{\Sigma_s}$ denotes the volume element on Σ_s , corresponding to the standard Euclidean structure on \mathbb{R}^{2d} .

Let us consider the following non-autonomous cubic wave kinetic integral operator $K(\tau)$, for any $\tau \geq 0$ sending a function $y_s, s \in \mathbb{R}^d$, to the function $K_s(\tau)y$, defined as

245
$$K_{s}(\tau)y = 4C_{d} \int_{\Sigma_{s}} \mu^{\Sigma_{s}} (\mathrm{d}s_{1}\mathrm{d}s_{2}) \Big(\mathcal{Z}^{4}y_{1}y_{2}y_{3} + \mathcal{Z}^{3}y_{1}y_{2}y_{4} - \mathcal{Z}^{2}y_{1}y_{3}y_{4} - \mathcal{Z}^{1}y_{2}y_{3}y_{4} \Big).$$
(1.35)

Here $y_j := y_{s_j}$ with $s_4 := s$ and $s_3 := s_1 + s_2 - s$, C_d is the constant from Theorem B below, the kernels $\mathcal{Z}^j = \mathcal{Z}^j(\tau; s_1, s_2, s_3, s_4)$ are given by formulas (4.14, 4.15) and satisfy $0 \leq \mathcal{Z}^j(\tau) \leq 1$. When $\tau \to \infty$, the operator $K(\tau)$ exponentially fast converges to a limiting kinetic integral operator $K(\infty)$, given by (1.35) with \mathcal{Z}^j replaced by $(\gamma_{s_1} + \gamma_{s_2} + \gamma_{s_3} + \gamma_{s_4})^{-1}$ for all j:

252
$$K_{s}(\infty)y = 4C_{d} \int_{\Sigma_{s}} \frac{\mu^{\Sigma_{s}}(\mathrm{d}s_{1}\mathrm{d}s_{2})}{\gamma_{s_{1}} + \gamma_{s_{2}} + \gamma_{s_{3}} + \gamma_{s_{4}}} \left(y_{1}y_{2}y_{3} + y_{1}y_{2}y_{4} - y_{1}y_{3}y_{4} - y_{2}y_{3}y_{4} \right).$$
(1.36)

It is similar to the standard four-wave kinetic operator of WT (e.g. see in [25]),
which has the form (1.35) with
$$\mathcal{Z}^{j} \equiv \text{const}$$
, but still is different from the latter
since $K(\infty)$ depends on the spectrum $\{\gamma_{s}\}$ of the dissipation operator $\mathfrak{A}^{.5}$

For $r \in \mathbb{R}$ we denote by $\mathcal{C}_r(\mathbb{R}^d)$ a space of continuous complex functions on \mathbb{R}^d with finite norm

$$|f|_r = \sup_{z \in \mathbb{R}^d} |f(z)| \langle z \rangle^r$$
, where $\langle z \rangle = \max(|z|, 1).$ (1.37)

⁵Earlier the kinetic operator $K(\infty)$ was heuristically obtained in [20].

In Sect. 5, following [7], we show that if r > d, then for any τ the operator $K(\tau)$ defines a continuous 3-homogeneous mapping $K(\tau) : \mathcal{C}_r(\mathbb{R}^d) \mapsto \mathcal{C}_{r+1}(\mathbb{R}^d)$, and for any $y \in \mathcal{C}_r(\mathbb{R}^d)$ the curve $\tau \mapsto K(\tau)(y)$ is Hölder continuous in $\mathcal{C}_r(\mathbb{R}^d)$.

263 Now consider the following damped/driven non-autonomous WKE

$$\dot{\mathfrak{z}}_s(\tau) = -2\gamma_s\mathfrak{z}_s + \varepsilon^2 K_s(\tau)(\mathfrak{z}) + 2b(s)^2, \qquad \mathfrak{z}(0) = 0, \tag{1.38}$$

where $\tau \geq 0$ and $s \in \mathbb{R}^d$. In Sect. 5 we prove that for small ε it has a unique solution $\mathfrak{z}_s(\tau)$, which can be written as $\mathfrak{z}_s(\tau) = \mathfrak{z}_s^0(\tau) + \varepsilon^2 \mathfrak{z}_s^1(\tau, \varepsilon)$, where $\mathfrak{z}_s^0, \mathfrak{z}_s^1 \sim$ 1 and \mathfrak{z}_s^0 solves the linear Eq. (1.38) $|_{\varepsilon=0}$. It is easy to see that \mathfrak{z}_s^0 equals the component $\mathfrak{n}_{s,L}^{(0)}$ of the energy spectrum $\mathfrak{n}_{s,L}$, given by (1.31), and we prove that \mathfrak{z}_s^1 is ε^4 -close to $\mathfrak{n}_{s,L}^{(2)}$ uniformly in τ . Then, in view of (1.32), the energy spectrum $\mathfrak{n}_{s,L}$ is ε^3 -close to the solution $\mathfrak{z}_s(\tau)$.

Below we denote by $C^{\#}(s)$ various positive functions of s which decay as $|s| \to \infty$ faster than any negative degree of |s|. These functions never depend on L, ε and τ . By $C^{\#}(s; p)$ we denote functions $C^{\#}(s)$ depending on a parameter p.

Theorem A (Main theorem). Let $d \geq 2$. Then, the energy spectrum $\mathbf{n}_{s,L}(\tau)$ of the quasisolution $\mathcal{A}_s(\tau)$ of (1.20), 1.21) satisfies the estimate $\mathbf{n}_{s,L}(\tau) \leq C^{\#}(s)$ and is ε^3 -close to the solution $\mathbf{j}_s(\tau)$ of WKE (1.38). Namely, under the scaling $\rho = \varepsilon L \chi_d(L)$, for any r there exists $\varepsilon_r \in (0, 1/2]$ such that for $0 < \varepsilon \leq \varepsilon_r$ we have

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$$|\mathfrak{n}_{,L}(\tau) - \mathfrak{z}_{,}(\tau)|_r \le C_r \varepsilon^3 \qquad \forall \, \tau \ge 0, \tag{1.39}$$

if $L \ge \varepsilon^{-2}$ for $d \ge 3$, and $L \ge e^{\varepsilon^{-1}}$ for d = 2.

See Theorem 5.9. Since the energy spectrum \mathfrak{n}_s is defined for $s \in \mathbb{Z}_L^d$ with finite L, then the norm in (1.39) is understood as $|f|_r = \sup_{z \in \mathbb{Z}_L^d} |f(z)| \langle z \rangle^r$.

Remark 1.2. If d = 2, the lower bound $L \ge e^{\varepsilon^{-1}}$ can be relaxed in the following sense. In Appendix D we explain that there is a bounded correction $f(\tau, L)$ which can be written explicitly, such that

$$\tau \qquad \left| \mathfrak{n}_{,L}(\tau) - \mathfrak{z}_{.}(\tau) - \frac{f(\tau,L)}{\ln L} \right|_{r} \le C_{r} \varepsilon^{3} \qquad \forall \tau \ge 0, \tag{1.40}$$

if $L \ge \varepsilon^{-6}$.

In Lemma 5.6 we show that in the vicinity of the unique steady state $\mathfrak{z}_{s}^{0} := b(s)^{2}/\gamma_{s}$ for the linear Eq. (1.38) $|_{\varepsilon=0}$, Eq. (1.38) $|_{\tau=\infty}$ with $\varepsilon \ll 1$ has a unique steady state $\mathfrak{z}^{\varepsilon} \in \mathcal{C}_{r}(\mathbb{R}^{d})$ and the latter is asymptotically stable. Jointly with Theorem A, this result implies the following asymptotic in time behaviour of the energy spectrum $\mathfrak{n}_{s,L}(\tau)$:

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$$|\mathfrak{n}_{,L}(\tau) - \mathfrak{z}^{\varepsilon}|_{r} \le C_{r}(e^{-\tau} + \varepsilon^{3}), \qquad \forall \tau \ge 0,$$
(1.41)

²⁹⁵ if L is as in Theorem A, see (5.19).

The cases $d \ge 3$ and d = 2 are similar, but should be treated separately. To shorten the presentation, we give a detailed proof of Theorem A only for $d \ge 3$, when

$$\chi_d = 1$$
 and $\rho = \varepsilon L$.

The proof for d = 2 can be obtained by a simple modification of the argument for $d \ge 3$. We sketch it in Appendix D. So from now on, except Sect. 2 which gives a brief account of the method of Feynman diagrams from [7,8], we assume that $d \ge 3$.

In paper [7] we examine the behaviour of the energy spectrum $\mathbf{n}_{s,L,\nu}(\tau)$ of a quasisolution to Eq. (1.18) under the limit (1.9), assuming that $\rho = \varepsilon \nu^{-1/2}$.⁶ We got there a similar result which states that $\mathbf{n}_{.,L,\nu}(\tau)$ is ε^4 -close to a solution of the damped/driven four-wave kinetic equation as in [25, Section 6.9.1] [in contrast with Eq. (1.38), the kinetic nonlinearity there does not depend on the dissipation \mathfrak{A} in Eq. (1.3)].

What next? In this work and in [7] we obtained wave kinetic limits for the 310 energy spectra of quasisolutions for the NLS Eq. (1.6) under limit (1.10) with 311 the scaling $\rho = \varepsilon L \chi_d(L)$ and limit (1.9) with the scaling $\rho = \varepsilon \nu^{-1/2}$. Our 312 next goal is to show that an exact solution $\mathfrak{a}_{\cdot}(\tau)$ of Eq. (1.18) is ε^3 -close 313 to its quasisolution $\mathcal{A}(\tau)$ (uniformly in $L \geq 2$ and $\tau \in [0,T]$, for any T > 1314 0). And that a solution of Eq. (1.6) is ε^3 -close to the quasisolution of the 315 equation (uniformly in ν , L and $\tau \in [0,T]$, if $L \geq \nu^{-2-\bar{\gamma}}, \bar{\gamma} > 0$). This 316 would imply that the energy spectra of solutions of Eq. (1.6) under limit (1.10)317 and limit (1.9) are ε^3 -close to solutions of the two WKE (namely, Eq. (1.38) 318 and the WKE from [7]). To prove this, say, for a solution $\mathfrak{a}_{\cdot}(\tau)$ of Eq. (1.18) 319 we consider the equation on any fixed time-interval [0, T] and regard it as a 320 nonlinear equation $F_T(\mathfrak{a}_{\cdot}(\cdot)) = 0$ for the unknown process $\mathfrak{a}_s(\tau)$. Then the 321 quasisolution \mathcal{A} satisfies the equation with a disparity $\lesssim \varepsilon^3$. By analogy with 322 some stochastic problems for nonlinear PDEs, recently successfully resolved 323 by the KAM-techniques (e.g. see [19]), we believe that KAM also applies to 324 the equation $F_T = 0$. Its application would imply that \mathfrak{a} is ε^3 -close to \mathcal{A} , as 325 stated. We also believe that analysis of the KAM-iterations which build \mathfrak{a} from 326 \mathcal{A} will show that the energy spectrum of the solution $\mathfrak{a}_{\tau}(\tau)$ of Eq. (1.18) under 327 the limit $L \to \infty$ converges to a solution of the WKE (1.38). A similar logic 328 should apply to the energy spectra of solutions for Eq. (1.6) under the limit 329 (1.9).330

331 1.4. Outline of The Proof: Feynman Diagrams and Number Theory

It is well understood that to write down formulas for the terms $\mathbf{n}_{s,L}^{(k)}$ of decompositions as (1.29) it is instrumental to use the language of Feynman diagrams. In application to similar problems this goes back at least to the works [11,12], and then was successfully used for the purposes of WT in [2–6,23] and other papers. We use this techniques in the form developed in [8] which gives a convenient presentation of the terms $\mathbf{n}_{s,L}^{(k)}$ [see (1.30)]. Namely, by iterating the

⁶In [7] the notation is slightly different: there we set $\rho = \varepsilon^{1/2} \nu^{-1/2}$.

Duhamel formula (1.25) we express $\mathfrak{a}^{(n)}(\tau)$ in terms of the Gaussian processes 338 $\mathfrak{a}_s^{(0)}$, and next evoking the Wick formula for moments of $\mathfrak{a}_s^{(0)}$ write the terms 339 $\mathfrak{n}_{s,L}^{(k)}$ as multiple sums. Then the just mentioned diagram techniques allows to 340 'integrate' these sums. That is, to write any $\mathfrak{n}_{s,L}^{(k)}$ as a sum over an intersections 341 of k-1 quadrics in $(\mathbb{Z}_L^d)^k$ in a form, convenient to pass to a limit as $L \to \infty$. 342 The term $\mathfrak{n}_{s,L}^{(2)}$ is a sum over a single quadric and may be analysed without the 343 diagram's machinery. This and some other similar terms play a leading role 344 in our analysis and dictate the form of the limiting WKE. The terms may be 345 written as sums 346

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$$G_s(\tau, L) = L^{2(1-d)} \sum_{\substack{z_1, z_2 \in \mathbb{Z}_L^d:\\z_1 \cdot z_2 = 0, \, z_1, z_2 \neq 0}} \Phi_s(\tau; z_1, z_2),$$
(1.42)

well known in works on WT. To study them under the limit $L \to \infty$ we make use of the celebrated circle method of Heath–Brown [14]. Since the result of [14] does not completely fit our purposes, we specified it in the accompanying paper [10] (also see [1, Section 5] for another specification of the Heath–Brown method, used for the purposes of WT). This implies

353 Theorem B. For any $L \ge 2$,

$$\left| G_s(\tau, L) - C_d \int_{\Sigma_0} \Phi_s(\tau; z_1, z_2) \, \mu^{\Sigma_0}(\mathrm{d} z_1 \mathrm{d} z_2) \right| \le K_d \, \frac{\|\Phi_s(\tau; \cdot)\|_{N_1, N_2}}{L^{d-5/2}},$$

where $\Sigma_0 = \{z_1, z_2 \in \mathbb{R}^d : z_1 \cdot z_2 = 0\}$, μ^{Σ_0} is the measure on it, defined by (1.34) with s = 0, C_d is a number-theoretical constant, satisfying $C_d \in (1, 1 + 2^{2-d})$, the norm $\|\cdot\|_{N_1,N_2}$ is defined in (3.3) and the constants $N_1, N_2 \in \mathbb{N}$ depend only on d.

In particular, the term $\mathfrak{n}_{s,L}^{(2)}(\tau)$ admits a limit when $L \to \infty$.

The terms $\mathfrak{n}_{s,L}^{(3)}$ and $\mathfrak{n}_{s,L}^{(4)}$ in (1.29) correspond to multiple intersections of quadrics, and the Heath–Brown method does not apply to them. Still the diagram technique allows to write these terms in a convenient compact form. Then next in Sect. 3 and Appendix A we use Theorem B jointly with another powerful result from the number theory—Bezout's theorem for finite fields—to prove⁷

366 **Theorem C.** For k = 3, 4, $|\mathbf{n}_{s,L}^{(k)}(\tau)| \le C^{\#}(s)$.

Theorems B and C imply (1.32). So to establish Theorem A it remains to show that the term $\mathfrak{n}_{s,L}^{\leq 2}(\tau) := \mathfrak{n}_{s,L}^{(0)}(\tau) + \varepsilon^2 \mathfrak{n}_{s,L}^{(2)}(\tau)$ (or equivalently its limit as $L \to \infty$, provided by Theorem B) can be well approximated by a solution of the WKE (1.38). To this end, following the lines of [7] (and the logic of the Krylov– Bogolyubov averaging) we consider increments $\Delta \mathfrak{n}_{s,L}^{\leq 2} := \mathfrak{n}_{s,L}^{\leq 2}(\tau + \theta) - \mathfrak{n}_{s,L}^{\leq 2}(\tau)$ and express them through the processes $\mathfrak{a}_m^{(0)}$ via the Duhamel formula (1.25)

 $^{^{7}}$ In fact, in Sect. 3 we prove an abstract result, more general than the theorem below; see there Theorem 3.2.

and the Wick theorem. Then the increments approximately take the form (1.42), and we use Theorem B to show that they are close to the r.h.s. of the WKE, multiplied by θ .

Although the computation of the increments $\Delta \mathfrak{n}_{s,L}^{\leq 2}$ is similar to that in 376 [7], now a mechanism leading to a WKE is rather different. Namely, in [7] com-377 ponents of the terms $\mathbf{n}_{sL}^{(k)}$ are approximated by formulas analogous to (1.42), 378 where the summation over the lattice $(\mathbb{Z}^d)^k = \{z\}$ is replaced by an integra-379 tion over \mathbb{R}^{dk} . The integrals in those formulae involve fast oscillating Gaussian 380 kernels. The zero sets of these kernels define quadrics, related to the quadrics 381 $\{(z_1, z_2) : z_1 \cdot z_2 = 0\}$ in (1.42). Due to the fast oscillations a crucial component 382 of the increments $\Delta \mathfrak{n}_{s,L}^{\leq 2}$ is given by the terms, associated with short-range cor-383 relations in τ of the processes $\mathfrak{a}_m^{(0)}(\tau)$. On the contrary, in the present situation 384 a crucial contribution is given by other terms, associated with long-time corre-385 lations of the processes $\mathfrak{a}_m^{(0)}(\tau)$, while the short-range correlations only give a 386 small correction, as it can be seen from computations of Appendix A.4. So, it 387 is natural that the kinetic integral in the WKE (1.38) depends on the viscosity 388 operator \mathfrak{A} , while that in the WKE in [7] does not. 389

Finally we note that, as we explain in Sect. 3.2, it is plausible that Theorem C holds for all $k \geq 3$. If so, then for $\rho = \varepsilon L$ the energy spectrum of a solution $\mathfrak{a}(\tau)$, written as (1.22), defines a formal series in ε , uniformly in $L \geq 2$. Then the partial sums of this series, made by the terms of order ε^m , $m \leq M$, with any fixed $M \geq 2$, also satisfy Theorem A with the constants C_r , depending on M. Cf. Conjecture 3.8.

Series Expansion: Approximating Equation and Diagrammatic Representation for Solutions

In this section, assuming that $d \ge 2$, we approximate processes (1.25) by more convenient processes $a^{(n)}$, and then obtain a compact and instrumental representation for their correlations in terms of Feynman diagrams (see Lemma 2.2), following [8, Sections 3–5]. This representation (as well as its analogy in [7,8]) used to estimate various disparity terms, related to quasisolutions $\mathcal{A}(\tau; L)$, see (1.26), and to their energy spectra.

Our presentation is sketchy, but missing details may be found in [8]. For a general discussion of the language of Feynman diagrams, see [18].

406 2.1. Approximate *a*-Equation

We start by considering an approximation of the original Eq. (1.20) by an equation, where the term $L^{-d}|\mathfrak{a}_s|^2\mathfrak{a}_s$ is removed:

$$\dot{a}_{s} + \gamma_{s} a_{s} = i\rho \mathcal{Y}_{s}(a) + b(s)\beta_{s}, \quad s \in \mathbb{Z}_{L}^{d},$$
$$\mathcal{Y}_{s}(a) = L^{-d} \sum_{1,2,3} \delta_{3s}^{\prime 12} \delta(\omega_{3s}^{12}) a_{1} a_{2} \bar{a}_{3}.$$
(2.1)

410 Similar to processes \mathfrak{a}_s , we decompose

$$a = a^{(0)} + \rho a^{(1)} + \dots$$
 (2.2)

Here $a^{(0)} = \mathfrak{a}^{(0)}$ and the processes $a_s^{(n)}(\tau)$ with $n \ge 1$ are built by the recursive formula (1.25) with the term $\mathfrak{a}_s^{n_1}\mathfrak{a}_s^{n_2}\bar{\mathfrak{a}}_s^{n_3}$ being dropped. That is, with the nonlinearity Y_s^{res} replaced by the \mathcal{Y}_s above.

Results of [8] together with Theorem 3.2 below (which is an abstract version of Theorem C from the introduction) imply

417 **Proposition 2.1.** For all $m, n \ge 0$, satisfying $N := m + n \le 4$,

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$$\left| \mathbb{E}a_{s}^{(m)}(\tau_{1})\bar{a}_{s}^{(n)}(\tau_{2}) - \mathbb{E}\mathfrak{a}_{s}^{(m)}(\tau_{1})\bar{\mathfrak{a}}_{s}^{(n)}(\tau_{2}) \right| \leq \frac{L^{-N-d+1}}{\chi_{d}(L)^{N-1}}C^{\#}(s;n,m), \quad (2.3)$$

419 uniformly in $\tau_1, \tau_2 \geq 0$.

We prove the proposition in Appendix C for $d \ge 3$ and discuss an adaptation of the proof to the case d = 2 in Appendix D. Doing that we use the relation $N := m + n \le 4$ only to apply Theorem 3.2 (or Theorem D.2 if d = 2). So, if the assertion (3.9) of the latter theorem holds for larger N's, then for those N's estimates (2.3) remains true as well [we believe that (3.9) is fulfilled for all N, see in Sect. 3.2].

Relations (2.3) imply that moments of processes $a_s^{(m)}(\tau)$ well approximate those of processes $\mathfrak{a}_s^{(m)}(\tau)$ as $L \to \infty$. Accordingly, from now on we will mostly study processes $a_s(\tau)$ and their decompositions (2.2).

429 2.2. Diagrams for Solutions

For what follows it is convenient to re-write operator \mathcal{Y} from (2.1), using a fictitious index s_4 :

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$$\mathcal{Y}_s(a) = L^{-d} \sum_{1,2,3,4} \delta_{34}^{\prime 12} \,\delta(\omega_{34}^{12}) \delta_4^s \,a_1 a_2 \bar{a}_3,$$

where δ_4^s is the Kronecker symbol. Then analogous of the expression (1.25) for $a^{(m)}, m \ge 1$, takes the form

$$a_{s}^{(m)}(\tau) = \sum_{\substack{m_{1}+m_{2}+m_{3}=m-1\\ L^{-d}\sum_{1,2,3,4}} \delta_{34}^{\prime 12} \,\delta(\omega_{34}^{12}) \,\delta_{4}^{s} \left(a_{1}^{(m_{1})}a_{2}^{(m_{2})}\bar{a}_{3}^{(m_{3})}\right)(l).$$

$$(2.4)$$

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We will call the objects as those in the r.h.s. of (2.4) sums, despite they involve integrating in dl. The r.h.s. of (2.4) contains several sums, corresponding to all admissible choices of numbers m_1, m_2, m_3 .

We apply Duhamel's formula (2.4) to the terms $a_{s_i}^{(m_i)}(l)$ in the right-hand side of (2.4) with $m_i > 0$, and iterate the procedure till $a_s^{(m)}(\tau)$ is expressed through the processes $a^{(0)}$ and $\bar{a}^{(0)}$. Then $a_s^{(m)}$ becomes represented as a finite sum of *sums*; we denote such sums by I_s . Below we will associate with each sum I_s an appropriately constructed diagram \mathfrak{D} . Thus, we will write $a_s^{(m)}(\tau)$ as

$$a_s^{(m)}(\tau) = \sum_{\mathfrak{D}\in\mathfrak{D}_m} I_s(\mathfrak{D};\tau), \qquad (2.5)$$

where \mathfrak{D}_m is a set of all diagrams, corresponding to the just explained representation of $a^{(m)}$ via the processes $a^{(0)}$ and $\bar{a}^{(0)}$. Similarly by $\overline{\mathfrak{D}}_n$ we denote the set of diagrams, parametrizing the terms in the sum, representing $\bar{a}^{(n)}(\tau)$ in a form, analogous to (2.5): $\bar{a}_s^{(n)}(\tau) = \sum_{\bar{\mathfrak{D}}\in\overline{\mathfrak{D}}_m} I_s(\bar{\mathfrak{D}};\tau)$.

2.2.1. Construction of the Sets of Diagrams \mathfrak{D}_m and $\overline{\mathfrak{D}}_n$. We start with discussing the set \mathfrak{D}_2 and the sums $I_s(\mathfrak{D})$ with $\mathfrak{D} \in \mathfrak{D}_2$.

When iterating the Duhamel formula (2.4) (or its complex conjugation) 452 for a *j*-th time, we will denote the corresponding time $l \in [0, \tau]$ by l_j and 453 will write the set of indices $\{s_1, s_2, s_3, s_4\}$ as $\{\xi_{2j-1}, \xi_{2j}, \sigma_{2j-1}, \sigma_{2j}\}$, where we 454 enumerate by ξ_i the indices of non-conjugated variables $a_{\epsilon'}^{(k)}$ in (2.4) and by 455 σ_i – those of conjugated variables $\bar{a}_{s''}^{(n)}$. We write the corresponding fictitious 456 index s_4 as σ_{2j} if we apply (2.4), or as ξ_{2j} if we apply the complex conjugation 457 of (2.4). More precisely, when applying (2.4) we denote $s_1 = \xi_{2j-1}, s_2 = \xi_{2j}$, 458 $s_3 = \sigma_{2j-1}$ and $s_4 = \sigma_{2j}$, and when applying its complex conjugation, we write 459 $s_1 = \sigma_{2j-1}, s_2 = \sigma_{2j}, s_3 = \xi_{2j-1}$ and $s_4 = \xi_{2j}$. We will abbreviate 460

$$\delta_j = \delta_{\sigma_{2j-1}\sigma_{2j}}^{\ell_{2j-1}\xi_{2j}} \quad \text{and} \quad \omega_j = \omega_{\sigma_{2j-1}\sigma_{2j}}^{\xi_{2j-1}\xi_{2j}}, \quad j \ge 1,$$
(2.6)

(these terms correspond to $\delta_{34}^{\prime 12}$ and ω_{34}^{12} in (2.4). We also set

$$_{0}=\sigma_{0}:=s. \tag{2.7}$$

Applying (2.4) to $a_s^{(2)}$ and using the notation above with j = 1, we find

$$a_{s}^{(2)}(\tau) = a_{\xi_{0}}^{(2)}(\tau) = \sum_{m_{1}+m_{2}+m_{3}=1} i \int_{0}^{\tau} \mathrm{d}l_{1} e^{-\gamma_{\xi_{0}}(\tau-l_{1})} L^{-d} \sum_{\xi_{1},\xi_{2},\sigma_{1},\sigma_{2}} \delta_{1} \delta(\omega_{1}) \,\delta_{\sigma_{2}}^{\xi_{0}} \big(a_{\xi_{1}}^{(m_{1})}a_{\xi_{2}}^{(m_{2})}\bar{a}_{\sigma_{1}}^{(m_{3})}\big)(l_{1}).$$

$$(2.8)$$

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Let us consider the summand with $m_1 = m_2 = 0$ and $m_3 = 1$. Applying the conjugated formula (2.4) to $\bar{a}_{\sigma_1}^{(1)}$ and using the introduced notation with j = 2, we get

$$\bar{a}_{\sigma_{1}}^{(1)}(l_{1}) = -i \int_{0}^{l_{1}} \mathrm{d}l_{2} \, e^{-\gamma_{\sigma_{1}}(l_{1}-l_{2})} \\ L^{-d} \sum_{\xi_{3},\xi_{4},\sigma_{3},\sigma_{4}} \delta_{2} \, \delta(\omega_{2}) \, \delta_{\xi_{4}}^{\sigma_{1}} \big(a_{\xi_{3}}^{(0)} \bar{a}_{\sigma_{3}}^{(0)} \bar{a}_{\sigma_{4}}^{(0)} \big) (l_{2}).$$

$$(2.9)$$

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Inserting (2.9) into the summand in (2.8) with $m_1 = m_2 = 0$ and $m_3 = 1$, we get a sum $I_s(\mathfrak{D}; \tau)$ which we associate with the diagram \mathfrak{D} from Fig. 1c; further on we will denote this diagram by \mathfrak{D}^c . The *non-conjugated* vertices $c_i^{(k)}$ of the diagram are associated with the variables $a_{\xi_i}^{(k)}$ in (2.8), (2.9); the corresponding



FIGURE 1. The set of diagrams \mathfrak{D}_2

to them indices are ξ_i . The *conjugated* vertices $\bar{c}_i^{(n)}$ are associated with the 475 variables $\bar{a}_{\sigma_j}^{(n)}$ and the corresponding indices are σ_j . In particular, the root $c_0^{(2)}$ is associated with $a_{\xi_0}^{(2)} = a_s^{(2)}$ and the corresponding index is ξ_0 . In the notation 476 477 $c_i^{(k)}$ and $\bar{c}_i^{(n)}$ we sometimes omit the upper indices k and n which we call the 478 degrees of the vertices $c_i^{(k)}$, $\bar{c}_j^{(n)}$. The vertices \bar{w}_2 and w_4 are called conjugated 479 (non-conjugated) virtual vertices, and the corresponding indices are σ_2 and ξ_4 ; 480 these vertices are associated with the Kronecker symbols $\delta_{\sigma_2}^{\xi_0}$ and $\delta_{\xi_4}^{\sigma_1}$ in (2.8) 481 and (2.9). Vertices which are not virtual are called *real*. Every edge of the 482 diagram couples a non-conjugated (conjugated) vertex $c_i^{(k)}$ $(\bar{c}_i^{(k)})$ of positive 483 degree $k \geq 1$ with a conjugated (non-conjugated) virtual vertex $\bar{w}_{i'}(w_{i'})$. It 484 is associated with an application of formula (2.4) (or its complex conjugation) 485 to the variable $a_{\xi_i}^{(k)}$ (or $\bar{a}_{\sigma_i}^{(k)}$), corresponding to the vertex $c_i^{(\vec{k})}$ (or $\bar{c}_i^{(\vec{k})}$). 486 The set of four vertices 487

$$c_{2j-1}, c_{2j}, \bar{c}_{2j-1}, \bar{w}_{2j}$$
 or $c_{2j-1}, w_{2j}, \bar{c}_{2j-1}, \bar{c}_{2j}$ (2.10)

(in dependence whether the virtual vertex is conjugated or not) to which corre-489 spond the indices $\xi_{2j-1}, \xi_{2j}, \sigma_{2j-1}, \sigma_{2j}$ is called the *j*-th block; the diagram \mathfrak{D}^c 490 has two blocks. The index i of a virtual vertex w_i (\bar{w}_i) is always pair, i = 2j. 491 Each block corresponds to an application of formula (2.4) (or its complex con-492 jugation) to its *parent*, i.e. to the vertex of positive degree coupled with the 493 virtual vertex of the block. The virtual vertex is conjugated if the parent is 494 non-conjugated and the other way round. The time variable l_i is associated 495 with the j-th block. 496

⁴⁹⁷ The *leaves* are the vertices of zero degree, that is, the vertices $c_i^{(0)}$ and ⁴⁹⁸ $\bar{c}_i^{(0)}$.

The diagrams from Fig. 1(a,b) correspond to the summands in (2.8) with $m_1 = 1, m_2 = m_3 = 0$ and $m_1 = m_3 = 0, m_2 = 1$; they are constructed by the same rules as the diagram \mathfrak{D}^c . The three diagrams from Fig. 1 form the set \mathfrak{D}_2 . The set of diagrams $\overline{\mathfrak{D}}_2$, corresponding to $\overline{a}_s^{(2)}(\tau)$, is obtained by conjugating the vertices in the three diagrams above and re-ordering the elements of each block in such a way that the pair of non-conjugated vertices is followed by the pair of conjugated vertices, i.e. the blocks have the form (2.10).

The sets \mathfrak{D}_m and $\overline{\mathfrak{D}}_n$ with arbitrary $m, n \geq 0$ and the diagrams which are their elements, are constructed similarly. Namely, the sets \mathfrak{D}_0 and $\overline{\mathfrak{D}}_0$ are trivial—they contain one diagram each, made by the root $c_0^{(0)}$ (or $\bar{c}_0^{(0)}$). The sets \mathfrak{D}_1 and $\overline{\mathfrak{D}}_1$ also contain only one diagram each; e.g. the diagram in \mathfrak{D}_1 consists of the root $c_1^{(0)}$, joint by an edge with \bar{w}_2 in the only block $B_1 = (c_1^{(0)}, c_2^{(0)}, \bar{c}_1^{(0)}, \bar{w}_2)$. Arbitrary sets \mathfrak{D}_m and $\overline{\mathfrak{D}}_n$ may be constructed by induction. Indeed, consider a process $a^{(m+1)}(\tau)$ with $m \ge 1$ and apply to it (2.4) with m := m + 1. In the r.h.s. of (2.4) the sum in m_1, m_2, m_3 contains (m+2)(m+1)/2 terms. Consider any one of them,

$$i \int_{0}^{t} \mathrm{d}l \, e^{-\gamma_{s}(\tau-l)} L^{-d} \sum_{1,2,3,4} \delta_{34}^{\prime 12} \, \delta(\omega_{34}^{12}) \, \delta_{4}^{s} \big(a_{1}^{(m_{1})} a_{2}^{(m_{2})} \bar{a}_{3}^{(m_{3})} \big)(l), \quad (2.11)$$

draw the block $B_1 = (c_1^{(m_1)}, c_2^{(m_2)}, \overline{c}_1^{(m_3)}, \overline{w}_2)$, and join \overline{w}_2 by an edge with the root $c_0^{(m+1)}$. Next consider the sets $\mathfrak{D}_{m_1}, \mathfrak{D}_{m_2}, \overline{\mathfrak{D}}_{m_3}$ and do the following:

- (a) Firstly, take \mathfrak{D}_{m_1} . If $m_1 = 0$, do nothing. Otherwise, choose any diagram $\mathfrak{D}^1 \in \mathfrak{D}_{m_1}$, place it below $c_1^{(m_1)}$ and identify its root with $c_1^{(m_1)}$. Do this for each diagram in \mathfrak{D}_{m_1} , thus obtaining $|\mathfrak{D}_{m_1}|$ diagrams with roots in $c_0^{(m+1)}$.
- (b) Then consider the set \mathfrak{D}_{m_2} and do the same with the just obtained $|\mathfrak{D}_{m_1}|$ diagrams, identifying their roots with the vertex $c_2^{(m_2)}$, and next—the set $\overline{\mathfrak{D}}_{m_3}$, identifying the roots with $\overline{c}_3^{(m_3)}$.
- (c) It remains to convert thus obtained $|\mathfrak{D}_{m_1}| \times |\mathfrak{D}_{m_2}| \times |\mathfrak{D}_{m_3}|$ diagrams to elements of the set \mathfrak{D}_{m+1} by re-numerating properly their blocks and accordingly re-numerating the vertices in the blocks as in (2.10). Do this by numerating the blocks from top to the bottom and from left to right, as in the examples above with m = 2.
- (d) Doing the same for all blocks, corresponding to all possible (m+2)(m+1)/2 terms (2.11), get the diagrams, forming the set \mathfrak{D}_{m+1} .

532 The set $\overline{\mathfrak{D}}_{m+1}$ is constructed inductively in the same way.

For further needs we note that due to the factors δ_{34}^{12} and $\delta_{s_4}^s$ in (2.4), the indices ξ_i, σ_j entering the formula for the sums $I_s(\mathfrak{D})$ from (2.5) satisfy the relations

(1)
$$\delta_{\sigma_{2j-1}\sigma_{2j}}^{\xi_{2j-1}\xi_{2j}} = 1 \ \forall j,$$

(2) indices ξ_i, σ_j corresponding to adjacent in \mathfrak{D} vertices are equal.

(2.12)

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538 2.3. Feynmann Diagrams for Expectations

The main objects we are interested in are the correlations $\mathbb{E}a_{s_1}^{(m)}(\tau_1)\bar{a}_{s_2}^{(n)}(\tau_2)$. It can be shown that they vanish if $s_1 \neq s_2$.⁸ To represent an expectation $\mathbb{E}a_s^{(m)}(\tau_1)\bar{a}_s^{(n)}(\tau_2)$, we consider the set of diagrams

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$$\mathfrak{D}_m\times\overline{\mathfrak{D}}_n:=\{\mathfrak{D}^1\sqcup\bar{\mathfrak{D}}^2:\,\mathfrak{D}^1\in\mathfrak{D}_m,\bar{\mathfrak{D}}^2\in\overline{\mathfrak{D}}_n\}.$$

Here a diagram $\mathfrak{D}^1 \sqcup \overline{\mathfrak{D}}^2$ is obtained by drawing \mathfrak{D}^1 and $\overline{\mathfrak{D}}^2$ side by side, where the blocks of \mathfrak{D}^1 are enumerated from 1 to m, while those of $\overline{\mathfrak{D}}^2$ together with

⁸As well vanish the correlations $\mathbb{E}a_{s_1}^{(m)}(\tau_1)a_{s_2}^{(n)}(\tau_2)$ and $\mathbb{E}\bar{a}_{s_1}^{(m)}(\tau_1)\bar{a}_{s_2}^{(n)}(\tau_2)$ for all s_1, s_2 .



FIGURE 2. A diagram from the set $\mathbf{a} \ \mathfrak{D}_2 \times \overline{\mathfrak{D}}_0$ and $\mathbf{b} \ \mathfrak{D}_1 \times \overline{\mathfrak{D}}_2$

the corresponding time variables l_j are enumerated from j = m + 1 to m + n. 545 The vertices together with the corresponding indices $\xi_{2j-1}, \xi_{2j}, \sigma_{2j-1}, \sigma_{2j}$ are 546 enumerated accordingly, see Fig. 2. The diagram $\mathfrak{D}^1 \sqcup \overline{\mathfrak{D}}^2$ has two roots $c_0^{(m)}$ and $\overline{c}_0^{(n)}$. For any $\mathfrak{D} = \mathfrak{D}^1 \sqcup \overline{\mathfrak{D}}^2$ consider 547 548

$$I_s(\mathfrak{D};\tau_1,\tau_2) = I_s(\mathfrak{D}^1;\tau_1)I_s(\bar{\mathfrak{D}}^2;\tau_2),$$

so that $a_s^{(m)}(\tau_1)\bar{a}_s^{(n)}(\tau_2) = \sum_{\mathfrak{D}\in\mathfrak{D}_m\times\overline{\mathfrak{D}}_n} I_s(\mathfrak{D};\tau_1,\tau_2)$. Our next task is to com-550 pute $\mathbb{E}I_s(\mathfrak{D})$ for each $\mathfrak{D} \in \mathfrak{D}_m \times \overline{\mathfrak{D}}_n$. 551

Randomness enters the term $I_s(\mathfrak{D})$ via the random variables $a_{\varepsilon_i}^{(0)}, \bar{a}_{\sigma_j}^{(0)}$, 552 corresponding to the leaves of the diagram $\mathfrak{D} = \mathfrak{D}^1 \sqcup \overline{\mathfrak{D}}^2$. They are Gaussian 553 with correlations 554

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$$\mathbb{E}a_{s}^{(0)}(l_{1})a_{s'}^{(0)}(l_{2}) = 0, \quad \mathbb{E}a_{s}^{(0)}(l_{1})\bar{a}_{s'}^{(0)}(l_{2}) = \delta_{s'}^{s} \operatorname{Corr}(\gamma_{s}, b(s), l_{1}, l_{2}),$$
556 (2.13)

where 557

where

$$\operatorname{Corr}(\gamma_s, b(s), l_1, l_2) = B_s \left(e^{-\gamma_s |l_1 - l_2|} - e^{-\gamma_s (l_1 + l_2)} \right), \quad B_s = \frac{b(s)^2}{\gamma_s}.$$

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So, the Wick theorem [18] implies that the expectation $\mathbb{E}I_s(\mathfrak{D})$ is given by a 560 sum over all Wick pairings of variables $a_{\xi_i}^{(0)}$, corresponding to non-conjugated 561 leaves $c_i^{(0)}$, with variables $\bar{a}_{\sigma_j}^{(0)}$ corresponding to conjugated leaves $\bar{c}_j^{(0)}$. More-562 over, the leaves $c_i^{(0)}$ and $\bar{c}_j^{(0)}$ should belong to different blocks since otherwise 563 the summand corresponding to such Wick pairing vanishes due to (1.15) and 564 item (1) in (2.12). We parametrize the sum over the Wick pairings by the 565 defined below set $\mathfrak{F}(\mathfrak{D})$ of Feynman diagrams. Denoting by $J_s(\mathfrak{F})$ a term (i.e. 566 a sum), corresponding to a specific Feynman diagram \mathfrak{F} , we have: 567

(2.14)

$$\mathbb{E}I_s(\mathfrak{D}) = \sum_{\mathfrak{F} \in \mathfrak{F}(\mathfrak{D})} J_s(\mathfrak{F}).$$

2.3.1. Definition of Feynman diagrams. To construct the set of Feynman di-569 agrams $\mathfrak{F}(\mathfrak{D})$, corresponding to some diagram $\mathfrak{D} = \mathfrak{D}^1 \sqcup \overline{\mathfrak{D}}^2$, we consider all 570 possible partitions of the set of leaves of \mathfrak{D} to non-intersecting pairs $(c_i^{(0)}, \bar{c}_j^{(0)})$, 571 such that the paired leaves $c_i^{(0)}$ and $\bar{c}_i^{(0)}$ do not belong to the same block. To 572



FIGURE 3. **a** A Feynman diagram \mathfrak{F} obtained from the diagram \mathfrak{D} in Fig. 2(a). **b** A cycle obtained from the Feynman diagram \mathfrak{F}

each such partition, we associate a diagram \mathfrak{F} obtained from \mathfrak{D} by joining with an edge the two leaves in every pair, see Fig. 3(a). So, in each diagram $\mathfrak{F} \in \mathfrak{F}(\mathfrak{D})$ all vertices of \mathfrak{D} are joint by edges, every edge couples a conjugated vertex with a non-conjugated in another block (or with a root), and every vertex belongs to exactly one edge.

Since $\mathbb{E}a_s^{(0)}\bar{a}_{s'}^{(0)} = 0$ if $s \neq s'$, the indices ξ_i, σ_j entering the formulas for sums $J_s(\mathfrak{F})$ satisfy (2.12), where in item (2) the diagram \mathfrak{D} is replaced by the Feynman diagram \mathfrak{F} ; below we denote these relations as $(2.12)_{\mathfrak{F}}$. In particular, due to the item (2), the vector of indices $\sigma = (\sigma_i)$ is a function of the vector $\xi = (\xi_j)$. Accordingly below we write $\sigma = \sigma_{\mathfrak{F}}(\xi)$. Let

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$$\mathfrak{F}_{m,n} = \bigcup_{\mathfrak{D} \in \mathfrak{D}_m \times \overline{\mathfrak{D}}_n} \mathfrak{F}(\mathfrak{D})$$

be the set of all Feynman diagrams associated with the product $a_s^{(m)} \bar{a}_s^{(n)}$. Each 585 diagram $\mathfrak{F} \in \mathfrak{F}_{m,n}$ has N := m+n blocks and 4N+2 vertices, including 2N+2586 leaves. Half of edges (and of leaves) are conjugated, while another half is not. 587 By construction a diagram $\mathfrak{F} \in \mathfrak{F}_{m,n}$ never pairs leaves from the same 588 block. This alone does not exclude that \mathfrak{F} is such that in $(2.12)_{\mathfrak{F}}$ the assump-589 tions (1) and (2) are incompatible since for some j we may have $\xi_{2i-1}, \xi_{2i} =$ 590 σ_{2i-1} or σ_{2i} once $\sigma = \sigma_{\mathfrak{F}}(\xi)$. Analysis shows that this cannot happen if 591 $m + n \leq 4$, but may happen if $m + n \geq 5$. Accordingly, we denote by 592

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$$\mathfrak{F}_{m,n}^{true} \subset \mathfrak{F}_{m,n}$$

the set of Feynman diagrams for which the set of indices ξ_i , σ_j satisfying the relations $(2.12)_{\mathfrak{F}}$ is not empty. For any diagram $\mathfrak{F} \notin \mathfrak{F}_{m,n}^{true}$, we have $J_s(\mathfrak{F}) = 0$ due to the factors $\delta_{34}^{\prime 12}$ and $\delta_{s_4}^s$ in (2.4).

597 2.4. Transformation, Resolving Linear Relations on Indices

Let us take a Feynman diagram $\mathfrak{F} \in \mathfrak{F}_{m,n}$, denote $N := m+n \ge 1$ and consider the sum $J_s(\mathfrak{F})$. The relations $(2.12)_{\mathfrak{F}}$ on indices $\xi_i, \sigma_j \in \mathbb{Z}_L^d, \ 0 \le i, j \le 2N, 9$

⁹Recall that $\xi_0 = \sigma_0 = s$.

entering the formula for $J_s(\mathfrak{F})$ are involved, which makes the sum $J_s(\mathfrak{F})$ difficult for further analysis. In [8] it was found a convenient way to "integrate the sums $J_s(\mathfrak{F})$ ", i.e. to parametrize the indices ξ_i, σ_j by N-vector $z = (z_1, \ldots, z_N)$ from a domain in $(\mathbb{Z}_L^d)^N$, free from any relations on its components. In this section we present this parametrization, referring the reader to [8] for a proof.

Since item (2) of $(2.12)_{\mathfrak{F}}$ is equivalent to the relation $\sigma = \sigma_{\mathfrak{F}}(\xi)$, it suffices to parametrize the set of *admissible* multi-indices ξ , i.e. of those ξ for which the multi-indices ξ and $\sigma = \sigma_{\mathfrak{F}}(\xi)$ satisfy item (1) in $(2.12)_{\mathfrak{F}}$. The construction starts with defining for each $\mathfrak{F} \in \mathfrak{F}_{m,n}$ a skew-symmetric $N \times N$ incidence matrix $\alpha^{\mathfrak{F}} = (\alpha_{ij}^{\mathfrak{F}})$ whose elements are integers from the set $\{0, +1, -1\}$. In terms of this matrix, we define the set of polyvectors

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$$\mathcal{Z}(\mathfrak{F}) = \{ z = (z_1, \dots, z_N) \in (\mathbb{Z}_L^d)^N : z_j \neq 0 \text{ and } (\alpha^{\mathfrak{F}} z)_j \neq 0 \quad \forall j \}.$$
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$$(2.15)$$

Here and below for an $M \times N$ -matrix A we denote by Az the polyvector with components $(Az)_j := \sum_i A_{ji} z_i \in \mathbb{Z}^d$.¹⁰ The matrix $\alpha^{\mathfrak{F}}$ has no zero rows and zero columns if and only if $\mathfrak{F} \in \mathfrak{F}_{m,n}^{true}$, and, accordingly, the set $\mathcal{Z}(\mathfrak{F})$ is nonempty if and only if $\mathfrak{F} \in \mathfrak{F}_{m,n}^{true}$. Next, it turns out that the vectors $z \in \mathcal{Z}(\mathfrak{F})$ may be used to parametrize the set of admissible indices ξ_i by means of an affine mapping

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$$\xi(z) = s + A^{\mathfrak{F}}z, \qquad z \in \mathcal{Z}(\mathfrak{F}). \tag{2.16}$$

Here $A^{\mathfrak{F}}$ is an $(2N+1) \times N$ -matrix, whose elements again are integers from the set $\{0, +1, -1\}$. Transformation (2.16) provides a presentation of the terms $J_s(\mathfrak{F})$, forming the correlation $\mathbb{E}a_s^{(m)}(\tau_1)\bar{a}_s^{(n)}(\tau_2)$, and so for the correlation itself. The corresponding result is proved in Theorem 5.5 of [8]. In our setting ts statement, where for the function $\theta(x,t)$ is chosen $\mathbb{I}_{\{0\}}(x)$ —the indicator function of the point x = 0—takes the following form:

Lemma 2.2. For any integers $m, n \ge 0$ satisfying $N = m + n \ge 1$, any $s \in \mathbb{Z}_L^d$ and $\tau_1, \tau_2 \ge 0$,

(1) for each $\mathfrak{F} \in \mathfrak{F}_{m,n}^{true}$ parametrization (2.16) (depending on s and \mathfrak{F}) is such that the quantity ω_j in (2.6), written in the z-coordinates, takes the form

$$\omega_j^{\mathfrak{F}}(z) = 2z_j \cdot \sum_{i=1}^N \alpha_{ji}^{\mathfrak{F}} z_i = 2z_j \cdot (\alpha^{\mathfrak{F}} z)_j.$$
(2.17)

632 (2) We have

$$L^{N}\mathbb{E}a_{s}^{(m)}(\tau_{1})\bar{a}_{s}^{(n)}(\tau_{2}) = \sum_{\mathfrak{F}\in\mathfrak{F}_{m,n}^{true}} c_{\mathfrak{F}}J_{s}(\tau_{1},\tau_{2};\mathfrak{F}), \qquad (2.18)$$

¹⁰That is, abusing notation we denote by A an operator in $(\mathbb{Z}^d)^M$ with the block-matrix $A \otimes \mathbb{1}$.

634 where the constants $c_{\mathfrak{F}} \in \{\pm 1, \pm i\}$ and

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$$J_{s}(\tau_{1},\tau_{2};\mathfrak{F}) = \int_{\mathbb{R}^{N}} \mathrm{d}l \, L^{N(1-d)} \sum_{z \in \mathcal{Z}(\mathfrak{F}), \ \omega_{j}^{\mathfrak{F}}(z) = 0 \ \forall j} F_{s}^{\mathfrak{F}}(\tau_{1},\tau_{2},l,z). \quad (2.19)$$

The density $F_s^{\mathfrak{F}}(\tau_1, \tau_2, l, z)$ is a real function, smooth in $(s, z) \in \mathbb{R}^d \times \mathbb{R}^{dN}$, satisfying

$$|\partial_s^{\mu} \partial_z^{\kappa} F_s^{\mathfrak{F}}(\tau_1, \tau_2, l, z)| \le C_{\mu,\kappa}^{\#}(s) C_{\mu,\kappa}^{\#}(z) e^{-\delta \left(\sum_{i=1}^m |\tau_1 - l_i| + \sum_{i=m+1}^N |\tau_2 - l_i|\right)}$$

$$(2.20)$$

640 with a suitable $\delta = \delta_N > 0$, for any vectors $\mu \in (\mathbb{N} \cup \{0\})^d$, $\kappa \in (\mathbb{N} \cup \{0\})^{dN}$ and any $s \in \mathbb{R}^d$, $z \in \mathbb{R}^{dN}$, $l \in \mathbb{R}^N$.

Let us briefly explain the way to construct the parametrization (2.16). We 642 first add to the Feynman diagram \mathfrak{F} dashed edges that couple non-conjugated 643 vertices with conjugated *inside* all blocks, as in Fig. 3(b). For each block there 644 are two ways of doing that. We prove that there exists a choice (possibly, 645 not unique) of a dashed edge in each block such that the diagram becomes 646 a cycle, as in Fig. 3b. Then, for each j we set $x_{2i-1} := \xi_{2i-1} - \sigma_{2i-1}$ and 647 $x_{2j} := \xi_{2j} - \sigma_{2j}$ or $x_{2j-1} = \xi_{2j-1} - \sigma_{2j}$ and $x_{2j} := \xi_{2j} - \sigma_{2j-1}$, according to 648 the choice of the dashed edges in the *j*-th block, where we substitute $\sigma = \sigma_{\mathfrak{F}}(\xi)$. 649 The fact that the Feynman diagram with added dashed edges forms a cycle 650 implies that the transformation $\xi \mapsto x$ is invertible. Item (1) of $(2.12)_{\mathfrak{F}}$ implies 651 that $x_{2i} = -x_{2i-1}$. Then we set $z_i := x_{2i-1}$ and get (2.16). The incidence 652 matrix $\alpha^{\mathfrak{F}}$ also is constructed in terms of this cycle. 653

Since the choice of the dashed edges in general is not unique, the parametrization $z \mapsto \xi$ is not unique as well. However, if $z' \mapsto \xi$ is another parametrization, obtained by the procedure above, and $\alpha^{\mathfrak{F}'}$ is the associated incidence matrix, then for each j we have either $z'_j(\xi) = z_j(\xi)$ or $z'_j(\xi) = (\alpha^{\mathfrak{F}} z(\xi))_j$. In the latter case we also have the symmetric relation $z_j(\xi) = (\alpha^{\mathfrak{F}'} z'(\xi))_j$.

Computing in (2.19) the integral over dl and using estimate (2.20), we obtain a form of integrals J_s , more convenient for some of the subsequent analysis:

Corollary 2.3. In terms of Lemma 2.2, the integrals J_s from (2.18) can be written as

$$J_s(\tau_1, \tau_2; \mathfrak{F}) = L^{N(1-d)} \sum_{z \in \mathcal{Z}(\mathfrak{F}), \ \omega_j^{\mathfrak{F}}(z) = 0 \ \forall j} \Phi_s^{\mathfrak{F}}(\tau_1, \tau_2, z),$$
(2.21)

where the real-valued functions $\Phi_s^{\mathfrak{F}}$ are Schwartz in (s, z) and satisfy

$$|\partial_s^{\mu} \partial_z^{\kappa} \Phi_s^{\mathfrak{F}}(\tau_1, \tau_2, z)| \le C_{\mu,\kappa}^{\#}(s) C_{\mu,\kappa}^{\#}(z), \qquad (2.22)$$

667 uniformly in $\tau_1, \tau_2 \geq 0$, for any vectors $\mu \in (\mathbb{N} \cup \{0\})^d$ and $\kappa \in (\mathbb{N} \cup \{0\})^{dN}$.

668 3. Main Estimates for The Sums

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In this section we focus on estimates for the sums (2.21) and on their dependence on L and N. We recall that $d \ge 3$. It is convenient to study the problem we consider in the following abstract setting. Let $\alpha = (\alpha_{ij}), N \ge 2$, be an $N \times N$ skew-symmetric matrix whose elements belong to the set $\{-1, 0, 1\}$, without zero lines and rows.¹¹ Consider a family of quadratic forms on $(\mathbb{R}^d)^N$

$$\omega_j(z) = z_j \cdot (\alpha z)_j, \qquad 1 \le j \le N,$$

where z is the polyvector (z_1, \ldots, z_N) , $z_j \in \mathbb{R}^d$, and $(\alpha z)_j := \sum_{i=1}^N \alpha_{ji} z_i$. Let us set

$$\mathcal{Z} = \{ z \in (\mathbb{Z}_L^d)^N : z_j \neq 0 \quad \text{and} \quad (\alpha z)_j \neq 0 \quad \forall j \}.$$
(3.1)

Let a function $\Phi : \mathbb{R}^{Nd} \to \mathbb{R}$ be sufficiently smooth and sufficiently fast decaying at infinity (see below for exact assumptions). Our goal is to study asymptotic as $L \to \infty$ behaviour of the sum

$$S_{L,N}(\Phi) := L^{N(1-d)} \sum_{z \in \mathcal{Z}: \ \omega_j(z) = 0 \ \forall j} \Phi(z).$$
(3.2)

For a function $f \in C^k(\mathbb{R}^m)$, $n_1 \in \mathbb{N} \cup \{0\}$ satisfying $n_1 \leq k$ and $n_2 \in \mathbb{R}$, we set

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$$||f||_{n_1,n_2} = \sup_{z \in \mathbb{R}^m} \max_{|\alpha| \le n_1} |\partial^{\alpha} f(z)| \langle z \rangle^{n_2}, \qquad \langle x \rangle := \max\{1, |x|\}.$$
(3.3)

The first crucial result concerns the case N = 2. Then $\omega_1(z) = -\omega_2(z) = \alpha_{12}z_1 \cdot z_2$ and $\alpha_{12} \neq 0$, so

$$S_{L,2}(\Phi) = L^{2(1-d)} \sum_{\substack{z \in \mathbb{Z}_{L}^{2d}: \ z_{1} \cdot z_{2} = 0\\ z_{1} \neq 0, \ z_{2} \neq 0}} \Phi(z).$$
(3.4)

688 Then we write the sum above as $\sum_{z_1 \cdot z_2 = 0} - \sum_{z_1 = 0 \text{ or } z_2 = 0}$. 689 Since $\left| L^{-d} \sum_{z \in \mathbb{Z}_L^{2d}: z_i = 0} \Phi(z) \right| \le C \|\Phi\|_{0, d+1}$ for i = 1, 2, we get

$$\left| S_{L,2}(\Phi) - L^{2(1-d)} \sum_{z \in \mathbb{Z}_{L}^{2d}: \ z_{1} \cdot z_{2} = 0} \Phi(z) \right| \le CL^{2-d} \|\Phi\|_{0,d+1}.$$
(3.5)

Now an asymptotic for the sum $S_{L,2}(\Phi)$ immediately follows from Theorem 1.3 in [10] where the dimension is 2d, $\varepsilon = 1/2$ and m = 0, by applying it to the sum $\sum_{z_1 \cdot z_2 = 0}$ in (3.5) (we recall that $d \geq 3$):

Theorem 3.1. Let $N_1(d) := 4d(4d^2 + 2d - 1)$ and $N_2(d) := N_1 + 6d + 4$. If $\|\Phi\|_{N_1,N_2} < \infty$, then there exist constants $C_d \in (1, 1 + 2^{2-d})$ and $K_d > 0$ such that

$$\left| S_{L,2}(\Phi) - C_d \int_{\Sigma_0} \Phi(z) \, \mu^{\Sigma_0}(\mathrm{d} z_1 \mathrm{d} z_2) \right| \le K_d \frac{\|\Phi\|_{N_1,N_2}}{L^{d-5/2}},\tag{3.6}$$

¹¹ The theorems below and their proofs remain valid as well for arbitrary skew-symmetric matrices with integer elements without zero lines and rows, but in this case the notation used in the proof becomes heavier.

where Σ_0 is the quadric $\{z \in \mathbb{R}^{2d} : z_1 \cdot z_2 = 0\}$ and the measure μ^{Σ_0} is given by (1.34) with s = 0.

In Appendix C of [10] we give the following explicit formula for the number-theoretical constants C_d :

$$C_d = \frac{\zeta(d-1)\zeta(4d-2)}{\zeta(d)\zeta(2d-2)},$$
(3.7)

where ζ is the Riemann zeta-function. Due to (3.7) C_d satisfies $1 < C_d < 1 + 2^{2-d}$, as is stated in the theorem. The integral in (3.6) converges if $\Phi(z)$ decays at infinity fast enough:

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$$\left| \int_{\Sigma_0} \Phi(z) \, \mu^{\Sigma_0}(\mathrm{d} z_1 \mathrm{d} z_2) \right| \le C_r \|\Phi\|_{0,r} \quad \text{if } r > 2d - 1, \tag{3.8}$$

⁷⁰⁷ see Proposition 3.5 in [7]. So, it converges under the theorem's assumptions.

From Theorem 3.1 another result can be deduced, whose proof is given in the next Sect. 3.1:

Theorem 3.2. For N = 2, 3, 4 there exist constants $C_{d,N}$ such that

$$|S_{L,N}(\Phi)| \le C_{d,N} \|\Phi\|_{0,\bar{N}},$$
(3.9)

for
$$\bar{N} := \lfloor N/2 \rfloor N_2(d) + (N-2)(d-1) + 1$$
, where N_2 is defined in Theorem 3.1.

Since in view of estimate (2.22) the functions $\Phi_s^{\mathfrak{F}}$ from Corollary 2.3 satisfy

$$\|\Phi_s^{\mathfrak{F}}(\tau_1, \tau_2, \cdot)\|_{n_1, n_2} \le C_{n_1, n_2}^{\#}(s), \qquad \forall n_1, n_2, \tag{3.10}$$

then the two theorems above apply to study correlations (2.18) with $N = m + n \leq 4$. In fact, in the case N = 2 the number of Feynmann diagrams is small and the corresponding correlations may be calculated directly without the machinery, developed in Sect. 2. In Example 3.4 which illustrates this computation, as well as in a number of situations below, we apply Theorem 3.1 in the following setting:

722 Corollary 3.3. Let

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$$\mathcal{S}_{L,2} = L^{2(1-d)} \sum_{1,2,3} \delta_{3s}^{\prime 12} \delta(\omega_{3s}^{12}) f_s(s_1, s_2, s_3; q),$$

where ω_{3s}^{12} is given by (1.17), $q \in \mathbb{R}^n$ is a parameter (in applications usually this will be the time) and $f_s(s_1, s_2, s_1 + s_2 - s; q)^{12}$ is a Schwartz function of (s_1, s_2, s) satisfying $|\partial_{(s_1, s_2, s)}^{\mu} f_s| \leq C_{\mu}^{\#}(s_1)C_{\mu}^{\#}(s_2)C_{\mu}^{\#}(s)$ uniformly in q, for any multi-index μ . Then

$$\left|\mathcal{S}_{L,2} - C_d \int_{\Sigma_s} f_s(s_1, s_2, s_1 + s_2 - s; q) \, \mu^{\Sigma_s}(\mathrm{d}s_1 \mathrm{d}s_2)\right| \le \frac{C^\#(s)}{L^{d-5/2}}, \quad (3.11)$$

uniformly in q, where Σ_s and μ^{Σ_s} are the quadric (1.33) and the measure (1.34) on it.

 $^{12} \mathrm{The}$ formula for s_3 comes from the relation $\delta_{3s}^{\prime 12} = 1.$

Proof. In the variables $z_1 = s_1 - s$, $z_2 = s_2 - s$ the quadratic form ω_{3s}^{12} with 731 $s_3 = s_1 + s_2 - s$ reads $\omega_{3s}^{12} = -2z_1 \cdot z_2$ [see (1.17)]. Then, taking into account that 732 the relation $\delta_{3s}^{\prime 12} = 1$ is equivalent to the relations $z_1, z_2 \neq 0$ and $s_3 = s_1 + s_2 - s_3$, 733 we find that the sum $S_{L,2}$ takes the form (3.4). Applying next Theorem 3.1 734 and changing in (3.6) back to the variables s_1, s_2 , we get (3.11). 735

Example 3.4. Let us calculate the asymptotic as $L \to \infty$ of $\mathbb{E}|a_s^{(1)}(\tau)|^2$. Ex-736 panding $a_s^{(1)}$ as in (2.4) and then using (2.13), we get: 737

$$\mathbb{E}|a_s^{(1)}(\tau)|^2 = 2L^{-2d} \sum_{1,2,3} \delta_{3s}^{\prime 12} \delta(\omega_{3s}^{12}) \int_0^{\tau} \mathrm{d}l_1 \int_0^{\tau} \mathrm{d}l_2 B_{123}$$
$$\times \prod_{j=1}^3 \left(e^{-\gamma_j |l_1 - l_2|} - e^{-\gamma_j (l_1 + l_2)} \right) e^{\gamma_s (l_1 + l_2 - 2\tau)}$$

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with $B_{123} = B_1 B_2 B_3$, where B_s is defined in (2.14). In the case of $\tau = \infty$ the 739 formula simplifies since by changing the integration variables as $r_j := \tau - l_j$ 740 and passing to the limit we get 741

$$\mathbb{E}|a_s^{(1)}(\infty)|^2 = 2L^{-2d} \sum_{1,2,3} \delta_{3s}^{\prime 12} \delta(\omega_{3s}^{12}) \int_0^\infty \mathrm{d}r_1 \int_0^\infty \mathrm{d}r_2$$
$$B_{123} e^{-(\gamma_1 + \gamma_2 + \gamma_3)|r_1 - r_2|} e^{-\gamma_s(r_1 + r_2)}$$
$$= \frac{2L^{-2d}}{\gamma_s} \sum_{1,2,3} \delta_{3s}^{\prime 12} \delta(\omega_{3s}^{12}) \frac{B_{123}}{\gamma_1 + \gamma_2 + \gamma_3 + \gamma_s}$$

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Then, by Corollary 3.3, 743

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$$\left| L^2 \mathbb{E} |a_s^{(1)}(\infty)|^2 - \frac{2C_d}{\gamma_s} \int_{\Sigma_s} \frac{B_{123} \ \mu^{\Sigma_s} (\mathrm{d}s_1 \mathrm{d}s_2)}{\gamma_1 + \gamma_2 + \gamma_3 + \gamma_s} \right| \le \frac{C^\#(s)}{L^{d-5/2}}, \quad s_3 := s_1 + s_2 - s.$$

3.1. Proof of Theorem 3.2 745

Let us define the geometric quadrics $Q_j := \{z \in (\mathbb{R}^d)^N : \omega_j(z) = 0\}$ and 746 consider their intersection $Q = \bigcap_{j=1}^{N} Q_j$. Note that $Q = \bigcap_{j=1}^{N-1} Q_j$ since the 747 skew symmetry of the matrix α implies $\omega_1 + \ldots + \omega_N = 0$. Denote by B_R^{Nd} the open cube $|z|_{\infty} < R$ in \mathbb{R}^{Nd} , where by $|\cdot|_{\infty}$ we denote the l_{∞} -norm. 748 749

Proposition 3.5. If $w : \mathbb{R}^{Nd} \mapsto \mathbb{R}$ is such that $|w|_{L_{\infty}} < \infty$ and $\operatorname{supp}(w) \subset B_{R}^{Nd}$, 750 where $R \ge 1$, then for N = 2, 3, 4 we have 751

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$$\sum_{z \in Q \cap \mathcal{Z}} w(z) \leq C(N, d) R^{\lfloor N/2 \rfloor N_2(d) + (N-2)(d-1)} L^{N(d-1)} |w|_{L_{\infty}}.$$
 (3.12)

Here N_2 is defined in Theorem 3.1 and \mathbb{Z} —in (3.1). 753

Proof. Below in this proof for any subset $\mathcal{Q} \subset \mathbb{R}^{md}$, we denote 754

$$\mathcal{Q}_L = \mathcal{Q} \cap \mathbb{Z}_L^{md}. \tag{3.13}$$

(3.14)

By suitably rearranging indices i and possibly multiplying ω_i by -1, ω_1 may 756 be assumed to be of the form $\omega_1(z) = z_1 \cdot \sum_i \alpha_{1i} z_i$ with $\alpha_{1N} = 1$. Define 757 $v = \sum_{i} \alpha_{1i} z_i$ so that 758

 $\omega_1(z) = z_1 \cdot v$ and $z_N = \alpha_{1N} z_N = v - \sum_{1 \le i \le N} \alpha_{1i} z_i$,

since $\alpha_{11} = 0$ by the skew symmetry of the matrix α . 760

For N > 2, fix $(z_1, v) \in \mathbb{R}^{2d}$. Then the remaining quadratic forms ω_j with 1 < j < N as functions of $(z_2, \ldots, z_{N-1}) \in \mathbb{R}^{(N-2)d}$ become polynomials q_j of 761 762 degree at most two, with no constant term. Namely 763

764
$$q_j(z_2, \dots, z_{N-1}; z_1, v) = z_j \cdot \left(\alpha_{j1} z_1 + \alpha_{jN} v + \sum_{1 < i < N} (\alpha_{ji} - \alpha_{jN} \alpha_{1i}) z_i \right).$$
765 (3.15)

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For 1 < j < N consider the sets 766

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$$\tilde{Q}_j(z_1, v) = \{(z_2, \dots, z_{N-1}) : q_j(z_2, \dots, z_{N-1}; z_1, v) = 0\} \subset \mathbb{R}^{(N-2)d}$$

and their intersection $\tilde{Q}(z_1, v) = \bigcap_{1 \le j \le N} \tilde{Q}_j(z_1, v)$. We denote $Q_1^0 = \{(z_1, v) \in Q_1^0 \}$ 768 \mathbb{R}^{2d} : $z_1 \cdot v = 0$ } (cf. (3.14)) and set 769

$$A_2 = \{ (z_1, v) \in \mathbb{R}^{2d} : z_1 \neq 0, v \neq 0 \}.$$
(3.16)

Since $|\alpha_{ij}| \leq 1$, then on the support of w we have $|(z_1, v)|_{\infty} \leq (N-1)R$. So, 771 recalling (3.13), for N > 2 we get 772

$$\sum_{z \in \mathcal{Z} \cap Q} w(z) \leq C(N,d) |w|_{L_{\infty}} \sum_{(z_1,v) \in Q_1^0 \cup \cap B_{(N-1)R}^{2d}} 1 \\
\times \sup_{(z_1,v) \in Q_1^0 \cup \cap A_2 \cap B_{(N-1)R}^{2d}} \sum_{(z_2,\dots,z_{N-1}) \in \tilde{Q}_L(z_1,v) \cap B_R^{(N-2)d}} 1.$$
774
$$(3.17)$$

For N = 2 the same estimate holds with the second line replaced by 1. 775

To estimate the sum in the first line, we take any smooth function $w_0(x) \geq 0$ 776 0, equal one for $x \leq 1$ and vanishing for $x \geq 2$. Then 777

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$$\sum_{(z_1,v)\in Q_{1\,L}^0\cap B_{(N-1)R}^{2d}} 1 \le \sum_{(z_1,v)\in Q_{1\,L}^0} w_R(z_1,v)$$

where $w_R(z_1, v) := w_0 \left((|(z_1, v)|/(N-1)R\sqrt{2d}) \right)$. Since for $R \ge 1$ and any $a \in$ 779 $\mathbb{N} \cup \{0\}, b \geq 0$ we have $||w_R||_{a,b} \leq C(a, b, N, d)R^b$, then in view of Theorem 3.1 780 and (3.8), 781

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$$\sum_{(z_1,v)\in Q_{1L}^0\cap B_{(N-1)R}^{2d}} 1 \le CL^{2(d-1)} \left[R^{2d} + R^{N_2}L^{-d+5/2} \right] \le C'L^{2(d-1)}R^{N_2},$$
(3.18)

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where C, C' depend on d, N, N_1 and N_2 . 784

To estimate the second line of (3.17), we use the following lemma, proved in Appendix A.

Lemma 3.6. Assume that the matrix α is irreducible. Then for N = 2, 3, 4, any $R \geq 1$ and any $(z_1, v) \in B^{2d}_{(N-1)R}$ satisfying $(z_1, v) \in Q^0_{1L} \cap A_2$ we have:

$$\left|\tilde{Q}_L(z_1, v) \cap B_R^{(N-2)d}\right| \le 2^{(N-2)d} (NRL)^{(N-2)(d-1)}.$$
(3.19)

This completes the proof of Proposition 3.5 in the case of irreducible matrix α : indeed, we get

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$$\left| \sum_{z \in \mathcal{Z} \cap Q} w(z) \right| \le C(N,d) |w|_{L_{\infty}} R^{N_2 + (N-2)(d-1)} L^{N(d-1)}.$$
(3.20)

If the matrix α is reducible, it can be reduced through permutations to a block diagonal matrix with m blocks which are irreducible square matrices of sizes N_i satisfying $\sum_i N_i = N$. Since $N_i \ge 2$ (otherwise there would be a zero row or column in α), $m \le \lfloor N/2 \rfloor$. Applying estimate (3.20) to each block, we get the assertion of the proposition.

Now we derive the theorem from the proposition. Let $\varphi_0(t) = \chi_{(-\infty,1]}(t)$ and for $k \ge 1$, $\varphi_k(t) = \chi_{(2^{k-1},2^k]}(t)$. Then $1 = \sum_k \varphi_k(t)$ and

$$\Phi = \sum_{k=0}^{\infty} f_k(z), \quad f_k(z) = \varphi_k(|z|_{\infty})\Phi(z).$$

Then supp $f_k \subset B_k = \{|z|_{\infty} \leq 2^k\}$ and $||f_k||_{\infty} \leq C2^{-k\bar{N}} ||\Phi||_{0,\bar{N}}$, for any \bar{N} . Therefore, by Proposition 3.5,

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$$|S_{L,N}(\Phi)| \le C(N,d) \|\Phi\|_{0,\bar{N}} \sum_{k=0}^{\infty} 2^{k(\lfloor N/2 \rfloor N_2 + (N-2)(d-1) - \bar{N})},$$

which converges if $\overline{N} > \lfloor N/2 \rfloor N_2 + (N-2)(d-1)$. This completes the proof of Theorem 3.2.

Remark 3.7. For any fixed vector (z_1, v) , $\tilde{Q}(z_1, v)$ is a real algebraic set in $\mathbb{R}^{(N-2)d}$ of codimension (N-2). If $\tilde{Q}(z_1, v)$ were a smooth manifold of that codimension, then estimate (3.19), modified by a multiplicative constant $C_{\tilde{Q}(z_1,v)}$, would be obvious. But $\tilde{Q}(z_1, v)$ is a stratified analytic manifold (with singularities), and to obtain for it a modified version of the estimate (3.19) as above, using analytical tools, seems to be a heavy job since we need a good control for the factor $C_{\tilde{Q}(z_1,v)}$. Instead in Appendix A we prove the lemma, using arithmetical tools.

814 3.2. On Extension of Theorem 3.2 to any N

The restriction on N in the statement of Theorem 3.2 comes from estimate (3.19) in Lemma 3.6, proved only for N = 3, 4. We know that for every N the system of polynomials $q_j(\cdot; z_1, v), 1 < j < N$, defining the set $\tilde{Q}_L(z_1, v)$ in Lemma 3.6, is linearly independent for any (z_1, v) and any irreducible incidence matrix α . Also we know that all polynomials $q_j(\cdot; z_1, v)$ are irreducible;

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see Lemmas A.9 and A.10 in Appendix A (there the independence and reducibility are understood over some specific algebraically closed field K, but the argument also works for K replaced by \mathbb{C}). These two facts certainly are insufficient to prove Lemma 3.6 for any N, but they naturally lead to

Conjecture 3.8. Under assumptions of Lemma 3.6, for any $N \ge 2$

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$$\left|\tilde{Q}_L(z_1,v) \cap B_R^{(N-2)d}\right| \le C(N,d)(RL)^{(N-2)(d-1)}.$$

One may try to prove this assertion using either arithmetical or analytical tools; cf. Appendix A and Remark 3.7. It is straightforward to see that, if the conjecture is true, then Theorem 3.2 holds for any N, so in view of Lemma 2.2 any expected value $L^N \mathbb{E} a_s^{(m)}(\tau_1) \bar{a}_s^{(n)}(\tau_2)$ admits a uniform in L upper bound.

4. Quasisolutions

In this section we start to study a quasisolution $A(\tau) = A(\tau; L)$ of Eq. (2.1) with $a_s(0) = 0$, which is the second-order truncations of series (2.2):

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$$A(\tau) = (A_s(\tau), s \in \mathbb{Z}_L^d), \qquad A_s(\tau) = a_s^{(0)}(\tau) + \rho a_s^{(1)}(\tau) + \rho^2 a_s^{(2)}(\tau).$$
834 (4.1)

835 We focus on its energy spectrum

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$$n_{s,L} = n_{s,L}(\tau) = \mathbb{E}|A_s(\tau)|^2, \quad s \in \mathbb{Z}_L^d,$$

$$(4.2)$$

when L is large and the parameter ρ is chosen to be $\rho = \varepsilon L$. Our goal is to show that it approximately satisfies a wave kinetic equation (WKE). Using Proposition 2.1, we will then show that the same applies to the quantities $\mathbf{n}_{s,L}$, considered in the Introduction.

The energy spectrum $n_{s,L}$ is a polynomial in ε of degree four,

$$n_{s,L} = n_{s,L}^{(0)} + \varepsilon \, n_{s,L}^{(1)} + \varepsilon^2 n_{s,L}^{(2)} + \varepsilon^3 n_{s,L}^{(3)} + \varepsilon^4 n_{s,L}^{(4)}, \quad s \in \mathbb{Z}_L^d, \tag{4.3}$$

where the terms $n_{s,L}^{(k)}(au)$ are defined by

$$n_{s,L}^{(k)}(\tau) = L^k \sum_{\substack{k_1+k_2=k\\0\le k_1, k_2\le 2}} \mathbb{E}a_s^{(k_1)}(\tau)\bar{a}_s^{(k_2)}(\tau).$$
(4.4)

845 By Corollary 2.3,

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the second moments $\mathbb{E}a_s^{(k_1)}\bar{a}_s^{(k_2)}$ naturally extend to a Schwartz function of $s \in \mathbb{R}^d$, (4.5)

given by (2.18), (2.21). Accordingly, from now on we always regard the second moments and the terms $n_{s,L}^{(k)}(\tau)$ as Schwartz functions of $s \in \mathbb{R}^d$.

As customary in WT, we aim at considering the limit of $n_{s,L}(\tau)$ as $L \rightarrow \infty$, that is, the limits of the terms $n_{s,L}^{(j)}$. The term $n_{s,L}^{(0)} = \mathfrak{n}_{s,L}^{(0)}$ is given by (1.31) and is *L*-independent, while by a direct computation we see that

$$n_{s,L}^{(1)} = 2\mathfrak{R}\mathbb{E}\bar{a}_s^{(0)}a_s^{(1)} = 0, \quad s \in \mathbb{R}^d,$$
(4.6)

[here we use (2.4), the Wick theorem and (1.15)]. Writing explicitly $n_{s,L}^{(i)}$ with 853 $2 \leq i \leq 4$, we find that 854

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$$n_{s,L}^{(2)} = L^2 \mathbb{E} \left(|a_s^{(1)}|^2 + 2 \Re \bar{a}_s^{(0)} a_s^{(2)} \right),$$

$$n_{s,L}^{(3)} = 2L^3 \Re \mathbb{E} \bar{a}_s^{(1)} a_s^{(2)}, \qquad n_{s,L}^{(4)} = L^4 \mathbb{E} |a_s^{(2)}|^2.$$
(4.7)

The function $\mathbb{R}^d \ni s \mapsto n_{s,L}^{(2)}(\tau)$ is made by two terms. By Corollary 2.3 with 856 N = 2, Theorem 3.1 applies to the both of them. Since $d \ge 3$, we get 857

$$|n_{s,L}^{(2)}(\tau) - n_s^{(2)}(\tau)| \le C^{\#}(s)/L^{1/2},$$
(4.8)

where 859

$$n_s^{(2)}(\tau) := C_d \left(\sum_{\mathfrak{F} \in \mathfrak{F}_{1,1}^{true}} + 2\mathfrak{R} \sum_{\mathfrak{F} \in \mathfrak{F}_{2,0}^{true}} \right) c_{\mathfrak{F}} \int_{\Sigma_0} \mu^{\Sigma_0} (\mathrm{d} z_1 \mathrm{d} z_2) \Phi_s^{\mathfrak{F}}(\tau, \tau, z),$$

and we have used estimate (3.10). Thus, we see that the processes $n_{s,L}^{(0)}$, $n_{s,L}^{(1)}$ 861 and $n_{s,L}^{(2)}$ admit the limits 862

$$n_s^{(j)}(\tau) := \lim_{L \to \infty} n_{s,L}^{(j)}(\tau;L).$$

The limits satisfy (4.8), and for all τ 864

 $\langle \alpha \rangle$

$$n_{s}^{(0)}(\tau) = B(s) \left(1 - e^{-2\gamma_{s}(\tau_{0} + \tau)} \right), \quad n_{s}^{(1)}(\tau) = 0, \quad |n_{s}^{(2)}(\tau)| \leq C^{\#}(s) ,$$

$$(4.9)$$

where the last inequality follows from Theorem 3.2. 867

We do not know if the terms $n_{s,L}^{(3)}, n_{s,L}^{(4)}$ admit limits as $L \to \infty$, but in 868 view of Corollary 2.3 both of them may be estimated through Theorem 3.2: 869

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$$|n_{s,L}^{(3)}(\tau)| \le C^{\#}(s), \quad |n_{s,L}^{(4)}(\tau)| \le C^{\#}(s), \quad (4.10)$$

uniformly in $L \ge 2$ and $\tau \ge 0$. We then decompose 871

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$$n_{s,L} = n_{s,L}^{\leq 2} + n_{s,L}^{\geq 3},$$

where 873

$$n_{s,L}^{\leq 2} = n_{s,L}^{(0)} + \varepsilon n_{s,L}^{(1)} + \varepsilon^2 n_{s,L}^{(2)} \quad \text{and} \quad n_{s,L}^{\geq 3} = \varepsilon^3 n_{s,L}^{(3)} + \varepsilon^4 n_{s,L}^{(4)}$$

(we recall that $n_{s,L}^{(1)} \equiv 0$), and similarly define 875

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$$n_s^{\leq 2} := n_s^{(0)} + \varepsilon^2 n_s^{(2)}$$

Due to (4.8), 877

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$$|n_s^{\le 2}(\tau) - n_{s,L}^{\le 2}(\tau)| \le C^{\#}(s)\varepsilon^2 L^{-1/2},$$
(4.11)

so by (4.10), 879

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$$|n_s^{\le 2}(\tau) - n_{s,L}(\tau)| \le C^{\#}(s)\varepsilon^2(L^{-1/2} + \varepsilon).$$
(4.12)

Thus, the cut energy spectrum $n_s^{\leq 2}$ governs the limiting as $L \to \infty$ behaviour 881 of the energy spectrum $n_{s,L}$ with precision $\varepsilon^3 C^{\#}(s)$, where we regard the 882 constant $\varepsilon \leq 1/2$ (which measures the size of solutions for (1.6) under the 883

proper scaling) as a fixed small parameter. Accordingly, our next goal is to show that $n_s^{\leq 2}(\tau)$ approximates the solution of a WKE.

4.1. Increments of the Energy Spectra $n_s^{\leq 2}$

In this section we will show that the process $n_s^{\leq 2}(\tau)$ approximately satisfies a WKE. We denote $s_4 := s, \gamma_j := \gamma_{s_j}$ and set

$$\gamma_{1234} = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4, \quad \vec{s} = (s_1, s_2, s_3, s_4) \in (\mathbb{R}^d)^4.$$
(4.13)

Now, for a fixed $\tau_0 \ge 0$ and for j = 1, 2, 3, 4 we define the functions $\mathcal{Z}^j(\tau_0) = \mathcal{Z}^j(\tau_0; \vec{s})$ as

$$\mathcal{Z}^{j}(\tau_{0};\vec{s}) := \int_{0}^{\tau_{0}} \mathrm{d}l \,\mathrm{e}^{-\gamma_{j}(\tau_{0}-l)} \prod_{\substack{m=1,2,3,4\\m\neq j}} \frac{\sinh(\gamma_{m}l)}{\sinh(\gamma_{m}\tau_{0})} \quad \text{if } \tau_{0} > 0, \quad (4.14)$$

and $\mathcal{Z}^{j}(0; \vec{s}) = 0$. Computing this integral, we get

$$\mathcal{Z}^{j}(\tau_{0};\vec{s}) = \left(\prod_{l\neq j} \frac{1}{1 - e^{-2\gamma_{l}\tau_{0}}}\right) \cdot \left[\frac{1 - e^{-\gamma_{1234}\tau_{0}}}{\gamma_{1234}} - \frac{e^{-2(\gamma_{1234} - \gamma_{j})\tau_{0}} - e^{-\gamma_{1234}\tau_{0}}}{2\gamma_{j} - \gamma_{1234}} + \sum_{l\neq j} \left(\frac{e^{-2(\gamma_{1234} - \gamma_{j} - \gamma_{l})\tau_{0}} - e^{-\gamma_{1234}\tau_{0}}}{2(\gamma_{j} + \gamma_{l}) - \gamma_{1234}} - \frac{e^{-2\gamma_{l}\tau_{0}} - e^{-\gamma_{1234}\tau_{0}}}{\gamma_{1234} - 2\gamma_{l}}\right)\right],$$

$$(4.15)$$

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where each fraction from the square brackets should be substituted by $\tau_0 e^{-\gamma_{1234}\tau_0}$ if its denominator vanishes.

For any real number r let $\mathcal{C}_r(\mathbb{R}^d)$ be the space of continuous complex functions on \mathbb{R}^d with the finite norm

$$|f|_r = |f(z)\langle z\rangle^r|_{L_{\infty}}.$$
(4.16)

901 We naturally extend this norm to $f \in L_{\infty}(\mathbb{R}^d)$ and set

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$$L_{\infty,r}(\mathbb{R}^d) = \left\{ f \in L_{\infty}(\mathbb{R}^d) : |f|_r < \infty \right\}.$$
(4.17)

903 Consider also the linear operator \mathcal{L} , given by

$$(\mathcal{L}v)(s) = 2\gamma_s v(s), \quad s \in \mathbb{R}^d.$$
(4.18)

Below we often write the value v(s) of a function v at $s \in \mathbb{R}^d$ as v_s and the function v itself as $(v_s, s \in \mathbb{R}^d)$. Now, for $v \in \mathcal{C}_r(\mathbb{R}^d)$, where r > d, and for $\tau_0 \ge 0, \tau \in (0, 1]$, we define the kinetic integral $K^{\tau}(\tau_0)(v) = (K_s^{\tau}(\tau_0)(v), s \in \mathbb{R}^d)$:

$$K^{\tau}(\tau_0)(v) = \int_0^{\tau} e^{-t\mathcal{L}} K(\tau_0)(v) \mathrm{d}t.$$
(4.19)

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Here the operator $K(\tau_0) = K^1(\tau_0) + \dots + K^4(\tau_0)$ sends a function $v = (v_s, s \in$ 910 \mathbb{R}^d) to the function 911

$$K_{s}(\tau_{0})(v) = 4C_{d} \int_{\Sigma_{s}} \mu^{\Sigma_{s}} (\mathrm{d}s_{1} \mathrm{d}s_{2}) \Big(\mathcal{Z}^{4}(\tau_{0}; \vec{s}) v_{1} v_{2} v_{3} \\ + \mathcal{Z}^{3}(\tau_{0}; \vec{s}) v_{1} v_{2} v_{4} - \mathcal{Z}^{2}(\tau_{0}; \vec{s}) v_{1} v_{3} v_{4} - \mathcal{Z}^{1}(\tau_{0}; \vec{s}) v_{2} v_{3} v_{4} \Big) \\ =: K_{s}^{4}(\tau_{0})(v) + K_{s}^{3}(\tau_{0})(v) + K_{s}^{2}(\tau_{0})(v) + K_{s}^{1}(\tau_{0})(v)$$

$$(4.20)$$

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(note the reversed signs for K^2 and K^3). Here $v_i := v(s_i)$, where $s_4 = s$ and 914 $s_3 := s_1 + s_2 - s_4$ (in view of the factor $\delta_{3s}^{\prime 12}$). While μ^{Σ_s} is the measure (1.34) 915 on the quadric $\Sigma_s = \{(s_1, s_2) \in \mathbb{R}^{2d} : (s_1 - s) \cdot (s_2 - s) = 0\}$. Computing the 916 integral in t in (4.19), we find 917

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$$K_s^{\tau}(\tau_0)(v) = \frac{1 - e^{-2\gamma_s \tau}}{2\gamma_s} K_s(\tau_0)(v) = \frac{1 - e^{-2\gamma_s \tau}}{2\gamma_s} \sum_{j=1}^4 K_s^j(\tau_0)(v). \quad (4.21)$$

We study the kinetic integral K^{τ} in Sect. 5 while now we formulate a result 919 which is the main step in deriving the wave kinetic limit. 920

Theorem 4.1. For any $0 < \tau \leq 1$ the function $(n_s^{\leq 2}, s \in \mathbb{R}^d)$ satisfies 921

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$$n^{\leq 2}(\tau_{0} + \tau) = e^{-\tau \mathcal{L}} n^{\leq 2}(\tau_{0}) + 2 \int_{0}^{\tau} e^{-t \mathcal{L}} b^{2} dt + \varepsilon^{2} K^{\tau}(\tau_{0}) (n^{\leq 2}(\tau_{0})) + \varepsilon^{2} \mathcal{R},$$
(4.22)

(4.23)

where $b^2 = (b^2(s), s \in \mathbb{R}^d)$ and the remainder $\mathcal{R}_s(\tau)$ satisfies 923 $|\mathcal{R}(\tau)|_r < C_r \tau \, (\tau + \varepsilon^2), \quad \forall r.$ 924

4.2. Proof of Theorem 4.1 925

We first fix a value for L and decompose the processes $\tau \mapsto a_s^{(i)}(\tau_0 + \tau)$, where 926 $\tau_0 \geq 0$ and $0 \leq \tau \leq 1$, as 927

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$$a_s^{(i)}(\tau_0 + \tau) = c_s^{(i)}(\tau; \tau_0) + \Delta a_s^{(i)}(\tau; \tau_0), \qquad i = 0, 1, 2, \quad s \in \mathbb{Z}_L^d.$$
 (4.24)
929 Here

929 Here

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$$c_s^{(i)}(\tau;\tau_0) = e^{-\gamma_s \tau} a_s^{(i)}(\tau_0)$$

and $\Delta a_s^{(i)}$ is defined via relation (4.24). Below we write $c_s^{(i)}(\tau;\tau_0)$ and $\Delta a_s^{(i)}(\tau;\tau_0)$ 931 as $c_s^{(i)}(\tau)$ and $\Delta a_s^{(i)}(\tau)$ since τ_0 is fixed. 932 Obviously,

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$$c(\tau) := c^{(0)}(\tau) + \rho c^{(1)}(\tau) + \rho^2 c^{(2)}(\tau)$$

with $\tau \geq 0$ being a solution of the linear equation $(2.1)_{\rho=0,b(s)\equiv 0}$, equal $A(\tau_0)$ 935 at $\tau = 0$, and $\Delta a(\tau) = \sum_{j=0}^{2} \rho^j \Delta a^{(j)}(\tau)$ equals $A(\tau_0 + \tau) - c(\tau)$. By (4.5), for 936 0 < i, j < 2937

the functions
$$\mathbb{E}c_s^{(i)}\bar{c}_s^{(j)}$$
, $\mathbb{E}c_s^{(i)}\Delta\bar{a}_s^{(j)}$, $\mathbb{E}\Delta a_s^{(i)}\Delta\bar{a}_s^{(j)}$ (4.25)

naturally extend to Schwartz functions of $s \in \mathbb{R}^d$.

939 Due to (4.6) and (4.7), 940 $e^{-2\gamma_s \tau} n_{s,L}^{\leq 2}(\tau_0) = \mathbb{E}|c_s^{(0)}(\tau)|^2 + \rho^2 \mathbb{E} (|c_s^{(1)}(\tau)|^2 + 2\Re \bar{c}_s^{(0)}(\tau) c_s^{(2)}(\tau)), \quad \forall s \in \mathbb{R}^d.$ 941 Then,

$$n_{s,L}^{\leq 2}(\tau_{0}+\tau) - e^{-2\gamma_{s}\tau} n_{s,L}^{\leq 2}(\tau_{0}) = \mathbb{E}\Big(|a_{s}^{(0)}(\tau_{0}+\tau)|^{2} - |c_{s}^{(0)}(\tau)|^{2} + \rho^{2} \big(|a_{s}^{(1)}(\tau_{0}+\tau)|^{2} - |c_{s}^{(1)}(\tau)|^{2} + 2\Re\big(a_{s}^{(2)}\bar{a}_{s}^{(0)}(\tau_{0}+\tau) - c_{s}^{(2)}\bar{c}_{s}^{(0)}(\tau)\big)\Big).$$

$$(4.26)$$

944 Let us set

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$$\mathcal{Y}_s(u,v,w) := L^{-d} \sum_{1,2,3} \delta_{3s}^{\prime 12} \delta(\omega_{3s}^{12}) u_1 v_2 \bar{w}_3.$$

Writing explicitly the processes $\Delta a_s^{(i)}(\tau), s \in \mathbb{Z}_L^d$, we find

$$\Delta a_{s}^{(0)}(\tau) = b(s) \int_{\tau_{0}}^{\tau_{0}+\tau} e^{-\gamma_{s}(\tau_{0}+\tau-l)} d\beta_{s}(l),$$

$$\Delta a_{s}^{(1)}(\tau) = i \int_{\tau_{0}}^{\tau_{0}+\tau} e^{-\gamma_{s}(\tau_{0}+\tau-l)} \mathcal{Y}_{s}(a^{(0)}) dl,$$

$$\Delta a_{s}^{(2)}(\tau) = i \int_{\tau_{0}}^{\tau_{0}+\tau} e^{-\gamma_{s}(\tau_{0}+\tau-l)} \left(\mathcal{Y}_{s}(a^{(0)}, a^{(0)}, a^{(1)}) + \mathcal{Y}_{s}(a^{(0)}, a^{(1)}, a^{(0)}) + \mathcal{Y}_{s}(a^{(1)}, a^{(0)}) \right) dl,$$

$$+ \mathcal{Y}_{s}(a^{(0)}, a^{(1)}, a^{(0)}) + \mathcal{Y}_{s}(a^{(1)}, a^{(0)}) \right) dl,$$

(4.27)

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where
$$a^{(i)} = a^{(i)}(l)$$
. Note that to get explicit formulas for $c_s^{(i)}(\tau)$, $i = 0, 1, 2$,
it suffices to replace in the r.h.s.'s of the relations in (4.27) the range of in-
tegration from $[\tau_0, \tau_0 + \tau]$ to $[0, \tau_0]$. For example, $c_s^{(0)}(\tau) = e^{-\gamma_s \tau} a_s^{(0)}(\tau_0) =$
 $b(s) \int_0^{\tau_0} e^{-\gamma_s(\tau_0 + \tau - l)} d\beta_s(l)$.

952 $b(s) J_0$ e $E_{\alpha} c_{s}(\tau)$. 953 Using that $\mathbb{E}c_s^{(i)}(\tau)\Delta \bar{a}_s^{(0)}(\tau) = \mathbb{E}c_s^{(i)}(\tau)\mathbb{E}\Delta \bar{a}_s^{(0)}(\tau) = 0$ for any i and s, we 954 obtain

955
$$\mathbb{E}\left(a_s^{(2)}\bar{a}_s^{(0)}(\tau_0+\tau) - c_s^{(2)}\bar{c}_s^{(0)}(\tau)\right) = \mathbb{E}\Delta a_s^{(2)}(\tau)\bar{a}_s^{(0)}(\tau_0+\tau), \quad (4.28)$$

and from (4.24) we get that

$$|a_s^{(1)}(\tau_0 + \tau)|^2 - |c_s^{(1)}(\tau)|^2 = |\Delta a_s^{(1)}(\tau)|^2 + 2\Re \Delta a_s^{(1)} \bar{c}_s^{(1)}(\tau),$$

$$\mathbb{E}(|a_s^{(0)}(\tau_0 + \tau)|^2 - |c_s^{(0)}(\tau)|^2) = \mathbb{E}|\Delta a_s^{(0)}(\tau)|^2.$$
(4.29)

Then, inserting (4.28) and (4.29) into (4.26) and using that $\rho = \varepsilon L$, we find

959
$$n_{s,L}^{\leq 2}(\tau_0 + \tau) - e^{-2\gamma_s \tau} n_{s,L}^{\leq 2}(\tau_0) = \mathbb{E} \left| \Delta a_s^{(0)}(\tau) \right|^2 + \varepsilon^2 Q_{s,L}(\tau_0, \tau), \quad s \in \mathbb{R}^d,$$

960 where

961

$$Q_{s,L}(\tau_0,\tau) := L^2 \Big(\mathbb{E} |\Delta a_s^{(1)}(\tau)|^2 + 2 \Re \mathbb{E} \big[\Delta a_s^{(1)}(\tau) \bar{c}_s^{(1)}(\tau) + \Delta a_s^{(2)}(\tau) \bar{a}_s^{(0)}(\tau_0 + \tau) \big] \Big),$$
(4.30)

 $_{962}$ and we recall (4.25). Since

963

965

$$\mathbb{E} \left| \Delta a_s^{(0)}(\tau) \right|^2 = \frac{b(s)^2}{\gamma_s} (1 - e^{-2\gamma_s \tau}) = 2 \int_0^\tau e^{-t\mathcal{L}} b^2(s) \, \mathrm{d}t,$$

964 then

$$n_{\cdot,L}^{\leq 2}(\tau_0+\tau) - e^{-\tau\mathcal{L}} n_{\cdot,L}^{\leq 2}(\tau_0) = 2 \int_0^\tau e^{-t\mathcal{L}} b^2 \,\mathrm{d}t + \varepsilon^2 Q_{\cdot,L}(\tau_0,\tau),$$

for $n_{:L}^{\leq 2} = (n_{s,L}^{\leq 2}, s \in \mathbb{R}^d)$. In order to pass to the limit $L \to \infty$ we recall the relation (4.11). Then the desired formula (4.22) is an immediate consequence of the assertion below:

969 Proposition 4.2. We have

970
$$\lim_{L \to \infty} Q_{s,L}(\tau_0, \tau) = K_s^{\tau}(\tau_0)(n^{\leq 2}(\tau_0)) + \mathcal{R}_s(\tau), \quad s \in \mathbb{R}^d,$$
(4.31)

971 where the remainder \mathcal{R} satisfies (4.23).

Proof. The first step in the proof of (4.31) is the following result, established in Appendix B:

974 Proposition 4.3. One has

975

$$\left|Q_{s,L}(\tau_0,\tau) - \mathcal{X}_{s,L}(\tau_0,\tau)\right| \le C^{\#}(s)\tau^2, \quad s \in \mathbb{R}^d,$$
(4.32)

976 where

977
$$\mathcal{X}_{s,L}(\tau_0,\tau) := 4L^{2(1-d)}\tau \sum_{1,2,3} \delta_{3s}^{\prime 12} \delta(\omega_{3s}^{12}) \left(\mathcal{Z}^4 n_1^{(0)} n_2^{(0)} n_3^{(0)} + \mathcal{Z}^3 n_1^{(0)} n_2^{(0)} n_s^{(0)} - \mathcal{Z}^1 n_2^{(0)} n_3^{(0)} n_s^{(0)} - \mathcal{Z}^2 n_1^{(0)} n_3^{(0)} n_s^{(0)} \right).$$
(4.33)

980 The terms $\mathcal{Z}^{j} = \mathcal{Z}^{j}(\tau_{0}; s_{1}, s_{2}, s_{3}, s)$ are defined by (4.14) and $n_{i}^{(0)} := n_{s_{i},L}^{(0)}(\tau_{0}),$ 981 $n_{s}^{(0)} := n_{s,L}^{(0)}(\tau_{0}).$

By (4.9) $n_i^{(0)} = n_{s_i}^{(0)}$ are Schwartz functions in s_i . Besides, the functions $\mathcal{Z}^j(\tau_0, \vec{s})$ have at most polynomial growth in \vec{s} together with their derivatives, uniformly in $\tau_0 \geq 0$:

Lemma 4.4. For any vector $\mu \in (\mathbb{N} \cup \{0\})^{4d}$, uniformly in $\tau_0 \geq 0$, we have $\left|\partial_{\vec{s}}^{\mu} \mathcal{Z}^{j}(\tau_0, \vec{s})\right| \leq P(\vec{s}; \mu)$, where $P(\vec{s}; \mu)$ has at most a polynomial growth in \vec{s} .

By the lemma, which is proven in Sect. B.7, $\mathcal{X}_{s,L}$ satisfies the hypotheses of Corollary 3.3. So

$$|\mathcal{X}_{s,L}(\tau_0,\tau) - \tau K_s(\tau_0)(n^{(0)})| \le C^{\#}(s)L^{-1/2}\tau.$$
(4.34)

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Next, note that $|n_s^{(0)}(\tau_0) - n_s^{\leq 2}(\tau_0)| \leq C^{\#}(s)\varepsilon^2$ due to (4.9). Then the estimate on the Lipschitz constants of the operators $K^j(t)$, given in (5.4), implies that

992
$$|K(\tau_0)(n^{(0)}(\tau_0)) - K(\tau_0)(n^{\leq 2}(\tau_0))|_r \leq C_r \varepsilon^2 \quad \forall r.$$

993 Hence,

994

$$|\tau K_s(\tau_0)(n^{(0)}(\tau_0)) - \tau K_s(\tau_0)(n^{\leq 2}(\tau_0))| \leq C^{\#}(s)\tau\varepsilon^2.$$
(4.35)

On the other hand, on account of the definition (4.21), for $0 \le \tau \le 1$ we have the bound

997
$$\left|\tau K_{s}(\tau_{0})(n^{\leq 2}) - K_{s}^{\tau}(\tau_{0})(n^{\leq 2})\right| \leq C\gamma_{s}\tau^{2}|K_{s}(\tau_{0})(n^{\leq 2})| \leq C^{\#}(s)\tau^{2},$$
 (4.36)

where the last inequality follows from the estimate of the norm of the operator $K^{j}(t)$, given in (5.3), and from (4.9).

Putting together (4.32), (4.34), (4.35), (4.36) and letting L grow to infinity, we conclude the proof.

1002 5. Kinetic Equation

1003 At this section we examine the wave kinetic equation

$$\dot{\mathfrak{z}}_s(\tau) = -(\mathcal{L}\mathfrak{z})_s + \varepsilon^2 K_s(\tau)(\mathfrak{z}) + 2b(s)^2, \qquad \tau \ge 0, \ \mathfrak{z}(0) = 0 \tag{5.1}$$

(\mathcal{L} is defined in (4.18) and the operator $K = \overline{K}^1 + \cdots + \overline{K}^4$ is defined in (4.20)), and next we derive from this analysis and (4.22) the proximity of $n_s^{\leq 2}(\tau)$ to a solution of (5.1). We will need the following result, which is Lemma 4.2 from [7]:

1009 Lemma 5.1. For j, l = 1, ..., 4 and $u^j \in \mathcal{C}_r(\mathbb{R}^d)$ consider the operators

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$$J_l(u^1, \dots, u^4)(s) = \int_{\Sigma_s} \mu^{\Sigma_s}(\mathrm{d}s_1 \mathrm{d}s_2) \prod_{i \neq l} u^i(s_i)$$

1011 (see (1.34)), where $s_4 = s$ and $s_3 = s_1 + s_2 - s$. Then for each l,

1012

$$|J_l(u^1, \dots, u^4)|_{r+1} \le C_r \prod_{i \ne l} |u^i|_r \quad \text{if } r > d.$$
(5.2)

1013 5.1. Kinetic Integrals

1014 We recall notation (4.13), (4.14).

1015 **Lemma 5.2.** For
$$j = 1, ..., 4$$
, any $\tau \ge 0$ and any $\vec{s} = (s_1, ..., s_4) \in (\mathbb{R}^d)^4$,

1016 (i)
$$0 \leq \mathcal{Z}^j(\tau; \vec{s}) \leq \min(\tau, 1/\gamma_{s_i}) \leq 1$$

1017 (*ii*)
$$|\mathcal{Z}^{j}(\tau; \vec{s}) - \mathcal{Z}(\infty; \vec{s})| \le Ce^{-2\tau}$$
, where $\mathcal{Z}(\infty; \vec{s}) = 1/\gamma_{1234}$.

¹⁰¹⁸ Proof. The first assertion follows from (4.14) since $\sinh(x)$ is an increasing ¹⁰¹⁹ non-negative function of $x \ge 0$, so in the integrand in (4.14) we have $0 \le \sinh(\gamma_m l) / \sinh(\gamma_m \tau') \le 1$. For $0 \le \tau \le 1$ the second estimate follows from the ¹⁰²¹ first one as

1022
$$|\mathcal{Z}^{j}(\tau;\vec{s}) - \mathcal{Z}(\infty;\vec{s})| \le |\mathcal{Z}^{j}(\tau;\vec{s})| + |\mathcal{Z}(\infty;\vec{s})|,$$

while for $\tau \ge 1$ it follows from (4.15) since $\gamma_{123j} - \gamma_j \ge 1$ and $\gamma_{1234} - \gamma_j - \gamma_l \ge 1$ for $j, l \in \{1, 2, 3, 4\}, j \ne l$.

Since the kernels \mathcal{Z}^{j} are non-negative by the first assertion of the lemma 1025 above, then denoting $\kappa_1 = \kappa_2 = 1$, $\kappa_3 = \kappa_4 = -1$ we achieve that the operators 1026 $\kappa_i K^j$, $1 \le j \le 4$, are positive (in the sense that they send positive functions to 1027 positive). Due to the first assertion of the lemma and (5.2), for any $\tau \geq 0$ they 1028 define positive 3-homogeneous mappings $\mathcal{C}_r(\mathbb{R}^d) \to \mathcal{C}_{r+1}(\mathbb{R}^d)$ if r > d, and 1029 $|\kappa_i K^j(\tau)(v)|_{r+1} = |K^j(\tau)(v)|_{r+1} \le C_r \min(\tau, 1)|v|_r^3, \quad j = 1, \dots, 4, \quad (5.3)$ 1030 for $\tau \geq 0$. So, the mappings $K^{j}(\tau)$ are locally Lipschitz: 1031 $|K^{j}(\tau)(v^{1}) - K^{j}(\tau)(v^{2})|_{r+1}$ 1032 $\leq 3C_r \min(\tau, 1)R^2 |v^1 - v^2|_r, \quad \text{if } |v^1|_r, |v^2|_r \leq R.$ (5.4)1033 Since for $j = 1, \ldots, 4$ and any $s \in \mathbb{R}^d$, 1034 • for non-negative functions $n, m \in L_{\infty,r}$ [see (4.17)] such that $m \leq n$ we 1035 have $\kappa_i K_s^j(\tau)(m) \leq \kappa_i K_s^j(\tau)(n) \leq \infty$, 1036 • $|K_s^j(\tau)(v)| \le \kappa_j K_s^j(\tau)(|v|) \le \infty$ for any complex function $v \in L_\infty$, 1037 • $|v_s| \leq |v|_r \langle s \rangle^{-r}$ for all $v \in L_{\infty,r}$, 1038 then the relations (5.3), (5.4) remain true for functions from $L_{\infty,r}$. 1039 **Lemma 5.3.** If $|s_l| \leq R$ for l = 1, ..., 4, then 1040 $\left|\frac{\partial}{\partial \tau} \mathcal{Z}^j(\tau; \vec{s})\right| \le C \gamma^0(R^2)$ (5.5)1041 (see (1.5)).1042 *Proof.* For any $m \in \{s_1, \ldots, s_4\}$ and $0 \le l \le \tau$, we have 1043 $\left|\frac{\partial}{\partial \tau} \frac{\sinh \gamma_m l}{\sinh \gamma_m \tau}\right| \le \gamma_m \frac{\cosh \gamma_m \tau}{\sinh \gamma_m \tau} \le \gamma_m C \max(1, 1/(\gamma_m \tau)).$ 1044 Considering separately the cases $\tau \ge 1$ and $0 \le \tau < 1$, using (4.14) and the 1045 estimate above we get the result. \square 1046 This lemma implies that for any $v \in \mathcal{C}_r(\mathbb{R}^d)$ and any j the curve $\tau \mapsto$ 1047 $K^{j}(\tau)(v) \in \mathcal{C}_{r}(\mathbb{R}^{d})$ is Hölder continuous:

1048 $K^{j}(\tau)(v) \in \mathcal{C}_{r}(\mathbb{R}^{n})$ is Holder continuous:

1049 Lemma 5.4. For any $\tau_0 \ge 0, \ 0 \le \tau \le 1, \ j = 1, \dots, 4$ and any r > d + 1,

$$|K^{j}(\tau_{0}+\tau)(v) - K^{j}(\tau_{0})(v)|_{r} \leq C_{r}|v|_{r}^{3}\tau^{\kappa_{*}} \qquad \forall v \in \mathcal{C}_{r}(\mathbb{R}^{d}),$$
(5.6)

1051 where $\kappa_* = 1/(1+2r_*)$.

1052 *Proof.* By the homogeneity we may assume that $|v|_r = 1$. For $R \ge 1$ let us set 1053 $v^R(s) = v(s)\chi_{|s| \le R} \in L_{\infty}$. Then

$$|v^{R}|_{r} \le 1, \quad |v - v^{R}|_{r-1} \le R^{-1}.$$
 (5.7)

Now let us write the increment $K^{j}(\tau_{0}+\tau)(v) - K^{j}(\tau_{0})(v)$ as

$$(K^{j}(\tau_{0}+\tau)(v) - K^{j}(\tau_{0}+\tau)(v^{R})) + (K^{j}(\tau_{0}+\tau)(v^{R}) - K^{j}(\tau_{0})(v^{R})) + (K^{j}(\tau_{0})(v^{R}) - K^{j}(\tau_{0})(v)) =: \Delta_{1} + \Delta_{2} + \Delta_{3}.$$

Recalling that (5.3) and (5.4) hold for functions from $L_{\infty,r'}$ with r' > d, we get from (5.7) that $|\Delta_1|_r + |\Delta_3|_r \le C_r R^{-1}$. To estimate Δ_2 , we set $\Delta_2^R = \Delta_2 \chi_{|s| \le R}$.

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Since by (5.3), $|\Delta_2|_{r+1} \leq 2C_r$, then $|\Delta_2 - \Delta_2^R|_r \leq 2C_r R^{-1}$. For |s| > R the function Δ_2^R vanishes, while for $|s| \leq R$ in view of Lemma 5.3 we have

$$\begin{aligned} |\Delta_{2s}^{R}| &= |\Delta_{2s}| = \left| K_{s}^{j}(\tau_{0} + \tau)(v^{R}) - K_{s}^{j}(\tau_{0})(v^{R}) \right| \\ &\leq C_{r} \int_{\Sigma_{s}} \mu_{s}(dv_{1}dv_{2}) \left| \mathcal{Z}^{j}(\tau_{0} + \tau; \vec{s}) - \mathcal{Z}^{j}(\tau_{0}; \vec{s}) \right| \frac{|v_{1}| \dots |v_{4}|}{|v_{j}|} \chi_{\{|s_{j}| \leq R \ \forall j\}} \\ &\leq C_{1r} \gamma^{0}(R^{2}) \tau \int_{\Sigma_{s}} \mu_{s}(dv_{1}dv_{2}) \frac{|v_{1}| \dots |v_{4}|}{|v_{j}|} \leq C_{2r} \langle s \rangle^{-r-1} \tau R^{2r_{*}}, \end{aligned}$$

where to get the last inequality we used (5.2). We have seen that the C_r -norm of the increment is bounded by $C_r(R^{-1} + \tau R^{2r_*})$, for any $R \ge 1$. Choosing $R = \tau^{-1/(1+2r_*)}$, we achieve (5.6).

1065 5.2. Kinetic Equation

Now we will apply the obtained results to the kinetic Eq. (5.1). Since the func-1066 tion $b(\cdot)^2 := \{b(s)^2\} \in \mathcal{C}_r(\mathbb{R}^d)$ for all r, since \mathcal{L} is the operator of multiplying by 1067 the function $2\gamma_s$ as in (1.4), 1.5), and the operator K satisfies (5.3), 5.4), then 1068 for small enough $\varepsilon > 0$ Eq. (5.1) has a unique solution, belonging to $\mathcal{C}_r(\mathbb{R}^d)$ for 1069 each r, which in a Lipschitz way depends on the r.h.s. of the equation, when 1070 the latter deviates from $b(\cdot)^2$. Namely, the following result, where X^r stands 1071 for the space $C(0,\infty;\mathcal{C}_r(\mathbb{R}^d))$, given the norm $|v(\cdot)|_{X^r} = \sup_{t>0} |v(t)|_r$, may 1072 be easily verified (a proof of a similar fact may be found in Section 4 of [7]). 1073

1074 Lemma 5.5. For any r > d,

- 1075 (1) There exists ε_* , depending on $b(\cdot), r$ and r_* , such that for $0 \le \varepsilon \le \varepsilon_*$ 1076 Eq. (5.1) has a unique solution $\mathfrak{z}(\tau)$, belonging to X^r . It satisfies
- 1077

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$$|\mathfrak{z}|_{X^r} \le C_r |b^2|_r. \tag{5.8}$$

1078 (2) If $\mathfrak{z}^{0}(\tau)$ is a solution of the linear Eq. $(5.1)|_{\varepsilon=0}$, then $|\mathfrak{z} - \mathfrak{z}^{0}|_{X^{r}} \leq C_{r}\varepsilon^{2}$. 1079 If a curve $\mathfrak{z}'(\tau)$ solves (5.1) with $2b(s)^{2}$ replaced by $2b(s)^{2} + \xi_{s}(t)$, where 1080 $\xi \in X^{r}$ and $|\xi|_{X^{r}} \leq 1$, then $|\mathfrak{z} - \mathfrak{z}'|_{X^{r}} \leq C_{r}|\xi|_{X^{r}}$.

The lemma's assertion holds as well for nonzero initial conditions $\mathfrak{z}(0) \in \mathcal{C}_r(\mathbb{R}^d)$ in (5.1), but we do not need this.

Let $K(\infty)$ be the operator, obtained from $K(\tau_0)$ by replacing in (4.20) the kernels $\mathcal{Z}^j(\tau_0; s), s \in \mathbb{R}^d$, by $\mathcal{Z}(\infty; \vec{s})$ (see Lemma 5.2). Let r > d and $\mathfrak{z}^{\varepsilon} \in \mathcal{C}_r(\mathbb{R}^d)$ be a solution of the limiting stationary equation

$$\mathcal{L}\mathfrak{z}^{\varepsilon} - \varepsilon^2 K(\infty)(\mathfrak{z}^{\varepsilon}) = 2b(\cdot)^2 \tag{5.9}$$

in the vicinity of $\mathcal{L}^{-1}(2b^2)$, existing for small ε by the inverse function theorem. Since $b^2(\cdot) \in \bigcap_r \mathcal{C}_r(\mathbb{R}^d)$ and, as in (5.3), the map $K(\infty)$ is one-smoothing, then decreasing ε_* if needed we achieve that $\mathfrak{z}^{\varepsilon} \in \bigcap_r \mathcal{C}_r(\mathbb{R}^d)$ for $\varepsilon \leq \varepsilon_*$ and

$$|\mathfrak{z}^{\varepsilon} - 2\mathcal{L}^{-1}(b^2)|_m \le C_m \varepsilon^2 \quad \forall \, m.$$
(5.10)

1091 Here and below the constants depend on b and r_* .

Let us consider the curve $w(t) = \mathfrak{z}(t) - \mathfrak{z}^{\varepsilon}$. It satisfies the equation

$$\dot{w} + \mathcal{L}(w) = \varepsilon^2 \left(K(t)(\mathfrak{z}) - K(\infty)(\mathfrak{z}^\varepsilon) \right) = \varepsilon^2 \left[\left(K(t)(\mathfrak{z}) - K(t)(\mathfrak{z}^\varepsilon) \right) - \left(K(t)(\mathfrak{z}^\varepsilon) - K(\infty)(\mathfrak{z}^\varepsilon) \right) \right]$$

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and $w(0) = -\mathfrak{z}^{\varepsilon}$. Denote $K(\tau)(\mathfrak{z}^{\varepsilon}) - K(\infty)(\mathfrak{z}^{\varepsilon}) =: -\eta(\tau)$. In view of Lemmas 5.2 and 5.1, $|\eta(\tau)|_r \leq C_r e^{-2\tau}$ for $\tau \geq 0$. Next, regarding the difference $K(\tau)(\mathfrak{z}(\tau)) - K(\tau)(\mathfrak{z}^{\varepsilon})$ as an operator, linear in $w(\tau) = \mathfrak{z}(\tau) - \mathfrak{z}^{\varepsilon}$ and quadratic in $(\mathfrak{z}(\tau), \mathfrak{z}^{\varepsilon})$, we write it as $\mathcal{K}(\tau)(w(\tau))$. Then by (5.4) and (5.8), $|\mathcal{K}(\tau)w|_{r+1} \leq C_r |w|_r, \forall r > d$. Finally, we substitute

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$$w(\tau) = v(\tau) + y(\tau), \quad v(\tau) = -e^{-\tau \mathcal{L}} \mathfrak{z}^{\varepsilon},$$

and rewrite the equation on w as an equation on y:

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$$\dot{y} + \mathcal{L}y = \varepsilon^2 \mathcal{K}(\tau) \big(v(\tau) + y(\tau) \big) + \varepsilon^2 \eta(\tau), \qquad y(0) = 0.$$

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$$y(t) = \varepsilon^2 \int_0^t e^{-(t-s)\mathcal{L}} \left[\mathcal{K}(s) \left(v(s) + y(s) \right) + \eta(s) \right] \mathrm{d}s.$$
(5.11)

Let Y^r be the space of continuous curves $y : \mathbb{R}_+ \to \mathcal{C}_r(\mathbb{R}^d)$, vanishing at zero, with finite norm $|y|_{Y^r} = \sup_{t \ge 0} e^t |y(t)|_r$.

1106 Let \mathfrak{B} be the linear operator

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$$\mathfrak{B}(y)(t) = \int_0^t e^{-(t-s)\mathcal{L}} \mathcal{K}(s)(y(s)) \,\mathrm{d}s.$$

1108 Then the equation for y may be written as

1109
$$y(t) = \varepsilon^2 \Big(\mathfrak{B}(y)(t) + \mathfrak{B}(v)(t) + \int_0^t e^{-(t-s)\mathcal{L}} \eta(s) \mathrm{d}s \Big).$$
(5.12)

1110 If $|\tilde{y}|_{Y^r} = 1$, then

1111
$$|\mathfrak{B}(\tilde{y}(t))|_{r+1} \le \int_0^t \left| e^{-(t-s)\mathcal{L}} \mathcal{K}(s)(\tilde{y}(s)) \right|_{r+1} \mathrm{d}s \le C'_r \int_0^t e^{-2(t-s)} e^{-s} \mathrm{d}s < C'_r e^{-t}$$

So $\mathfrak{B}: Y^r \to Y^{r+1}$ is a bounded linear operator if r > d, and accordingly the operator (id $-\varepsilon^2 \mathfrak{B}$) is a linear isomorphism of Y^r if r > d and ε is sufficiently small. It easy to see that $\mathfrak{B}(v)$ and $\int_0^t e^{-(t-s)\mathcal{L}}\eta(s)ds$ both belong to all spaces Y^r . Then in view of (5.12), $|y|_{Y^{r+1}} \leq C\varepsilon^2$. Since the operator \mathfrak{B} is 1-smoothing, then by induction we get that y belongs to all spaces Y^r . We have proved that

1117 **Lemma 5.6.** The solution $\mathfrak{z}(\tau)$, constructed in Lemma 5.5, may be written as

$$\mathfrak{z}(\tau) = (id - e^{-\tau \mathcal{L}})\mathfrak{z}^{\varepsilon} + y(\tau), \quad where \quad |y(\tau)|_r \le C_r \varepsilon^2 e^{-\tau} \quad \forall \tau \ge 0, \ \forall r.$$

1119 Here $\mathfrak{z}^{\varepsilon}$ is the stationary solution, defined in (5.9) and satisfying (5.10).

1120 5.3. Energy Spectra of Quasisolutions and Kinetic Equation

In this section we prove our main result. Namely, we show that the energy spectrum (4.2) of the quasisolution $n_{s,L}(\tau) = \mathbb{E}|A_s(\tau)|^2$ of Eq. (2.1) with large *L* is ε^3 -close to the solution $\mathfrak{z}(\tau)$ of the WKE (5.1), constructed in Lemmas 5.5, 5.6. By (4.12), it suffices to prove this for $n_{s,L}$ replaced by $n_s^{\leq 2}$. Let us denote $w_s(\tau) = n_s^{\leq 2}(\tau) - \mathfrak{z}_s(\tau)$; then $w_s(0) = 0$. Recall that ε_* is defined in Lemma 5.5. 1127 **Lemma 5.7.** If r > d + 1 and $\varepsilon \leq C_{1r}^{-1} \leq \varepsilon_*$ for an appropriate constant C_{1r} , 1128 then for any $\tau_0 \geq 0$ and $0 < \tau \leq 1/2$,

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$$|w(\tau_0 + \tau)|_r \le (1 - \tau/2)|w(\tau_0)|_r + C_{2r}\tau\varepsilon^2(\tau^{\kappa_*} + \varepsilon^2),$$
 (5.13)

1130 where $\kappa_* = 1/(1+2r_*)$.

1131 Proof. Since by (5.1)

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$$\mathfrak{z}(\tau_0+\tau) = e^{-\tau\mathcal{L}}\mathfrak{z}(\tau_0) + 2\int_0^\tau e^{-t\mathcal{L}}b^2\mathrm{d}t + \varepsilon^2\int_{\tau_0}^{\tau_0+\tau} e^{-(\tau_0+\tau-t)\mathcal{L}}K(t)(\mathfrak{z}(t))\mathrm{d}t,$$

1133 then in view of (4.22) and (4.19)

$$w(\tau + \tau_0) = e^{-\tau \mathcal{L}} w(\tau_0) + \varepsilon^2 \Delta + \mathcal{R}, \qquad (5.14)$$

1135 where \mathcal{R} is as in (4.22) and

1136
$$\Delta = \int_{\tau_0}^{\tau_0 + \tau} e^{-(\tau_0 + \tau - t)\mathcal{L}} \Big(K(\tau_0)(n^{\leq 2}(\tau_0)) - K(t)(\mathfrak{z}(t)) \Big) \mathrm{d}t.$$

1137 Note that in view of Lemma 5.5 and estimates (4.9),

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$$|n^{\leq 2}(\tau)|_r, \ |\mathfrak{z}(\tau)|_r \leq C_r \quad \text{for all } \tau \text{ and all } r,$$
(5.15)

1139 with suitable constants C_r . Let us re-write Δ as follows:

$$\begin{split} \Delta &= \int_{\tau_0}^{\tau_0 + \tau} e^{-(\tau_0 + \tau - t)\mathcal{L}} \big(K(\tau_0) (n^{\leq 2}(\tau_0)) - K(\tau_0) (\mathfrak{z}(\tau_0)) \big) \mathrm{d}t \\ &+ \int_{\tau_0}^{\tau_0 + \tau} e^{-(\tau_0 + \tau - t)\mathcal{L}} \big(K(\tau_0) (\mathfrak{z}(\tau_0)) - K(t) (\mathfrak{z}(\tau_0)) \big) \mathrm{d}t \\ &+ \int_{\tau_0}^{\tau_0 + \tau} e^{-(\tau_0 + \tau - t)\mathcal{L}} \big(K(t) (\mathfrak{z}(\tau_0)) - K(t) (\mathfrak{z}(t)) \big) \mathrm{d}t =: \Delta^1 + \Delta^2 + \Delta^3. \end{split}$$

 J_{τ_0}

1141 By (5.4) and (5.15), $|\Delta^1|_r \leq C_r \tau |w(\tau_0)|_r$. Similar,

1142
$$|\Delta^3|_r \le C_r \tau \sup_{\tau_0 \le t \le \tau_0 + \tau} |\mathfrak{z}(t) - \mathfrak{z}(\tau_0)|_r \le C_r \tau^2$$

1143 since $|\mathfrak{z}(t) - \mathfrak{z}(\tau_0)|_r \leq \int_{\tau_0}^t |-\mathcal{L}\mathfrak{z}(l) + \varepsilon^2 K(l)(\mathfrak{z}(l)) + 2b^2|_r dl$ and $|\mathfrak{z}(t)|_{r+r_*} \leq C'_r$ by 1144 (5.15). Now let us consider Δ^2 . By Lemma 5.4, $|K(\tau_0)(\mathfrak{z}(\tau_0)) - K(t)(\mathfrak{z}(\tau_0))|_r \leq$ 1145 $C_r(t-\tau_0)^{\kappa_*}$. So $\Delta^2 \leq C_r \int_0^\tau t^{\kappa_*} dt = C'_r \tau^{1+\kappa_*}$.

1146 Since $\mathcal{L} \geq 2\mathbb{1}$ and $\tau \leq 1/2$, then $|e^{-\tau \mathcal{L}}w(\tau_0)|_r \leq (1-\tau)|w(\tau_0)|_r$. Now 1147 (5.14), (4.23) and the bounds on Δ^j imply that

1148
$$|w(\tau_0 + \tau)|_r \le (1 - \tau)|w(\tau_0)|_r + C_r \varepsilon^2 \tau (|w(\tau_0)|_r + \tau + \tau^{\kappa_*} + (\tau + \varepsilon^2)),$$

1149 and (5.13) follows if $C_{1r}^{-1} \ll 1.$

For any $0 < \tau \le 1/2$, any N and for k = 0, ..., N let us set $w_k = |w(k\tau)|_r$. Let the function $k \to w_k$ attain its maximum at a point k which we write as

1152 $k := k_0 + 1$. If $k_0 + 1 = 0$, then $w_k \equiv 0$. Otherwise, in view of (5.13) we have

$$w_{k_0} \le w_{k_0+1} \le (1-\tau/2)w_{k_0} + C_{2r}\tau\varepsilon^2(\tau^{\kappa_*}+\varepsilon^2).$$

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1154 So, $w_{k_0} \leq 2C_{2r}\varepsilon^2(\tau^{\kappa_*}+\varepsilon^2)$ and

1155
$$\max_{0 \le k \le N} |w(k\tau)|_r = w_{k_0+1} \le 3C_{2r}\varepsilon^2(\tau^{\kappa_*} + \varepsilon^2)$$

¹¹⁵⁶ since $\tau \leq 1/2$. Applying again (5.13) with $\tau_0 = k\tau$ and τ replaced by any ¹¹⁵⁷ $\bar{\tau} \in (0, \tau)$, and using that in the formula above N is any, we get that $|w(t)|_r \leq$ ¹¹⁵⁸ $4C_{2r}\varepsilon^2(\tau^{\kappa_*} + \varepsilon^2)$, for any $t \geq 0$. Sending $\tau \to 0$ (and estimating norms $|\cdot|_r$ ¹¹⁵⁹ with r < d + 2 via $|\cdot|_{d+2}$) and then using (4.12), we finally get

Theorem 5.8. For any r there exist positive constants C_{1r}, C_{2r}, C_{3r} such that if $\varepsilon \leq C_{1r}^{-1}$, then

$$\sup_{\tau \ge 0} |n^{\le 2}(\tau) - \mathfrak{z}(\tau)|_r \le C_{2r} \varepsilon^4$$
(5.16)

1163 and if $L \geq \varepsilon^{-2}$, then

$$\sup_{\tau \ge 0} |n_{\cdot,L}(\tau) - \mathfrak{z}(\tau)|_r \le C_{3r}\varepsilon^3.$$
(5.17)

Relation (5.17) together with Lemma 5.6 gives a control over the longtime behaviour of the spectra of quasisolutions of (2.1) in terms of the stationary solution $\mathfrak{z}_{\varepsilon}$ of the limiting kinetic equation (see (5.9)):

1168
$$|n_{\cdot,L}(\tau) - \mathfrak{z}^{\varepsilon}|_r \le C_r(e^{-\tau} + \varepsilon^3), \quad \forall \tau \ge 0.$$

By Proposition 2.1 with $d \ge 3$ this result and (5.17) extend to the spectra of quasisolutions of (1.20), defined in (1.27), as expressed in

Theorem 5.9. For any r there exist positive constants C_{4r}, C_{5r} such that if $\varepsilon \leq C_{4r}^{-1}$ and $L \geq \varepsilon^{-2}$, then

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1164

$$\sup_{\tau \ge 0} |\mathfrak{n}_{,L}(\tau) - \mathfrak{z}(\tau)|_r \le C_{4r}\varepsilon^3, \qquad (5.18)$$

1175

$$|\mathfrak{n}_{\cdot,L}(\tau) - \mathfrak{z}^{\varepsilon}|_r \le C_{5r}(e^{-\tau} + \varepsilon^3), \qquad \forall \tau \ge 0.$$
(5.19)

Relation (5.16) extends to the energy spectra of quasisolutions of (1.20) analogously.

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1185 Appendix A. Proof of Lemma 3.6

In this appendix we suppose that the dimension d satisfies $d \ge 2$.

1187 A.1. Idea of the Proof and General Setting

In Lemma 3.6 (up to an obvious scaling) we have to estimate the number 1188 of integer points on a quadric inside a large box. The idea is to embed the 1189 integral points of the box in an affine space over a large finite field and then 1190 apply powerful algebraic geometry techniques to estimate the needed number 1191 (note that this identification of bounded integers with elements of a finite 1192 field is ubiquitous in coding theory and combinatorics). It is possible mainly 1193 due to the fact that this techniques permits to count points defined over a 1194 finite field using some geometric information (essentially the dimension, the 1195 degree and irredundant decomposition) on the corresponding algebraic set over 1196 the algebraic closure of our finite field. We begin with recalling some basic 1197 definitions and results concerning such algebraic sets (see, for example, the 1198 first chapter of the book [27]). 1199

Affine algebraic sets. Let us fix an algebraically closed field K. Let $\mathbb{A}^m = K^m$ be the *m*-dimensional affine space over K, and let $F_1, \ldots, F_s \in K[T_1, \ldots, T_m]$ be nonzero polynomials. Then an *affine algebraic set* (AAS) X is just the set of common zeros of these polynomials:

$$X = \{(a_1, \dots, a_m) \in K^m : F_1(a_1, \dots, a_m) = \dots = F_s(a_1, \dots, a_m) = 0\}.$$
(A.1)

Irreducibility. An AAS X is *reducible* if $X = X_1 \cup X_2$ with two non-empty AAS X_1, X_2 s.t. $X_1 \neq X, X_2 \neq X$. If it is not the case, X is called *irreducible*, or an *affine algebraic variety* (see [27], Section I.3.1).

Theorem A.1 (Irredundant decomposition). Any non-empty AAS X can be
presented as

1211

$$X = X_1 \cup \ldots \cup X_l \tag{A.2}$$

for irreducible X_1, \ldots, X_l such that $X_i \not\subset X_j$ for $i \neq j$. The decomposition is unique up to order.

This decomposition is especially simple for a hypersurface X, i.e. when 1214 in (A.1) s = 1. Then $F = F_1(T_1, \ldots, T_m) = \prod_{j=1}^l Q_j$ for irreducible polynomials 1215 Q_i which are uniquely defined up to multiplicative constants and permutation 1216 since the ring $K[T_1, \ldots, T_m]$ is a unique factorization domain, see, for example, 1217 Chapter IV of [24], and then $X_j = \{Q_j = 0\}$. This uniqueness is true under the 1218 condition which we can and will suppose to hold, namely, that the polynomial 1219 P does not have multiple divisors, i.e. all $Q_i, j = 1, \ldots, l$ are distinct. For 1220 further references we formulate a corollary of the unique factorization property 1221 (see [27], Section I.3.1) 1222

1223 **Lemma A.2.** (i). If X and Y are hypersurfaces, then X = Y if and only if 1224 the corresponding polynomials P_X and P_Y are proportional. Moreover if 1225 Y is irreducible and $X \subseteq Y$, then X = Y. (ii). If deg $P_X = 2$, then there are exactly two possibilities: either X irreducible (in this case it cannot contain a hyperplane), or $X = X_1 \cup X_2$ for two affine hyperplanes, defined by affine linear polynomials l_1 and l_2 , and $P_X = l_1 l_2$.

Dimension. One can define the dimension $r = \dim X \in \{0, 1, \ldots, m\}$ as follows: dim $X = \max\{\dim X_i, i = 1, \ldots, l\}$ for (A.2), and for an irreducible AAS X

1232
$$\dim X = \max\{r : X = X_0 \supset X_1 \supset \ldots \supset X_r \neq \emptyset\},$$

where all X_i , i = 0, ..., r are irreducible AAS and all inclusions are strict. The codimension of X is codim $X = m - \dim X$.

1235 In particular, if dim X = m then $X = \mathbb{A}^m$ (indeed, if $X \subset \mathbb{A}^m, X \neq \mathbb{A}^m$ 1236 the definition implies that dim $X < \dim \mathbb{A}^m = m$) and if dim X = 0 then X is 1237 a finite set. The codimension of X as in (A.1) is at most s.

From the definition we get immediately (see [27], Section I.6.2)

Lemma A.3. If $Y \subset X, Y \neq X$ and X is irreducible, then dim $X > \dim Y$, codim $X < \operatorname{codim} Y$.

1241 **Degree.** Let $X \subset \mathbb{A}^m$ be a non-empty AAS, dim X = r. Then its *degree* deg X 1242 is defined as follows:

1243 $\deg X = \max\{\text{cardinality of } X \cap L : \dim (X \cap L) = 0\},\$

0

1244 where $L \subset \mathbb{A}^m$ is an affine plane with dim L = m - r.

Lemma A.4. If X is a hypersurface (i.e. in (A.1) s = 1), then $\operatorname{codim} X = 1$ and $\deg X = \deg F_1$.

1247 The famous *Bezout theorem* in its the most elementary setting over the 1248 field $\mathbb C$ states that

1249

$$\log X \le \prod_{i=1}^s \deg F_i.$$

1250 A.2. Finite Fields' Bezout Theorem

From now on the field K is the algebraic closure \mathbb{F}_p of a finite field \mathbb{F}_p , where *p* is a large prime number (see [24], Section V.5).

We will use a version of Bezout's theorem over finite fields which can be deduced from its general form, e.g. [13], and is also explicitly stated and proved in [22, Corollary 2.2].

Theorem A.5. Let $K = \overline{\mathbb{F}}_p$ and the AAS X in (A.1) is such that $F_j \in \mathbb{F}_p[T_1, \dots, T_m]$, deg $F_j = d_j$, $j = 1, \dots, s$, and dim X = r. Then

$$|X \cap \mathbb{F}_p^m| \le p^r \prod_{i=1}^s d_i.$$

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1259 A.3. Preliminary Result

Let $q_1, \ldots, q_s, s \ge 1$, be polynomials of degree at most two in $m \ge s$ variables, $q_i \in \mathbb{Z}[X_1, \ldots, X_m]$, with $q_i(0) = 0, i = 1, \ldots, s$. Consider the geometric quadrics $Q_i = \{x \in \mathbb{R}^m : q_i(x) = 0\}$ and their intersection $Q = \bigcap_{i=1}^s Q_i$. The latter is not empty since $0 \in Q$.

1264 Let $B_M^m \subset \mathbb{R}^m$ be an open cube $\{|x|_{\infty} < M\}$ with some $M \ge 1$. Consider 1265 the set

1266
$$S_m(M,Q) = Q \cap \mathbb{Z}^m \cap B_M^m$$

Let p be a prime and $q_i^{(p)} \in \mathbb{F}_p[X_1, \ldots, X_m]$ denote the polynomials q_i mod p over the finite field \mathbb{F}_p . Consider the sets

1269
$$Q_i^{(p)} = \{ x \in K^m : q_i^{(p)}(x) = 0 \}$$

and their intersection $Q^{(p)} = \bigcap_{i=1}^{s} Q_i^{(p)}$ (recall that now $K = \overline{\mathbb{F}}_p$ is the algebraic closure of \mathbb{F}_p). We will be interested mainly in the cardinality of $Q^{(p)}(\mathbb{F}_p) :=$ $Q^{(p)} \cap \mathbb{F}_p^m$ as a tool to estimate $|S_m(M,Q)|$.

1273 **Proposition A.6.** Let $M \ge 1$ and suppose that a prime p > 2M satisfies p = 2M(1 + r(M)), where r(M) > 0. Suppose also that $\deg q_i \le 2$ for each i and that the AAS $Q^{(p)}$ is of dimension m - s (and of codimension s, that is, the s quadrics $Q_i^{(p)}$ intersect properly):

$$\dim Q^{(p)} = m - s. \tag{A.3}$$

1278 Then

1277

$$|S_m(M,Q)| \le 2^m (1+r(M))^{m-s} M^{m-s} .$$
(A.4)

By Bertrand's postulate, for any $M \ge 1$ there is a p satisfying 2M , and when applying Proposition A.6 we will always chose

1282

$$r(M) < 1. \tag{A.5}$$

1283 Moreover, by the Prime Number Theorem, for large M one can chose r(M) = o(1).

1285 Proof of Proposition A.6. Let $\Pi: S_m(M,Q) \longrightarrow \mathbb{F}_p^m$ be defined by

1286
$$\Pi(x_1,\ldots,x_m) = (x_1 \mod p,\ldots,x_m \mod p)$$

¹²⁸⁷ Then Π is injective and its image is contained in $Q^{(p)} \cap \mathbb{F}_p^m \subset \mathbb{F}_p^m$. Indeed, the ¹²⁸⁸ last assertion is clear and the injectivity is established as follows: if

1289
$$(x'_1 \mod p, \dots, x'_m \mod p) = (x_1 \mod p, \dots, x_m \mod p)$$

1290 but $x' \neq x$, then for some $i \in \{1, ..., m\}$ we have $x'_i \mod p = x_i \mod p$, but 1291 $x'_i \neq x_i$. Consequently, $|x_i - x'_i| \geq p > 2M$ which contradicts the condition 1292 $x_i, x'_i \in B^m_M$. Applying then Theorem A.5 to $X = Q^{(p)}$, we get the conclusion 1293 since

1294
$$|S_m(M,Q)| \le |Q^{(p)}(\mathbb{F}_p)| \le 2^s p^{m-s} = 2^m (1+r(M))^{m-s} M^{m-s}$$

1296 A.4. Main Estimate for N = 3 and 4

Now we pass to the proof of Lemma 3.6 and denote $|\tilde{Q}_L(z_1, v) \cap B_R^{(N-2)d}| = s(R, \tilde{Q}, L)$. Consider the set

1299
$$S'(R,\tilde{\mathcal{Q}},L) = \tilde{\mathcal{Q}} \cap \mathbb{Z}^{(N-2)d} \cap B_{RL}^{(N-2)d}, \qquad \tilde{\mathcal{Q}} = \tilde{Q}(Lz_1,Lv),$$

and denote by $s'(R, \tilde{Q}, L)$ its cardinality. Then $s'(R, \tilde{Q}, L) = s(R, \tilde{Q}, L)$ since the map $(z_2, \ldots, z_{N-1}) \mapsto (Lz_2, \ldots, Lz_{N-1})$ is a bijection between the sets $\tilde{Q}_L(z_1, v) \cap B_R^{(N-2)d}$ and $S'(R, \tilde{Q}, L)$.

Let us estimate $s'(R, \tilde{Q}, L)$ through Proposition A.6 with m = (N-2)dand s = N-2, where N = 3 or 4. To this end it suffices to find $M \ge RL$ and p > 2M such that assumption (A.3) is fulfilled for any $(z_1, v) \in B^{2d}_{(N-1)R}$ satisfying $(z_1, v) \in Q^0_{1L} \cap A_2$. Lemma A.7 below establishes this for M =NRL/2 and any p > 2M. Then, applying (A.4) with r(M) < 1 (see (A.5)), we conclude the proof of Lemma 3.6.

For a prime p and $a, b \in \mathbb{F}_p^d$, let us consider algebraic sets $\tilde{Q}_j^{(p)}$ over 1310 $K = \bar{\mathbb{F}}_p$:

1311
$$\tilde{Q}_{j}^{(p)}(a,b) := \{(z_{2},\ldots,z_{N-2}) \in K^{(N-2)d} : q_{j}^{(p)}(z_{2},\ldots,z_{N-2};a,b) = 0\},\$$

where $q_j^{(p)}(z_2, \ldots, z_{N-2}; a, b)$ are the residues modulo p of the polynomials $q_j(z_2, \ldots, z_{N-2}; a, b)$, defined by (3.15). We set $\tilde{Q}^{(p)} = \bigcap_{1 < j < N} \tilde{Q}_j^{(p)}$ for the intersection of the algebraic sets.

1315 Lemma A.7. Let $N \in \{3,4\}$, $(z_1,v) \in Q_{1L}^0 \cap A_2$ (see (3.16)) and let p be a 1316 prime satisfying $p > \max(|Lz_1|_{\infty}, |Lv|_{\infty})$. Then

1317

dim
$$\tilde{Q}^{(p)}(Lz_1, Lv) = (N-2)(d-1).$$
 (A.6)

The assumption $p > \max(|Lz_1|_{\infty}, |Lv|_{\infty})$ ensures that Lz_1 and Lv are different from zero in K^d . In particular, for $(z_1, v) \in B^{2d}_{(N-1)R}$ this assumptions is satisfied if p > 2M with M = NRL/2.

1321 Proof of Lemma A.7. Let N = 3. Then N - 2 = 1 and $\tilde{Q}^{(p)}$ is given by the 1322 unique equation $q_2^{(p)}(z_2; Lz_1, Lv) = 0$, for a fixed (z_1, v) . By Lemma A.9 the 1323 equation is non-trivial, so the conclusion follows from Lemma A.4.

N = 4. The codimension of the intersection of two quadrics is at most two. We have to show that it is two (and not one). The result will follow from the next three lemmas.

1327 **Lemma A.8.** Let $Q_1 = {\tilde{q}_1 = 0}, Q_2 = {\tilde{q}_2 = 0}$ be two linearly independent 1328 quadrics over K. Then the codimension of $Q_1 \cap Q_2$ is one if and only if \tilde{q}_1 and 1329 \tilde{q}_2 have a mutual affine linear factor l(x).

1330 Proof. Let the codimension of the intersection be one. In this case if one of 1331 Q_1, Q_2 is irreducible, then $Q_1 = Q_2$ by Lemma A.3 with $Y = Q_1 \cap Q_2$. How-1332 ever, this is impossible by Lemma A.2. (i) since \tilde{q}_1 and \tilde{q}_2 are independent.

Therefore, by Lemma A.2. (ii) $Q_1 = H_1 \cup H_2$ and $Q_2 = H'_1 \cup H'_2$, with hyper-1333 planes H_1, \ldots, H'_2 . If all $H_i \cap H'_i$ are of codimension two, then 1334

1335
$$\operatorname{codim} Q_1 \cap Q_2 = \operatorname{codim} \left(\cup (H_i \cap H'_j) \right) = \min(\operatorname{codim} H_i \cap H'_j) = 2.$$

Therefore, at least one of $H_i \cap H'_i$ is of codimension one and then we have 1336 $\ker(l(x)) = H_i \cap H'_i \subset \mathcal{Q}_1 \cap \mathcal{Q}_2$ for an affine linear l(x). Hence, l(x) divides 1337 both \tilde{q}_1 and \tilde{q}_2 by Lemma A.2. (ii). 1338

The inverse statement is obvious. 1339

Lemma A.9. For any N > 2, if the matrix α is irreducible and $(z_1, v) \in Q_{1L}^0 \cap$ 1340 A_2 is such that $Lz_1, Lv \neq 0$ in K^d , then the polynomials $q_i^{(p)}(\cdot, Lz_1, Lv)$, 1341 1 < j < N are linearly independent over K. In particular, each $q_i^{(p)}$ is a 1342 nonzero polynomial. 1343

Proof. Consider a linear combination $\sum_{1 \le j \le N} c_j q_j^{(p)}$. By the homogeneity in 1344 (z_2,\ldots,z_{N-1}) , it vanishes identically if and only if 1345

$$\sum_{1 < j < N} c_j z_j \cdot (\alpha_{j1}(Lz_1) + \alpha_{jN}(Lv)) \equiv 0,$$

$$\sum_{1 < i,j < N} c_j (\alpha_{j1} - \alpha_{jN}\alpha_{1i}) z_j \cdot z_i \equiv 0.$$
(A.7)

1

Arguing by induction and using that the matrix α is irreducible, we construct 1347 a partition $E_0, \ldots, E_M, M \ge 1$, of the set $\{1, \ldots, N\}$ such that $E_0 = \{1, N\}$ 1348 and for $n \ge 1$, 1349

1350
$$E_n = \{j : \alpha_{jl} = 0 \ \forall l \in E_{n'}, \ n' \le n-2, \ \text{and} \ \exists l' \in E_{n-1} \ \text{such that} \ \alpha_{jl'} \ne 0\}.$$

Since $(z_1, v) \in Q_{1L}^0$ and $Lz_1, Lv \neq 0$ in K^d , then the term in brackets in the 1351 first line of (A.7) is not identically zero for each $j \in E_1$, so $c_j = 0$ for every 1352 $j \in E_1$. Using this in the second line of (A.7), we get: 1353

$$\sum_{n=2}^{M} \sum_{m=n-1}^{M} \sum_{j \in E_n} \sum_{i \in E_m} c_j \alpha_{ji} z_j \cdot z_i \equiv 0.$$

This relation holds if and only if $(c_i - c_i)\alpha_{ji} = 0$ for all $j \in E_n, 2 \leq n \leq M$, 1355 and $i \in E_m$, $n-1 \leq m \leq M$. We know that $c_j = 0$ if $j \in E_1$. Starting from 1356 n=2 and arguing by induction in n, we find that if $c_i=0$ for all $i \in E_{n-1}$, 1357 then $c_j = 0$ for all $j \in E_n$. Indeed, for any $j \in E_n$ there exists at least one 1358 $i \in E_{n-1}$ such that $\alpha_{ji} \neq 0$ by the definition of E_i , so relation $(c_j - c_i)\alpha_{ji} = 0$ 1359 implies that $c_j = 0$ if $j \in E_n$. That is, $c_j \equiv 0$. \square 1360

Lemma A.10. For any N > 2, if the matrix α is irreducible and $(z_1, v) \in$ 1361 $Q_{1L}^0 \cap A_2$ is such that $Lz_1, Lv \neq 0$ in K^d , then the polynomials $q_i^{(p)}(\cdot, Lz_1, Lv)$, 1362 1 < j < N, are irreducible. 1363

Proof. Each polynomial $q_i^{(p)}$ has degree one or two. If its degree is one the 1364 assertion is obvious. Now let the degree be two. Note that in view of (3.15)1365 $q_j^{(p)}$ can be written as the scalar products $q_j^{(p)} = z_j \cdot l_j(z_2, \dots, z_{N-1}; z_1, v) \mod d$ 1366

p, where l_j are surjective affine functions $l_j : K^{d(N-2)} \longrightarrow K^d$. But such scalar 1367 product cannot vanish for $d \geq 2 > 1$ on a hyperplane $H \subset K^{d(N-2)}$ which 1368 by Lemma A.2. ii) would be the case for a reducible quadric. Indeed, only two 1369 cases can occur: 1370

- (a) The coefficient α of z_i in l_i is nonzero, or 1371
- (b) It is zero but then the coefficient β of some other z_i is nonzero. 1372

In case (a) take the two-dimensional plane $P(x_1, x_2)$ in the whole space, gener-1373 ated by two orthogonal vectors from the z_i -space, where the first basis vector 1374 is parallel to $\alpha_{1j}z_1 + \alpha_{Nj}v \neq 0$ (this vector is nonzero since $(z_1, v) \in Q_{1L}^0$ and 1375 $Lz_1, Lv \neq 0$ in K^d , and for the case a) we have $\alpha_{1i}, \alpha_{Ni} \neq 0$. Then the restric-1376 tion of $q_j^{(p)} = 0$ on P is $\alpha(x_1^2 + x_2^2) + c_1 x_1 = 0$ with $c_1 \neq 0$, which is isomorphic to $x_1^2 + x_2^2 = C \neq 0$. This plane quadric in $P(x_1, x_2)$ cannot contain $P(x_1, x_2)$ 1377 1378 $\cap H$ (a line or the whole $P(x_1, x_2)$). Indeed, otherwise, supposing by symme-1379 try that the quadric contains a line $x_1 = ax_2 + b$, we would have that the 1380 polynomial 1381

1382
$$a^2x_2^2 + 2abx_2 + b^2 + x_2^2 - C$$

1383 $= (a^2 + 1)x_2^2 + 2abx_2 + b^2 - C$

vanishes identically. This implies ab = 0, and if a = 0 then the term 1384 $(a^{2}+1)x_{2}^{2} = x_{2}^{2} \neq 0$, while for b = 0 the term $b^{2} - C = -C \neq 0$. 1385

Similarly, in case b) we take the four-dimensional vector subspace P'1386 generated by the two first basis vectors in the z_i space and the two first basis 1387 vectors in the z_i space. The restriction of $q_i^{(p)} = 0$ on P' is then $\beta(x_1y_1 + y_1)$ 1388 $(x_2y_2) + c_1x_1 + c_2x_2 = 0$, isomorphic to $x_1y_1 + x_2y_2 = C$ which cannot contain 1389 $P'(x_1, x_2, y_1, y_2) \cap H$. Indeed, else, supposing by symmetry that $P'(x_1, x_2, y_1, y_2)$ 1390 $\cap H \supset \{x_1 = a_1x_2 + b_1y_1 + b_2y_2 - c\}$ we get that the following quadratic function 1391 of x_2, y_1, y_2 : 1392

1393
$$(a_1x_2 + b_1y_1 + b_2y_2 - c) \quad y_1 + x_2y_2 - C$$

1394 $= a_1x_2y_1 + b_1y_1^2 + b_2y_1y_2 - cy_1 + x_2y_2 - C$

vanishes identically, which is clearly wrong. 1395

End of the proof of Lemma A.7. Since each $q_j^{(p)}$ is a nonzero polynomial of degree one or two, then to prove Lemma A.7 we have to consider three cases. In 1396 1397 the first case both polynomials $q_2^{(p)}$ and $q_3^{(p)}$ are linear. Then the codimension 1398 of the intersection $\tilde{Q}^{(p)}$ is two since they are linearly independent. In the sec-1399 ond case both $q_2^{(p)}$ and $q_3^{(p)}$ are quadratic. Then, according to Lemma A.8, the 1400 codimension still is two since the polynomials are irreducible by Lemma A.10. 1401 Finally in the last case, when one polynomial is linear and another one is qua-1402 dratic, the assertion is clear since then the AAS in question is an intersection 1403 of a quadratic irreducible surface with a hyperplane. Thus, its codimension is 1404 two by Lemma A.2. (ii). 1405

Remark A.11. The proof of Lemma A.7 follows from three lemmas. Two of 1406 them are valid for any N > 2, but Lemma A.8 holds only for N = 4 (and 1407

tautologically holds for smaller N). Still the bi-linear (or linear) nature of the 1408 polynomials $q_i^{(p)}$ and direct analysis of the AAS $\tilde{Q}^{(p)}$, jointly with the two 1409 lemmas, valid for any N > 2, allow to prove by hand Lemma A.7 for "not too 1410 high" values of N, and thus, to prove for those N's Theorem 3.2. Unfortunately, 1411 for the moment we cannot prove the theorem for all N > 2; cf. Conjecture 3.8. 1412

Appendix B. Proof of Proposition 4.3 and Lemma 4.4 1413

We prove Proposition 4.3 in Sects. B.1-B.6 and Lemma 4.4 in Sect. B.7. 1414

B.1. Beginning of the Proof of Proposition 4.3 1415

The proof of the proposition is somewhat cumbersome since we have to con-1416 sider a number of different terms and different cases. During the proof we will 1417 often skip the upper index (0), so by writing a and a_s we will mean $a^{(0)}$ and $a_s^{(0)}$. 1418 We will also skip the dependence on τ_0 by writing $c_s^{(i)}(\tau;\tau_0)$ and $\Delta a_s^{(i)}(\tau;\tau_0)$ 1419 as $c_s^{(i)}(\tau)$ and $\Delta a_s^{(i)}(\tau)$. Besides, for a complex function $(w_{s_1,\ldots,s_k}, s_j \in \mathbb{Z}_L^d)$ we 1420 denote 1421

22
$$\sum_{s_1,...,s_k \in \mathbb{Z}_L^d} w_{s_1,...,s_k} = L^{-kd} \sum_{s_1,...,s_k \in \mathbb{Z}_L^d} w_{s_1,...,s_k} ,$$

and we introduce the symmetrization 1423

1424
$$\mathcal{Y}_{s}^{sym}(u,v,w;t) = \frac{L^{-d}}{3} \sum_{1,2,3} \delta_{3s}^{\prime 12} \delta(\omega_{3s}^{12}) \left(u_1 v_2 \bar{w}_3 + v_1 w_2 \bar{u}_3 + w_1 u_2 \bar{v}_3 \right).$$

We recall that $Q_{s,L}$ is given by formula (4.30) and first consider the term 1425 $\mathbb{E}\Delta a_s^{(2)}(\tau)\bar{a}_s(\tau_0+\tau)$. Inserting the identity $a^{(1)}(\tau_0+l)=c^{(1)}(l)+\Delta a^{(1)}(l)$ into 1426 formula (4.27) for $\Delta a_s^{(2)}$, we obtain 1427

1428
$$\mathbb{E}\Delta a_s^{(2)}(\tau)\bar{a}_s(\tau_0+\tau) = N_s + \widetilde{N}_s$$

where 1429

1430
$$N_s := i \mathbb{E} \Big(\bar{a}_s(\tau_0 + \tau) \int_0^\tau e^{-\gamma_s(\tau - l)} 3 \mathcal{Y}_s^{sym}(a(\tau_0 + l), a(\tau_0 + l), \Delta a^{(1)}(l)) \, \mathrm{d}l \Big)$$
1431 (B.1)

1432

1433
$$\widetilde{N}_s := i \mathbb{E}\Big(\bar{a}_s(\tau_0 + \tau) \int_0^\tau e^{-\gamma_s(\tau - l)} 3\mathcal{Y}_s^{sym}(a(\tau_0 + l), a(\tau_0 + l), c^{(1)}(l)) \,\mathrm{d}l\Big).$$

Thus, 1434

and

14

14

Thus,

$$Q_{s,L} = L^2 \left(\mathbb{E} |\Delta a_s^{(1)}(\tau)|^2 + 2\Re N_s + 2\Re \mathbb{E} \Delta a_s^{(1)}(\tau) \bar{c}_s^{(1)}(\tau) + 2\Re \widetilde{N}_s \right), \quad s \in \mathbb{R}^d.$$
(B.2)

We will analyse the four terms above term by term. 1437

B.2. The First Term of $Q_{s,L}$ in (**B.2**) 1438

Due to (4.27), we have 1439

1440
$$\mathbb{E}|\Delta a_{s}^{(1)}(\tau)|^{2} = \mathbb{E}\int_{\tau_{0}}^{\tau_{0}+\tau} \mathrm{d}l \int_{\tau_{0}}^{\tau_{0}+\tau} \mathrm{d}l' \, e^{-\gamma_{s}(2\tau_{0}+2\tau-l-l')} \mathcal{Y}_{s}(a(l)) \, \overline{\mathcal{Y}_{s}(a(l'))}.$$
1441 (B.3)

Writing the functions \mathcal{Y}_s explicitly and applying the Wick theorem, in view of 1442 (2.13) we find 1443

1444
$$\mathbb{E}|\Delta a_s^{(1)}(\tau)|^2 = 2L^{-2d} \sum_{1,2} \delta_{3s}^{\prime 12} \delta(\omega_{3s}^{12}) \int_{\tau_0}^{\tau_0+\tau} \mathrm{d}l \int_{\tau_0}^{\tau_0+\tau} \mathrm{d}l' \, e^{-\gamma_s(2\tau_0+2\tau-l-l')}$$

$$\mathbb{E}a_1(l)\bar{a}_1(l')\mathbb{E}a_2(l)\bar{a}_2(l')\mathbb{E}\bar{a}_3(l)a_3(l')$$

and note that 1447

1448
$$\int_{\tau_0}^{\tau_0+\tau} \mathrm{d}l \int_{\tau_0}^{\tau_0+\tau} \mathrm{d}l' \, e^{-\gamma_s(2\tau_0+2\tau-l-l')} \le \tau^2.$$

On account of (2.13), we can bound 1449

1450
$$\mathbb{E}|\Delta a_s^{(1)}(\tau)|^2 \le 2\tau^2 \sum_{1,2} \delta_{3s}^{\prime 12} \delta(\omega_{3s}^{12}) B_{123},$$

where $B_{123} = B_1 B_2 B_3$. Since B_{123} with $s_3 = s - s_1 - s_2$ is a Schwartz function 1451 of s, s_1, s_2 then Theorem 3.2 with N = 2 applies and we find 1452

1453
$$\mathbb{E}|\Delta a_s^{(1)}(\tau)|^2 \le C^{\#}(s)L^{-2}\tau^2.$$
(B.4)

B.3. The Second Term of $Q_{s,L}$ in (**B.2**) 1454

To study the term $2\Re N_s$, we use the same strategy as above. Namely, express-1455 ing in (B.1) the function $3\mathcal{Y}_s^{sym}$ via \mathcal{Y}_s , we write N_s as $N_s = N_s^1 + 2N_s^2$, $s \in \mathbb{R}^d$, 1456 where 1457

1458
$$N_s^1 = i \mathbb{E}\Big(\bar{a}_s(\tau_0 + \tau) \int_0^\tau e^{-\gamma_s(\tau - l)} \mathcal{Y}_s(a(\tau_0 + l), a(\tau_0 + l), \Delta a^{(1)}(l)) \,\mathrm{d}l\Big),$$

1459
$$N_s^2 = i \mathbb{E} \Big(\bar{a}_s(\tau_0 + \tau) \int_0^t e^{-\gamma_s(\tau - l)} \mathcal{Y}_s(\Delta a^{(1)}(l), a(\tau_0 + l), a(\tau_0 + l)) \, \mathrm{d}l \Big).$$

Term N_s^1 . Writing explicitly the function \mathcal{Y}_s and then $\Delta \bar{a}_3^{(1)}$, we get

1462
$$N_s^1 = i L^{-d} \sum_{1,2} \delta_{3s}^{\prime 12} \delta(\omega_{3s}^{12}) \int_0^{\tau} \mathrm{d}l \, e^{-\gamma_s(\tau-l)}$$

1463
$$\times \mathbb{E} \left(a_1(\tau_0 + l) a_2(\tau_0 + l) \Delta \bar{a}_3^{(1)}(l) \bar{a}_s(\tau_0 + \tau) \right)^{\tau_1}$$

$$= L^{-2d} \sum_{1,2} \sum_{1',2'} \delta_{3s}^{\prime 12} \delta_{3'3}^{\prime 1'2'} \delta(\omega_{3s}^{12}) \delta(\omega_{3'3}^{1'2'}) \int_0^\tau \mathrm{d}l \, \int_0^l \mathrm{d}l' \, e^{-\gamma_s(\tau-l)} e^{-\gamma_3(l-l')}$$

1465
$$\times \mathbb{E} \big(a_1(\tau_0+l)a_2(\tau_0+l)\bar{a}_{1'}(\tau_0+l')\bar{a}_{2'}(\tau_0+l')a_{3'}(\tau_0+l')\bar{a}_s(\tau_0+\tau) \big).$$
(B.5)

1466

By the Wick theorem, we need to take the summation only over $s_{1'}, s_{2'}, s_{3'}$ 1467 satisfying $s_{1'} = s_1$, $s_{2'} = s_2$, $s_{3'} = s$ or $s_{1'} = s_2$, $s_{2'} = s_1$, $s_{3'} = s$. Since in both cases we get $\delta_{3'3}^{1'2'} = \delta_{3s}^{\prime 12}$ and $\omega_{3'3}^{1'2'} = \omega_{3s}^{12}$, we find 1468 1469

1470
$$N_{s}^{1} = 2 \sum_{1,2} \delta_{3s}^{\prime 12} \delta(\omega_{3s}^{12}) \int_{0}^{\tau} \mathrm{d}l \int_{0}^{l} \mathrm{d}l' \, e^{-\gamma_{s}(\tau-l)-\gamma_{3}(l-l')} \times \mathbb{E}a_{1}(\tau_{0}+l)\bar{a}_{1}(\tau_{0}+l') \, \mathbb{E}a_{2}(\tau_{0}+l)\bar{a}_{2}(\tau_{0}+l') \, \mathbb{E}a_{s}(\tau_{0}+l')\bar{a}_{s}(\tau_{0}+\tau).$$

Arguing as in Sect. B.2 we find 1473

1474

 $|N_{2}^{1}| < C^{\#}(s)L^{-2}\tau^{2}.$ (B.6)

Term N_s^2 . By literally repeating the argument we have applied to N_s^1 , we 1475 find that 1476

1477
$$N_{s}^{2} = i L^{-d} \sum_{1,2} \delta_{3s}^{\prime 12} \delta(\omega_{3s}^{12}) \int_{0}^{\tau} \mathrm{d}l \, e^{-\gamma_{s}(\tau-l)}$$
1478
$$\times \mathbb{E} \left(\Delta a_{1}^{(1)}(l) a_{2}(\tau_{0}+l) \bar{a}_{3}(\tau_{0}+l) \bar{a}_{s}(\tau_{0}+\tau) \right)$$

1478

$$= -L^{-2d} \sum_{1,2} \sum_{1',2'} \delta_{3s}^{\prime 12} \delta_{3'1}^{\prime 1'2'} \delta(\omega_{3s}^{12}) \delta(\omega_{3'1}^{1'2'}) \int_0^\tau \mathrm{d}l \int_0^l \mathrm{d}l' \, e^{-\gamma_s(\tau-l)} e^{-\gamma_1(l-l')}$$

$$\times \mathbb{E} \Big(a_{1'}(\tau_0 + l') a_{2'}(\tau_0 + l') \bar{a}_{3'}(\tau_0 + l') a_2(\tau_0 + l) \bar{a}_3(\tau_0 + l) \bar{a}_s(\tau_0 + \tau) \Big).$$

By the Wick theorem, we should take summation either under the condition 1482 $s_{1'} = s_3, s_{2'} = s, s_{3'} = s_2$ or $s_{1'} = s, s_{2'} = s_3, s_{3'} = s_2$. Since in both cases $\delta_{3'1}^{\prime 1'2'} = \delta_{3s}^{\prime 12}$ and $\omega_{3'1}^{1'2'} = -\omega_{3s}^{12}$, then 1483 1484

1485
$$N_{s}^{2} = -2 \sum_{1,2} \delta_{3s}^{\prime 12} \delta(\omega_{3s}^{12}) \int_{0}^{\tau} \mathrm{d}l \int_{0}^{l} \mathrm{d}l' e^{-\gamma_{s}(\tau-l)} e^{-\gamma_{1}(l-l')}$$

1486
$$\mathbb{E}a_{2}(\tau_{0}+l) \bar{a}_{2}(\tau_{0}+l') \mathbb{E}a_{3}(\tau_{0}+l') \bar{a}_{3}(\tau_{0}+l) \mathbb{E}a_{s}(\tau_{0}+l') \bar{a}_{s}(\tau_{0}+\tau) .$$

1487

1489

Again we get 1488

 $|c_s^2| \le C^{\#}(s)L^{-2}\tau^2$ (B.8)

(B.7)

B.4. The Third Term of $Q_{s,L}$ in (**B.2**) 1490

We have 1491

1492
$$\mathbb{E}\Delta a_s^{(1)}\bar{c}_s^{(1)}(\tau) = \mathbb{E}\int_{\tau_0}^{\tau_0+\tau} e^{-\gamma_s(\tau_0+\tau-l)}\mathcal{Y}_s(a(l))\,\mathrm{d}l\int_0^{\tau_0} e^{-\gamma_s(\tau_0+\tau-l')}\overline{\mathcal{Y}_s(a(l'))}\,\mathrm{d}l'.$$

This expression coincides with (B.3) in which the integral $\int_{\tau_0}^{\tau_0+\tau} dl'$ is replaced 1493 by $\int_0^{\tau_0} dl'$. Then, 1494

1495
$$\mathbb{E}\Delta a_{s}^{(1)}(\tau)\bar{c}_{s}^{(1)}(\tau) = 2\sum_{1,2} \delta_{3s}^{\prime 12} \delta(\omega_{3s}^{12}) \int_{\tau_{0}}^{\tau_{0}+\tau} \mathrm{d}l \int_{0}^{\tau_{0}} \mathrm{d}l' \, e^{-\gamma_{s}(2\tau_{0}+2\tau-l-l')} \\ \mathbb{E}a_{1}(l)\bar{a}_{1}(l') \, \mathbb{E}a_{2}(l)\bar{a}_{2}(l') \, \mathbb{E}\bar{a}_{3}(l)a_{3}(l'),$$

1486
$$\mathbb{E}a_1(l)\bar{a}_1(l')\mathbb{E}a_2(l)\bar{a}_2(l')\mathbb{E}\bar{a}_3$$

Expressing the correlations $\mathbb{E}a_i(l)\bar{a}_i(l')$ through (2.13), we get 1498

1499
$$\mathbb{E}\Delta a_{s}^{(1)}(\tau)\bar{c}_{s}^{(1)}(\tau) = 2\sum_{1,2}\delta_{3s}^{\prime 12}\delta(\omega_{3s}^{12})B_{123}\int_{0}^{\tau}\mathrm{d}l\,e^{-2\gamma_{s}(\tau-l)-\gamma_{123s}l}$$

1500
$$\int_{0}^{\tau_{0}} \mathrm{d}l' \, e^{-\gamma_{123s}\tau_{0}} e^{\gamma_{s}l'} \prod_{j=1,2,3} \left(e^{\gamma_{j}l'} - e^{-\gamma_{j}l'} \right)$$

For the integral in the first line, we have 1502

$$T_s := \int_0^\tau \mathrm{d}l \, e^{-2\gamma_s(\tau-l)-\gamma_{123s}l} = \begin{cases} \tau e^{-2\gamma_s\tau} & \text{if } 2\gamma_s = \gamma_{123s} \\ \frac{e^{-2\gamma_s\tau}-e^{-\gamma_{123s}\tau}}{\gamma_{123s}-2\gamma_s} & \text{elsewhere} \end{cases}.$$
 (B.9)

For the integral in the second line, let us denote 1504

1505
$$\mathcal{T}^{j} := \int_{0}^{\tau_{0}} \mathrm{d}l \, e^{-\gamma_{123s}\tau_{0}} e^{\gamma_{j}l} \prod_{k \neq j} \left(e^{\gamma_{k}l} - e^{-\gamma_{k}l} \right), \tag{B.10}$$

where $j, k \in \{1, 2, 3, s\}$. Then, 1506

1507
$$0 \le \mathcal{T}^j \le 1/\gamma_{123s}.$$
 (B.11)

Due to (B.9) and (B.10) we get 1508

1509
$$\mathbb{E}\Delta a_s^{(1)}(\tau)\bar{c}_s^{(1)}(\tau) = 2\sum_{1,2} \delta_{3s}^{\prime 12} \delta(\omega_{3s}^{12}) B_{123} T_s \mathcal{T}^s.$$
(B.12)

B.5. The Fourth Term of $Q_{s,L}$ in (**B.2**) 1510

To study the term $2\Re \widetilde{N}_s$, as in Sect. B.3, we write \widetilde{N}_s as $\widetilde{N}_s = \widetilde{N}_s^1 + 2\widetilde{N}_s^2$, $s \in$ 1511 \mathbb{R}^d , where 1512

1513
$$\widetilde{N}_{s}^{1} = i \mathbb{E} \Big(\bar{a}_{s}(\tau_{0} + \tau) \int_{0}^{\tau} e^{-\gamma_{s}(\tau - l)} \mathcal{Y}_{s}(a(\tau_{0} + l), a(\tau_{0} + l), c^{(1)}(l)) \, \mathrm{d}l \Big),$$

 $\widetilde{N}_{s}^{2} = i \mathbb{E}\Big(\bar{a}_{s}(\tau_{0}+\tau) \int_{0}^{\cdot} e^{-\gamma_{s}(\tau-l)} \mathcal{Y}_{s}(c^{(1)}(l), a(\tau_{0}+l), a(\tau_{0}+l)) \,\mathrm{d}l\Big).$ 5

Term \widetilde{N}_s^1 . Writing explicitly the function \mathcal{Y}_s and then $\bar{c}^{(1)}$, we get

1517
$$\widetilde{N}_{s}^{1} = L^{-2d} \sum_{1,2} \sum_{1',2'} \delta_{3s}^{\prime 12} \delta_{3'3}^{\prime 1'2'} \delta(\omega_{3s}^{12}) \delta(\omega_{3'3}^{1'2'}) \int_{0}^{\tau} \mathrm{d}l \int_{-\tau_{0}}^{0} \mathrm{d}l' \, e^{-\gamma_{s}(\tau-l)} e^{-\gamma_{3}(l-l')}$$

$$\times \mathbb{E} \Big(a_1(\tau_0+l) a_2(\tau_0+l) \bar{a}_{1'}(\tau_0+l') \bar{a}_{2'}(\tau_0+l') a_{3'}(\tau_0+l') \bar{a}_s(\tau_0+\tau) \Big).$$

Again, this is the same expression as (B.5), with the integration over dl' ranging 1520 from $-\tau_0$ to 0 instead of from 0 to l. Thus, by the Wick theorem, we obtain 1521

1522
$$\widetilde{N}_{s}^{1} = 2 \sum_{1,2} \delta_{3s}^{\prime 12} \delta(\omega_{3s}^{12}) \int_{0}^{\tau} \mathrm{d}l \int_{-\tau_{0}}^{0} \mathrm{d}l' \, e^{-\gamma_{s}(\tau-l)-\gamma_{3}(l-l')} \times \mathbb{E}a_{1}(\tau_{0}+l)\bar{a}_{1}(\tau_{0}+l') \mathbb{E}a_{2}(\tau_{0}+l)\bar{a}_{2}(\tau_{0}+l') \mathbb{E}a_{s}(\tau_{0}+l')\bar{a}_{s}(\tau_{0}+\tau).$$

 $\frac{1523}{1524}$

Following the line of Sect. B.4, we express the correlations through (2.13) and 1525 get 1526

$$\widetilde{N}_{s}^{1} = 2 \sum_{1,2} \delta_{3s}^{\prime 12} \delta(\omega_{3s}^{12}) B_{12s} \int_{0}^{\tau} \mathrm{d}l \, e^{-2\gamma_{s}(\tau-l)-\gamma_{123s}l} \\ \int_{0}^{\tau_{0}} \mathrm{d}l^{\prime} \, e^{-\gamma_{123s}\tau_{0}} e^{\gamma_{3}l^{\prime}} \left(e^{\gamma_{s}l^{\prime}} - e^{-\gamma_{s}l^{\prime}} \right) \prod_{j=1,2} \left(e^{\gamma_{j}l^{\prime}} - e^{-\gamma_{j}l^{\prime}} \right) \\ = 2 \sum_{1,2} \delta_{3s}^{\prime 12} \delta(\omega_{3s}^{12}) B_{12s} T_{s} T^{3}.$$
(B.13)

1528

1529 1530

Term \widetilde{N}_s^2 . Literally repeating the argument which we have applied to \widetilde{N}_s^1 , 1531 we find that the term \tilde{N}_s^2 is given by the same expression as (B.7) with the integral \int_0^l replaced by $\int_{-\tau_0}^0$: 1532 1533

1534
$$\widetilde{N}_{s}^{2} = -2 \sum_{1,2} \delta_{3s}^{\prime 12} \delta(\omega_{3s}^{12}) \int_{0}^{\tau} \mathrm{d}l \int_{-\tau_{0}}^{0} \mathrm{d}l' \, e^{-\gamma_{s}(\tau-l)} e^{-\gamma_{1}(l-l')} \\ \mathbb{E}a_{2}(\tau_{0}+l) \bar{a}_{2}(\tau_{0}+l') \, \mathbb{E}a_{3}(\tau_{0}+l') \bar{a}_{3}(\tau_{0}+l) \, \mathbb{E}a_{s}(\tau_{0}+l') \bar{a}_{s}(\tau_{0}+\tau).$$

 $\frac{1535}{1536}$

$$\widetilde{N}_{s}^{2} = -2 \sum_{1,2} \delta_{3s}^{\prime 12} \delta(\omega_{3s}^{12}) B_{23s} T_{s} \mathcal{T}^{1}.$$
(B.14)

B.6. End of the Proof 1539

Inserting formulas (B.12), (B.13) and (B.14), as well as (B.4), (B.6), (B.8) in 1540 (B.2), we get 1541

$$\left| Q_{s,L} - 4L^2 T_s \sum_{1,2} \delta_{3s}^{\prime 12} \delta(\omega_{3s}^{12}) \left(B_{123} T^s + B_{12s} T^3 - 2B_{23s} T^1 \right) \right| \le C^{\#}(s) \tau^2.$$
1543
(B.15)

1543

Note that the terms \mathcal{Z}^{j} defined in (4.14) can be written as 1544

$$\mathcal{Z}^{j} = \frac{\mathcal{T}^{j}}{\prod_{k \neq j} (1 - e^{-2\gamma_{k}\tau_{0}})}.$$
(B.16)

1545

The relations (2.13)-(2.14) imply that for any permutation
$$(k_1, k_2, k_3, k_4)$$
 of
(1,2,3,s) we have $B_{k_1k_2k_3} = n_{k_1}^{(0)} n_{k_2}^{(0)} n_{k_3}^{(0)} / \prod_{m=k_1,k_2,k_3} (1 - e^{-2\gamma_m \tau_0})$, where
 $n_{k_i}^{(0)} = n_{k_i,L}^{(0)}(\tau_0)$. Together with (B.16), this implies

1549
$$B_{k_1k_2k_3}\mathcal{T}^{k_4} = \mathcal{Z}^{k_4} n_{k_1}^{(0)} n_{k_2}^{(0)} n_{k_3}^{(0)}.$$
(B.17)

By symmetry, the term $2B_{23s}\mathcal{T}^1$ in (B.15) can be replaced by $B_{23s}\mathcal{T}^1 + B_{13s}\mathcal{T}^2$. 1550 Then, inserting (B.17) in (B.15) we get 1551

$$|Q_{s,L} - \mathcal{X}_s| \le \left| 4L^2(T_s - \tau) \sum_{1,2} \delta_{3s}^{\prime 12} \delta(\omega_{3s}^{12}) \left(\mathcal{T}^s B_{123} + \mathcal{T}^3 B_{12s} - 2\mathcal{T}^s B_{23s} \right) \right| + C^{\#}(s)\tau^2,$$

with \mathcal{X}_s defined in (4.33). Finally, we point out that $|(T_s - \tau)\mathcal{T}^j| \leq 3\tau^2$, due to (B.11) and since $|T_s - \tau| \leq 3\tau^2 \gamma_{123s}$. So, the bound

1555
$$\left| L^2(T_s - \tau) \left| \sum_{1,2} \delta_{3s}^{\prime 12} \delta(\omega_{3s}^{12}) \left(\mathcal{T}^s B_{123} + \mathcal{T}^3 B_{12s} - 2\mathcal{T}^s B_{23s} \right) \right| \le C^{\#}(s) \tau^2,$$

is a consequence of Theorem 3.2. This concludes the proof of Proposition 4.3.

1557 B.7. Proof of Lemma 4.4

Note that $\partial_{s_j} f(\gamma_j) = f'(\gamma_j) \partial_{s_j} \gamma_j$, where $\partial_{s_j} \gamma_j$ (as well as higher-order derivatives of γ_j) have at most polynomial growth at infinity. Then, using the definition (4.14) of \mathcal{Z}^j we find

1561
$$\left|\partial_{\vec{s}}^{\mu} \mathcal{Z}^{j}(\tau_{0}, \vec{s})\right| = \sum_{n_{1}+n_{2}+n_{3}+n_{s}=1}^{|\mu|} P(\vec{s}; n_{1}, \dots, n_{s}) \int_{0}^{\tau_{0}} \mathrm{d}l \, (\tau_{0}-l)^{n_{j}} \mathrm{e}^{-\gamma_{j}(\tau_{0}-l)}$$

1562 $\prod_{m \neq j} \frac{d^{n_{m}}}{d\gamma_{m}^{n_{m}}} \left(\frac{\sinh(\gamma_{m}l)}{\sinh(\gamma_{m}\tau_{0})}\right),$ (B.18)

where $P(\vec{s}; a_1, \ldots, a_n)$ denotes a function of \vec{s} , dependent on parameters (a_1, \ldots, a_n), having at most a polynomial growth at infinity. Using the relation $\frac{\sinh(\gamma l)}{\sinh(\gamma \tau_0)} = \frac{e^{-\gamma(\tau_0-l)} - e^{-\gamma(l+\tau_0)}}{1 - e^{-2\gamma\tau_0}}$ we find by induction that

$$\frac{d^n}{d\gamma^n} \left(\frac{\sinh(\gamma l)}{\sinh(\gamma \tau_0)} \right) = \sum_{k+m+p=n} c_{k,m,p} I_{k,m,p}(l,\tau_0,\gamma),$$

1567 where $c_{k,m,p}$ are constants,

1568
$$I_{k,m,p} = \left((\tau_0 - l)^k e^{-\gamma(\tau_0 - l)} - (l + \tau_0)^k e^{-\gamma(l + \tau_0)} \right) \frac{\tau_0^{m+p} e^{-2\gamma m \tau_0}}{(1 - e^{-2\gamma \tau_0})^{m+1}}$$
1569 (B.19)

and $p \neq 0$ only if $m \neq 0$. For $\tau_0 \geq \gamma^{-1}$ the terms $I_{k,m,p}$ are bounded in absolute values by absolute constants $C_{k,m,p}$, where we recall that $0 \leq l \leq \tau_0$ and $\gamma \geq 1$. Let now $\tau_0 \leq \gamma^{-1}$. In this case, since k + m + p = n,

1573
$$|I_{k,m,p}| \le 2 \frac{(l+\tau_0)^k \tau_0^{m+p}}{(1-e^{-2\gamma\tau_0})^{m+1}} \le 2^{k+1} \frac{\tau_0^n}{(1-e^{-2\gamma\tau_0})^{m+1}}.$$

So, in the case $m \le n-1$ we have $|I_{k,m,p}| \le C_{k,m,p}$ uniformly in $\tau_0 \le \gamma^{-1}$. If m = n (so k = p = 0) we use another estimate, following from (B.19):

1576
$$|I_{k,m,p}| \le C \frac{e^{-\gamma(\tau_0-l)} - e^{-\gamma(l+\tau_0)}}{(1-e^{-2\gamma\tau_0})^{m+1}} \tau_0^m = C\tau_0^m e^{-\gamma(\tau_0-l)} \frac{1-e^{-2\gamma l}}{(1-e^{-2\gamma\tau_0})^{m+1}} \le C_{k,m,p}$$

1577 uniformly in $\tau_0 \leq \gamma^{-1}$.

¹⁵⁷⁸ We have seen that the product in (B.18) is bounded uniformly in \vec{s} , l and ¹⁵⁷⁹ τ_0 , so the integral over l is also bounded uniformly in \vec{s} and τ_0 .

1580 Appendix C. Proof of Proposition 2.1

The proof uses the theory of Feynman diagrams, presented in Sect. 2. For N = 0 the assertion is trivial. For $N \ge 1$ in Proposition 8.7 of [8] it is proven that $\mathbb{E}(\mathfrak{a}_s^{(m)}(\tau_1)\bar{\mathfrak{a}}_s^{(n)}(\tau_2) - a_s^{(m)}(\tau_1)\bar{\mathfrak{a}}_s^{(n)}(\tau_2))$ equals to

$$\sum_{\mathfrak{F}\in\mathfrak{F}_{m,n}^+\setminus\mathfrak{F}_{m,n}} c_{\mathfrak{F}}\mathcal{J}_s(\mathfrak{F}) + \sum_{\mathfrak{F}\in\mathfrak{F}_{m,n}} c_{\mathfrak{F}}\mathcal{J}_s^2(\mathfrak{F}), \qquad (C.1)$$

where $\mathfrak{F}_{m,n}^+$ is a certain (finite) set of *extended* Feynman diagrams,¹³ $c_{\mathfrak{F}}$ is a complex number of unit norm and $\mathcal{J}_s(\mathfrak{F})$, $\mathcal{J}_s^2(\mathfrak{F})$ are sums, similar to (2.19). In Section 8.6.3 of [8] are established the following bounds for these sums:

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$$|\mathcal{J}_{s}^{2}(\mathfrak{F})| \leq C^{\#}(s)L^{-Nd} \sum_{\substack{z \in \mathcal{Z}^{+}(\mathfrak{F}):\\z_{j}=0 \text{ for some } j,\\\omega^{\mathfrak{F}}(z)=0 \forall 1 \leq k \leq N}} C^{\#}(z), \qquad (C.2)$$

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$$\mathcal{Z}^{+}(\mathfrak{F}) = \left\{ z \in (\mathbb{Z}_{L}^{d})^{N} : z_{k} \neq 0 \Leftrightarrow \sum_{i=1}^{N} \alpha_{ki}^{\mathfrak{F}} z_{i} \neq 0 \quad \forall 1 \leq k \leq N \right\}$$

while the quadratic forms $\omega_k^{\mathfrak{F}}$ and the skew-symmetric matrix $\alpha^{\mathfrak{F}}$ are defined in Sect. 2.4. Note that possibly the diagram \mathfrak{F} does not belong to the set $\mathfrak{F}_{m,n}^{true}$, so that the matrix $\alpha^{\mathfrak{F}}$ may have zero columns and lines. On the other hand,

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$$|\mathcal{J}_{s}(\mathfrak{F})| \leq C^{\#}(s)L^{-Nd} \sum_{\substack{z \in \widetilde{Z}^{+}(\mathfrak{F}):\\ \widetilde{\omega}^{\mathfrak{F}}(z) = 0 \,\forall 1 \leq k \leq \widetilde{N}}} C^{\#}(z). \tag{C.3}$$

Here $\widetilde{N} = \widetilde{N}(\mathfrak{F}) < N$, quadratic forms $\widetilde{\omega}_{k}^{\mathfrak{F}}(z)$ are defined by relations (2.17), where N is replaced by \widetilde{N} and the matrix $(\alpha_{ij}^{\mathfrak{F}})$ – by a certain $\widetilde{N} \times \widetilde{N}$ -matrix $(\widetilde{\alpha}_{ji}^{\mathfrak{F}})$, also satisfying $\widetilde{\alpha}_{ji}^{\mathfrak{F}} = -\widetilde{\alpha}_{ij}^{\mathfrak{F}} \in \{0, \pm 1\}$ for all i, j. Accordingly the set $\widetilde{\mathcal{Z}}^{+}(\mathfrak{F}) \subset (\mathbb{Z}_{L}^{d})^{\widetilde{N}}$ is defined as $\mathcal{Z}^{+}(\mathfrak{F})$ above, but with N and $\alpha_{ij}^{\mathfrak{F}}$ replaced by \widetilde{N} and $\widetilde{\alpha}_{ij}^{\mathfrak{F}}$.

We first show that the term $\mathcal{J}_s^2(\mathfrak{F})$ is bounded by the r.h.s. of (2.3). To this end we write $\mathcal{Z}^+(\mathfrak{F}) = \bigcup_{\mathcal{K}} \mathcal{Z}_{\mathcal{K}}$, where the union is taken over all subsets $\mathcal{K} \subset \{1, \ldots, N\}$ and

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$$\mathcal{Z}_{\mathcal{K}}(\mathfrak{F}) = \left\{ z : z_k = \sum_{i=1}^N \alpha_{ki}^{\mathfrak{F}} z_i = 0 \ \forall k \in \mathcal{K} \text{ and } z_k \neq 0, \sum_{i=1}^N \alpha_{ki}^{\mathfrak{F}} z_i \neq 0 \ \forall k \notin \mathcal{K} \right\}.$$

1604 Then the r.h.s. of (C.2) takes the form

$$C^{\#}(s)L^{-Nd}\sum_{\substack{\mathcal{K}\neq\varnothing}\\\omega_{k}^{\mathfrak{F}}(z)=0\,\forall 1\leq k\leq N}}\sum_{\substack{z\in\mathcal{Z}_{\mathcal{K}}(\mathfrak{F}):\\\omega_{k}^{\mathfrak{F}}(z)=0\,\forall 1\leq k\leq N}}C^{\#}(z).$$
(C.4)

¹³These diagrams are defined similarly to the Feynman diagrams from Sect. 2.3.1, but now we allow to couple leaves not only from different blocks but also from the same block.

Note that on the set $\mathcal{Z}_{\mathcal{K}}(\mathfrak{F})$ we have $\omega_k(z) = 0$ for all $k \in \mathcal{K}$ and $\omega_k(z) =$ 1606 $2z_k \cdot \sum_{i \notin \mathcal{K}} \alpha_{ki}^{\mathfrak{F}} z_i$ for $k \notin \mathcal{K}$. Thus, the sum over z in (C.4) takes the form of 1607 the sum in (3.2), where $z = (z_i)_{i \notin \mathcal{K}}$ and N is replaced by 1608

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$$N - \varkappa, \quad \varkappa = \# \mathcal{K}.$$

We recall that $N \leq 4$ and $\mathcal{K} \neq \emptyset$, so that $N - \varkappa$ takes values 0, 1, 2 or 3. 1610 For the sets \mathcal{K} satisfying $N - \varkappa = 0$ we have $\mathcal{Z}_{\mathcal{K}}(\mathfrak{F}) = \{0\}$, so the sum (C.3) 1611 is bounded by $C^{\#}(s)L^{-Nd}$. Since the matrix $\alpha^{\mathfrak{F}}$ is skew-symmetric, then in 1612 the case $N - \varkappa = 1$ we have $\mathcal{Z}_{\mathcal{K}}(\mathfrak{F}) = \emptyset$, so the sum (C.3) vanishes. When 1613 $N - \varkappa = 2$ or 3 we apply Theorem 3.2 and see that the sum over z in (C.4) is 1614 bounded by $CL^{-(N-\varkappa)(1-d)}$. So 1615

$$|\mathcal{J}_{s}^{2}(\mathfrak{F})| \leq C^{\#}(s)L^{-Nd} \sum_{\kappa \neq \varnothing} L^{-(N-\varkappa)(1-d)} = C^{\#}(s) \sum_{\kappa \neq \varnothing} L^{-N+\varkappa(1-d)} \leq C_{1}^{\#}(s)L^{-N+1-d}.$$

Same argument implies that the r.h.s. of (C.3) also is bounded by the 1617 quantity $C^{\#}(s)L^{-N+1-d}$ (note that decomposing the r.h.s. of (C.3) as in (C.4) 1618 we get a new term with $\mathcal{K} = \emptyset$, but for it $N - \varkappa = N \leq 4$ and Theorem 3.2 1619 still applies). 1620

Appendix D. Case d = 21621

A difference between the cases $d \geq 3$ and d = 2 comes from Theorem 3.1 1622 since in the asymptotic, given by the latter, an additional log-factor appears 1623 when d = 2. To handle it we redefine the sum in (3.2), defining $S_{L,N}(\Phi)$, by 1624 multiplying it by $(\ln L)^{-N/2}$. So when $d = 2 S_{L,N}$ takes the form 1625

$$S_{L,N}(\Phi) := \frac{L^{N(1-d)}}{(\ln L)^{N/2}} \sum_{z \in \mathcal{Z}: \ \omega_j(z) = 0 \ \forall j} \Phi(z).$$
(D.1)

Accordingly the (d = 2)-analogy of (3.5) reads 1627

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$$\left| S_{L,2}(\Phi) - \frac{L^{2(1-d)}}{\ln L} \sum_{z \in \mathbb{Z}_{L}^{2d}: z_{1} \cdot z_{2} = 0} \Phi(z) \right| \leq \frac{CL^{2-d}}{\ln L} \|\Phi\|_{0,d+1} = \frac{C}{\ln L} \|\Phi\|_{0,3}.$$
(D.2)

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This approximation, jointly with a modification of the Heath–Brown result 1630 from [14], given in Theorem 1.4 of [10], implies the following version of Theo-1631 rem **3.1** for d = 2: 1632

Theorem D.1. Let d = 2. Then there exist constants $N_1, N_2 > 4$ such that if 1633 $\|\Phi\|_{N_1,N_2} < \infty,$ 1634

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$$\left| S_{L,2}(\Phi) - C_2 \int_{\Sigma_0} \Phi(z) \, \mu^{\Sigma_0}(\mathrm{d} z_1 \mathrm{d} z_2) \right| \le K_2 \frac{\|\Phi\|_{N_1,N_2}}{\ln L},\tag{D.3}$$

where $C_2 > 0$ is a number-theoretical constant and $K_2 > 0$. 1636

Note that estimate (3.8) stays true when d = 2. 1637

Theorem D.2. In the case d = 2 assertion of Theorem 3.2 remains true, if the sum $S_{L,N}$ is defined as in (D.1) and N_2 is the constant from Theorem D.1.

Proof. The only difference with the proof of Theorem 3.2 comes from estimate 1640 (3.18) since the latter is obtained by applying Theorem 3.1, and in the case 1641 d = 2 we should apply Theorem D.1 instead. Namely, now the r.h.s. of (3.18) 1642 takes the form $CL^{2(d-1)} \ln L [R^{2d} + R^{N_2} (\ln L)^{-1}] \leq C' R^{N_2} L^{2(d-1)} \ln L$. Since 1643 Lemma 3.6 remains unchanged, then for d = 2 the r.h.s. of estimate (3.20), 1644 which holds for irreducible matrices α , should be multiplied by $\ln L$. In the case 1645 of reducible matrix α we apply the latter estimate to each irreducible block, 1646 which gives the factor $(\ln L)^{\lfloor N/2 \rfloor}$ in the r.h.s. of (3.12), since the number of 1647 blocks does not exceed |N/2|. However, the final estimate of Theorem 3.2 1648 remains unchanged because of the factor $(\ln L)^{-N/2}$ in the definition (D.1) of 1649 the sum $S_{L,N}$. \Box 1650

Since in the case d = 2 we choose $\rho = \varepsilon L / \sqrt{\ln L}$, then the terms $n_{s,L}^{(k)}$ are 1651 given by formula (4.4), multiplied by $(\ln L)^{-k/2}$. The proof of Proposition 2.1 1652 is analogous to that presented in Appendix C for d = 2. The only difference 1653 being the use of Theorem D.2 in place of Theorem 3.2. Lemma 2.2 remains 1654 unchanged, so the correlations $L^{k}(\ln L)^{-k/2}\mathbb{E}a_{s}^{(m)}\bar{a}_{s}^{(n)}(\tau_{2}), m+n=k$, are 1655 given by formula (2.18), multiplied by $(\ln L)^{-k/2}$ (recall that the sum of these 1656 correlations makes $n_{s,L}^{(k)}$). We see that the correlations take the form (D.1), so 1657 Theorems D.1 and D.2 apply to study them. 1658

The rest of the proof of Theorem 5.8 literally repeats that for the case $d \geq 3$, except the appearance of the $(\ln L)^{-k/2}$ factors, coming from the new definition of ρ . Now the estimates, using Theorem 3.1, should be relaxed since the estimate provided by Theorem D.1 is slightly weaker than that of Theorem 3.1. In particular, in the r.h.s. of (4.8), (4.11) and (4.12) the factor $L^{-1/2}$ should be replaced by $(\ln L)^{-1}$. This results in the stronger lower bound for Lin Theorem 5.8: now it is $L \geq e^{\varepsilon^{-1}}$ instead of $L \geq \varepsilon^{-2}$ (see Theorem A).

Theorem 5.9, as Theorem 5.8, remains unchanged, except the lower bound for *L* which is modified as above. Indeed, the theorem follows from Theorem 5.8 and Proposition 2.1, and the term $\chi_2(L)^{-N+1} = (\ln L)^{(N-1)/2}$, appearing in estimate (2.3) for d = 2 does not change the assertion of Theorem 5.8.

1670 D.1. Discussion of Remark 1.2

In fact, Theorem 1.4 from [10] provides more delicate information about $S_{L,2}$ than what is stated in Theorem D.1. Namely, if d = 2 then due to [10],

$$\left|\frac{L^{2(1-d)}}{\ln L}\sum_{z:\ z_1\cdot z_2=0}\Phi(z) - C_2\int_{\Sigma_0}\Phi(z)\,\mu^{\Sigma_0}(\mathrm{d} z_1\mathrm{d} z_2) - \frac{\sigma_1^{\Phi}(L)}{\ln L}\right| \le C\frac{\|\Phi\|_{N_1,N_2}}{L^{1/6}},$$

where σ_1^{Φ} is a certain function satisfying $|\sigma_1^{\Phi}(L)| \leq C_1 \|\Phi\|_{N_1,N_2}$, uniformly in L. See [10] for an explicit (but complicated) formula for σ_1^{Φ} . Consequently,

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$$\left| S_{L,2}(\Phi) - C_2 \int_{\Sigma_0} \Phi(z) \, \mu^{\Sigma_0}(\mathrm{d} z_1 \mathrm{d} z_2) - \frac{\widetilde{\sigma}_1^{\Phi}(L)}{\ln L} \right| \le C \frac{\|\Phi\|_{N_1,N_2}}{L^{1/6}},$$

1677 where

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$$\tilde{\sigma}_1^{\Phi}(L) := \sigma_1^{\Phi}(L) - L^{2(1-d)} \sum_{z: z_1=0 \text{ or } z_2=0} \Phi(z)$$

1679 still satisfies $|\widetilde{\sigma}_1^{\Phi}(L)| \leq C \|\Phi\|_{N_1,N_2}$ in view of (D.2). Then estimate (4.12) 1680 refines as

$$\left| n_s^{\leq 2} - n_{s,L} - \frac{f(\tau,L)}{\ln L} \right| \le C^{\#}(s)\varepsilon^2(L^{-1/6} + \varepsilon),$$
 (D.4)

where $f(\tau, L) := \tilde{\sigma}_1^{\Phi(\tau)}(L)$ and $\Phi(\tau)$ is the function satisfying $n_{s,L}^{\leq 2}(\tau) = S_{L,2}(\Phi(\tau))$ that comes from Corollary 2.3. By (2.22) and the estimate for $\tilde{\sigma}_1^{\Phi}$ above, the function $f(\tau, L)$ is bounded uniformly in τ . The rest of the proofs of Theorems 5.8 and 5.9 remain unchanged while the estimate (D.4) leads to the assertion of the remark.

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