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► **To cite this version:**

Andrey Dymov, Sergei Kuksin, Alberto Maiocchi, Sergei Vlăduț. The Large-Period Limit for Equations of Discrete Turbulence. *Annales Henri Poincaré*, 2023, 24 (11), pp.3685-3739. 10.1007/s00023-023-01366-2 . hal-04463211

HAL Id: hal-04463211

<https://hal.science/hal-04463211>

Submitted on 16 Feb 2024

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The Large-Period Limit for Equations of Discrete Turbulence

Andrey Dymov, Sergei Kuksin , Alberto Maiocchi and Sergei Vlăduț

Abstract. We consider the damped/driven cubic NLS equation on the torus of a large period L with a small nonlinearity of size λ , a properly scaled random forcing and dissipation. We examine its solutions under the subsequent limit when first $\lambda \rightarrow 0$ and then $L \rightarrow \infty$. The first limit, called the limit of discrete turbulence, is known to exist, and in this work we study the second limit $L \rightarrow \infty$ for solutions to the equations of discrete turbulence. Namely, we decompose the solutions to formal series in amplitude and study the second-order truncation of this series. We prove that the energy spectrum of the truncated solutions becomes close to solutions of a damped/driven nonlinear wave kinetic equation. Kinetic nonlinearity of the latter is similar to that which usually appears in works on wave turbulence, but is different from it (in particular, it is non-autonomous). Apart from tools from analysis and stochastic analysis, our work uses two powerful results from the number theory.

1. Introduction

1.1. The Setting

In this paper we continue the study of the Zakharov–L’vov stochastic model for wave turbulence (WT), initiated in [7, 8]; see also a survey [9]. We start by recalling the classical and the Zakharov–L’vov stochastic settings of WT. See the introduction to [7] for more detailed discussions of the two models.

Classical setting. Let $\mathbb{T}_L^d = \mathbb{R}^d / (L\mathbb{Z}^d)$ be the d -dimensional torus, $d \geq 2$, of period $L \geq 2$. We denote by $\|u\|$ the normalized L_2 -norm of a complex function u on \mathbb{T}_L^d , $\|u\|^2 = L^{-d} \int_{\mathbb{T}_L^d} |u(x)|^2 dx$, and write the Fourier series of u in the form

$$u(x) = L^{-d/2} \sum_{s \in \mathbb{Z}_L^d} v_s e^{2\pi i s \cdot x}, \quad \mathbb{Z}_L^d = L^{-1} \mathbb{Z}^d. \tag{1.1}$$

Here the vector of Fourier coefficients $v = (v_s)_{s \in \mathbb{Z}_L^d}$ is given by the Fourier transform of $u(x)$,

$$v = \mathcal{F}(u), \quad v_s = L^{-d/2} \int_{\mathbb{T}_L^d} u(x) e^{-2\pi i s \cdot x} dx \quad \text{for } s \in \mathbb{Z}_L^d,$$

so the Parseval identity takes form $\|u\|^2 = L^{-d} \sum_{s \in \mathbb{Z}_L^d} |v_s|^2$. We will study solutions $u(t, x)$ whose norms satisfy $\|u(t, \cdot)\| \sim 1$ as $L \rightarrow \infty$. This makes the chosen in (1.1) scaling of Fourier series convenient for our purposes.

We consider the cubic NLS equation with modified nonlinearity

$$\frac{\partial}{\partial t} u + i\Delta u - i\lambda (|u|^2 - 2\|u\|^2)u = 0, \quad x \in \mathbb{T}_L^d, \tag{1.2}$$

where $u = u(t, x)$, $\Delta = (2\pi)^{-2} \sum_{j=1}^d (\partial^2 / \partial x_j^2)$ and $\lambda \in (0, 1]$ is a small parameter. The modification of the nonlinearity by the term $2i\lambda \|u\|^2 u$ keeps the main features of the standard cubic NLS equation, reducing some non-crucial technicalities; see the introduction to [7].

The objective of WT is to study solutions of (1.2) under the limit $L \rightarrow \infty$ and $\lambda \rightarrow 0$ on long time intervals. There are plenty of physical works containing some different (but consistent) approaches to the limit; many references may be found in [25, 26, 30]. Despite the strong interest in physical and mathematical communities to the addressed questions, significant progress in the rigorous justification of the physical predictions was achieved only recently [1–6, 15, 16, 23]. See, for example, the introductions to [3, 6, 7] for discussions of the obtained results.

Zakharov–L’vov setting. When studying Eq. (1.2), members of the WT community talk about “pumping energy to low modes and dissipating it in high modes”. To make this rigorous, following Zakharov–L’vov [29], in the present paper as well as in [7, 8] we consider the NLS equation (1.2) damped by a (hyper) viscosity and driven by a random force:

$$\frac{\partial}{\partial t} u + i\Delta u - i\lambda (|u|^2 - 2\|u\|^2)u = -\nu \mathfrak{A}(u) + \sqrt{\nu} \frac{\partial}{\partial t} \eta^\omega(t, x). \tag{1.3}$$

Here $\nu \in (0, 1/2]$ is another small parameter, which should be properly agreed with λ and L . The dissipative linear operator \mathfrak{A} is defined as

$$\mathfrak{A}(u(x)) = L^{-d/2} \sum_{s \in \mathbb{Z}_L^d} \gamma_s v_s e^{2\pi i s \cdot x}, \quad v = \mathcal{F}(u), \quad \gamma_s = \gamma^0(|s|^2), \tag{1.4}$$

where $|s|$ stands for the Euclidean norm of a vector s and $\gamma^0(y)$ is a smooth real increasing function of $y > 0$, satisfying¹

$$\gamma^0 \geq 1 \quad \text{and} \quad c(1+y)^{r^*} \leq \gamma^0(y) \leq C(1+y)^{r^*} \quad \forall y > 0. \tag{1.5}$$

¹ For example, if $\gamma_s = (1+|s|^2)^{r^*}$, then $\mathfrak{A} = (1-\Delta)^{r^*}$. In particular, we can take $\mathfrak{A} = 1-\Delta$.

62 The exponents $r_* > 0$ and c, C are positive constants. We also assume that

63 *all derivatives of γ^0 have at most polynomial growths at infinity.*

64 The random noise η^ω is given by a Fourier series

$$65 \quad \eta^\omega(t, x) = L^{-d/2} \sum_{s \in \mathbb{Z}_L^d} b(s) \beta_s^\omega(t) e^{2\pi i s \cdot x},$$

66 where $\{\beta_s(t), s \in \mathbb{Z}_L^d\}$ are standard independent complex Wiener processes²
 67 and $b(s)$ is a Schwartz function on $\mathbb{R}^d \supset \mathbb{Z}_L^d$.³

68 Solutions $u(\tau)$ of (1.3) are random processes in the space $H = L_2(\mathbb{T}_L^d, \mathbb{C})$,
 69 equipped with the norm $\|\cdot\|$. If r_* is sufficiently big in terms of d , Eq. 1.3 is
 70 known to be well posed, see Theorem 1.1 below and a discussion after its for-
 71 mulation. Moreover, Ito's formula shows that $\mathbb{E}\|u(\tau)\|^2$ is bounded uniformly
 72 in τ and L, ν, λ , once $\mathbb{E}\|u(0)\|^2$ is bounded uniformly in these parameters, see
 73 in [7].

74 We will study the equation on time intervals of order ν^{-1} . So, it is con-
 75 venient to pass from t to the slow time $\tau = \nu t$ and write Eq. (1.3) as

$$76 \quad \begin{aligned} \dot{u} + i\nu^{-1} \Delta u - i\rho(|u|^2 - 2\|u\|^2)u &= -\mathfrak{A}(u) + \dot{\eta}^\omega(\tau, x), \\ \eta^\omega(\tau, x) &= L^{-d/2} \sum_{s \in \mathbb{Z}_L^d} b(s) \beta_s^\omega(\tau) e^{2\pi i s \cdot x}. \end{aligned} \quad (1.6)$$

77 Here $\rho = \lambda\nu^{-1}$, the upper dot stands for $d/d\tau$ and $\{\beta_s(\tau), s \in \mathbb{Z}_L^d\}$ is another
 78 set of standard independent complex Wiener processes. Below we use ρ, ν and
 79 L as parameters of the equation.

80 In the context of Eq. (1.6), the objective of WT is to study its solutions
 81 $u(\tau)$ when

$$82 \quad L \rightarrow \infty \quad \text{and} \quad \nu \rightarrow 0, \quad (1.7)$$

83 while $\rho = \rho(\nu, L)$ is scaled appropriately, mostly paying attention to their
 84 *energy spectra*

$$85 \quad N_s(\tau) := \mathbb{E}|v_s(\tau)|^2, \quad \text{where} \quad v(\tau) = \mathcal{F}(u(\tau)). \quad (1.8)$$

86 Exact meaning of the limit (1.7) is unclear since no relation between the pa-
 87 rameters ν and L is postulated by the theory.

88 Motivated by physical works, in the present paper, as in [7,8], we study
 89 formal decompositions in ρ of solutions to Eq. (1.6) and of their energy spectra
 90 N_s under the limit (1.7). See the introduction to [7] for a discussion of our
 91 motivation, and see below Sect. 4. In [7,8] we understand the limit (1.7) as

$$92 \quad \text{first } L \rightarrow \infty \text{ and then } \nu \rightarrow 0, \text{ or } L \gg \nu^{-2} \text{ while } \nu \rightarrow 0. \quad (1.9)$$

93 There we have shown that principal terms of the decomposition of N_s in ρ
 94 have a non-trivial limiting behaviour, provided that ρ is scaled as $\rho \sim \nu^{-1/2}$,

² i.e. $\beta_s = \beta_s^1 + i\beta_s^2$, where $\{\beta_s^j, s \in \mathbb{Z}_L^d, j = 1, 2\}$ are standard independent real Wiener processes.

³ Often it is assumed that the intensity $b(s)$ of the noise η^ω is non-negative, but we do not impose this condition. Note that if $b(s) \equiv 0$, then our results become trivial since below we will provide (1.3) with the zero initial conditions.

95 governed by a nonlinear *wave kinetic equation* (WKE) with added dissipation
 96 and a constant forcing. The WKE coincides with that, arising in physical
 97 works, so this result agrees well with the predictions of the WT.

98 In the present paper we are interested in the opposite order of limits,
 99 which rarely appears in physical works:

$$100 \quad \text{firstly } \nu \rightarrow 0 \quad \text{and then } L \rightarrow \infty. \quad (1.10)$$

101 Roughly speaking, our main result is that under the double limit (1.10) the
 102 behaviour of principal terms of the decomposition in ρ for the energy spectrum
 103 N_s is governed by a *modified* WKE. The latter is similar to the WKE arising
 104 in [7, 8] and physical papers, but is different from them. The scaling of ρ now
 105 is $\rho \sim L \chi_d(L)$, where

$$106 \quad \chi_d(L) \equiv 1 \quad \text{if } d \geq 3 \quad \text{and} \quad \chi_d(L) = (\ln L)^{-1/2} \quad \text{if } d = 2. \quad (1.11)$$

107 To the best of our knowledge, this WKE did not appear in the literature be-
 108 fore. For the proof we start with the result obtained in [17, 21], where the
 109 limiting as $\nu \rightarrow 0$ behaviour of Eq. (1.6) is examined (while L and ρ are kept
 110 fixed). Then we pass to the limit as $L \rightarrow \infty$, following the approach of [7, 8]
 111 and using the developed there tools, such as a specific Feynman diagram pre-
 112 sentation. Another key ingredient of the proof is an obtained in [10] refinement
 113 of the Heath–Brown circle method for quadratic forms [14], and certain upper
 114 bounds for the number of integer points on intersections of quadrics. In the
 115 next subsections we describe our results and methods in more detail.

116 In [20] a similar result concerning the iterated limit (1.10) was found
 117 heuristically; however, there ρ was scaled as $\rho \sim \sqrt{L}$. The present paper shows
 118 that the correct scaling is different: $\rho \sim L \chi_d(L)$.

119 Similar regimes, when $L \rightarrow \infty$ slowly while $\nu > 0$ fast decays to zero,
 120 were studied in [1, 16]. However the elegant description of the limit, obtained
 121 there, is far from the prediction of WT. The works [1, 16] should rather be
 122 regarded as a kind of averaging (similar to that of Krylov–Bogolyubov) since
 123 the considered there time scale is much shorter than the characteristic time
 124 scale of WT. Note that in [1] a similar to ours [10] refinement of the Heath–
 125 Brown method also is crucially used.

126 1.2. The Limit of Discrete Turbulence

127 We first consider the limit

$$128 \quad \nu \rightarrow 0 \quad \text{while } L \text{ and } \rho \text{ stay fixed.} \quad (1.12)$$

129 It is known as *the limit of discrete turbulence* (see [25, Section 10]) and has been
 130 successfully studied in [17, 21]. To explain the result, let us take the Fourier
 131 transform of Eq. (1.6):

$$132 \quad \dot{v}_s - i\nu^{-1}|s|^2 v_s + \gamma_s v_s = i\rho L^{-d} \left(\sum_{1,2,3} \delta_{3s}^{\prime 12} v_1 v_2 \bar{v}_3 - |v_s|^2 v_s \right) + b(s) \dot{\beta}_s$$

133
134 (1.13)

135 for $s \in \mathbb{Z}_L^d$. Here, as it is common in WT, v_j abbreviates v_{s_j} , $\sum_{1,2,3}$ stands for
 136 $\sum_{s_1, s_2, s_3 \in \mathbb{Z}_L^d}$, and

$$137 \quad \delta_{3_s}^{\prime 12} = \delta_{s_3 s}^{\prime s_1 s_2} := \begin{cases} 1, & \text{if } s_1 + s_2 = s_3 + s \text{ and } \{s_1, s_2\} \neq \{s_3, s\}, \\ 0, & \text{otherwise.} \end{cases} \quad (1.14)$$

138 Note that

$$139 \quad \text{if } \delta_{3_s}^{\prime 12} = 1, \text{ then } \{s_1, s_2\} \cap \{s_3, s\} = \emptyset. \quad (1.15)$$

140 We pass to the *interaction representation*,

$$141 \quad \mathbf{a}_s(\tau) = v_s(\tau) e^{-i\nu^{-1}\tau|s|^2}, \quad s \in \mathbb{Z}_L^d, \quad (1.16)$$

142 and denote

$$143 \quad \omega_{3_s}^{12} = \omega_{s_3 s}^{s_1 s_2} := |s_1|^2 + |s_2|^2 - |s_3|^2 - |s|^2 = -2(s_1 - s) \cdot (s_2 - s), \quad (1.17)$$

144 where the last equality holds if $\delta_{3_s}^{\prime 12} = 1$ since then $s_3 = s_1 + s_2 - s$. Then Eq.
 145 (1.13) takes the form

$$146 \quad \begin{aligned} \dot{\mathbf{a}}_s + \gamma_s \mathbf{a}_s &= i\rho Y_s(\mathbf{a}, \nu^{-1}\tau) + b(s)\dot{\beta}_s, \quad s \in \mathbb{Z}_L^d, \\ Y_s(\mathbf{a}, t) &= L^{-d} \left(\sum_{1,2,3} \delta_{3_s}^{\prime 12} \mathbf{a}_1 \mathbf{a}_2 \bar{\mathbf{a}}_3 e^{it\omega_{3_s}^{12}} - |\mathbf{a}_s|^2 \mathbf{a}_s \right), \end{aligned} \quad (1.18)$$

147 where $\{\beta_s\}$ is yet another set of standard independent complex Wiener pro-
 148 cesses (and, again \mathbf{a}_j stands for \mathbf{a}_{s_j}). Note that the energy spectra of solutions
 149 to Eqs. (1.13) and (1.18) coincide:

$$150 \quad N_s(\tau) = \mathbb{E}|v_s(\tau)|^2 = \mathbb{E}|\mathbf{a}_s(\tau)|^2. \quad (1.19)$$

151 Sometimes we will write N_s and \mathbf{a}_s as $N_s(\tau; \nu, L)$ and $\mathbf{a}_s(\tau; \nu, L)$. The limit-
 152 ing dynamics in Eq. (1.18) under the limit (1.12) is governed by the *effective*
 153 *equation* of discrete turbulence. The latter has the form (1.18) with the mod-
 154 ified nonlinearity Y^{res} , in which the sum is taken only over resonant vectors
 155 s_1, s_2, s_3 :

$$156 \quad \begin{aligned} \dot{\mathbf{a}}_s + \gamma_s \mathbf{a}_s &= i\rho Y_s^{res}(\mathbf{a}) + b(s)\dot{\beta}_s, \quad s \in \mathbb{Z}_L^d, \\ Y_s^{res}(\mathbf{a}) &= L^{-d} \left(\sum_{1,2,3} \delta_{3_s}^{\prime 12} \delta(\omega_{3_s}^{12}) \mathbf{a}_1 \mathbf{a}_2 \bar{\mathbf{a}}_3 - |\mathbf{a}_s|^2 \mathbf{a}_s \right). \end{aligned} \quad (1.20)$$

157 Here $\delta(\omega_{3_s}^{12}) = 1$ if $\omega_{3_s}^{12} = 0$ and $\delta(\omega_{3_s}^{12}) = 0$ otherwise. The following result is
 158 proven in [17, 21] (concerning the well posedness of (1.18) also see [28]).

159 **Theorem 1.1.** *If $\mathfrak{A}(u) = -\Delta u + u$ and $d \leq 3$, then Eqs. (1.18) and (1.20) are*
 160 *well posed. Under the limit (1.12), on time intervals of order 1,*

- 161 (i) *A solution $\mathbf{a}^\nu(\tau)$ of (1.18) converges in distribution to a solution $\mathbf{a}^0(\tau)$*
 162 *of (1.20) once they have the same deterministic initial data at $\tau = 0$;*
 163 (ii) *The energy spectrum $\mathbb{E}|\mathbf{a}_s^\nu(\tau)|^2 = N_s(\tau; \nu, L)$ converges to the energy*
 164 *spectrum $\mathbb{E}|\mathbf{a}_s^0(\tau)|^2$.*

165 It is very likely that the assertions of the theorem stays true with a similar
 166 proof for any d if \mathfrak{A} is the operator (1.4, 1.5) with a sufficiently large r_* , and
 167 in [21] this indeed is proved, provided that Eq. (1.18) is known to be well
 168 posed. It is also shown in [17, 21] that if under the assumptions of Theorem 1.1
 169 Eqs. (1.18) and (1.20) are mixing, then the stationary measure for the former
 170 converges to that for the latter as $\nu \rightarrow 0$. The mixing property for Eq. (1.18)
 171 is established in [28] for the case when all numbers $b(s)$ are nonzero and $d = 1$.
 172 It is plausible that a variation of the argument in [28] allows to establish the
 173 mixing property for both Eqs. (1.18) and (1.20) if in (1.5) r_* is big in terms of
 174 d and $b(s) \neq 0$ for all s .

175 **1.3. The Main Result**

176 In view of Theorem 1.1, to understand behaviour of the energy spectrum (1.19)
 177 of Eq. (1.18) under the limit (1.10), it remains to study that of the energy
 178 spectrum $\mathbb{E}|\mathbf{a}_s(\tau; L)|^2$ of the effective Eq. (1.20) under the limit $L \rightarrow \infty$.
 179 Instead, following the logic of [7], we study the energy spectrum corresponding
 180 to a principal part of a decomposition in ρ for the solutions $\mathbf{a}_s(\tau; L)$ of Eq.
 181 (1.20).

182 *Quasisolutions and their energy spectra.* To simplify presentation we assume
 183 that initially the system was at rest, i.e. supplement Eq. (1.20) with the zero
 184 initial condition

$$185 \quad \mathbf{a}_s(0) = 0 \quad \forall s \in \mathbb{Z}_L^d. \quad (1.21)$$

186 We formally decompose the corresponding solution of (1.20) in ρ ,

$$187 \quad \mathbf{a}(\tau) = \mathbf{a}^{(0)}(\tau) + \rho \mathbf{a}^{(1)}(\tau) + \rho^2 \mathbf{a}^{(2)}(\tau) + \dots, \quad \mathbf{a}^{(k)}(0) = 0, \quad (1.22)$$

188 $\mathbf{a}^{(k)}(\tau) = \mathbf{a}^{(k)}(\tau; L)$. The process $\mathbf{a}^{(0)}(\tau)$ satisfies the linear equation

$$189 \quad \dot{\mathbf{a}}_s^{(0)} + \gamma_s \mathbf{a}_s^{(0)} = b(s) \dot{\beta}_s, \quad s \in \mathbb{Z}_L^d.$$

190 So it is Gaussian,

$$191 \quad \mathbf{a}_s^{(0)}(\tau) = b(s) \int_0^\tau e^{-\gamma_s(\tau-l)} d\beta_s(l), \quad (1.23)$$

192 and its components $\{\mathbf{a}_s^{(0)}\}$ are independent. The process $\mathbf{a}^{(1)}$ satisfies

$$193 \quad \dot{\mathbf{a}}_s^{(1)} + \gamma_s \mathbf{a}_s^{(1)} = iY_s^{res}(\mathbf{a}^{(0)}),$$

194 so that

$$195 \quad \mathbf{a}_s^{(1)}(\tau) = iL^{-d} \int_0^\tau e^{-\gamma_s(\tau-l)} \left(\sum_{1,2,3} \delta_{3s}^{l12} \delta(\omega_{3s}^{12}) (\mathbf{a}_1^{(0)} \mathbf{a}_2^{(0)} \bar{\mathbf{a}}_3^{(0)}) - |\mathbf{a}_s^{(0)}|^2 \mathbf{a}_s^{(0)} \right) (l) dl \quad (1.24)$$

197 is a Wiener chaos of third order (see [18]). Similar for $n \geq 1$,

$$198 \quad \mathbf{a}_s^{(n)}(\tau) = iL^{-d} \sum_{n_1+n_2+n_3=n-1} \int_0^\tau e^{-\gamma_s(\tau-l)} \\ \times \left(\sum_{1,2,3} \delta_{3s}^{l12} \delta(\omega_{3s}^{12}) (\mathbf{a}_1^{(n_1)} \mathbf{a}_2^{(n_2)} \bar{\mathbf{a}}_3^{(n_3)}) - \mathbf{a}_s^{(n_1)} \mathbf{a}_s^{(n_2)} \bar{\mathbf{a}}_s^{(n_3)} \right) (l) dl, \\ 199 \quad (1.25)$$

200 is a Wiener chaos of order $2n + 1$.

201 Next we consider the quadratic truncation of the series (1.22),

$$202 \quad \mathcal{A}_s(\tau; L) = \mathcal{A}_s(\tau) = \mathbf{a}_s^{(0)}(\tau) + \rho \mathbf{a}_s^{(1)}(\tau) + \rho^2 \mathbf{a}_s^{(2)}(\tau), \quad (1.26)$$

203 which we call the *quasisolution*⁴ of the effective Eqs. (1.20), (1.21). It is tradi-
204 tional in WT to analyse the quasisolution instead of the solution itself, postu-
205 lating that the former well approximates the latter; see introduction to [7] for
206 a discussion. The goal of the present paper is to study the behaviour of the
207 energy spectrum of $\mathcal{A}(\tau)$,

$$208 \quad \mathbf{n}_{s,L}(\tau) = \mathbb{E} |\mathcal{A}_s(\tau; L)|^2, \quad s \in \mathbb{Z}_L^d, \quad (1.27)$$

209 as $L \rightarrow \infty$. Our results, formulated below, show that under this limit the energy
210 spectrum $\mathbf{n}_{s,L}(\tau)$ has a non-trivial behaviour (i.e. stays finite and behaves
211 differently from $\mathbb{E} |\mathbf{a}_s^{(0)}|^2$) only if $\rho \sim L \chi_d(L)$, where χ_d is defined in (1.11).
212 Accordingly, from now on we assume that

$$213 \quad \rho = \varepsilon L \chi_d(L), \quad (1.28)$$

214 where $0 < \varepsilon \leq 1/2$ is a small but fixed constant (see a few lines below for its
215 discussion). Then the energy spectrum $\mathbf{n}_{s,L}$ expands as

$$216 \quad \mathbf{n}_{s,L}(\tau) = \mathbf{n}_{s,L}^{(0)}(\tau) + \varepsilon \mathbf{n}_{s,L}^{(1)}(\tau) + \varepsilon^2 \mathbf{n}_{s,L}^{(2)}(\tau) + \varepsilon^3 \mathbf{n}_{s,L}^{(3)}(\tau) + \varepsilon^4 \mathbf{n}_{s,L}^{(4)}(\tau), \\ 217 \quad (1.29)$$

218 $s \in \mathbb{Z}_L^d$, where

$$219 \quad \mathbf{n}_{s,L}^{(k)}(\tau) = (L \chi_d(L))^k \sum_{\substack{k_1+k_2=k \\ 0 \leq k_1, k_2 \leq 2}} \mathbb{E} \mathbf{a}_s^{(k_1)}(\tau) \bar{\mathbf{a}}_s^{(k_2)}(\tau). \quad (1.30)$$

220 In particular, by (1.23)

$$221 \quad \mathbf{n}_{s,L}^{(0)}(\tau) = \mathbb{E} |\mathbf{a}_s^{(0)}(\tau; L)|^2 = \frac{b(s)^2}{\gamma_s} (1 - e^{-2\gamma_s \tau}), \quad s \in \mathbb{Z}_L^d, \quad (1.31)$$

222 and a simple computation shows that $\mathbf{n}_{s,L}^{(1)}(\tau) \equiv 0$. For higher-order terms, we
223 prove that

$$224 \quad \mathbf{n}_{s,L}^{(2)} \sim 1 \quad \text{and} \quad |\mathbf{n}_{s,L}^{(3)}|, |\mathbf{n}_{s,L}^{(4)}| \lesssim 1 \quad \text{as } L \rightarrow \infty \text{ uniformly in } \tau \geq 0; \\ 225 \quad (1.32)$$

⁴By analogy with the *quasimodes* in the spectral theory of the Shrödinger operator.

226 see a discussion in the next subsection. Thus, the parameter ε measures the
 227 properly scaled amplitude of the solutions, and indeed it should be small for the
 228 methodology of WT to apply. Then, the term $\varepsilon^2 \mathbf{n}_{s,L}^{(2)}$ is the crucial non-trivial
 229 component of the energy spectrum $\mathbf{n}_{s,L}$, while the terms $\varepsilon^3 \mathbf{n}_{s,L}^{(3)}$, $\varepsilon^4 \mathbf{n}_{s,L}^{(4)}$ are
 230 perturbative. This well agrees with the prediction of physical works concerning
 231 various models of WT.

232 *Wave kinetic equation.* In view of (1.32), to study the limiting as $L \rightarrow \infty$
 233 behaviour of the energy spectrum $\mathbf{n}_{s,L}(\tau)$ up to an error of size ε^3 it remains
 234 to investigate the behaviour of its principal component $\mathbf{n}_{s,L}^{(0)}(\tau) + \varepsilon^2 \mathbf{n}_{s,L}^{(2)}(\tau)$. We
 235 show that the latter is governed by a WKE. To state the result, let us consider
 236 the *resonant quadric*

$$237 \quad \Sigma_s = \{(s_1, s_2) \in \mathbb{R}^{2d} : (s_1 - s) \cdot (s_2 - s) = 0\}, \quad (1.33)$$

238 cf. (1.17), and a measure μ^{Σ_s} on it, given by

$$239 \quad \mu^{\Sigma_s}(ds_1 ds_2) = (|s_1 - s|^2 + |s_2 - s|^2)^{-1/2} ds_1 ds_2 |_{\Sigma_s}, \quad (1.34)$$

240 where $ds_1 ds_2 |_{\Sigma_s}$ denotes the volume element on Σ_s , corresponding to the
 241 standard Euclidean structure on \mathbb{R}^{2d} .

242 Let us consider the following non-autonomous cubic *wave kinetic integral*
 243 operator $K(\tau)$, for any $\tau \geq 0$ sending a function y_s , $s \in \mathbb{R}^d$, to the function
 244 $K_s(\tau)y$, defined as

$$245 \quad K_s(\tau)y = 4C_d \int_{\Sigma_s} \mu^{\Sigma_s}(ds_1 ds_2) \left(\mathcal{Z}^4 y_1 y_2 y_3 \right. \\ 246 \quad \left. + \mathcal{Z}^3 y_1 y_2 y_4 - \mathcal{Z}^2 y_1 y_3 y_4 - \mathcal{Z}^1 y_2 y_3 y_4 \right). \quad (1.35)$$

247 Here $y_j := y_{s_j}$ with $s_4 := s$ and $s_3 := s_1 + s_2 - s$, C_d is the constant from
 248 Theorem B below, the kernels $\mathcal{Z}^j = \mathcal{Z}^j(\tau; s_1, s_2, s_3, s_4)$ are given by formulas
 249 (4.14, 4.15) and satisfy $0 \leq \mathcal{Z}^j(\tau) \leq 1$. When $\tau \rightarrow \infty$, the operator $K(\tau)$
 250 exponentially fast converges to a limiting kinetic integral operator $K(\infty)$, given
 251 by (1.35) with \mathcal{Z}^j replaced by $(\gamma_{s_1} + \gamma_{s_2} + \gamma_{s_3} + \gamma_{s_4})^{-1}$ for all j :

$$252 \quad K_s(\infty)y = 4C_d \int_{\Sigma_s} \frac{\mu^{\Sigma_s}(ds_1 ds_2)}{\gamma_{s_1} + \gamma_{s_2} + \gamma_{s_3} + \gamma_{s_4}} \\ 253 \quad \left(y_1 y_2 y_3 + y_1 y_2 y_4 - y_1 y_3 y_4 - y_2 y_3 y_4 \right). \quad (1.36)$$

254 It is similar to the standard four-wave kinetic operator of WT (e.g. see in [25]),
 255 which has the form (1.35) with $\mathcal{Z}^j \equiv \text{const}$, but still is different from the latter
 256 since $K(\infty)$ depends on the spectrum $\{\gamma_s\}$ of the dissipation operator \mathfrak{A} .⁵

257 For $r \in \mathbb{R}$ we denote by $\mathcal{C}_r(\mathbb{R}^d)$ a space of continuous complex functions
 258 on \mathbb{R}^d with finite norm

$$259 \quad \|f\|_r = \sup_{z \in \mathbb{R}^d} |f(z)| \langle z \rangle^r, \quad \text{where } \langle z \rangle = \max(|z|, 1). \quad (1.37)$$

⁵Earlier the kinetic operator $K(\infty)$ was heuristically obtained in [20].

260 In Sect. 5, following [7], we show that if $r > d$, then for any τ the operator $K(\tau)$
 261 defines a continuous 3-homogeneous mapping $K(\tau) : \mathcal{C}_r(\mathbb{R}^d) \mapsto \mathcal{C}_{r+1}(\mathbb{R}^d)$, and
 262 for any $y \in \mathcal{C}_r(\mathbb{R}^d)$ the curve $\tau \mapsto K(\tau)(y)$ is Hölder continuous in $\mathcal{C}_r(\mathbb{R}^d)$.

263 Now consider the following damped/driven non-autonomous WKE

$$264 \quad \dot{\mathfrak{z}}_s(\tau) = -2\gamma_s \mathfrak{z}_s + \varepsilon^2 K_s(\tau)(\mathfrak{z}) + 2b(s)^2, \quad \mathfrak{z}(0) = 0, \quad (1.38)$$

265 where $\tau \geq 0$ and $s \in \mathbb{R}^d$. In Sect. 5 we prove that for small ε it has a unique
 266 solution $\mathfrak{z}_s(\tau)$, which can be written as $\mathfrak{z}_s(\tau) = \mathfrak{z}_s^0(\tau) + \varepsilon^2 \mathfrak{z}_s^1(\tau, \varepsilon)$, where $\mathfrak{z}_s^0, \mathfrak{z}_s^1 \sim$
 267 1 and \mathfrak{z}_s^0 solves the linear Eq. (1.38)| $_{\varepsilon=0}$. It is easy to see that \mathfrak{z}_s^0 equals the
 268 component $\mathfrak{n}_{s,L}^{(0)}$ of the energy spectrum $\mathfrak{n}_{s,L}$, given by (1.31), and we prove
 269 that \mathfrak{z}_s^1 is ε^4 -close to $\mathfrak{n}_{s,L}^{(2)}$ uniformly in τ . Then, in view of (1.32), the energy
 270 spectrum $\mathfrak{n}_{s,L}$ is ε^3 -close to the solution $\mathfrak{z}_s(\tau)$.

271 Below we denote by $C^\#(s)$ various positive functions of s which decay as
 272 $|s| \rightarrow \infty$ faster than any negative degree of $|s|$. These functions never depend on
 273 L, ε and τ . By $C^\#(s; p)$ we denote functions $C^\#(s)$ depending on a parameter
 274 p .

275 **Theorem A** (Main theorem). *Let $d \geq 2$. Then, the energy spectrum $\mathfrak{n}_{s,L}(\tau)$ of*
 276 *the quasiresolution $\mathcal{A}_s(\tau)$ of (1.20), (1.21) satisfies the estimate $\mathfrak{n}_{s,L}(\tau) \leq C^\#(s)$*
 277 *and is ε^3 -close to the solution $\mathfrak{z}_s(\tau)$ of WKE (1.38). Namely, under the scaling*
 278 *$\rho = \varepsilon L \chi_d(L)$, for any r there exists $\varepsilon_r \in (0, 1/2]$ such that for $0 < \varepsilon \leq \varepsilon_r$ we*
 279 *have*

$$280 \quad |\mathfrak{n}_{\cdot,L}(\tau) - \mathfrak{z}_{\cdot}(\tau)|_r \leq C_r \varepsilon^3 \quad \forall \tau \geq 0, \quad (1.39)$$

281 if $L \geq \varepsilon^{-2}$ for $d \geq 3$, and $L \geq e^{\varepsilon^{-1}}$ for $d = 2$.

282 See Theorem 5.9. Since the energy spectrum \mathfrak{n}_s is defined for $s \in \mathbb{Z}_L^d$ with
 283 finite L , then the norm in (1.39) is understood as $|f|_r = \sup_{z \in \mathbb{Z}_L^d} |f(z)| \langle z \rangle^r$.

284 *Remark 1.2.* If $d = 2$, the lower bound $L \geq e^{\varepsilon^{-1}}$ can be relaxed in the following
 285 sense. In Appendix D we explain that there is a bounded correction $f(\tau, L)$
 286 which can be written explicitly, such that

$$287 \quad \left| \mathfrak{n}_{\cdot,L}(\tau) - \mathfrak{z}_{\cdot}(\tau) - \frac{f(\tau, L)}{\ln L} \right|_r \leq C_r \varepsilon^3 \quad \forall \tau \geq 0, \quad (1.40)$$

288 if $L \geq \varepsilon^{-6}$.

289 In Lemma 5.6 we show that in the vicinity of the unique steady state
 290 $\mathfrak{z}_s^0 := b(s)^2/\gamma_s$ for the linear Eq. (1.38)| $_{\varepsilon=0}$, Eq. (1.38)| $_{\tau=\infty}$ with $\varepsilon \ll 1$ has a
 291 unique steady state $\mathfrak{z}^\varepsilon \in \mathcal{C}_r(\mathbb{R}^d)$ and the latter is asymptotically stable. Jointly
 292 with Theorem A, this result implies the following asymptotic in time behaviour
 293 of the energy spectrum $\mathfrak{n}_{s,L}(\tau)$:

$$294 \quad |\mathfrak{n}_{\cdot,L}(\tau) - \mathfrak{z}^\varepsilon|_r \leq C_r (e^{-\tau} + \varepsilon^3), \quad \forall \tau \geq 0, \quad (1.41)$$

295 if L is as in Theorem A, see (5.19).

296 The cases $d \geq 3$ and $d = 2$ are similar, but should be treated separately.
 297 To shorten the presentation, we give a detailed proof of Theorem A only for
 298 $d \geq 3$, when

$$299 \quad \chi_d = 1 \quad \text{and} \quad \rho = \varepsilon L.$$

300 The proof for $d = 2$ can be obtained by a simple modification of the argument
 301 for $d \geq 3$. We sketch it in Appendix D. So from now on, except Sect. 2 which
 302 gives a brief account of the method of Feynman diagrams from [7, 8], we assume
 303 that $d \geq 3$.

304 In paper [7] we examine the behaviour of the energy spectrum $\mathbf{n}_{s,L,\nu}(\tau)$ of
 305 a quasisolution to Eq. (1.18) under the limit (1.9), assuming that $\rho = \varepsilon\nu^{-1/2}$.⁶
 306 We got there a similar result which states that $\mathbf{n}_{s,L,\nu}(\tau)$ is ε^4 -close to a solution
 307 of the damped/driven four-wave kinetic equation as in [25, Section 6.9.1] [in
 308 contrast with Eq. (1.38)], the kinetic nonlinearity there does not depend on the
 309 dissipation \mathfrak{A} in Eq. (1.3)].

310 *What next?* In this work and in [7] we obtained wave kinetic limits for the
 311 energy spectra of quasisolutions for the NLS Eq. (1.6) under limit (1.10) with
 312 the scaling $\rho = \varepsilon L \chi_d(L)$ and limit (1.9) with the scaling $\rho = \varepsilon\nu^{-1/2}$. Our
 313 next goal is to show that an exact solution $\mathbf{a}(\tau)$ of Eq. (1.18) is ε^3 -close
 314 to its quasisolution $\mathcal{A}(\tau)$ (uniformly in $L \geq 2$ and $\tau \in [0, T]$, for any $T >$
 315 0). And that a solution of Eq. (1.6) is ε^3 -close to the quasisolution of the
 316 equation (uniformly in ν, L and $\tau \in [0, T]$, if $L \geq \nu^{-2-\bar{\gamma}}$, $\bar{\gamma} > 0$). This
 317 would imply that the energy spectra of solutions of Eq. (1.6) under limit (1.10)
 318 and limit (1.9) are ε^3 -close to solutions of the two WKE (namely, Eq. (1.38)
 319 and the WKE from [7]). To prove this, say, for a solution $\mathbf{a}(\tau)$ of Eq. (1.18)
 320 we consider the equation on any fixed time-interval $[0, T]$ and regard it as a
 321 nonlinear equation $F_T(\mathbf{a}(\cdot)) = 0$ for the unknown process $\mathbf{a}_s(\tau)$. Then the
 322 quasisolution \mathcal{A} satisfies the equation with a disparity $\lesssim \varepsilon^3$. By analogy with
 323 some stochastic problems for nonlinear PDEs, recently successfully resolved
 324 by the KAM-techniques (e.g. see [19]), we believe that KAM also applies to
 325 the equation $F_T = 0$. Its application would imply that \mathbf{a} is ε^3 -close to \mathcal{A} , as
 326 stated. We also believe that analysis of the KAM-iterations which build \mathbf{a} from
 327 \mathcal{A} will show that the energy spectrum of the solution $\mathbf{a}(\tau)$ of Eq. (1.18) under
 328 the limit $L \rightarrow \infty$ converges to a solution of the WKE (1.38). A similar logic
 329 should apply to the energy spectra of solutions for Eq. (1.6) under the limit
 330 (1.9).

331 1.4. Outline of The Proof: Feynman Diagrams and Number Theory

332 It is well understood that to write down formulas for the terms $\mathbf{n}_{s,L}^{(k)}$ of decom-
 333 positions as (1.29) it is instrumental to use the language of Feynman diagrams.
 334 In application to similar problems this goes back at least to the works [11, 12],
 335 and then was successfully used for the purposes of WT in [2–6, 23] and other
 336 papers. We use this techniques in the form developed in [8] which gives a con-
 337 venient presentation of the terms $\mathbf{n}_{s,L}^{(k)}$ [see (1.30)]. Namely, by iterating the

⁶In [7] the notation is slightly different: there we set $\rho = \varepsilon^{1/2}\nu^{-1/2}$.

338 Duhamel formula (1.25) we express $\mathfrak{a}^{(n)}(\tau)$ in terms of the Gaussian processes
 339 $\mathfrak{a}_s^{(0)}$, and next evoking the Wick formula for moments of $\mathfrak{a}_s^{(0)}$ write the terms
 340 $\mathfrak{n}_{s,L}^{(k)}$ as multiple sums. Then the just mentioned diagram techniques allows to
 341 ‘integrate’ these sums. That is, to write any $\mathfrak{n}_{s,L}^{(k)}$ as a sum over an intersections
 342 of $k - 1$ quadrics in $(\mathbb{Z}_L^d)^k$ in a form, convenient to pass to a limit as $L \rightarrow \infty$.
 343 The term $\mathfrak{n}_{s,L}^{(2)}$ is a sum over a single quadric and may be analysed without the
 344 diagram’s machinery. This and some other similar terms play a leading role
 345 in our analysis and dictate the form of the limiting WKE. The terms may be
 346 written as sums

$$347 \quad G_s(\tau, L) = L^{2(1-d)} \sum_{\substack{z_1, z_2 \in \mathbb{Z}_L^d: \\ z_1 \cdot z_2 = 0, z_1, z_2 \neq 0}} \Phi_s(\tau; z_1, z_2), \quad (1.42)$$

348 well known in works on WT. To study them under the limit $L \rightarrow \infty$ we make
 349 use of the celebrated circle method of Heath–Brown [14]. Since the result of
 350 [14] does not completely fit our purposes, we specified it in the accompanying
 351 paper [10] (also see [1, Section 5] for another specification of the Heath–Brown
 352 method, used for the purposes of WT). This implies

353 **Theorem B.** *For any $L \geq 2$,*

$$354 \quad \left| G_s(\tau, L) - C_d \int_{\Sigma_0} \Phi_s(\tau; z_1, z_2) \mu^{\Sigma_0}(dz_1 dz_2) \right| \leq K_d \frac{\|\Phi_s(\tau; \cdot)\|_{N_1, N_2}}{L^{d-5/2}},$$

355 where $\Sigma_0 = \{z_1, z_2 \in \mathbb{R}^d : z_1 \cdot z_2 = 0\}$, μ^{Σ_0} is the measure on it, defined by
 356 (1.34) with $s = 0$, C_d is a number-theoretical constant, satisfying $C_d \in (1, 1 +$
 357 $2^{2-d})$, the norm $\|\cdot\|_{N_1, N_2}$ is defined in (3.3) and the constants $N_1, N_2 \in \mathbb{N}$
 358 depend only on d .

359 In particular, the term $\mathfrak{n}_{s,L}^{(2)}(\tau)$ admits a limit when $L \rightarrow \infty$.

360 The terms $\mathfrak{n}_{s,L}^{(3)}$ and $\mathfrak{n}_{s,L}^{(4)}$ in (1.29) correspond to multiple intersections
 361 of quadrics, and the Heath–Brown method does not apply to them. Still the
 362 diagram technique allows to write these terms in a convenient compact form.
 363 Then next in Sect. 3 and Appendix A we use Theorem B jointly with another
 364 powerful result from the number theory—Bezout’s theorem for finite fields—to
 365 prove ⁷

366 **Theorem C.** *For $k = 3, 4$, $|\mathfrak{n}_{s,L}^{(k)}(\tau)| \leq C^\#(s)$.*

367 Theorems B and C imply (1.32). So to establish Theorem A it remains to
 368 show that the term $\mathfrak{n}_{s,L}^{\leq 2}(\tau) := \mathfrak{n}_{s,L}^{(0)}(\tau) + \varepsilon^2 \mathfrak{n}_{s,L}^{(2)}(\tau)$ (or equivalently its limit as
 369 $L \rightarrow \infty$, provided by Theorem B) can be well approximated by a solution of the
 370 WKE (1.38). To this end, following the lines of [7] (and the logic of the Krylov–
 371 Bogolyubov averaging) we consider increments $\Delta \mathfrak{n}_{s,L}^{\leq 2} := \mathfrak{n}_{s,L}^{\leq 2}(\tau + \theta) - \mathfrak{n}_{s,L}^{\leq 2}(\tau)$
 372 and express them through the processes $\mathfrak{a}_m^{(0)}$ via the Duhamel formula (1.25)

⁷ In fact, in Sect. 3 we prove an abstract result, more general than the theorem below; see there Theorem 3.2.

373 and the Wick theorem. Then the increments approximately take the form
 374 (1.42), and we use Theorem B to show that they are close to the r.h.s. of the
 375 WKE, multiplied by θ .

376 Although the computation of the increments $\Delta \mathbf{n}_{s,L}^{\leq 2}$ is similar to that in
 377 [7], now a mechanism leading to a WKE is rather different. Namely, in [7] com-
 378 ponents of the terms $\mathbf{n}_{s,L}^{(k)}$ are approximated by formulas analogous to (1.42),
 379 where the summation over the lattice $(\mathbb{Z}^d)^k = \{z\}$ is replaced by an integra-
 380 tion over \mathbb{R}^{dk} . The integrals in those formulae involve fast oscillating Gaussian
 381 kernels. The zero sets of these kernels define quadrics, related to the quadrics
 382 $\{(z_1, z_2) : z_1 \cdot z_2 = 0\}$ in (1.42). Due to the fast oscillations a crucial component
 383 of the increments $\Delta \mathbf{n}_{s,L}^{\leq 2}$ is given by the terms, associated with short-range cor-
 384 relations in τ of the processes $\mathbf{a}_m^{(0)}(\tau)$. On the contrary, in the present situation
 385 a crucial contribution is given by other terms, associated with long-time correla-
 386 tions of the processes $\mathbf{a}_m^{(0)}(\tau)$, while the short-range correlations only give a
 387 small correction, as it can be seen from computations of Appendix A.4. So, it
 388 is natural that the kinetic integral in the WKE (1.38) depends on the viscosity
 389 operator \mathfrak{A} , while that in the WKE in [7] does not.

390 Finally we note that, as we explain in Sect. 3.2, it is plausible that The-
 391 orem C holds for all $k \geq 3$. If so, then for $\rho = \varepsilon L$ the energy spectrum of
 392 a solution $\mathbf{a}(\tau)$, written as (1.22), defines a formal series in ε , uniformly in
 393 $L \geq 2$. Then the partial sums of this series, made by the terms of order ε^m ,
 394 $m \leq M$, with any fixed $M \geq 2$, also satisfy Theorem A with the constants C_r ,
 395 depending on M . Cf. Conjecture 3.8.

396 2. Series Expansion: Approximating Equation and 397 Diagrammatic Representation for Solutions

398 In this section, assuming that $d \geq 2$, we approximate processes (1.25) by more
 399 convenient processes $a^{(n)}$, and then obtain a compact and instrumental repre-
 400 sentation for their correlations in terms of Feynman diagrams (see Lemma 2.2),
 401 following [8, Sections 3–5]. This representation (as well as its analogy in [7, 8])
 402 is used to estimate various disparity terms, related to quasisolutions $\mathcal{A}(\tau; L)$,
 403 see (1.26), and to their energy spectra.

404 Our presentation is sketchy, but missing details may be found in [8]. For
 405 a general discussion of the language of Feynman diagrams, see [18].

406 2.1. Approximate a -Equation

407 We start by considering an approximation of the original Eq. (1.20) by an
 408 equation, where the term $L^{-d} |\mathbf{a}_s|^2 \mathbf{a}_s$ is removed:

$$\begin{aligned}
 \dot{a}_s + \gamma_s a_s &= i\rho \mathcal{Y}_s(a) + b(s) \dot{\beta}_s, \quad s \in \mathbb{Z}^d, \\
 \mathcal{Y}_s(a) &= L^{-d} \sum_{1,2,3} \delta_{3s}^{r12} \delta(\omega_{3s}^{12}) a_1 a_2 \bar{a}_3.
 \end{aligned}
 \tag{2.1}$$

410 Similar to processes \mathbf{a}_s , we decompose

$$411 \quad a = a^{(0)} + \rho a^{(1)} + \dots \quad (2.2)$$

412 Here $a^{(0)} = \mathbf{a}^{(0)}$ and the processes $a_s^{(n)}(\tau)$ with $n \geq 1$ are built by the recur-
 413 sive formula (1.25) with the term $\mathbf{a}_s^{n_1} \mathbf{a}_s^{n_2} \bar{\mathbf{a}}_s^{n_3}$ being dropped. That is, with the
 414 nonlinearity Y_s^{res} replaced by the \mathcal{Y}_s above.

415 Results of [8] together with Theorem 3.2 below (which is an abstract
 416 version of Theorem C from the introduction) imply

417 **Proposition 2.1.** For all $m, n \geq 0$, satisfying $N := m + n \leq 4$,

$$418 \quad \left| \mathbb{E} a_s^{(m)}(\tau_1) \bar{a}_s^{(n)}(\tau_2) - \mathbb{E} \mathbf{a}_s^{(m)}(\tau_1) \bar{\mathbf{a}}_s^{(n)}(\tau_2) \right| \leq \frac{L^{-N-d+1}}{\chi_d(L)^{N-1}} C^\#(s; n, m), \quad (2.3)$$

419 uniformly in $\tau_1, \tau_2 \geq 0$.

420 We prove the proposition in Appendix C for $d \geq 3$ and discuss an adap-
 421 tation of the proof to the case $d = 2$ in Appendix D. Doing that we use the
 422 relation $N := m + n \leq 4$ only to apply Theorem 3.2 (or Theorem D.2 if $d = 2$).
 423 So, if the assertion (3.9) of the latter theorem holds for larger N 's, then for
 424 those N 's estimates (2.3) remains true as well [we believe that (3.9) is fulfilled
 425 for all N , see in Sect. 3.2].

426 Relations (2.3) imply that moments of processes $a_s^{(m)}(\tau)$ well approximate
 427 those of processes $\mathbf{a}_s^{(m)}(\tau)$ as $L \rightarrow \infty$. Accordingly, from now on we will mostly
 428 study processes $a_s(\tau)$ and their decompositions (2.2).

429 2.2. Diagrams for Solutions

430 For what follows it is convenient to re-write operator \mathcal{Y} from (2.1), using a
 431 fictitious index s_4 :

$$432 \quad \mathcal{Y}_s(a) = L^{-d} \sum_{1,2,3,4} \delta_{34}^{12} \delta(\omega_{34}^{12}) \delta_4^s a_1 a_2 \bar{a}_3,$$

433 where δ_4^s is the Kronecker symbol. Then analogous of the expression (1.25) for
 434 $a^{(m)}$, $m \geq 1$, takes the form

$$435 \quad a_s^{(m)}(\tau) = \sum_{m_1+m_2+m_3=m-1} i \int_0^\tau dl e^{-\gamma_s(\tau-l)} \quad (2.4)$$

$$L^{-d} \sum_{1,2,3,4} \delta_{34}^{12} \delta(\omega_{34}^{12}) \delta_4^s (a_1^{(m_1)} a_2^{(m_2)} \bar{a}_3^{(m_3)})(l).$$

436 We will call the objects as those in the r.h.s. of (2.4) *sums*, despite they involve
 437 integrating in dl . The r.h.s. of (2.4) contains several sums, corresponding to
 438 all admissible choices of numbers m_1, m_2, m_3 .

439 We apply Duhamel's formula (2.4) to the terms $a_{s_i}^{(m_i)}(l)$ in the right-hand
 440 side of (2.4) with $m_i > 0$, and iterate the procedure till $a_s^{(m)}(\tau)$ is expressed
 441 through the processes $a^{(0)}$ and $\bar{a}^{(0)}$. Then $a_s^{(m)}$ becomes represented as a finite
 442 sum of *sums*; we denote such sums by I_s . Below we will associate with each

443 sum I_s an appropriately constructed diagram \mathfrak{D} . Thus, we will write $a_s^{(m)}(\tau)$
 444 as

445
$$a_s^{(m)}(\tau) = \sum_{\mathfrak{D} \in \mathfrak{D}_m} I_s(\mathfrak{D}; \tau), \tag{2.5}$$

446 where \mathfrak{D}_m is a set of all diagrams, corresponding to the just explained repre-
 447 sentation of $a^{(m)}$ via the processes $a^{(0)}$ and $\bar{a}^{(0)}$. Similarly by $\overline{\mathfrak{D}}_n$ we denote
 448 the set of diagrams, parametrizing the terms in the sum, representing $\bar{a}^{(n)}(\tau)$
 449 in a form, analogous to (2.5): $\bar{a}_s^{(n)}(\tau) = \sum_{\overline{\mathfrak{D}} \in \overline{\mathfrak{D}}_n} I_s(\overline{\mathfrak{D}}; \tau)$.

450 **2.2.1. Construction of the Sets of Diagrams \mathfrak{D}_m and $\overline{\mathfrak{D}}_n$.** We start with dis-
 451 cussing the set \mathfrak{D}_2 and the sums $I_s(\mathfrak{D})$ with $\mathfrak{D} \in \mathfrak{D}_2$.

452 When iterating the Duhamel formula (2.4) (or its complex conjugation)
 453 for a j -th time, we will denote the corresponding time $l \in [0, \tau]$ by l_j and
 454 will write the set of indices $\{s_1, s_2, s_3, s_4\}$ as $\{\xi_{2j-1}, \xi_{2j}, \sigma_{2j-1}, \sigma_{2j}\}$, where we
 455 enumerate by ξ_i the indices of non-conjugated variables $a_{s_i}^{(k)}$ in (2.4) and by
 456 σ_i – those of conjugated variables $\bar{a}_{s_i}^{(n)}$. We write the corresponding fictitious
 457 index s_4 as σ_{2j} if we apply (2.4), or as ξ_{2j} if we apply the complex conjugation
 458 of (2.4). More precisely, when applying (2.4) we denote $s_1 = \xi_{2j-1}$, $s_2 = \xi_{2j}$,
 459 $s_3 = \sigma_{2j-1}$ and $s_4 = \sigma_{2j}$, and when applying its complex conjugation, we write
 460 $s_1 = \sigma_{2j-1}$, $s_2 = \sigma_{2j}$, $s_3 = \xi_{2j-1}$ and $s_4 = \xi_{2j}$. We will abbreviate

461
$$\delta_j = \delta'_{\sigma_{2j-1}\sigma_{2j}}^{\xi_{2j-1}\xi_{2j}} \quad \text{and} \quad \omega_j = \omega_{\sigma_{2j-1}\sigma_{2j}}^{\xi_{2j-1}\xi_{2j}}, \quad j \geq 1, \tag{2.6}$$

462 [these terms correspond to δ_{34}^{12} and ω_{34}^{12} in (2.4)]. We also set

463
$$\xi_0 = \sigma_0 := s. \tag{2.7}$$

464 Applying (2.4) to $a_s^{(2)}$ and using the notation above with $j = 1$, we find

465
$$a_s^{(2)}(\tau) = a_{\xi_0}^{(2)}(\tau) = \sum_{m_1+m_2+m_3=1} i \int_0^\tau dl_1 e^{-\gamma_{\xi_0}(\tau-l_1)} L^{-d} \sum_{\xi_1, \xi_2, \sigma_1, \sigma_2} \delta_1 \delta(\omega_1) \delta_{\sigma_2}^{\xi_0} (a_{\xi_1}^{(m_1)} a_{\xi_2}^{(m_2)} \bar{a}_{\sigma_1}^{(m_3)})(l_1). \tag{2.8}$$

467 Let us consider the summand with $m_1 = m_2 = 0$ and $m_3 = 1$. Applying the
 468 conjugated formula (2.4) to $\bar{a}_{\sigma_1}^{(1)}$ and using the introduced notation with $j = 2$,
 469 we get

470
$$\bar{a}_{\sigma_1}^{(1)}(l_1) = -i \int_0^{l_1} dl_2 e^{-\gamma_{\sigma_1}(l_1-l_2)} L^{-d} \sum_{\xi_3, \xi_4, \sigma_3, \sigma_4} \delta_2 \delta(\omega_2) \delta_{\xi_4}^{\sigma_1} (a_{\xi_3}^{(0)} \bar{a}_{\sigma_3}^{(0)} \bar{a}_{\sigma_4}^{(0)})(l_2). \tag{2.9}$$

471 Inserting (2.9) into the summand in (2.8) with $m_1 = m_2 = 0$ and $m_3 = 1$, we
 472 get a sum $I_s(\mathfrak{D}; \tau)$ which we associate with the diagram \mathfrak{D} from Fig. 1c; further
 473 on we will denote this diagram by \mathfrak{D}^c . The *non-conjugated* vertices $c_i^{(k)}$ of the
 474 diagram are associated with the variables $a_{\xi_i}^{(k)}$ in (2.8), (2.9); the corresponding

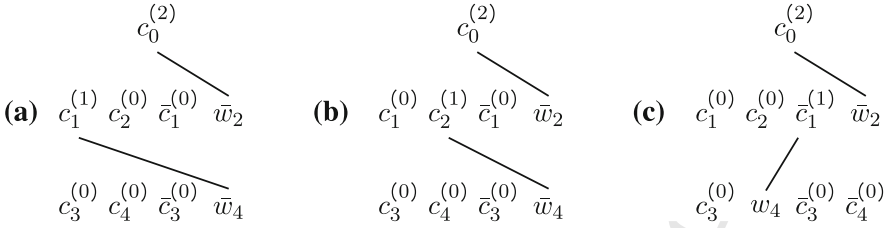


FIGURE 1. The set of diagrams \mathfrak{D}_2

475 to them indices are ξ_i . The *conjugated* vertices $\bar{c}_j^{(n)}$ are associated with the
 476 variables $\bar{a}_{\sigma_j}^{(n)}$ and the corresponding indices are σ_j . In particular, the root $c_0^{(2)}$
 477 is associated with $a_{\xi_0}^{(2)} = a_s^{(2)}$ and the corresponding index is ξ_0 . In the notation
 478 $c_i^{(k)}$ and $\bar{c}_j^{(n)}$ we sometimes omit the upper indices k and n which we call the
 479 *degrees* of the vertices $c_i^{(k)}$, $\bar{c}_j^{(n)}$. The vertices w_2 and w_4 are called conjugated
 480 (non-conjugated) *virtual* vertices, and the corresponding indices are σ_2 and ξ_4 ;
 481 these vertices are associated with the Kronecker symbols $\delta_{\sigma_2}^{\xi_0}$ and $\delta_{\xi_4}^{\sigma_1}$ in (2.8)
 482 and (2.9). Vertices which are not virtual are called *real*. Every edge of the
 483 diagram couples a non-conjugated (conjugated) vertex $c_i^{(k)}$ ($\bar{c}_i^{(k)}$) of positive
 484 degree $k \geq 1$ with a conjugated (non-conjugated) virtual vertex $w_{i'}$ ($w_{i'}$). It
 485 is associated with an application of formula (2.4) (or its complex conjugation)
 486 to the variable $a_{\xi_i}^{(k)}$ (or $\bar{a}_{\sigma_i}^{(k)}$), corresponding to the vertex $c_i^{(k)}$ (or $\bar{c}_i^{(k)}$).
 487

The set of four vertices

$$488 \quad c_{2j-1}, c_{2j}, \bar{c}_{2j-1}, \bar{w}_{2j} \quad \text{or} \quad c_{2j-1}, w_{2j}, \bar{c}_{2j-1}, \bar{c}_{2j} \quad (2.10)$$

489 (in dependence whether the virtual vertex is conjugated or not) to which corre-
 490 spond the indices $\xi_{2j-1}, \xi_{2j}, \sigma_{2j-1}, \sigma_{2j}$ is called *the j-th block*; the diagram \mathfrak{D}^c
 491 has two blocks. The index i of a virtual vertex w_i (\bar{w}_i) is always pair, $i = 2j$.
 492 Each block corresponds to an application of formula (2.4) (or its complex con-
 493 jugation) to its *parent*, i.e. to the vertex of positive degree coupled with the
 494 virtual vertex of the block. The virtual vertex is conjugated if the parent is
 495 non-conjugated and the other way round. The *time variable* l_j is associated
 496 with the j -th block.

497 The *leaves* are the vertices of zero degree, that is, the vertices $c_i^{(0)}$ and
 498 $\bar{c}_j^{(0)}$.

499 The diagrams from Fig. 1(a,b) correspond to the summands in (2.8) with
 500 $m_1 = 1$, $m_2 = m_3 = 0$ and $m_1 = m_3 = 0$, $m_2 = 1$; they are constructed by the
 501 same rules as the diagram \mathfrak{D}^c . The three diagrams from Fig. 1 form the set \mathfrak{D}_2 .
 502 The set of diagrams $\bar{\mathfrak{D}}_2$, corresponding to $\bar{a}_s^{(2)}(\tau)$, is obtained by conjugating
 503 the vertices in the three diagrams above and re-ordering the elements of each
 504 block in such a way that the pair of non-conjugated vertices is followed by the
 505 pair of conjugated vertices, i.e. the blocks have the form (2.10).

506 The sets \mathfrak{D}_m and $\bar{\mathfrak{D}}_n$ with arbitrary $m, n \geq 0$ and the diagrams which
 507 are their elements, are constructed similarly. Namely, the sets \mathfrak{D}_0 and $\bar{\mathfrak{D}}_0$

508 are trivial—they contain one diagram each, made by the root $c_0^{(0)}$ (or $\bar{c}_0^{(0)}$).
 509 The sets \mathfrak{D}_1 and $\bar{\mathfrak{D}}_1$ also contain only one diagram each; e.g. the diagram
 510 in \mathfrak{D}_1 consists of the root $c_1^{(0)}$, joint by an edge with \bar{w}_2 in the only block
 511 $B_1 = (c_1^{(0)}, c_2^{(0)}, \bar{c}_1^{(0)}, \bar{w}_2)$. Arbitrary sets \mathfrak{D}_m and $\bar{\mathfrak{D}}_n$ may be constructed by
 512 induction. Indeed, consider a process $a^{(m+1)}(\tau)$ with $m \geq 1$ and apply to it
 513 (2.4) with $m := m + 1$. In the r.h.s. of (2.4) the sum in m_1, m_2, m_3 contains
 514 $(m + 2)(m + 1)/2$ terms. Consider any one of them,

$$515 \quad i \int_0^\tau dl e^{-\gamma_s(\tau-l)} L^{-d} \sum_{1,2,3,4} \delta_{34}^{12} \delta(\omega_{34}^{12}) \delta_4^s (a_1^{(m_1)} a_2^{(m_2)} \bar{a}_3^{(m_3)})(l), \quad (2.11)$$

516 draw the block $B_1 = (c_1^{(m_1)}, c_2^{(m_2)}, \bar{c}_1^{(m_3)}, \bar{w}_2)$, and join \bar{w}_2 by an edge with the
 517 root $c_0^{(m+1)}$. Next consider the sets $\mathfrak{D}_{m_1}, \mathfrak{D}_{m_2}, \bar{\mathfrak{D}}_{m_3}$ and do the following:

- 518 (a) Firstly, take \mathfrak{D}_{m_1} . If $m_1 = 0$, do nothing. Otherwise, choose any diagram
 519 $\mathfrak{D}^1 \in \mathfrak{D}_{m_1}$, place it below $c_1^{(m_1)}$ and identify its root with $c_1^{(m_1)}$. Do this
 520 for each diagram in \mathfrak{D}_{m_1} , thus obtaining $|\mathfrak{D}_{m_1}|$ diagrams with roots in
 521 $c_0^{(m+1)}$.
- 522 (b) Then consider the set \mathfrak{D}_{m_2} and do the same with the just obtained $|\mathfrak{D}_{m_1}|$
 523 diagrams, identifying their roots with the vertex $c_2^{(m_2)}$, and next—the
 524 set $\bar{\mathfrak{D}}_{m_3}$, identifying the roots with $\bar{c}_3^{(m_3)}$.
- 525 (c) It remains to convert thus obtained $|\mathfrak{D}_{m_1}| \times |\mathfrak{D}_{m_2}| \times |\bar{\mathfrak{D}}_{m_3}|$ diagrams to
 526 elements of the set \mathfrak{D}_{m+1} by re-numerating properly their blocks and
 527 accordingly re-numerating the vertices in the blocks as in (2.10). Do this
 528 by numerating the blocks from top to the bottom and from left to right,
 529 as in the examples above with $m = 2$.
- 530 (d) Doing the same for all blocks, corresponding to all possible $(m + 2)(m +$
 531 $1)/2$ terms (2.11), get the diagrams, forming the set \mathfrak{D}_{m+1} .

532 The set $\bar{\mathfrak{D}}_{m+1}$ is constructed inductively in the same way.

533 For further needs we note that due to the factors δ_{34}^{12} and $\delta_{s_4}^s$ in (2.4),
 534 the indices ξ_i, σ_j entering the formula for the sums $I_s(\mathfrak{D})$ from (2.5) satisfy
 535 the relations

$$536 \quad (1) \delta_{\sigma_{2j-1}\sigma_{2j}}^{\xi_{2j-1}\xi_{2j}} = 1 \quad \forall j,$$

$$537 \quad (2) \text{ indices } \xi_i, \sigma_j \text{ corresponding to adjacent in } \mathfrak{D} \text{ vertices are equal.} \quad (2.12)$$

538 2.3. Feynmann Diagrams for Expectations

539 The main objects we are interested in are the correlations $\mathbb{E}a_{s_1}^{(m)}(\tau_1)\bar{a}_{s_2}^{(n)}(\tau_2)$.
 540 It can be shown that they vanish if $s_1 \neq s_2$.⁸ To represent an expectation
 541 $\mathbb{E}a_s^{(m)}(\tau_1)\bar{a}_s^{(n)}(\tau_2)$, we consider the set of diagrams

$$542 \quad \mathfrak{D}_m \times \bar{\mathfrak{D}}_n := \{\mathfrak{D}^1 \sqcup \bar{\mathfrak{D}}^2 : \mathfrak{D}^1 \in \mathfrak{D}_m, \bar{\mathfrak{D}}^2 \in \bar{\mathfrak{D}}_n\}.$$

543 Here a diagram $\mathfrak{D}^1 \sqcup \bar{\mathfrak{D}}^2$ is obtained by drawing \mathfrak{D}^1 and $\bar{\mathfrak{D}}^2$ side by side, where
 544 the blocks of \mathfrak{D}^1 are enumerated from 1 to m , while those of $\bar{\mathfrak{D}}^2$ together with

⁸As well vanish the correlations $\mathbb{E}a_{s_1}^{(m)}(\tau_1)a_{s_2}^{(n)}(\tau_2)$ and $\mathbb{E}\bar{a}_{s_1}^{(m)}(\tau_1)\bar{a}_{s_2}^{(n)}(\tau_2)$ for all s_1, s_2 .

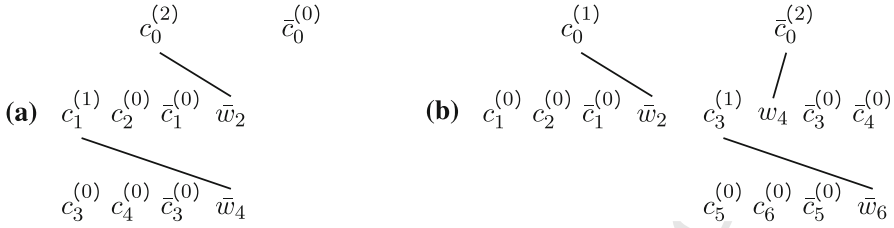


FIGURE 2. A diagram from the set **a** $\mathfrak{D}_2 \times \overline{\mathfrak{D}}_0$ and **b** $\mathfrak{D}_1 \times \overline{\mathfrak{D}}_2$

545 the corresponding time variables l_j are enumerated from $j = m + 1$ to $m + n$.
 546 The vertices together with the corresponding indices $\xi_{2j-1}, \xi_{2j}, \sigma_{2j-1}, \sigma_{2j}$ are
 547 enumerated accordingly, see Fig. 2. The diagram $\mathfrak{D}^1 \sqcup \overline{\mathfrak{D}}^2$ has two roots $c_0^{(m)}$
 548 and $\bar{c}_0^{(n)}$. For any $\mathfrak{D} = \mathfrak{D}^1 \sqcup \overline{\mathfrak{D}}^2$ consider

549
$$I_s(\mathfrak{D}; \tau_1, \tau_2) = I_s(\mathfrak{D}^1; \tau_1) I_s(\overline{\mathfrak{D}}^2; \tau_2),$$

550 so that $a_s^{(m)}(\tau_1) \bar{a}_s^{(n)}(\tau_2) = \sum_{\mathfrak{D} \in \mathfrak{D}_m \times \overline{\mathfrak{D}}_n} I_s(\mathfrak{D}; \tau_1, \tau_2)$. Our next task is to com-
 551 pute $\mathbb{E} I_s(\mathfrak{D})$ for each $\mathfrak{D} \in \mathfrak{D}_m \times \overline{\mathfrak{D}}_n$.

552 Randomness enters the term $I_s(\mathfrak{D})$ via the random variables $a_{\xi_i}^{(0)}, \bar{a}_{\sigma_j}^{(0)}$,
 553 corresponding to the leaves of the diagram $\mathfrak{D} = \mathfrak{D}^1 \sqcup \overline{\mathfrak{D}}^2$. They are Gaussian
 554 with correlations

555
$$\mathbb{E} a_s^{(0)}(l_1) a_{s'}^{(0)}(l_2) = 0, \quad \mathbb{E} a_s^{(0)}(l_1) \bar{a}_{s'}^{(0)}(l_2) = \delta_{s'}^s \text{Corr}(\gamma_s, b(s), l_1, l_2),$$

 556 (2.13)

557 where

558
$$\text{Corr}(\gamma_s, b(s), l_1, l_2) = B_s (e^{-\gamma_s |l_1 - l_2|} - e^{-\gamma_s (l_1 + l_2)}), \quad B_s = \frac{b(s)^2}{\gamma_s}.$$

 559 (2.14)

560 So, the Wick theorem [18] implies that the expectation $\mathbb{E} I_s(\mathfrak{D})$ is given by a
 561 sum over all Wick pairings of variables $a_{\xi_i}^{(0)}$, corresponding to non-conjugated
 562 leaves $c_i^{(0)}$, with variables $\bar{a}_{\sigma_j}^{(0)}$ corresponding to conjugated leaves $\bar{c}_j^{(0)}$. More-
 563 over, the leaves $c_i^{(0)}$ and $\bar{c}_j^{(0)}$ should belong to different blocks since otherwise
 564 the summand corresponding to such Wick pairing vanishes due to (1.15) and
 565 item (1) in (2.12). We parametrize the sum over the Wick pairings by the
 566 defined below set $\mathfrak{F}(\mathfrak{D})$ of Feynman diagrams. Denoting by $J_s(\mathfrak{F})$ a term (i.e.
 567 a sum), corresponding to a specific Feynman diagram \mathfrak{F} , we have:

568
$$\mathbb{E} I_s(\mathfrak{D}) = \sum_{\mathfrak{F} \in \mathfrak{F}(\mathfrak{D})} J_s(\mathfrak{F}).$$

569 **2.3.1. Definition of Feynman diagrams.** To construct the set of Feynman di-
 570 agrams $\mathfrak{F}(\mathfrak{D})$, corresponding to some diagram $\mathfrak{D} = \mathfrak{D}^1 \sqcup \overline{\mathfrak{D}}^2$, we consider all
 571 possible partitions of the set of leaves of \mathfrak{D} to non-intersecting pairs $(c_i^{(0)}, \bar{c}_j^{(0)})$,
 572 such that the paired leaves $c_i^{(0)}$ and $\bar{c}_j^{(0)}$ do not belong to the same block. To

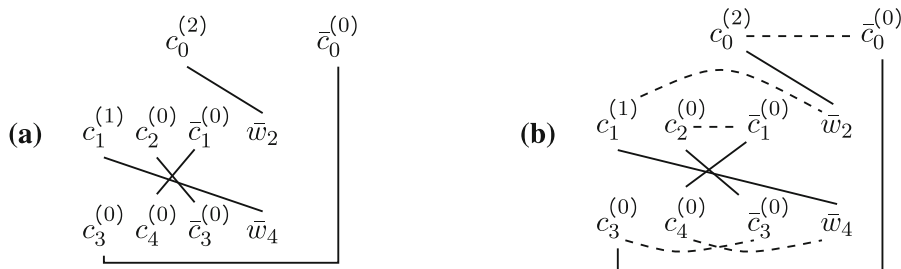


FIGURE 3. **a** A Feynman diagram \mathfrak{F} obtained from the diagram \mathfrak{D} in Fig. 2(a). **b** A cycle obtained from the Feynman diagram \mathfrak{F}

573 each such partition, we associate a diagram \mathfrak{F} obtained from \mathfrak{D} by joining
 574 with an edge the two leaves in every pair, see Fig. 3(a). So, in each diagram
 575 $\mathfrak{F} \in \mathfrak{F}(\mathfrak{D})$ all vertices of \mathfrak{D} are joint by edges, every edge couples a conjugated
 576 vertex with a non-conjugated in another block (or with a root), and every
 577 vertex belongs to exactly one edge.

578 Since $\mathbb{E}a_s^{(0)}\bar{a}_{s'}^{(0)} = 0$ if $s \neq s'$, the indices ξ_i, σ_j entering the formulas for
 579 sums $J_s(\mathfrak{F})$ satisfy (2.12), where in item (2) the diagram \mathfrak{D} is replaced by the
 580 Feynman diagram \mathfrak{F} ; below we denote these relations as (2.12) $_{\mathfrak{F}}$. In particular,
 581 due to the item (2), the vector of indices $\sigma = (\sigma_i)$ is a function of the vector
 582 $\xi = (\xi_j)$. Accordingly below we write $\sigma = \sigma_{\mathfrak{F}}(\xi)$.

583 Let

584
$$\mathfrak{F}_{m,n} = \bigcup_{\mathfrak{D} \in \mathfrak{D}_m \times \bar{\mathfrak{D}}_n} \mathfrak{F}(\mathfrak{D})$$

585 be the set of all Feynman diagrams associated with the product $a_s^{(m)}\bar{a}_s^{(n)}$. Each
 586 diagram $\mathfrak{F} \in \mathfrak{F}_{m,n}$ has $N := m+n$ blocks and $4N+2$ vertices, including $2N+2$
 587 leaves. Half of edges (and of leaves) are conjugated, while another half is not.

588 By construction a diagram $\mathfrak{F} \in \mathfrak{F}_{m,n}$ never pairs leaves from the same
 589 block. This alone does not exclude that \mathfrak{F} is such that in (2.12) $_{\mathfrak{F}}$ the assump-
 590 tions (1) and (2) are incompatible since for some j we may have $\xi_{2j-1}, \xi_{2j} =$
 591 σ_{2j-1} or σ_{2j} once $\sigma = \sigma_{\mathfrak{F}}(\xi)$. Analysis shows that this cannot happen if
 592 $m+n \leq 4$, but may happen if $m+n \geq 5$. Accordingly, we denote by

593
$$\mathfrak{F}_{m,n}^{true} \subset \mathfrak{F}_{m,n}$$

594 the set of Feynman diagrams for which the set of indices ξ_i, σ_j satisfying the
 595 relations (2.12) $_{\mathfrak{F}}$ is not empty. For any diagram $\mathfrak{F} \notin \mathfrak{F}_{m,n}^{true}$, we have $J_s(\mathfrak{F}) = 0$
 596 due to the factors δ_{34}^{12} and δ_{s4}^s in (2.4).

597 **2.4. Transformation, Resolving Linear Relations on Indices**

598 Let us take a Feynman diagram $\mathfrak{F} \in \mathfrak{F}_{m,n}$, denote $N := m+n \geq 1$ and consider
 599 the sum $J_s(\mathfrak{F})$. The relations (2.12) $_{\mathfrak{F}}$ on indices $\xi_i, \sigma_j \in \mathbb{Z}_L^d$, $0 \leq i, j \leq 2N$,⁹

⁹Recall that $\xi_0 = \sigma_0 = s$.

600 entering the formula for $J_s(\mathfrak{F})$ are involved, which makes the sum $J_s(\mathfrak{F})$ difficult
 601 for further analysis. In [8] it was found a convenient way to “integrate the sums
 602 $J_s(\mathfrak{F})$ ”, i.e. to parametrize the indices ξ_i, σ_j by N -vector $z = (z_1, \dots, z_N)$ from
 603 a domain in $(\mathbb{Z}_L^d)^N$, free from any relations on its components. In this section
 604 we present this parametrization, referring the reader to [8] for a proof.

605 Since item (2) of (2.12) $_{\mathfrak{F}}$ is equivalent to the relation $\sigma = \sigma_{\mathfrak{F}}(\xi)$, it suffices
 606 to parametrize the set of *admissible* multi-indices ξ , i.e. of those ξ for which
 607 the multi-indices ξ and $\sigma = \sigma_{\mathfrak{F}}(\xi)$ satisfy item (1) in (2.12) $_{\mathfrak{F}}$. The construction
 608 starts with defining for each $\mathfrak{F} \in \mathfrak{F}_{m,n}$ a skew-symmetric $N \times N$ *incidence*
 609 matrix $\alpha^{\mathfrak{F}} = (\alpha_{ij}^{\mathfrak{F}})$ whose elements are integers from the set $\{0, +1, -1\}$. In
 610 terms of this matrix, we define the set of polyvectors

$$611 \quad \mathcal{Z}(\mathfrak{F}) = \{z = (z_1, \dots, z_N) \in (\mathbb{Z}_L^d)^N : z_j \neq 0 \text{ and } (\alpha^{\mathfrak{F}}z)_j \neq 0 \quad \forall j\}. \quad (2.15)$$

613 Here and below for an $M \times N$ -matrix A we denote by Az the polyvector with
 614 components $(Az)_j := \sum_i A_{ji}z_i \in \mathbb{Z}^d$.¹⁰ The matrix $\alpha^{\mathfrak{F}}$ has no zero rows and
 615 zero columns if and only if $\mathfrak{F} \in \mathfrak{F}_{m,n}^{true}$, and, accordingly, the set $\mathcal{Z}(\mathfrak{F})$ is non-
 616 empty if and only if $\mathfrak{F} \in \mathfrak{F}_{m,n}^{true}$. Next, it turns out that the vectors $z \in \mathcal{Z}(\mathfrak{F})$
 617 may be used to parametrize the set of admissible indices ξ_i by means of an
 618 affine mapping

$$619 \quad \xi(z) = s + A^{\mathfrak{F}}z, \quad z \in \mathcal{Z}(\mathfrak{F}). \quad (2.16)$$

620 Here $A^{\mathfrak{F}}$ is an $(2N + 1) \times N$ -matrix, whose elements again are integers from
 621 the set $\{0, +1, -1\}$. Transformation (2.16) provides a presentation of the terms
 622 $J_s(\mathfrak{F})$, forming the correlation $\mathbb{E}a_s^{(m)}(\tau_1)\bar{a}_s^{(n)}(\tau_2)$, and so for the correlation
 623 itself. The corresponding result is proved in Theorem 5.5 of [8]. In our setting
 624 its statement, where for the function $\theta(x, t)$ is chosen $\mathbb{I}_{\{0\}}(x)$ —the indicator
 625 function of the point $x = 0$ —takes the following form:

626 **Lemma 2.2.** *For any integers $m, n \geq 0$ satisfying $N = m + n \geq 1$, any $s \in \mathbb{Z}_L^d$
 627 and $\tau_1, \tau_2 \geq 0$,*

628 (1) *for each $\mathfrak{F} \in \mathfrak{F}_{m,n}^{true}$ parametrization (2.16) (depending on s and \mathfrak{F}) is
 629 such that the quantity ω_j in (2.6), written in the z -coordinates, takes the
 630 form*

$$631 \quad \omega_j^{\mathfrak{F}}(z) = 2z_j \cdot \sum_{i=1}^N \alpha_{ji}^{\mathfrak{F}}z_i = 2z_j \cdot (\alpha^{\mathfrak{F}}z)_j. \quad (2.17)$$

632 (2) *We have*

$$633 \quad L^N \mathbb{E}a_s^{(m)}(\tau_1)\bar{a}_s^{(n)}(\tau_2) = \sum_{\mathfrak{F} \in \mathfrak{F}_{m,n}^{true}} c_{\mathfrak{F}} J_s(\tau_1, \tau_2; \mathfrak{F}), \quad (2.18)$$

¹⁰That is, abusing notation we denote by A an operator in $(\mathbb{Z}^d)^M$ with the block-matrix $A \otimes \mathbb{1}$.

634 where the constants $c_{\mathfrak{F}} \in \{\pm 1, \pm i\}$ and

$$635 \quad J_s(\tau_1, \tau_2; \mathfrak{F}) = \int_{\mathbb{R}^N} dl L^{N(1-d)} \sum_{z \in \mathcal{Z}(\mathfrak{F}), \omega_j^{\mathfrak{F}}(z)=0 \forall j} F_s^{\mathfrak{F}}(\tau_1, \tau_2, l, z). \quad (2.19)$$

636 The density $F_s^{\mathfrak{F}}(\tau_1, \tau_2, l, z)$ is a real function, smooth in $(s, z) \in \mathbb{R}^d \times$
 637 \mathbb{R}^{dN} , satisfying

$$638 \quad |\partial_s^\mu \partial_z^\kappa F_s^{\mathfrak{F}}(\tau_1, \tau_2, l, z)| \leq C_{\mu, \kappa}^\#(s) C_{\mu, \kappa}^\#(z) e^{-\delta(\sum_{i=1}^m |\tau_1 - l_i| + \sum_{i=m+1}^N |\tau_2 - l_i|)}$$

639 (2.20)

640 with a suitable $\delta = \delta_N > 0$, for any vectors $\mu \in (\mathbb{N} \cup \{0\})^d$, $\kappa \in (\mathbb{N} \cup$
 641 $\{0\})^{dN}$ and any $s \in \mathbb{R}^d$, $z \in \mathbb{R}^{dN}$, $l \in \mathbb{R}^N$.

642 Let us briefly explain the way to construct the parametrization (2.16). We
 643 first add to the Feynman diagram \mathfrak{F} dashed edges that couple non-conjugated
 644 vertices with conjugated *inside* all blocks, as in Fig. 3(b). For each block there
 645 are two ways of doing that. We prove that there exists a choice (possibly,
 646 not unique) of a dashed edge in each block such that the diagram becomes
 647 a cycle, as in Fig. 3b. Then, for each j we set $x_{2j-1} := \xi_{2j-1} - \sigma_{2j-1}$ and
 648 $x_{2j} := \xi_{2j} - \sigma_{2j}$ or $x_{2j-1} = \xi_{2j-1} - \sigma_{2j}$ and $x_{2j} := \xi_{2j} - \sigma_{2j-1}$, according to
 649 the choice of the dashed edges in the j -th block, where we substitute $\sigma = \sigma_{\mathfrak{F}}(\xi)$.
 650 The fact that the Feynman diagram with added dashed edges forms a cycle
 651 implies that the transformation $\xi \mapsto x$ is invertible. Item (1) of (2.12) $_{\mathfrak{F}}$ implies
 652 that $x_{2j} = -x_{2j-1}$. Then we set $z_j := x_{2j-1}$ and get (2.16). The incidence
 653 matrix $\alpha^{\mathfrak{F}}$ also is constructed in terms of this cycle.

654 Since the choice of the dashed edges in general is not unique, the parametri-
 655 zation $z \mapsto \xi$ is not unique as well. However, if $z' \mapsto \xi$ is another parametriza-
 656 tion, obtained by the procedure above, and $\alpha^{\mathfrak{F}'}$ is the associated incidence
 657 matrix, then for each j we have either $z'_j(\xi) = z_j(\xi)$ or $z'_j(\xi) = (\alpha^{\mathfrak{F}'} z(\xi))_j$. In
 658 the latter case we also have the symmetric relation $z_j(\xi) = (\alpha^{\mathfrak{F}'} z'(\xi))_j$.

659 Computing in (2.19) the integral over dl and using estimate (2.20), we
 660 obtain a form of integrals J_s , more convenient for some of the subsequent
 661 analysis:

662 **Corollary 2.3.** *In terms of Lemma 2.2, the integrals J_s from (2.18) can be*
 663 *written as*

$$664 \quad J_s(\tau_1, \tau_2; \mathfrak{F}) = L^{N(1-d)} \sum_{z \in \mathcal{Z}(\mathfrak{F}), \omega_j^{\mathfrak{F}}(z)=0 \forall j} \Phi_s^{\mathfrak{F}}(\tau_1, \tau_2, z), \quad (2.21)$$

665 where the real-valued functions $\Phi_s^{\mathfrak{F}}$ are Schwartz in (s, z) and satisfy

$$666 \quad |\partial_s^\mu \partial_z^\kappa \Phi_s^{\mathfrak{F}}(\tau_1, \tau_2, z)| \leq C_{\mu, \kappa}^\#(s) C_{\mu, \kappa}^\#(z), \quad (2.22)$$

667 uniformly in $\tau_1, \tau_2 \geq 0$, for any vectors $\mu \in (\mathbb{N} \cup \{0\})^d$ and $\kappa \in (\mathbb{N} \cup \{0\})^{dN}$.

668 **3. Main Estimates for The Sums**

669 In this section we focus on estimates for the sums (2.21) and on their dependence
 670 on L and N . We recall that $d \geq 3$. It is convenient to study the problem
 671 we consider in the following abstract setting. Let $\alpha = (\alpha_{ij})$, $N \geq 2$, be an
 672 $N \times N$ skew-symmetric matrix whose elements belong to the set $\{-1, 0, 1\}$,
 673 without zero lines and rows.¹¹ Consider a family of quadratic forms on $(\mathbb{R}^d)^N$

674
$$\omega_j(z) = z_j \cdot (\alpha z)_j, \quad 1 \leq j \leq N,$$

675 where z is the polyvector (z_1, \dots, z_N) , $z_j \in \mathbb{R}^d$, and $(\alpha z)_j := \sum_{i=1}^N \alpha_{ji} z_i$. Let
 676 us set

677
$$\mathcal{Z} = \{z \in (\mathbb{Z}_L^d)^N : z_j \neq 0 \text{ and } (\alpha z)_j \neq 0 \ \forall j\}. \quad (3.1)$$

678 Let a function $\Phi : \mathbb{R}^{Nd} \rightarrow \mathbb{R}$ be sufficiently smooth and sufficiently fast decaying
 679 at infinity (see below for exact assumptions). Our goal is to study
 680 asymptotic as $L \rightarrow \infty$ behaviour of the sum

681
$$S_{L,N}(\Phi) := L^{N(1-d)} \sum_{z \in \mathcal{Z}: \omega_j(z)=0 \ \forall j} \Phi(z). \quad (3.2)$$

682 For a function $f \in C^k(\mathbb{R}^m)$, $n_1 \in \mathbb{N} \cup \{0\}$ satisfying $n_1 \leq k$ and $n_2 \in \mathbb{R}$,
 683 we set

684
$$\|f\|_{n_1, n_2} = \sup_{z \in \mathbb{R}^m} \max_{|\alpha| \leq n_1} |\partial^\alpha f(z)| \langle z \rangle^{n_2}, \quad \langle x \rangle := \max\{1, |x|\}. \quad (3.3)$$

685 The first crucial result concerns the case $N = 2$. Then $\omega_1(z) = -\omega_2(z) =$
 686 $\alpha_{12} z_1 \cdot z_2$ and $\alpha_{12} \neq 0$, so

687
$$S_{L,2}(\Phi) = L^{2(1-d)} \sum_{\substack{z \in \mathbb{Z}_L^{2d}: z_1 \cdot z_2 = 0 \\ z_1 \neq 0, z_2 \neq 0}} \Phi(z). \quad (3.4)$$

688 Then we write the sum above as $\sum_{z_1 \cdot z_2 = 0} - \sum_{z_1 = 0 \text{ or } z_2 = 0}$.

689 Since $\left| L^{-d} \sum_{z \in \mathbb{Z}_L^{2d}: z_i = 0} \Phi(z) \right| \leq C \|\Phi\|_{0, d+1}$ for $i = 1, 2$, we get

690
$$\left| S_{L,2}(\Phi) - L^{2(1-d)} \sum_{z \in \mathbb{Z}_L^{2d}: z_1 \cdot z_2 = 0} \Phi(z) \right| \leq CL^{2-d} \|\Phi\|_{0, d+1}. \quad (3.5)$$

691 Now an asymptotic for the sum $S_{L,2}(\Phi)$ immediately follows from Theorem 1.3
 692 in [10] where the dimension is $2d$, $\varepsilon = 1/2$ and $m = 0$, by applying it to the
 693 sum $\sum_{z_1 \cdot z_2 = 0}$ in (3.5) (we recall that $d \geq 3$):

694 **Theorem 3.1.** *Let $N_1(d) := 4d(4d^2 + 2d - 1)$ and $N_2(d) := N_1 + 6d + 4$. If*
 695 *$\|\Phi\|_{N_1, N_2} < \infty$, then there exist constants $C_d \in (1, 1 + 2^{2-d})$ and $K_d > 0$ such*
 696 *that*

697
$$\left| S_{L,2}(\Phi) - C_d \int_{\Sigma_0} \Phi(z) \mu^{\Sigma_0}(dz_1 dz_2) \right| \leq K_d \frac{\|\Phi\|_{N_1, N_2}}{L^{d-5/2}}, \quad (3.6)$$

¹¹ The theorems below and their proofs remain valid as well for arbitrary skew-symmetric matrices with integer elements without zero lines and rows, but in this case the notation used in the proof becomes heavier.

698 where Σ_0 is the quadric $\{z \in \mathbb{R}^{2d} : z_1 \cdot z_2 = 0\}$ and the measure μ^{Σ_0} is given
 699 by (1.34) with $s = 0$.

700 In Appendix C of [10] we give the following explicit formula for the
 701 number-theoretical constants C_d :

$$702 \quad C_d = \frac{\zeta(d-1)\zeta(4d-2)}{\zeta(d)\zeta(2d-2)}, \tag{3.7}$$

703 where ζ is the Riemann zeta-function. Due to (3.7) C_d satisfies $1 < C_d <$
 704 $1 + 2^{2-d}$, as is stated in the theorem. The integral in (3.6) converges if $\Phi(z)$
 705 decays at infinity fast enough:

$$706 \quad \left| \int_{\Sigma_0} \Phi(z) \mu^{\Sigma_0}(dz_1 dz_2) \right| \leq C_r \|\Phi\|_{0,r} \quad \text{if } r > 2d-1, \tag{3.8}$$

707 see Proposition 3.5 in [7]. So, it converges under the theorem's assumptions.

708 From Theorem 3.1 another result can be deduced, whose proof is given
 709 in the next Sect. 3.1:

710 **Theorem 3.2.** *For $N = 2, 3, 4$ there exist constants $C_{d,N}$ such that*

$$711 \quad |S_{L,N}(\Phi)| \leq C_{d,N} \|\Phi\|_{0,\bar{N}}, \tag{3.9}$$

712 for $\bar{N} := \lfloor N/2 \rfloor N_2(d) + (N-2)(d-1) + 1$, where N_2 is defined in Theorem 3.1.

713 Since in view of estimate (2.22) the functions $\Phi_s^{\mathfrak{F}}$ from Corollary 2.3
 714 satisfy

$$715 \quad \|\Phi_s^{\mathfrak{F}}(\tau_1, \tau_2, \cdot)\|_{n_1, n_2} \leq C_{n_1, n_2}^{\#}(s), \quad \forall n_1, n_2, \tag{3.10}$$

716 then the two theorems above apply to study correlations (2.18) with $N =$
 717 $m + n \leq 4$. In fact, in the case $N = 2$ the number of Feynmann diagrams is
 718 small and the corresponding correlations may be calculated directly without
 719 the machinery, developed in Sect. 2. In Example 3.4 which illustrates this com-
 720 putation, as well as in a number of situations below, we apply Theorem 3.1 in
 721 the following setting:

722 **Corollary 3.3.** *Let*

$$723 \quad \mathcal{S}_{L,2} = L^{2(1-d)} \sum_{1,2,3} \delta_{3s}^{12} \delta(\omega_{3s}^{12}) f_s(s_1, s_2, s_3; q),$$

724 where ω_{3s}^{12} is given by (1.17), $q \in \mathbb{R}^n$ is a parameter (in applications usually
 725 this will be the time) and $f_s(s_1, s_2, s_1 + s_2 - s; q)^{12}$ is a Schwartz function of
 726 (s_1, s_2, s) satisfying $|\partial_{(s_1, s_2, s)}^{\mu} f_s| \leq C_{\mu}^{\#}(s_1) C_{\mu}^{\#}(s_2) C_{\mu}^{\#}(s)$ uniformly in q , for
 727 any multi-index μ . Then

$$728 \quad \left| \mathcal{S}_{L,2} - C_d \int_{\Sigma_s} f_s(s_1, s_2, s_1 + s_2 - s; q) \mu^{\Sigma_s}(ds_1 ds_2) \right| \leq \frac{C^{\#}(s)}{L^{d-5/2}}, \tag{3.11}$$

729 uniformly in q , where Σ_s and μ^{Σ_s} are the quadric (1.33) and the measure
 730 (1.34) on it.

¹²The formula for s_3 comes from the relation $\delta_{3s}^{12} = 1$.

731 *Proof.* In the variables $z_1 = s_1 - s$, $z_2 = s_2 - s$ the quadratic form ω_{3s}^{12} with
 732 $s_3 = s_1 + s_2 - s$ reads $\omega_{3s}^{12} = -2z_1 \cdot z_2$ [see (1.17)]. Then, taking into account that
 733 the relation $\delta_{3s}^{12} = 1$ is equivalent to the relations $z_1, z_2 \neq 0$ and $s_3 = s_1 + s_2 - s$,
 734 we find that the sum $\mathcal{S}_{L,2}$ takes the form (3.4). Applying next Theorem 3.1
 735 and changing in (3.6) back to the variables s_1, s_2 , we get (3.11). \square

736 *Example 3.4.* Let us calculate the asymptotic as $L \rightarrow \infty$ of $\mathbb{E}|a_s^{(1)}(\tau)|^2$. Ex-
 737 panding $a_s^{(1)}$ as in (2.4) and then using (2.13), we get:

$$738 \quad \mathbb{E}|a_s^{(1)}(\tau)|^2 = 2L^{-2d} \sum_{1,2,3} \delta_{3s}^{12} \delta(\omega_{3s}^{12}) \int_0^\tau dl_1 \int_0^\tau dl_2 B_{123} \\ \times \prod_{j=1}^3 \left(e^{-\gamma_j |l_1 - l_2|} - e^{-\gamma_j (l_1 + l_2)} \right) e^{\gamma_s (l_1 + l_2 - 2\tau)}$$

739 with $B_{123} = B_1 B_2 B_3$, where B_s is defined in (2.14). In the case of $\tau = \infty$ the
 740 formula simplifies since by changing the integration variables as $r_j := \tau - l_j$
 741 and passing to the limit we get

$$742 \quad \mathbb{E}|a_s^{(1)}(\infty)|^2 = 2L^{-2d} \sum_{1,2,3} \delta_{3s}^{12} \delta(\omega_{3s}^{12}) \int_0^\infty dr_1 \int_0^\infty dr_2 \\ B_{123} e^{-(\gamma_1 + \gamma_2 + \gamma_3) |r_1 - r_2|} e^{-\gamma_s (r_1 + r_2)} \\ = \frac{2L^{-2d}}{\gamma_s} \sum_{1,2,3} \delta_{3s}^{12} \delta(\omega_{3s}^{12}) \frac{B_{123}}{\gamma_1 + \gamma_2 + \gamma_3 + \gamma_s}.$$

743 Then, by Corollary 3.3,

$$744 \quad \left| L^2 \mathbb{E}|a_s^{(1)}(\infty)|^2 - \frac{2C_d}{\gamma_s} \int_{\Sigma_s} \frac{B_{123} \mu^{\Sigma_s}(ds_1 ds_2)}{\gamma_1 + \gamma_2 + \gamma_3 + \gamma_s} \right| \leq \frac{C^\#(s)}{L^{d-5/2}}, \quad s_3 := s_1 + s_2 - s.$$

745 3.1. Proof of Theorem 3.2

746 Let us define the geometric quadrics $Q_j := \{z \in (\mathbb{R}^d)^N : \omega_j(z) = 0\}$ and
 747 consider their intersection $Q = \cap_{j=1}^N Q_j$. Note that $Q = \cap_{j=1}^{N-1} Q_j$ since the
 748 skew symmetry of the matrix α implies $\omega_1 + \dots + \omega_N = 0$. Denote by B_R^{Nd} the
 749 open cube $|z|_\infty < R$ in \mathbb{R}^{Nd} , where by $|\cdot|_\infty$ we denote the l_∞ -norm.

750 **Proposition 3.5.** *If $w : \mathbb{R}^{Nd} \mapsto \mathbb{R}$ is such that $|w|_{L^\infty} < \infty$ and $\text{supp}(w) \subset B_R^{Nd}$,*
 751 *where $R \geq 1$, then for $N = 2, 3, 4$ we have*

$$752 \quad \left| \sum_{z \in Q \cap \mathcal{Z}} w(z) \right| \leq C(N, d) R^{[N/2] N_2(d) + (N-2)(d-1)} L^{N(d-1)} |w|_{L^\infty}. \quad (3.12)$$

753 Here N_2 is defined in Theorem 3.1 and \mathcal{Z} —in (3.1).

754 *Proof.* Below in this proof for any subset $Q \subset \mathbb{R}^{md}$, we denote

$$755 \quad Q_L = Q \cap \mathbb{Z}_L^{md}. \quad (3.13)$$

756 By suitably rearranging indices i and possibly multiplying ω_i by -1 , ω_1 may
 757 be assumed to be of the form $\omega_1(z) = z_1 \cdot \sum_i \alpha_{1i} z_i$ with $\alpha_{1N} = 1$. Define
 758 $v = \sum_i \alpha_{1i} z_i$ so that

$$759 \quad \omega_1(z) = z_1 \cdot v \quad \text{and} \quad z_N = \alpha_{1N} z_N = v - \sum_{1 < i < N} \alpha_{1i} z_i, \quad (3.14)$$

760 since $\alpha_{11} = 0$ by the skew symmetry of the matrix α .

761 For $N > 2$, fix $(z_1, v) \in \mathbb{R}^{2d}$. Then the remaining quadratic forms ω_j with
 762 $1 < j < N$ as functions of $(z_2, \dots, z_{N-1}) \in \mathbb{R}^{(N-2)d}$ become polynomials q_j of
 763 degree at most two, with no constant term. Namely

$$764 \quad q_j(z_2, \dots, z_{N-1}; z_1, v) = z_j \cdot \left(\alpha_{j1} z_1 + \alpha_{jN} v + \sum_{1 < i < N} (\alpha_{ji} - \alpha_{jN} \alpha_{1i}) z_i \right). \quad (3.15)$$

765 For $1 < j < N$ consider the sets

$$766 \quad \tilde{Q}_j(z_1, v) = \{(z_2, \dots, z_{N-1}) : q_j(z_2, \dots, z_{N-1}; z_1, v) = 0\} \subset \mathbb{R}^{(N-2)d},$$

767 and their intersection $\tilde{Q}(z_1, v) = \cap_{1 < j < N} \tilde{Q}_j(z_1, v)$. We denote $Q_1^0 = \{(z_1, v) \in$
 768 $\mathbb{R}^{2d} : z_1 \cdot v = 0\}$ (cf. (3.14)) and set

$$769 \quad A_2 = \{(z_1, v) \in \mathbb{R}^{2d} : z_1 \neq 0, v \neq 0\}. \quad (3.16)$$

770 Since $|\alpha_{ij}| \leq 1$, then on the support of w we have $|(z_1, v)|_\infty \leq (N - 1)R$. So,
 771 recalling (3.13), for $N > 2$ we get

$$772 \quad \left| \sum_{z \in \mathcal{Z} \cap Q} w(z) \right| \leq C(N, d) |w|_{L^\infty} \sum_{(z_1, v) \in Q_{1L}^0 \cap B_{(N-1)R}^{2d}} 1$$

$$773 \quad \times \sup_{(z_1, v) \in Q_{1L}^0 \cap A_2 \cap B_{(N-1)R}^{2d}} \sum_{(z_2, \dots, z_{N-1}) \in \tilde{Q}_L(z_1, v) \cap B_R^{(N-2)d}} 1. \quad (3.17)$$

774 For $N = 2$ the same estimate holds with the second line replaced by 1.

775 To estimate the sum in the first line, we take any smooth function $w_0(x) \geq$
 776 0 , equal one for $x \leq 1$ and vanishing for $x \geq 2$. Then

$$777 \quad \sum_{(z_1, v) \in Q_{1L}^0 \cap B_{(N-1)R}^{2d}} 1 \leq \sum_{(z_1, v) \in Q_{1L}^0} w_R(z_1, v),$$

778 where $w_R(z_1, v) := w_0\left(\frac{(|(z_1, v)| / (N - 1)R\sqrt{2d})}{2}\right)$. Since for $R \geq 1$ and any $a \in$
 779 $\mathbb{N} \cup \{0\}$, $b \geq 0$ we have $\|w_R\|_{a,b} \leq C(a, b, N, d)R^b$, then in view of Theorem 3.1
 780 and (3.8),

$$781 \quad \sum_{(z_1, v) \in Q_{1L}^0 \cap B_{(N-1)R}^{2d}} 1 \leq CL^{2(d-1)} [R^{2d} + R^{N_2} L^{-d+5/2}] \leq C' L^{2(d-1)} R^{N_2}, \quad (3.18)$$

782 where C, C' depend on d, N, N_1 and N_2 .

785 To estimate the second line of (3.17), we use the following lemma, proved
 786 in Appendix A.

787 **Lemma 3.6.** *Assume that the matrix α is irreducible. Then for $N = 2, 3, 4$,*
 788 *any $R \geq 1$ and any $(z_1, v) \in B_{(N-1)R}^{2d}$ satisfying $(z_1, v) \in Q_{1L}^0 \cap A_2$ we have:*

789
$$|\tilde{Q}_L(z_1, v) \cap B_R^{(N-2)d}| \leq 2^{(N-2)d} (NRL)^{(N-2)(d-1)}. \quad (3.19)$$

790 This completes the proof of Proposition 3.5 in the case of irreducible
 791 matrix α : indeed, we get

792
$$\left| \sum_{z \in \mathcal{Z} \cap Q} w(z) \right| \leq C(N, d) |w|_{L_\infty} R^{N_2 + (N-2)(d-1)} L^{N(d-1)}. \quad (3.20)$$

793 If the matrix α is reducible, it can be reduced through permutations to a block
 794 diagonal matrix with m blocks which are irreducible square matrices of sizes
 795 N_i satisfying $\sum_i N_i = N$. Since $N_i \geq 2$ (otherwise there would be a zero row
 796 or column in α), $m \leq \lfloor N/2 \rfloor$. Applying estimate (3.20) to each block, we get
 797 the assertion of the proposition. \square

798 Now we derive the theorem from the proposition. Let $\varphi_0(t) = \chi_{(-\infty, 1]}(t)$
 799 and for $k \geq 1$, $\varphi_k(t) = \chi_{(2^{k-1}, 2^k]}(t)$. Then $1 = \sum_k \varphi_k(t)$ and

800
$$\Phi = \sum_{k=0}^{\infty} f_k(z), \quad f_k(z) = \varphi_k(|z|_\infty) \Phi(z).$$

801 Then $\text{supp } f_k \subset B_k = \{|z|_\infty \leq 2^k\}$ and $\|f_k\|_\infty \leq C 2^{-k\bar{N}} \|\Phi\|_{0, \bar{N}}$, for any \bar{N} .
 802 Therefore, by Proposition 3.5,

803
$$|S_{L, N}(\Phi)| \leq C(N, d) \|\Phi\|_{0, \bar{N}} \sum_{k=0}^{\infty} 2^{k(\lfloor N/2 \rfloor N_2 + (N-2)(d-1) - \bar{N})},$$

804 which converges if $\bar{N} > \lfloor N/2 \rfloor N_2 + (N-2)(d-1)$. This completes the proof
 805 of Theorem 3.2. \square

806 *Remark 3.7.* For any fixed vector (z_1, v) , $\tilde{Q}(z_1, v)$ is a real algebraic set in
 807 $\mathbb{R}^{(N-2)d}$ of codimension $(N-2)$. If $\tilde{Q}(z_1, v)$ were a smooth manifold of that
 808 codimension, then estimate (3.19), modified by a multiplicative constant
 809 $C_{\tilde{Q}(z_1, v)}$, would be obvious. But $\tilde{Q}(z_1, v)$ is a stratified analytic manifold (with
 810 singularities), and to obtain for it a modified version of the estimate (3.19) as
 811 above, using analytical tools, seems to be a heavy job since we need a good
 812 control for the factor $C_{\tilde{Q}(z_1, v)}$. Instead in Appendix A we prove the lemma,
 813 using arithmetical tools.

814 **3.2. On Extension of Theorem 3.2 to any N**

815 The restriction on N in the statement of Theorem 3.2 comes from estimate
 816 (3.19) in Lemma 3.6, proved only for $N = 3, 4$. We know that for every N
 817 the system of polynomials $q_j(\cdot; z_1, v), 1 < j < N$, defining the set $\tilde{Q}_L(z_1, v)$
 818 in Lemma 3.6, is linearly independent for any (z_1, v) and any irreducible inci-
 819 dence matrix α . Also we know that all polynomials $q_j(\cdot; z_1, v)$ are irreducible;

820 see Lemmas A.9 and A.10 in Appendix A (there the independence and reducibility are understood over some specific algebraically closed field K , but the argument also works for K replaced by \mathbb{C}). These two facts certainly are 822 insufficient to prove Lemma 3.6 for any N , but they naturally lead to

824 **Conjecture 3.8.** *Under assumptions of Lemma 3.6, for any $N \geq 2$*

825
$$|\tilde{Q}_L(z_1, v) \cap B_R^{(N-2)d}| \leq C(N, d)(RL)^{(N-2)(d-1)}.$$

826 One may try to prove this assertion using either arithmetical or analytical 827 tools; cf. Appendix A and Remark 3.7. It is straightforward to see that, if the 828 conjecture is true, then Theorem 3.2 holds for any N , so in view of Lemma 2.2 829 any expected value $L^N \mathbb{E} a_s^{(m)}(\tau_1) \bar{a}_s^{(n)}(\tau_2)$ admits a uniform in L upper bound.

830 **4. Quasisolutions**

831 In this section we start to study a quasisolution $A(\tau) = A(\tau; L)$ of Eq. (2.1) 832 with $a_s(0) = 0$, which is the second-order truncations of series (2.2):

833
$$A(\tau) = (A_s(\tau), s \in \mathbb{Z}_L^d), \quad A_s(\tau) = a_s^{(0)}(\tau) + \rho a_s^{(1)}(\tau) + \rho^2 a_s^{(2)}(\tau). \tag{4.1}$$

835 We focus on its energy spectrum

836
$$n_{s,L} = n_{s,L}(\tau) = \mathbb{E}|A_s(\tau)|^2, \quad s \in \mathbb{Z}_L^d, \tag{4.2}$$

837 when L is large and the parameter ρ is chosen to be $\rho = \varepsilon L$. Our goal is 838 to show that it approximately satisfies a wave kinetic equation (WKE). Using 839 Proposition 2.1, we will then show that the same applies to the quantities $n_{s,L}$, 840 considered in the Introduction.

841 The energy spectrum $n_{s,L}$ is a polynomial in ε of degree four,

842
$$n_{s,L} = n_{s,L}^{(0)} + \varepsilon n_{s,L}^{(1)} + \varepsilon^2 n_{s,L}^{(2)} + \varepsilon^3 n_{s,L}^{(3)} + \varepsilon^4 n_{s,L}^{(4)}, \quad s \in \mathbb{Z}_L^d, \tag{4.3}$$

843 where the terms $n_{s,L}^{(k)}(\tau)$ are defined by

844
$$n_{s,L}^{(k)}(\tau) = L^k \sum_{\substack{k_1+k_2=k \\ 0 \leq k_1, k_2 \leq 2}} \mathbb{E} a_s^{(k_1)}(\tau) \bar{a}_s^{(k_2)}(\tau). \tag{4.4}$$

845 By Corollary 2.3,

846
$$\text{the second moments } \mathbb{E} a_s^{(k_1)} \bar{a}_s^{(k_2)} \tag{4.5}$$

naturally extend to a Schwartz function of $s \in \mathbb{R}^d$,

847 given by (2.18), (2.21). Accordingly, from now on we always regard the second 848 moments and the terms $n_{s,L}^{(k)}(\tau)$ as Schwartz functions of $s \in \mathbb{R}^d$.

849 As customary in WT, we aim at considering the limit of $n_{s,L}(\tau)$ as $L \rightarrow$ 850 ∞ , that is, the limits of the terms $n_{s,L}^{(j)}$. The term $n_{s,L}^{(0)} = n_{s,L}^{(0)}$ is given by (1.31) 851 and is L -independent, while by a direct computation we see that

852
$$n_{s,L}^{(1)} = 2\Re \mathbb{E} \bar{a}_s^{(0)} a_s^{(1)} = 0, \quad s \in \mathbb{R}^d, \tag{4.6}$$

853 [here we use (2.4), the Wick theorem and (1.15)]. Writing explicitly $n_{s,L}^{(i)}$ with
 854 $2 \leq i \leq 4$, we find that

$$855 \quad \begin{aligned} n_{s,L}^{(2)} &= L^2 \mathbb{E}(|a_s^{(1)}|^2 + 2\Re \bar{a}_s^{(0)} a_s^{(2)}), \\ n_{s,L}^{(3)} &= 2L^3 \Re \mathbb{E} \bar{a}_s^{(1)} a_s^{(2)}, \quad n_{s,L}^{(4)} = L^4 \mathbb{E}|a_s^{(2)}|^2. \end{aligned} \quad (4.7)$$

856 The function $\mathbb{R}^d \ni s \mapsto n_{s,L}^{(2)}(\tau)$ is made by two terms. By Corollary 2.3 with
 857 $N = 2$, Theorem 3.1 applies to the both of them. Since $d \geq 3$, we get

$$858 \quad |n_{s,L}^{(2)}(\tau) - n_s^{(2)}(\tau)| \leq C^\#(s)/L^{1/2}, \quad (4.8)$$

859 where

$$860 \quad n_s^{(2)}(\tau) := C_d \left(\sum_{\mathfrak{F} \in \mathfrak{F}_{1,1}^{true}} + 2\Re \sum_{\mathfrak{F} \in \mathfrak{F}_{2,0}^{true}} \right) c_{\mathfrak{F}} \int_{\Sigma_0} \mu^{\Sigma_0}(dz_1 dz_2) \Phi_s^{\mathfrak{F}}(\tau, \tau, z),$$

861 and we have used estimate (3.10). Thus, we see that the processes $n_{s,L}^{(0)}$, $n_{s,L}^{(1)}$
 862 and $n_{s,L}^{(2)}$ admit the limits

$$863 \quad n_s^{(j)}(\tau) := \lim_{L \rightarrow \infty} n_{s,L}^{(j)}(\tau; L).$$

864 The limits satisfy (4.8), and for all τ

$$865 \quad n_s^{(0)}(\tau) = B(s)(1 - e^{-2\gamma_s(\tau_0 + \tau)}), \quad n_s^{(1)}(\tau) = 0, \quad |n_s^{(2)}(\tau)| \leq C^\#(s), \quad (4.9)$$

867 where the last inequality follows from Theorem 3.2.

868 We do not know if the terms $n_{s,L}^{(3)}$, $n_{s,L}^{(4)}$ admit limits as $L \rightarrow \infty$, but in
 869 view of Corollary 2.3 both of them may be estimated through Theorem 3.2:

$$870 \quad |n_{s,L}^{(3)}(\tau)| \leq C^\#(s), \quad |n_{s,L}^{(4)}(\tau)| \leq C^\#(s), \quad (4.10)$$

871 uniformly in $L \geq 2$ and $\tau \geq 0$. We then decompose

$$872 \quad n_{s,L} = n_{s,L}^{\leq 2} + n_{s,L}^{\geq 3},$$

873 where

$$874 \quad n_{s,L}^{\leq 2} = n_{s,L}^{(0)} + \varepsilon n_{s,L}^{(1)} + \varepsilon^2 n_{s,L}^{(2)} \quad \text{and} \quad n_{s,L}^{\geq 3} = \varepsilon^3 n_{s,L}^{(3)} + \varepsilon^4 n_{s,L}^{(4)}$$

875 (we recall that $n_{s,L}^{(1)} \equiv 0$), and similarly define

$$876 \quad n_s^{\leq 2} := n_s^{(0)} + \varepsilon^2 n_s^{(2)}.$$

877 Due to (4.8),

$$878 \quad |n_{s,L}^{\leq 2}(\tau) - n_s^{\leq 2}(\tau)| \leq C^\#(s)\varepsilon^2 L^{-1/2}, \quad (4.11)$$

879 so by (4.10),

$$880 \quad |n_{s,L}^{\leq 2}(\tau) - n_{s,L}(\tau)| \leq C^\#(s)\varepsilon^2(L^{-1/2} + \varepsilon). \quad (4.12)$$

881 Thus, the cut energy spectrum $n_s^{\leq 2}$ governs the limiting as $L \rightarrow \infty$ behaviour
 882 of the energy spectrum $n_{s,L}$ with precision $\varepsilon^3 C^\#(s)$, where we regard the
 883 constant $\varepsilon \leq 1/2$ (which measures the size of solutions for (1.6) under the

884 proper scaling) as a fixed small parameter. Accordingly, our next goal is to
 885 show that $n_s^{\leq 2}(\tau)$ approximates the solution of a WKE.

886 **4.1. Increments of the Energy Spectra $n_s^{\leq 2}$**

887 In this section we will show that the process $n_s^{\leq 2}(\tau)$ approximately satisfies a
 888 WKE. We denote $s_4 := s$, $\gamma_j := \gamma_{s_j}$ and set

889
$$\gamma_{1234} = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4, \quad \vec{s} = (s_1, s_2, s_3, s_4) \in (\mathbb{R}^d)^4. \quad (4.13)$$

890 Now, for a fixed $\tau_0 \geq 0$ and for $j = 1, 2, 3, 4$ we define the functions
 891 $\mathcal{Z}^j(\tau_0) = \mathcal{Z}^j(\tau_0; \vec{s})$ as

892
$$\mathcal{Z}^j(\tau_0; \vec{s}) := \int_0^{\tau_0} dl e^{-\gamma_j(\tau_0-l)} \prod_{\substack{m=1,2,3,4 \\ m \neq j}} \frac{\sinh(\gamma_m l)}{\sinh(\gamma_m \tau_0)} \quad \text{if } \tau_0 > 0, \quad (4.14)$$

893 and $\mathcal{Z}^j(0; \vec{s}) = 0$. Computing this integral, we get

894
$$\begin{aligned} \mathcal{Z}^j(\tau_0; \vec{s}) = & \left(\prod_{l \neq j} \frac{1}{1 - e^{-2\gamma_l \tau_0}} \right) \cdot \left[\frac{1 - e^{-\gamma_{1234} \tau_0}}{\gamma_{1234}} - \frac{e^{-2(\gamma_{1234} - \gamma_j) \tau_0} - e^{-\gamma_{1234} \tau_0}}{2\gamma_j - \gamma_{1234}} \right. \\ & \left. + \sum_{l \neq j} \left(\frac{e^{-2(\gamma_{1234} - \gamma_j - \gamma_l) \tau_0} - e^{-\gamma_{1234} \tau_0}}{2(\gamma_j + \gamma_l) - \gamma_{1234}} - \frac{e^{-2\gamma_l \tau_0} - e^{-\gamma_{1234} \tau_0}}{\gamma_{1234} - 2\gamma_l} \right) \right], \end{aligned} \quad (4.15)$$

896 where each fraction from the square brackets should be substituted by $\tau_0 e^{-\gamma_{1234} \tau_0}$
 897 if its denominator vanishes.

898 For any real number r let $\mathcal{C}_r(\mathbb{R}^d)$ be the space of continuous complex
 899 functions on \mathbb{R}^d with the finite norm

900
$$\|f\|_r = \|f(z)\langle z \rangle^r\|_{L_\infty}. \quad (4.16)$$

901 We naturally extend this norm to $f \in L_\infty(\mathbb{R}^d)$ and set

902
$$L_{\infty,r}(\mathbb{R}^d) = \{f \in L_\infty(\mathbb{R}^d) : \|f\|_r < \infty\}. \quad (4.17)$$

903 Consider also the linear operator \mathcal{L} , given by

904
$$(\mathcal{L}v)(s) = 2\gamma_s v(s), \quad s \in \mathbb{R}^d. \quad (4.18)$$

905 Below we often write the value $v(s)$ of a function v at $s \in \mathbb{R}^d$ as v_s and the
 906 function v itself as $(v_s, s \in \mathbb{R}^d)$. Now, for $v \in \mathcal{C}_r(\mathbb{R}^d)$, where $r > d$, and for
 907 $\tau_0 \geq 0$, $\tau \in (0, 1]$, we define the kinetic integral $K^\tau(\tau_0)(v) = (K_s^\tau(\tau_0))(v)$, $s \in$
 908 \mathbb{R}^d :

909
$$K^\tau(\tau_0)(v) = \int_0^\tau e^{-t\mathcal{L}} K(\tau_0)(v) dt. \quad (4.19)$$

910 Here the operator $K(\tau_0) = K^1(\tau_0) + \dots + K^4(\tau_0)$ sends a function $v = (v_s, s \in$
 911 $\mathbb{R}^d)$ to the function

$$\begin{aligned}
 K_s(\tau_0)(v) &= 4C_d \int_{\Sigma_s} \mu^{\Sigma_s}(ds_1 ds_2) \left(\mathcal{Z}^4(\tau_0; \vec{s}) v_1 v_2 v_3 \right. \\
 &\quad \left. + \mathcal{Z}^3(\tau_0; \vec{s}) v_1 v_2 v_4 - \mathcal{Z}^2(\tau_0; \vec{s}) v_1 v_3 v_4 - \mathcal{Z}^1(\tau_0; \vec{s}) v_2 v_3 v_4 \right) \\
 &=: K_s^4(\tau_0)(v) + K_s^3(\tau_0)(v) + K_s^2(\tau_0)(v) + K_s^1(\tau_0)(v)
 \end{aligned} \tag{4.20}$$

913
 914 (note the reversed signs for K^2 and K^3). Here $v_j := v(s_j)$, where $s_4 = s$ and
 915 $s_3 := s_1 + s_2 - s_4$ (in view of the factor δ_{3s}^{12}). While μ^{Σ_s} is the measure (1.34)
 916 on the quadric $\Sigma_s = \{(s_1, s_2) \in \mathbb{R}^{2d} : (s_1 - s) \cdot (s_2 - s) = 0\}$. Computing the
 917 integral in t in (4.19), we find

$$K_s^\tau(\tau_0)(v) = \frac{1 - e^{-2\gamma_s \tau}}{2\gamma_s} K_s(\tau_0)(v) = \frac{1 - e^{-2\gamma_s \tau}}{2\gamma_s} \sum_{j=1}^4 K_s^j(\tau_0)(v). \tag{4.21}$$

919 We study the kinetic integral K^τ in Sect. 5 while now we formulate a result
 920 which is the main step in deriving the wave kinetic limit.

921 **Theorem 4.1.** *For any $0 < \tau \leq 1$ the function $(n_s^{\leq 2}, s \in \mathbb{R}^d)$ satisfies*

$$\begin{aligned}
 n^{\leq 2}(\tau_0 + \tau) &= e^{-\tau \mathcal{L}} n^{\leq 2}(\tau_0) + 2 \int_0^\tau e^{-t\mathcal{L}} b^2 dt \\
 &\quad + \varepsilon^2 K^\tau(\tau_0)(n^{\leq 2}(\tau_0)) + \varepsilon^2 \mathcal{R},
 \end{aligned} \tag{4.22}$$

923 where $b^2 = (b^2(s), s \in \mathbb{R}^d)$ and the remainder $\mathcal{R}_s(\tau)$ satisfies

$$|\mathcal{R}(\tau)|_r \leq C_r \tau (\tau + \varepsilon^2), \quad \forall r. \tag{4.23}$$

925 **4.2. Proof of Theorem 4.1**

926 We first fix a value for L and decompose the processes $\tau \mapsto a_s^{(i)}(\tau_0 + \tau)$, where
 927 $\tau_0 \geq 0$ and $0 \leq \tau \leq 1$, as

$$a_s^{(i)}(\tau_0 + \tau) = c_s^{(i)}(\tau; \tau_0) + \Delta a_s^{(i)}(\tau; \tau_0), \quad i = 0, 1, 2, \quad s \in \mathbb{Z}_L^d. \tag{4.24}$$

929 Here

$$c_s^{(i)}(\tau; \tau_0) = e^{-\gamma_s \tau} a_s^{(i)}(\tau_0)$$

931 and $\Delta a_s^{(i)}$ is defined via relation (4.24). Below we write $c_s^{(i)}(\tau; \tau_0)$ and $\Delta a_s^{(i)}(\tau; \tau_0)$
 932 as $c_s^{(i)}(\tau)$ and $\Delta a_s^{(i)}(\tau)$ since τ_0 is fixed.

933 Obviously,

$$c(\tau) := c^{(0)}(\tau) + \rho c^{(1)}(\tau) + \rho^2 c^{(2)}(\tau)$$

935 with $\tau \geq 0$ being a solution of the linear equation (2.1) $_{\rho=0, b(s)=0}$, equal $A(\tau_0)$
 936 at $\tau = 0$, and $\Delta a(\tau) = \sum_{j=0}^2 \rho^j \Delta a^{(j)}(\tau)$ equals $A(\tau_0 + \tau) - c(\tau)$. By (4.5), for
 937 $0 \leq i, j \leq 2$

$$\begin{aligned}
 &\text{the functions } \mathbb{E} c_s^{(i)} \bar{c}_s^{(j)}, \quad \mathbb{E} c_s^{(i)} \Delta \bar{a}_s^{(j)}, \quad \mathbb{E} \Delta a_s^{(i)} \Delta \bar{a}_s^{(j)} \\
 &\text{naturally extend to Schwartz functions of } s \in \mathbb{R}^d.
 \end{aligned} \tag{4.25}$$

939 Due to (4.6) and (4.7),

940
$$e^{-2\gamma_s\tau} n_{s,L}^{\leq 2}(\tau_0) = \mathbb{E}|c_s^{(0)}(\tau)|^2 + \rho^2 \mathbb{E}(|c_s^{(1)}(\tau)|^2 + 2\Re \bar{c}_s^{(0)}(\tau)c_s^{(2)}(\tau)), \quad \forall s \in \mathbb{R}^d.$$

941 Then,

942
$$\begin{aligned} n_{s,L}^{\leq 2}(\tau_0 + \tau) - e^{-2\gamma_s\tau} n_{s,L}^{\leq 2}(\tau_0) &= \mathbb{E} \left(|a_s^{(0)}(\tau_0 + \tau)|^2 - |c_s^{(0)}(\tau)|^2 \right. \\ &+ \rho^2 (|a_s^{(1)}(\tau_0 + \tau)|^2 - |c_s^{(1)}(\tau)|^2 \\ &\left. + 2\Re(a_s^{(2)}\bar{a}_s^{(0)}(\tau_0 + \tau) - c_s^{(2)}\bar{c}_s^{(0)}(\tau)) \right). \end{aligned}$$

943 (4.26)

944 Let us set

945
$$\mathcal{Y}_s(u, v, w) := L^{-d} \sum_{1,2,3} \delta_{3s}^{12} \delta(\omega_{3s}^{12}) u_1 v_2 \bar{w}_3.$$

946 Writing explicitly the processes $\Delta a_s^{(i)}(\tau)$, $s \in \mathbb{Z}_L^d$, we find

947
$$\begin{aligned} \Delta a_s^{(0)}(\tau) &= b(s) \int_{\tau_0}^{\tau_0+\tau} e^{-\gamma_s(\tau_0+\tau-l)} d\beta_s(l), \\ \Delta a_s^{(1)}(\tau) &= i \int_{\tau_0}^{\tau_0+\tau} e^{-\gamma_s(\tau_0+\tau-l)} \mathcal{Y}_s(a^{(0)}) dl, \\ \Delta a_s^{(2)}(\tau) &= i \int_{\tau_0}^{\tau_0+\tau} e^{-\gamma_s(\tau_0+\tau-l)} \left(\mathcal{Y}_s(a^{(0)}, a^{(0)}, a^{(1)}) \right. \\ &\quad \left. + \mathcal{Y}_s(a^{(0)}, a^{(1)}, a^{(0)}) + \mathcal{Y}_s(a^{(1)}, a^{(0)}, a^{(0)}) \right) dl, \end{aligned}$$

948 (4.27)

949 where $a^{(i)} = a^{(i)}(l)$. Note that to get explicit formulas for $c_s^{(i)}(\tau)$, $i = 0, 1, 2$,
 950 it suffices to replace in the r.h.s.'s of the relations in (4.27) the range of
 951 integration from $[\tau_0, \tau_0 + \tau]$ to $[0, \tau_0]$. For example, $c_s^{(0)}(\tau) = e^{-\gamma_s\tau} a_s^{(0)}(\tau_0) =$
 952 $b(s) \int_0^{\tau_0} e^{-\gamma_s(\tau_0+\tau-l)} d\beta_s(l)$.

953 Using that $\mathbb{E}c_s^{(i)}(\tau)\Delta\bar{a}_s^{(0)}(\tau) = \mathbb{E}c_s^{(i)}(\tau)\mathbb{E}\Delta\bar{a}_s^{(0)}(\tau) = 0$ for any i and s , we
 954 obtain

955
$$\mathbb{E}(a_s^{(2)}\bar{a}_s^{(0)}(\tau_0 + \tau) - c_s^{(2)}\bar{c}_s^{(0)}(\tau)) = \mathbb{E}\Delta a_s^{(2)}(\tau)\bar{a}_s^{(0)}(\tau_0 + \tau), \quad (4.28)$$

956 and from (4.24) we get that

957
$$\begin{aligned} |a_s^{(1)}(\tau_0 + \tau)|^2 - |c_s^{(1)}(\tau)|^2 &= |\Delta a_s^{(1)}(\tau)|^2 + 2\Re \Delta a_s^{(1)}(\tau)\bar{c}_s^{(1)}(\tau), \\ \mathbb{E}(|a_s^{(0)}(\tau_0 + \tau)|^2 - |c_s^{(0)}(\tau)|^2) &= \mathbb{E}|\Delta a_s^{(0)}(\tau)|^2. \end{aligned}$$

958 (4.29)

958 Then, inserting (4.28) and (4.29) into (4.26) and using that $\rho = \varepsilon L$, we find

959
$$n_{s,L}^{\leq 2}(\tau_0 + \tau) - e^{-2\gamma_s\tau} n_{s,L}^{\leq 2}(\tau_0) = \mathbb{E}|\Delta a_s^{(0)}(\tau)|^2 + \varepsilon^2 Q_{s,L}(\tau_0, \tau), \quad s \in \mathbb{R}^d,$$

960 where

$$961 \quad Q_{s,L}(\tau_0, \tau) := L^2 \left(\mathbb{E} |\Delta a_s^{(1)}(\tau)|^2 + 2\Re \mathbb{E} [\Delta a_s^{(1)}(\tau) \bar{c}_s^{(1)}(\tau) \right. \\ \left. + \Delta a_s^{(2)}(\tau) \bar{a}_s^{(0)}(\tau_0 + \tau) \right], \quad (4.30)$$

962 and we recall (4.25). Since

$$963 \quad \mathbb{E} |\Delta a_s^{(0)}(\tau)|^2 = \frac{b(s)^2}{\gamma_s} (1 - e^{-2\gamma_s \tau}) = 2 \int_0^\tau e^{-t\mathcal{L}} b^2(s) dt,$$

964 then

$$965 \quad n_{s,L}^{\leq 2}(\tau_0 + \tau) - e^{-\tau\mathcal{L}} n_{s,L}^{\leq 2}(\tau_0) = 2 \int_0^\tau e^{-t\mathcal{L}} b^2 dt + \varepsilon^2 Q_{s,L}(\tau_0, \tau),$$

966 for $n_{s,L}^{\leq 2} = (n_{s,L}^{\leq 2}, s \in \mathbb{R}^d)$. In order to pass to the limit $L \rightarrow \infty$ we recall the
967 relation (4.11). Then the desired formula (4.22) is an immediate consequence
968 of the assertion below:

969 **Proposition 4.2.** *We have*

$$970 \quad \lim_{L \rightarrow \infty} Q_{s,L}(\tau_0, \tau) = K_s^\tau(\tau_0)(n^{\leq 2}(\tau_0)) + \mathcal{R}_s(\tau), \quad s \in \mathbb{R}^d, \quad (4.31)$$

971 where the remainder \mathcal{R} satisfies (4.23).

972 *Proof.* The first step in the proof of (4.31) is the following result, established
973 in Appendix B:

974 **Proposition 4.3.** *One has*

$$975 \quad |Q_{s,L}(\tau_0, \tau) - \mathcal{X}_{s,L}(\tau_0, \tau)| \leq C^\#(s)\tau^2, \quad s \in \mathbb{R}^d, \quad (4.32)$$

976 where

$$977 \quad \mathcal{X}_{s,L}(\tau_0, \tau) := 4L^{2(1-d)}\tau \sum_{1,2,3} \delta_{3s}^{12} \delta(\omega_{3s}^{12}) (\mathcal{Z}^4 n_1^{(0)} n_2^{(0)} n_3^{(0)} + \mathcal{Z}^3 n_1^{(0)} n_2^{(0)} n_s^{(0)} \\ 978 \quad - \mathcal{Z}^1 n_2^{(0)} n_3^{(0)} n_s^{(0)} - \mathcal{Z}^2 n_1^{(0)} n_3^{(0)} n_s^{(0)}). \quad (4.33)$$

980 The terms $\mathcal{Z}^j = \mathcal{Z}^j(\tau_0; s_1, s_2, s_3, s)$ are defined by (4.14) and $n_i^{(0)} := n_{s_i,L}^{(0)}$,
981 $n_s^{(0)} := n_{s,L}^{(0)}$.

982 By (4.9) $n_i^{(0)} = n_{s_i}^{(0)}$ are Schwartz functions in s_i . Besides, the functions
983 $\mathcal{Z}^j(\tau_0, \vec{s})$ have at most polynomial growth in \vec{s} together with their derivatives,
984 uniformly in $\tau_0 \geq 0$:

985 **Lemma 4.4.** *For any vector $\mu \in (\mathbb{N} \cup \{0\})^{4d}$, uniformly in $\tau_0 \geq 0$, we have*
986 $|\partial_s^\mu \mathcal{Z}^j(\tau_0, \vec{s})| \leq P(\vec{s}; \mu)$, where $P(\vec{s}; \mu)$ has at most a polynomial growth in \vec{s} .

987 By the lemma, which is proven in Sect. B.7, $\mathcal{X}_{s,L}$ satisfies the hypotheses
988 of Corollary 3.3. So

$$989 \quad |\mathcal{X}_{s,L}(\tau_0, \tau) - \tau K_s(\tau_0)(n^{(0)})| \leq C^\#(s)L^{-1/2}\tau. \quad (4.34)$$

990 Next, note that $|n_s^{(0)}(\tau_0) - n_s^{\leq 2}(\tau_0)| \leq C^\#(s)\varepsilon^2$ due to (4.9). Then the estimate
 991 on the Lipschitz constants of the operators $K^j(t)$, given in (5.4), implies that

$$992 \quad |K(\tau_0)(n^{(0)}(\tau_0)) - K(\tau_0)(n^{\leq 2}(\tau_0))|_r \leq C_r \varepsilon^2 \quad \forall r.$$

993 Hence,

$$994 \quad |\tau K_s(\tau_0)(n^{(0)}(\tau_0)) - \tau K_s(\tau_0)(n^{\leq 2}(\tau_0))| \leq C^\#(s)\tau\varepsilon^2. \quad (4.35)$$

995 On the other hand, on account of the definition (4.21), for $0 \leq \tau \leq 1$ we have
 996 the bound

$$997 \quad |\tau K_s(\tau_0)(n^{\leq 2}) - K_s^\tau(\tau_0)(n^{\leq 2})| \leq C\gamma_s\tau^2|K_s(\tau_0)(n^{\leq 2})| \leq C^\#(s)\tau^2, \quad (4.36)$$

998 where the last inequality follows from the estimate of the norm of the operator
 999 $K^j(t)$, given in (5.3), and from (4.9).

1000 Putting together (4.32), (4.34), (4.35), (4.36) and letting L grow to in-
 1001 finity, we conclude the proof. \square

1002 5. Kinetic Equation

1003 At this section we examine the wave kinetic equation

$$1004 \quad \dot{\mathfrak{z}}_s(\tau) = -(\mathcal{L}\mathfrak{z})_s + \varepsilon^2 K_s(\tau)(\mathfrak{z}) + 2b(s)^2, \quad \tau \geq 0, \quad \mathfrak{z}(0) = 0 \quad (5.1)$$

1005 (\mathcal{L} is defined in (4.18) and the operator $K = K^1 + \dots + K^4$ is defined in (4.20)),
 1006 and next we derive from this analysis and (4.22) the proximity of $n_s^{\leq 2}(\tau)$ to
 1007 a solution of (5.1). We will need the following result, which is Lemma 4.2
 1008 from [7]:

1009 **Lemma 5.1.** For $j, l = 1, \dots, 4$ and $w^j \in \mathcal{C}_r(\mathbb{R}^d)$ consider the operators

$$1010 \quad J_l(u^1, \dots, u^4)(s) = \int_{\Sigma_s} \mu^{\Sigma_s}(ds_1 ds_2) \prod_{i \neq l} u^i(s_i)$$

1011 (see (1.34)), where $s_4 = s$ and $s_3 = s_1 + s_2 - s$. Then for each l ,

$$1012 \quad |J_l(u^1, \dots, u^4)|_{r+1} \leq C_r \prod_{i \neq l} |u^i|_r \quad \text{if } r > d. \quad (5.2)$$

1013 5.1. Kinetic Integrals

1014 We recall notation (4.13), (4.14).

1015 **Lemma 5.2.** For $j = 1, \dots, 4$, any $\tau \geq 0$ and any $\vec{s} = (s_1, \dots, s_4) \in (\mathbb{R}^d)^4$,

$$1016 \quad (i) \quad 0 \leq \mathcal{Z}^j(\tau; \vec{s}) \leq \min(\tau, 1/\gamma_{s_j}) \leq 1,$$

$$1017 \quad (ii) \quad |\mathcal{Z}^j(\tau; \vec{s}) - \mathcal{Z}(\infty; \vec{s})| \leq C e^{-2\tau}, \text{ where } \mathcal{Z}(\infty; \vec{s}) = 1/\gamma_{1234}.$$

1018 *Proof.* The first assertion follows from (4.14) since $\sinh(x)$ is an increasing
 1019 non-negative function of $x \geq 0$, so in the integrand in (4.14) we have $0 \leq$
 1020 $\sinh(\gamma_m l) / \sinh(\gamma_m \tau') \leq 1$. For $0 \leq \tau \leq 1$ the second estimate follows from the
 1021 first one as

$$1022 \quad |\mathcal{Z}^j(\tau; \vec{s}) - \mathcal{Z}(\infty; \vec{s})| \leq |\mathcal{Z}^j(\tau; \vec{s})| + |\mathcal{Z}(\infty; \vec{s})|,$$

1023 while for $\tau \geq 1$ it follows from (4.15) since $\gamma_{123j} - \gamma_j \geq 1$ and $\gamma_{1234} - \gamma_j - \gamma_l \geq 1$
 1024 for $j, l \in \{1, 2, 3, 4\}, j \neq l$. \square

1025 Since the kernels \mathcal{Z}^j are non-negative by the first assertion of the lemma
 1026 above, then denoting $\kappa_1 = \kappa_2 = 1, \kappa_3 = \kappa_4 = -1$ we achieve that the operators
 1027 $\kappa_j K^j, 1 \leq j \leq 4$, are positive (in the sense that they send positive functions to
 1028 positive). Due to the first assertion of the lemma and (5.2), for any $\tau \geq 0$ they
 1029 define positive 3-homogeneous mappings $\mathcal{C}_r(\mathbb{R}^d) \rightarrow \mathcal{C}_{r+1}(\mathbb{R}^d)$ if $r > d$, and

$$1030 |\kappa_j K^j(\tau)(v)|_{r+1} = |K^j(\tau)(v)|_{r+1} \leq C_r \min(\tau, 1) |v|_r^3, \quad j = 1, \dots, 4, \quad (5.3)$$

1031 for $\tau \geq 0$. So, the mappings $K^j(\tau)$ are locally Lipschitz:

$$1032 |K^j(\tau)(v^1) - K^j(\tau)(v^2)|_{r+1} \\
 1033 \leq 3C_r \min(\tau, 1) R^2 |v^1 - v^2|_r, \quad \text{if } |v^1|_r, |v^2|_r \leq R. \quad (5.4)$$

1034 Since for $j = 1, \dots, 4$ and any $s \in \mathbb{R}^d$,

- 1035 • for non-negative functions $n, m \in L_{\infty, r}$ [see (4.17)] such that $m \leq n$ we
 1036 have $\kappa_j K_s^j(\tau)(m) \leq \kappa_j K_s^j(\tau)(n) \leq \infty$,
- 1037 • $|K_s^j(\tau)(v)| \leq \kappa_j K_s^j(\tau)(|v|) \leq \infty$ for any complex function $v \in L_{\infty}$,
- 1038 • $|v_s| \leq |v|_r \langle s \rangle^{-r}$ for all $v \in L_{\infty, r}$,

1039 then the relations (5.3), (5.4) remain true for functions from $L_{\infty, r}$.

1040 **Lemma 5.3.** *If $|s_l| \leq R$ for $l = 1, \dots, 4$, then*

$$1041 \left| \frac{\partial}{\partial \tau} \mathcal{Z}^j(\tau; \vec{s}) \right| \leq C \gamma^0(R^2) \quad (5.5)$$

1042 (see (1.5)).

1043 *Proof.* For any $m \in \{s_1, \dots, s_4\}$ and $0 \leq l \leq \tau$, we have

$$1044 \left| \frac{\partial}{\partial \tau} \frac{\sinh \gamma_m l}{\sinh \gamma_m \tau} \right| \leq \gamma_m \frac{\cosh \gamma_m \tau}{\sinh \gamma_m \tau} \leq \gamma_m C \max(1, 1/(\gamma_m \tau)).$$

1045 Considering separately the cases $\tau \geq 1$ and $0 \leq \tau < 1$, using (4.14) and the
 1046 estimate above we get the result. \square

1047 This lemma implies that for any $v \in \mathcal{C}_r(\mathbb{R}^d)$ and any j the curve $\tau \mapsto$
 1048 $K^j(\tau)(v) \in \mathcal{C}_r(\mathbb{R}^d)$ is Hölder continuous:

1049 **Lemma 5.4.** *For any $\tau_0 \geq 0, 0 \leq \tau \leq 1, j = 1, \dots, 4$ and any $r > d + 1$,*

$$1050 |K^j(\tau_0 + \tau)(v) - K^j(\tau_0)(v)|_r \leq C_r |v|_r^3 \tau^{\kappa_*} \quad \forall v \in \mathcal{C}_r(\mathbb{R}^d), \quad (5.6)$$

1051 where $\kappa_* = 1/(1 + 2r_*)$.

1052 *Proof.* By the homogeneity we may assume that $|v|_r = 1$. For $R \geq 1$ let us set
 1053 $v^R(s) = v(s) \chi_{|s| \leq R} \in L_{\infty}$. Then

$$1054 |v^R|_r \leq 1, \quad |v - v^R|_{r-1} \leq R^{-1}. \quad (5.7)$$

1055 Now let us write the increment $K^j(\tau_0 + \tau)(v) - K^j(\tau_0)(v)$ as

$$1056 (K^j(\tau_0 + \tau)(v) - K^j(\tau_0 + \tau)(v^R)) + (K^j(\tau_0 + \tau)(v^R) - K^j(\tau_0)(v^R)) \\
 + (K^j(\tau_0)(v^R) - K^j(\tau_0)(v)) =: \Delta_1 + \Delta_2 + \Delta_3.$$

1057 Recalling that (5.3) and (5.4) hold for functions from $L_{\infty, r'}$ with $r' > d$, we get
 1058 from (5.7) that $|\Delta_1|_r + |\Delta_3|_r \leq C_r R^{-1}$. To estimate Δ_2 , we set $\Delta_2^R = \Delta_2 \chi_{|s| \leq R}$.

1059 Since by (5.3), $|\Delta_2|_{r+1} \leq 2C_r$, then $|\Delta_2 - \Delta_2^R|_r \leq 2C_r R^{-1}$. For $|s| > R$ the
 1060 function Δ_2^R vanishes, while for $|s| \leq R$ in view of Lemma 5.3 we have

$$\begin{aligned} |\Delta_2^R|_s &= |\Delta_{2s}| = |K_s^j(\tau_0 + \tau)(v^R) - K_s^j(\tau_0)(v^R)| \\ &\leq C_r \int_{\Sigma_s} \mu_s(dv_1 dv_2) |\mathcal{Z}^j(\tau_0 + \tau; \vec{s}) - \mathcal{Z}^j(\tau_0; \vec{s})| \frac{|v_1| \cdots |v_4|}{|v_j|} \chi_{\{|s_j| \leq R \ \forall j\}} \\ &\leq C_{1r} \gamma^0(R^2) \tau \int_{\Sigma_s} \mu_s(dv_1 dv_2) \frac{|v_1| \cdots |v_4|}{|v_j|} \leq C_{2r} \langle s \rangle^{-r-1} \tau R^{2r*}, \end{aligned}$$

1062 where to get the last inequality we used (5.2). We have seen that the C_r -norm
 1063 of the increment is bounded by $C_r(R^{-1} + \tau R^{2r*})$, for any $R \geq 1$. Choosing
 1064 $R = \tau^{-1/(1+2r*)}$, we achieve (5.6). \square

1065 **5.2. Kinetic Equation**

1066 Now we will apply the obtained results to the kinetic Eq. (5.1). Since the func-
 1067 tion $b(\cdot)^2 := \{b(s)^2\} \in C_r(\mathbb{R}^d)$ for all r , since \mathcal{L} is the operator of multiplying by
 1068 the function $2\gamma_s$ as in (1.4, 1.5), and the operator K satisfies (5.3), (5.4), then
 1069 for small enough $\varepsilon > 0$ Eq. (5.1) has a unique solution, belonging to $C_r(\mathbb{R}^d)$ for
 1070 each r , which in a Lipschitz way depends on the r.h.s. of the equation, when
 1071 the latter deviates from $b(\cdot)^2$. Namely, the following result, where X^r stands
 1072 for the space $C(0, \infty; C_r(\mathbb{R}^d))$, given the norm $\|v(\cdot)\|_{X^r} = \sup_{t \geq 0} \|v(t)\|_r$, may
 1073 be easily verified (a proof of a similar fact may be found in Section 4 of [7]).

1074 **Lemma 5.5.** *For any $r > d$,*

- 1075 (1) *There exists ε_* , depending on $b(\cdot), r$ and r_* , such that for $0 \leq \varepsilon \leq \varepsilon_*$*
 1076 *Eq. (5.1) has a unique solution $\mathfrak{z}(\tau)$, belonging to X^r . It satisfies*

$$1077 \|\mathfrak{z}\|_{X^r} \leq C_r \|b^2\|_r. \tag{5.8}$$

- 1078 (2) *If $\mathfrak{z}^0(\tau)$ is a solution of the linear Eq. (5.1)| $_{\varepsilon=0}$, then $\|\mathfrak{z} - \mathfrak{z}^0\|_{X^r} \leq C_r \varepsilon^2$.*
 1079 *If a curve $\mathfrak{z}'(\tau)$ solves (5.1) with $2b(s)^2$ replaced by $2b(s)^2 + \xi_s(t)$, where*
 1080 *$\xi \in X^r$ and $\|\xi\|_{X^r} \leq 1$, then $\|\mathfrak{z} - \mathfrak{z}'\|_{X^r} \leq C_r \|\xi\|_{X^r}$.*

1081 The lemma's assertion holds as well for nonzero initial conditions $\mathfrak{z}(0) \in$
 1082 $C_r(\mathbb{R}^d)$ in (5.1), but we do not need this.

1083 Let $K(\infty)$ be the operator, obtained from $K(\tau_0)$ by replacing in (4.20)
 1084 the kernels $\mathcal{Z}^j(\tau_0; s)$, $s \in \mathbb{R}^d$, by $\mathcal{Z}(\infty; \vec{s})$ (see Lemma 5.2). Let $r > d$ and
 1085 $\mathfrak{z}^\varepsilon \in C_r(\mathbb{R}^d)$ be a solution of the limiting stationary equation

$$1086 \mathcal{L}\mathfrak{z}^\varepsilon - \varepsilon^2 K(\infty)(\mathfrak{z}^\varepsilon) = 2b(\cdot)^2 \tag{5.9}$$

1087 in the vicinity of $\mathcal{L}^{-1}(2b^2)$, existing for small ε by the inverse function theorem.
 1088 Since $b^2(\cdot) \in \bigcap_r C_r(\mathbb{R}^d)$ and, as in (5.3), the map $K(\infty)$ is one-smoothing, then
 1089 decreasing ε_* if needed we achieve that $\mathfrak{z}^\varepsilon \in \bigcap_r C_r(\mathbb{R}^d)$ for $\varepsilon \leq \varepsilon_*$ and

$$1090 \|\mathfrak{z}^\varepsilon - 2\mathcal{L}^{-1}(b^2)\|_m \leq C_m \varepsilon^2 \quad \forall m. \tag{5.10}$$

1091 Here and below the constants depend on b and r_* .

1092 Let us consider the curve $w(t) = \mathfrak{z}(t) - \mathfrak{z}^\varepsilon$. It satisfies the equation

$$\begin{aligned} 1093 \dot{w} + \mathcal{L}(w) &= \varepsilon^2 (K(t)(\mathfrak{z}) - K(\infty)(\mathfrak{z}^\varepsilon)) \\ &= \varepsilon^2 [(K(t)(\mathfrak{z}) - K(t)(\mathfrak{z}^\varepsilon)) - (K(t)(\mathfrak{z}^\varepsilon) - K(\infty)(\mathfrak{z}^\varepsilon))] \end{aligned}$$

1094 and $w(0) = -\mathfrak{z}^\varepsilon$. Denote $K(\tau)(\mathfrak{z}^\varepsilon) - K(\infty)(\mathfrak{z}^\varepsilon) =: -\eta(\tau)$. In view of Lem-
 1095 mas 5.2 and 5.1, $|\eta(\tau)|_r \leq C_r e^{-2\tau}$ for $\tau \geq 0$. Next, regarding the differ-
 1096 ence $K(\tau)(\mathfrak{z}(\tau)) - K(\tau)(\mathfrak{z}^\varepsilon)$ as an operator, linear in $w(\tau) = \mathfrak{z}(\tau) - \mathfrak{z}^\varepsilon$ and
 1097 quadratic in $(\mathfrak{z}(\tau), \mathfrak{z}^\varepsilon)$, we write it as $\mathcal{K}(\tau)(w(\tau))$. Then by (5.4) and (5.8),
 1098 $|\mathcal{K}(\tau)w|_{r+1} \leq C_r |w|_r, \forall r > d$. Finally, we substitute

$$1099 \quad w(\tau) = v(\tau) + y(\tau), \quad v(\tau) = -e^{-\tau\mathcal{L}}\mathfrak{z}^\varepsilon,$$

1100 and rewrite the equation on w as an equation on y :

$$1101 \quad \dot{y} + \mathcal{L}y = \varepsilon^2 \mathcal{K}(\tau)(v(\tau) + y(\tau)) + \varepsilon^2 \eta(\tau), \quad y(0) = 0.$$

1102 Or

$$1103 \quad y(t) = \varepsilon^2 \int_0^t e^{-(t-s)\mathcal{L}} [\mathcal{K}(s)(v(s) + y(s)) + \eta(s)] ds. \quad (5.11)$$

1104 Let Y^r be the space of continuous curves $y : \mathbb{R}_+ \rightarrow \mathcal{C}_r(\mathbb{R}^d)$, vanishing at zero,
 1105 with finite norm $|y|_{Y^r} = \sup_{t \geq 0} e^t |y(t)|_r$.

1106 Let \mathfrak{B} be the linear operator

$$1107 \quad \mathfrak{B}(y)(t) = \int_0^t e^{-(t-s)\mathcal{L}} \mathcal{K}(s)(y(s)) ds.$$

1108 Then the equation for y may be written as

$$1109 \quad y(t) = \varepsilon^2 \left(\mathfrak{B}(y)(t) + \mathfrak{B}(v)(t) + \int_0^t e^{-(t-s)\mathcal{L}} \eta(s) ds \right). \quad (5.12)$$

1110 If $|\tilde{y}|_{Y^r} = 1$, then

$$1111 \quad |\mathfrak{B}(\tilde{y}(t))|_{r+1} \leq \int_0^t |e^{-(t-s)\mathcal{L}} \mathcal{K}(s)(\tilde{y}(s))|_{r+1} ds \leq C'_r \int_0^t e^{-2(t-s)} e^{-s} ds < C'_r e^{-t}.$$

1112 So $\mathfrak{B} : Y^r \rightarrow Y^{r+1}$ is a bounded linear operator if $r > d$, and accordingly the
 1113 operator $(\text{id} - \varepsilon^2 \mathfrak{B})$ is a linear isomorphism of Y^r if $r > d$ and ε is sufficiently
 1114 small. It easy to see that $\mathfrak{B}(v)$ and $\int_0^t e^{-(t-s)\mathcal{L}} \eta(s) ds$ both belong to all spaces
 1115 Y^r . Then in view of (5.12), $|y|_{Y^{r+1}} \leq C\varepsilon^2$. Since the operator \mathfrak{B} is 1-smoothing,
 1116 then by induction we get that y belongs to all spaces Y^r . We have proved that

1117 **Lemma 5.6.** *The solution $\mathfrak{z}(\tau)$, constructed in Lemma 5.5, may be written as*

$$1118 \quad \mathfrak{z}(\tau) = (\text{id} - e^{-\tau\mathcal{L}})\mathfrak{z}^\varepsilon + y(\tau), \quad \text{where } |y(\tau)|_r \leq C_r \varepsilon^2 e^{-\tau} \quad \forall \tau \geq 0, \forall r.$$

1119 Here \mathfrak{z}^ε is the stationary solution, defined in (5.9) and satisfying (5.10).

1120 5.3. Energy Spectra of Quasisolutions and Kinetic Equation

1121 In this section we prove our main result. Namely, we show that the energy
 1122 spectrum (4.2) of the quasisolution $n_{s,L}(\tau) = \mathbb{E}|A_s(\tau)|^2$ of Eq. (2.1) with
 1123 large L is ε^3 -close to the solution $\mathfrak{z}(\tau)$ of the WKE (5.1), constructed in Lem-
 1124 mas 5.5, 5.6. By (4.12), it suffices to prove this for $n_{s,L}$ replaced by $n_s^{\leq 2}$. Let
 1125 us denote $w_s(\tau) = n_s^{\leq 2}(\tau) - \mathfrak{z}_s(\tau)$; then $w_s(0) = 0$. Recall that ε_* is defined in
 1126 Lemma 5.5.

1127 **Lemma 5.7.** *If $r > d + 1$ and $\varepsilon \leq C_{1r}^{-1} \leq \varepsilon_*$ for an appropriate constant C_{1r} ,*
 1128 *then for any $\tau_0 \geq 0$ and $0 < \tau \leq 1/2$,*

1129
$$|w(\tau_0 + \tau)|_r \leq (1 - \tau/2)|w(\tau_0)|_r + C_{2r}\tau\varepsilon^2(\tau^{\kappa_*} + \varepsilon^2), \tag{5.13}$$

1130 where $\kappa_* = 1/(1 + 2r_*)$.

1131 *Proof.* Since by (5.1)

1132
$$\mathfrak{z}(\tau_0 + \tau) = e^{-\tau\mathcal{L}}\mathfrak{z}(\tau_0) + 2 \int_0^\tau e^{-t\mathcal{L}}b^2 dt + \varepsilon^2 \int_{\tau_0}^{\tau_0+\tau} e^{-(\tau_0+\tau-t)\mathcal{L}}K(t)(\mathfrak{z}(t))dt,$$

1133 then in view of (4.22) and (4.19)

1134
$$w(\tau + \tau_0) = e^{-\tau\mathcal{L}}w(\tau_0) + \varepsilon^2\Delta + \mathcal{R}, \tag{5.14}$$

1135 where \mathcal{R} is as in (4.22) and

1136
$$\Delta = \int_{\tau_0}^{\tau_0+\tau} e^{-(\tau_0+\tau-t)\mathcal{L}} \left(K(\tau_0)(n^{\leq 2}(\tau_0)) - K(t)(\mathfrak{z}(t)) \right) dt.$$

1137 Note that in view of Lemma 5.5 and estimates (4.9),

1138
$$|n^{\leq 2}(\tau)|_r, |\mathfrak{z}(\tau)|_r \leq C_r \quad \text{for all } \tau \text{ and all } r, \tag{5.15}$$

1139 with suitable constants C_r . Let us re-write Δ as follows:

1140
$$\begin{aligned} \Delta &= \int_{\tau_0}^{\tau_0+\tau} e^{-(\tau_0+\tau-t)\mathcal{L}} (K(\tau_0)(n^{\leq 2}(\tau_0)) - K(\tau_0)(\mathfrak{z}(\tau_0))) dt \\ &+ \int_{\tau_0}^{\tau_0+\tau} e^{-(\tau_0+\tau-t)\mathcal{L}} (K(\tau_0)(\mathfrak{z}(\tau_0)) - K(t)(\mathfrak{z}(\tau_0))) dt \\ &+ \int_{\tau_0}^{\tau_0+\tau} e^{-(\tau_0+\tau-t)\mathcal{L}} (K(t)(\mathfrak{z}(\tau_0)) - K(t)(\mathfrak{z}(t))) dt =: \Delta^1 + \Delta^2 + \Delta^3. \end{aligned}$$

1141 By (5.4) and (5.15), $|\Delta^1|_r \leq C_r\tau|w(\tau_0)|_r$. Similar,

1142
$$|\Delta^3|_r \leq C_r\tau \sup_{\tau_0 \leq t \leq \tau_0+\tau} |\mathfrak{z}(t) - \mathfrak{z}(\tau_0)|_r \leq C_r\tau^2$$

1143 since $|\mathfrak{z}(t) - \mathfrak{z}(\tau_0)|_r \leq \int_{\tau_0}^t |-\mathcal{L}\mathfrak{z}(l) + \varepsilon^2 K(l)(\mathfrak{z}(l)) + 2b^2|_r dl$ and $|\mathfrak{z}(t)|_{r+r_*} \leq C'_r$ by
 1144 (5.15). Now let us consider Δ^2 . By Lemma 5.4, $|K(\tau_0)(\mathfrak{z}(\tau_0)) - K(t)(\mathfrak{z}(\tau_0))|_r \leq$
 1145 $C_r(t - \tau_0)^{\kappa_*}$. So $\Delta^2 \leq C_r \int_0^\tau t^{\kappa_*} dt = C'_r\tau^{1+\kappa_*}$.

1146 Since $\mathcal{L} \geq 2\mathbb{1}$ and $\tau \leq 1/2$, then $|e^{-\tau\mathcal{L}}w(\tau_0)|_r \leq (1 - \tau)|w(\tau_0)|_r$. Now
 1147 (5.14), (4.23) and the bounds on Δ^j imply that

1148
$$|w(\tau_0 + \tau)|_r \leq (1 - \tau)|w(\tau_0)|_r + C_r\varepsilon^2\tau(|w(\tau_0)|_r + \tau + \tau^{\kappa_*} + (\tau + \varepsilon^2)),$$

1149 and (5.13) follows if $C_{1r}^{-1} \ll 1$. □

1150 For any $0 < \tau \leq 1/2$, any N and for $k = 0, \dots, N$ let us set $w_k = |w(k\tau)|_r$.
 1151 Let the function $k \rightarrow w_k$ attain its maximum at a point k which we write as
 1152 $k := k_0 + 1$. If $k_0 + 1 = 0$, then $w_k \equiv 0$. Otherwise, in view of (5.13) we have

1153
$$w_{k_0} \leq w_{k_0+1} \leq (1 - \tau/2)w_{k_0} + C_{2r}\tau\varepsilon^2(\tau^{\kappa_*} + \varepsilon^2).$$

1154 So, $w_{k_0} \leq 2C_{2r}\varepsilon^2(\tau^{\kappa_*} + \varepsilon^2)$ and

1155
$$\max_{0 \leq k \leq N} |w(k\tau)|_r = w_{k_0+1} \leq 3C_{2r}\varepsilon^2(\tau^{\kappa_*} + \varepsilon^2)$$

1156 since $\tau \leq 1/2$. Applying again (5.13) with $\tau_0 = k\tau$ and τ replaced by any
 1157 $\bar{\tau} \in (0, \tau)$, and using that in the formula above N is any, we get that $|w(t)|_r \leq$
 1158 $4C_{2r}\varepsilon^2(\tau^{\kappa_*} + \varepsilon^2)$, for any $t \geq 0$. Sending $\tau \rightarrow 0$ (and estimating norms $|\cdot|_r$
 1159 with $r < d + 2$ via $|\cdot|_{d+2}$) and then using (4.12), we finally get

1160 **Theorem 5.8.** *For any r there exist positive constants C_{1r}, C_{2r}, C_{3r} such that*
 1161 *if $\varepsilon \leq C_{1r}^{-1}$, then*

1162
$$\sup_{\tau \geq 0} |n^{\leq 2}(\tau) - \mathfrak{z}(\tau)|_r \leq C_{2r}\varepsilon^4 \tag{5.16}$$

1163 and if $L \geq \varepsilon^{-2}$, then

1164
$$\sup_{\tau \geq 0} |n_{\cdot, L}(\tau) - \mathfrak{z}(\tau)|_r \leq C_{3r}\varepsilon^3. \tag{5.17}$$

1165 Relation (5.17) together with Lemma 5.6 gives a control over the long-
 1166 time behaviour of the spectra of quasisolutions of (2.1) in terms of the station-
 1167 ary solution \mathfrak{z}_ε of the limiting kinetic equation (see (5.9)):

1168
$$|n_{\cdot, L}(\tau) - \mathfrak{z}^\varepsilon|_r \leq C_r(e^{-\tau} + \varepsilon^3), \quad \forall \tau \geq 0.$$

1169 By Proposition 2.1 with $d \geq 3$ this result and (5.17) extend to the spectra of
 1170 quasisolutions of (1.20), defined in (1.27), as expressed in

1171 **Theorem 5.9.** *For any r there exist positive constants C_{4r}, C_{5r} such that if*
 1172 *$\varepsilon \leq C_{4r}^{-1}$ and $L \geq \varepsilon^{-2}$, then*

1173
$$\sup_{\tau \geq 0} |n_{\cdot, L}(\tau) - \mathfrak{z}(\tau)|_r \leq C_{4r}\varepsilon^3, \tag{5.18}$$

1174
$$|n_{\cdot, L}(\tau) - \mathfrak{z}^\varepsilon|_r \leq C_{5r}(e^{-\tau} + \varepsilon^3), \quad \forall \tau \geq 0. \tag{5.19}$$

1176 Relation (5.16) extends to the energy spectra of quasisolutions of (1.20)
 1177 analogously.

1178 **Funding** The work of AD on Sects. (1, 2, 3) was supported by the Russian Sci-
 1179 ence Foundation under Grant no.23-11-00150. The work of AD on the other
 1180 sections and work of AM and SK on the whole paper was supported by the
 1181 Ministry of Science and Higher Education of the Russian Federation (mega-
 1182 grant No. 075-15-2022-1115). All authors have contributed equally in this work

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 1184 tional claims in published maps and institutional affiliations.

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1185 **Appendix A. Proof of Lemma 3.6**1186 In this appendix we suppose that the dimension d satisfies $d \geq 2$.1187 **A.1. Idea of the Proof and General Setting**

1188 In Lemma 3.6 (up to an obvious scaling) we have to estimate the number
 1189 of integer points on a quadric inside a large box. The idea is to embed the
 1190 integral points of the box in an affine space over a large finite field and then
 1191 apply powerful algebraic geometry techniques to estimate the needed number
 1192 (note that this identification of bounded integers with elements of a finite
 1193 field is ubiquitous in coding theory and combinatorics). It is possible mainly
 1194 due to the fact that this techniques permits to count points defined over a
 1195 finite field using some geometric information (essentially the dimension, the
 1196 degree and irredundant decomposition) on the corresponding algebraic set over
 1197 the algebraic closure of our finite field. We begin with recalling some basic
 1198 definitions and results concerning such algebraic sets (see, for example, the
 1199 first chapter of the book [27]).

1200 **Affine algebraic sets.** Let us fix an algebraically closed field K . Let $\mathbb{A}^m = K^m$
 1201 be the m -dimensional affine space over K , and let $F_1, \dots, F_s \in K[T_1, \dots, T_m]$
 1202 be nonzero polynomials. Then an *affine algebraic set* (AAS) X is just the set
 1203 of common zeros of these polynomials:

$$1204 \quad X = \{(a_1, \dots, a_m) \in K^m : F_1(a_1, \dots, a_m) = \dots = F_s(a_1, \dots, a_m) = 0\}. \quad (A.1)$$

1206 **Irreducibility.** An AAS X is *reducible* if $X = X_1 \cup X_2$ with two non-empty
 1207 AAS X_1, X_2 s.t. $X_1 \neq X, X_2 \neq X$. If it is not the case, X is called *irreducible*,
 1208 or an *affine algebraic variety* (see [27], Section I.3.1).

1209 **Theorem A.1** (Irredundant decomposition). *Any non-empty AAS X can be*
 1210 *presented as*

$$1211 \quad X = X_1 \cup \dots \cup X_l \quad (A.2)$$

1212 *for irreducible X_1, \dots, X_l such that $X_i \not\subseteq X_j$ for $i \neq j$. The decomposition is*
 1213 *unique up to order.*

1214 This decomposition is especially simple for a hypersurface X , i.e. when
 1215 in (A.1) $s = 1$. Then $F = F_1(T_1, \dots, T_m) = \prod_{j=1}^l Q_j$ for irreducible polynomials
 1216 Q_j which are uniquely defined up to multiplicative constants and permutation
 1217 since the ring $K[T_1, \dots, T_m]$ is a unique factorization domain, see, for example,
 1218 Chapter IV of [24], and then $X_j = \{Q_j = 0\}$. This uniqueness is true under the
 1219 condition which we can and will suppose to hold, namely, that the polynomial
 1220 P does not have multiple divisors, i.e. all $Q_j, j = 1, \dots, l$ are distinct. For
 1221 further references we formulate a corollary of the unique factorization property
 1222 (see [27], Section I.3.1)

1223 **Lemma A.2.** (i). *If X and Y are hypersurfaces, then $X = Y$ if and only if*
 1224 *the corresponding polynomials P_X and P_Y are proportional. Moreover if*
 1225 *Y is irreducible and $X \subseteq Y$, then $X = Y$.*

1226 (ii). If $\deg P_X = 2$, then there are exactly two possibilities: either X irre-
 1227 ducible (in this case it cannot contain a hyperplane), or $X = X_1 \cup X_2$
 1228 for two affine hyperplanes, defined by affine linear polynomials l_1 and l_2 ,
 1229 and $P_X = l_1 l_2$.

1230 **Dimension.** One can define the *dimension* $r = \dim X \in \{0, 1, \dots, m\}$ as follows:
 1231 $\dim X = \max\{\dim X_i, i = 1, \dots, l\}$ for (A.2), and for an irreducible AAS X

$$1232 \quad \dim X = \max\{r : X = X_0 \supset X_1 \supset \dots \supset X_r \neq \emptyset\},$$

1233 where all $X_i, i = 0, \dots, r$ are irreducible AAS and all inclusions are strict. The
 1234 *codimension* of X is $\text{codim } X = m - \dim X$.

1235 In particular, if $\dim X = m$ then $X = \mathbb{A}^m$ (indeed, if $X \subset \mathbb{A}^m, X \neq \mathbb{A}^m$
 1236 the definition implies that $\dim X < \dim \mathbb{A}^m = m$) and if $\dim X = 0$ then X is
 1237 a finite set. The codimension of X as in (A.1) is at most s .

1238 From the definition we get immediately (see [27], Section I.6.2)

1239 **Lemma A.3.** *If $Y \subset X, Y \neq X$ and X is irreducible, then $\dim X > \dim Y$,*
 1240 *$\text{codim } X < \text{codim } Y$.*

1241 **Degree.** Let $X \subset \mathbb{A}^m$ be a non-empty AAS, $\dim X = r$. Then its *degree* $\deg X$
 1242 is defined as follows:

$$1243 \quad \deg X = \max\{\text{cardinality of } X \cap L : \dim(X \cap L) = 0\},$$

1244 where $L \subset \mathbb{A}^m$ is an affine plane with $\dim L = m - r$.

1245 **Lemma A.4.** *If X is a hypersurface (i.e. in (A.1) $s = 1$), then $\text{codim } X = 1$*
 1246 *and $\deg X = \deg F_1$.*

1247 The famous *Bezout theorem* in its the most elementary setting over the
 1248 field \mathbb{C} states that

$$1249 \quad \deg X \leq \prod_{i=1}^s \deg F_i.$$

1250 A.2. Finite Fields' Bezout Theorem

1251 From now on the field K is the algebraic closure $\bar{\mathbb{F}}_p$ of a finite field \mathbb{F}_p , where
 1252 p is a large prime number (see [24], Section V.5).

1253 We will use a version of Bezout's theorem over finite fields which can
 1254 be deduced from its general form, e.g. [13], and is also explicitly stated and
 1255 proved in [22, Corollary 2.2].

1256 **Theorem A.5.** *Let $K = \bar{\mathbb{F}}_p$ and the AAS X in (A.1) is such that $F_j \in \mathbb{F}_p[T_1,$
 1257 $\dots, T_m]$, $\deg F_j = d_j, j = 1, \dots, s$, and $\dim X = r$. Then*

$$1258 \quad |X \cap \mathbb{F}_p^m| \leq p^r \prod_{i=1}^s d_i.$$

1259 **A.3. Preliminary Result**

1260 Let q_1, \dots, q_s , $s \geq 1$, be polynomials of degree at most two in $m \geq s$ vari-
 1261 ables, $q_i \in \mathbb{Z}[X_1, \dots, X_m]$, with $q_i(0) = 0$, $i = 1, \dots, s$. Consider the geometric
 1262 quadrics $Q_i = \{x \in \mathbb{R}^m : q_i(x) = 0\}$ and their intersection $Q = \cap_{i=1}^s Q_i$. The
 1263 latter is not empty since $0 \in Q$.

1264 Let $B_M^m \subset \mathbb{R}^m$ be an open cube $\{|x|_\infty < M\}$ with some $M \geq 1$. Consider
 1265 the set

1266
$$S_m(M, Q) = Q \cap \mathbb{Z}^m \cap B_M^m.$$

1267 Let p be a prime and $q_i^{(p)} \in \mathbb{F}_p[X_1, \dots, X_m]$ denote the polynomials q_i
 1268 mod p over the finite field \mathbb{F}_p . Consider the sets

1269
$$Q_i^{(p)} = \{x \in K^m : q_i^{(p)}(x) = 0\}$$

1270 and their intersection $Q^{(p)} = \cap_{i=1}^s Q_i^{(p)}$ (recall that now $K = \overline{\mathbb{F}}_p$ is the algebraic
 1271 closure of \mathbb{F}_p). We will be interested mainly in the cardinality of $Q^{(p)}(\mathbb{F}_p) :=$
 1272 $Q^{(p)} \cap \mathbb{F}_p^m$ as a tool to estimate $|S_m(M, Q)|$.

1273 **Proposition A.6.** *Let $M \geq 1$ and suppose that a prime $p > 2M$ satisfies $p =$
 1274 $2M(1 + r(M))$, where $r(M) > 0$. Suppose also that $\deg q_i \leq 2$ for each i and
 1275 that the AAS $Q^{(p)}$ is of dimension $m - s$ (and of codimension s , that is, the
 1276 s quadrics $Q_i^{(p)}$ intersect properly):*

1277
$$\dim Q^{(p)} = m - s. \tag{A.3}$$

1278 *Then*

1279
$$|S_m(M, Q)| \leq 2^m (1 + r(M))^{m-s} M^{m-s}. \tag{A.4}$$

1280 By Bertrand's postulate, for any $M \geq 1$ there is a p satisfying $2M < p <$
 1281 $4M$, and when applying Proposition A.6 we will always chose

1282
$$r(M) < 1. \tag{A.5}$$

1283 Moreover, by the Prime Number Theorem, for large M one can chose $r(M) =$
 1284 $o(1)$.

1285 *Proof of Proposition A.6.* Let $\Pi : S_m(M, Q) \rightarrow \mathbb{F}_p^m$ be defined by

1286
$$\Pi(x_1, \dots, x_m) = (x_1 \pmod p, \dots, x_m \pmod p).$$

1287 Then Π is injective and its image is contained in $Q^{(p)} \cap \mathbb{F}_p^m \subset \mathbb{F}_p^m$. Indeed, the
 1288 last assertion is clear and the injectivity is established as follows: if

1289
$$(x'_1 \pmod p, \dots, x'_m \pmod p) = (x_1 \pmod p, \dots, x_m \pmod p)$$

1290 but $x' \neq x$, then for some $i \in \{1, \dots, m\}$ we have $x'_i \pmod p = x_i \pmod p$, but
 1291 $x'_i \neq x_i$. Consequently, $|x_i - x'_i| \geq p > 2M$ which contradicts the condition
 1292 $x_i, x'_i \in B_M^m$. Applying then Theorem A.5 to $X = Q^{(p)}$, we get the conclusion
 1293 since

1294
$$|S_m(M, Q)| \leq |Q^{(p)}(\mathbb{F}_p)| \leq 2^s p^{m-s} = 2^m (1 + r(M))^{m-s} M^{m-s}.$$

1295 □

1296 **A.4. Main Estimate for $N = 3$ and 4**

1297 Now we pass to the proof of Lemma 3.6 and denote $|\tilde{Q}_L(z_1, v) \cap B_R^{(N-2)d}| =$
 1298 $s(R, \tilde{Q}, L)$. Consider the set

1299
$$S'(R, \tilde{Q}, L) = \tilde{Q} \cap \mathbb{Z}^{(N-2)d} \cap B_{RL}^{(N-2)d}, \quad \tilde{Q} = \tilde{Q}(Lz_1, Lv),$$

1300 and denote by $s'(R, \tilde{Q}, L)$ its cardinality. Then $s'(R, \tilde{Q}, L) = s(R, \tilde{Q}, L)$ since
 1301 the map $(z_2, \dots, z_{N-1}) \mapsto (Lz_2, \dots, Lz_{N-1})$ is a bijection between the sets
 1302 $\tilde{Q}_L(z_1, v) \cap B_R^{(N-2)d}$ and $S'(R, \tilde{Q}, L)$.

1303 Let us estimate $s'(R, \tilde{Q}, L)$ through Proposition A.6 with $m = (N - 2)d$
 1304 and $s = N - 2$, where $N = 3$ or 4. To this end it suffices to find $M \geq RL$
 1305 and $p > 2M$ such that assumption (A.3) is fulfilled for any $(z_1, v) \in B_{(N-1)R}^{2d}$
 1306 satisfying $(z_1, v) \in Q_{1L}^0 \cap A_2$. Lemma A.7 below establishes this for $M =$
 1307 $NRL/2$ and any $p > 2M$. Then, applying (A.4) with $r(M) < 1$ (see (A.5)),
 1308 we conclude the proof of Lemma 3.6.

1309 For a prime p and $a, b \in \mathbb{F}_p^d$, let us consider algebraic sets $\tilde{Q}_j^{(p)}$ over
 1310 $K = \overline{\mathbb{F}}_p$:

1311
$$\tilde{Q}_j^{(p)}(a, b) := \{(z_2, \dots, z_{N-2}) \in K^{(N-2)d} : q_j^{(p)}(z_2, \dots, z_{N-2}; a, b) = 0\},$$

1312 where $q_j^{(p)}(z_2, \dots, z_{N-2}; a, b)$ are the residues modulo p of the polynomials
 1313 $q_j(z_2, \dots, z_{N-2}; a, b)$, defined by (3.15). We set $\tilde{Q}^{(p)} = \cap_{1 < j < N} \tilde{Q}_j^{(p)}$ for the
 1314 intersection of the algebraic sets.

1315 **Lemma A.7.** *Let $N \in \{3, 4\}$, $(z_1, v) \in Q_{1L}^0 \cap A_2$ (see (3.16)) and let p be a*
 1316 *prime satisfying $p > \max(|Lz_1|_\infty, |Lv|_\infty)$. Then*

1317
$$\dim \tilde{Q}^{(p)}(Lz_1, Lv) = (N - 2)(d - 1). \tag{A.6}$$

1318 The assumption $p > \max(|Lz_1|_\infty, |Lv|_\infty)$ ensures that Lz_1 and Lv are
 1319 different from zero in K^d . In particular, for $(z_1, v) \in B_{(N-1)R}^{2d}$ this assumptions
 1320 is satisfied if $p > 2M$ with $M = NRL/2$.

1321 *Proof of Lemma A.7.* Let $N = 3$. Then $N - 2 = 1$ and $\tilde{Q}^{(p)}$ is given by the
 1322 unique equation $q_2^{(p)}(z_2; Lz_1, Lv) = 0$, for a fixed (z_1, v) . By Lemma A.9 the
 1323 equation is non-trivial, so the conclusion follows from Lemma A.4.

1324 $N = 4$. The codimension of the intersection of two quadrics is at most
 1325 two. We have to show that it is two (and not one). The result will follow from
 1326 the next three lemmas.

1327 **Lemma A.8.** *Let $\mathcal{Q}_1 = \{\tilde{q}_1 = 0\}$, $\mathcal{Q}_2 = \{\tilde{q}_2 = 0\}$ be two linearly independent*
 1328 *quadrics over K . Then the codimension of $\mathcal{Q}_1 \cap \mathcal{Q}_2$ is one if and only if \tilde{q}_1 and*
 1329 *\tilde{q}_2 have a mutual affine linear factor $l(x)$.*

1330 *Proof.* Let the codimension of the intersection be one. In this case if one of
 1331 $\mathcal{Q}_1, \mathcal{Q}_2$ is irreducible, then $\mathcal{Q}_1 = \mathcal{Q}_2$ by Lemma A.3 with $Y = \mathcal{Q}_1 \cap \mathcal{Q}_2$. How-
 1332 ever, this is impossible by Lemma A.2. (i) since \tilde{q}_1 and \tilde{q}_2 are independent.

1333 Therefore, by Lemma A.2. (ii) $\mathcal{Q}_1 = H_1 \cup H_2$ and $\mathcal{Q}_2 = H'_1 \cup H'_2$, with hyper-
 1334 planes H_1, \dots, H'_2 . If all $H_i \cap H'_j$ are of codimension two, then

1335
$$\text{codim } \mathcal{Q}_1 \cap \mathcal{Q}_2 = \text{codim } (\cup(H_i \cap H'_j)) = \min(\text{codim } H_i \cap H'_j) = 2.$$

1336 Therefore, at least one of $H_i \cap H'_j$ is of codimension one and then we have
 1337 $\ker(l(x)) = H_i \cap H'_j \subset \mathcal{Q}_1 \cap \mathcal{Q}_2$ for an affine linear $l(x)$. Hence, $l(x)$ divides
 1338 both \tilde{q}_1 and \tilde{q}_2 by Lemma A.2. (ii).

1339 The inverse statement is obvious. □

1340 **Lemma A.9.** *For any $N > 2$, if the matrix α is irreducible and $(z_1, v) \in Q_{1L}^0 \cap$
 1341 A_2 is such that $Lz_1, Lv \neq 0$ in K^d , then the polynomials $q_j^{(p)}(\cdot, Lz_1, Lv)$,
 1342 $1 < j < N$ are linearly independent over K . In particular, each $q_j^{(p)}$ is a
 1343 nonzero polynomial.*

1344 *Proof.* Consider a linear combination $\sum_{1 < j < N} c_j q_j^{(p)}$. By the homogeneity in
 1345 (z_2, \dots, z_{N-1}) , it vanishes identically if and only if

1346
$$\begin{aligned} \sum_{1 < j < N} c_j z_j \cdot (\alpha_{j1}(Lz_1) + \alpha_{jN}(Lv)) &\equiv 0, \\ \sum_{1 < i, j < N} c_j (\alpha_{ji} - \alpha_{jN}\alpha_{1i}) z_j \cdot z_i &\equiv 0. \end{aligned} \tag{A.7}$$

1347 Arguing by induction and using that the matrix α is irreducible, we construct
 1348 a partition E_0, \dots, E_M , $M \geq 1$, of the set $\{1, \dots, N\}$ such that $E_0 = \{1, N\}$
 1349 and for $n \geq 1$,

1350
$$E_n = \{j : \alpha_{jl} = 0 \forall l \in E_{n'}, n' \leq n - 2, \text{ and } \exists l' \in E_{n-1} \text{ such that } \alpha_{jv} \neq 0\}.$$

1351 Since $(z_1, v) \in Q_{1L}^0$ and $Lz_1, Lv \neq 0$ in K^d , then the term in brackets in the
 1352 first line of (A.7) is not identically zero for each $j \in E_1$, so $c_j = 0$ for every
 1353 $j \in E_1$. Using this in the second line of (A.7), we get:

1354
$$\sum_{n=2}^M \sum_{m=n-1}^M \sum_{j \in E_n} \sum_{i \in E_m} c_j \alpha_{ji} z_j \cdot z_i \equiv 0.$$

1355 This relation holds if and only if $(c_j - c_i)\alpha_{ji} = 0$ for all $j \in E_n$, $2 \leq n \leq M$,
 1356 and $i \in E_m$, $n - 1 \leq m \leq M$. We know that $c_j = 0$ if $j \in E_1$. Starting from
 1357 $n = 2$ and arguing by induction in n , we find that if $c_i = 0$ for all $i \in E_{n-1}$,
 1358 then $c_j = 0$ for all $j \in E_n$. Indeed, for any $j \in E_n$ there exists at least one
 1359 $i \in E_{n-1}$ such that $\alpha_{ji} \neq 0$ by the definition of E_i , so relation $(c_j - c_i)\alpha_{ji} = 0$
 1360 implies that $c_j = 0$ if $j \in E_n$. That is, $c_j \equiv 0$. □

1361 **Lemma A.10.** *For any $N > 2$, if the matrix α is irreducible and $(z_1, v) \in$
 1362 $Q_{1L}^0 \cap A_2$ is such that $Lz_1, Lv \neq 0$ in K^d , then the polynomials $q_j^{(p)}(\cdot, Lz_1, Lv)$,
 1363 $1 < j < N$, are irreducible.*

1364 *Proof.* Each polynomial $q_j^{(p)}$ has degree one or two. If its degree is one the
 1365 assertion is obvious. Now let the degree be two. Note that in view of (3.15)
 1366 $q_j^{(p)}$ can be written as the scalar products $q_j^{(p)} = z_j \cdot l_j(z_2, \dots, z_{N-1}; z_1, v) \text{ mod}$

1367 p , where l_j are surjective affine functions $l_j : K^{d(N-2)} \longrightarrow K^d$. But such scalar
 1368 product cannot vanish for $d \geq 2 > 1$ on a hyperplane $H \subset K^{d(N-2)}$ which
 1369 by Lemma A.2. ii) would be the case for a reducible quadric. Indeed, only two
 1370 cases can occur:

- 1371 (a) The coefficient α of z_j in l_j is nonzero, or
- 1372 (b) It is zero but then the coefficient β of some other z_i is nonzero.

1373 In case (a) take the two-dimensional plane $P(x_1, x_2)$ in the whole space, gener-
 1374 ated by two orthogonal vectors from the z_j -space, where the first basis vector
 1375 is parallel to $\alpha_{1j}z_1 + \alpha_{Nj}v \neq 0$ (this vector is nonzero since $(z_1, v) \in Q_{1L}^0$ and
 1376 $Lz_1, Lv \neq 0$ in K^d , and for the case a) we have $\alpha_{1j}, \alpha_{Nj} \neq 0$). Then the restric-
 1377 tion of $q_j^{(p)} = 0$ on P is $\alpha(x_1^2 + x_2^2) + c_1x_1 = 0$ with $c_1 \neq 0$, which is isomorphic
 1378 to $x_1^2 + x_2^2 = C \neq 0$. This plane quadric in $P(x_1, x_2)$ cannot contain $P(x_1, x_2)$
 1379 $\cap H$ (a line or the whole $P(x_1, x_2)$). Indeed, otherwise, supposing by symme-
 1380 try that the quadric contains a line $x_1 = ax_2 + b$, we would have that the
 1381 polynomial

$$1382 \quad a^2x_2^2 + 2abx_2 + b^2 + x_2^2 - C$$

$$1383 \quad = (a^2 + 1)x_2^2 + 2abx_2 + b^2 - C$$

1384 vanishes identically. This implies $ab = 0$, and if $a = 0$ then the term
 1385 $(a^2 + 1)x_2^2 = x_2^2 \neq 0$, while for $b = 0$ the term $b^2 - C = -C \neq 0$.

1386 Similarly, in case b) we take the four-dimensional vector subspace P'
 1387 generated by the two first basis vectors in the z_j space and the two first basis
 1388 vectors in the z_i space. The restriction of $q_j^{(p)} = 0$ on P' is then $\beta(x_1y_1 +$
 1389 $x_2y_2) + c_1x_1 + c_2x_2 = 0$, isomorphic to $x_1y_1 + x_2y_2 = C$ which cannot contain
 1390 $P'(x_1, x_2, y_1, y_2) \cap H$. Indeed, else, supposing by symmetry that $P'(x_1, x_2, y_1, y_2)$
 1391 $\cap H \supset \{x_1 = a_1x_2 + b_1y_1 + b_2y_2 - c\}$ we get that the following quadratic function
 1392 of x_2, y_1, y_2 :

$$1393 \quad (a_1x_2 + b_1y_1 + b_2y_2 - c) y_1 + x_2y_2 - C$$

$$1394 \quad = a_1x_2y_1 + b_1y_1^2 + b_2y_1y_2 - cy_1 + x_2y_2 - C$$

1395 vanishes identically, which is clearly wrong. □

1396 *End of the proof of Lemma A.7.* Since each $q_j^{(p)}$ is a nonzero polynomial of de-
 1397 gree one or two, then to prove Lemma A.7 we have to consider three cases. In
 1398 the first case both polynomials $q_2^{(p)}$ and $q_3^{(p)}$ are linear. Then the codimension
 1399 of the intersection $\tilde{Q}^{(p)}$ is two since they are linearly independent. In the sec-
 1400 ond case both $q_2^{(p)}$ and $q_3^{(p)}$ are quadratic. Then, according to Lemma A.8, the
 1401 codimension still is two since the polynomials are irreducible by Lemma A.10.
 1402 Finally in the last case, when one polynomial is linear and another one is qua-
 1403adratic, the assertion is clear since then the AAS in question is an intersection
 1404 of a quadratic irreducible surface with a hyperplane. Thus, its codimension is
 1405 two by Lemma A.2. (ii).

1406 *Remark A.11.* The proof of Lemma A.7 follows from three lemmas. Two of
 1407 them are valid for any $N > 2$, but Lemma A.8 holds only for $N = 4$ (and

1408 tautologically holds for smaller N). Still the bi-linear (or linear) nature of the
 1409 polynomials $q_j^{(p)}$ and direct analysis of the AAS $\tilde{Q}^{(p)}$, jointly with the two
 1410 lemmas, valid for any $N > 2$, allow to prove by hand Lemma A.7 for “not too
 1411 high” values of N , and thus, to prove for those N ’s Theorem 3.2. Unfortunately,
 1412 for the moment we cannot prove the theorem for all $N > 2$; cf. Conjecture 3.8.

1413 **Appendix B. Proof of Proposition 4.3 and Lemma 4.4**

1414 We prove Proposition 4.3 in Sects. B.1-B.6 and Lemma 4.4 in Sect. B.7.

1415 **B.1. Beginning of the Proof of Proposition 4.3**

1416 The proof of the proposition is somewhat cumbersome since we have to con-
 1417 sider a number of different terms and different cases. During the proof we will
 1418 often skip the upper index (0), so by writing a and a_s we will mean $a^{(0)}$ and $a_s^{(0)}$.
 1419 We will also skip the dependence on τ_0 by writing $c_s^{(i)}(\tau; \tau_0)$ and $\Delta a_s^{(i)}(\tau; \tau_0)$
 1420 as $c_s^{(i)}(\tau)$ and $\Delta a_s^{(i)}(\tau)$. Besides, for a complex function ($w_{s_1, \dots, s_k}, s_j \in \mathbb{Z}_L^d$) we
 1421 denote

1422
$$\sum_{s_1, \dots, s_k \in \mathbb{Z}_L^d} w_{s_1, \dots, s_k} = L^{-kd} \sum_{s_1, \dots, s_k \in \mathbb{Z}_L^d} w_{s_1, \dots, s_k},$$

1423 and we introduce the symmetrization

1424
$$\mathcal{Y}_s^{sym}(u, v, w; t) = \frac{L^{-d}}{3} \sum_{1,2,3} \delta_{3s}^{12} \delta(\omega_{3s}^{12}) (u_1 v_2 \bar{w}_3 + v_1 w_2 \bar{u}_3 + w_1 u_2 \bar{v}_3).$$

1425 We recall that $Q_{s,L}$ is given by formula (4.30) and first consider the term
 1426 $\mathbb{E} \Delta a_s^{(2)}(\tau) \bar{a}_s(\tau_0 + \tau)$. Inserting the identity $a^{(1)}(\tau_0 + l) = c^{(1)}(l) + \Delta a^{(1)}(l)$ into
 1427 formula (4.27) for $\Delta a_s^{(2)}$, we obtain

1428
$$\mathbb{E} \Delta a_s^{(2)}(\tau) \bar{a}_s(\tau_0 + \tau) = N_s + \tilde{N}_s,$$

1429 where

1430
$$N_s := i \mathbb{E} \left(\bar{a}_s(\tau_0 + \tau) \int_0^\tau e^{-\gamma_s(\tau-l)} \mathfrak{Z} \mathcal{Y}_s^{sym}(a(\tau_0 + l), a(\tau_0 + l), \Delta a^{(1)}(l)) dl \right)$$

 1431 (B.1)

1432 and

1433
$$\tilde{N}_s := i \mathbb{E} \left(\bar{a}_s(\tau_0 + \tau) \int_0^\tau e^{-\gamma_s(\tau-l)} \mathfrak{Z} \mathcal{Y}_s^{sym}(a(\tau_0 + l), a(\tau_0 + l), c^{(1)}(l)) dl \right).$$

1434 Thus,

1435
$$Q_{s,L} = L^2 \left(\mathbb{E} |\Delta a_s^{(1)}(\tau)|^2 + 2\Re N_s \right.$$

 1436
$$\left. + 2\Re \mathbb{E} \Delta a_s^{(1)}(\tau) \bar{c}_s^{(1)}(\tau) + 2\Re \tilde{N}_s \right), \quad s \in \mathbb{R}^d. \quad (B.2)$$

1437 We will analyse the four terms above term by term.

1438 **B.2. The First Term of $Q_{s,L}$ in (B.2)**

1439 Due to (4.27), we have

$$1440 \quad \mathbb{E}|\Delta a_s^{(1)}(\tau)|^2 = \mathbb{E} \int_{\tau_0}^{\tau_0+\tau} dl \int_{\tau_0}^{\tau_0+\tau} dl' e^{-\gamma_s(2\tau_0+2\tau-l-l')} \mathcal{Y}_s(a(l)) \overline{\mathcal{Y}_s(a(l'))}. \tag{B.3}$$

1441

1442 Writing the functions \mathcal{Y}_s explicitly and applying the Wick theorem, in view of
1443 (2.13) we find

$$1444 \quad \mathbb{E}|\Delta a_s^{(1)}(\tau)|^2 = 2L^{-2d} \sum_{1,2} \delta_{3s}^{\prime 12} \delta(\omega_{3s}^{12}) \int_{\tau_0}^{\tau_0+\tau} dl \int_{\tau_0}^{\tau_0+\tau} dl' e^{-\gamma_s(2\tau_0+2\tau-l-l')} \\ 1445 \quad \mathbb{E}a_1(l)\bar{a}_1(l') \mathbb{E}a_2(l)\bar{a}_2(l') \mathbb{E}a_3(l)a_3(l'),$$

1447 and note that

$$1448 \quad \int_{\tau_0}^{\tau_0+\tau} dl \int_{\tau_0}^{\tau_0+\tau} dl' e^{-\gamma_s(2\tau_0+2\tau-l-l')} \leq \tau^2.$$

1449 On account of (2.13), we can bound

$$1450 \quad \mathbb{E}|\Delta a_s^{(1)}(\tau)|^2 \leq 2\tau^2 \sum_{1,2} \delta_{3s}^{\prime 12} \delta(\omega_{3s}^{12}) B_{123},$$

1451 where $B_{123} = B_1 B_2 B_3$. Since B_{123} with $s_3 = s - s_1 - s_2$ is a Schwartz function
1452 of s, s_1, s_2 then Theorem 3.2 with $N = 2$ applies and we find

$$1453 \quad \mathbb{E}|\Delta a_s^{(1)}(\tau)|^2 \leq C^\#(s) L^{-2} \tau^2. \tag{B.4}$$

1454 **B.3. The Second Term of $Q_{s,L}$ in (B.2)**

1455 To study the term $2\mathfrak{R}N_s$, we use the same strategy as above. Namely, expressing
1456 in (B.1) the function $3\mathfrak{Y}_s^{sym}$ via \mathcal{Y}_s , we write N_s as $N_s = N_s^1 + 2N_s^2$, $s \in \mathbb{R}^d$,
1457 where

$$1458 \quad N_s^1 = i \mathbb{E} \left(\bar{a}_s(\tau_0 + \tau) \int_0^\tau e^{-\gamma_s(\tau-l)} \mathcal{Y}_s(a(\tau_0 + l), a(\tau_0 + l), \Delta a^{(1)}(l)) dl \right),$$

$$1459 \quad N_s^2 = i \mathbb{E} \left(\bar{a}_s(\tau_0 + \tau) \int_0^\tau e^{-\gamma_s(\tau-l)} \mathcal{Y}_s(\Delta a^{(1)}(l), a(\tau_0 + l), a(\tau_0 + l)) dl \right).$$

1460

1461 *Term N_s^1 .* Writing explicitly the function \mathcal{Y}_s and then $\Delta \bar{a}_3^{(1)}$, we get

$$1462 \quad N_s^1 = i L^{-d} \sum_{1,2} \delta_{3s}^{\prime 12} \delta(\omega_{3s}^{12}) \int_0^\tau dl e^{-\gamma_s(\tau-l)} \\ 1463 \quad \times \mathbb{E}(a_1(\tau_0 + l)a_2(\tau_0 + l)\Delta \bar{a}_3^{(1)}(l)\bar{a}_s(\tau_0 + \tau)) \\ 1464 \quad = L^{-2d} \sum_{1,2} \sum_{1',2'} \delta_{3s}^{\prime 12} \delta_{3'3'}^{\prime 1'2'} \delta(\omega_{3s}^{12}) \delta(\omega_{3'3'}^{1'2'}) \int_0^\tau dl \int_0^l dl' e^{-\gamma_s(\tau-l)} e^{-\gamma_s(l-l')} \\ 1465 \quad \times \mathbb{E}(a_1(\tau_0 + l)a_2(\tau_0 + l)\bar{a}_1(\tau_0 + l')\bar{a}_2(\tau_0 + l')a_3(\tau_0 + l')\bar{a}_s(\tau_0 + \tau)). \tag{B.5}$$

1466

1467 By the Wick theorem, we need to take the summation only over $s_{1'}, s_{2'}, s_{3'}$
 1468 satisfying $s_{1'} = s_1, s_{2'} = s_2, s_{3'} = s$ or $s_{1'} = s_2, s_{2'} = s_1, s_{3'} = s$. Since in
 1469 both cases we get $\delta_{3'3}^{1'2'} = \delta_{3s}^{12}$ and $\omega_{3'3}^{1'2'} = \omega_{3s}^{12}$, we find

$$1470 \quad N_s^1 = 2 \sum_{1,2} \delta_{3s}^{12} \delta(\omega_{3s}^{12}) \int_0^\tau dl \int_0^l dl' e^{-\gamma_s(\tau-l)-\gamma_3(l-l')} \\ 1471 \quad \times \mathbb{E}a_1(\tau_0 + l)\bar{a}_1(\tau_0 + l') \mathbb{E}a_2(\tau_0 + l)\bar{a}_2(\tau_0 + l') \mathbb{E}a_s(\tau_0 + l')\bar{a}_s(\tau_0 + \tau).$$

1473 Arguing as in Sect. B.2 we find

$$1474 \quad |N_s^1| \leq C^\#(s)L^{-2}\tau^2. \tag{B.6}$$

1475 *Term N_s^2 .* By literally repeating the argument we have applied to N_s^1 , we
 1476 find that

$$1477 \quad N_s^2 = iL^{-d} \sum_{1,2} \delta_{3s}^{12} \delta(\omega_{3s}^{12}) \int_0^\tau dl e^{-\gamma_s(\tau-l)} \\ 1478 \quad \times \mathbb{E}(\Delta a_1^{(1)}(l)a_2(\tau_0 + l)\bar{a}_3(\tau_0 + l)\bar{a}_s(\tau_0 + \tau)) \\ 1479 \quad = -L^{-2d} \sum_{1,2} \sum_{1',2'} \delta_{3s}^{12} \delta_{3'1}^{1'2'} \delta(\omega_{3s}^{12}) \delta(\omega_{3'1}^{1'2'}) \int_0^\tau dl \int_0^l dl' e^{-\gamma_s(\tau-l)} e^{-\gamma_1(l-l')} \\ 1480 \quad \times \mathbb{E}(a_{1'}(\tau_0 + l')a_{2'}(\tau_0 + l')\bar{a}_{3'}(\tau_0 + l')a_2(\tau_0 + l)\bar{a}_3(\tau_0 + l)\bar{a}_s(\tau_0 + \tau)).$$

1482 By the Wick theorem, we should take summation either under the condition
 1483 $s_{1'} = s_3, s_{2'} = s, s_{3'} = s_2$ or $s_{1'} = s, s_{2'} = s_3, s_{3'} = s_2$. Since in both cases
 1484 $\delta_{3'1}^{1'2'} = \delta_{3s}^{12}$ and $\omega_{3'1}^{1'2'} = -\omega_{3s}^{12}$, then

$$1485 \quad N_s^2 = -2 \sum_{1,2} \delta_{3s}^{12} \delta(\omega_{3s}^{12}) \int_0^\tau dl \int_0^l dl' e^{-\gamma_s(\tau-l)} e^{-\gamma_1(l-l')} \\ 1486 \quad \mathbb{E}a_2(\tau_0 + l)\bar{a}_2(\tau_0 + l') \mathbb{E}a_3(\tau_0 + l')\bar{a}_3(\tau_0 + l) \mathbb{E}a_s(\tau_0 + l')\bar{a}_s(\tau_0 + \tau). \tag{B.7}$$

1488 Again we get

$$1489 \quad |N_s^2| \leq C^\#(s)L^{-2}\tau^2. \tag{B.8}$$

1490 **B.4. The Third Term of $Q_{s,L}$ in (B.2)**

1491 We have

$$1492 \quad \mathbb{E}\Delta a_s^{(1)}\bar{c}_s^{(1)}(\tau) = \mathbb{E} \int_{\tau_0}^{\tau_0+\tau} e^{-\gamma_s(\tau_0+\tau-l)} \mathcal{Y}_s(a(l)) dl \int_0^{\tau_0} e^{-\gamma_s(\tau_0+\tau-l')} \overline{\mathcal{Y}_s(a(l'))} dl'.$$

1493 This expression coincides with (B.3) in which the integral $\int_{\tau_0}^{\tau_0+\tau} dl'$ is replaced
 1494 by $\int_0^{\tau_0} dl'$. Then,

$$1495 \quad \mathbb{E}\Delta a_s^{(1)}(\tau)\bar{c}_s^{(1)}(\tau) = 2 \sum_{1,2} \delta_{3s}^{12} \delta(\omega_{3s}^{12}) \int_{\tau_0}^{\tau_0+\tau} dl \int_0^{\tau_0} dl' e^{-\gamma_s(2\tau_0+2\tau-l-l')} \\ 1496 \quad \mathbb{E}a_1(l)\bar{a}_1(l') \mathbb{E}a_2(l)\bar{a}_2(l') \mathbb{E}\bar{a}_3(l)a_3(l'),$$

1498 Expressing the correlations $\mathbb{E}a_j(l)\bar{a}_j(l')$ through (2.13), we get

$$1499 \quad \mathbb{E}\Delta a_s^{(1)}(\tau)\bar{c}_s^{(1)}(\tau) = 2 \sum_{1,2} \delta_{3s}^{\prime 12} \delta(\omega_{3s}^{12}) B_{123} \int_0^\tau dl e^{-2\gamma_s(\tau-l)-\gamma_{123s}l}$$

$$1500 \quad \int_0^{\tau_0} dl' e^{-\gamma_{123s}\tau_0} e^{\gamma_s l'} \prod_{j=1,2,3} \left(e^{\gamma_j l'} - e^{-\gamma_j l'} \right).$$

1502 For the integral in the first line, we have

$$1503 \quad T_s := \int_0^\tau dl e^{-2\gamma_s(\tau-l)-\gamma_{123s}l} = \begin{cases} \tau e^{-2\gamma_s\tau} & \text{if } 2\gamma_s = \gamma_{123s} \\ \frac{e^{-2\gamma_s\tau} - e^{-\gamma_{123s}\tau}}{\gamma_{123s} - 2\gamma_s} & \text{elsewhere} \end{cases}. \quad (\text{B.9})$$

1504 For the integral in the second line, let us denote

$$1505 \quad \mathcal{T}^j := \int_0^{\tau_0} dl e^{-\gamma_{123s}\tau_0} e^{\gamma_j l} \prod_{k \neq j} \left(e^{\gamma_k l} - e^{-\gamma_k l} \right), \quad (\text{B.10})$$

1506 where $j, k \in \{1, 2, 3, s\}$. Then,

$$1507 \quad 0 \leq \mathcal{T}^j \leq 1/\gamma_{123s}. \quad (\text{B.11})$$

1508 Due to (B.9) and (B.10) we get

$$1509 \quad \mathbb{E}\Delta a_s^{(1)}(\tau)\bar{c}_s^{(1)}(\tau) = 2 \sum_{1,2} \delta_{3s}^{\prime 12} \delta(\omega_{3s}^{12}) B_{123} T_s \mathcal{T}^s. \quad (\text{B.12})$$

1510 B.5. The Fourth Term of $Q_{s,L}$ in (B.2)

1511 To study the term $2\mathfrak{R}\tilde{N}_s$, as in Sect. B.3, we write \tilde{N}_s as $\tilde{N}_s = \tilde{N}_s^1 + 2\tilde{N}_s^2$, $s \in$
1512 \mathbb{R}^d , where

$$1513 \quad \tilde{N}_s^1 = i \mathbb{E} \left(\bar{a}_s(\tau_0 + \tau) \int_0^\tau e^{-\gamma_s(\tau-l)} \mathcal{Y}_s(a(\tau_0 + l), a(\tau_0 + l), c^{(1)}(l)) dl \right),$$

$$1514 \quad \tilde{N}_s^2 = i \mathbb{E} \left(\bar{a}_s(\tau_0 + \tau) \int_0^\tau e^{-\gamma_s(\tau-l)} \mathcal{Y}_s(c^{(1)}(l), a(\tau_0 + l), a(\tau_0 + l)) dl \right).$$

1516 Term \tilde{N}_s^1 . Writing explicitly the function \mathcal{Y}_s and then $\bar{c}^{(1)}$, we get

$$1517 \quad \tilde{N}_s^1 = L^{-2d} \sum_{1,2} \sum_{1',2'} \delta_{3s}^{\prime 12} \delta_{3'3}^{\prime 1'2'} \delta(\omega_{3s}^{12}) \delta(\omega_{3'3}^{1'2'}) \int_0^\tau dl \int_{-\tau_0}^0 dl' e^{-\gamma_s(\tau-l)} e^{-\gamma_3(l-l')}$$

$$1518 \quad \times \mathbb{E}(a_1(\tau_0 + l)a_2(\tau_0 + l)\bar{a}_{1'}(\tau_0 + l')\bar{a}_{2'}(\tau_0 + l')a_{3'}(\tau_0 + l')\bar{a}_s(\tau_0 + \tau)).$$

1520 Again, this is the same expression as (B.5), with the integration over dl' ranging
1521 from $-\tau_0$ to 0 instead of from 0 to l . Thus, by the Wick theorem, we obtain

$$1522 \quad \tilde{N}_s^1 = 2 \sum_{1,2} \delta_{3s}^{\prime 12} \delta(\omega_{3s}^{12}) \int_0^\tau dl \int_{-\tau_0}^0 dl' e^{-\gamma_s(\tau-l)-\gamma_3(l-l')}$$

$$1523 \quad \times \mathbb{E}a_1(\tau_0 + l)\bar{a}_1(\tau_0 + l') \mathbb{E}a_2(\tau_0 + l)\bar{a}_2(\tau_0 + l') \mathbb{E}a_s(\tau_0 + l')\bar{a}_s(\tau_0 + \tau).$$

1525 Following the line of Sect. B.4, we express the correlations through (2.13) and
 1526 get

$$\begin{aligned}
 \tilde{N}_s^1 &= 2 \sum_{1,2} \delta_{3s}^{\prime 12} \delta(\omega_{3s}^{12}) B_{12s} \int_0^\tau dl e^{-2\gamma_s(\tau-l) - \gamma_{123s}l} \\
 &\int_0^{\tau_0} dl' e^{-\gamma_{123s}\tau_0} e^{\gamma_3 l'} \left(e^{\gamma_s l'} - e^{-\gamma_s l'} \right) \prod_{j=1,2} \left(e^{\gamma_j l'} - e^{-\gamma_j l'} \right) \\
 &= 2 \sum_{1,2} \delta_{3s}^{\prime 12} \delta(\omega_{3s}^{12}) B_{12s} T_s \mathcal{T}^3.
 \end{aligned} \tag{B.13}$$

1531 *Term \tilde{N}_s^2 .* Literally repeating the argument which we have applied to \tilde{N}_s^1 ,
 1532 we find that the term \tilde{N}_s^2 is given by the same expression as (B.7) with the
 1533 integral \int_0^l replaced by $\int_{-\tau_0}^0$:

$$\begin{aligned}
 \tilde{N}_s^2 &= -2 \sum_{1,2} \delta_{3s}^{\prime 12} \delta(\omega_{3s}^{12}) \int_0^\tau dl \int_{-\tau_0}^0 dl' e^{-\gamma_s(\tau-l)} e^{-\gamma_1(l-l')} \\
 &\mathbb{E}a_2(\tau_0 + l) \bar{a}_2(\tau_0 + l') \mathbb{E}a_3(\tau_0 + l') \bar{a}_3(\tau_0 + l) \mathbb{E}a_s(\tau_0 + l') \bar{a}_s(\tau_0 + \tau).
 \end{aligned}$$

1536 Again we get

$$\tilde{N}_s^2 = -2 \sum_{1,2} \delta_{3s}^{\prime 12} \delta(\omega_{3s}^{12}) B_{23s} T_s \mathcal{T}^1. \tag{B.14}$$

1539 **B.6. End of the Proof**

1540 Inserting formulas (B.12), (B.13) and (B.14), as well as (B.4), (B.6), (B.8) in
 1541 (B.2), we get

$$\left| Q_{s,L} - 4L^2 T_s \sum_{1,2} \delta_{3s}^{\prime 12} \delta(\omega_{3s}^{12}) (B_{123} \mathcal{T}^s + B_{12s} \mathcal{T}^3 - 2B_{23s} \mathcal{T}^1) \right| \leq C^\#(s) \tau^2. \tag{B.15}$$

1544 Note that the terms \mathcal{Z}^j defined in (4.14) can be written as

$$\mathcal{Z}^j = \frac{\mathcal{T}^j}{\prod_{k \neq j} (1 - e^{-2\gamma_k \tau_0})}. \tag{B.16}$$

1546 The relations (2.13)-(2.14) imply that for any permutation (k_1, k_2, k_3, k_4) of
 1547 $(1, 2, 3, s)$ we have $B_{k_1 k_2 k_3} = n_{k_1}^{(0)} n_{k_2}^{(0)} n_{k_3}^{(0)} / \prod_{m=k_1, k_2, k_3} (1 - e^{-2\gamma_m \tau_0})$, where
 1548 $n_{k_i}^{(0)} = n_{k_i, L}^{(0)}(\tau_0)$. Together with (B.16), this implies

$$B_{k_1 k_2 k_3} \mathcal{T}^{k_4} = \mathcal{Z}^{k_4} n_{k_1}^{(0)} n_{k_2}^{(0)} n_{k_3}^{(0)}. \tag{B.17}$$

1550 By symmetry, the term $2B_{23s} \mathcal{T}^1$ in (B.15) can be replaced by $B_{23s} \mathcal{T}^1 + B_{13s} \mathcal{T}^2$.
 1551 Then, inserting (B.17) in (B.15) we get

$$\begin{aligned}
 |Q_{s,L} - \mathcal{X}_s| &\leq \left| 4L^2 (T_s - \tau) \sum_{1,2} \delta_{3s}^{\prime 12} \delta(\omega_{3s}^{12}) (T^s B_{123} + T^3 B_{12s} - 2T^s B_{23s}) \right| \\
 &+ C^\#(s) \tau^2,
 \end{aligned}$$

1553 with \mathcal{X}_s defined in (4.33). Finally, we point out that $|(T_s - \tau)T^j| \leq 3\tau^2$, due
 1554 to (B.11) and since $|T_s - \tau| \leq 3\tau^2\gamma_{123s}$. So, the bound

$$1555 \quad \left| L^2(T_s - \tau) \sum_{1,2} \delta_{3s}^{\prime 12} \delta(\omega_{3s}^{12}) (T^s B_{123} + T^3 B_{12s} - 2T^s B_{23s}) \right| \leq C^\#(s)\tau^2,$$

1556 is a consequence of Theorem 3.2. This concludes the proof of Proposition 4.3.

1557 B.7. Proof of Lemma 4.4

1558 Note that $\partial_{s_j} f(\gamma_j) = f'(\gamma_j) \partial_{s_j} \gamma_j$, where $\partial_{s_j} \gamma_j$ (as well as higher-order deriva-
 1559 tives of γ_j) have at most polynomial growth at infinity. Then, using the defini-
 1560 tion (4.14) of \mathcal{Z}^j we find

$$1561 \quad \left| \partial_{\vec{s}}^\mu \mathcal{Z}^j(\tau_0, \vec{s}) \right| = \sum_{n_1+n_2+n_3+n_s=|\mu|} P(\vec{s}; n_1, \dots, n_s) \int_0^{\tau_0} dl (\tau_0 - l)^{n_j} e^{-\gamma_j(\tau_0 - l)} \\ 1562 \quad \prod_{m \neq j} \frac{d^{n_m}}{d\gamma_m^{n_m}} \left(\frac{\sinh(\gamma_m l)}{\sinh(\gamma_m \tau_0)} \right), \quad (\text{B.18})$$

1563 where $P(\vec{s}; a_1, \dots, a_n)$ denotes a function of \vec{s} , dependent on parameters
 1564 (a_1, \dots, a_n) , having at most a polynomial growth at infinity. Using the re-
 1565 lation $\frac{\sinh(\gamma l)}{\sinh(\gamma \tau_0)} = \frac{e^{-\gamma(\tau_0 - l)} - e^{-\gamma(l + \tau_0)}}{1 - e^{-2\gamma\tau_0}}$ we find by induction that

$$1566 \quad \frac{d^n}{d\gamma^n} \left(\frac{\sinh(\gamma l)}{\sinh(\gamma \tau_0)} \right) = \sum_{k+m+p=n} c_{k,m,p} I_{k,m,p}(l, \tau_0, \gamma),$$

1567 where $c_{k,m,p}$ are constants,

$$1568 \quad I_{k,m,p} = \left((\tau_0 - l)^k e^{-\gamma(\tau_0 - l)} - (l + \tau_0)^k e^{-\gamma(l + \tau_0)} \right) \frac{\tau_0^{m+p} e^{-2\gamma m \tau_0}}{(1 - e^{-2\gamma\tau_0})^{m+1}} \\ 1569 \quad (\text{B.19})$$

1570 and $p \neq 0$ only if $m \neq 0$. For $\tau_0 \geq \gamma^{-1}$ the terms $I_{k,m,p}$ are bounded in
 1571 absolute values by absolute constants $C_{k,m,p}$, where we recall that $0 \leq l \leq \tau_0$
 1572 and $\gamma \geq 1$. Let now $\tau_0 \leq \gamma^{-1}$. In this case, since $k + m + p = n$,

$$1573 \quad |I_{k,m,p}| \leq 2 \frac{(l + \tau_0)^k \tau_0^{m+p}}{(1 - e^{-2\gamma\tau_0})^{m+1}} \leq 2^{k+1} \frac{\tau_0^n}{(1 - e^{-2\gamma\tau_0})^{m+1}}.$$

1574 So, in the case $m \leq n - 1$ we have $|I_{k,m,p}| \leq C_{k,m,p}$ uniformly in $\tau_0 \leq \gamma^{-1}$. If
 1575 $m = n$ (so $k = p = 0$) we use another estimate, following from (B.19):

$$1576 \quad |I_{k,m,p}| \leq C \frac{e^{-\gamma(\tau_0 - l)} - e^{-\gamma(l + \tau_0)}}{(1 - e^{-2\gamma\tau_0})^{m+1}} \tau_0^m = C \tau_0^m e^{-\gamma(\tau_0 - l)} \frac{1 - e^{-2\gamma l}}{(1 - e^{-2\gamma\tau_0})^{m+1}} \leq C_{k,m,p},$$

1577 uniformly in $\tau_0 \leq \gamma^{-1}$.

1578 We have seen that the product in (B.18) is bounded uniformly in \vec{s} , l and
 1579 τ_0 , so the integral over l is also bounded uniformly in \vec{s} and τ_0 .

1580 **Appendix C. Proof of Proposition 2.1**

1581 The proof uses the theory of Feynman diagrams, presented in Sect. 2. For
 1582 $N = 0$ the assertion is trivial. For $N \geq 1$ in Proposition 8.7 of [8] it is proven
 1583 that $\mathbb{E}(\mathbf{a}_s^{(m)}(\tau_1)\bar{\mathbf{a}}_s^{(n)}(\tau_2) - a_s^{(m)}(\tau_1)\bar{a}_s^{(n)}(\tau_2))$ equals to

$$1584 \sum_{\mathfrak{F} \in \mathfrak{F}_{m,n}^+ \setminus \mathfrak{F}_{m,n}} c_{\mathfrak{F}} \mathcal{J}_s(\mathfrak{F}) + \sum_{\mathfrak{F} \in \tilde{\mathfrak{F}}_{m,n}} c_{\mathfrak{F}} \mathcal{J}_s^2(\mathfrak{F}), \tag{C.1}$$

1585 where $\mathfrak{F}_{m,n}^+$ is a certain (finite) set of *extended* Feynman diagrams,¹³ $c_{\mathfrak{F}}$ is a
 1586 complex number of unit norm and $\mathcal{J}_s(\mathfrak{F})$, $\mathcal{J}_s^2(\mathfrak{F})$ are sums, similar to (2.19).
 1587 In Section 8.6.3 of [8] are established the following bounds for these sums:

$$1588 |\mathcal{J}_s^2(\mathfrak{F})| \leq C^\#(s)L^{-Nd} \sum_{\substack{z \in \mathcal{Z}^+(\mathfrak{F}): \\ z_j = 0 \text{ for some } j, \\ \omega_k^{\mathfrak{F}}(z) = 0 \forall 1 \leq k \leq N}} C^\#(z), \tag{C.2}$$

1589 where

$$1590 \mathcal{Z}^+(\mathfrak{F}) = \left\{ z \in (\mathbb{Z}_L^d)^N : z_k \neq 0 \Leftrightarrow \sum_{i=1}^N \alpha_{ki}^{\mathfrak{F}} z_i \neq 0 \quad \forall 1 \leq k \leq N \right\}$$

1591 while the quadratic forms $\omega_k^{\mathfrak{F}}$ and the skew-symmetric matrix $\alpha^{\mathfrak{F}}$ are defined
 1592 in Sect. 2.4. Note that possibly the diagram \mathfrak{F} does not belong to the set $\mathfrak{F}_{m,n}^{true}$,
 1593 so that the matrix $\alpha^{\mathfrak{F}}$ may have zero columns and lines. On the other hand,

$$1594 |\mathcal{J}_s(\mathfrak{F})| \leq C^\#(s)L^{-Nd} \sum_{\substack{z \in \tilde{\mathcal{Z}}^+(\mathfrak{F}): \\ \tilde{\omega}_k^{\mathfrak{F}}(z) = 0 \forall 1 \leq k \leq \tilde{N}}} C^\#(z). \tag{C.3}$$

1595 Here $\tilde{N} = \tilde{N}(\mathfrak{F}) < N$, quadratic forms $\tilde{\omega}_k^{\mathfrak{F}}(z)$ are defined by relations (2.17),
 1596 where N is replaced by \tilde{N} and the matrix $(\alpha_{ij}^{\mathfrak{F}})$ – by a certain $\tilde{N} \times \tilde{N}$ -matrix
 1597 $(\tilde{\alpha}_{ij}^{\mathfrak{F}})$, also satisfying $\tilde{\alpha}_{ij}^{\mathfrak{F}} = -\tilde{\alpha}_{ji}^{\mathfrak{F}} \in \{0, \pm 1\}$ for all i, j . Accordingly the set
 1598 $\tilde{\mathcal{Z}}^+(\mathfrak{F}) \subset (\mathbb{Z}_L^d)^{\tilde{N}}$ is defined as $\mathcal{Z}^+(\mathfrak{F})$ above, but with N and $\alpha_{ij}^{\mathfrak{F}}$ replaced by
 1599 \tilde{N} and $\tilde{\alpha}_{ij}^{\mathfrak{F}}$.

1600 We first show that the term $\mathcal{J}_s^2(\mathfrak{F})$ is bounded by the r.h.s. of (2.3). To
 1601 this end we write $\mathcal{Z}^+(\mathfrak{F}) = \cup_{\mathcal{K}} \mathcal{Z}_{\mathcal{K}}$, where the union is taken over all subsets
 1602 $\mathcal{K} \subset \{1, \dots, N\}$ and

$$1603 \mathcal{Z}_{\mathcal{K}}(\mathfrak{F}) = \left\{ z : z_k = \sum_{i=1}^N \alpha_{ki}^{\mathfrak{F}} z_i = 0 \forall k \in \mathcal{K} \text{ and } z_k \neq 0, \sum_{i=1}^N \alpha_{ki}^{\mathfrak{F}} z_i \neq 0 \forall k \notin \mathcal{K} \right\}.$$

1604 Then the r.h.s. of (C.2) takes the form

$$1605 C^\#(s)L^{-Nd} \sum_{\mathcal{K} \neq \emptyset} \sum_{\substack{z \in \mathcal{Z}_{\mathcal{K}}(\mathfrak{F}): \\ \omega_k^{\mathfrak{F}}(z) = 0 \forall 1 \leq k \leq N}} C^\#(z). \tag{C.4}$$

¹³These diagrams are defined similarly to the Feynman diagrams from Sect. 2.3.1, but now we allow to couple leaves not only from different blocks but also from the same block.

1606 Note that on the set $\mathcal{Z}_{\mathcal{K}}(\mathfrak{F})$ we have $\omega_k(z) = 0$ for all $k \in \mathcal{K}$ and $\omega_k(z) =$
 1607 $2z_k \cdot \sum_{i \notin \mathcal{K}} \alpha_{ki}^{\mathfrak{F}} z_i$ for $k \notin \mathcal{K}$. Thus, the sum over z in (C.4) takes the form of
 1608 the sum in (3.2), where $z = (z_j)_{j \notin \mathcal{K}}$ and N is replaced by

$$1609 \quad N - \varkappa, \quad \varkappa = \#\mathcal{K}.$$

1610 We recall that $N \leq 4$ and $\mathcal{K} \neq \emptyset$, so that $N - \varkappa$ takes values 0, 1, 2 or 3.
 1611 For the sets \mathcal{K} satisfying $N - \varkappa = 0$ we have $\mathcal{Z}_{\mathcal{K}}(\mathfrak{F}) = \{0\}$, so the sum (C.3)
 1612 is bounded by $C^{\#}(s)L^{-Nd}$. Since the matrix $\alpha^{\mathfrak{F}}$ is skew-symmetric, then in
 1613 the case $N - \varkappa = 1$ we have $\mathcal{Z}_{\mathcal{K}}(\mathfrak{F}) = \emptyset$, so the sum (C.3) vanishes. When
 1614 $N - \varkappa = 2$ or 3 we apply Theorem 3.2 and see that the sum over z in (C.4) is
 1615 bounded by $CL^{-(N-\varkappa)(1-d)}$. So

$$1616 \quad |\mathcal{J}_s^2(\mathfrak{F})| \leq C^{\#}(s)L^{-Nd} \sum_{\mathcal{K} \neq \emptyset} L^{-(N-\varkappa)(1-d)} = C^{\#}(s) \sum_{\mathcal{K} \neq \emptyset} L^{-N+\varkappa(1-d)} \\ \leq C_1^{\#}(s)L^{-N+1-d}.$$

1617 Same argument implies that the r.h.s. of (C.3) also is bounded by the
 1618 quantity $C^{\#}(s)L^{-N+1-d}$ (note that decomposing the r.h.s. of (C.3) as in (C.4)
 1619 we get a new term with $\mathcal{K} = \emptyset$, but for it $N - \varkappa = N \leq 4$ and Theorem 3.2
 1620 still applies).

1621 Appendix D. Case $d = 2$

1622 A difference between the cases $d \geq 3$ and $d = 2$ comes from Theorem 3.1
 1623 since in the asymptotic, given by the latter, an additional log-factor appears
 1624 when $d = 2$. To handle it we redefine the sum in (3.2), defining $S_{L,N}(\Phi)$, by
 1625 multiplying it by $(\ln L)^{-N/2}$. So when $d = 2$ $S_{L,N}$ takes the form

$$1626 \quad S_{L,N}(\Phi) := \frac{L^{N(1-d)}}{(\ln L)^{N/2}} \sum_{z \in \mathcal{Z}: \omega_j(z)=0 \forall j} \Phi(z). \quad (\text{D.1})$$

1627 Accordingly the ($d = 2$)-analogy of (3.5) reads

$$1628 \quad \left| S_{L,2}(\Phi) - \frac{L^{2(1-d)}}{\ln L} \sum_{z \in \mathbb{Z}_L^{2d}: z_1 \cdot z_2 = 0} \Phi(z) \right| \leq \frac{CL^{2-d}}{\ln L} \|\Phi\|_{0,d+1} = \frac{C}{\ln L} \|\Phi\|_{0,3}. \\ 1629 \quad (\text{D.2})$$

1630 This approximation, jointly with a modification of the Heath–Brown result
 1631 from [14], given in Theorem 1.4 of [10], implies the following version of Theorem
 1632 3.1 for $d = 2$:

1633 **Theorem D.1.** *Let $d = 2$. Then there exist constants $N_1, N_2 > 4$ such that if*
 1634 $\|\Phi\|_{N_1, N_2} < \infty$,

$$1635 \quad \left| S_{L,2}(\Phi) - C_2 \int_{\Sigma_0} \Phi(z) \mu^{\Sigma_0}(dz_1 dz_2) \right| \leq K_2 \frac{\|\Phi\|_{N_1, N_2}}{\ln L}, \quad (\text{D.3})$$

1636 where $C_2 > 0$ is a number-theoretical constant and $K_2 > 0$.

1637 Note that estimate (3.8) stays true when $d = 2$.

1638 **Theorem D.2.** *In the case $d = 2$ assertion of Theorem 3.2 remains true, if the*
 1639 *sum $S_{L,N}$ is defined as in (D.1) and N_2 is the constant from Theorem D.1.*

1640 *Proof.* The only difference with the proof of Theorem 3.2 comes from estimate
 1641 (3.18) since the latter is obtained by applying Theorem 3.1, and in the case
 1642 $d = 2$ we should apply Theorem D.1 instead. Namely, now the r.h.s. of (3.18)
 1643 takes the form $CL^{2(d-1)} \ln L [R^{2d} + R^{N_2} (\ln L)^{-1}] \leq C' R^{N_2} L^{2(d-1)} \ln L$. Since
 1644 Lemma 3.6 remains unchanged, then for $d = 2$ the r.h.s. of estimate (3.20),
 1645 which holds for irreducible matrices α , should be multiplied by $\ln L$. In the case
 1646 of reducible matrix α we apply the latter estimate to each irreducible block,
 1647 which gives the factor $(\ln L)^{\lfloor N/2 \rfloor}$ in the r.h.s. of (3.12), since the number of
 1648 blocks does not exceed $\lfloor N/2 \rfloor$. However, the final estimate of Theorem 3.2
 1649 remains unchanged because of the factor $(\ln L)^{-N/2}$ in the definition (D.1) of
 1650 the sum $S_{L,N}$. □

1651 Since in the case $d = 2$ we choose $\rho = \varepsilon L / \sqrt{\ln L}$, then the terms $n_{s,L}^{(k)}$ are
 1652 given by formula (4.4), multiplied by $(\ln L)^{-k/2}$. The proof of Proposition 2.1
 1653 is analogous to that presented in Appendix C for $d = 2$. The only difference
 1654 being the use of Theorem D.2 in place of Theorem 3.2. Lemma 2.2 remains
 1655 unchanged, so the correlations $L^k (\ln L)^{-k/2} \mathbb{E} a_s^{(m)} \bar{a}_s^{(n)}(\tau_2)$, $m + n = k$, are
 1656 given by formula (2.18), multiplied by $(\ln L)^{-k/2}$ (recall that the sum of these
 1657 correlations makes $n_{s,L}^{(k)}$). We see that the correlations take the form (D.1), so
 1658 Theorems D.1 and D.2 apply to study them.

1659 The rest of the proof of Theorem 5.8 literally repeats that for the case
 1660 $d \geq 3$, except the appearance of the $(\ln L)^{-k/2}$ factors, coming from the new
 1661 definition of ρ . Now the estimates, using Theorem 3.1, should be relaxed since
 1662 the estimate provided by Theorem D.1 is slightly weaker than that of Theo-
 1663 rem 3.1. In particular, in the r.h.s. of (4.8), (4.11) and (4.12) the factor $L^{-1/2}$
 1664 should be replaced by $(\ln L)^{-1}$. This results in the stronger lower bound for L
 1665 in Theorem 5.8: now it is $L \geq e^{\varepsilon^{-1}}$ instead of $L \geq \varepsilon^{-2}$ (see Theorem A).

1666 Theorem 5.9, as Theorem 5.8, remains unchanged, except the lower bound
 1667 for L which is modified as above. Indeed, the theorem follows from Theorem 5.8
 1668 and Proposition 2.1, and the term $\chi_2(L)^{-N+1} = (\ln L)^{(N-1)/2}$, appearing in
 1669 estimate (2.3) for $d = 2$ does not change the assertion of Theorem 5.8.

1670 **D.1. Discussion of Remark 1.2**

1671 In fact, Theorem 1.4 from [10] provides more delicate information about $S_{L,2}$
 1672 than what is stated in Theorem D.1. Namely, if $d = 2$ then due to [10],

$$1673 \left| \frac{L^{2(1-d)}}{\ln L} \sum_{z: z_1, z_2=0} \Phi(z) - C_2 \int_{\Sigma_0} \Phi(z) \mu^{\Sigma_0}(dz_1 dz_2) - \frac{\sigma_1^\Phi(L)}{\ln L} \right| \leq C \frac{\|\Phi\|_{N_1, N_2}}{L^{1/6}},$$

1674 where σ_1^Φ is a certain function satisfying $|\sigma_1^\Phi(L)| \leq C_1 \|\Phi\|_{N_1, N_2}$, uniformly in
 1675 L . See [10] for an explicit (but complicated) formula for σ_1^Φ . Consequently,

$$1676 \left| S_{L,2}(\Phi) - C_2 \int_{\Sigma_0} \Phi(z) \mu^{\Sigma_0}(dz_1 dz_2) - \frac{\tilde{\sigma}_1^\Phi(L)}{\ln L} \right| \leq C \frac{\|\Phi\|_{N_1, N_2}}{L^{1/6}},$$

1677 where

$$1678 \quad \tilde{\sigma}_1^\Phi(L) := \sigma_1^\Phi(L) - L^{2(1-d)} \sum_{z: z_1=0 \text{ or } z_2=0} \Phi(z)$$

1679 still satisfies $|\tilde{\sigma}_1^\Phi(L)| \leq C\|\Phi\|_{N_1, N_2}$ in view of (D.2). Then estimate (4.12)
1680 refines as

$$1681 \quad \left| n_s^{\leq 2} - n_{s,L} - \frac{f(\tau, L)}{\ln L} \right| \leq C^\#(s)\varepsilon^2(L^{-1/6} + \varepsilon), \quad (\text{D.4})$$

1682 where $f(\tau, L) := \tilde{\sigma}_1^{\Phi(\tau)}(L)$ and $\Phi(\tau)$ is the function satisfying $n_{s,L}^{\leq 2}(\tau) =$
1683 $S_{L,2}(\Phi(\tau))$ that comes from Corollary 2.3. By (2.22) and the estimate for $\tilde{\sigma}_1^\Phi$
1684 above, the function $f(\tau, L)$ is bounded uniformly in τ . The rest of the proofs
1685 of Theorems 5.8 and 5.9 remain unchanged while the estimate (D.4) leads to
1686 the assertion of the remark.

1687 References

- 1688 [1] Buckmaster, T., Germain, P., Hani, Z., Shatah, J.: Effective dynamics of the
1689 nonlinear Schrödinger equation on large domains. *Comm. Pure Appl. Math.* **71**,
1690 1407–1460 (2018)
- 1691 [2] Buckmaster, T., Germain, P., Hani, Z., Shatah, J.: Onset of the wave turbulence
1692 description of the longtime behaviour of the nonlinear Schrödinger equation.
1693 *Invent. Math.* **225**, 787–855 (2021)
- 1694 [3] Collot, C., Germain, P.: On the derivation of the homogeneous kinetic wave
1695 equation, (2019), [arXiv:1912.10368](https://arxiv.org/abs/1912.10368)
- 1696 [4] Collot, C., Germain, P.: Derivation of the homogeneous kinetic wave equation:
1697 longer time scales, (2020), [arXiv:2007.03508](https://arxiv.org/abs/2007.03508)
- 1698 [5] Deng, Y., Hani, Z.: On the derivation of the wave kinetic equation for NLS,
1699 (2019). *Forum Math., Pi* **9**, 1–37 (2021)
- 1700 [6] Deng, Y., Hani, Z.: Full derivation of the wave kinetic equation. *Invent. Math.*
1701 **233**, 543–724 (2023)
- 1702 [7] Dymov, A., Kuksin, S.: Formal expansions in stochastic model for wave turbu-
1703 lence 1: kinetic limit. *Comm. Math. Phys.* **382**, 951–1014 (2021)
- 1704 [8] Dymov, A., Kuksin, S.: Formal expansions in stochastic model for wave turbu-
1705 lence 2: method of diagram decomposition. *J. Stat. Phys.* **190**, 3 (2023)
- 1706 [9] Dymov, A., Kuksin, S.: On the Zakharov-L’vov stochastic model for wave tur-
1707 bulence. *Dokl. Math.* **101**, 102–109 (2020)
- 1708 [10] Dymov, A., Kuksin, S., Maiocchi, A., Vlăduț, S.: A refinement of Heath-Brown’s
1709 theorem on quadratic forms. *Sbornik: Mathematics* (2023)
- 1710 [11] Erdős, L., Yau, H.-T.: Linear Boltzmann equation as the weak coupling limit of a
1711 random Schrödinger equation. *Commun. Pure Appl. Math.* **53**, 667–735 (2000)
- 1712 [12] Erdős, L., Schlein, B., Yau, H.-T.: Derivation of the cubic non-linear Schrödinger
1713 equation from quantum dynamics of many-body systems. *Invent. Math.* **167**,
1714 515–614 (2007)
- 1715 [13] Fulton, W.: *Intersection Theory*. Springer, Berlin (1998)
- 1716 [14] Heath-Brown, D.R.: A new form of the circle method, and its application to
1717 quadratic forms. *J. Reine Angew. Math.* **481**, 149–206 (1996)

- 1718 [15] Faou, E.: Linearized wave turbulence convergence results for three-wave systems.
1719 Comm. Math. Physics **378**, 807–849 (2020)
- 1720 [16] Faou, E., Germain, P., Hani, Z.: The weakly nonlinear large-box limit of the 2D
1721 cubic nonlinear Schrödinger equation. J. Amer. Math. Soc. **29**, 915–982 (2016)
- 1722 [17] Huang, G., Kuksin, S., Maiocchi, A.: Time-averaging for weakly nonlinear CGL
1723 equations with arbitrary potentials. Fields Inst. Commun. **75**, 323–349 (2015)
- 1724 [18] Janson, S.: Gaussian Hilbert Spaces. Cambridge University Press, Cambridge
1725 (1997)
- 1726 [19] Kuksin, S.: Ergodicity, mixing and KAM, Séminaire Laurent Schwartz EDP et
1727 applications, 2018-2019, Exp. No. 8, 9 (2019); [https://doi.org/10.5802/slsedp.](https://doi.org/10.5802/slsedp.128)
1728 [128](https://doi.org/10.5802/slsedp.128)
- 1729 [20] Kuksin, S., Maiocchi, A.: Derivation of a wave kinetic equation from the
1730 resonant-averaged stochastic NLS equation. Physica D **309**, 65–70 (2015)
- 1731 [21] Kuksin, S., Maiocchi, A.: Resonant averaging for small solutions of stochastic
1732 NLS equations. Proc. Royal Soc. Edinburgh **147A**, 1–38 (2017)
- 1733 [22] Lachaud, G., Rolland, R.: On the number of points of algebraic sets over finite
1734 fields. J. Pure Appl. Algebra **219**, 5117–5136 (2015)
- 1735 [23] Lukkarinen, J., Spohn, H.: Weakly nonlinear Schrödinger equation with random
1736 initial data. Invent. Math. **183**, 79–188 (2015)
- 1737 [24] Lang, S.: Algebra. Springer, New York (2002)
- 1738 [25] Nazarenko, S.: Wave Turbulence. Springer, Heidelberg (2011)
- 1739 [26] Newell, A.C., Rumpf, B.: Wave Turbulence. Annu. Rev. Fluid Mech. **43**, 59–78
1740 (2011)
- 1741 [27] Shafarevitch, I.: Basic algebraic geometry. Springer, Berlin (1994)
- 1742 [28] Shirikyan, A.: Ergodicity for a class of Markov processes and applications to
1743 randomly forced PDE's II. Discrete Contin. Dynam. Syst., Ser. A **6**, 911–926
1744 (2006)
- 1745 [29] Zakharov, V., L'vov, V.: Statistical description of nonlinear wave fields. Radio-
1746 phys. Quan. Electron. **18**, 1084–1097 (1975)
- 1747 [30] Zakharov, V., L'vov, V., Falkovich, G.: Kolmogorov Spectra of Turbulence.
1748 Springer, Heidelberg (1992)

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1790 Communicated by Claude-Alain Pillet.

1791 Received: January 13, 2022.

1792 Accepted: August 28, 2023.