HAL open science

The Large-Period Limit for Equations of Discrete Turbulence

Andrey Dymov, Sergei Kuksin, Alberto Maiocchi, Sergei Vlăduţ

▶ **To cite this version:**

Andrey Dymov, Sergei Kuksin, Alberto Maiocchi, Sergei Vlăduţ. The Large-Period Limit for Equations of Discrete Turbulence. Annales Henri Poincaré, 2023, 24 (11), pp.3685-3739. 10.1007/s00023-023-01366-2 . hal-04463211

HAL Id: hal-04463211

https://hal.science/hal-04463211v1

Submitted on 16 Feb 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Ann. Henri Poincaré *Online First*
© 2023 Springer Nature Switzerland AG
https://doi.org/10.1007/s00023-023-01366-2

Annales Henri Poincaré

The Large-Period Limit for Equations of Discrete Turbulence

Andrey Dymov, Sergei Kuksin, Alberto Maiocchi and Sergei Vlăduţ

Abstract. We consider the damped/driven cubic NLS equation on the torus of a large period L with a small nonlinearity of size λ, a properly scaled random forcing and dissipation. We examine its solutions under the subsequent limit when first $\lambda \to 0$ and then $L \to \infty$. The first limit, called the limit of discrete turbulence, is known to exist, and in this work we study the second limit $L \to \infty$ for solutions to the equations of discrete turbulence. Namely, we decompose the solutions to formal series in amplitude and study the second-order truncation of this series. We prove that the energy spectrum of the truncated solutions becomes close to solutions of a damped/driven nonlinear wave kinetic equation. Kinetic nonlinearity of the latter is similar to that which usually appears in works on wave turbulence, but is different from it (in particular, it is non-autonomous). Apart from tools from analysis and stochastic analysis, our work uses two powerful results from the number theory.

1. Introduction

1.1. The Setting

In this paper we continue the study of the Zakharov–L'vov stochastic model for wave turbulence (WT), initiated in [7,8]; see also a survey [9]. We start by recalling the classical and the Zakharov–L'vov stochastic settings of WT. See the introduction to [7] for more detailed discussions of the two models.

Classical setting. Let $\mathbb{T}^d_L = \mathbb{R}^d/(L\mathbb{Z}^d)$ be the d-dimensional torus, $d \geq 2$, of period $L \geq 2$. We denote by $\|u\|$ the normalized L_2-norm of a complex function u on $\mathbb{T}^d_L$, $\|u\|^2 = L^{-d} \int_{\mathbb{T}^d_L} |u(x)|^2 \, \mathrm{d}x$, and write the Fourier series of u in the form

Revised Proof

$$u(x) = L^{-d/2} \sum_{s\in\mathbb{Z}^d_L} v_s e^{2\pi i s\cdot x}, \qquad \mathbb{Z}^d_L = L^{-1}\mathbb{Z}^d. \tag{1.1}$$

Here the vector of Fourier coefficients $v = (v_s)_{s\in\mathbb{Z}^d_L}$ is given by the Fourier transform of $u(x)$,

$$v = \mathcal{F}(u), \quad v_s = L^{-d/2} \int_{\mathbb{T}^d_L} u(x) e^{-2\pi i s\cdot x}\,\mathrm{d}x \quad \text{for } s \in \mathbb{Z}^d_L,$$

so the Parseval identity takes form $\|u\|^2 = L^{-d}\sum_{s\in\mathbb{Z}^d_L}|v_s|^2$. We will study solutions $u(t,x)$ whose norms satisfy $\|u(t,\cdot)\| \sim 1$ as $L \to \infty$. This makes the chosen in (1.1) scaling of Fourier series convenient for our purposes.

We consider the cubic NLS equation with modified nonlinearity

$$\frac{\partial}{\partial t}u + i\Delta u - i\lambda\left(|u|^2 - 2\|u\|^2\right)u = 0, \qquad x \in \mathbb{T}^d_L, \tag{1.2}$$

where $u = u(t,x)$, $\Delta = (2\pi)^{-2}\sum_{j=1}^d(\partial^2/\partial x_j^2)$ and $\lambda \in (0,1]$ is a small parameter. The modification of the nonlinearity by the term $2i\lambda\|u\|^2u$ keeps the main features of the standard cubic NLS equation, reducing some non-crucial technicalities; see the introduction to [7].

The objective of WT is to study solutions of (1.2) under the limit $L \to \infty$ and $\lambda \to 0$ on long time intervals. There are plenty of physical works containing some different (but consistent) approaches to the limit; many references may be found in [25,26,30]. Despite the strong interest in physical and mathematical communities to the addressed questions, significant progress in the rigorous justification of the physical predictions was achieved only recently [1–6,15,16,23]. See, for example, the introductions to [3,6,7] for discussions of the obtained results.

Zakharov–L'vov setting. When studying Eq. (1.2), members of the WT community talk about "pumping energy to low modes and dissipating it in high modes". To make this rigorous, following Zakharov–L'vov [29], in the present paper as well as in [7,8] we consider the NLS equation (1.2) damped by a (hyper) viscosity and driven by a random force:

$$\frac{\partial}{\partial t}u + i\Delta u - i\lambda\left(|u|^2 - 2\|u\|^2\right)u = -\nu\mathfrak{A}(u) + \sqrt{\nu}\frac{\partial}{\partial t}\eta^\omega(t,x). \tag{1.3}$$

Here $\nu \in (0,1/2]$ is another small parameter, which should be properly agreed with λ and L. The dissipative linear operator $\mathfrak{A}$ is defined as

$$\mathfrak{A}(u(x)) = L^{-d/2}\sum_{s\in\mathbb{Z}^d_L}\gamma_s v_s e^{2\pi i s\cdot x}, \quad v = \mathcal{F}(u), \;\; \gamma_s = \gamma^0(|s|^2), \tag{1.4}$$

where $|s|$ stands for the Euclidean norm of a vector s and $\gamma^0(y)$ is a smooth real increasing function of $y > 0$, satisfying [1]

$$\gamma^0 \geq 1 \;\text{ and }\; c(1+y)^{r_*} \leq \gamma^0(y) \leq C(1+y)^{r_*} \qquad \forall\, y > 0. \tag{1.5}$$

[1] For example, if $\gamma_s = (1+|s|^2)^{r*}$, then $\mathfrak{A} = (1-\Delta)^{r*}$. In particular, we can take $\mathfrak{A} = 1-\Delta$.

Reviewed Proof

The exponents $r_* > 0$ and c, C are positive constants. We also assume that

all derivatives of γ^0 have at most polynomial growths at infinity.

The random noise η^ω is given by a Fourier series

$$\eta^\omega(t,x) = L^{-d/2} \sum_{s\in\mathbb{Z}_L^d} b(s)\beta_s^\omega(t)e^{2\pi i s\cdot x},$$

where $\{\beta_s(t), s \in \mathbb{Z}_L^d\}$ are standard independent complex Wiener processes [2] and $b(s)$ is a Schwartz function on $\mathbb{R}^d \supset \mathbb{Z}_L^d$.[3]

Solutions $u(\tau)$ of (1.3) are random processes in the space $H = L_2(\mathbb{T}_L^d, \mathbb{C})$, equipped with the norm $\|\cdot\|$. If r_* is sufficiently big in terms of d, Eq. 1.3 is known to be well posed, see Theorem 1.1 below and a discussion after its formulation. Moreover, Ito's formula shows that $\mathbb{E}\|u(\tau)\|^2$ is bounded uniformly in τ and L, ν, λ, once $\mathbb{E}\|u(0)\|^2$ is bounded uniformly in these parameters, see in [7].

We will study the equation on time intervals of order ν^{-1}. So, it is convenient to pass from t to the slow time $\tau = \nu t$ and write Eq. (1.3) as

$$\begin{aligned} \dot u + i\nu^{-1}\Delta u - i\rho\left(|u|^2 - 2\|u\|^2\right)u &= -\mathfrak{A}(u) + \dot\eta^\omega(\tau,x), \\ \eta^\omega(\tau,x) &= L^{-d/2}\sum_{s\in\mathbb{Z}_L^d} b(s)\beta_s^\omega(\tau)e^{2\pi i s\cdot x}. \end{aligned} \tag{1.6}$$

Here $\rho = \lambda\nu^{-1}$, the upper dot stands for $d/\mathrm{d}\tau$ and $\{\beta_s(\tau),\ s \in \mathbb{Z}_L^d\}$ is another set of standard independent complex Wiener processes. Below we use ρ, ν and L as parameters of the equation.

In the context of Eq. (1.6), the objective of WT is to study its solutions $u(\tau)$ when

$$L \to \infty \quad \text{and} \quad \nu \to 0, \tag{1.7}$$

while $\rho = \rho(\nu, L)$ is scaled appropriately, mostly paying attention to their *energy spectra*

$$N_s(\tau) := \mathbb{E}|v_s(\tau)|^2, \quad \text{where} \quad v(\tau) = \mathcal{F}(u(\tau)). \tag{1.8}$$

Exact meaning of the limit (1.7) is unclear since no relation between the parameters ν and L is postulated by the theory.

Motivated by physical works, in the present paper, as in [7,8], we study formal decompositions in ρ of solutions to Eq. (1.6) and of their energy spectra N_s under the limit (1.7). See the introduction to [7] for a discussion of our motivation, and see below Sect. 4. In [7,8] we understand the limit (1.7) as

$$\text{first } L \to \infty \text{ and then } \nu \to 0, \text{ or } L \gg \nu^{-2} \text{ while } \nu \to 0. \tag{1.9}$$

There we have shown that principal terms of the decomposition of N_s in ρ have a non-trivial limiting behaviour, provided that ρ is scaled as $\rho \sim \nu^{-1/2}$,

[2] i.e. $\beta_s = \beta_s^1 + i\beta_s^2$, where $\{\beta_s^j, s \in \mathbb{Z}_L^d, j = 1, 2\}$ are standard independent real Wiener processes.

[3] Often it is assumed that the intensity $b(s)$ of the noise η^ω is non-negative, but we do not impose this condition. Note that if $b(s) \equiv 0$, then our results become trivial since below we will provide (1.3) with the zero initial conditions.

Revised Proof

Journal: 23 Article No.: 1366 TYPESET DISK LE CP Disp.:2023/9/13 Pages: 55

governed by a nonlinear *wave kinetic equation* (WKE) with added dissipation and a constant forcing. The WKE coincides with that, arising in physical works, so this result agrees well with the predictions of the WT.

In the present paper we are interested in the opposite order of limits, which rarely appears in physical works:

$$\text{firstly} \quad \nu \to 0 \quad \text{and then} \quad L \to \infty. \tag{1.10}$$

Roughly speaking, our main result is that under the double limit (1.10) the behaviour of principal terms of the decomposition in ρ for the energy spectrum N_s is governed by a *modified* WKE. The latter is similar to the WKE arising in [7,8] and physical papers, but is different from them. The scaling of ρ now is $\rho \sim L\,\chi_d(L)$, where

$$\chi_d(L) \equiv 1 \quad \text{if} \quad d \geq 3 \quad \text{and} \quad \chi_d(L) = (\ln L)^{-1/2} \quad \text{if} \quad d = 2. \tag{1.11}$$

To the best of our knowledge, this WKE did not appear in the literature before. For the proof we start with the result obtained in [17,21], where the limiting as $\nu \to 0$ behaviour of Eq. (1.6) is examined (while L and ρ are kept fixed). Then we pass to the limit as $L \to \infty$, following the approach of [7,8] and using the developed there tools, such as a specific Feynman diagram presentation. Another key ingredient of the proof is an obtained in [10] refinement of the Heath–Brown circle method for quadratic forms [14], and certain upper bounds for the number of integer points on intersections of quadrics. In the next subsections we describe our results and methods in more detail.

In [20] a similar result concerning the iterated limit (1.10) was found heuristically; however, there ρ was scaled as $\rho \sim \sqrt{L}$. The present paper shows that the correct scaling is different: $\rho \sim L\,\chi_d(L)$.

Similar regimes, when $L \to \infty$ slowly while $\nu > 0$ fast decays to zero, were studied in [1,16]. However the elegant description of the limit, obtained there, is far from the prediction of WT. The works [1,16] should rather be regarded as a kind of averaging (similar to that of Krylov–Bogolyubov) since the considered there time scale is much shorter than the characteristic time scale of WT. Note that in [1] a similar to ours [10] refinement of the Heath–Brown method also is crucially used.

1.2. The Limit of Discrete Turbulence

We first consider the limit

$$\nu \to 0 \quad \text{while } L \text{ and } \rho \text{ stay fixed.} \tag{1.12}$$

It is known as *the limit of discrete turbulence* (see [25, Section 10]) and has been successfully studied in [17,21]. To explain the result, let us take the Fourier transform of Eq. (1.6):

$$\dot{v}_s - i\nu^{-1}|s|^2 v_s + \gamma_s v_s = i\rho L^{-d}\left(\sum_{1,2,3} \delta'^{12}_{3s} v_1 v_2 \bar{v}_3 - |v_s|^2 v_s\right) + b(s)\dot{\beta}_s \tag{1.13}$$

Revised Proof

for $s \in \mathbb{Z}_L^d$. Here, as it is common in WT, v_j abbreviates v_{s_j}, $\sum_{1,2,3}$ stands for $\sum_{s_1,s_2,s_3\in\mathbb{Z}_L^d}$, and

$$\delta'^{12}_{3s} = \delta'^{s_1 s_2}_{\ \ s_3 s} := \begin{cases} 1, & \text{if } s_1+s_2=s_3+s \text{ and } \{s_1,s_2\}\neq\{s_3,s\}, \\ 0, & \text{otherwise.} \end{cases} \tag{1.14}$$

Note that

$$\text{if } \delta'^{12}_{3s} = 1, \quad \text{then} \quad \{s_1,s_2\}\cap\{s_3,s\} = \varnothing. \tag{1.15}$$

We pass to the *interaction representation*,

$$\mathfrak{a}_s(\tau) = v_s(\tau)e^{-i\nu^{-1}\tau|s|^2}, \quad s\in\mathbb{Z}_L^d, \tag{1.16}$$

and denote

$$\omega^{12}_{3s} = \omega^{s_1 s_2}_{s_3 s} := |s_1|^2+|s_2|^2-|s_3|^2-|s|^2 = -2(s_1-s)\cdot(s_2-s), \tag{1.17}$$

where the last equality holds if $\delta'^{12}_{3s}=1$ since then $s_3=s_1+s_2-s$. Then Eq. (1.13) takes the form

$$\begin{aligned} &\dot{\mathfrak{a}}_s + \gamma_s\mathfrak{a}_s = i\rho Y_s(\mathfrak{a},\nu^{-1}\tau) + b(s)\dot{\beta}_s, \qquad s\in\mathbb{Z}_L^d, \\ &Y_s(\mathfrak{a},t) = L^{-d}\left(\sum_{1,2,3}\delta'^{12}_{3s}\mathfrak{a}_1\mathfrak{a}_2\bar{\mathfrak{a}}_3 e^{it\omega^{12}_{3s}} - |\mathfrak{a}_s|^2\mathfrak{a}_s\right), \end{aligned} \tag{1.18}$$

where $\{\beta_s\}$ is yet another set of standard independent complex Wiener processes (and, again $\mathfrak{a}_j$ stands for $\mathfrak{a}_{s_j}$). Note that the energy spectra of solutions to Eqs. (1.13) and (1.18) coincide:

$$N_s(\tau) = \mathbb{E}|v_s(\tau)|^2 = \mathbb{E}|\mathfrak{a}_s(\tau)|^2. \tag{1.19}$$

Sometimes we will write N_s and $\mathfrak{a}_s$ as $N_s(\tau;\nu,L)$ and $\mathfrak{a}_s(\tau;\nu,L)$. The limiting dynamics in Eq. (1.18) under the limit (1.12) is governed by the *effective equation* of discrete turbulence. The latter has the form (1.18) with the modified nonlinearity Y^{res}, in which the sum is taken only over resonant vectors s_1,s_2,s_3:

$$\begin{aligned} &\dot{\mathfrak{a}}_s + \gamma_s\mathfrak{a}_s = i\rho Y^{res}_s(\mathfrak{a}) + b(s)\dot{\beta}_s, \quad s\in\mathbb{Z}_L^d, \\ &Y^{res}_s(\mathfrak{a}) = L^{-d}\left(\sum_{1,2,3}\delta'^{12}_{3s}\delta(\omega^{12}_{3s})\mathfrak{a}_1\mathfrak{a}_2\bar{\mathfrak{a}}_3 - |\mathfrak{a}_s|^2\mathfrak{a}_s\right). \end{aligned} \tag{1.20}$$

Here $\delta(\omega^{12}_{3s})=1$ if $\omega^{12}_{3s}=0$ and $\delta(\omega^{12}_{3s})=0$ otherwise. The following result is proven in [17,21] (concerning the well posedness of (1.18) also see [28]).

Theorem 1.1. *If $\mathfrak{A}(u) = -\Delta u + u$ and $d\le 3$, then Eqs. (1.18) and (1.20) are well posed. Under the limit (1.12), on time intervals of order 1,*

(i) A solution $\mathfrak{a}^\nu(\tau)$ of (1.18) converges in distribution to a solution $\mathfrak{a}^0(\tau)$ of (1.20) once they have the same deterministic initial data at $\tau=0$;

(ii) The energy spectrum $\mathbb{E}|\mathfrak{a}^\nu_s(\tau)|^2 = N_s(\tau;\nu,L)$ converges to the energy spectrum $\mathbb{E}|\mathfrak{a}^0_s(\tau)|^2$.

Revised Proof

It is very likely that the assertions of the theorem stays true with a similar proof for any d if $\mathfrak{A}$ is the operator (1.4, 1.5) with a sufficiently large r_*, and in [21] this indeed is proved, provided that Eq. (1.18) is known to be well posed. It is also shown in [17, 21] that if under the assumptions of Theorem 1.1 Eqs. (1.18) and (1.20) are mixing, then the stationary measure for the former converges to that for the latter as $\nu \to 0$. The mixing property for Eq. (1.18) is established in [28] for the case when all numbers $b(s)$ are nonzero and $d = 1$. It is plausible that a variation of the argument in [28] allows to establish the mixing property for both Eqs. (1.18) and (1.20) if in (1.5) r_* is big in terms of d and $b(s) \neq 0$ for all s.

1.3. The Main Result

In view of Theorem 1.1, to understand behaviour of the energy spectrum (1.19) of Eq. (1.18) under the limit (1.10), it remains to study that of the energy spectrum $\mathbb{E}|\mathfrak{a}_s(\tau; L)|^2$ of the effective Eq. (1.20) under the limit $L \to \infty$. Instead, following the logic of [7], we study the energy spectrum corresponding to a principal part of a decomposition in ρ for the solutions $\mathfrak{a}_s(\tau; L)$ of Eq. (1.20).

Quasisolutions and their energy spectra. To simplify presentation we assume that initially the system was at rest, i.e. supplement Eq. (1.20) with the zero initial condition

$$\mathfrak{a}_s(0) = 0 \qquad \forall s \in \mathbb{Z}_L^d. \tag{1.21}$$

We formally decompose the corresponding solution of (1.20) in ρ,

$$\mathfrak{a}(\tau) = \mathfrak{a}^{(0)}(\tau) + \rho\mathfrak{a}^{(1)}(\tau) + \rho^2\mathfrak{a}^{(2)}(\tau) + \dots, \qquad \mathfrak{a}^{(k)}(0) = 0, \tag{1.22}$$

$\mathfrak{a}^{(k)}(\tau) = \mathfrak{a}^{(k)}(\tau; L)$. The process $\mathfrak{a}^{(0)}(\tau)$ satisfies the linear equation

$$\dot{\mathfrak{a}}_s^{(0)} + \gamma_s\mathfrak{a}_s^{(0)} = b(s)\dot{\beta}_s, \quad s \in \mathbb{Z}_L^d.$$

So it is Gaussian,

$$\mathfrak{a}_s^{(0)}(\tau) = b(s)\int_0^\tau e^{-\gamma_s(\tau-l)}d\beta_s(l), \tag{1.23}$$

and its components $\{\mathfrak{a}_s^{(0)}\}$ are independent. The process $\mathfrak{a}^{(1)}$ satisfies

$$\dot{\mathfrak{a}}_s^{(1)} + \gamma_s\mathfrak{a}_s^{(1)} = iY_s^{res}(\mathfrak{a}^{(0)}),$$

so that

$$\mathfrak{a}_s^{(1)}(\tau) = iL^{-d}\int_0^\tau e^{-\gamma_s(\tau-l)} \left(\sum_{1,2,3}\delta_{3s}'^{12}\delta(\omega_{3s}^{12})(\mathfrak{a}_1^{(0)}\mathfrak{a}_2^{(0)}\bar{\mathfrak{a}}_3^{(0)}) - |\mathfrak{a}_s^{(0)}|^2\mathfrak{a}_s^{(0)}\right)(l)\mathrm{d}l \tag{1.24}$$

Revised Proof

is a Wiener chaos of third order (see [18]). Similar for $n \geq 1$,

$$\mathfrak{a}_s^{(n)}(\tau) = iL^{-d} \sum_{n_1+n_2+n_3=n-1} \int_0^\tau e^{-\gamma_s(\tau-l)} \times \left(\sum_{1,2,3} \delta_{3s}'^{12} \delta(\omega_{3s}^{12}) \big(\mathfrak{a}_1^{(n_1)} \mathfrak{a}_2^{(n_2)} \bar{\mathfrak{a}}_3^{(n_3)}\big) - \mathfrak{a}_s^{(n_1)} \mathfrak{a}_s^{(n_2)} \bar{\mathfrak{a}}_s^{(n_3)} \right)(l)\, \mathrm{d}l, \tag{1.25}$$

is a Wiener chaos of order $2n+1$.

Next we consider the quadratic truncation of the series (1.22),

$$\mathcal{A}_s(\tau; L) = \mathcal{A}_s(\tau) = \mathfrak{a}_s^{(0)}(\tau) + \rho \mathfrak{a}_s^{(1)}(\tau) + \rho^2 \mathfrak{a}_s^{(2)}(\tau), \tag{1.26}$$

which we call the *quasisolution*[4] of the effective Eqs. (1.20), 1.21). It is traditional in WT to analyse the quasisolution instead of the solution itself, postulating that the former well approximates the latter; see introduction to [7] for a discussion. The goal of the present paper is to study the behaviour of the energy spectrum of $\mathcal{A}(\tau)$,

$$\mathfrak{n}_{s,L}(\tau) = \mathbb{E}|\mathcal{A}_s(\tau; L)|^2, \quad s \in \mathbb{Z}_L^d, \tag{1.27}$$

as $L \to \infty$. Our results, formulated below, show that under this limit the energy spectrum $\mathfrak{n}_{s,L}(\tau)$ has a non-trivial behaviour (i.e. stays finite and behaves differently from $\mathbb{E}|\mathfrak{a}_s^{(0)}|^2$) only if $\rho \sim L\,\chi_d(L)$, where χ_d is defined in (1.11). Accordingly, from now on we assume that

$$\rho = \varepsilon L \chi_d(L), \tag{1.28}$$

where $0 < \varepsilon \leq 1/2$ is a small but fixed constant (see a few lines below for its discussion). Then the energy spectrum $\mathfrak{n}_{s,L}$ expands as

$$\mathfrak{n}_{s,L}(\tau) = \mathfrak{n}_{s,L}^{(0)}(\tau) + \varepsilon\, \mathfrak{n}_{s,L}^{(1)}(\tau) + \varepsilon^2 \mathfrak{n}_{s,L}^{(2)}(\tau) + \varepsilon^3 \mathfrak{n}_{s,L}^{(3)}(\tau) + \varepsilon^4 \mathfrak{n}_{s,L}^{(4)}(\tau), \tag{1.29}$$

$s \in \mathbb{Z}_L^d$, where

$$\mathfrak{n}_{s,L}^{(k)}(\tau) = \big(L\chi_d(L)\big)^k \sum_{\substack{k_1+k_2=k \\ 0 \leq k_1, k_2 \leq 2}} \mathbb{E}\mathfrak{a}_s^{(k_1)}(\tau) \bar{\mathfrak{a}}_s^{(k_2)}(\tau). \tag{1.30}$$

In particular, by (1.23)

$$\mathfrak{n}_{s,L}^{(0)}(\tau) = \mathbb{E}|\mathfrak{a}_s^{(0)}(\tau; L)|^2 = \frac{b(s)^2}{\gamma_s}\big(1 - e^{-2\gamma_s \tau}\big), \qquad s \in \mathbb{Z}_L^d, \tag{1.31}$$

and a simple computation shows that $\mathfrak{n}_{s,L}^{(1)}(\tau) \equiv 0$. For higher-order terms, we prove that

$$\mathfrak{n}_{s,L}^{(2)} \sim 1 \quad \text{and} \quad |\mathfrak{n}_{s,L}^{(3)}|,\, |\mathfrak{n}_{s,L}^{(4)}| \lesssim 1 \quad \text{as } L \to \infty \text{ uniformly in } \tau \geq 0; \tag{1.32}$$

[4]By analogy with the *quasimodes* in the spectral theory of the Shrödinger operator.

Revised Proof

see a discussion in the next subsection. Thus, the parameter ε measures the properly scaled amplitude of the solutions, and indeed it should be small for the methodology of WT to apply. Then, the term $\varepsilon^2\mathfrak{n}^{(2)}_{s,L}$ is the crucial non-trivial component of the energy spectrum $\mathfrak{n}_{s,L}$, while the terms $\varepsilon^3\mathfrak{n}^{(3)}_{s,L}$, $\varepsilon^4\mathfrak{n}^{(4)}_{s,L}$ are perturbative. This well agrees with the prediction of physical works concerning various models of WT.

Wave kinetic equation. In view of (1.32), to study the limiting as $L \to \infty$ behaviour of the energy spectrum $\mathfrak{n}_{s,L}(\tau)$ up to an error of size ε^3 it remains to investigate the behaviour of its principal component $\mathfrak{n}^{(0)}_{s,L}(\tau)+\varepsilon^2\mathfrak{n}^{(2)}_{s,L}(\tau)$. We show that the latter is governed by a WKE. To state the result, let us consider the *resonant quadric*

$$\Sigma_s = \left\{(s_1, s_2) \in \mathbb{R}^{2d} : (s_1 - s)\cdot(s_2 - s) = 0\right\}, \tag{1.33}$$

cf. (1.17), and a measure μ^{Σ_s} on it, given by

$$\mu^{\Sigma_s}(\mathrm{d}s_1\mathrm{d}s_2) = \left(|s_1 - s|^2 + |s_2 - s|^2\right)^{-1/2}\mathrm{d}s_1\mathrm{d}s_2\,|_{\Sigma_s}, \tag{1.34}$$

where $\mathrm{d}s_1\mathrm{d}s_2\,|_{\Sigma_s}$ denotes the volume element on Σ_s, corresponding to the standard Euclidean structure on $\mathbb{R}^{2d}$.

Let us consider the following non-autonomous cubic *wave kinetic integral* operator $K(\tau)$, for any $\tau \geq 0$ sending a function y_s, $s \in \mathbb{R}^d$, to the function $K_s(\tau)y$, defined as

$$\begin{aligned} K_s(\tau)y = 4C_d \int_{\Sigma_s} \mu^{\Sigma_s}(\mathrm{d}s_1\mathrm{d}s_2)\Big(&\mathcal{Z}^4 y_1y_2y_3 \\ &+\mathcal{Z}^3 y_1y_2y_4 - \mathcal{Z}^2 y_1y_3y_4 - \mathcal{Z}^1 y_2y_3y_4\Big). \end{aligned} \tag{1.35}$$

Here $y_j := y_{s_j}$ with $s_4 := s$ and $s_3 := s_1 + s_2 - s$, C_d is the constant from Theorem B below, the kernels $\mathcal{Z}^j = \mathcal{Z}^j(\tau; s_1, s_2, s_3, s_4)$ are given by formulas (4.14, 4.15) and satisfy $0 \leq \mathcal{Z}^j(\tau) \leq 1$. When $\tau \to \infty$, the operator $K(\tau)$ exponentially fast converges to a limiting kinetic integral operator $K(\infty)$, given by (1.35) with $\mathcal{Z}^j$ replaced by $(\gamma_{s_1} + \gamma_{s_2} + \gamma_{s_3} + \gamma_{s_4})^{-1}$ for all j:

$$\begin{aligned} K_s(\infty)y = 4C_d \int_{\Sigma_s} &\frac{\mu^{\Sigma_s}(\mathrm{d}s_1\mathrm{d}s_2)}{\gamma_{s_1} + \gamma_{s_2} + \gamma_{s_3} + \gamma_{s_4}} \\ &\Big(y_1y_2y_3 + y_1y_2y_4 - y_1y_3y_4 - y_2y_3y_4\Big). \end{aligned} \tag{1.36}$$

It is similar to the standard four-wave kinetic operator of WT (e.g. see in [25]), which has the form (1.35) with $\mathcal{Z}^j \equiv \mathrm{const}$, but still is different from the latter since $K(\infty)$ depends on the spectrum $\{\gamma_s\}$ of the dissipation operator $\mathfrak{A}$.[5]

For $r \in \mathbb{R}$ we denote by $\mathcal{C}_r(\mathbb{R}^d)$ a space of continuous complex functions on $\mathbb{R}^d$ with finite norm

$$|f|_r = \sup_{z\in\mathbb{R}^d} |f(z)|\langle z\rangle^r, \quad \text{where} \quad \langle z\rangle = \max(|z|, 1). \tag{1.37}$$

[5]Earlier the kinetic operator $K(\infty)$ was heuristically obtained in [20].

Revised Proof

In Sect. 5, following [7], we show that if $r > d$, then for any τ the operator $K(\tau)$ defines a continuous 3-homogeneous mapping $K(\tau) : \mathcal{C}_r(\mathbb{R}^d) \mapsto \mathcal{C}_{r+1}(\mathbb{R}^d)$, and for any $y \in \mathcal{C}_r(\mathbb{R}^d)$ the curve $\tau \mapsto K(\tau)(y)$ is Hölder continuous in $\mathcal{C}_r(\mathbb{R}^d)$.

Now consider the following damped/driven non-autonomous WKE

$$\dot{\mathfrak{z}}_s(\tau) = -2\gamma_s \mathfrak{z}_s + \varepsilon^2 K_s(\tau)(\mathfrak{z}) + 2b(s)^2, \qquad \mathfrak{z}(0) = 0, \tag{1.38}$$

where $\tau \geq 0$ and $s \in \mathbb{R}^d$. In Sect. 5 we prove that for small ε it has a unique solution $\mathfrak{z}_s(\tau)$, which can be written as $\mathfrak{z}_s(\tau) = \mathfrak{z}_s^0(\tau) + \varepsilon^2 \mathfrak{z}_s^1(\tau, \varepsilon)$, where $\mathfrak{z}_s^0$, $\mathfrak{z}_s^1 \sim 1$ and $\mathfrak{z}_s^0$ solves the linear Eq. (1.38)$|_{\varepsilon=0}$. It is easy to see that $\mathfrak{z}_s^0$ equals the component $\mathfrak{n}_{s,L}^{(0)}$ of the energy spectrum $\mathfrak{n}_{s,L}$, given by (1.31), and we prove that $\mathfrak{z}_s^1$ is ε^4-close to $\mathfrak{n}_{s,L}^{(2)}$ uniformly in τ. Then, in view of (1.32), the energy spectrum $\mathfrak{n}_{s,L}$ is ε^3-close to the solution $\mathfrak{z}_s(\tau)$.

Below we denote by $C^{\#}(s)$ various positive functions of s which decay as $|s| \to \infty$ faster than any negative degree of $|s|$. These functions never depend on L, ε and τ. By $C^{\#}(s; p)$ we denote functions $C^{\#}(s)$ depending on a parameter p.

Theorem A (Main theorem). *Let $d \geq 2$. Then, the energy spectrum $\mathfrak{n}_{s,L}(\tau)$ of the quasisolution $\mathcal{A}_s(\tau)$ of (1.20), 1.21) satisfies the estimate $\mathfrak{n}_{s,L}(\tau) \leq C^{\#}(s)$ and is ε^3-close to the solution $\mathfrak{z}_s(\tau)$ of WKE (1.38). Namely, under the scaling $\rho = \varepsilon L \chi_d(L)$, for any r there exists $\varepsilon_r \in (0, 1/2]$ such that for $0 < \varepsilon \leq \varepsilon_r$ we have*

$$|\mathfrak{n}_{\cdot,L}(\tau) - \mathfrak{z}_\cdot(\tau)|_r \leq C_r \varepsilon^3 \qquad \forall\, \tau \geq 0, \tag{1.39}$$

if $L \geq \varepsilon^{-2}$ for $d \geq 3$, and $L \geq e^{\varepsilon^{-1}}$ for $d = 2$.

See Theorem 5.9. Since the energy spectrum $\mathfrak{n}_s$ is defined for $s \in \mathbb{Z}_L^d$ with finite L, then the norm in (1.39) is understood as $|f|_r = \sup_{z \in \mathbb{Z}_L^d} |f(z)| \langle z \rangle^r$.

Remark 1.2. If $d = 2$, the lower bound $L \geq e^{\varepsilon^{-1}}$ can be relaxed in the following sense. In Appendix D we explain that there is a bounded correction $f(\tau, L)$ which can be written explicitly, such that

$$\left| \mathfrak{n}_{\cdot,L}(\tau) - \mathfrak{z}_\cdot(\tau) - \frac{f(\tau, L)}{\ln L} \right|_r \leq C_r \varepsilon^3 \qquad \forall\, \tau \geq 0, \tag{1.40}$$

if $L \geq \varepsilon^{-6}$.

In Lemma 5.6 we show that in the vicinity of the unique steady state $\mathfrak{z}_s^0 := b(s)^2/\gamma_s$ for the linear Eq. (1.38)$|_{\varepsilon=0}$, Eq. (1.38)$|_{\tau=\infty}$ with $\varepsilon \ll 1$ has a unique steady state $\mathfrak{z}^\varepsilon \in \mathcal{C}_r(\mathbb{R}^d)$ and the latter is asymptotically stable. Jointly with Theorem A, this result implies the following asymptotic in time behaviour of the energy spectrum $\mathfrak{n}_{s,L}(\tau)$:

$$|\mathfrak{n}_{\cdot,L}(\tau) - \mathfrak{z}_\cdot^\varepsilon|_r \leq C_r(e^{-\tau} + \varepsilon^3), \qquad \forall\, \tau \geq 0, \tag{1.41}$$

if L is as in Theorem A, see (5.19).

Revised Proof

The cases $d \geq 3$ and $d = 2$ are similar, but should be treated separately. To shorten the presentation, we give a detailed proof of Theorem A only for $d \geq 3$, when

$$\chi_d = 1 \quad \text{and} \quad \rho = \varepsilon L.$$

The proof for $d = 2$ can be obtained by a simple modification of the argument for $d \geq 3$. We sketch it in Appendix D. So from now on, except Sect. 2 which gives a brief account of the method of Feynman diagrams from [7,8], we assume that $d \geq 3$.

In paper [7] we examine the behaviour of the energy spectrum $\mathfrak{n}_{s,L,\nu}(\tau)$ of a quasisolution to Eq. (1.18) under the limit (1.9), assuming that $\rho = \varepsilon\nu^{-1/2}$.[6] We got there a similar result which states that $\mathfrak{n}_{\cdot,L,\nu}(\tau)$ is ε^4-close to a solution of the damped/driven four-wave kinetic equation as in [25, Section 6.9.1] [in contrast with Eq. (1.38), the kinetic nonlinearity there does not depend on the dissipation $\mathfrak{A}$ in Eq. (1.3)].

What next? In this work and in [7] we obtained wave kinetic limits for the energy spectra of quasisolutions for the NLS Eq. (1.6) under limit (1.10) with the scaling $\rho = \varepsilon L\, \chi_d(L)$ and limit (1.9) with the scaling $\rho = \varepsilon\nu^{-1/2}$. Our next goal is to show that an exact solution $\mathfrak{a}_\cdot(\tau)$ of Eq. (1.18) is ε^3-close to its quasisolution $\mathcal{A}_\cdot(\tau)$ (uniformly in $L \geq 2$ and $\tau \in [0, T]$, for any $T > 0$). And that a solution of Eq. (1.6) is ε^3-close to the quasisolution of the equation (uniformly in ν, L and $\tau \in [0, T]$, if $L \geq \nu^{-2-\bar{\gamma}}$, $\bar{\gamma} > 0$). This would imply that the energy spectra of solutions of Eq. (1.6) under limit (1.10) and limit (1.9) are ε^3-close to solutions of the two WKE (namely, Eq. (1.38) and the WKE from [7]). To prove this, say, for a solution $\mathfrak{a}_\cdot(\tau)$ of Eq. (1.18) we consider the equation on any fixed time-interval $[0, T]$ and regard it as a nonlinear equation $F_T(\mathfrak{a}_\cdot(\cdot)) = 0$ for the unknown process $\mathfrak{a}_s(\tau)$. Then the quasisolution $\mathcal{A}$ satisfies the equation with a disparity $\lesssim \varepsilon^3$. By analogy with some stochastic problems for nonlinear PDEs, recently successfully resolved by the KAM-techniques (e.g. see [19]), we believe that KAM also applies to the equation $F_T = 0$. Its application would imply that $\mathfrak{a}$ is ε^3-close to $\mathcal{A}$, as stated. We also believe that analysis of the KAM-iterations which build $\mathfrak{a}$ from $\mathcal{A}$ will show that the energy spectrum of the solution $\mathfrak{a}_\cdot(\tau)$ of Eq. (1.18) under the limit $L \to \infty$ converges to a solution of the WKE (1.38). A similar logic should apply to the energy spectra of solutions for Eq. (1.6) under the limit (1.9).

1.4. Outline of The Proof: Feynman Diagrams and Number Theory

It is well understood that to write down formulas for the terms $\mathfrak{n}^{(k)}_{s,L}$ of decompositions as (1.29) it is instrumental to use the language of Feynman diagrams. In application to similar problems this goes back at least to the works [11,12], and then was successfully used for the purposes of WT in [2–6,23] and other papers. We use this techniques in the form developed in [8] which gives a convenient presentation of the terms $\mathfrak{n}^{(k)}_{s,L}$ [see (1.30)]. Namely, by iterating the

[6] In [7] the notation is slightly different: there we set $\rho = \varepsilon^{1/2}\nu^{-1/2}$.

Revised Proof

Duhamel formula (1.25) we express $\mathfrak{a}^{(n)}(\tau)$ in terms of the Gaussian processes $\mathfrak{a}_s^{(0)}$, and next evoking the Wick formula for moments of $\mathfrak{a}_s^{(0)}$ write the terms $\mathfrak{n}_{s,L}^{(k)}$ as multiple sums. Then the just mentioned diagram techniques allows to 'integrate' these sums. That is, to write any $\mathfrak{n}_{s,L}^{(k)}$ as a sum over an intersections of $k-1$ quadrics in $(\mathbb{Z}_L^d)^k$ in a form, convenient to pass to a limit as $L \to \infty$. The term $\mathfrak{n}_{s,L}^{(2)}$ is a sum over a single quadric and may be analysed without the diagram's machinery. This and some other similar terms play a leading role in our analysis and dictate the form of the limiting WKE. The terms may be written as sums

$$G_s(\tau, L) = L^{2(1-d)} \sum_{\substack{z_1, z_2 \in \mathbb{Z}_L^d: \\ z_1 \cdot z_2 = 0,\, z_1, z_2 \neq 0}} \Phi_s(\tau; z_1, z_2), \tag{1.42}$$

well known in works on WT. To study them under the limit $L \to \infty$ we make use of the celebrated circle method of Heath–Brown [14]. Since the result of [14] does not completely fit our purposes, we specified it in the accompanying paper [10] (also see [1, Section 5] for another specification of the Heath–Brown method, used for the purposes of WT). This implies

Theorem B. *For any $L \geq 2$,*

$$\left| G_s(\tau, L) - C_d \int_{\Sigma_0} \Phi_s(\tau; z_1, z_2)\, \mu^{\Sigma_0}(\mathrm{d}z_1 \mathrm{d}z_2) \right| \leq K_d \, \frac{\|\Phi_s(\tau; \cdot)\|_{N_1, N_2}}{L^{d-5/2}},$$

where $\Sigma_0 = \{z_1, z_2 \in \mathbb{R}^d : z_1 \cdot z_2 = 0\}$, μ^{Σ_0} is the measure on it, defined by (1.34) with $s = 0$, C_d is a number-theoretical constant, satisfying $C_d \in (1, 1 + 2^{2-d})$, the norm $\|\cdot\|_{N_1,N_2}$ is defined in (3.3) and the constants $N_1, N_2 \in \mathbb{N}$ depend only on d.

In particular, the term $\mathfrak{n}_{s,L}^{(2)}(\tau)$ admits a limit when $L \to \infty$.

The terms $\mathfrak{n}_{s,L}^{(3)}$ and $\mathfrak{n}_{s,L}^{(4)}$ in (1.29) correspond to multiple intersections of quadrics, and the Heath–Brown method does not apply to them. Still the diagram technique allows to write these terms in a convenient compact form. Then next in Sect. 3 and Appendix A we use Theorem B jointly with another powerful result from the number theory—Bezout's theorem for finite fields—to prove [7]

Theorem C. *For $k = 3, 4$, $|\mathfrak{n}_{s,L}^{(k)}(\tau)| \leq C^{\#}(s)$.*

Theorems B and C imply (1.32). So to establish Theorem A it remains to show that the term $\mathfrak{n}_{s,L}^{<2}(\tau) := \mathfrak{n}_{s,L}^{(0)}(\tau) + \varepsilon^2 \mathfrak{n}_{s,L}^{(2)}(\tau)$ (or equivalently its limit as $L \to \infty$, provided by Theorem B) can be well approximated by a solution of the WKE (1.38). To this end, following the lines of [7] (and the logic of the Krylov–Bogolyubov averaging) we consider increments $\Delta\mathfrak{n}_{s,L}^{<2} := \mathfrak{n}_{s,L}^{<2}(\tau + \theta) - \mathfrak{n}_{s,L}^{<2}(\tau)$ and express them through the processes $\mathfrak{a}_m^{(0)}$ via the Duhamel formula (1.25)

[7] In fact, in Sect. 3 we prove an abstract result, more general than the theorem below; see there Theorem 3.2.

Revised Proof

and the Wick theorem. Then the increments approximately take the form (1.42), and we use Theorem B to show that they are close to the r.h.s. of the WKE, multiplied by θ.

Although the computation of the increments $\Delta\mathfrak{n}_{s,L}^{\leq 2}$ is similar to that in [7], now a mechanism leading to a WKE is rather different. Namely, in [7] components of the terms $\mathfrak{n}_{s,L}^{(k)}$ are approximated by formulas analogous to (1.42), where the summation over the lattice $(\mathbb{Z}^d)^k = \{z\}$ is replaced by an integration over $\mathbb{R}^{dk}$. The integrals in those formulae involve fast oscillating Gaussian kernels. The zero sets of these kernels define quadrics, related to the quadrics $\{(z_1, z_2) : z_1 \cdot z_2 = 0\}$ in (1.42). Due to the fast oscillations a crucial component of the increments $\Delta\mathfrak{n}_{s,L}^{\leq 2}$ is given by the terms, associated with short-range correlations in τ of the processes $\mathfrak{a}_m^{(0)}(\tau)$. On the contrary, in the present situation a crucial contribution is given by other terms, associated with long-time correlations of the processes $\mathfrak{a}_m^{(0)}(\tau)$, while the short-range correlations only give a small correction, as it can be seen from computations of Appendix A.4. So, it is natural that the kinetic integral in the WKE (1.38) depends on the viscosity operator $\mathfrak{A}$, while that in the WKE in [7] does not.

Finally we note that, as we explain in Sect. 3.2, it is plausible that Theorem C holds for all $k \geq 3$. If so, then for $\rho = \varepsilon L$ the energy spectrum of a solution $\mathfrak{a}(\tau)$, written as (1.22), defines a formal series in ε, uniformly in $L \geq 2$. Then the partial sums of this series, made by the terms of order ε^m, $m \leq M$, with any fixed $M \geq 2$, also satisfy Theorem A with the constants C_r, depending on M. Cf. Conjecture 3.8.

2. Series Expansion: Approximating Equation and Diagrammatic Representation for Solutions

In this section, assuming that $d \geq 2$, we approximate processes (1.25) by more convenient processes $a^{(n)}$, and then obtain a compact and instrumental representation for their correlations in terms of Feynman diagrams (see Lemma 2.2), following [8, Sections 3–5]. This representation (as well as its analogy in [7,8]) is used to estimate various disparity terms, related to quasisolutions $\mathcal{A}(\tau; L)$, see (1.26), and to their energy spectra.

Our presentation is sketchy, but missing details may be found in [8]. For a general discussion of the language of Feynman diagrams, see [18].

2.1. Approximate a-Equation

We start by considering an approximation of the original Eq. (1.20) by an equation, where the term $L^{-d}|\mathfrak{a}_s|^2\mathfrak{a}_s$ is removed:

$$\begin{aligned} \dot{a}_s + \gamma_s a_s &= i\rho\mathcal{Y}_s(a) + b(s)\dot{\beta}_s, \quad s \in \mathbb{Z}_L^d, \\ \mathcal{Y}_s(a) &= L^{-d} \sum_{1,2,3} \delta_{3s}'^{12} \delta(\omega_{3s}^{12}) a_1 a_2 \bar{a}_3. \end{aligned} \tag{2.1}$$

Similar to processes $\mathfrak{a}_s$, we decompose

$$a = a^{(0)} + \rho a^{(1)} + \dots . \tag{2.2}$$

Here $a^{(0)} = \mathfrak{a}^{(0)}$ and the processes $a_s^{(n)}(\tau)$ with $n \geq 1$ are built by the recursive formula (1.25) with the term $\mathfrak{a}_s^{n_1}\mathfrak{a}_s^{n_2}\bar{\mathfrak{a}}_s^{n_3}$ being dropped. That is, with the nonlinearity Y_s^{res} replaced by the $\mathcal{Y}_s$ above.

Results of [8] together with Theorem 3.2 below (which is an abstract version of Theorem C from the introduction) imply

Proposition 2.1. *For all $m, n \geq 0$, satisfying $N := m + n \leq 4$,*

$$\left|\mathbb{E}a_s^{(m)}(\tau_1)\bar{a}_s^{(n)}(\tau_2) - \mathbb{E}\mathfrak{a}_s^{(m)}(\tau_1)\bar{\mathfrak{a}}_s^{(n)}(\tau_2)\right| \leq \frac{L^{-N-d+1}}{\chi_d(L)^{N-1}} C^{\#}(s; n, m), \tag{2.3}$$

uniformly in τ_1, $\tau_2 \geq 0$.

We prove the proposition in Appendix C for $d \geq 3$ and discuss an adaptation of the proof to the case $d = 2$ in Appendix D. Doing that we use the relation $N := m+n \leq 4$ only to apply Theorem 3.2 (or Theorem D.2 if $d = 2$). So, if the assertion (3.9) of the latter theorem holds for larger N's, then for those N's estimates (2.3) remains true as well [we believe that (3.9) is fulfilled for all N, see in Sect. 3.2].

Relations (2.3) imply that moments of processes $a_s^{(m)}(\tau)$ well approximate those of processes $\mathfrak{a}_s^{(m)}(\tau)$ as $L \to \infty$. Accordingly, from now on we will mostly study processes $a_s(\tau)$ and their decompositions (2.2).

2.2. Diagrams for Solutions

For what follows it is convenient to re-write operator $\mathcal{Y}$ from (2.1), using a fictitious index s_4:

$$\mathcal{Y}_s(a) = L^{-d} \sum_{1,2,3,4} \delta_{34}'^{12}\, \delta(\omega_{34}^{12}) \delta_4^s\, a_1 a_2 \bar{a}_3,$$

where δ_4^s is the Kronecker symbol. Then analogous of the expression (1.25) for $a^{(m)}$, $m \geq 1$, takes the form

$$\begin{aligned} a_s^{(m)}(\tau) = \sum_{m_1+m_2+m_3=m-1} & i \int_0^\tau \mathrm{d}l\, e^{-\gamma_s(\tau-l)} \\ & L^{-d} \sum_{1,2,3,4} \delta_{34}'^{12}\, \delta(\omega_{34}^{12})\, \delta_4^s \big(a_1^{(m_1)} a_2^{(m_2)} \bar{a}_3^{(m_3)}\big)(l). \end{aligned} \tag{2.4}$$

We will call the objects as those in the r.h.s. of (2.4) *sums*, despite they involve integrating in $\mathrm{d}l$. The r.h.s. of (2.4) contains several sums, corresponding to all admissible choices of numbers m_1, m_2, m_3.

We apply Duhamel's formula (2.4) to the terms $a_{s_i}^{(m_i)}(l)$ in the right-hand side of (2.4) with $m_i > 0$, and iterate the procedure till $a_s^{(m)}(\tau)$ is expressed through the processes $a^{(0)}$ and $\bar{a}^{(0)}$. Then $a_s^{(m)}$ becomes represented as a finite sum of *sums*; we denote such sums by I_s. Below we will associate with each

Revised Proof

sum I_s an appropriately constructed diagram $\mathfrak{D}$. Thus, we will write $a_s^{(m)}(\tau)$ as

$$a_s^{(m)}(\tau) = \sum_{\mathfrak{D}\in\mathfrak{D}_m} I_s(\mathfrak{D};\tau), \tag{2.5}$$

where $\mathfrak{D}_m$ is a set of all diagrams, corresponding to the just explained representation of $a^{(m)}$ via the processes $a^{(0)}$ and $\bar{a}^{(0)}$. Similarly by $\overline{\mathfrak{D}}_n$ we denote the set of diagrams, parametrizing the terms in the sum, representing $\bar{a}^{(n)}(\tau)$ in a form, analogous to (2.5): $\bar{a}_s^{(n)}(\tau) = \sum_{\bar{\mathfrak{D}}\in\overline{\mathfrak{D}}_m} I_s(\bar{\mathfrak{D}};\tau)$.

2.2.1. Construction of the Sets of Diagrams $\mathfrak{D}_m$ and $\overline{\mathfrak{D}}_n$. We start with discussing the set $\mathfrak{D}_2$ and the sums $I_s(\mathfrak{D})$ with $\mathfrak{D}\in\mathfrak{D}_2$.

When iterating the Duhamel formula (2.4) (or its complex conjugation) for a j-th time, we will denote the corresponding time $l\in[0,\tau]$ by l_j and will write the set of indices $\{s_1,s_2,s_3,s_4\}$ as $\{\xi_{2j-1},\xi_{2j},\sigma_{2j-1},\sigma_{2j}\}$, where we enumerate by ξ_i the indices of non-conjugated variables $a_{s'}^{(k)}$ in (2.4) and by σ_i – those of conjugated variables $\bar{a}_{s''}^{(n)}$. We write the corresponding fictitious index s_4 as σ_{2j} if we apply (2.4), or as ξ_{2j} if we apply the complex conjugation of (2.4). More precisely, when applying (2.4) we denote $s_1=\xi_{2j-1}$, $s_2=\xi_{2j}$, $s_3=\sigma_{2j-1}$ and $s_4=\sigma_{2j}$, and when applying its complex conjugation, we write $s_1=\sigma_{2j-1}$, $s_2=\sigma_{2j}$, $s_3=\xi_{2j-1}$ and $s_4=\xi_{2j}$. We will abbreviate

$$\delta_j = \delta'^{\xi_{2j-1}\xi_{2j}}_{\sigma_{2j-1}\sigma_{2j}} \quad\text{and}\quad \omega_j = \omega^{\xi_{2j-1}\xi_{2j}}_{\sigma_{2j-1}\sigma_{2j}}, \quad j\ge 1, \tag{2.6}$$

[these terms correspond to δ'^{12}_{34} and ω^{12}_{34} in (2.4)]. We also set

$$\xi_0=\sigma_0:=s. \tag{2.7}$$

Applying (2.4) to $a_s^{(2)}$ and using the notation above with $j=1$, we find

$$\begin{aligned} a_s^{(2)}(\tau) = a_{\xi_0}^{(2)}(\tau) = &\sum_{m_1+m_2+m_3=1} i\int_0^\tau \mathrm{d}l_1\, e^{-\gamma_{\xi_0}(\tau-l_1)} \\ &L^{-d}\sum_{\xi_1,\xi_2,\sigma_1,\sigma_2} \delta_1\,\delta(\omega_1)\,\delta^{\xi_0}_{\sigma_2}\big(a^{(m_1)}_{\xi_1}a^{(m_2)}_{\xi_2}\bar{a}^{(m_3)}_{\sigma_1}\big)(l_1). \end{aligned} \tag{2.8}$$

Let us consider the summand with $m_1=m_2=0$ and $m_3=1$. Applying the conjugated formula (2.4) to $\bar{a}^{(1)}_{\sigma_1}$ and using the introduced notation with $j=2$, we get

$$\begin{aligned} \bar{a}^{(1)}_{\sigma_1}(l_1) = -i\int_0^{l_1} \mathrm{d}l_2\, e^{-\gamma_{\sigma_1}(l_1-l_2)} \\ L^{-d}\sum_{\xi_3,\xi_4,\sigma_3,\sigma_4} \delta_2\,\delta(\omega_2)\,\delta^{\sigma_1}_{\xi_4}\big(a^{(0)}_{\xi_3}\bar{a}^{(0)}_{\sigma_3}\bar{a}^{(0)}_{\sigma_4}\big)(l_2). \end{aligned} \tag{2.9}$$

Inserting (2.9) into the summand in (2.8) with $m_1=m_2=0$ and $m_3=1$, we get a sum $I_s(\mathfrak{D};\tau)$ which we associate with the diagram $\mathfrak{D}$ from Fig. 1c; further on we will denote this diagram by $\mathfrak{D}^c$. The *non-conjugated* vertices $c_i^{(k)}$ of the diagram are associated with the variables $a^{(k)}_{\xi_i}$ in (2.8), (2.9); the corresponding

Revised Proof

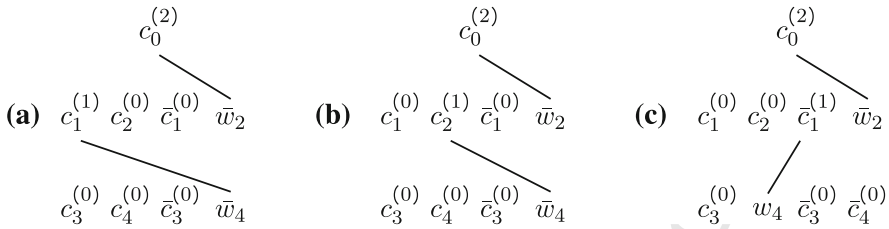

FIGURE 1. The set of diagrams $\mathfrak{D}_2$

to them indices are ξ_i. The *conjugated* vertices $\bar{c}_j^{(n)}$ are associated with the variables $\bar{a}_{\sigma_j}^{(n)}$ and the corresponding indices are σ_j. In particular, the root $c_0^{(2)}$ is associated with $a_{\xi_0}^{(2)} = a_s^{(2)}$ and the corresponding index is ξ_0. In the notation $c_i^{(k)}$ and $\bar{c}_j^{(n)}$ we sometimes omit the upper indices k and n which we call the *degrees* of the vertices $c_i^{(k)}$, $\bar{c}_j^{(n)}$. The vertices $\bar{w}_2$ and w_4 are called conjugated (non-conjugated) *virtual* vertices, and the corresponding indices are σ_2 and ξ_4; these vertices are associated with the Kronecker symbols $\delta_{\sigma_2}^{\xi_0}$ and $\delta_{\xi_4}^{\sigma_1}$ in (2.8) and (2.9). Vertices which are not virtual are called *real*. Every edge of the diagram couples a non-conjugated (conjugated) vertex $c_i^{(k)}$ ($\bar{c}_i^{(k)}$) of positive degree $k \geq 1$ with a conjugated (non-conjugated) virtual vertex $\bar{w}_{i'}$ ($w_{i'}$). It is associated with an application of formula (2.4) (or its complex conjugation) to the variable $a_{\xi_i}^{(k)}$ (or $\bar{a}_{\sigma_i}^{(k)}$), corresponding to the vertex $c_i^{(k)}$ (or $\bar{c}_i^{(k)}$).

The set of four vertices

$$c_{2j-1}, c_{2j}, \bar{c}_{2j-1}, \bar{w}_{2j} \quad \text{or} \quad c_{2j-1}, w_{2j}, \bar{c}_{2j-1}, \bar{c}_{2j} \tag{2.10}$$

(in dependence whether the virtual vertex is conjugated or not) to which correspond the indices $\xi_{2j-1}, \xi_{2j}, \sigma_{2j-1}, \sigma_{2j}$ is called *the j-th block*; the diagram $\mathfrak{D}^c$ has two blocks. The index i of a virtual vertex w_i ($\bar{w}_i$) is always pair, $i = 2j$. Each block corresponds to an application of formula (2.4) (or its complex conjugation) to its *parent*, i.e. to the vertex of positive degree coupled with the virtual vertex of the block. The virtual vertex is conjugated if the parent is non-conjugated and the other way round. The *time variable* l_j is associated with the j-th block.

The *leaves* are the vertices of zero degree, that is, the vertices $c_i^{(0)}$ and $\bar{c}_j^{(0)}$.

The diagrams from Fig. 1(a,b) correspond to the summands in (2.8) with $m_1 = 1$, $m_2 = m_3 = 0$ and $m_1 = m_3 = 0$, $m_2 = 1$; they are constructed by the same rules as the diagram $\mathfrak{D}^c$. The three diagrams from Fig. 1 form the set $\mathfrak{D}_2$. The set of diagrams $\overline{\mathfrak{D}}_2$, corresponding to $\bar{a}_s^{(2)}(\tau)$, is obtained by conjugating the vertices in the three diagrams above and re-ordering the elements of each block in such a way that the pair of non-conjugated vertices is followed by the pair of conjugated vertices, i.e. the blocks have the form (2.10).

The sets $\mathfrak{D}_m$ and $\overline{\mathfrak{D}}_n$ with arbitrary $m, n \geq 0$ and the diagrams which are their elements, are constructed similarly. Namely, the sets $\mathfrak{D}_0$ and $\overline{\mathfrak{D}}_0$

Revised Proof

are trivial—they contain one diagram each, made by the root $c_0^{(0)}$ (or $\bar{c}_0^{(0)}$). The sets $\mathfrak{D}_1$ and $\overline{\mathfrak{D}}_1$ also contain only one diagram each; e.g. the diagram in $\mathfrak{D}_1$ consists of the root $c_1^{(0)}$, joint by an edge with $\bar{w}_2$ in the only block $B_1 = (c_1^{(0)}, c_2^{(0)}, \bar{c}_1^{(0)}, \bar{w}_2)$. Arbitrary sets $\mathfrak{D}_m$ and $\overline{\mathfrak{D}}_n$ may be constructed by induction. Indeed, consider a process $a^{(m+1)}(\tau)$ with $m \geq 1$ and apply to it (2.4) with $m := m+1$. In the r.h.s. of (2.4) the sum in m_1, m_2, m_3 contains $(m+2)(m+1)/2$ terms. Consider any one of them,

$$i \int_0^\tau \mathrm{d}l\, e^{-\gamma_s(\tau-l)} L^{-d} \sum_{1,2,3,4} \delta'^{12}_{34}\, \delta(\omega^{12}_{34})\, \delta^s_4 \big(a_1^{(m_1)} a_2^{(m_2)} \bar{a}_3^{(m_3)}\big)(l), \tag{2.11}$$

draw the block $B_1 = (c_1^{(m_1)}, c_2^{(m_2)}, \bar{c}_1^{(m_3)}, \bar{w}_2)$, and join $\bar{w}_2$ by an edge with the root $c_0^{(m+1)}$. Next consider the sets $\mathfrak{D}_{m_1}, \mathfrak{D}_{m_2}, \overline{\mathfrak{D}}_{m_3}$ and do the following:

(a) Firstly, take $\mathfrak{D}_{m_1}$. If $m_1 = 0$, do nothing. Otherwise, choose any diagram $\mathfrak{D}^1 \in \mathfrak{D}_{m_1}$, place it below $c_1^{(m_1)}$ and identify its root with $c_1^{(m_1)}$. Do this for each diagram in $\mathfrak{D}_{m_1}$, thus obtaining $|\mathfrak{D}_{m_1}|$ diagrams with roots in $c_0^{(m+1)}$.

(b) Then consider the set $\mathfrak{D}_{m_2}$ and do the same with the just obtained $|\mathfrak{D}_{m_1}|$ diagrams, identifying their roots with the vertex $c_2^{(m_2)}$, and next—the set $\overline{\mathfrak{D}}_{m_3}$, identifying the roots with $\bar{c}_3^{(m_3)}$.

(c) It remains to convert thus obtained $|\mathfrak{D}_{m_1}| \times |\mathfrak{D}_{m_2}| \times |\mathfrak{D}_{m_3}|$ diagrams to elements of the set $\mathfrak{D}_{m+1}$ by re-numerating properly their blocks and accordingly re-numerating the vertices in the blocks as in (2.10). Do this by numerating the blocks from top to the bottom and from left to right, as in the examples above with $m = 2$.

(d) Doing the same for all blocks, corresponding to all possible $(m+2)(m+1)/2$ terms (2.11), get the diagrams, forming the set $\mathfrak{D}_{m+1}$.

The set $\overline{\mathfrak{D}}_{m+1}$ is constructed inductively in the same way.

For further needs we note that due to the factors δ'^{12}_{34} and $\delta^s_{s_4}$ in (2.4), the indices ξ_i, σ_j entering the formula for the sums $I_s(\mathfrak{D})$ from (2.5) satisfy the relations

$$\begin{aligned} &(1)\ \delta'^{\xi_{2j-1}\xi_{2j}}_{\sigma_{2j-1}\sigma_{2j}} = 1\ \forall j, \\ &(2)\ \text{indices } \xi_i, \sigma_j \text{ corresponding to adjacent in } \mathfrak{D} \text{ vertices are equal.} \end{aligned} \tag{2.12}$$

2.3. Feynmann Diagrams for Expectations

The main objects we are interested in are the correlations $\mathbb{E} a_{s_1}^{(m)}(\tau_1) \bar{a}_{s_2}^{(n)}(\tau_2)$. It can be shown that they vanish if $s_1 \neq s_2$.[8] To represent an expectation $\mathbb{E} a_s^{(m)}(\tau_1) \bar{a}_s^{(n)}(\tau_2)$, we consider the set of diagrams

$$\mathfrak{D}_m \times \overline{\mathfrak{D}}_n := \{\mathfrak{D}^1 \sqcup \bar{\mathfrak{D}}^2 : \mathfrak{D}^1 \in \mathfrak{D}_m, \bar{\mathfrak{D}}^2 \in \overline{\mathfrak{D}}_n\}.$$

Here a diagram $\mathfrak{D}^1 \sqcup \bar{\mathfrak{D}}^2$ is obtained by drawing $\mathfrak{D}^1$ and $\bar{\mathfrak{D}}^2$ side by side, where the blocks of $\mathfrak{D}^1$ are enumerated from 1 to m, while those of $\bar{\mathfrak{D}}^2$ together with

[8] As well vanish the correlations $\mathbb{E} a_{s_1}^{(m)}(\tau_1) a_{s_2}^{(n)}(\tau_2)$ and $\mathbb{E} \bar{a}_{s_1}^{(m)}(\tau_1) \bar{a}_{s_2}^{(n)}(\tau_2)$ for all s_1, s_2.

Revised Proof

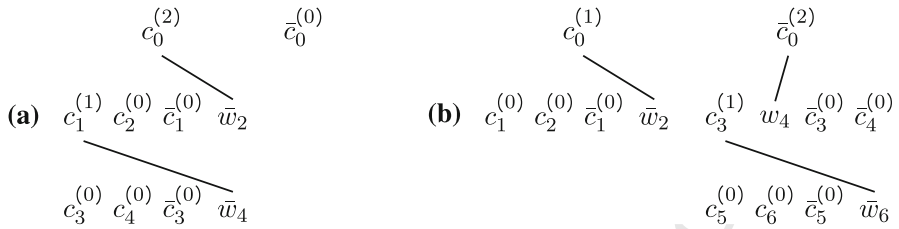

FIGURE 2. A diagram from the set **a** $\mathfrak{D}_2 \times \overline{\mathfrak{D}}_0$ and **b** $\mathfrak{D}_1 \times \overline{\mathfrak{D}}_2$

the corresponding time variables l_j are enumerated from $j = m+1$ to $m+n$. The vertices together with the corresponding indices $\xi_{2j-1}, \xi_{2j}, \sigma_{2j-1}, \sigma_{2j}$ are enumerated accordingly, see Fig. 2. The diagram $\mathfrak{D}^1 \sqcup \bar{\mathfrak{D}}^2$ has two roots $c_0^{(m)}$ and $\bar{c}_0^{(n)}$. For any $\mathfrak{D} = \mathfrak{D}^1 \sqcup \bar{\mathfrak{D}}^2$ consider

$$I_s(\mathfrak{D}; \tau_1, \tau_2) = I_s(\mathfrak{D}^1; \tau_1) I_s(\bar{\mathfrak{D}}^2; \tau_2),$$

so that $a_s^{(m)}(\tau_1)\bar{a}_s^{(n)}(\tau_2) = \sum_{\mathfrak{D} \in \mathfrak{D}_m \times \overline{\mathfrak{D}}_n} I_s(\mathfrak{D}; \tau_1, \tau_2)$. Our next task is to compute $\mathbb{E} I_s(\mathfrak{D})$ for each $\mathfrak{D} \in \mathfrak{D}_m \times \overline{\mathfrak{D}}_n$.

Randomness enters the term $I_s(\mathfrak{D})$ via the random variables $a_{\xi_i}^{(0)}$, $\bar{a}_{\sigma_j}^{(0)}$, corresponding to the leaves of the diagram $\mathfrak{D} = \mathfrak{D}^1 \sqcup \bar{\mathfrak{D}}^2$. They are Gaussian with correlations

$$\mathbb{E} a_s^{(0)}(l_1) a_{s'}^{(0)}(l_2) = 0, \quad \mathbb{E} a_s^{(0)}(l_1) \bar{a}_{s'}^{(0)}(l_2) = \delta_{s'}^s \operatorname{Corr}(\gamma_s, b(s), l_1, l_2), \tag{2.13}$$

where

$$\operatorname{Corr}(\gamma_s, b(s), l_1, l_2) = B_s \big(e^{-\gamma_s |l_1 - l_2|} - e^{-\gamma_s (l_1 + l_2)} \big), \quad B_s = \frac{b(s)^2}{\gamma_s}. \tag{2.14}$$

So, the Wick theorem [18] implies that the expectation $\mathbb{E} I_s(\mathfrak{D})$ is given by a sum over all Wick pairings of variables $a_{\xi_i}^{(0)}$, corresponding to non-conjugated leaves $c_i^{(0)}$, with variables $\bar{a}_{\sigma_j}^{(0)}$ corresponding to conjugated leaves $\bar{c}_j^{(0)}$. Moreover, the leaves $c_i^{(0)}$ and $\bar{c}_j^{(0)}$ should belong to different blocks since otherwise the summand corresponding to such Wick pairing vanishes due to (1.15) and item (1) in (2.12). We parametrize the sum over the Wick pairings by the defined below set $\mathfrak{F}(\mathfrak{D})$ of Feynman diagrams. Denoting by $J_s(\mathfrak{F})$ a term (i.e. a sum), corresponding to a specific Feynman diagram $\mathfrak{F}$, we have:

$$\mathbb{E} I_s(\mathfrak{D}) = \sum_{\mathfrak{F} \in \mathfrak{F}(\mathfrak{D})} J_s(\mathfrak{F}).$$

2.3.1. Definition of Feynman diagrams. To construct the set of Feynman diagrams $\mathfrak{F}(\mathfrak{D})$, corresponding to some diagram $\mathfrak{D} = \mathfrak{D}^1 \sqcup \bar{\mathfrak{D}}^2$, we consider all possible partitions of the set of leaves of $\mathfrak{D}$ to non-intersecting pairs $(c_i^{(0)}, \bar{c}_j^{(0)})$, such that the paired leaves $c_i^{(0)}$ and $\bar{c}_j^{(0)}$ do not belong to the same block. To

Revised Proof

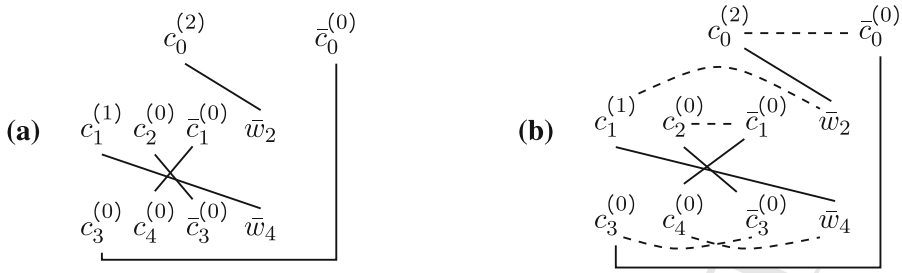

FIGURE 3. **a** A Feynman diagram $\mathfrak{F}$ obtained from the diagram $\mathfrak{D}$ in Fig. 2(a). **b** A cycle obtained from the Feynman diagram $\mathfrak{F}$

each such partition, we associate a diagram $\mathfrak{F}$ obtained from $\mathfrak{D}$ by joining with an edge the two leaves in every pair, see Fig. 3(a). So, in each diagram $\mathfrak{F} \in \mathfrak{F}(\mathfrak{D})$ all vertices of $\mathfrak{D}$ are joint by edges, every edge couples a conjugated vertex with a non-conjugated in another block (or with a root), and every vertex belongs to exactly one edge.

Since $\mathbb{E}a_s^{(0)}\bar{a}_{s'}^{(0)} = 0$ if $s \neq s'$, the indices ξ_i, σ_j entering the formulas for sums $J_s(\mathfrak{F})$ satisfy (2.12), where in item (2) the diagram $\mathfrak{D}$ is replaced by the Feynman diagram $\mathfrak{F}$; below we denote these relations as $(2.12)_{\mathfrak{F}}$. In particular, due to the item (2), the vector of indices $\sigma = (\sigma_i)$ is a function of the vector $\xi = (\xi_j)$. Accordingly below we write $\sigma = \sigma_{\mathfrak{F}}(\xi)$.

Let

$$\mathfrak{F}_{m,n} = \bigcup_{\mathfrak{D} \in \mathfrak{D}_m \times \overline{\mathfrak{D}}_n} \mathfrak{F}(\mathfrak{D})$$

be the set of all Feynman diagrams associated with the product $a_s^{(m)}\bar{a}_s^{(n)}$. Each diagram $\mathfrak{F} \in \mathfrak{F}_{m,n}$ has $N := m+n$ blocks and $4N+2$ vertices, including $2N+2$ leaves. Half of edges (and of leaves) are conjugated, while another half is not.

By construction a diagram $\mathfrak{F} \in \mathfrak{F}_{m,n}$ never pairs leaves from the same block. This alone does not exclude that $\mathfrak{F}$ is such that in $(2.12)_{\mathfrak{F}}$ the assumptions (1) and (2) are incompatible since for some j we may have $\xi_{2j-1}, \xi_{2j} = \sigma_{2j-1}$ or σ_{2j} once $\sigma = \sigma_{\mathfrak{F}}(\xi)$. Analysis shows that this cannot happen if $m+n \leq 4$, but may happen if $m+n \geq 5$. Accordingly, we denote by

$$\mathfrak{F}_{m,n}^{true} \subset \mathfrak{F}_{m,n}$$

the set of Feynman diagrams for which the set of indices ξ_i, σ_j satisfying the relations $(2.12)_{\mathfrak{F}}$ is not empty. For any diagram $\mathfrak{F} \notin \mathfrak{F}_{m,n}^{true}$, we have $J_s(\mathfrak{F}) = 0$ due to the factors $\delta_{34}^{\prime 12}$ and $\delta_{s_4}^{s}$ in (2.4).

2.4. Transformation, Resolving Linear Relations on Indices

Let us take a Feynman diagram $\mathfrak{F} \in \mathfrak{F}_{m,n}$, denote $N := m+n \geq 1$ and consider the sum $J_s(\mathfrak{F})$. The relations $(2.12)_{\mathfrak{F}}$ on indices $\xi_i, \sigma_j \in \mathbb{Z}_L^d$, $0 \leq i, j \leq 2N$,[9]

[9]Recall that $\xi_0 = \sigma_0 = s$.

Revised Proof

entering the formula for $J_s(\mathfrak{F})$ are involved, which makes the sum $J_s(\mathfrak{F})$ difficult for further analysis. In [8] it was found a convenient way to "integrate the sums $J_s(\mathfrak{F})$", i.e. to parametrize the indices ξ_i, σ_j by N-vector $z = (z_1, \dots, z_N)$ from a domain in $(\mathbb{Z}_L^d)^N$, free from any relations on its components. In this section we present this parametrization, referring the reader to [8] for a proof.

Since item (2) of $(2.12)_{\mathfrak{F}}$ is equivalent to the relation $\sigma = \sigma_{\mathfrak{F}}(\xi)$, it suffices to parametrize the set of *admissible* multi-indices ξ, i.e. of those ξ for which the multi-indices ξ and $\sigma = \sigma_{\mathfrak{F}}(\xi)$ satisfy item (1) in $(2.12)_{\mathfrak{F}}$. The construction starts with defining for each $\mathfrak{F} \in \mathfrak{F}_{m,n}$ a skew-symmetric $N \times N$ *incidence* matrix $\alpha^{\mathfrak{F}} = (\alpha_{ij}^{\mathfrak{F}})$ whose elements are integers from the set $\{0, +1, -1\}$. In terms of this matrix, we define the set of polyvectors

$$\mathcal{Z}(\mathfrak{F}) = \{z = (z_1, \dots, z_N) \in (\mathbb{Z}_L^d)^N : z_j \neq 0 \text{ and } (\alpha^{\mathfrak{F}} z)_j \neq 0 \quad \forall j\}. \tag{2.15}$$

Here and below for an $M \times N$-matrix A we denote by Az the polyvector with components $(Az)_j := \sum_i A_{ji} z_i \in \mathbb{Z}^d$.[10] The matrix $\alpha^{\mathfrak{F}}$ has no zero rows and zero columns if and only if $\mathfrak{F} \in \mathfrak{F}_{m,n}^{true}$, and, accordingly, the set $\mathcal{Z}(\mathfrak{F})$ is non-empty if and only if $\mathfrak{F} \in \mathfrak{F}_{m,n}^{true}$. Next, it turns out that the vectors $z \in \mathcal{Z}(\mathfrak{F})$ may be used to parametrize the set of admissible indices ξ_i by means of an affine mapping

$$\xi(z) = s + A^{\mathfrak{F}} z, \qquad z \in \mathcal{Z}(\mathfrak{F}). \tag{2.16}$$

Here $A^{\mathfrak{F}}$ is an $(2N+1) \times N$-matrix, whose elements again are integers from the set $\{0, +1, -1\}$. Transformation (2.16) provides a presentation of the terms $J_s(\mathfrak{F})$, forming the correlation $\mathbb{E} a_s^{(m)}(\tau_1) \bar{a}_s^{(n)}(\tau_2)$, and so for the correlation itself. The corresponding result is proved in Theorem 5.5 of [8]. In our setting its statement, where for the function $\theta(x,t)$ is chosen $\mathbb{I}_{\{0\}}(x)$—the indicator function of the point $x = 0$—takes the following form:

Lemma 2.2. *For any integers $m, n \geq 0$ satisfying $N = m + n \geq 1$, any $s \in \mathbb{Z}_L^d$ and $\tau_1, \tau_2 \geq 0$,*

(1) for each $\mathfrak{F} \in \mathfrak{F}_{m,n}^{true}$ parametrization (2.16) (depending on s and $\mathfrak{F}$) is such that the quantity ω_j in (2.6), written in the z-coordinates, takes the form

$$\omega_j^{\mathfrak{F}}(z) = 2z_j \cdot \sum_{i=1}^{N} \alpha_{ji}^{\mathfrak{F}} z_i = 2z_j \cdot (\alpha^{\mathfrak{F}} z)_j. \tag{2.17}$$

(2) We have

$$L^N \mathbb{E} a_s^{(m)}(\tau_1) \bar{a}_s^{(n)}(\tau_2) = \sum_{\mathfrak{F} \in \mathfrak{F}_{m,n}^{true}} c_{\mathfrak{F}} J_s(\tau_1, \tau_2; \mathfrak{F}), \tag{2.18}$$

[10] That is, abusing notation we denote by A an operator in $(\mathbb{Z}^d)^M$ with the block-matrix $A \otimes \mathbb{1}$.

Revised Proof

where the constants $c_{\mathfrak{F}} \in \{\pm 1, \pm i\}$ *and*

$$J_s(\tau_1, \tau_2; \mathfrak{F}) = \int_{\mathbb{R}^N} \mathrm{d}l\, L^{N(1-d)} \sum_{z \in \mathcal{Z}(\mathfrak{F}),\ \omega_j^{\mathfrak{F}}(z)=0\,\forall j} F_s^{\mathfrak{F}}(\tau_1, \tau_2, l, z). \tag{2.19}$$

The density $F_s^{\mathfrak{F}}(\tau_1, \tau_2, l, z)$ *is a real function, smooth in* $(s, z) \in \mathbb{R}^d \times \mathbb{R}^{dN}$*, satisfying*

$$|\partial_s^\mu \partial_z^\kappa F_s^{\mathfrak{F}}(\tau_1, \tau_2, l, z)| \le C^{\#}_{\mu,\kappa}(s) C^{\#}_{\mu,\kappa}(z)\, e^{-\delta\left(\sum_{i=1}^m |\tau_1 - l_i| + \sum_{i=m+1}^N |\tau_2 - l_i|\right)} \tag{2.20}$$

with a suitable $\delta = \delta_N > 0$*, for any vectors* $\mu \in (\mathbb{N} \cup \{0\})^d$*,* $\kappa \in (\mathbb{N} \cup \{0\})^{dN}$ *and any* $s \in \mathbb{R}^d$*,* $z \in \mathbb{R}^{dN}$*,* $l \in \mathbb{R}^N$*.*

Let us briefly explain the way to construct the parametrization (2.16). We first add to the Feynman diagram $\mathfrak{F}$ dashed edges that couple non-conjugated vertices with conjugated *inside* all blocks, as in Fig. 3(b). For each block there are two ways of doing that. We prove that there exists a choice (possibly, not unique) of a dashed edge in each block such that the diagram becomes a cycle, as in Fig. 3b. Then, for each j we set $x_{2j-1} := \xi_{2j-1} - \sigma_{2j-1}$ and $x_{2j} := \xi_{2j} - \sigma_{2j}$ or $x_{2j-1} = \xi_{2j-1} - \sigma_{2j}$ and $x_{2j} := \xi_{2j} - \sigma_{2j-1}$, according to the choice of the dashed edges in the j-th block, where we substitute $\sigma = \sigma_{\mathfrak{F}}(\xi)$. The fact that the Feynman diagram with added dashed edges forms a cycle implies that the transformation $\xi \mapsto x$ is invertible. Item (1) of $(2.12)_{\mathfrak{F}}$ implies that $x_{2j} = -x_{2j-1}$. Then we set $z_j := x_{2j-1}$ and get (2.16). The incidence matrix $\alpha^{\mathfrak{F}}$ also is constructed in terms of this cycle.

Since the choice of the dashed edges in general is not unique, the parametrization $z \mapsto \xi$ is not unique as well. However, if $z' \mapsto \xi$ is another parametrization, obtained by the procedure above, and $\alpha^{\mathfrak{F}\prime}$ is the associated incidence matrix, then for each j we have either $z'_j(\xi) = z_j(\xi)$ or $z'_j(\xi) = (\alpha^{\mathfrak{F}} z(\xi))_j$. In the latter case we also have the symmetric relation $z_j(\xi) = (\alpha^{\mathfrak{F}\prime} z'(\xi))_j$.

Computing in (2.19) the integral over $\mathrm{d}l$ and using estimate (2.20), we obtain a form of integrals J_s, more convenient for some of the subsequent analysis:

Corollary 2.3. *In terms of Lemma 2.2, the integrals* J_s *from (2.18) can be written as*

$$J_s(\tau_1, \tau_2; \mathfrak{F}) = L^{N(1-d)} \sum_{z \in \mathcal{Z}(\mathfrak{F}),\ \omega_j^{\mathfrak{F}}(z)=0\,\forall j} \Phi_s^{\mathfrak{F}}(\tau_1, \tau_2, z), \tag{2.21}$$

where the real-valued functions $\Phi_s^{\mathfrak{F}}$ *are Schwartz in* (s, z) *and satisfy*

$$|\partial_s^\mu \partial_z^\kappa \Phi_s^{\mathfrak{F}}(\tau_1, \tau_2, z)| \le C^{\#}_{\mu,\kappa}(s) C^{\#}_{\mu,\kappa}(z), \tag{2.22}$$

uniformly in $\tau_1, \tau_2 \ge 0$*, for any vectors* $\mu \in (\mathbb{N} \cup \{0\})^d$ *and* $\kappa \in (\mathbb{N} \cup \{0\})^{dN}$*.*

Revised Proof

3. Main Estimates for The Sums

In this section we focus on estimates for the sums (2.21) and on their dependence on L and N. We recall that $d \geq 3$. It is convenient to study the problem we consider in the following abstract setting. Let $\alpha = (\alpha_{ij})$, $N \geq 2$, be an $N \times N$ skew-symmetric matrix whose elements belong to the set $\{-1, 0, 1\}$, without zero lines and rows.[11] Consider a family of quadratic forms on $(\mathbb{R}^d)^N$

$$\omega_j(z) = z_j \cdot (\alpha z)_j, \qquad 1 \leq j \leq N,$$

where z is the polyvector $(z_1, \ldots, z_N)$, $z_j \in \mathbb{R}^d$, and $(\alpha z)_j := \sum_{i=1}^N \alpha_{ji} z_i$. Let us set

$$\mathcal{Z} = \{z \in (\mathbb{Z}_L^d)^N : z_j \neq 0 \quad \text{and} \quad (\alpha z)_j \neq 0 \quad \forall j\}. \tag{3.1}$$

Let a function $\Phi : \mathbb{R}^{Nd} \to \mathbb{R}$ be sufficiently smooth and sufficiently fast decaying at infinity (see below for exact assumptions). Our goal is to study asymptotic as $L \to \infty$ behaviour of the sum

$$S_{L,N}(\Phi) := L^{N(1-d)} \sum_{z \in \mathcal{Z}:\ \omega_j(z)=0\, \forall j} \Phi(z). \tag{3.2}$$

For a function $f \in C^k(\mathbb{R}^m)$, $n_1 \in \mathbb{N} \cup \{0\}$ satisfying $n_1 \leq k$ and $n_2 \in \mathbb{R}$, we set

$$\|f\|_{n_1,n_2} = \sup_{z \in \mathbb{R}^m} \max_{|\alpha| \leq n_1} |\partial^\alpha f(z)| \langle z \rangle^{n_2}, \qquad \langle x \rangle := \max\{1, |x|\}. \tag{3.3}$$

The first crucial result concerns the case $N = 2$. Then $\omega_1(z) = -\omega_2(z) = \alpha_{12} z_1 \cdot z_2$ and $\alpha_{12} \neq 0$, so

$$S_{L,2}(\Phi) = L^{2(1-d)} \sum_{\substack{z \in \mathbb{Z}_L^{2d}:\ z_1 \cdot z_2 = 0 \\ z_1 \neq 0,\ z_2 \neq 0}} \Phi(z). \tag{3.4}$$

Then we write the sum above as $\sum_{z_1 \cdot z_2 = 0} - \sum_{z_1 = 0 \text{ or } z_2 = 0}$. Since $\left| L^{-d} \sum_{z \in \mathbb{Z}_L^{2d}:\ z_i = 0} \Phi(z) \right| \leq C \|\Phi\|_{0,d+1}$ for $i = 1, 2$, we get

$$\left| S_{L,2}(\Phi) - L^{2(1-d)} \sum_{z \in \mathbb{Z}_L^{2d}:\ z_1 \cdot z_2 = 0} \Phi(z) \right| \leq C L^{2-d} \|\Phi\|_{0,d+1}. \tag{3.5}$$

Now an asymptotic for the sum $S_{L,2}(\Phi)$ immediately follows from Theorem 1.3 in [10] where the dimension is $2d$, $\varepsilon = 1/2$ and $m = 0$, by applying it to the sum $\sum_{z_1 \cdot z_2 = 0}$ in (3.5) (we recall that $d \geq 3$):

Theorem 3.1. *Let $N_1(d) := 4d(4d^2 + 2d - 1)$ and $N_2(d) := N_1 + 6d + 4$. If $\|\Phi\|_{N_1,N_2} < \infty$, then there exist constants $C_d \in (1, 1 + 2^{2-d})$ and $K_d > 0$ such that*

$$\left| S_{L,2}(\Phi) - C_d \int_{\Sigma_0} \Phi(z)\, \mu^{\Sigma_0}(\mathrm{d}z_1 \mathrm{d}z_2) \right| \leq K_d \frac{\|\Phi\|_{N_1,N_2}}{L^{d-5/2}}, \tag{3.6}$$

[11] The theorems below and their proofs remain valid as well for arbitrary skew-symmetric matrices with integer elements without zero lines and rows, but in this case the notation used in the proof becomes heavier.

Revised Proof

where Σ_0 is the quadric $\{z \in \mathbb{R}^{2d} : z_1 \cdot z_2 = 0\}$ and the measure μ^{Σ_0} is given by (1.34) with $s = 0$.

In Appendix C of [10] we give the following explicit formula for the number-theoretical constants C_d:

$$C_d = \frac{\zeta(d-1)\zeta(4d-2)}{\zeta(d)\zeta(2d-2)}, \tag{3.7}$$

where ζ is the Riemann zeta-function. Due to (3.7) C_d satisfies $1 < C_d < 1 + 2^{2-d}$, as is stated in the theorem. The integral in (3.6) converges if $\Phi(z)$ decays at infinity fast enough:

$$\Big| \int_{\Sigma_0} \Phi(z)\, \mu^{\Sigma_0}(\mathrm{d}z_1 \mathrm{d}z_2) \Big| \le C_r \|\Phi\|_{0,r} \quad \text{if } r > 2d-1, \tag{3.8}$$

see Proposition 3.5 in [7]. So, it converges under the theorem's assumptions.

From Theorem 3.1 another result can be deduced, whose proof is given in the next Sect. 3.1:

Theorem 3.2. *For $N = 2, 3, 4$ there exist constants $C_{d,N}$ such that*

$$\big| S_{L,N}(\Phi) \big| \le C_{d,N} \|\Phi\|_{0,\bar{N}}, \tag{3.9}$$

for $\bar{N} := \lfloor N/2 \rfloor N_2(d) + (N-2)(d-1) + 1$, where N_2 is defined in Theorem 3.1.

Since in view of estimate (2.22) the functions $\Phi_s^{\mathfrak{F}}$ from Corollary 2.3 satisfy

$$\|\Phi_s^{\mathfrak{F}}(\tau_1, \tau_2, \cdot)\|_{n_1,n_2} \le C^{\#}_{n_1,n_2}(s), \qquad \forall n_1, n_2, \tag{3.10}$$

then the two theorems above apply to study correlations (2.18) with $N = m + n \le 4$. In fact, in the case $N = 2$ the number of Feynmann diagrams is small and the corresponding correlations may be calculated directly without the machinery, developed in Sect. 2. In Example 3.4 which illustrates this computation, as well as in a number of situations below, we apply Theorem 3.1 in the following setting:

Corollary 3.3. *Let*

$$\mathcal{S}_{L,2} = L^{2(1-d)} \sum_{1,2,3} \delta'^{12}_{3s} \delta(\omega^{12}_{3s}) f_s(s_1, s_2, s_3; q),$$

where ω^{12}_{3s} is given by (1.17), $q \in \mathbb{R}^n$ is a parameter (in applications usually this will be the time) and $f_s(s_1, s_2, s_1 + s_2 - s; q)$[12] is a Schwartz function of (s_1, s_2, s) satisfying $|\partial^{\mu}_{(s_1,s_2,s)} f_s| \le C^{\#}_{\mu}(s_1) C^{\#}_{\mu}(s_2) C^{\#}_{\mu}(s)$ uniformly in q, for any multi-index μ. Then

$$\Big| \mathcal{S}_{L,2} - C_d \int_{\Sigma_s} f_s(s_1, s_2, s_1 + s_2 - s; q)\, \mu^{\Sigma_s}(\mathrm{d}s_1 \mathrm{d}s_2) \Big| \le \frac{C^{\#}(s)}{L^{d-5/2}}, \tag{3.11}$$

uniformly in q, where Σ_s and μ^{Σ_s} are the quadric (1.33) and the measure (1.34) on it.

[12] The formula for s_3 comes from the relation $\delta'^{12}_{3s} = 1$.

Revised Proof

Proof. In the variables $z_1 = s_1 - s$, $z_2 = s_2 - s$ the quadratic form ω_{3s}^{12} with $s_3 = s_1+s_2-s$ reads $\omega_{3s}^{12} = -2z_1 \cdot z_2$ [see (1.17)]. Then, taking into account that the relation $\delta_{3s}'^{12} = 1$ is equivalent to the relations $z_1, z_2 \neq 0$ and $s_3 = s_1+s_2-s$, we find that the sum $\mathcal{S}_{L,2}$ takes the form (3.4). Applying next Theorem 3.1 and changing in (3.6) back to the variables s_1, s_2, we get (3.11). □

Example 3.4. Let us calculate the asymptotic as $L \to \infty$ of $\mathbb{E}|a_s^{(1)}(\tau)|^2$. Expanding $a_s^{(1)}$ as in (2.4) and then using (2.13), we get:

$$\mathbb{E}|a_s^{(1)}(\tau)|^2 = 2L^{-2d} \sum_{1,2,3} \delta_{3s}'^{12} \delta(\omega_{3s}^{12}) \int_0^\tau \mathrm{d}l_1 \int_0^\tau \mathrm{d}l_2 B_{123} \times \prod_{j=1}^{3} \left(e^{-\gamma_j |l_1 - l_2|} - e^{-\gamma_j (l_1+l_2)} \right) e^{\gamma_s (l_1+l_2-2\tau)}$$

with $B_{123} = B_1 B_2 B_3$, where B_s is defined in (2.14). In the case of $\tau = \infty$ the formula simplifies since by changing the integration variables as $r_j := \tau - l_j$ and passing to the limit we get

$$\mathbb{E}|a_s^{(1)}(\infty)|^2 = 2L^{-2d} \sum_{1,2,3} \delta_{3s}'^{12} \delta(\omega_{3s}^{12}) \int_0^\infty \mathrm{d}r_1 \int_0^\infty \mathrm{d}r_2 \, B_{123} \, e^{-(\gamma_1+\gamma_2+\gamma_3)|r_1-r_2|} e^{-\gamma_s (r_1+r_2)} = \frac{2L^{-2d}}{\gamma_s} \sum_{1,2,3} \delta_{3s}'^{12} \delta(\omega_{3s}^{12}) \frac{B_{123}}{\gamma_1+\gamma_2+\gamma_3+\gamma_s}.$$

Then, by Corollary 3.3,

$$\left| L^2 \mathbb{E}|a_s^{(1)}(\infty)|^2 - \frac{2C_d}{\gamma_s} \int_{\Sigma_s} \frac{B_{123} \, \mu^{\Sigma_s}(\mathrm{d}s_1 \mathrm{d}s_2)}{\gamma_1+\gamma_2+\gamma_3+\gamma_s} \right| \leq \frac{C^{\#}(s)}{L^{d-5/2}}, \quad s_3 := s_1 + s_2 - s.$$

3.1. Proof of Theorem 3.2

Let us define the geometric quadrics $Q_j := \{z \in (\mathbb{R}^d)^N : \omega_j(z) = 0\}$ and consider their intersection $Q = \cap_{j=1}^N Q_j$. Note that $Q = \cap_{j=1}^{N-1} Q_j$ since the skew symmetry of the matrix α implies $\omega_1 + \ldots + \omega_N = 0$. Denote by B_R^{Nd} the open cube $|z|_\infty < R$ in $\mathbb{R}^{Nd}$, where by $|\cdot|_\infty$ we denote the l_∞-norm.

Proposition 3.5. *If $w : \mathbb{R}^{Nd} \mapsto \mathbb{R}$ is such that $|w|_{L_\infty} < \infty$ and $\mathrm{supp}(w) \subset B_R^{Nd}$, where $R \geq 1$, then for $N = 2, 3, 4$ we have*

$$\left| \sum_{z \in Q \cap \mathcal{Z}} w(z) \right| \leq C(N,d) R^{\lfloor N/2 \rfloor N_2(d) + (N-2)(d-1)} L^{N(d-1)} |w|_{L_\infty}. \tag{3.12}$$

Here N_2 is defined in Theorem 3.1 and $\mathcal{Z}$—in (3.1).

Proof. Below in this proof for any subset $\mathcal{Q} \subset \mathbb{R}^{md}$, we denote

$$\mathcal{Q}_L = \mathcal{Q} \cap \mathbb{Z}_L^{md}. \tag{3.13}$$

Revised Proof

By suitably rearranging indices i and possibly multiplying ω_i by -1, ω_1 may be assumed to be of the form $\omega_1(z) = z_1 \cdot \sum_i \alpha_{1i} z_i$ with $\alpha_{1N} = 1$. Define $v = \sum_i \alpha_{1i} z_i$ so that

$$\omega_1(z) = z_1 \cdot v \quad \text{and} \quad z_N = \alpha_{1N} z_N = v - \sum_{1<i<N} \alpha_{1i} z_i, \tag{3.14}$$

since $\alpha_{11} = 0$ by the skew symmetry of the matrix α.

For $N > 2$, fix $(z_1, v) \in \mathbb{R}^{2d}$. Then the remaining quadratic forms ω_j with $1 < j < N$ as functions of $(z_2, \dots, z_{N-1}) \in \mathbb{R}^{(N-2)d}$ become polynomials q_j of degree at most two, with no constant term. Namely

$$q_j(z_2, \dots, z_{N-1}; z_1, v) = z_j \cdot \left(\alpha_{j1} z_1 + \alpha_{jN} v + \sum_{1<i<N} (\alpha_{ji} - \alpha_{jN}\alpha_{1i}) z_i \right). \tag{3.15}$$

For $1 < j < N$ consider the sets

$$\tilde{Q}_j(z_1, v) = \{(z_2, \dots, z_{N-1}) : q_j(z_2, \dots, z_{N-1}; z_1, v) = 0\} \subset \mathbb{R}^{(N-2)d},$$

and their intersection $\tilde{Q}(z_1, v) = \cap_{1<j<N} \tilde{Q}_j(z_1, v)$. We denote $Q_1^0 = \{(z_1, v) \in \mathbb{R}^{2d} : z_1 \cdot v = 0\}$ (cf. (3.14)) and set

$$A_2 = \{(z_1, v) \in \mathbb{R}^{2d} : z_1 \neq 0, v \neq 0\}. \tag{3.16}$$

Since $|\alpha_{ij}| \le 1$, then on the support of w we have $|(z_1, v)|_\infty \le (N-1)R$. So, recalling (3.13), for $N > 2$ we get

$$\begin{aligned} \left| \sum_{z \in \mathcal{Z} \cap Q} w(z) \right| &\le C(N, d) |w|_{L_\infty} \sum_{(z_1, v) \in Q^0_{1\,L} \cap B^{2d}_{(N-1)R}} 1 \\ &\quad \times \sup_{(z_1, v) \in Q^0_{1\,L} \cap A_2 \cap B^{2d}_{(N-1)R}} \sum_{(z_2, \dots, z_{N-1}) \in \tilde{Q}_L(z_1, v) \cap B^{(N-2)d}_R} 1 . \end{aligned} \tag{3.17}$$

For $N = 2$ the same estimate holds with the second line replaced by 1.

To estimate the sum in the first line, we take any smooth function $w_0(x) \ge 0$, equal one for $x \le 1$ and vanishing for $x \ge 2$. Then

$$\sum_{(z_1, v) \in Q^0_{1\,L} \cap B^{2d}_{(N-1)R}} 1 \le \sum_{(z_1, v) \in Q^0_{1\,L}} w_R(z_1, v),$$

where $w_R(z_1, v) := w_0\Big(|(z_1, v)| / (N-1) R \sqrt{2d} \Big)$. Since for $R \ge 1$ and any $a \in \mathbb{N} \cup \{0\}$, $b \ge 0$ we have $\|w_R\|_{a,b} \le C(a, b, N, d) R^b$, then in view of Theorem 3.1 and (3.8),

$$\sum_{(z_1, v) \in Q^0_{1\,L} \cap B^{2d}_{(N-1)R}} 1 \le C L^{2(d-1)} \left[R^{2d} + R^{N_2} L^{-d+5/2} \right] \le C' L^{2(d-1)} R^{N_2}, \tag{3.18}$$

where C, C' depend on d, N, N_1 and N_2.

Revised Proof

To estimate the second line of (3.17), we use the following lemma, proved in Appendix A.

Lemma 3.6. *Assume that the matrix α is irreducible. Then for $N = 2, 3, 4$, any $R \geq 1$ and any $(z_1, v) \in B^{2d}_{(N-1)R}$ satisfying $(z_1, v) \in Q^0_{1\,L} \cap A_2$ we have:*

$$\left|\tilde{Q}_L(z_1, v) \cap B_R^{(N-2)d}\right| \leq 2^{(N-2)d}(NRL)^{(N-2)(d-1)}. \tag{3.19}$$

This completes the proof of Proposition 3.5 in the case of irreducible matrix α: indeed, we get

$$\left|\sum_{z \in \mathcal{Z} \cap Q} w(z)\right| \leq C(N, d)|w|_{L_\infty} R^{N_2+(N-2)(d-1)} L^{N(d-1)}. \tag{3.20}$$

If the matrix α is reducible, it can be reduced through permutations to a block diagonal matrix with m blocks which are irreducible square matrices of sizes N_i satisfying $\sum_i N_i = N$. Since $N_i \geq 2$ (otherwise there would be a zero row or column in α), $m \leq \lfloor N/2 \rfloor$. Applying estimate (3.20) to each block, we get the assertion of the proposition. □

Now we derive the theorem from the proposition. Let $\varphi_0(t) = \chi_{(-\infty,1]}(t)$ and for $k \geq 1$, $\varphi_k(t) = \chi_{(2^{k-1},2^k]}(t)$. Then $1 = \sum_k \varphi_k(t)$ and

$$\Phi = \sum_{k=0}^{\infty} f_k(z), \quad f_k(z) = \varphi_k(|z|_\infty)\Phi(z).$$

Then $\operatorname{supp} f_k \subset B_k = \{|z|_\infty \leq 2^k\}$ and $\|f_k\|_\infty \leq C2^{-k\bar{N}}\|\Phi\|_{0,\bar{N}}$, for any $\bar{N}$. Therefore, by Proposition 3.5,

$$|S_{L,N}(\Phi)| \leq C(N, d)\|\Phi\|_{0,\bar{N}} \sum_{k=0}^{\infty} 2^{k(\lfloor N/2 \rfloor N_2+(N-2)(d-1)-\bar{N})},$$

which converges if $\bar{N} > \lfloor N/2 \rfloor N_2 + (N-2)(d-1)$. This completes the proof of Theorem 3.2. □

Remark 3.7. For any fixed vector (z_1, v), $\tilde{Q}(z_1, v)$ is a real algebraic set in $\mathbb{R}^{(N-2)d}$ of codimension $(N-2)$. If $\tilde{Q}(z_1, v)$ were a smooth manifold of that codimension, then estimate (3.19), modified by a multiplicative constant $C_{\tilde{Q}(z_1,v)}$, would be obvious. But $\tilde{Q}(z_1, v)$ is a stratified analytic manifold (with singularities), and to obtain for it a modified version of the estimate (3.19) as above, using analytical tools, seems to be a heavy job since we need a good control for the factor $C_{\tilde{Q}(z_1,v)}$. Instead in Appendix A we prove the lemma, using arithmetical tools.

3.2. On Extension of Theorem 3.2 to any N

The restriction on N in the statement of Theorem 3.2 comes from estimate (3.19) in Lemma 3.6, proved only for $N = 3, 4$. We know that for every N the system of polynomials $q_j(\cdot; z_1, v), 1 < j < N$, defining the set $\tilde{Q}_L(z_1, v)$ in Lemma 3.6, is linearly independent for any (z_1, v) and any irreducible incidence matrix α. Also we know that all polynomials $q_j(\cdot; z_1, v)$ are irreducible;

Revised Proof

see Lemmas A.9 and A.10 in Appendix A (there the independence and reducibility are understood over some specific algebraically closed field K, but the argument also works for K replaced by $\mathbb{C}$). These two facts certainly are insufficient to prove Lemma 3.6 for any N, but they naturally lead to

Conjecture 3.8. *Under assumptions of Lemma 3.6, for any $N \geq 2$*

$$\left|\tilde{Q}_L(z_1, v) \cap B_R^{(N-2)d}\right| \leq C(N,d)(RL)^{(N-2)(d-1)}.$$

One may try to prove this assertion using either arithmetical or analytical tools; cf. Appendix A and Remark 3.7. It is straightforward to see that, if the conjecture is true, then Theorem 3.2 holds for any N, so in view of Lemma 2.2 any expected value $L^N \mathbb{E} a_s^{(m)}(\tau_1)\bar{a}_s^{(n)}(\tau_2)$ admits a uniform in L upper bound.

4. Quasisolutions

In this section we start to study a quasisolution $A(\tau) = A(\tau; L)$ of Eq. (2.1) with $a_s(0) = 0$, which is the second-order truncations of series (2.2):

$$A(\tau) = (A_s(\tau), s \in \mathbb{Z}_L^d), \qquad A_s(\tau) = a_s^{(0)}(\tau) + \rho a_s^{(1)}(\tau) + \rho^2 a_s^{(2)}(\tau). \tag{4.1}$$

We focus on its energy spectrum

$$n_{s,L} = n_{s,L}(\tau) = \mathbb{E}|A_s(\tau)|^2, \quad s \in \mathbb{Z}_L^d, \tag{4.2}$$

when L is large and the parameter ρ is chosen to be $\rho = \varepsilon L$. Our goal is to show that it approximately satisfies a wave kinetic equation (WKE). Using Proposition 2.1, we will then show that the same applies to the quantities $\mathfrak{n}_{s,L}$, considered in the Introduction.

The energy spectrum $n_{s,L}$ is a polynomial in ε of degree four,

$$n_{s,L} = n_{s,L}^{(0)} + \varepsilon\, n_{s,L}^{(1)} + \varepsilon^2 n_{s,L}^{(2)} + \varepsilon^3 n_{s,L}^{(3)} + \varepsilon^4 n_{s,L}^{(4)}, \quad s \in \mathbb{Z}_L^d, \tag{4.3}$$

where the terms $n_{s,L}^{(k)}(\tau)$ are defined by

$$n_{s,L}^{(k)}(\tau) = L^k \sum_{\substack{k_1+k_2=k \\ 0\leq k_1,k_2\leq 2}} \mathbb{E} a_s^{(k_1)}(\tau)\bar{a}_s^{(k_2)}(\tau). \tag{4.4}$$

By Corollary 2.3,

$$\begin{gathered}\text{the second moments } \mathbb{E} a_s^{(k_1)}\bar{a}_s^{(k_2)} \\ \text{naturally extend to a Schwartz function of } s \in \mathbb{R}^d,\end{gathered} \tag{4.5}$$

given by (2.18), (2.21). Accordingly, from now on we always regard the second moments and the terms $n_{s,L}^{(k)}(\tau)$ as Schwartz functions of $s \in \mathbb{R}^d$.

As customary in WT, we aim at considering the limit of $n_{s,L}(\tau)$ as $L \to \infty$, that is, the limits of the terms $n_{s,L}^{(j)}$. The term $n_{s,L}^{(0)} = \mathfrak{n}_{s,L}^{(0)}$ is given by (1.31) and is L-independent, while by a direct computation we see that

$$n_{s,L}^{(1)} = 2\Re\mathbb{E}\bar{a}_s^{(0)} a_s^{(1)} = 0, \quad s \in \mathbb{R}^d, \tag{4.6}$$

Revised Proof

[here we use (2.4), the Wick theorem and (1.15)]. Writing explicitly $n_{s,L}^{(i)}$ with $2 \le i \le 4$, we find that

$$\begin{aligned} n_{s,L}^{(2)} &= L^2\mathbb{E}\big(|a_s^{(1)}|^2 + 2\Re\bar{a}_s^{(0)}a_s^{(2)}\big), \\ n_{s,L}^{(3)} &= 2L^3\Re\mathbb{E}\bar{a}_s^{(1)}a_s^{(2)}, \qquad n_{s,L}^{(4)} = L^4\mathbb{E}|a_s^{(2)}|^2. \end{aligned} \tag{4.7}$$

The function $\mathbb{R}^d \ni s \mapsto n_{s,L}^{(2)}(\tau)$ is made by two terms. By Corollary 2.3 with $N = 2$, Theorem 3.1 applies to the both of them. Since $d \ge 3$, we get

$$|n_{s,L}^{(2)}(\tau) - n_s^{(2)}(\tau)| \le C^{\#}(s)/L^{1/2}, \tag{4.8}$$

where

$$n_s^{(2)}(\tau) := C_d\left(\sum_{\mathfrak{F}\in\mathfrak{F}_{1,1}^{true}} + 2\Re\sum_{\mathfrak{F}\in\mathfrak{F}_{2,0}^{true}}\right)c_{\mathfrak{F}}\int_{\Sigma_0}\mu^{\Sigma_0}(\mathrm{d}z_1\mathrm{d}z_2)\Phi_s^{\mathfrak{F}}(\tau,\tau,z),$$

and we have used estimate (3.10). Thus, we see that the processes $n_{s,L}^{(0)}$, $n_{s,L}^{(1)}$ and $n_{s,L}^{(2)}$ admit the limits

$$n_s^{(j)}(\tau) := \lim_{L\to\infty} n_{s,L}^{(j)}(\tau; L).$$

The limits satisfy (4.8), and for all τ

$$n_s^{(0)}(\tau) = B(s)\big(1 - e^{-2\gamma_s(\tau_0+\tau)}\big), \quad n_s^{(1)}(\tau) = 0, \quad |n_s^{(2)}(\tau)| \le C^{\#}(s)\,, \tag{4.9}$$

where the last inequality follows from Theorem 3.2.

We do not know if the terms $n_{s,L}^{(3)}, n_{s,L}^{(4)}$ admit limits as $L \to \infty$, but in view of Corollary 2.3 both of them may be estimated through Theorem 3.2:

$$|n_{s,L}^{(3)}(\tau)| \le C^{\#}(s), \quad |n_{s,L}^{(4)}(\tau)| \le C^{\#}(s), \tag{4.10}$$

uniformly in $L \ge 2$ and $\tau \ge 0$. We then decompose

$$n_{s,L} = n_{s,L}^{\le 2} + n_{s,L}^{\ge 3},$$

where

$$n_{s,L}^{\le 2} = n_{s,L}^{(0)} + \varepsilon n_{s,L}^{(1)} + \varepsilon^2 n_{s,L}^{(2)} \quad \text{and} \quad n_{s,L}^{\ge 3} = \varepsilon^3 n_{s,L}^{(3)} + \varepsilon^4 n_{s,L}^{(4)}$$

(we recall that $n_{s,L}^{(1)} \equiv 0$), and similarly define

$$n_s^{\le 2} := n_s^{(0)} + \varepsilon^2 n_s^{(2)}.$$

Due to (4.8),

$$|n_s^{\le 2}(\tau) - n_{s,L}^{\le 2}(\tau)| \le C^{\#}(s)\varepsilon^2 L^{-1/2}, \tag{4.11}$$

so by (4.10),

$$|n_s^{\le 2}(\tau) - n_{s,L}(\tau)| \le C^{\#}(s)\varepsilon^2(L^{-1/2} + \varepsilon). \tag{4.12}$$

Thus, the cut energy spectrum $n_s^{\le 2}$ governs the limiting as $L \to \infty$ behaviour of the energy spectrum $n_{s,L}$ with precision $\varepsilon^3 C^{\#}(s)$, where we regard the constant $\varepsilon \le 1/2$ (which measures the size of solutions for (1.6) under the

Revised Proof

proper scaling) as a fixed small parameter. Accordingly, our next goal is to show that $n_s^{\leq 2}(\tau)$ approximates the solution of a WKE.

4.1. Increments of the Energy Spectra $n_s^{\leq 2}$

In this section we will show that the process $n_s^{\leq 2}(\tau)$ approximately satisfies a WKE. We denote $s_4 := s$, $\gamma_j := \gamma_{s_j}$ and set

$$\gamma_{1234} = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4, \quad \vec{s} = (s_1, s_2, s_3, s_4) \in (\mathbb{R}^d)^4. \tag{4.13}$$

Now, for a fixed $\tau_0 \geq 0$ and for $j = 1, 2, 3, 4$ we define the functions $\mathcal{Z}^j(\tau_0) = \mathcal{Z}^j(\tau_0; \vec{s})$ as

$$\mathcal{Z}^j(\tau_0; \vec{s}) := \int_0^{\tau_0} \mathrm{d}l \, \mathrm{e}^{-\gamma_j(\tau_0 - l)} \prod_{\substack{m=1,2,3,4 \\ m \neq j}} \frac{\sinh(\gamma_m l)}{\sinh(\gamma_m \tau_0)} \quad \text{if } \tau_0 > 0, \tag{4.14}$$

and $\mathcal{Z}^j(0; \vec{s}) = 0$. Computing this integral, we get

$$\begin{aligned} \mathcal{Z}^j(\tau_0; \vec{s}) = \left(\prod_{l \neq j} \frac{1}{1 - e^{-2\gamma_l \tau_0}} \right) \cdot \Bigg[& \frac{1 - e^{-\gamma_{1234}\tau_0}}{\gamma_{1234}} - \frac{e^{-2(\gamma_{1234} - \gamma_j)\tau_0} - e^{-\gamma_{1234}\tau_0}}{2\gamma_j - \gamma_{1234}} \\ & + \sum_{l \neq j} \left(\frac{e^{-2(\gamma_{1234} - \gamma_j - \gamma_l)\tau_0} - e^{-\gamma_{1234}\tau_0}}{2(\gamma_j + \gamma_l) - \gamma_{1234}} - \frac{e^{-2\gamma_l \tau_0} - e^{-\gamma_{1234}\tau_0}}{\gamma_{1234} - 2\gamma_l} \right) \Bigg], \end{aligned} \tag{4.15}$$

where each fraction from the square brackets should be substituted by $\tau_0 e^{-\gamma_{1234}\tau_0}$ if its denominator vanishes.

For any real number r let $\mathcal{C}_r(\mathbb{R}^d)$ be the space of continuous complex functions on $\mathbb{R}^d$ with the finite norm

$$|f|_r = |f(z)\langle z \rangle^r|_{L_\infty}. \tag{4.16}$$

We naturally extend this norm to $f \in L_\infty(\mathbb{R}^d)$ and set

$$L_{\infty,r}(\mathbb{R}^d) = \left\{ f \in L_\infty(\mathbb{R}^d) : |f|_r < \infty \right\}. \tag{4.17}$$

Consider also the linear operator $\mathcal{L}$, given by

$$(\mathcal{L}v)(s) = 2\gamma_s v(s), \quad s \in \mathbb{R}^d. \tag{4.18}$$

Below we often write the value $v(s)$ of a function v at $s \in \mathbb{R}^d$ as v_s and the function v itself as $(v_s,\, s \in \mathbb{R}^d)$. Now, for $v \in \mathcal{C}_r(\mathbb{R}^d)$, where $r > d$, and for $\tau_0 \geq 0$, $\tau \in (0, 1]$, we define the kinetic integral $K^\tau(\tau_0)(v) = (K_s^\tau(\tau_0)(v),\, s \in \mathbb{R}^d)$:

$$K^\tau(\tau_0)(v) = \int_0^\tau e^{-t\mathcal{L}} K(\tau_0)(v) \mathrm{d}t. \tag{4.19}$$

Revised Proof

Here the operator $K(\tau_0) = K^1(\tau_0) + \cdots + K^4(\tau_0)$ sends a function $v = (v_s, s \in \mathbb{R}^d)$ to the function

$$\begin{aligned} K_s(\tau_0)(v) &= 4C_d \int_{\Sigma_s} \mu^{\Sigma_s}(\mathrm{d}s_1 \mathrm{d}s_2)\Big(\mathcal{Z}^4(\tau_0;\vec{s})v_1v_2v_3 \\ &\qquad + \mathcal{Z}^3(\tau_0;\vec{s})v_1v_2v_4 - \mathcal{Z}^2(\tau_0;\vec{s})v_1v_3v_4 - \mathcal{Z}^1(\tau_0;\vec{s})v_2v_3v_4\Big) \\ &=: K_s^4(\tau_0)(v) + K_s^3(\tau_0)(v) + K_s^2(\tau_0)(v) + K_s^1(\tau_0)(v) \end{aligned} \tag{4.20}$$

(note the reversed signs for K^2 and K^3). Here $v_j := v(s_j)$, where $s_4 = s$ and $s_3 := s_1 + s_2 - s_4$ (in view of the factor δ'^{12}_{3s}). While μ^{Σ_s} is the measure (1.34) on the quadric $\Sigma_s = \{(s_1, s_2) \in \mathbb{R}^{2d} : (s_1 - s) \cdot (s_2 - s) = 0\}$. Computing the integral in t in (4.19), we find

$$K_s^\tau(\tau_0)(v) = \frac{1 - e^{-2\gamma_s \tau}}{2\gamma_s} K_s(\tau_0)(v) = \frac{1 - e^{-2\gamma_s \tau}}{2\gamma_s} \sum_{j=1}^4 K_s^j(\tau_0)(v). \tag{4.21}$$

We study the kinetic integral K^τ in Sect. 5 while now we formulate a result which is the main step in deriving the wave kinetic limit.

Theorem 4.1. *For any $0 < \tau \le 1$ the function $(n_s^{\le 2}, s \in \mathbb{R}^d)$ satisfies*

$$\begin{aligned} n^{\le 2}(\tau_0 + \tau) &= e^{-\tau\mathcal{L}} n^{\le 2}(\tau_0) + 2\int_0^\tau e^{-t\mathcal{L}} b^2 \,\mathrm{d}t \\ &\qquad + \varepsilon^2 K^\tau(\tau_0)(n^{\le 2}(\tau_0)) + \varepsilon^2 \mathcal{R}, \end{aligned} \tag{4.22}$$

where $b^2 = (b^2(s), s \in \mathbb{R}^d)$ and the remainder $\mathcal{R}_s(\tau)$ satisfies

$$|\mathcal{R}(\tau)|_r \le C_r \tau(\tau + \varepsilon^2), \quad \forall r. \tag{4.23}$$

4.2. Proof of Theorem 4.1

We first fix a value for L and decompose the processes $\tau \mapsto a_s^{(i)}(\tau_0 + \tau)$, where $\tau_0 \ge 0$ and $0 \le \tau \le 1$, as

$$a_s^{(i)}(\tau_0 + \tau) = c_s^{(i)}(\tau; \tau_0) + \Delta a_s^{(i)}(\tau; \tau_0), \qquad i = 0, 1, 2, \quad s \in \mathbb{Z}_L^d. \tag{4.24}$$

Here

$$c_s^{(i)}(\tau; \tau_0) = e^{-\gamma_s \tau} a_s^{(i)}(\tau_0)$$

and $\Delta a_s^{(i)}$ is defined via relation (4.24). Below we write $c_s^{(i)}(\tau; \tau_0)$ and $\Delta a_s^{(i)}(\tau; \tau_0)$ as $c_s^{(i)}(\tau)$ and $\Delta a_s^{(i)}(\tau)$ since τ_0 is fixed.

Obviously,

$$c(\tau) := c^{(0)}(\tau) + \rho c^{(1)}(\tau) + \rho^2 c^{(2)}(\tau)$$

with $\tau \ge 0$ being a solution of the linear equation $(2.1)_{\rho=0, b(s)\equiv 0}$, equal $A(\tau_0)$ at $\tau = 0$, and $\Delta a(\tau) = \sum_{j=0}^2 \rho^j \Delta a^{(j)}(\tau)$ equals $A(\tau_0 + \tau) - c(\tau)$. By (4.5), for $0 \le i, j \le 2$

$$\begin{gathered} \text{the functions } \mathbb{E} c_s^{(i)} \bar{c}_s^{(j)}, \quad \mathbb{E} c_s^{(i)} \Delta \bar{a}_s^{(j)}, \quad \mathbb{E} \Delta a_s^{(i)} \Delta \bar{a}_s^{(j)} \\ \text{naturally extend to Schwartz functions of } s \in \mathbb{R}^d. \end{gathered} \tag{4.25}$$

Revised Proof

Due to (4.6) and (4.7),

$$e^{-2\gamma_s\tau} n^{\le 2}_{s,L}(\tau_0) = \mathbb{E}|c^{(0)}_s(\tau)|^2 + \rho^2\mathbb{E}\big(|c^{(1)}_s(\tau)|^2 + 2\Re\bar{c}^{(0)}_s(\tau)c^{(2)}_s(\tau)\big), \quad \forall\, s\in\mathbb{R}^d.$$

Then,

$$\begin{aligned} n^{\le 2}_{s,L}(\tau_0+\tau) - e^{-2\gamma_s\tau} n^{\le 2}_{s,L}(\tau_0) = \mathbb{E}\Big(&|a^{(0)}_s(\tau_0+\tau)|^2 - |c^{(0)}_s(\tau)|^2 \\ &+ \rho^2\big(|a^{(1)}_s(\tau_0+\tau)|^2 - |c^{(1)}_s(\tau)|^2 \\ &+ 2\Re\big(a^{(2)}_s\bar{a}^{(0)}_s(\tau_0+\tau) - c^{(2)}_s\bar{c}^{(0)}_s(\tau)\big)\Big). \end{aligned} \tag{4.26}$$

Let us set

$$\mathcal{Y}_s(u,v,w) := L^{-d}\sum_{1,2,3} \delta'^{12}_{3s}\delta(\omega^{12}_{3s})u_1v_2\bar{w}_3.$$

Writing explicitly the processes $\Delta a^{(i)}_s(\tau)$, $s\in\mathbb{Z}^d_L$, we find

$$\begin{aligned} \Delta a^{(0)}_s(\tau) &= b(s)\int_{\tau_0}^{\tau_0+\tau} e^{-\gamma_s(\tau_0+\tau-l)}\, d\beta_s(l), \\ \Delta a^{(1)}_s(\tau) &= i\int_{\tau_0}^{\tau_0+\tau} e^{-\gamma_s(\tau_0+\tau-l)}\mathcal{Y}_s(a^{(0)})\,\mathrm{d}l, \\ \Delta a^{(2)}_s(\tau) &= i\int_{\tau_0}^{\tau_0+\tau} e^{-\gamma_s(\tau_0+\tau-l)}\Big(\mathcal{Y}_s(a^{(0)},a^{(0)},a^{(1)}) \\ &\qquad + \mathcal{Y}_s(a^{(0)},a^{(1)},a^{(0)}) + \mathcal{Y}_s(a^{(1)},a^{(0)},a^{(0)})\Big)\mathrm{d}l, \end{aligned} \tag{4.27}$$

where $a^{(i)} = a^{(i)}(l)$. Note that to get explicit formulas for $c^{(i)}_s(\tau)$, $i=0,1,2$, it suffices to replace in the r.h.s.'s of the relations in (4.27) the range of integration from $[\tau_0,\tau_0+\tau]$ to $[0,\tau_0]$. For example, $c^{(0)}_s(\tau) = e^{-\gamma_s\tau}a^{(0)}_s(\tau_0) = b(s)\int_0^{\tau_0} e^{-\gamma_s(\tau_0+\tau-l)}\, d\beta_s(l)$.

Using that $\mathbb{E}c^{(i)}_s(\tau)\Delta\bar{a}^{(0)}_s(\tau) = \mathbb{E}c^{(i)}_s(\tau)\,\mathbb{E}\Delta\bar{a}^{(0)}_s(\tau) = 0$ for any i and s, we obtain

$$\mathbb{E}\big(a^{(2)}_s\bar{a}^{(0)}_s(\tau_0+\tau) - c^{(2)}_s\bar{c}^{(0)}_s(\tau)\big) = \mathbb{E}\Delta a^{(2)}_s(\tau)\bar{a}^{(0)}_s(\tau_0+\tau), \tag{4.28}$$

and from (4.24) we get that

$$\begin{aligned} |a^{(1)}_s(\tau_0+\tau)|^2 - |c^{(1)}_s(\tau)|^2 &= |\Delta a^{(1)}_s(\tau)|^2 + 2\Re\Delta a^{(1)}_s\bar{c}^{(1)}_s(\tau), \\ \mathbb{E}\big(|a^{(0)}_s(\tau_0+\tau)|^2 - |c^{(0)}_s(\tau)|^2\big) &= \mathbb{E}|\Delta a^{(0)}_s(\tau)|^2. \end{aligned} \tag{4.29}$$

Then, inserting (4.28) and (4.29) into (4.26) and using that $\rho = \varepsilon L$, we find

$$n^{\le 2}_{s,L}(\tau_0+\tau) - e^{-2\gamma_s\tau} n^{\le 2}_{s,L}(\tau_0) = \mathbb{E}\big|\Delta a^{(0)}_s(\tau)\big|^2 + \varepsilon^2 Q_{s,L}(\tau_0,\tau), \quad s\in\mathbb{R}^d,$$

Revised Proof

where

$$Q_{s,L}(\tau_0,\tau) := L^2\Big(\mathbb{E}|\Delta a_s^{(1)}(\tau)|^2 + 2\Re\mathbb{E}\big[\Delta a_s^{(1)}(\tau)\bar{c}_s^{(1)}(\tau) + \Delta a_s^{(2)}(\tau)\bar{a}_s^{(0)}(\tau_0+\tau)\big]\Big), \tag{4.30}$$

and we recall (4.25). Since

$$\mathbb{E}\big|\Delta a_s^{(0)}(\tau)\big|^2 = \frac{b(s)^2}{\gamma_s}(1-e^{-2\gamma_s\tau}) = 2\int_0^\tau e^{-t\mathcal{L}}b^2(s)\,\mathrm{d}t,$$

then

$$n_{\cdot,L}^{\leq 2}(\tau_0+\tau) - e^{-\tau\mathcal{L}}n_{\cdot,L}^{\leq 2}(\tau_0) = 2\int_0^\tau e^{-t\mathcal{L}}b^2\,\mathrm{d}t + \varepsilon^2 Q_{\cdot,L}(\tau_0,\tau),$$

for $n_{\cdot,L}^{\leq 2} = (n_{s,L}^{\leq 2},\ s\in\mathbb{R}^d)$. In order to pass to the limit $L\to\infty$ we recall the relation (4.11). Then the desired formula (4.22) is an immediate consequence of the assertion below:

Proposition 4.2. *We have*

$$\lim_{L\to\infty} Q_{s,L}(\tau_0,\tau) = K_s^\tau(\tau_0)(n^{\leq 2}(\tau_0)) + \mathcal{R}_s(\tau), \quad s\in\mathbb{R}^d, \tag{4.31}$$

where the remainder $\mathcal{R}$ satisfies (4.23).

Proof. The first step in the proof of (4.31) is the following result, established in Appendix B:

Proposition 4.3. *One has*

$$\big|Q_{s,L}(\tau_0,\tau) - \mathcal{X}_{s,L}(\tau_0,\tau)\big| \leq C^{\#}(s)\tau^2, \quad s\in\mathbb{R}^d, \tag{4.32}$$

where

$$\mathcal{X}_{s,L}(\tau_0,\tau) := 4L^{2(1-d)}\tau\sum_{1,2,3}\delta'^{12}_{3s}\delta(\omega^{12}_{3s})\big(\mathcal{Z}^4 n_1^{(0)}n_2^{(0)}n_3^{(0)} + \mathcal{Z}^3 n_1^{(0)}n_2^{(0)}n_s^{(0)} - \mathcal{Z}^1 n_2^{(0)}n_3^{(0)}n_s^{(0)} - \mathcal{Z}^2 n_1^{(0)}n_3^{(0)}n_s^{(0)}\big). \tag{4.33}$$

The terms $\mathcal{Z}^j = \mathcal{Z}^j(\tau_0; s_1,s_2,s_3,s)$ are defined by (4.14) and $n_i^{(0)} := n_{s_i,L}^{(0)}(\tau_0)$, $n_s^{(0)} := n_{s,L}^{(0)}(\tau_0)$.

By (4.9) $n_i^{(0)} = n_{s_i}^{(0)}$ are Schwartz functions in s_i. Besides, the functions $\mathcal{Z}^j(\tau_0,\vec{s})$ have at most polynomial growth in $\vec{s}$ together with their derivatives, uniformly in $\tau_0\geq 0$:

Lemma 4.4. *For any vector $\mu\in(\mathbb{N}\cup\{0\})^{4d}$, uniformly in $\tau_0\geq 0$, we have $\big|\partial_{\vec{s}}^\mu\mathcal{Z}^j(\tau_0,\vec{s})\big| \leq P(\vec{s};\mu)$, where $P(\vec{s};\mu)$ has at most a polynomial growth in $\vec{s}$.*

By the lemma, which is proven in Sect. B.7, $\mathcal{X}_{s,L}$ satisfies the hypotheses of Corollary 3.3. So

$$|\mathcal{X}_{s,L}(\tau_0,\tau) - \tau K_s(\tau_0)(n^{(0)})| \leq C^{\#}(s)L^{-1/2}\tau. \tag{4.34}$$

Revised Proof

Next, note that $|n_s^{(0)}(\tau_0)-n_s^{\leq 2}(\tau_0)| \leq C^{\#}(s)\varepsilon^2$ due to (4.9). Then the estimate on the Lipschitz constants of the operators $K^j(t)$, given in (5.4), implies that

$$\left|K(\tau_0)(n^{(0)}(\tau_0)) - K(\tau_0)(n^{\leq 2}(\tau_0))\right|_r \leq C_r\varepsilon^2 \quad \forall r.$$

Hence,

$$|\tau K_s(\tau_0)(n^{(0)}(\tau_0)) - \tau K_s(\tau_0)(n^{\leq 2}(\tau_0))| \leq C^{\#}(s)\tau\varepsilon^2. \tag{4.35}$$

On the other hand, on account of the definition (4.21), for $0 \leq \tau \leq 1$ we have the bound

$$\left|\tau K_s(\tau_0)(n^{\leq 2}) - K_s^{\tau}(\tau_0)(n^{\leq 2})\right| \leq C\gamma_s\tau^2|K_s(\tau_0)(n^{\leq 2})| \leq C^{\#}(s)\tau^2, \tag{4.36}$$

where the last inequality follows from the estimate of the norm of the operator $K^j(t)$, given in (5.3), and from (4.9).

Putting together (4.32), (4.34), (4.35), (4.36) and letting L grow to infinity, we conclude the proof. □

5. Kinetic Equation

At this section we examine the wave kinetic equation

$$\dot{\mathfrak{z}}_s(\tau) = -(\mathcal{L}\mathfrak{z})_s + \varepsilon^2 K_s(\tau)(\mathfrak{z}) + 2b(s)^2, \qquad \tau \geq 0, \;\; \mathfrak{z}(0) = 0 \tag{5.1}$$

($\mathcal{L}$ is defined in (4.18) and the operator $K = K^1 + \cdots + K^4$ is defined in (4.20)), and next we derive from this analysis and (4.22) the proximity of $n_s^{\leq 2}(\tau)$ to a solution of (5.1). We will need the following result, which is Lemma 4.2 from [7]:

Lemma 5.1. *For $j, l = 1, \ldots, 4$ and $u^j \in \mathcal{C}_r(\mathbb{R}^d)$ consider the operators*

$$J_l(u^1, \ldots, u^4)(s) = \int_{\Sigma_s} \mu^{\Sigma_s}(\mathrm{d}s_1 \mathrm{d}s_2) \prod_{i \neq l} u^i(s_i)$$

(see (1.34)), where $s_4 = s$ and $s_3 = s_1 + s_2 - s$. Then for each l,

$$|J_l(u^1, \ldots, u^4)|_{r+1} \leq C_r \prod_{i \neq l} |u^i|_r \qquad \text{if } r > d. \tag{5.2}$$

5.1. Kinetic Integrals

We recall notation (4.13), (4.14).

Lemma 5.2. *For $j = 1, \ldots 4$, any $\tau \geq 0$ and any $\vec{s} = (s_1, \ldots, s_4) \in (\mathbb{R}^d)^4$,*

(i) $0 \leq \mathcal{Z}^j(\tau; \vec{s}) \leq \min(\tau, 1/\gamma_{s_j}) \leq 1$,

(ii) $|\mathcal{Z}^j(\tau; \vec{s}) - \mathcal{Z}(\infty; \vec{s})| \leq Ce^{-2\tau}$, *where* $\mathcal{Z}(\infty; \vec{s}) = 1/\gamma_{1234}$.

Proof. The first assertion follows from (4.14) since $\sinh(x)$ is an increasing non-negative function of $x \geq 0$, so in the integrand in (4.14) we have $0 \leq \sinh(\gamma_m l)/\sinh(\gamma_m \tau') \leq 1$. For $0 \leq \tau \leq 1$ the second estimate follows from the first one as

$$|\mathcal{Z}^j(\tau; \vec{s}) - \mathcal{Z}(\infty; \vec{s})| \leq |\mathcal{Z}^j(\tau; \vec{s})| + |\mathcal{Z}(\infty; \vec{s})|,$$

while for $\tau \geq 1$ it follows from (4.15) since $\gamma_{123j} - \gamma_j \geq 1$ and $\gamma_{1234} - \gamma_j - \gamma_l \geq 1$ for $j, l \in \{1, 2, 3, 4\}, j \neq l$. □

Revised Proof

Since the kernels $\mathcal{Z}^j$ are non-negative by the first assertion of the lemma above, then denoting $\kappa_1 = \kappa_2 = 1$, $\kappa_3 = \kappa_4 = -1$ we achieve that the operators $\kappa_j K^j$, $1 \le j \le 4$, are positive (in the sense that they send positive functions to positive). Due to the first assertion of the lemma and (5.2), for any $\tau \ge 0$ they define positive 3-homogeneous mappings $\mathcal{C}_r(\mathbb{R}^d) \to \mathcal{C}_{r+1}(\mathbb{R}^d)$ if $r > d$, and

$$|\kappa_j K^j(\tau)(v)|_{r+1} = |K^j(\tau)(v)|_{r+1} \le C_r \min(\tau, 1)|v|_r^3, \quad j = 1, \dots, 4, \tag{5.3}$$

for $\tau \ge 0$. So, the mappings $K^j(\tau)$ are locally Lipschitz:

$$\begin{aligned} &|K^j(\tau)(v^1) - K^j(\tau)(v^2)|_{r+1} \\ &\quad \le 3C_r \min(\tau, 1)R^2|v^1 - v^2|_r, \quad \text{if } |v^1|_r, |v^2|_r \le R. \end{aligned} \tag{5.4}$$

Since for $j = 1, \dots, 4$ and any $s \in \mathbb{R}^d$,

- for non-negative functions $n, m \in L_{\infty,r}$ [see (4.17)] such that $m \le n$ we have $\kappa_j K_s^j(\tau)(m) \le \kappa_j K_s^j(\tau)(n) \le \infty$,
- $|K_s^j(\tau)(v)| \le \kappa_j K_s^j(\tau)(|v|) \le \infty$ for any complex function $v \in L_\infty$,
- $|v_s| \le |v|_r \langle s \rangle^{-r}$ for all $v \in L_{\infty,r}$,

then the relations (5.3), (5.4) remain true for functions from $L_{\infty,r}$.

Lemma 5.3. *If $|s_l| \le R$ for $l = 1, \dots, 4$, then*

$$\left| \frac{\partial}{\partial \tau} \mathcal{Z}^j(\tau; \vec{s}) \right| \le C\gamma^0(R^2) \tag{5.5}$$

(see (1.5)).

Proof. For any $m \in \{s_1, \dots, s_4\}$ and $0 \le l \le \tau$, we have

$$\left| \frac{\partial}{\partial \tau} \frac{\sinh \gamma_m l}{\sinh \gamma_m \tau} \right| \le \gamma_m \frac{\cosh \gamma_m \tau}{\sinh \gamma_m \tau} \le \gamma_m C \max(1, 1/(\gamma_m \tau)).$$

Considering separately the cases $\tau \ge 1$ and $0 \le \tau < 1$, using (4.14) and the estimate above we get the result. □

This lemma implies that for any $v \in \mathcal{C}_r(\mathbb{R}^d)$ and any j the curve $\tau \mapsto K^j(\tau)(v) \in \mathcal{C}_r(\mathbb{R}^d)$ is Hölder continuous:

Lemma 5.4. *For any $\tau_0 \ge 0$, $0 \le \tau \le 1$, $j = 1, \dots, 4$ and any $r > d + 1$,*

$$|K^j(\tau_0 + \tau)(v) - K^j(\tau_0)(v)|_r \le C_r |v|_r^3 \tau^{\kappa_*} \qquad \forall\, v \in \mathcal{C}_r(\mathbb{R}^d), \tag{5.6}$$

where $\kappa_* = 1/(1 + 2r_*)$.

Proof. By the homogeneity we may assume that $|v|_r = 1$. For $R \ge 1$ let us set $v^R(s) = v(s)\chi_{|s| \le R} \in L_\infty$. Then

$$|v^R|_r \le 1, \quad |v - v^R|_{r-1} \le R^{-1}. \tag{5.7}$$

Now let us write the increment $K^j(\tau_0 + \tau)(v) - K^j(\tau_0)(v)$ as

$$\begin{aligned} &\big(K^j(\tau_0 + \tau)(v) - K^j(\tau_0 + \tau)(v^R)\big) + \big(K^j(\tau_0 + \tau)(v^R) - K^j(\tau_0)(v^R)\big) \\ &\quad + \big(K^j(\tau_0)(v^R) - K^j(\tau_0)(v)\big) =: \Delta_1 + \Delta_2 + \Delta_3. \end{aligned}$$

Recalling that (5.3) and (5.4) hold for functions from $L_{\infty,r'}$ with $r' > d$, we get from (5.7) that $|\Delta_1|_r + |\Delta_3|_r \le C_r R^{-1}$. To estimate Δ_2, we set $\Delta_2^R = \Delta_2 \chi_{|s| \le R}$.

Revised Proof

Since by (5.3), $|\Delta_2|_{r+1} \le 2C_r$, then $|\Delta_2 - \Delta_2^R|_r \le 2C_r R^{-1}$. For $|s| > R$ the function Δ_2^R vanishes, while for $|s| \le R$ in view of Lemma 5.3 we have

$$\begin{aligned}|\Delta_{2\,s}^R| = |\Delta_{2s}| &= \big|K_s^j(\tau_0+\tau)(v^R) - K_s^j(\tau_0)(v^R)\big| \\ &\le C_r \int_{\Sigma_s} \mu_s(dv_1dv_2)\big|\mathcal{Z}^j(\tau_0+\tau;\vec{s}) - \mathcal{Z}^j(\tau_0;\vec{s})\big| \frac{|v_1|\dots|v_4|}{|v_j|}\chi_{\{|s_j|\le R\ \forall j\}} \\ &\le C_{1r}\gamma^0(R^2)\tau \int_{\Sigma_s} \mu_s(dv_1dv_2)\frac{|v_1|\dots|v_4|}{|v_j|} \le C_{2r}\langle s\rangle^{-r-1}\tau R^{2r_*},\end{aligned}$$

where to get the last inequality we used (5.2). We have seen that the $\mathcal{C}_r$-norm of the increment is bounded by $C_r(R^{-1} + \tau R^{2r_*})$, for any $R \ge 1$. Choosing $R = \tau^{-1/(1+2r_*)}$, we achieve (5.6). □

5.2. Kinetic Equation

Now we will apply the obtained results to the kinetic Eq. (5.1). Since the function $b(\cdot)^2 := \{b(s)^2\} \in \mathcal{C}_r(\mathbb{R}^d)$ for all r, since $\mathcal{L}$ is the operator of multiplying by the function $2\gamma_s$ as in (1.4), 1.5), and the operator K satisfies (5.3), 5.4), then for small enough $\varepsilon > 0$ Eq. (5.1) has a unique solution, belonging to $\mathcal{C}_r(\mathbb{R}^d)$ for each r, which in a Lipschitz way depends on the r.h.s. of the equation, when the latter deviates from $b(\cdot)^2$. Namely, the following result, where X^r stands for the space $C(0,\infty;\mathcal{C}_r(\mathbb{R}^d))$, given the norm $|v(\cdot)|_{X^r} = \sup_{t\ge 0}|v(t)|_r$, may be easily verified (a proof of a similar fact may be found in Section 4 of [7]).

Lemma 5.5. *For any $r > d$,*

(1) There exists ε_*, depending on $b(\cdot), r$ and r_*, such that for $0 \le \varepsilon \le \varepsilon_*$ Eq. (5.1) has a unique solution $\mathfrak{z}(\tau)$, belonging to X^r. It satisfies

$$|\mathfrak{z}|_{X^r} \le C_r|b^2|_r. \tag{5.8}$$

(2) If $\mathfrak{z}^0(\tau)$ is a solution of the linear Eq. (5.1)$|_{\varepsilon=0}$, then $|\mathfrak{z} - \mathfrak{z}^0|_{X^r} \le C_r\varepsilon^2$. If a curve $\mathfrak{z}'(\tau)$ solves (5.1) with $2b(s)^2$ replaced by $2b(s)^2 + \xi_s(t)$, where $\xi \in X^r$ and $|\xi|_{X^r} \le 1$, then $|\mathfrak{z} - \mathfrak{z}'|_{X^r} \le C_r|\xi|_{X^r}$.

The lemma's assertion holds as well for nonzero initial conditions $\mathfrak{z}(0) \in \mathcal{C}_r(\mathbb{R}^d)$ in (5.1), but we do not need this.

Let $K(\infty)$ be the operator, obtained from $K(\tau_0)$ by replacing in (4.20) the kernels $\mathcal{Z}^j(\tau_0;s)$, $s \in \mathbb{R}^d$, by $\mathcal{Z}(\infty;\vec{s})$ (see Lemma 5.2). Let $r > d$ and $\mathfrak{z}^\varepsilon \in \mathcal{C}_r(\mathbb{R}^d)$ be a solution of the limiting stationary equation

$$\mathcal{L}\mathfrak{z}^\varepsilon - \varepsilon^2 K(\infty)(\mathfrak{z}^\varepsilon) = 2b(\cdot)^2 \tag{5.9}$$

in the vicinity of $\mathcal{L}^{-1}(2b^2)$, existing for small ε by the inverse function theorem. Since $b^2(\cdot) \in \bigcap_r \mathcal{C}_r(\mathbb{R}^d)$ and, as in (5.3), the map $K(\infty)$ is one-smoothing, then decreasing ε_* if needed we achieve that $\mathfrak{z}^\varepsilon \in \bigcap_r \mathcal{C}_r(\mathbb{R}^d)$ for $\varepsilon \le \varepsilon_*$ and

$$|\mathfrak{z}^\varepsilon - 2\mathcal{L}^{-1}(b^2)|_m \le C_m\varepsilon^2 \quad \forall\, m. \tag{5.10}$$

Here and below the constants depend on b and r_*.

Let us consider the curve $w(t) = \mathfrak{z}(t) - \mathfrak{z}^\varepsilon$. It satisfies the equation

$$\begin{aligned}\dot{w} + \mathcal{L}(w) &= \varepsilon^2\big(K(t)(\mathfrak{z}) - K(\infty)(\mathfrak{z}^\varepsilon)\big) \\ &= \varepsilon^2\big[\big(K(t)(\mathfrak{z}) - K(t)(\mathfrak{z}^\varepsilon)\big) - \big(K(t)(\mathfrak{z}^\varepsilon) - K(\infty)(\mathfrak{z}^\varepsilon)\big)\big]\end{aligned}$$

Revised Proof

and $w(0) = -\mathfrak{z}^\varepsilon$. Denote $K(\tau)(\mathfrak{z}^\varepsilon) - K(\infty)(\mathfrak{z}^\varepsilon) =: -\eta(\tau)$. In view of Lemmas 5.2 and 5.1, $|\eta(\tau)|_r \le C_r e^{-2\tau}$ for $\tau \ge 0$. Next, regarding the difference $K(\tau)(\mathfrak{z}(\tau)) - K(\tau)(\mathfrak{z}^\varepsilon)$ as an operator, linear in $w(\tau) = \mathfrak{z}(\tau) - \mathfrak{z}^\varepsilon$ and quadratic in $(\mathfrak{z}(\tau), \mathfrak{z}^\varepsilon)$, we write it as $\mathcal{K}(\tau)(w(\tau))$. Then by (5.4) and (5.8), $|\mathcal{K}(\tau)w|_{r+1} \le C_r|w|_r$, $\forall r > d$. Finally, we substitute

$$w(\tau) = v(\tau) + y(\tau), \quad v(\tau) = -e^{-\tau\mathcal{L}}\mathfrak{z}^\varepsilon,$$

and rewrite the equation on w as an equation on y:

$$\dot{y} + \mathcal{L}y = \varepsilon^2\mathcal{K}(\tau)\big(v(\tau) + y(\tau)\big) + \varepsilon^2\eta(\tau), \qquad y(0) = 0.$$

Or

$$y(t) = \varepsilon^2 \int_0^t e^{-(t-s)\mathcal{L}}\big[\mathcal{K}(s)\big(v(s) + y(s)\big) + \eta(s)\big]\mathrm{d}s. \tag{5.11}$$

Let Y^r be the space of continuous curves $y : \mathbb{R}_+ \to \mathcal{C}_r(\mathbb{R}^d)$, vanishing at zero, with finite norm $|y|_{Y^r} = \sup_{t\ge 0} e^t|y(t)|_r$.

Let $\mathfrak{B}$ be the linear operator

$$\mathfrak{B}(y)(t) = \int_0^t e^{-(t-s)\mathcal{L}}\mathcal{K}(s)(y(s))\,\mathrm{d}s.$$

Then the equation for y may be written as

$$y(t) = \varepsilon^2\Big(\mathfrak{B}(y)(t) + \mathfrak{B}(v)(t) + \int_0^t e^{-(t-s)\mathcal{L}}\eta(s)\mathrm{d}s\Big). \tag{5.12}$$

If $|\tilde{y}|_{Y^r} = 1$, then

$$|\mathfrak{B}(\tilde{y}(t))|_{r+1} \le \int_0^t \big|e^{-(t-s)\mathcal{L}}\mathcal{K}(s)(\tilde{y}(s))\big|_{r+1}\mathrm{d}s \le C_r' \int_0^t e^{-2(t-s)}e^{-s}\mathrm{d}s < C_r' e^{-t}.$$

So $\mathfrak{B} : Y^r \to Y^{r+1}$ is a bounded linear operator if $r > d$, and accordingly the operator $(\mathrm{id} - \varepsilon^2\mathfrak{B})$ is a linear isomorphism of Y^r if $r > d$ and ε is sufficiently small. It easy to see that $\mathfrak{B}(v)$ and $\int_0^t e^{-(t-s)\mathcal{L}}\eta(s)\mathrm{d}s$ both belong to all spaces Y^r. Then in view of (5.12), $|y|_{Y^{r+1}} \le C\varepsilon^2$. Since the operator $\mathfrak{B}$ is 1-smoothing, then by induction we get that y belongs to all spaces Y^r. We have proved that

Lemma 5.6. *The solution $\mathfrak{z}(\tau)$, constructed in Lemma 5.5, may be written as*

$$\mathfrak{z}(\tau) = (id - e^{-\tau\mathcal{L}})\mathfrak{z}^\varepsilon + y(\tau), \quad \textit{where } |y(\tau)|_r \le C_r\varepsilon^2 e^{-\tau} \ \ \forall\,\tau \ge 0,\ \forall\, r.$$

Here $\mathfrak{z}^\varepsilon$ is the stationary solution, defined in (5.9) and satisfying (5.10).

5.3. Energy Spectra of Quasisolutions and Kinetic Equation

In this section we prove our main result. Namely, we show that the energy spectrum (4.2) of the quasisolution $n_{s,L}(\tau) = \mathbb{E}|A_s(\tau)|^2$ of Eq. (2.1) with large L is ε^3-close to the solution $\mathfrak{z}(\tau)$ of the WKE (5.1), constructed in Lemmas 5.5, 5.6. By (4.12), it suffices to prove this for $n_{s,L}$ replaced by $n_s^{\le 2}$. Let us denote $w_s(\tau) = n_s^{\le 2}(\tau) - \mathfrak{z}_s(\tau)$; then $w_s(0) = 0$. Recall that ε_* is defined in Lemma 5.5.

Revised Proof

Lemma 5.7. *If $r > d+1$ and $\varepsilon \le C_{1r}^{-1} \le \varepsilon_*$ for an appropriate constant C_{1r}, then for any $\tau_0 \ge 0$ and $0 < \tau \le 1/2$,*

$$|w(\tau_0+\tau)|_r \le (1-\tau/2)|w(\tau_0)|_r + C_{2r}\tau\varepsilon^2(\tau^{\kappa_*}+\varepsilon^2), \tag{5.13}$$

where $\kappa_* = 1/(1+2r_*)$.

Proof. Since by (5.1)

$$\mathfrak{z}(\tau_0+\tau) = e^{-\tau\mathcal{L}}\mathfrak{z}(\tau_0) + 2\int_0^\tau e^{-t\mathcal{L}}b^2\mathrm{d}t + \varepsilon^2\int_{\tau_0}^{\tau_0+\tau} e^{-(\tau_0+\tau-t)\mathcal{L}}K(t)(\mathfrak{z}(t))\mathrm{d}t,$$

then in view of (4.22) and (4.19)

$$w(\tau+\tau_0) = e^{-\tau\mathcal{L}}w(\tau_0) + \varepsilon^2\Delta + \mathcal{R}, \tag{5.14}$$

where $\mathcal{R}$ is as in (4.22) and

$$\Delta = \int_{\tau_0}^{\tau_0+\tau} e^{-(\tau_0+\tau-t)\mathcal{L}}\Big(K(\tau_0)(n^{\le 2}(\tau_0)) - K(t)(\mathfrak{z}(t))\Big)\mathrm{d}t.$$

Note that in view of Lemma 5.5 and estimates (4.9),

$$|n^{\le 2}(\tau)|_r,\ |\mathfrak{z}(\tau)|_r \le C_r \quad \text{for all } \tau \text{ and all } r, \tag{5.15}$$

with suitable constants C_r. Let us re-write Δ as follows:

$$\begin{aligned}\Delta = &\int_{\tau_0}^{\tau_0+\tau} e^{-(\tau_0+\tau-t)\mathcal{L}}\big(K(\tau_0)(n^{\le 2}(\tau_0)) - K(\tau_0)(\mathfrak{z}(\tau_0))\big)\mathrm{d}t\\ &+ \int_{\tau_0}^{\tau_0+\tau} e^{-(\tau_0+\tau-t)\mathcal{L}}\big(K(\tau_0)(\mathfrak{z}(\tau_0)) - K(t)(\mathfrak{z}(\tau_0))\big)\mathrm{d}t\\ &+ \int_{\tau_0}^{\tau_0+\tau} e^{-(\tau_0+\tau-t)\mathcal{L}}\big(K(t)(\mathfrak{z}(\tau_0)) - K(t)(\mathfrak{z}(t))\big)\mathrm{d}t =: \Delta^1+\Delta^2+\Delta^3.\end{aligned}$$

By (5.4) and (5.15), $|\Delta^1|_r \le C_r\tau|w(\tau_0)|_r$. Similar,

$$|\Delta^3|_r \le C_r\tau \sup_{\tau_0\le t\le\tau_0+\tau} |\mathfrak{z}(t)-\mathfrak{z}(\tau_0)|_r \le C_r\tau^2$$

since $|\mathfrak{z}(t)-\mathfrak{z}(\tau_0)|_r \le \int_{\tau_0}^t |-\mathcal{L}\mathfrak{z}(l)+\varepsilon^2K(l)(\mathfrak{z}(l))+2b^2|_r\mathrm{d}l$ and $|\mathfrak{z}(t)|_{r+r_*} \le C_r'$ by (5.15). Now let us consider Δ^2. By Lemma 5.4, $|K(\tau_0)(\mathfrak{z}(\tau_0))-K(t)(\mathfrak{z}(\tau_0))|_r \le C_r(t-\tau_0)^{\kappa_*}$. So $\Delta^2 \le C_r\int_0^\tau t^{\kappa_*}\mathrm{d}t = C_r'\tau^{1+\kappa_*}$.

Since $\mathcal{L} \ge 2\mathbb{1}$ and $\tau \le 1/2$, then $|e^{-\tau\mathcal{L}}w(\tau_0)|_r \le (1-\tau)|w(\tau_0)|_r$. Now (5.14), (4.23) and the bounds on Δ^j imply that

$$|w(\tau_0+\tau)|_r \le (1-\tau)|w(\tau_0)|_r + C_r\varepsilon^2\tau\big(|w(\tau_0)|_r + \tau + \tau^{\kappa_*} + (\tau+\varepsilon^2)\big),$$

and (5.13) follows if $C_{1r}^{-1} \ll 1$. □

For any $0 < \tau \le 1/2$, any N and for $k = 0,\dots,N$ let us set $w_k = |w(k\tau)|_r$. Let the function $k \to w_k$ attain its maximum at a point k which we write as $k := k_0+1$. If $k_0+1 = 0$, then $w_k \equiv 0$. Otherwise, in view of (5.13) we have

$$w_{k_0} \le w_{k_0+1} \le (1-\tau/2)w_{k_0} + C_{2r}\tau\varepsilon^2(\tau^{\kappa_*}+\varepsilon^2).$$

Revised Proof

So, $w_{k_0} \le 2C_{2r}\varepsilon^2(\tau^{\kappa_*} + \varepsilon^2)$ and

$$\max_{0\le k\le N} |w(k\tau)|_r = w_{k_0+1} \le 3C_{2r}\varepsilon^2(\tau^{\kappa_*} + \varepsilon^2)$$

since $\tau \le 1/2$. Applying again (5.13) with $\tau_0 = k\tau$ and τ replaced by any $\bar\tau \in (0,\tau)$, and using that in the formula above N is any, we get that $|w(t)|_r \le 4C_{2r}\varepsilon^2(\tau^{\kappa_*} + \varepsilon^2)$, for any $t \ge 0$. Sending $\tau \to 0$ (and estimating norms $|\cdot|_r$ with $r < d+2$ via $|\cdot|_{d+2}$) and then using (4.12), we finally get

Theorem 5.8. *For any r there exist positive constants C_{1r}, C_{2r}, C_{3r} such that if $\varepsilon \le C_{1r}^{-1}$, then*

$$\sup_{\tau\ge 0} |n^{\le 2}(\tau) - \mathfrak{z}(\tau)|_r \le C_{2r}\varepsilon^4 \tag{5.16}$$

and if $L \ge \varepsilon^{-2}$, then

$$\sup_{\tau\ge 0} |n_{\cdot,L}(\tau) - \mathfrak{z}(\tau)|_r \le C_{3r}\varepsilon^3. \tag{5.17}$$

Relation (5.17) together with Lemma 5.6 gives a control over the long-time behaviour of the spectra of quasisolutions of (2.1) in terms of the stationary solution $\mathfrak{z}_\varepsilon$ of the limiting kinetic equation (see (5.9)):

$$|n_{\cdot,L}(\tau) - \mathfrak{z}^\varepsilon|_r \le C_r(e^{-\tau} + \varepsilon^3), \qquad \forall\, \tau \ge 0.$$

By Proposition 2.1 with $d \ge 3$ this result and (5.17) extend to the spectra of quasisolutions of (1.20), defined in (1.27), as expressed in

Theorem 5.9. *For any r there exist positive constants C_{4r}, C_{5r} such that if $\varepsilon \le C_{4r}^{-1}$ and $L \ge \varepsilon^{-2}$, then*

$$\sup_{\tau\ge 0} |\mathfrak{n}_{\cdot,L}(\tau) - \mathfrak{z}(\tau)|_r \le C_{4r}\varepsilon^3\,, \tag{5.18}$$

$$|\mathfrak{n}_{\cdot,L}(\tau) - \mathfrak{z}^\varepsilon|_r \le C_{5r}(e^{-\tau} + \varepsilon^3), \qquad \forall\, \tau \ge 0\,. \tag{5.19}$$

Relation (5.16) extends to the energy spectra of quasisolutions of (1.20) analogously.

Funding The work of AD on Sects. (1, 2, 3) was supported by the Russian Science Foundation under Grant no.23-11-00150. The work of AD on the other sections and work of AM and SK on the whole paper was supported by the Ministry of Science and Higher Education of the Russian Federation (mega-grant No. 075-15-2022-1115). All authors have contributed equally in this work

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.

Revised Proof

Appendix A. Proof of Lemma 3.6

In this appendix we suppose that the dimension d satisfies $d \geq 2$.

A.1. Idea of the Proof and General Setting

In Lemma 3.6 (up to an obvious scaling) we have to estimate the number of integer points on a quadric inside a large box. The idea is to embed the integral points of the box in an affine space over a large finite field and then apply powerful algebraic geometry techniques to estimate the needed number (note that this identification of bounded integers with elements of a finite field is ubiquitous in coding theory and combinatorics). It is possible mainly due to the fact that this techniques permits to count points defined over a finite field using some geometric information (essentially the dimension, the degree and irredundant decomposition) on the corresponding algebraic set over the algebraic closure of our finite field. We begin with recalling some basic definitions and results concerning such algebraic sets (see, for example, the first chapter of the book [27]).

Affine algebraic sets. Let us fix an algebraically closed field K. Let $\mathbb{A}^m = K^m$ be the m-dimensional affine space over K, and let $F_1, \ldots, F_s \in K[T_1, \ldots, T_m]$ be nonzero polynomials. Then an *affine algebraic set* (AAS) X is just the set of common zeros of these polynomials:

$$X = \{(a_1, \ldots, a_m) \in K^m : F_1(a_1, \ldots, a_m) = \ldots = F_s(a_1, \ldots, a_m) = 0\}. \tag{A.1}$$

Irreducibility. An AAS X is *reducible* if $X = X_1 \cup X_2$ with two non-empty AAS X_1, X_2 s.t. $X_1 \neq X, X_2 \neq X$. If it is not the case, X is called *irreducible,* or an *affine algebraic variety* (see [27], Section I.3.1).

Theorem A.1 (Irredundant decomposition). *Any non-empty AAS X can be presented as*

$$X = X_1 \cup \ldots \cup X_l \tag{A.2}$$

for irreducible $X_1, \ldots, X_l$ such that $X_i \not\subset X_j$ for $i \neq j$. The decomposition is unique up to order.

This decomposition is especially simple for a hypersurface X, i.e. when in (A.1) $s = 1$. Then $F = F_1(T_1, \ldots T_m) = \Pi_{j=1}^{l} Q_j$ for irreducible polynomials Q_j which are uniquely defined up to multiplicative constants and permutation since the ring $K[T_1, \ldots, T_m]$ is a unique factorization domain, see, for example, Chapter IV of [24], and then $X_j = \{Q_j = 0\}$. This uniqueness is true under the condition which we can and will suppose to hold, namely, that the polynomial P does not have multiple divisors, i.e. all $Q_j, j = 1, \ldots, l$ are distinct. For further references we formulate a corollary of the unique factorization property (see [27], Section I.3.1)

Lemma A.2. *(i). If X and Y are hypersurfaces, then $X = Y$ if and only if the corresponding polynomials P_X and P_Y are proportional. Moreover if Y is irreducible and $X \subseteq Y$, then $X = Y$.*

Revised Proof

(ii). If $\deg P_X = 2$*, then there are exactly two possibilities: either* X *irreducible (in this case it cannot contain a hyperplane), or* $X = X_1 \cup X_2$ *for two affine hyperplanes, defined by affine linear polynomials* l_1 *and* l_2*, and* $P_X = l_1 l_2$*.*

Dimension. One can define the *dimension* $r = \dim X \in \{0, 1, \ldots, m\}$ as follows: $\dim X = \max\{\dim X_i, i = 1, \ldots, l\}$ for (A.2), and for an irreducible AAS X

$$\dim X = \max\{r : X = X_0 \supset X_1 \supset \ldots \supset X_r \neq \varnothing\},$$

where all $X_i, i = 0, \ldots, r$ are irreducible AAS and all inclusions are strict. The *codimension* of X is $\operatorname{codim} X = m - \dim X$.

In particular, if $\dim X = m$ then $X = \mathbb{A}^m$ (indeed, if $X \subset \mathbb{A}^m, X \neq \mathbb{A}^m$ the definition implies that $\dim X < \dim \mathbb{A}^m = m$) and if $\dim X = 0$ then X is a finite set. The codimension of X as in (A.1) is at most s.

From the definition we get immediately (see [27], Section I.6.2)

Lemma A.3. *If* $Y \subset X, Y \neq X$ *and* X *is irreducible, then* $\dim X > \dim Y$*,* $\operatorname{codim} X < \operatorname{codim} Y$*.*

Degree. Let $X \subset \mathbb{A}^m$ be a non-empty AAS, $\dim X = r$. Then its *degree* $\deg X$ is defined as follows:

$$\deg X = \max\{\text{cardinality of } X \cap L : \dim\big(X \cap L\big) = 0\},$$

where $L \subset \mathbb{A}^m$ is an affine plane with $\dim L = m - r$.

Lemma A.4. *If* X *is a hypersurface (i.e. in (A.1)* $s = 1$*), then* $\operatorname{codim} X = 1$ *and* $\deg X = \deg F_1$*.*

The famous *Bezout theorem* in its the most elementary setting over the field $\mathbb{C}$ states that

$$\deg X \leq \Pi_{i=1}^{s} \deg F_i.$$

A.2. Finite Fields' Bezout Theorem

From now on the field K is the algebraic closure $\bar{\mathbb{F}}_p$ of a finite field $\mathbb{F}_p$, where p is a large prime number (see [24], Section V.5).

We will use a version of Bezout's theorem over finite fields which can be deduced from its general form, e.g. [13], and is also explicitly stated and proved in [22, Corollary 2.2].

Theorem A.5. *Let* $K = \bar{\mathbb{F}}_p$ *and the AAS* X *in (A.1) is such that* $F_j \in \mathbb{F}_p[T_1, \ldots, T_m]$*,* $\deg F_j = d_j$*,* $j = 1, \ldots, s$*, and* $\dim X = r$*. Then*

$$|X \cap \mathbb{F}_p^m| \leq p^r \Pi_{i=1}^{s} d_i.$$

Revised Proof

A.3. Preliminary Result

Let $q_1, \dots, q_s$, $s \geq 1$, be polynomials of degree at most two in $m \geq s$ variables, $q_i \in \mathbb{Z}[X_1, \dots, X_m]$, with $q_i(0) = 0$, $i = 1, ..., s$. Consider the geometric quadrics $Q_i = \{x \in \mathbb{R}^m : q_i(x) = 0\}$ and their intersection $Q = \cap_{i=1}^s Q_i$. The latter is not empty since $0 \in Q$.

Let $B_M^m \subset \mathbb{R}^m$ be an open cube $\{|x|_\infty < M\}$ with some $M \geq 1$. Consider the set

$$S_m(M, Q) = Q \cap \mathbb{Z}^m \cap B_M^m.$$

Let p be a prime and $q_i^{(p)} \in \mathbb{F}_p[X_1, \dots, X_m]$ denote the polynomials q_i mod p over the finite field $\mathbb{F}_p$. Consider the sets

$$Q_i^{(p)} = \{x \in K^m : q_i^{(p)}(x) = 0\}$$

and their intersection $Q^{(p)} = \cap_{i=1}^s Q_i^{(p)}$ (recall that now $K = \overline{\mathbb{F}}_p$ is the algebraic closure of $\mathbb{F}_p$). We will be interested mainly in the cardinality of $Q^{(p)}(\mathbb{F}_p) := Q^{(p)} \cap \mathbb{F}_p^m$ as a tool to estimate $|S_m(M, Q)|$.

Proposition A.6. *Let $M \geq 1$ and suppose that a prime $p > 2M$ satisfies $p = 2\,M(1 + r(M))$, where $r(M) > 0$. Suppose also that $\deg q_i \leq 2$ for each i and that the AAS $Q^{(p)}$ is of dimension $m - s$ (and of codimension s, that is, the s quadrics $Q_i^{(p)}$ intersect properly):*

$$\dim Q^{(p)} = m - s. \tag{A.3}$$

Then

$$|S_m(M, Q)| \leq 2^m \big(1 + r(M)\big)^{m-s} M^{m-s}. \tag{A.4}$$

By Bertrand's postulate, for any $M \geq 1$ there is a p satisfying $2M < p < 4M$, and when applying Proposition A.6 we will always chose

$$r(M) < 1. \tag{A.5}$$

Moreover, by the Prime Number Theorem, for large M one can chose $r(M) = o(1)$.

Proof of Proposition A.6. Let $\Pi : S_m(M, Q) \longrightarrow \mathbb{F}_p^m$ be defined by

$$\Pi(x_1, \dots, x_m) = (x_1 \mod p, \dots, x_m \mod p).$$

Then Π is injective and its image is contained in $Q^{(p)} \cap \mathbb{F}_p^m \subset \mathbb{F}_p^m$. Indeed, the last assertion is clear and the injectivity is established as follows: if

$$(x_1' \mod p, \dots, x_m' \mod p) = (x_1 \mod p, \dots, x_m \mod p)$$

but $x' \neq x$, then for some $i \in \{1, ..., m\}$ we have $x_i' \mod p = x_i \mod p$, but $x_i' \neq x_i$. Consequently, $|x_i - x_i'| \geq p > 2M$ which contradicts the condition $x_i, x_i' \in B_M^m$. Applying then Theorem A.5 to $X = Q^{(p)}$, we get the conclusion since

$$|S_m(M, Q)| \leq |Q^{(p)}(\mathbb{F}_p)| \leq 2^s p^{m-s} = 2^m \big(1 + r(M)\big)^{m-s} M^{m-s}.$$

□

Revised Proof

A.4. Main Estimate for $N = 3$ and 4

Now we pass to the proof of Lemma 3.6 and denote $\left|\tilde{Q}_L(z_1, v) \cap B_R^{(N-2)d}\right| = s(R, \tilde{Q}, L)$. Consider the set

$$S'\big(R, \tilde{\mathcal{Q}}, L\big) = \tilde{\mathcal{Q}} \cap \mathbb{Z}^{(N-2)d} \cap B_{RL}^{(N-2)d}, \qquad \tilde{\mathcal{Q}} = \tilde{Q}(Lz_1, Lv),$$

and denote by $s'(R, \tilde{\mathcal{Q}}, L)$ its cardinality. Then $s'(R, \tilde{\mathcal{Q}}, L) = s(R, \tilde{Q}, L)$ since the map $(z_2, \ldots, z_{N-1}) \mapsto (Lz_2, \ldots, Lz_{N-1})$ is a bijection between the sets $\tilde{Q}_L(z_1, v) \cap B_R^{(N-2)d}$ and $S'(R, \tilde{\mathcal{Q}}, L)$.

Let us estimate $s'(R, \tilde{\mathcal{Q}}, L)$ through Proposition A.6 with $m = (N-2)d$ and $s = N-2$, where $N = 3$ or 4. To this end it suffices to find $M \geq RL$ and $p > 2\,M$ such that assumption (A.3) is fulfilled for any $(z_1, v) \in B_{(N-1)R}^{2d}$ satisfying $(z_1, v) \in Q_{1\,L}^0 \cap A_2$. Lemma A.7 below establishes this for $M = NRL/2$ and any $p > 2M$. Then, applying (A.4) with $r(M) < 1$ (see (A.5)), we conclude the proof of Lemma 3.6.

For a prime p and $a, b \in \mathbb{F}_p^d$, let us consider algebraic sets $\tilde{Q}_j^{(p)}$ over $K = \bar{\mathbb{F}}_p$:

$$\tilde{Q}_j^{(p)}(a, b) := \{(z_2, \ldots, z_{N-2}) \in K^{(N-2)d} : q_j^{(p)}(z_2, \ldots, z_{N-2}; a, b) = 0\},$$

where $q_j^{(p)}(z_2, \ldots, z_{N-2}; a, b)$ are the residues modulo p of the polynomials $q_j(z_2, \ldots, z_{N-2}; a, b)$, defined by (3.15). We set $\tilde{Q}^{(p)} = \cap_{1<j<N} \tilde{Q}_j^{(p)}$ for the intersection of the algebraic sets.

Lemma A.7. *Let $N \in \{3, 4\}$, $(z_1, v) \in Q_{1\,L}^0 \cap A_2$ (see (3.16)) and let p be a prime satisfying $p > \max(|Lz_1|_\infty, |Lv|_\infty)$. Then*

$$\dim \tilde{Q}^{(p)}(Lz_1, Lv) = (N-2)(d-1). \tag{A.6}$$

The assumption $p > \max(|Lz_1|_\infty, |Lv|_\infty)$ ensures that Lz_1 and Lv are different from zero in K^d. In particular, for $(z_1, v) \in B_{(N-1)R}^{2d}$ this assumptions is satisfied if $p > 2M$ with $M = NRL/2$.

Proof of Lemma A.7. Let $N = 3$. Then $N - 2 = 1$ and $\tilde{Q}^{(p)}$ is given by the unique equation $q_2^{(p)}(z_2; Lz_1, Lv) = 0$, for a fixed (z_1, v). By Lemma A.9 the equation is non-trivial, so the conclusion follows from Lemma A.4.

$N = 4$. The codimension of the intersection of two quadrics is at most two. We have to show that it is two (and not one). The result will follow from the next three lemmas.

Lemma A.8. *Let $\mathcal{Q}_1 = \{\tilde{q}_1 = 0\}, \mathcal{Q}_2 = \{\tilde{q}_2 = 0\}$ be two linearly independent quadrics over K. Then the codimension of $\mathcal{Q}_1 \cap \mathcal{Q}_2$ is one if and only if $\tilde{q}_1$ and $\tilde{q}_2$ have a mutual affine linear factor $l(x)$.*

Proof. Let the codimension of the intersection be one. In this case if one of $\mathcal{Q}_1, \mathcal{Q}_2$ is irreducible, then $\mathcal{Q}_1 = \mathcal{Q}_2$ by Lemma A.3 with $Y = \mathcal{Q}_1 \cap \mathcal{Q}_2$. However, this is impossible by Lemma A.2. (i) since $\tilde{q}_1$ and $\tilde{q}_2$ are independent.

Revised Proof

Therefore, by Lemma A.2. (ii) $\mathcal{Q}_1 = H_1 \cup H_2$ and $\mathcal{Q}_2 = H_1' \cup H_2'$, with hyperplanes $H_1, \dots, H_2'$. If all $H_i \cap H_j'$ are of codimension two, then

$$\operatorname{codim} Q_1 \cap Q_2 = \operatorname{codim} \big(\cup (H_i \cap H_j')\big) = \min(\operatorname{codim} H_i \cap H_j') = 2.$$

Therefore, at least one of $H_i \cap H_j'$ is of codimension one and then we have $\ker(l(x)) = H_i \cap H_j' \subset \mathcal{Q}_1 \cap \mathcal{Q}_2$ for an affine linear $l(x)$. Hence, $l(x)$ divides both $\tilde{q}_1$ and $\tilde{q}_2$ by Lemma A.2. (ii).

The inverse statement is obvious. □

Lemma A.9. *For any $N > 2$, if the matrix α is irreducible and $(z_1, v) \in Q^0_{1\,L} \cap A_2$ is such that Lz_1, $Lv \neq 0$ in K^d, then the polynomials $q_j^{(p)}(\cdot, Lz_1, Lv)$, $1 < j < N$ are linearly independent over K. In particular, each $q_j^{(p)}$ is a nonzero polynomial.*

Proof. Consider a linear combination $\sum_{1<j<N} c_j q_j^{(p)}$. By the homogeneity in $(z_2, \dots, z_{N-1})$, it vanishes identically if and only if

$$\begin{aligned} &\sum_{1<j<N} c_j z_j \cdot (\alpha_{j1}(Lz_1) + \alpha_{jN}(Lv)) \equiv 0, \\ &\sum_{1<i,j<N} c_j(\alpha_{ji} - \alpha_{jN}\alpha_{1i}) z_j \cdot z_i \equiv 0. \end{aligned} \tag{A.7}$$

Arguing by induction and using that the matrix α is irreducible, we construct a partition $E_0, \dots, E_M$, $M \geq 1$, of the set $\{1, \dots, N\}$ such that $E_0 = \{1, N\}$ and for $n \geq 1$,

$$E_n = \{j : \alpha_{jl} = 0 \,\forall\, l \in E_{n'},\ n' \leq n-2, \text{ and } \exists\, l' \in E_{n-1} \text{ such that } \alpha_{jl'} \neq 0\}.$$

Since $(z_1, v) \in Q^0_{1\,L}$ and Lz_1, $Lv \neq 0$ in K^d, then the term in brackets in the first line of (A.7) is not identically zero for each $j \in E_1$, so $c_j = 0$ for every $j \in E_1$. Using this in the second line of (A.7), we get:

$$\sum_{n=2}^{M} \sum_{m=n-1}^{M} \sum_{j \in E_n} \sum_{i \in E_m} c_j \alpha_{ji} z_j \cdot z_i \equiv 0.$$

This relation holds if and only if $(c_j - c_i)\alpha_{ji} = 0$ for all $j \in E_n$, $2 \leq n \leq M$, and $i \in E_m$, $n-1 \leq m \leq M$. We know that $c_j = 0$ if $j \in E_1$. Starting from $n = 2$ and arguing by induction in n, we find that if $c_i = 0$ for all $i \in E_{n-1}$, then $c_j = 0$ for all $j \in E_n$. Indeed, for any $j \in E_n$ there exists at least one $i \in E_{n-1}$ such that $\alpha_{ji} \neq 0$ by the definition of E_i, so relation $(c_j - c_i)\alpha_{ji} = 0$ implies that $c_j = 0$ if $j \in E_n$. That is, $c_j \equiv 0$. □

Lemma A.10. *For any $N > 2$, if the matrix α is irreducible and $(z_1, v) \in Q^0_{1\,L} \cap A_2$ is such that Lz_1, $Lv \neq 0$ in K^d, then the polynomials $q_j^{(p)}(\cdot, Lz_1, Lv)$, $1 < j < N$, are irreducible.*

Proof. Each polynomial $q_j^{(p)}$ has degree one or two. If its degree is one the assertion is obvious. Now let the degree be two. Note that in view of (3.15) $q_j^{(p)}$ can be written as the scalar products $q_j^{(p)} = z_j \cdot l_j(z_2, \dots, z_{N-1}; z_1, v)$ mod

Revised Proof

p, where l_j are surjective affine functions $l_j : K^{d(N-2)} \longrightarrow K^d$. But such scalar product cannot vanish for $d \geq 2 > 1$ on a hyperplane $H \subset K^{d(N-2)}$ which by Lemma A.2. ii) would be the case for a reducible quadric. Indeed, only two cases can occur:

(a) The coefficient α of z_j in l_j is nonzero, or
(b) It is zero but then the coefficient β of some other z_i is nonzero.

In case (a) take the two-dimensional plane $P(x_1, x_2)$ in the whole space, generated by two orthogonal vectors from the z_j-space, where the first basis vector is parallel to $\alpha_{1j} z_1 + \alpha_{Nj} v \neq 0$ (this vector is nonzero since $(z_1, v) \in \bar{Q}^0_{1\,L}$ and Lz_1, $Lv \neq 0$ in K^d, and for the case a) we have $\alpha_{1j}, \alpha_{Nj} \neq 0$). Then the restriction of $q_j^{(p)} = 0$ on P is $\alpha(x_1^2 + x_2^2) + c_1 x_1 = 0$ with $c_1 \neq 0$, which is isomorphic to $x_1^2 + x_2^2 = C \neq 0$. This plane quadric in $P(x_1, x_2)$ cannot contain $P(x_1, x_2) \cap H$ (a line or the whole $P(x_1, x_2)$). Indeed, otherwise, supposing by symmetry that the quadric contains a line $x_1 = ax_2 + b$, we would have that the polynomial

$$\begin{aligned} &a^2 x_2^2 + 2abx_2 + b^2 + x_2^2 - C \\ &\quad = (a^2+1)x_2^2 + 2abx_2 + b^2 - C \end{aligned}$$

vanishes identically. This implies $ab = 0$, and if $a = 0$ then the term $(a^2+1)x_2^2 = x_2^2 \neq 0$, while for $b = 0$ the term $b^2 - C = -C \neq 0$.

Similarly, in case b) we take the four-dimensional vector subspace P' generated by the two first basis vectors in the z_j space and the two first basis vectors in the z_i space. The restriction of $q_j^{(p)} = 0$ on P' is then $\beta(x_1 y_1 + x_2 y_2) + c_1 x_1 + c_2 x_2 = 0$, isomorphic to $x_1 y_1 + x_2 y_2 = C$ which cannot contain $P'(x_1, x_2, y_1, y_2) \cap H$. Indeed, else, supposing by symmetry that $P'(x_1, x_2, y_1, y_2) \cap H \supset \{x_1 = a_1 x_2 + b_1 y_1 + b_2 y_2 - c\}$ we get that the following quadratic function of x_2, y_1, y_2:

$$\begin{aligned} (a_1 x_2 + b_1 y_1 + b_2 y_2 - c)\ \ & y_1 + x_2 y_2 - C \\ &= a_1 x_2 y_1 + b_1 y_1^2 + b_2 y_1 y_2 - c y_1 + x_2 y_2 - C \end{aligned}$$

vanishes identically, which is clearly wrong. □

End of the proof of Lemma A.7. Since each $q_j^{(p)}$ is a nonzero polynomial of degree one or two, then to prove Lemma A.7 we have to consider three cases. In the first case both polynomials $q_2^{(p)}$ and $q_3^{(p)}$ are linear. Then the codimension of the intersection $\tilde{Q}^{(p)}$ is two since they are linearly independent. In the second case both $q_2^{(p)}$ and $q_3^{(p)}$ are quadratic. Then, according to Lemma A.8, the codimension still is two since the polynomials are irreducible by Lemma A.10. Finally in the last case, when one polynomial is linear and another one is quadratic, the assertion is clear since then the AAS in question is an intersection of a quadratic irreducible surface with a hyperplane. Thus, its codimension is two by Lemma A.2. (ii).

Remark A.11. The proof of Lemma A.7 follows from three lemmas. Two of them are valid for any $N > 2$, but Lemma A.8 holds only for $N = 4$ (and

Revised Proof

tautologically holds for smaller N). Still the bi-linear (or linear) nature of the polynomials $q_j^{(p)}$ and direct analysis of the AAS $\tilde{Q}^{(p)}$, jointly with the two lemmas, valid for any $N > 2$, allow to prove by hand Lemma A.7 for "not too high" values of N, and thus, to prove for those N's Theorem 3.2. Unfortunately, for the moment we cannot prove the theorem for all $N > 2$; cf. Conjecture 3.8.

Appendix B. Proof of Proposition 4.3 and Lemma 4.4

We prove Proposition 4.3 in Sects. B.1-B.6 and Lemma 4.4 in Sect. B.7.

B.1. Beginning of the Proof of Proposition 4.3

The proof of the proposition is somewhat cumbersome since we have to consider a number of different terms and different cases. During the proof we will often skip the upper index (0), so by writing a and a_s we will mean $a^{(0)}$ and $a_s^{(0)}$. We will also skip the dependence on τ_0 by writing $c_s^{(i)}(\tau;\tau_0)$ and $\Delta a_s^{(i)}(\tau;\tau_0)$ as $c_s^{(i)}(\tau)$ and $\Delta a_s^{(i)}(\tau)$. Besides, for a complex function $(w_{s_1,\dots,s_k}, s_j \in \mathbb{Z}_L^d)$ we denote

$$\mathop{\sum\!\!\!\!\!\!\int}_{s_1,\dots,s_k\in\mathbb{Z}_L^d} w_{s_1,\dots,s_k} = L^{-kd} \sum_{s_1,\dots,s_k\in\mathbb{Z}_L^d} w_{s_1,\dots,s_k}\,,$$

and we introduce the symmetrization

$$\mathcal{Y}_s^{sym}(u,v,w;t) = \frac{L^{-d}}{3} \sum_{1,2,3} \delta'^{12}_{3s}\delta(\omega^{12}_{3s})\,(u_1v_2\bar{w}_3 + v_1w_2\bar{u}_3 + w_1u_2\bar{v}_3)\,.$$

We recall that $Q_{s,L}$ is given by formula (4.30) and first consider the term $\mathbb{E}\Delta a_s^{(2)}(\tau)\bar{a}_s(\tau_0+\tau)$. Inserting the identity $a^{(1)}(\tau_0+l) = c^{(1)}(l) + \Delta a^{(1)}(l)$ into formula (4.27) for $\Delta a_s^{(2)}$, we obtain

$$\mathbb{E}\Delta a_s^{(2)}(\tau)\bar{a}_s(\tau_0+\tau) = N_s + \widetilde{N}_s,$$

where

$$N_s := i\,\mathbb{E}\Big(\bar{a}_s(\tau_0+\tau)\int_0^\tau e^{-\gamma_s(\tau-l)}3\mathcal{Y}_s^{sym}(a(\tau_0+l), a(\tau_0+l), \Delta a^{(1)}(l))\,\mathrm{d}l\Big) \tag{B.1}$$

and

$$\widetilde{N}_s := i\,\mathbb{E}\Big(\bar{a}_s(\tau_0+\tau)\int_0^\tau e^{-\gamma_s(\tau-l)}3\mathcal{Y}_s^{sym}(a(\tau_0+l), a(\tau_0+l), c^{(1)}(l))\,\mathrm{d}l\Big).$$

Thus,

$$\begin{aligned} Q_{s,L} = L^2\Big(&\mathbb{E}|\Delta a_s^{(1)}(\tau)|^2 + 2\Re N_s \\ &+ 2\Re\mathbb{E}\Delta a_s^{(1)}(\tau)\bar{c}_s^{(1)}(\tau) + 2\Re\widetilde{N}_s\Big)\,, \quad s\in\mathbb{R}^d. \end{aligned} \tag{B.2}$$

We will analyse the four terms above term by term.

Revised Proof

B.2. The First Term of $Q_{s,L}$ in (B.2)

Due to (4.27), we have

$$\mathbb{E}|\Delta a_s^{(1)}(\tau)|^2 = \mathbb{E}\int_{\tau_0}^{\tau_0+\tau} \mathrm{d}l \int_{\tau_0}^{\tau_0+\tau} \mathrm{d}l'\, e^{-\gamma_s(2\tau_0+2\tau-l-l')}\mathcal{Y}_s(a(l))\,\overline{\mathcal{Y}_s(a(l'))}. \tag{B.3}$$

Writing the functions $\mathcal{Y}_s$ explicitly and applying the Wick theorem, in view of (2.13) we find

$$\mathbb{E}|\Delta a_s^{(1)}(\tau)|^2 = 2L^{-2d}\sum_{1,2}\delta_{3s}'^{12}\delta(\omega_{3s}^{12})\int_{\tau_0}^{\tau_0+\tau} \mathrm{d}l \int_{\tau_0}^{\tau_0+\tau} \mathrm{d}l'\, e^{-\gamma_s(2\tau_0+2\tau-l-l')}$$
$$\mathbb{E}a_1(l)\bar{a}_1(l')\,\mathbb{E}a_2(l)\bar{a}_2(l')\,\mathbb{E}\bar{a}_3(l)a_3(l'),$$

and note that

$$\int_{\tau_0}^{\tau_0+\tau} \mathrm{d}l \int_{\tau_0}^{\tau_0+\tau} \mathrm{d}l'\, e^{-\gamma_s(2\tau_0+2\tau-l-l')} \le \tau^2.$$

On account of (2.13), we can bound

$$\mathbb{E}|\Delta a_s^{(1)}(\tau)|^2 \le 2\tau^2 \sum_{1,2}\delta_{3s}'^{12}\delta(\omega_{3s}^{12})B_{123},$$

where $B_{123} = B_1B_2B_3$. Since B_{123} with $s_3 = s - s_1 - s_2$ is a Schwartz function of s, s_1, s_2 then Theorem 3.2 with $N = 2$ applies and we find

$$\mathbb{E}|\Delta a_s^{(1)}(\tau)|^2 \le C^{\#}(s)L^{-2}\tau^2. \tag{B.4}$$

B.3. The Second Term of $Q_{s,L}$ in (B.2)

To study the term $2\Re N_s$, we use the same strategy as above. Namely, expressing in (B.1) the function $3\mathcal{Y}_s^{sym}$ via $\mathcal{Y}_s$, we write N_s as $N_s = N_s^1 + 2N_s^2,\ s \in \mathbb{R}^d$, where

$$N_s^1 = i\,\mathbb{E}\Big(\bar{a}_s(\tau_0+\tau)\int_0^\tau e^{-\gamma_s(\tau-l)}\mathcal{Y}_s(a(\tau_0+l), a(\tau_0+l), \Delta a^{(1)}(l))\,\mathrm{d}l\Big),$$
$$N_s^2 = i\,\mathbb{E}\Big(\bar{a}_s(\tau_0+\tau)\int_0^\tau e^{-\gamma_s(\tau-l)}\mathcal{Y}_s(\Delta a^{(1)}(l), a(\tau_0+l), a(\tau_0+l))\,\mathrm{d}l\Big).$$

Term N_s^1. Writing explicitly the function $\mathcal{Y}_s$ and then $\Delta\bar{a}_3^{(1)}$, we get

$$N_s^1 = i\,L^{-d}\sum_{1,2}\delta_{3s}'^{12}\delta(\omega_{3s}^{12})\int_0^\tau \mathrm{d}l\, e^{-\gamma_s(\tau-l)}$$
$$\times\,\mathbb{E}\big(a_1(\tau_0+l)a_2(\tau_0+l)\Delta\bar{a}_3^{(1)}(l)\bar{a}_s(\tau_0+\tau)\big)$$
$$= L^{-2d}\sum_{1,2}\sum_{1',2'}\delta_{3s}'^{12}\delta_{3'3}'^{1'2'}\delta(\omega_{3s}^{12})\delta(\omega_{3'3}^{1'2'})\int_0^\tau \mathrm{d}l\int_0^l \mathrm{d}l'\, e^{-\gamma_s(\tau-l)}e^{-\gamma_3(l-l')}$$
$$\times\,\mathbb{E}\big(a_1(\tau_0+l)a_2(\tau_0+l)\bar{a}_{1'}(\tau_0+l')\bar{a}_{2'}(\tau_0+l')a_{3'}(\tau_0+l')\bar{a}_s(\tau_0+\tau)\big). \tag{B.5}$$

Revised Proof

By the Wick theorem, we need to take the summation only over $s_{1'}, s_{2'}, s_{3'}$ satisfying $s_{1'} = s_1$, $s_{2'} = s_2$, $s_{3'} = s$ or $s_{1'} = s_2$, $s_{2'} = s_1$, $s_{3'} = s$. Since in both cases we get $\delta'^{1'2'}_{3'3} = \delta'^{12}_{3s}$ and $\omega^{1'2'}_{3'3} = \omega^{12}_{3s}$, we find

$$N^1_s = 2 \sum_{1,2} \delta'^{12}_{3s} \delta(\omega^{12}_{3s}) \int_0^\tau \mathrm{d}l \int_0^l \mathrm{d}l'\, e^{-\gamma_s(\tau-l)-\gamma_3(l-l')} \times \mathbb{E} a_1(\tau_0+l)\bar{a}_1(\tau_0+l')\, \mathbb{E} a_2(\tau_0+l)\bar{a}_2(\tau_0+l')\, \mathbb{E} a_s(\tau_0+l')\bar{a}_s(\tau_0+\tau).$$

Arguing as in Sect. B.2 we find

$$|N^1_s| \le C^{\#}(s) L^{-2}\tau^2. \tag{B.6}$$

Term N^2_s. By literally repeating the argument we have applied to N^1_s, we find that

$$\begin{aligned} N^2_s &= i\, L^{-d} \sum_{1,2} \delta'^{12}_{3s} \delta(\omega^{12}_{3s}) \int_0^\tau \mathrm{d}l\, e^{-\gamma_s(\tau-l)} \\ &\quad \times \mathbb{E}\big(\Delta a^{(1)}_1(l) a_2(\tau_0+l)\bar{a}_3(\tau_0+l)\bar{a}_s(\tau_0+\tau)\big) \\ &= -L^{-2d} \sum_{1,2} \sum_{1',2'} \delta'^{12}_{3s} \delta'^{1'2'}_{3'1} \delta(\omega^{12}_{3s}) \delta(\omega^{1'2'}_{3'1}) \int_0^\tau \mathrm{d}l \int_0^l \mathrm{d}l'\, e^{-\gamma_s(\tau-l)} e^{-\gamma_1(l-l')} \\ &\quad \times \mathbb{E}\big(a_{1'}(\tau_0+l')a_{2'}(\tau_0+l')\bar{a}_{3'}(\tau_0+l')a_2(\tau_0+l)\bar{a}_3(\tau_0+l)\bar{a}_s(\tau_0+\tau)\big). \end{aligned}$$

By the Wick theorem, we should take summation either under the condition $s_{1'} = s_3$, $s_{2'} = s$, $s_{3'} = s_2$ or $s_{1'} = s$, $s_{2'} = s_3$, $s_{3'} = s_2$. Since in both cases $\delta'^{1'2'}_{3'1} = \delta'^{12}_{3s}$ and $\omega^{1'2'}_{3'1} = -\omega^{12}_{3s}$, then

$$\begin{aligned} N^2_s &= -2 \sum_{1,2} \delta'^{12}_{3s} \delta(\omega^{12}_{3s}) \int_0^\tau \mathrm{d}l \int_0^l \mathrm{d}l'\, e^{-\gamma_s(\tau-l)} e^{-\gamma_1(l-l')} \\ &\quad \mathbb{E} a_2(\tau_0+l)\bar{a}_2(\tau_0+l')\, \mathbb{E} a_3(\tau_0+l')\bar{a}_3(\tau_0+l)\, \mathbb{E} a_s(\tau_0+l')\bar{a}_s(\tau_0+\tau)\,. \end{aligned} \tag{B.7}$$

Again we get

$$|N^2_s| \le C^{\#}(s) L^{-2}\tau^2. \tag{B.8}$$

B.4. The Third Term of $Q_{s,L}$ in (B.2)

We have

$$\mathbb{E}\Delta a^{(1)}_s \bar{c}^{(1)}_s(\tau) = \mathbb{E} \int_{\tau_0}^{\tau_0+\tau} e^{-\gamma_s(\tau_0+\tau-l)} \mathcal{Y}_s(a(l))\, \mathrm{d}l \int_0^{\tau_0} e^{-\gamma_s(\tau_0+\tau-l')} \overline{\mathcal{Y}_s(a(l'))}\, \mathrm{d}l'.$$

This expression coincides with (B.3) in which the integral $\int_{\tau_0}^{\tau_0+\tau} \mathrm{d}l'$ is replaced by $\int_0^{\tau_0} \mathrm{d}l'$. Then,

$$\mathbb{E}\Delta a^{(1)}_s(\tau)\bar{c}^{(1)}_s(\tau) = 2 \sum_{1,2} \delta'^{12}_{3s} \delta(\omega^{12}_{3s}) \int_{\tau_0}^{\tau_0+\tau} \mathrm{d}l \int_0^{\tau_0} \mathrm{d}l'\, e^{-\gamma_s(2\tau_0+2\tau-l-l')} \mathbb{E} a_1(l)\bar{a}_1(l')\, \mathbb{E} a_2(l)\bar{a}_2(l')\, \mathbb{E}\bar{a}_3(l)a_3(l'),$$

Revised Proof

Expressing the correlations $\mathbb{E}a_j(l)\bar{a}_j(l')$ through (2.13), we get

$$\mathbb{E}\Delta a_s^{(1)}(\tau)\bar{c}_s^{(1)}(\tau) = 2\sum_{1,2}{}^{\circledast}\,\delta_{3s}'^{12}\delta(\omega_{3s}^{12})B_{123}\int_0^\tau \mathrm{d}l\, e^{-2\gamma_s(\tau-l)-\gamma_{123s}l}$$
$$\int_0^{\tau_0}\mathrm{d}l'\, e^{-\gamma_{123s}\tau_0}e^{\gamma_s l'}\prod_{j=1,2,3}\left(e^{\gamma_j l'}-e^{-\gamma_j l'}\right).$$

For the integral in the first line, we have

$$T_s := \int_0^\tau \mathrm{d}l\, e^{-2\gamma_s(\tau-l)-\gamma_{123s}l} = \begin{cases} \tau e^{-2\gamma_s\tau} & \text{if } 2\gamma_s=\gamma_{123s} \\ \frac{e^{-2\gamma_s\tau}-e^{-\gamma_{123s}\tau}}{\gamma_{123s}-2\gamma_s} & \text{elsewhere}\end{cases}. \tag{B.9}$$

For the integral in the second line, let us denote

$$\mathcal{T}^j := \int_0^{\tau_0}\mathrm{d}l\, e^{-\gamma_{123s}\tau_0}e^{\gamma_j l}\prod_{k\neq j}\left(e^{\gamma_k l}-e^{-\gamma_k l}\right), \tag{B.10}$$

where $j,k\in\{1,2,3,s\}$. Then,

$$0\le \mathcal{T}^j \le 1/\gamma_{123s}. \tag{B.11}$$

Due to (B.9) and (B.10) we get

$$\mathbb{E}\Delta a_s^{(1)}(\tau)\bar{c}_s^{(1)}(\tau) = 2\sum_{1,2}{}^{\circledast}\,\delta_{3s}'^{12}\delta(\omega_{3s}^{12})B_{123}T_s\mathcal{T}^s. \tag{B.12}$$

B.5. The Fourth Term of $Q_{s,L}$ in (B.2)

To study the term $2\Re\widetilde{N}_s$, as in Sect. B.3, we write $\widetilde{N}_s$ as $\widetilde{N}_s=\widetilde{N}_s^1+2\widetilde{N}_s^2$, $s\in\mathbb{R}^d$, where

$$\widetilde{N}_s^1 = i\,\mathbb{E}\Big(\bar{a}_s(\tau_0+\tau)\int_0^\tau e^{-\gamma_s(\tau-l)}\mathcal{Y}_s(a(\tau_0+l),a(\tau_0+l),c^{(1)}(l))\,\mathrm{d}l\Big),$$
$$\widetilde{N}_s^2 = i\,\mathbb{E}\Big(\bar{a}_s(\tau_0+\tau)\int_0^\tau e^{-\gamma_s(\tau-l)}\mathcal{Y}_s(c^{(1)}(l),a(\tau_0+l),a(\tau_0+l))\,\mathrm{d}l\Big).$$

Term $\widetilde{N}_s^1$. Writing explicitly the function $\mathcal{Y}_s$ and then $\bar{c}^{(1)}$, we get

$$\widetilde{N}_s^1 = L^{-2d}\sum_{1,2}\sum_{1',2'}\delta_{3s}'^{12}\delta_{3'3}'^{1'2'}\delta(\omega_{3s}^{12})\delta(\omega_{3'3}^{1'2'})\int_0^\tau \mathrm{d}l\int_{-\tau_0}^0\mathrm{d}l'\, e^{-\gamma_s(\tau-l)}e^{-\gamma_3(l-l')}$$
$$\times\,\mathbb{E}\big(a_1(\tau_0+l)a_2(\tau_0+l)\bar{a}_{1'}(\tau_0+l')\bar{a}_{2'}(\tau_0+l')a_{3'}(\tau_0+l')\bar{a}_s(\tau_0+\tau)\big).$$

Again, this is the same expression as (B.5), with the integration over $\mathrm{d}l'$ ranging from $-\tau_0$ to 0 instead of from 0 to l. Thus, by the Wick theorem, we obtain

$$\widetilde{N}_s^1 = 2\sum_{1,2}{}^{\circledast}\,\delta_{3s}'^{12}\delta(\omega_{3s}^{12})\int_0^\tau \mathrm{d}l\int_{-\tau_0}^0\mathrm{d}l'\, e^{-\gamma_s(\tau-l)-\gamma_3(l-l')}$$
$$\times\,\mathbb{E}a_1(\tau_0+l)\bar{a}_1(\tau_0+l')\,\mathbb{E}a_2(\tau_0+l)\bar{a}_2(\tau_0+l')\,\mathbb{E}a_s(\tau_0+l')\bar{a}_s(\tau_0+\tau).$$

Revised Proof

Following the line of Sect. B.4, we express the correlations through (2.13) and get

$$\widetilde{N}_s^1 = 2 \sum_{1,2} \delta_{3s}'^{12} \delta(\omega_{3s}^{12}) B_{12s} \int_0^\tau \mathrm{d}l\, e^{-2\gamma_s(\tau-l)-\gamma_{123s}l}$$
$$\int_0^{\tau_0} \mathrm{d}l'\, e^{-\gamma_{123s}\tau_0} e^{\gamma_3 l'} \left(e^{\gamma_s l'} - e^{-\gamma_s l'}\right) \prod_{j=1,2} \left(e^{\gamma_j l'} - e^{-\gamma_j l'}\right)$$
$$= 2 \sum_{1,2} \delta_{3s}'^{12} \delta(\omega_{3s}^{12}) B_{12s} T_s \mathcal{T}^3. \tag{B.13}$$

Term $\widetilde{N}_s^2$. Literally repeating the argument which we have applied to $\widetilde{N}_s^1$, we find that the term $\widetilde{N}_s^2$ is given by the same expression as (B.7) with the integral $\int_0^l$ replaced by $\int_{-\tau_0}^0$:

$$\widetilde{N}_s^2 = -2 \sum_{1,2} \delta_{3s}'^{12} \delta(\omega_{3s}^{12}) \int_0^\tau \mathrm{d}l \int_{-\tau_0}^0 \mathrm{d}l'\, e^{-\gamma_s(\tau-l)} e^{-\gamma_1(l-l')}$$
$$\mathbb{E}a_2(\tau_0+l)\bar{a}_2(\tau_0+l')\, \mathbb{E}a_3(\tau_0+l')\bar{a}_3(\tau_0+l)\, \mathbb{E}a_s(\tau_0+l')\bar{a}_s(\tau_0+\tau).$$

Again we get

$$\widetilde{N}_s^2 = -2 \sum_{1,2} \delta_{3s}'^{12} \delta(\omega_{3s}^{12}) B_{23s} T_s \mathcal{T}^1. \tag{B.14}$$

B.6. End of the Proof

Inserting formulas (B.12), (B.13) and (B.14), as well as (B.4), (B.6), (B.8) in (B.2), we get

$$\left| Q_{s,L} - 4L^2 T_s \sum_{1,2} \delta_{3s}'^{12} \delta(\omega_{3s}^{12}) \left(B_{123}\mathcal{T}^s + B_{12s}\mathcal{T}^3 - 2B_{23s}\mathcal{T}^1\right) \right| \le C^{\#}(s)\tau^2. \tag{B.15}$$

Note that the terms $\mathcal{Z}^j$ defined in (4.14) can be written as

$$\mathcal{Z}^j = \frac{\mathcal{T}^j}{\prod_{k\neq j}(1-e^{-2\gamma_k\tau_0})}. \tag{B.16}$$

The relations (2.13)-(2.14) imply that for any permutation (k_1,k_2,k_3,k_4) of $(1,2,3,s)$ we have $B_{k_1k_2k_3} = n_{k_1}^{(0)} n_{k_2}^{(0)} n_{k_3}^{(0)} / \prod_{m=k_1,k_2,k_3}(1-e^{-2\gamma_m\tau_0})$, where $n_{k_i}^{(0)} = n_{k_i,L}^{(0)}(\tau_0)$. Together with (B.16), this implies

$$B_{k_1k_2k_3}\mathcal{T}^{k_4} = \mathcal{Z}^{k_4} n_{k_1}^{(0)} n_{k_2}^{(0)} n_{k_3}^{(0)}. \tag{B.17}$$

By symmetry, the term $2B_{23s}\mathcal{T}^1$ in (B.15) can be replaced by $B_{23s}\mathcal{T}^1 + B_{13s}\mathcal{T}^2$. Then, inserting (B.17) in (B.15) we get

$$|Q_{s,L} - \mathcal{X}_s| \le \left| 4L^2(T_s-\tau) \sum_{1,2} \delta_{3s}'^{12} \delta(\omega_{3s}^{12}) \left(\mathcal{T}^s B_{123} + \mathcal{T}^3 B_{12s} - 2\mathcal{T}^s B_{23s}\right) \right|$$
$$+ C^{\#}(s)\tau^2,$$

Revised Proof

with $\mathcal{X}_s$ defined in (4.33). Finally, we point out that $|(T_s-\tau)\mathcal{T}^j| \le 3\tau^2$, due to (B.11) and since $|T_s-\tau| \le 3\tau^2\gamma_{123s}$. So, the bound

$$\left| L^2(T_s-\tau) \sum_{1,2} \delta'^{12}_{3s}\delta(\omega^{12}_{3s}) \left(\mathcal{T}^s B_{123} \ + \mathcal{T}^3 B_{12s} - 2\mathcal{T}^s B_{23s}\right) \right| \le C^{\#}(s)\tau^2,$$

is a consequence of Theorem 3.2. This concludes the proof of Proposition 4.3.

B.7. Proof of Lemma 4.4

Note that $\partial_{s_j} f(\gamma_j) = f'(\gamma_j)\partial_{s_j}\gamma_j$, where $\partial_{s_j}\gamma_j$ (as well as higher-order derivatives of γ_j) have at most polynomial growth at infinity. Then, using the definition (4.14) of $\mathcal{Z}^j$ we find

$$\begin{aligned} \left|\partial^{\mu}_{\vec{s}}\mathcal{Z}^j(\tau_0,\vec{s})\right| \ = & \sum_{n_1+n_2+n_3+n_s=1}^{|\mu|} P(\vec{s};n_1,\dots,n_s)\int_0^{\tau_0} \mathrm{d}l\,(\tau_0-l)^{n_j}\mathrm{e}^{-\gamma_j(\tau_0-l)} \\ & \prod_{m\neq j}\frac{d^{n_m}}{d\gamma_m^{n_m}}\left(\frac{\sinh(\gamma_m l)}{\sinh(\gamma_m\tau_0)}\right), \end{aligned} \tag{B.18}$$

where $P(\vec{s};a_1,\dots,a_n)$ denotes a function of $\vec{s}$, dependent on parameters $(a_1,\dots,a_n)$, having at most a polynomial growth at infinity. Using the relation $\frac{\sinh(\gamma l)}{\sinh(\gamma\tau_0)} = \frac{e^{-\gamma(\tau_0-l)}-e^{-\gamma(l+\tau_0)}}{1-e^{-2\gamma\tau_0}}$ we find by induction that

$$\frac{d^n}{d\gamma^n}\left(\frac{\sinh(\gamma l)}{\sinh(\gamma\tau_0)}\right) = \sum_{k+m+p=n} c_{k,m,p} I_{k,m,p}(l,\tau_0,\gamma),$$

where $c_{k,m,p}$ are constants,

$$I_{k,m,p} = \left((\tau_0-l)^k e^{-\gamma(\tau_0-l)} - (l+\tau_0)^k e^{-\gamma(l+\tau_0)}\right)\frac{\tau_0^{m+p}e^{-2\gamma m\tau_0}}{(1-e^{-2\gamma\tau_0})^{m+1}} \tag{B.19}$$

and $p\neq 0$ only if $m\neq 0$. For $\tau_0 \ge \gamma^{-1}$ the terms $I_{k,m,p}$ are bounded in absolute values by absolute constants $C_{k,m,p}$, where we recall that $0\le l\le\tau_0$ and $\gamma\ge 1$. Let now $\tau_0\le\gamma^{-1}$. In this case, since $k+m+p=n$,

$$|I_{k,m,p}| \le 2\,\frac{(l+\tau_0)^k\tau_0^{m+p}}{(1-e^{-2\gamma\tau_0})^{m+1}} \le 2^{k+1}\,\frac{\tau_0^n}{(1-e^{-2\gamma\tau_0})^{m+1}}.$$

So, in the case $m\le n-1$ we have $|I_{k,m,p}|\le C_{k,m,p}$ uniformly in $\tau_0\le\gamma^{-1}$. If $m=n$ (so $k=p=0$) we use another estimate, following from (B.19):

$$|I_{k,m,p}| \le C\,\frac{e^{-\gamma(\tau_0-l)}-e^{-\gamma(l+\tau_0)}}{(1-e^{-2\gamma\tau_0})^{m+1}}\tau_0^m = C\tau_0^m\,e^{-\gamma(\tau_0-l)}\,\frac{1-e^{-2\gamma l}}{(1-e^{-2\gamma\tau_0})^{m+1}} \le C_{k,m,p},$$

uniformly in $\tau_0\le\gamma^{-1}$.

We have seen that the product in (B.18) is bounded uniformly in $\vec{s}$, l and τ_0, so the integral over l is also bounded uniformly in $\vec{s}$ and τ_0.

Revised Proof

Appendix C. Proof of Proposition 2.1

The proof uses the theory of Feynman diagrams, presented in Sect. 2. For $N = 0$ the assertion is trivial. For $N \geq 1$ in Proposition 8.7 of [8] it is proven that $\mathbb{E}\big(\mathfrak{a}_s^{(m)}(\tau_1)\bar{\mathfrak{a}}_s^{(n)}(\tau_2) - a_s^{(m)}(\tau_1)\bar{a}_s^{(n)}(\tau_2)\big)$ equals to

$$\sum_{\mathfrak{F}\in\mathfrak{F}^+_{m,n}\setminus\mathfrak{F}_{m,n}} c_{\mathfrak{F}}\,\mathcal{J}_s(\mathfrak{F}) + \sum_{\mathfrak{F}\in\mathfrak{F}_{m,n}} c_{\mathfrak{F}}\,\mathcal{J}_s^2(\mathfrak{F}), \tag{C.1}$$

where $\mathfrak{F}^+_{m,n}$ is a certain (finite) set of *extended* Feynman diagrams,[13] $c_{\mathfrak{F}}$ is a complex number of unit norm and $\mathcal{J}_s(\mathfrak{F})$, $\mathcal{J}_s^2(\mathfrak{F})$ are sums, similar to (2.19). In Section 8.6.3 of [8] are established the following bounds for these sums:

$$|\mathcal{J}_s^2(\mathfrak{F})| \leq C^{\#}(s)L^{-Nd} \sum_{\substack{z\in\mathcal{Z}^+(\mathfrak{F}):\\ z_j=0 \text{ for some } j,\\ \omega_k^{\mathfrak{F}}(z)=0\,\forall 1\leq k\leq N}} C^{\#}(z), \tag{C.2}$$

where

$$\mathcal{Z}^+(\mathfrak{F}) = \left\{ z\in(\mathbb{Z}_L^d)^N:\ z_k\neq 0 \Leftrightarrow \sum_{i=1}^N \alpha_{ki}^{\mathfrak{F}} z_i \neq 0 \quad \forall 1\leq k\leq N \right\}$$

while the quadratic forms $\omega_k^{\mathfrak{F}}$ and the skew-symmetric matrix $\alpha^{\mathfrak{F}}$ are defined in Sect. 2.4. Note that possibly the diagram $\mathfrak{F}$ does not belong to the set $\mathfrak{F}^{true}_{m,n}$, so that the matrix $\alpha^{\mathfrak{F}}$ may have zero columns and lines. On the other hand,

$$|\mathcal{J}_s(\mathfrak{F})| \leq C^{\#}(s)L^{-Nd} \sum_{\substack{z\in\widetilde{\mathcal{Z}}^+(\mathfrak{F}):\\ \widetilde{\omega}_k^{\mathfrak{F}}(z)=0\,\forall 1\leq k\leq \widetilde{N}}} C^{\#}(z). \tag{C.3}$$

Here $\widetilde{N} = \widetilde{N}(\mathfrak{F}) < N$, quadratic forms $\widetilde{\omega}_k^{\mathfrak{F}}(z)$ are defined by relations (2.17), where N is replaced by $\widetilde{N}$ and the matrix $(\alpha_{ij}^{\mathfrak{F}})$ – by a certain $\widetilde{N}\times\widetilde{N}$-matrix $(\widetilde{\alpha}_{ji}^{\mathfrak{F}})$, also satisfying $\widetilde{\alpha}_{ji}^{\mathfrak{F}} = -\widetilde{\alpha}_{ij}^{\mathfrak{F}} \in \{0,\pm 1\}$ for all i,j. Accordingly the set $\widetilde{\mathcal{Z}}^+(\mathfrak{F}) \subset (\mathbb{Z}_L^d)^{\widetilde{N}}$ is defined as $\mathcal{Z}^+(\mathfrak{F})$ above, but with N and $\alpha_{ij}^{\mathfrak{F}}$ replaced by $\widetilde{N}$ and $\widetilde{\alpha}_{ij}^{\mathfrak{F}}$.

We first show that the term $\mathcal{J}_s^2(\mathfrak{F})$ is bounded by the r.h.s. of (2.3). To this end we write $\mathcal{Z}^+(\mathfrak{F}) = \cup_{\mathcal{K}}\mathcal{Z}_{\mathcal{K}}$, where the union is taken over all subsets $\mathcal{K}\subset\{1,\dots,N\}$ and

$$\mathcal{Z}_{\mathcal{K}}(\mathfrak{F}) = \left\{ z:\ z_k = \sum_{i=1}^N \alpha_{ki}^{\mathfrak{F}} z_i = 0\ \forall k\in\mathcal{K} \ \text{ and } \ z_k\neq 0,\ \sum_{i=1}^N \alpha_{ki}^{\mathfrak{F}} z_i \neq 0\ \forall k\notin\mathcal{K} \right\}.$$

Then the r.h.s. of (C.2) takes the form

$$C^{\#}(s)L^{-Nd} \sum_{\mathcal{K}\neq\varnothing} \sum_{\substack{z\in\mathcal{Z}_{\mathcal{K}}(\mathfrak{F}):\\ \omega_k^{\mathfrak{F}}(z)=0\,\forall 1\leq k\leq N}} C^{\#}(z). \tag{C.4}$$

[13] These diagrams are defined similarly to the Feynman diagrams from Sect. 2.3.1, but now we allow to couple leaves not only from different blocks but also from the same block.

Revised Proof

Note that on the set $\mathcal{Z}_{\mathcal{K}}(\mathfrak{F})$ we have $\omega_k(z) = 0$ for all $k \in \mathcal{K}$ and $\omega_k(z) = 2z_k \cdot \sum_{i\notin\mathcal{K}} \alpha^{\mathfrak{F}}_{ki} z_i$ for $k \notin \mathcal{K}$. Thus, the sum over z in (C.4) takes the form of the sum in (3.2), where $z = (z_j)_{j\notin\mathcal{K}}$ and N is replaced by

$$N - \varkappa, \quad \varkappa = \#\mathcal{K}.$$

We recall that $N \le 4$ and $\mathcal{K} \ne \varnothing$, so that $N - \varkappa$ takes values 0, 1, 2 or 3. For the sets $\mathcal{K}$ satisfying $N - \varkappa = 0$ we have $\mathcal{Z}_{\mathcal{K}}(\mathfrak{F}) = \{0\}$, so the sum (C.3) is bounded by $C^{\#}(s)L^{-Nd}$. Since the matrix $\alpha^{\mathfrak{F}}$ is skew-symmetric, then in the case $N - \varkappa = 1$ we have $\mathcal{Z}_{\mathcal{K}}(\mathfrak{F}) = \varnothing$, so the sum (C.3) vanishes. When $N - \varkappa = 2$ or 3 we apply Theorem 3.2 and see that the sum over z in (C.4) is bounded by $CL^{-(N-\varkappa)(1-d)}$. So

$$\begin{aligned}|\mathcal{J}^2_s(\mathfrak{F})| \le C^{\#}(s)L^{-Nd} \sum_{\mathcal{K}\ne\varnothing} L^{-(N-\varkappa)(1-d)} &= C^{\#}(s) \sum_{\mathcal{K}\ne\varnothing} L^{-N+\varkappa(1-d)} \\ &\le C^{\#}_1(s)L^{-N+1-d}.\end{aligned}$$

Same argument implies that the r.h.s. of (C.3) also is bounded by the quantity $C^{\#}(s)L^{-N+1-d}$ (note that decomposing the r.h.s. of (C.3) as in (C.4) we get a new term with $\mathcal{K} = \varnothing$, but for it $N - \varkappa = N \le 4$ and Theorem 3.2 still applies).

Appendix D. Case $d = 2$

A difference between the cases $d \ge 3$ and $d = 2$ comes from Theorem 3.1 since in the asymptotic, given by the latter, an additional log-factor appears when $d = 2$. To handle it we redefine the sum in (3.2), defining $S_{L,N}(\Phi)$, by multiplying it by $(\ln L)^{-N/2}$. So when $d = 2$ $S_{L,N}$ takes the form

$$S_{L,N}(\Phi) := \frac{L^{N(1-d)}}{(\ln L)^{N/2}} \sum_{z\in\mathcal{Z}:\ \omega_j(z)=0\,\forall j} \Phi(z). \tag{D.1}$$

Accordingly the $(d = 2)$-analogy of (3.5) reads

$$\left|S_{L,2}(\Phi) - \frac{L^{2(1-d)}}{\ln L} \sum_{z\in\mathbb{Z}^{2d}_L:\ z_1\cdot z_2=0} \Phi(z)\right| \le \frac{CL^{2-d}}{\ln L}\|\Phi\|_{0,d+1} = \frac{C}{\ln L}\|\Phi\|_{0,3}. \tag{D.2}$$

This approximation, jointly with a modification of the Heath–Brown result from [14], given in Theorem 1.4 of [10], implies the following version of Theorem 3.1 for $d = 2$:

Theorem D.1. *Let $d = 2$. Then there exist constants $N_1, N_2 > 4$ such that if $\|\Phi\|_{N_1,N_2} < \infty$,*

$$\left|S_{L,2}(\Phi) - C_2 \int_{\Sigma_0} \Phi(z)\, \mu^{\Sigma_0}(\mathrm{d}z_1\mathrm{d}z_2)\right| \le K_2 \frac{\|\Phi\|_{N_1,N_2}}{\ln L}, \tag{D.3}$$

where $C_2 > 0$ is a number-theoretical constant and $K_2 > 0$.

Note that estimate (3.8) stays true when $d = 2$.

Revised Proof

Theorem D.2. *In the case $d = 2$ assertion of Theorem 3.2 remains true, if the sum $S_{L,N}$ is defined as in (D.1) and N_2 is the constant from Theorem D.1.*

Proof. The only difference with the proof of Theorem 3.2 comes from estimate (3.18) since the latter is obtained by applying Theorem 3.1, and in the case $d = 2$ we should apply Theorem D.1 instead. Namely, now the r.h.s. of (3.18) takes the form $CL^{2(d-1)} \ln L\big[R^{2d} + R^{N_2}(\ln L)^{-1}\big] \le C' R^{N_2} L^{2(d-1)} \ln L$. Since Lemma 3.6 remains unchanged, then for $d = 2$ the r.h.s. of estimate (3.20), which holds for irreducible matrices α, should be multiplied by $\ln L$. In the case of reducible matrix α we apply the latter estimate to each irreducible block, which gives the factor $(\ln L)^{\lfloor N/2 \rfloor}$ in the r.h.s. of (3.12), since the number of blocks does not exceed $\lfloor N/2 \rfloor$. However, the final estimate of Theorem 3.2 remains unchanged because of the factor $(\ln L)^{-N/2}$ in the definition (D.1) of the sum $S_{L,N}$. □

Since in the case $d = 2$ we choose $\rho = \varepsilon L/\sqrt{\ln L}$, then the terms $n_{s,L}^{(k)}$ are given by formula (4.4), multiplied by $(\ln L)^{-k/2}$. The proof of Proposition 2.1 is analogous to that presented in Appendix C for $d = 2$. The only difference being the use of Theorem D.2 in place of Theorem 3.2. Lemma 2.2 remains unchanged, so the correlations $L^k(\ln L)^{-k/2}\mathbb{E}a_s^{(m)}\bar{a}_s^{(n)}(\tau_2)$, $m + n = k$, are given by formula (2.18), multiplied by $(\ln L)^{-k/2}$ (recall that the sum of these correlations makes $n_{s,L}^{(k)}$). We see that the correlations take the form (D.1), so Theorems D.1 and D.2 apply to study them.

The rest of the proof of Theorem 5.8 literally repeats that for the case $d \ge 3$, except the appearance of the $(\ln L)^{-k/2}$ factors, coming from the new definition of ρ. Now the estimates, using Theorem 3.1, should be relaxed since the estimate provided by Theorem D.1 is slightly weaker than that of Theorem 3.1. In particular, in the r.h.s. of (4.8), (4.11) and (4.12) the factor $L^{-1/2}$ should be replaced by $(\ln L)^{-1}$. This results in the stronger lower bound for L in Theorem 5.8: now it is $L \ge e^{\varepsilon^{-1}}$ instead of $L \ge \varepsilon^{-2}$ (see Theorem A).

Theorem 5.9, as Theorem 5.8, remains unchanged, except the lower bound for L which is modified as above. Indeed, the theorem follows from Theorem 5.8 and Proposition 2.1, and the term $\chi_2(L)^{-N+1} = (\ln L)^{(N-1)/2}$, appearing in estimate (2.3) for $d = 2$ does not change the assertion of Theorem 5.8.

D.1. Discussion of Remark 1.2

In fact, Theorem 1.4 from [10] provides more delicate information about $S_{L,2}$ than what is stated in Theorem D.1. Namely, if $d = 2$ then due to [10],

$$\left| \frac{L^{2(1-d)}}{\ln L} \sum_{z:\ z_1 \cdot z_2 = 0} \Phi(z) - C_2 \int_{\Sigma_0} \Phi(z)\, \mu^{\Sigma_0}(\mathrm{d}z_1 \mathrm{d}z_2) - \frac{\sigma_1^{\Phi}(L)}{\ln L} \right| \le C \frac{\|\Phi\|_{N_1,N_2}}{L^{1/6}},$$

where σ_1^{Φ} is a certain function satisfying $|\sigma_1^{\Phi}(L)| \le C_1 \|\Phi\|_{N_1,N_2}$, uniformly in L. See [10] for an explicit (but complicated) formula for σ_1^{Φ}. Consequently,

$$\left| S_{L,2}(\Phi) - C_2 \int_{\Sigma_0} \Phi(z)\, \mu^{\Sigma_0}(\mathrm{d}z_1 \mathrm{d}z_2) - \frac{\widetilde{\sigma}_1^{\Phi}(L)}{\ln L} \right| \le C \frac{\|\Phi\|_{N_1,N_2}}{L^{1/6}},$$

Revised Proof

where

$$\widetilde{\sigma}_1^{\Phi}(L) := \sigma_1^{\Phi}(L) - L^{2(1-d)} \sum_{z:\ z_1=0 \text{ or } z_2=0} \Phi(z)$$

still satisfies $|\widetilde{\sigma}_1^{\Phi}(L)| \leq C\|\Phi\|_{N_1,N_2}$ in view of (D.2). Then estimate (4.12) refines as

$$\left| n_s^{\leq 2} - n_{s,L} - \frac{f(\tau, L)}{\ln L} \right| \leq C^{\#}(s)\varepsilon^2(L^{-1/6} + \varepsilon), \tag{D.4}$$

where $f(\tau, L) := \widetilde{\sigma}_1^{\Phi(\tau)}(L)$ and $\Phi(\tau)$ is the function satisfying $n_{s,L}^{\leq 2}(\tau) = S_{L,2}(\Phi(\tau))$ that comes from Corollary 2.3. By (2.22) and the estimate for $\widetilde{\sigma}_1^{\Phi}$ above, the function $f(\tau, L)$ is bounded uniformly in τ. The rest of the proofs of Theorems 5.8 and 5.9 remain unchanged while the estimate (D.4) leads to the assertion of the remark.

References

[1] Buckmaster, T., Germain, P., Hani, Z., Shatah, J.: Effective dynamics of the nonlinear Schrödinger equation on large domains. Comm. Pure Appl. Math. **71**, 1407–1460 (2018)

[2] Buckmaster, T., Germain, P., Hani, Z., Shatah, J.: Onset of the wave turbulence description of the longtime behaviour of the nonlinear Schrödinger equation. Invent. Math. **225**, 787–855 (2021)

[3] Collot, C., Germain, P.: On the derivation of the homogeneous kinetic wave equation, (2019), arXiv:1912.10368

[4] Collot, C., Germain, P.: Derivation of the homogeneous kinetic wave equation: longer time scales, (2020), arXiv:2007.03508

[5] Deng, Y., Hani, Z.: On the derivation of the wave kinetic equation for NLS, (2019). Forum Math., Pi **9**, 1–37 (2021)

[6] Deng, Y., Hani, Z.: Full derivation of the wave kinetic equation. Invent. Math. **233**, 543–724 (2023)

[7] Dymov, A., Kuksin, S.: Formal expansions in stochastic model for wave turbulence 1: kinetic limit. Comm. Math. Phys. **382**, 951–1014 (2021)

[8] Dymov, A., Kuksin, S.: Formal expansions in stochastic model for wave turbulence 2: method of diagram decomposition. J. Stat. Phys. **190**, 3 (2023)

[9] Dymov, A., Kuksin, S.: On the Zakharov-L'vov stochastic model for wave turbulence. Dokl. Math. **101**, 102–109 (2020)

[10] Dymov, A., Kuksin, S., Maiocchi, A., Vlăduţ, S.: A refinement of Heath-Brown's theorem on quadratic forms. Sbornik: Mathematics (2023)

[11] Erdös, L., Yau, H.T.: Linear Boltzmann equation as the weak coupling limit of a random Schrödinger equation. Commun. Pure Appl. Math. **53**, 667–735 (2000)

[12] Erdös, L., Schlein, B., Yau, H.-T.: Derivation of the cubic non-linear Schrödinger equation from quantum dynamics of many-body systems. Invent. Math. **167**, 515–614 (2007)

[13] Fulton, W.: Intersection Theory. Springer, Berlin (1998)

[14] Heath-Brown, D.R.: A new form of the circle method, and its application to quadratic forms. J. Reine Angew. Math. **481**, 149–206 (1996)

Revised Proof

[15] Faou, E.: Linearized wave turbulence convergence results for three-wave systems. Comm. Math. Physics **378**, 807–849 (2020)

[16] Faou, E., Germain, P., Hani, Z.: The weakly nonlinear large-box limit of the 2D cubic nonlinear Schrödinger equation. J. Amer. Math. Soc. **29**, 915–982 (2016)

[17] Huang, G., Kuksin, S., Maiocchi, A.: Time-averaging for weakly nonlinear CGL equations with arbitrary potentials. Fields Inst. Commun. **75**, 323–349 (2015)

[18] Janson, S.: Gaussian Hilbert Spaces. Cambridge University Press, Cambridge (1997)

[19] Kuksin, S.: Ergodicity, mixing and KAM, Séminaire Laurent Schwartz EDP et applications, 2018-2019, Exp. No. 8, 9 (2019); https://doi.org/10.5802/slsedp.128

[20] Kuksin, S., Maiocchi, A.: Derivation of a wave kinetic equation from the resonant-averaged stochastic NLS equation. Physica D **309**, 65–70 (2015)

[21] Kuksin, S., Maiocchi, A.: Resonant averaging for small solutions of stochastic NLS equations. Proc. Royal Soc. Edinburgh **147A**, 1–38 (2017)

[22] Lachaud, G., Rolland, R.: On the number of points of algebraic sets over finite fields. J. Pure Appl. Algebra **219**, 5117–5136 (2015)

[23] Lukkarinen, J., Spohn, H.: Weakly nonlinear Schrödinger equation with random initial data. Invent. Math. **183**, 79–188 (2015)

[24] Lang, S.: Algebra. Springer, New York (2002)

[25] Nazarenko, S.: Wave Turbulence. Springer, Heidelberg (2011)

[26] Newell, A.C., Rumpf, B.: Wave Turbulence. Annu. Rev. Fluid Mech. **43**, 59–78 (2011)

[27] Shafarevitch, I.: Basic algebraic geometry. Springer, Berlin (1994)

[28] Shirikyan, A.: Ergodicity for a class of Markov processes and applications to randomly forced PDE's II. Discrete Contin. Dynam. Syst., Ser. A **6**, 911–926 (2006)

[29] Zakharov, V., L'vov, V.: Statistical description of nonlinear wave fields. Radiophys. Quan. Electron. **18**, 1084–1097 (1975)

[30] Zakharov, V., L'vov, V., Falkovich, G.: Kolmogorov Spectra of Turbulence. Springer, Heidelberg (1992)

Andrey Dymov
Steklov Mathematical Institute of RAS
Moscow
Russia
e-mail: dymov@mi-ras.ru

and

Revised Proof

Skolkovo Institute of Science and Technology
Skolkovo
Russia

Andrey Dymov and Sergei Kuksin
Peoples' Friendship University of Russia (RUDN University)
Moscow
Russia

Sergei Kuksin
CNRS, IMJ-PRG
Université Paris Cité and Sorbonne Université
75013 Paris
France

and

Steklov Mathematical Institute of Russian Academy of Sciences
Moscow
Russia
e-mail: sergei.kuksin@imj-prg.fr

Alberto Maiocchi
Dipartimento di Matematica e Applicazioni - Edificio U5
Università degli Studi di Milano-Bicocca
via Roberto Cozzi, 55
20125 Milan
Italy
e-mail: alberto.maiocchi@unimib.it

Sergei Vlăduţ
CNRS, Centrale Marseille, I2M UMR 7373
Aix Marseille Université
13453 Marseille
France
e-mail: serge.vladuts@univ-amu.fr

and

IITP RAS
19 B. Karetnyi
Moscow
Russia

Communicated by Claude-Alain Pillet.
Received: January 13, 2022.
Accepted: August 28, 2023.

Revised Proof