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FORMAL EXPANSIONS IN STOCHASTIC MODEL FOR WAVE TURBULENCE 2: METHOD OF DIAGRAM DECOMPOSITION

ANDREY DYMOV AND SERGEI KUKSIN

Abstract. In this paper we continue to study small amplitude solutions of the damped cubic NLS equation, driven by a random force (the study was initiated in our previous work [9] and continued in [11]). We write solutions of the equation as formal series in the amplitude and discuss the behaviour of this series under the wave turbulence limit, when the amplitude goes to zero, while the space-period goes to infinity.

1. Introduction and results

We continue the study of the nonlinear cubic Shrödinger equation (NLS) on a torus of large period, dumped by a viscosity and driven by a random force, initiated in our papers [9, 11]. Results of this work are crucially used in [9, 11] and are of independent interest for further study of wave turbulence (WT) in the equation which we consider and in similar ones. In particular, our approach applies to the NLS equations with higher order nonlinearities.

Below we recall the setup, state the main theorem and then discuss its relevance for works [9, 11] and for the theory of WT. The theorem's proof is outlined in Section 2 and is given in details in Sections 3-7. In Section 8 we establish some lemmas, used in the proof, and discuss a number of related results.

For a general theory of wave turbulence we refer to works [20, 21, 23], and for an additional discussion of the relevance of our results for WT – to the introduction in [9].

1.1. **The setting.** Let $\mathbb{T}^d_L = \mathbb{R}^d/(L\mathbb{Z}^d)$ be a d-dimensional torus, $d \geq 1$, of period $L \geq 1$. We denote by $\|u\|$ the normalized L_2-norm of a complex function u on $\mathbb{T}^d_L$, $\|u\|^2 = L^{-d}\int_{\mathbb{T}^d_L} |u(x)|^2\,dx$, and write the Fourier series of u as [1]

$$u(x) = L^{-d/2}\sum\nolimits_{s\in\mathbb{Z}^d_L} v_s e^{2\pi i s\cdot x}, \qquad \mathbb{Z}^d_L = L^{-1}\mathbb{Z}^d. \tag{1.1}$$

[1] The scaling factor $L^{-d/2}$ is convenient for our calculations.

Here the vector of Fourier coefficients $v = (v_s)_{s\in\mathbb{Z}^d_L}$ is given by the Fourier transform of $u(x)$,

$$v = \mathcal{F}(u), \quad v_s = L^{-d/2}\int_{\mathbb{T}^d_L} u(x)e^{-2\pi i s\cdot x}\,dx \quad \text{for } s\in\mathbb{Z}^d_L.$$

The Parseval identity reeds $\|u\|^2 = L^{-d}\sum_{s\in\mathbb{Z}^d_L}|v_s|^2$. We will study solutions $u(t,x)$ whose norms satisfy $\|u(t,\cdot)\|\sim 1$ as $L\to\infty$.

Our goal is to study the cubic NLS equation with modified nonlinearity

$$\frac{\partial}{\partial t}u + i\Delta u - i\lambda\left(|u|^2 - 2\|u\|^2\right)u = 0, \qquad x\in\mathbb{T}^d_L, \tag{1.2}$$

where $u = u(t,x)$, $\Delta = (2\pi)^{-2}\sum_{j=1}^d(\partial^2/\partial x_j^2)$ and $\lambda\in(0,1]$ is a small parameter. The modification of the nonlinearity by the term $2i\lambda\|u\|^2u$ keeps the main features of the standard cubic NLS equation, reducing some non-crucial technicalities, see the introduction to [9] for a detailed discussion.

The objective of WT is to study solutions of (1.2) under the wave turbulence limit $L\to\infty$ and $\lambda\to 0$ on long time intervals, usually while "pumping the energy to low modes and dissipating it in high modes". To make this rigorous, following Zakharov-L'vov [22], we consider the NLS equation (1.2) dumped by a (hyper) viscosity and driven by a random force:

$$\frac{\partial}{\partial t}u + i\Delta u - i\lambda\left(|u|^2 - 2\|u\|^2\right)u = -\nu\mathfrak{A}(u) + \sqrt{\nu}\frac{\partial}{\partial t}\eta^\omega(t,x). \tag{1.3}$$

Here $\nu\in(0,1/2]$ is a small parameter and $\mathfrak{A}$ is the dissipative linear operator

$$\mathfrak{A}(u(x)) = L^{-d/2}\sum_{s\in\mathbb{Z}^d_L}\gamma_s v_s e^{2\pi i s\cdot x}, \quad v = \mathcal{F}(u), \quad \gamma_s = \gamma^0(|s|^2). \tag{1.4}$$

The real function $\gamma^0(y)\geq 1$ is smooth and has at most a polynomial growth at infinity, together with all its derivatives. [2] The random noise η^ω is given by the Fourier series

$$\eta^\omega(t,x) = L^{-d/2}\sum_{s\in\mathbb{Z}^d_L} b(s)\beta^\omega_s(t)e^{2\pi i s\cdot x},$$

where $\{\beta_s(t), s\in\mathbb{Z}^d_L\}$ are standard independent complex Wiener processes [3] and $b(x)$ is a Schwartz function on $\mathbb{R}^d\supset\mathbb{Z}^d_L$. [4] If the function $\gamma^0(y)$ grows at infinity as $|y|^{r_*}$ with r_* sufficiently big, then eq. (1.3) is well posed. In difference with [9, 11], where we assumed $d\geq 2$, in this work we allow $d=1$ and impose less restrictions on the function γ^0.

[2] For example $\mathfrak{A} = (1-\Delta)^{r^*}$ for some $r^*\geq 0$.

[3] i.e. $\beta_s = \beta^1_s + i\beta^2_s$, where $\{\beta^j_s, s\in\mathbb{Z}^d_L, j=1,2\}$ are standard independent real Wiener processes.

[4] Often it is assumed that the intensity $b(x)$ of the noise η^ω is non-negative, but we do not impose this condition. Still note that if $b(x)\equiv 0$, then our results become trivial since below we will provide (1.3) with the zero initial conditions.

We will study the equation on time intervals of order ν^{-1}, so it is convenient to pass from t to the slow time $\tau = \nu t$. Then eq. (1.3) takes the form

$$\begin{aligned} \dot u + i\nu^{-1}\Delta u - i\rho\big(|u|^2 - 2\|u\|^2\big)u &= -\mathfrak{A}(u) + \dot\eta^\omega(\tau,x), \\ \eta^\omega(\tau,x) &= L^{-d/2}\sum\nolimits_{s\in\mathbb{Z}^d_L} b(s)\beta^\omega_s(\tau)e^{2\pi i s\cdot x}, \end{aligned} \tag{1.5}$$

where $\rho = \lambda\nu^{-1}$, the upper-dot stands for $d/d\tau$ and $\{\beta_s(\tau),\, s\in\mathbb{Z}^d_L\}$ is another set of standard independent complex Wiener processes. Below we use ρ, ν and L as the parameters of the equation.

In the context of eq. (1.5), the objective of WT is to study its solutions u when

$$L\to\infty \quad\text{and}\quad \nu\to 0, \tag{1.6}$$

as well as main characteristics of the solutions such as the *energy spectrum*

$$n_s(\tau) := \mathbb{E}|v_s(\tau)|^2, \qquad s\in\mathbb{Z}^d_L, \quad \text{where } v = \mathcal{F}(u). \tag{1.7}$$

In [9,11] and in this paper we study formal decompositions in ρ of solutions to eq. (1.5), regarding ρ as an independent parameter, and those of the energy spectrum n_s. In Section 1.3, where we present the results of [9] and [11], we assume that $\rho\sim\nu^{-1/2}$ or $\rho\sim L$ correspondingly, since due to [9] and [11] exactly under these scalings the quadratic in ρ part of the decomposition of the energy spectrum has a non-trivial behaviour under the limit (1.6), taken in the regimes, considered in [9] and [11].

1.2. **The result.** Let us take the Fourier transform of eq. (1.5):

$$\dot v_s - i\nu^{-1}|s|^2 v_s + \gamma_s v_s = i\rho L^{-d}\Big(\sum\nolimits_{s_1,s_2,s_3\in\mathbb{Z}^d_L}\delta^{\prime 12}_{3s} v_{s_1}v_{s_2}\bar v_{s_3} - |v_s|^2 v_s\Big) + b(s)\dot\beta_s\,, \tag{1.8}$$

$s\in\mathbb{Z}^d_L$, where $|s|$ stands for the Euclidean norm of a vector s and

$$\delta^{\prime 12}_{3s} = \delta^{\prime s_1 s_2}_{\ \ s_3 s} = \begin{cases} 1, & \text{if } s_1+s_2 = s_3+s \text{ and } \{s_1,s_2\}\neq\{s_3,s\}\,,\\ 0, & \text{otherwise.}\end{cases} \tag{1.9}$$

Note that

$$\text{if } \delta^{\prime 12}_{3s} = 1, \text{ then } \{s_1,s_2\}\cap\{s_3,s\}\neq\varnothing. \tag{1.10}$$

In view of the factor $\delta^{\prime 12}_{3s}$, in the sum in (1.8) the variable s_3 is a function of s_1, s_2, s. That is why below we write the sum as

$$\sum\nolimits_{s_1,s_2,s_3\in\mathbb{Z}^d_L}\delta^{\prime 12}_{3s} =: \sum\nolimits_{s_1}\sum\nolimits_{s_2}\delta^{\prime 12}_{3s}.$$

We fix $0\le T\le +\infty$ and study equation (1.5)=(1.8) with zero initial condition at $\tau = -T$, so

$$v_s(-T) = 0 \quad\text{for all}\quad s\in\mathbb{Z}^d_L. \tag{1.11}$$

Let us write solutions of (1.8), (1.11) as formal series in ρ: [5]

$$v_s = \sum_{i=0}^{\infty} \rho^i v_s^{(i)}, \qquad v_s^{(i)}(-T) = 0, \tag{1.12}$$

where $v_s^{(i)} = v_s^{(i)}(\tau;\nu,L)$. The processes $v_s^{(0)}$ satisfy the linear equations

$$\dot v_s^{(0)}(\tau) - i\nu^{-1}|s|^2 v_s^{(0)}(\tau) + \gamma_s v_s^{(0)}(\tau) = b(s)\dot\beta_s(\tau), \qquad v_s^{(0)}(-T) = 0, \tag{1.13}$$

so these are independent Gaussian processes. The linear in ρ term $v_s^{(1)}$ satisfies

$$\dot v_s^{(1)} - i\nu^{-1}|s|^2 v_s^{(1)} + \gamma_s v_s^{(1)} = iL^{-d}\Big(\sum\nolimits_{s_1,s_2,s_3} \delta_{3s}^{I12} v_{s_1}^{(0)} v_{s_2}^{(0)} \bar v_{s_3}^{(0)} - |v_s^{(0)}|^2 v_s^{(0)}\Big), \tag{1.14}$$

and $v_s^{(1)}(-T) = 0$. Other terms $v_s^{(n)}$ can be found recursively in a similar way.

Our goal is to study correlations of the terms of series (1.12) under the limit (1.6). Repeating the argument of Lemma 2.3 from [9] it is straightforward to show that $\mathbb{E} v_s^{(m)}(\tau_1) v_{s'}^{(n)}(\tau_2) = 0$ for any $s, s' \in \mathbb{Z}_L^d$ and that $\mathbb{E} v_s^{(m)}(\tau_1)\bar v_{s'}^{(n)}(\tau_2) = 0$ for $s \neq s'$. Our main result, given below in Theorem 1.1, establishes upper bounds for the correlations with $s = s'$. Its particular case is crucially used in [9], see below in Section 1.3. Our second main result is Theorems 5.7, 5.9, which are steps in the proof of Theorem 1.1. There we develop an instrumental representation of the correlations above in a form of explicit sums, convenient for further analysis. This result is crucially used in [11].

To state Theorem 1.1 we use the function

$$\chi_d^N(\nu) = \begin{cases} -\ln\nu & \text{if} \quad (N,d) = (3,2) \text{ or } (2,1) \\ 1 & \text{otherwise} \end{cases} \tag{1.15}$$

(we recall that $0 < \nu \le 1/2$). For an $x \in \mathbb{R}$ we denote $\lceil x \rceil = \min\{n \in \mathbb{Z} : n \ge x\}$; and denote by $C^{\#}(\cdot)$ various positive continuous functions on $\mathbb{R}^k$, $k \in \mathbb{N}$, fast decaying at infinity:

$$C^{\#}(x) \le c_n (1 + |x|)^{-n} \quad \forall x \in \mathbb{R}^k, \tag{1.16}$$

for every $n \in \mathbb{N}$, with a suitable constant $c_n > 0$.

Theorem 1.1. *Assume that $d \ge 2$. Then for any integers $m, n \ge 0$ there exists a function $C^{\#}$ such that for each $s \in \mathbb{Z}_L^d$ and $\tau_1, \tau_2 \ge -T$ we have*

$$\big|\mathbb{E} v_s^{(m)}(\tau_1)\bar v_s^{(n)}(\tau_2)\big| \le C^{\#}(s)\big(\nu^{-2}L^{-2} + \nu^{\min(\lceil N/2\rceil, d)}\chi_d^N(\nu)\big), \tag{1.17}$$

[5] As we mentioned above, in paper [9] we choose $\rho \sim \nu^{-1/2}$ to be large. However, the first d coefficients $v_s^{(i)}$ of series (1.12) are in fact of order $\lesssim \nu^{i/2}$ and we expect that the others also satisfy this estimate, see in Section 1.4. So there is a hope that the series converges. A similar situation takes place in paper [11], where we assume $\rho \sim L$.

where $N = m + n$. For $d = 1$ the assertion above remains true if we modify the r.h.s. of (1.17) by adding the term $C^{\#}(s)L^{-1}$.

In our work [9] estimate (1.17) (with $d \geq 2$) is used in the situation when $\nu^{-2}L^{-2} \leq \nu^{\min(\lceil N/2\rceil,d)}$ – then it implies that $\mathbb{E}|v_s^{(m)}(\tau)|^2 \leq C^{\#}(s)\nu^{\min(m,d)}$, uniformly in L. The above restriction on the parameters well agrees with a postulate, usually accepted by the WT community, that in (1.6) L should go fast to infinity when $\nu \to 0$.

To prove the theorem we approximate, with accuracy $\nu^{-2}L^{-2}$, the expectation $\mathbb{E}v_s^{(m)}\bar{v}_s^{(n)}$ by a finite sum of integrals, independent from L and parametrized by a suitable class $\mathfrak{F}_{m,n}$ of Feynman diagrams (the cardinality of the set $\mathfrak{F}_{m,n}$ grows factorially with $m + n$). Next we study the behaviour of each integral as $\nu \ll 1$. To this end, in Section 5, for every diagram $\mathfrak{F}$ we find a special coordinate system such that in these coordinates the integral, corresponding to $\mathfrak{F}$, has the simple explicit form:

$$\tilde{J}_s(\tau_1,\tau_2;\mathfrak{F}) = \int_{\mathbb{R}^N} dl \int_{\mathbb{R}^{dN}} dz\, F_s^{\mathfrak{F}}(\tau_1,\tau_2;l,z)e^{i\nu^{-1}\Omega^{\mathfrak{F}}(l,z)} \tag{1.18}$$

with explicit functions $\Omega^{\mathfrak{F}}$ and $F_s^{\mathfrak{F}}$, see Theorem 5.9. Here the phase function $\Omega^{\mathfrak{F}}$ is a real quadratic form in z (usually degenerate) which linearly depends on l. The density function $F_s^{\mathfrak{F}}$ is Schwartz in (s,z), while as a function of l it is piecewise smooth and fast decays as $|l| \to \infty$. Then in Sections 6 and 7 we show that

$$|\tilde{J}_s(\mathfrak{F})| \leq C^{\#}(s)\nu^{\min(\lceil N/2\rceil,d)}\chi_d^N(\nu) \qquad \forall \mathfrak{F},$$

which implies (1.17). More precisely, there we prove abstract Theorem 6.2 where we get an upper estimate for a class of fast oscillating integrals of the form (1.18).

The diagram decomposition and transformation of coordinates from Sections 5 can be rather straightforwardly generalized to NLS equations with higher order polynomial nonlinearities. Together with abstract Theorem 6.2 this allows to generalize Theorem 1.1 for these equations.

Behaviour of energy spectra of small-amplitude solutions for nonlinear equations when the space-period of the solution grows to infinity has been intensively studied by physicists since the 1960's. Last decade this problem draw attention of mathematicians starting [19], see [1–6, 14, 15] and discussions in introduction to [3, 6, 9]. In some of the mentioned works decompositions of energy spectra to series of the form (1.12) where examined, evoking the techniques of Feynman diagrams, inspired by the works [12, 13] (where the diagram techniques were used to analyse expansions of solutions in different, but related situations). Our diagram presentation is based on similar ideas, but is rather different from those in the mentioned works. In particular, as explained above, our approach leads to an explicit presentation of the correlations $\mathbb{E}v_s^{(m)}\bar{v}_s^{(n)}$ which may be treated by asymptotical methods, developed in the present paper and in [9, 11].

In the next two subsections we briefly present the results of [9, 11] and explain the role of the present paper for the research started in [9, 11].

1.3. **Kinetic limit.** Let $u(\tau, x)$ be a solution of equation (1.5), so that $v(\tau) = \mathcal{F}\big(u(\tau)\big)$ is a solution of (1.8), and $n_s(\tau)$ be its energy spectrum (see (1.7)). One of objective of WT is to study the behaviour of the energy spectrum under the limits (1.6), with a properly scaled ρ, see [20, 22, 23]. The order of limits is not quite clear but, as we mentioned above, in WT it is usually accepted that L grows to infinity fast when ν goes to zero (for a discussion of other possibilities see introductions to [9] and [11] and see below). Accordingly in [9] we assume that

$$L \geq \nu^{-2-\delta} \quad \text{for some} \quad \delta > 0, \tag{1.19}$$

or that first $L \to \infty$ and then $\nu \to 0$. It is traditional in WT to study the quadratic in ρ truncation of series (1.12), postulating that it well approximates small amplitude solutions. Thus motivated, in [9] we considered the process

$$\tilde{v}_s(\tau) := v_s^{(0)}(\tau) + \rho v_s^{(1)}(\tau) + \rho^2 v_s^{(2)}(\tau), \quad s \in \mathbb{Z}_L^d, \tag{1.20}$$

which we called a *quasisolution* of equation (1.8), (1.11), and examined its energy spectrum $\tilde{n} = (\tilde{n}_s)_{s\in\mathbb{Z}_L^d}$, $\tilde{n}_s = \mathbb{E}|\tilde{v}_s|^2$,

$$\tilde{n}_s(\tau) = n_s^{(0)}(\tau) + \rho n_s^{(1)}(\tau) + \rho^2 n_s^{(2)}(\tau) + \rho^3 n_s^{(3)}(\tau) + \rho^4 n_s^{(4)}(\tau), \tag{1.21}$$

where

$$n_s^{(k)} = \sum_{\substack{0\leq k_1,k_2\leq 2:\\ k_1+k_2=k}} \mathbb{E} v_s^{(k_1)} \bar{v}_s^{(k_2)}.$$

Assuming $d \geq 2$, we proved that $n_s^{(2)} \sim \nu$ as $\nu \to 0$ uniformly in L satisfying (1.19), while a simple computation showed that $n_s^{(0)} \sim 1$ and $n_s^{(1)} = 0$. This indicates that the right scaling for ρ is such that $\rho^2\nu \sim 1$ as $\nu \to 0$, and accordingly in [9] we chose ρ to be of the form

$$\rho = \sqrt{\varepsilon}\nu^{-1/2}, \quad \text{so that} \quad \rho^2\nu = \varepsilon. \tag{1.22}$$

Here $0 < \varepsilon \leq 1$ is a fixed small number (independent from ν and L). Using estimate (1.17) under the scaling (1.19), (1.22) we found that $|\rho^3 n_s^{(3)}|$, $|\rho^4 n_s^{(4)}| \lesssim \varepsilon^2$. Accordingly,

$$\tilde{n}_s = n_s^{(0)} + \varepsilon \tilde{n}_s^{(2)}(\varepsilon), \quad \text{where} \quad \tilde{n}_s^{(2)} = \nu^{-1} n_s^{(2)} + O(\varepsilon) \sim 1$$

when $\nu \to 0$, since $n_s^{(2)} \sim \nu$.

Thus, the parameter ε measures the properly scaled amplitude of oscillations, described by eq. (1.5), so indeed it should be small for the methodology of WT apply to the solutions. Next we establish that for ν small in terms

of ε the energy spectrum $\tilde{n}_s(\tau)$ is ε^2-close to a function $m(\tau, s)$, which is a unique solution of the damped/driven *wave kinetic equation*:

$$\dot{m}(\tau, s) = -2\gamma_s m(\tau, s) + 2b(s)^2 + \varepsilon K(m(\tau, \cdot))(s), \quad s \in \mathbb{R}^d; \tag{1.23}$$

$m(-T) = 0$. Here $K(m)(s)$ is the *wave kinetic integral*

$$2\pi \int_{{}^s\Sigma_*} \frac{ds_1\, ds_2\, |_{{}^s\Sigma_*}\ m_1 m_2 m_3 m_s}{\sqrt{|s_1 - s|^2 + |s_2 - s|^2}} \left(\frac{1}{m_s} + \frac{1}{m_3} - \frac{1}{m_1} - \frac{1}{m_2} \right),$$

where $m_i = m(s_i)$, $m_s = m(s)$ with $s_3 = s_1 + s_2 - s$ (cf. (1.9)). The set ${}^s\Sigma_*$ is the resonant surface (cf. (2.5)-(2.6))

$${}^s\Sigma_* = \{(s_1, s_2) \in \mathbb{R}^{2d} : |s_1|^2 + |s_2|^2 = |s_3|^2 + |s|^2, \quad s_3 = s_1 + s_2 - s\}, \tag{1.24}$$

and $ds_1\, ds_2\, |_{{}^s\Sigma_*}$ stands for the volume element on ${}^s\Sigma_*$, corresponding to the euclidean structure in $\mathbb{R}^{2d}$. This is exactly the wave kinetic integral which appears in physical works on WT to describe the 4–waves interaction, see [23], p. 71 and [20], p. 91.

In particular from what was said it follows that in the regime (1.19), the scaling $\rho \sim \nu^{-1/2}$ is the only scaling of ρ when the energy spectrum admits a non-trivial kinetic limit of order one; see [9] for a more detailed discussion.

In [11] we consider the opposite to (1.19) order of limits, when first $\nu \to 0$ and then $L \to \infty$. As in [9], we study the energy spectrum (1.21) of the quasisolution, but now analysis of the term $n_s^{(2)}$ shows that the correct scaling is $\rho = \sqrt{\varepsilon} L$.[6] In this setting, crucially using Theorem 5.7, we establish a result, similar to the discussed above that in [9], but the limiting WKE (1.23) should be modified. In particular, the WKE in [11] is non-autonomous.

Results of the present paper also are used in our work in progress, where by means of the KAM-technique we are aiming to show that the results in [9,11] remain true for the energy spectrum of the exact solution for eq. (1.5), under the corresponding limits (and not only for the quasisolution). See subsection *What next?* in [11, Section 1.3].

1.4. **Discussion of results. Higher order truncations.** Let us come back to the decomposition (1.12) and accordingly write the energy spectrum n_s, defined in (1.7), as formal series in ρ:

$$n_s = \sum_{k=0}^{\infty} \rho^k n_s^k, \qquad n_s^k(\tau) = \sum_{k_1 + k_2 = k} \mathbb{E} v_s^{(k_1)}(\tau) \bar{v}_s^{(k_2)}(\tau). \tag{1.25}$$

Here the terms n_s^0, n_s^1 and n_s^2 coincide with $n_s^{(0)}$, $n_s^{(1)}$ and $n_s^{(2)}$ in (1.21), while the terms n_s^3 and n_s^4 are slightly different (but still satisfy the estimate (1.26)

[6] In [11] the notation is slightly different: ε is written as ε^2. Besides when $d = 2$, ρ should be multiplied by the factor $(\ln L)^{-1/2}$.

below). According to Theorem 1.1,

(1.26)
$$|n_s^k| \le C^{\#}(s)\big((\nu^{-2}L^{-2} + \nu^{\min(\lceil k/2\rceil,d)}\chi_d^k(\nu)\big) \le C_1^{\#}(s)\big(\nu^{-2}L^{-2} + \nu^{\min(k/2,d)}\big),$$

where for definiteness we assumed that $d \ge 2$. For $k \le 2d$ in the exponent in r.h.s. the minimum with d can be removed, so (1.26) takes the form

$$|n_s^k| \le C^{\#}(s)\big(\nu^{-2}L^{-2} + \nu^{k/2}\big). \tag{1.27}$$

In particular, (1.27) holds for $k \le 4$ since $d \ge 2$. Using estimate (1.27) with $k \le 2d$ it is straightforward to show that under the scaling (1.22) the kinetic limit for quasisolution, established in [9] and discussed in the previous section, takes place for the truncation $\sum_{i=0}^m \rho^i v_s^{(i)}$ of series (1.12) not only of order $m = 2$ as in (1.20) but of any order $2 \le m \le d$ (with the same kinetic equation (1.23) and at least with the same accuracy ε^2, once L is sufficiently large in terms of ν^{-1}).

If it was true that (1.27) is valid for any k, so that $|\rho^k n_s^k| \lesssim \varepsilon^{k/2}$ under the scaling (1.22) with $L \gg \nu^{-1}$, then the kinetic limit would hold for truncations of any order $m \ge 2$. On the contrary, if for some k we have $|n^k|_1 := L^{-d}\sum_s |n_s^k| \gg \nu^{k/2}$, then this is not the case since the term $\rho^k n^k$ explodes as $\nu \to 0$.

The question if estimate (1.27) holds for high k is complicated. Namely, estimate (1.26) follows from (1.17), and in Theorem 1.1 we established the latter by showing that each integral $\tilde{J}_s(\mathfrak{F})$ is bounded by the r.h.s. of (1.17). We are able to prove that this estimate is optimal at least for some integrals $\tilde{J}_s(\mathfrak{F})$ in the sense that for some of them the minimum with d in the exponent cannot be removed, so for these terms estimate (1.27) fails. However, in the sum $\sum_{\mathfrak{F}} \tilde{J}_s(\mathfrak{F})$ that approximates $\mathbb{E}v_s^{(m)}\bar{v}_s^{(n)}$ there are some non-trivial cancellations in a few first orders of its decomposition in ν which could lead to the validity of (1.27). In Section 8.1 we discuss this phenomenon in more detail.

The arXiv version of this paper [10], which is longer than the present one, contains some additional results and examples (not needed for the proofs of our main statements). Also we give in [10] a simple but rather long and tedious proof of Lemma 4.1 below which we omit in the present version of the paper for the sake of brevity.

Notation. By $\mathbb{Z}_+^n$ we denote the set of integer n-vectors with non-negative components. For vectors in $\mathbb{R}^n$ we denote by $|v|$ their Euclidean norms and by $v \cdot u$ – the Euclidean scalar product. By $C^{\#}, C_1^{\#}, \widetilde{C}_1^{\#}, \dots$ we denote various positive continuous functions satisfying (1.16) for any $n \in \mathbb{N}$. If we want to indicate that a function $C^{\#}$ depends on a parameter m, we write it as $C_m^{\#}$. The functions $C^{\#}, C_1^{\#}, \dots$ and constants $C, C_1, \dots$ never depend on the parameters $\nu, L, \rho, \varepsilon$ and the moments of time $T, \tau, \tau_1, \dots$ unless otherwise stated. In Section 1.4 of [9] it is shown that for any function $C^{\#}(x, y)$, where

$(x,y) \in \mathbb{R}^{d_1+d_2}$, $d_1, d_2 \geq 1$, there exist functions $C_1^{\#}(x)$, $C_2^{\#}(y)$ such that

$$C^{\#}(x,y) \leq C_1^{\#}(x) C_2^{\#}(y). \tag{1.28}$$

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2. Preliminaries and outline of the proof

2.1. **Approximate equation.** We start by getting rid in the r.h.s. of eq. (1.8) of the term $-i\rho L^{-d}|v_s|^2 v_s$, which is small but rather inconvenient for analysis. With this term being dropped, equation (1.8), (1.11) reads

$$\begin{aligned} &\dot{y}_s - i\nu^{-1}|s|^2 y_s + \gamma_s y_s \\ &\quad = i\rho L^{-d} \sum\nolimits_{s_1,s_2,s_3 \in \mathbb{Z}_L^d} \delta_{3s}^{\prime 12} y_{s_1} y_{s_2} \bar{y}_{s_3} + b(s)\dot{\beta}_s\,, \quad y_s(-T) = 0, \ \ s \in \mathbb{Z}_L^d. \end{aligned} \tag{2.1}$$

As in (1.12), we decompose its solution y_s into a formal series in ρ:

$$y_s = \sum_{i=0}^{\infty} \rho^i y_s^{(i)}, \qquad y_s^{(i)}(-T) = 0. \tag{2.2}$$

Then $\{y_s^{(0)} = v_s^{(0)}\}$ are independent Gaussian processes, satisfying eq. (1.13), $y_s^{(1)}(\tau)$ satisfies eq. (1.14) without the term $|v_s^{(0)}|^2 v_s^{(0)}$, etc.

Correlations of the processes $v_s^{(i)}$ and $y_s^{(i)}$ are L^{-d}-close:

Proposition 2.1. *For any $m, n \geq 0$ there exists a function $C^{\#}$ such that for each $s \in \mathbb{Z}_L^d$ and $\tau_1, \tau_2 \geq -T$,*

$$\left|\mathbb{E} v_s^{(m)}(\tau_1) \bar{v}_s^{(n)}(\tau_2) - \mathbb{E} y_s^{(m)}(\tau_1) \bar{y}_s^{(n)}(\tau_2)\right| \leq L^{-d} C^{\#}(s). \tag{2.3}$$

Proof of the proposition is straightforward, so we give it only in the arXiv version of this paper [10]. There, for the needs of [11], we adopt the method of Feynman diagrams developed for eq. (2.1) to eq. (1.8), and get a stronger result than (2.3). Alternatively, (2.3) immediately follows from Proposition 2.1 in [9] and the Cauchy inequality.

In view of Proposition 2.1 it suffices to establish inequality (1.17) (and its 1d analogy) when the processes $v_s^{(m)}$ are replaced by $y_s^{(m)}$. Accordingly below we study the latter instead of the former.

2.2. **Interaction representation.** It is convenient to work in the *interaction representation*, that is in the a-variables defined as

$$a_s(\tau) = y_s(\tau) e^{-i\nu^{-1}\tau|s|^2}, \quad s \in \mathbb{Z}_L^d. \tag{2.4}$$

We set

$$\omega_{3s}^{12} = \omega_{s_3 s}^{s_1 s_2} = |s_1|^2 + |s_2|^2 - |s_3|^2 - |s|^2, \tag{2.5}$$

(cf. (1.24)). Then eq. (2.1), written in the a-variables reads

$$\begin{aligned} &\dot a_s + \gamma_s a_s = i\rho \mathcal{Y}_s(a, \nu^{-1}\tau) + b(s)\dot\beta_s, \qquad a_s(-T) = 0, \\ &\mathcal{Y}_s(a,t) = L^{-d} \sum_{s_1,s_2,s_3\in\mathbb{Z}^d_L} \delta^{\prime 12}_{3s} a_{s_1} a_{s_2} \bar a_{s_3} e^{it\omega^{12}_{3s}}, \qquad s\in\mathbb{Z}^d_L, \end{aligned} \tag{2.6}$$

where $\{\beta_s\}$ is another set of standard independent complex Wiener processes. Formal expansion (2.2) takes the form

$$a_s = \sum_{j=0}^{\infty} \rho^j a_s^{(j)}, \qquad a_s^{(j)}(-T) = 0,$$

where $a_s^{(j)}(\tau) = y_s^{(j)}(\tau) e^{-i\nu^{-1}\tau|s|^2}$. For a process $a_s^{(j)}(\tau)$ we call the integer $j \ge 0$ its *degree*,

$$\deg a_s^{(j)}(\tau) = \deg \bar a_s^{(j)}(\tau) = j, \tag{2.7}$$

and call the vector $s\in\mathbb{Z}^d_L$ its *index*. Since $a_s^{(k)}(\tau)$ differs from $y_s^{(k)}(\tau)$ by the factor $e^{-i\nu^{-1}\tau|s|^2}$, then $|\mathbb{E} a_s^{(m)} \bar a_s^{(n)}| = |\mathbb{E} y_s^{(m)} \bar y_s^{(n)}|$. Thus, in view of (2.3), it suffices to establish Theorem 1.1 with the processes $v_s^{(m)}$ replaced by $a_s^{(m)}$:

Theorem 2.2. *Assume that $d \ge 2$. Then for any integers $m, n \ge 0$ there exists a function $C^{\#}$ such that for $s\in\mathbb{Z}^d_L$ and any $\tau_1,\tau_2 \ge -T$,*

$$\left|\mathbb{E} a_s^{(m)}(\tau_1) \bar a_s^{(n)}(\tau_2)\right| \le C^{\#}(s)\big(\nu^{-2}L^{-2} + \nu^{\min(\lceil N/2\rceil, d)} \chi_d^N(\nu)\big), \tag{2.8}$$

where $N = m+n$ and χ_d^N is defined in (1.15). If $d = 1$, the assertion above holds if we modify the r.h.s. of (2.8) by adding the term $C^{\#}(s)L^{-1}$. Moreover, the correlation $\mathbb{E} a_s^{(m)}(\tau_1)\bar a_s^{(n)}(\tau_2)$ extends to a Schwartz function of $s\in\mathbb{R}^d \supset \mathbb{Z}^d_L$ which satisfies the same estimate (2.8).

Note that for $N = 1$ the l.h.s. of (2.8) vanishes, see in Section 4.2 and Lemma 2.3 in [9].

The processes $a_s^{(i)}$ can be computed inductively. Namely, $a_s^{(0)}$ satisfies the linear equation

$$\dot a_s^{(0)}(\tau) + \gamma_s a_s^{(0)}(\tau) = b(s)\dot\beta_s(\tau), \qquad a_s^{(0)}(-T) = 0\,,$$

so

$$a_s^{(0)}(\tau) = b(s)\int_{-T}^{\tau} e^{-\gamma_s(\tau - l)} d\beta_s(l), \tag{2.9}$$

and $\{a_s^{(0)}(\tau), s\in\mathbb{Z}^d_L\}$, are independent Gaussian processes. The process $a^{(1)}$ satisfies

$$\dot a_s^{(1)}(\tau) + \gamma_s a_s^{(1)}(\tau) = i\mathcal{Y}_s(a^{(0)}(\tau), \nu^{-1}\tau), \qquad a_s^{(1)}(-T) = 0,$$

so

$$a_s^{(1)}(\tau) = i\int_{-T}^{\tau} e^{-\gamma_s(\tau-l)} \mathcal{Y}_s(a^{(0)}(l), \nu^{-1} l)\, dl\,, \tag{2.10}$$

is a Wiener chaos of third order (see [17]). For what follows, it is convenient to introduce in (2.6) a fictitious index s_4 and write the function $\mathcal{Y}_s$ from (2.6) as

$$\mathcal{Y}_s(a,t) = L^{-d} \sum_{s_1,\dots,s_4 \in \mathbb{Z}_L^d} \delta_{34}^{\prime 12}\, \delta_{s_4}^{s}\, a_{s_1} a_{s_2} \bar{a}_{s_3}\, \theta(\omega_{34}^{12}, t),$$

where $\delta_{s_4}^{s}$ is the Kronecker symbol and we denote

$$\theta(x,t) = e^{itx}. \tag{2.11}$$

Then (2.10) takes the form

$$a_s^{(1)}(\tau) = i \int_{-T}^{\tau} L^{-d} \sum_{s_1,\dots,s_4} \delta_{34}^{\prime 12}\, e^{-\gamma_{s_4}(\tau - l)}\, \delta_{s_4}^{s} (a_{s_1}^{(0)} a_{s_2}^{(0)} \bar{a}_{s_3}^{(0)})(l)\, \theta(\omega_{34}^{12}, \nu^{-1} l)\, dl. \tag{2.12}$$

Similar, for $m \geq 1$,

$$\begin{aligned} a_s^{(m)}(\tau) = & \sum_{m_1+m_2+m_3=m-1} i \int_{-T}^{\tau} dl \\ & \times L^{-d} \sum_{s_1,\dots,s_4} \delta_{34}^{\prime 12}\, e^{-\gamma_{s_4}(\tau - l)}\, \delta_{s_4}^{s} \big(a_{s_1}^{(m_1)} a_{s_2}^{(m_2)} \bar{a}_{s_3}^{(m_3)}\big)(l)\, \theta(\omega_{34}^{12}, \nu^{-1} l). \end{aligned} \tag{2.13}$$

This is a Wiener chaos of order $2m+1$.

In what follows, till the end of Section 5.3, we do not use the explicit form (2.11) of the function θ (except the next section, where we give an outline of the proof of Theorem 2.2). Instead we assume that the processes $a_s^{(m)}$ are given by eq. (2.13), where

$$\theta : \mathbb{R}^2 \mapsto \mathbb{C} \text{ is a bounded measurable function.} \tag{2.14}$$

In particular, we may choose for θ the function $\theta(x,t) = \mathbb{I}_{\{0\}}(x)$, where $\mathbb{I}_{\{0\}}(x)$ is the indicator function of the point $x = 0$ (in this case θ does not depend on t). This choice of θ is used in our paper [11], where we use that in some sense a properly scaled function $(s_1, s_2, s_3, s_4) \mapsto \exp(i\nu^{-1}\tau\omega_{34}^{12})$ with $\tau \neq 0$, regarded as a "generalised function on $(\mathbb{Z}_L^d)^4$", converges to $\mathbb{I}_{\{0\}}(\omega_{34}^{12})$ as $\nu \to 0$.

We will use correlations of the process $a_s^{(0)}$. By (2.9), for any $\tau_1, \tau_2 \geq -T$ and $s, s' \in \mathbb{Z}_L^d$,

$$\begin{aligned} & \mathbb{E} a_s^{(0)}(\tau_1) a_{s'}^{(0)}(\tau_2) = 0, \\ & \mathbb{E} a_s^{(0)}(\tau_1) \bar{a}_{s'}^{(0)}(\tau_2) = \delta_{s'}^{s} \frac{b(s)^2}{\gamma_s} \big(e^{-\gamma_s|\tau_1-\tau_2|} - e^{-\gamma_s(2T+\tau_1+\tau_2)}\big) \end{aligned} \tag{2.15}$$

(see (2.8) in [9] for the calculation).

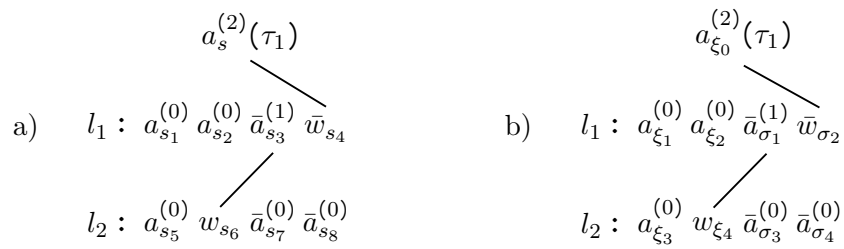

FIGURE 1. A diagram $\mathfrak{D} \in \mathfrak{D}_2$. The notation "$l_i$:" means that the vertices $a^{(k)}$, $\bar a^{(k)}$ situated opposite to it are taken at the time l_i.

2.3. **Outline of the proof of Theorem 2.2.** We divide a quite detailed outline of the proof to four steps, approximately corresponding to Sections 3, 4, 5 and 6+7. In the outline we assume that θ takes the form (2.11).

Step 1. Let us write the variable $a_s^{(m)}(\tau_1)$ in the form (2.13) and then apply the Duhamel formula (2.13) to the variables $a_{s_i}^{(m_i)}(l)$ in its r.h.s. with the degrees $m_i \geq 1$ (see (2.7)). Iterating the procedure, we represent $a_s^{(m)}(\tau_1)$ by a sum of terms of the form

$$\tilde I_s = \int \cdots \int L^{-md} \sum_{s_1,\dots,s_{4m}} (\dots)\, dl_1 \dots dl_m; \tag{2.16}$$

here and below we call terms of this form *sums*. The integrating zone in (2.16) is a convex polyhedron in $[-T,\tau_1]^m$ and the summation is taken over the vectors $s_1,\dots,s_{4m} \in \mathbb{Z}_L^d$ which are subject to certain linear relations, following from the factors δ_{34}^{l12} and $\delta_{s_4}^{s}$ in (2.13). The summand in brackets is a product of functions $e^{-\gamma_{s'}(l_k-l_r)}$, $\exp(i\nu^{-1}\omega_{s_3' s_4'}^{s_1' s_2'} l_q)$ and the processes $[a_{s''}^{(0)}(l_j)]^*$ where a^* is either a or $\bar a$, with various indices $1 \leq k,r,q,j \leq m$ and $s', s_i', s'' \in \{s_1,\dots,s_{4m}\}$. This product is of degree $2m+1$ with respect to the process $a^{(0)}$ (see (2.12) for the case $m=1$).

To each sum $\tilde I_s$ as in (2.16) we associate a diagram $\mathfrak{D}$ as in fig. 1(a), constructed according to a rule, explained below; we then write $\tilde I_s = \tilde I_s(\mathfrak{D})$. The root of a diagram $\mathfrak{D}$, associated with a sum $\tilde I_s$, is the vertex $a_s^{(m)}(\tau_1)$. If $a_{s'}^{(p)}(t)$, $p \geq 1$, is any vertex of the diagram $\mathfrak{D}$, then it generates a *block* of four vertices, three of which correspond to a choice of the three terms $a^{(m_1)}, a^{(m_2)}, \bar a^{(m_3)}$ in the decomposition (2.13) of $a_{s'}^{(p)}(t)$. These three are called *real vertices*, and the vertex $a_{s'}^{(p)}$ is called the *parent* of the block. The fourth vertex of the block, denoted by $\bar w_{s''}$, is called the *conjugated virtual vertex*. It corresponds to the factor $\delta_{s''}^{s'}$ in (2.13), is coupled with the parent $a_{s'}^{(p)}(t)$ by an edge and one can think that its role is to couple the block

with its parent. Similarly, each vertex $\bar{a}^{(p)}_{s'}(t)$ with degree $p \geq 1$ generates a block, where the three real vertices correspond to a choice of the terms $\bar{a}^{(m_1)}, \bar{a}^{(m_2)}, a^{(m_3)}$ in the formula, conjugated to (2.13), and the fourth vertex $w_{s''}$ is the *non-conjugated virtual vertex* (corresponding to factor $\delta^{s'}_{s''}$).

We denote by $\mathfrak{D}_m$ the set of all diagrams $\mathfrak{D}$ associated with different terms $\tilde{I}_s$ from (2.16), sum of which gives $a^{(m)}_s(\tau_1)$. [7] Then in every diagram $\mathfrak{D} \in \mathfrak{D}_m$ there are m blocks, and in each block there are two conjugated and two non-conjugated vertices. By convention, in each block we draw and enumerate first the non-conjugated vertices and then the conjugated. The virtual vertex is always positioned at the second or forth place, depending whether it is conjugated or not, see fig. 1(a).

We denote by $\xi_1, \dots, \xi_{2m}$ the indices $s_j \in \mathbb{Z}^d_L$, enumerating the non-conjugated vertices, and by $\sigma_1, \dots, \sigma_{2m}$ – those enumerating the conjugated vertices. Then, setting $\xi_0 = s$, the diagram from fig. 1(a) takes the form as in fig. 1(b). For more examples see fig. 2, 3(a-c), where we write c instead of a and omit indices ξ and σ.

Now the linear relations, imposed on the multi-indices ξ_i, σ_j with $i, j \geq 1$ in a sum (2.16) take the following compact form:

$$1)\ \delta^{\xi_{2j-1}\xi_{2j}}_{\sigma_{2j-1}\sigma_{2j}} = 1\ \forall j \quad 2)\ \text{indices of adjacent in } \mathfrak{D} \text{ vertices are equal,} \tag{2.17}$$

where the first relation comes from the factor δ^{12}_{34} in (2.13) while the second one – from the factor $\delta^{s}_{s_4}$. Moreover, we see that in this notation the frequency of a fast rotating exponent, hidden in the term $(\dots)$ in (2.16) and arising from the factor $\theta(\omega^{12}_{34}, \nu^{-1}l) = e^{i\nu^{-1}\omega^{12}_{34}l}$ in (2.13), also takes the compact form $i\nu^{-1}\sum_{j=1}^{m} \omega^{\xi_{2j-1}\xi_{2j}}_{\sigma_{2j-1}\sigma_{2j}}\, l_j$.

We have seen that

$$a^{(m)}_s(\tau_1) = \sum_{\mathfrak{D}\in\mathfrak{D}_m} \tilde{I}_s(\mathfrak{D}),$$

where each sum $\tilde{I}_s(\mathfrak{D})$ has the form (2.16). Similarly, $\bar{a}^{(n)}_s(\tau_2) = \sum_{\bar{\mathfrak{D}}\in\overline{\mathfrak{D}}_n} \tilde{I}_s(\bar{\mathfrak{D}})$, where $\overline{\mathfrak{D}}_n$ is the set of diagrams with the root $\bar{a}^{(n)}_{\sigma_0}(\tau_2)$, $\sigma_0 := s$, constructed by the same inductive rule as diagrams from the set $\mathfrak{D}_m$ (see fig. 3(d)). [8]

Finally, we introduce the set $\mathfrak{D}_m \times \overline{\mathfrak{D}}_n$ of diagrams $\mathfrak{D} = \mathfrak{D}^1 \sqcup \bar{\mathfrak{D}}^2$, which are obtained by drawing side by side pairs $\mathfrak{D}^1$ and $\bar{\mathfrak{D}}^2$ with various $\mathfrak{D}^1 \in \mathfrak{D}_m$ and $\bar{\mathfrak{D}}^2 \in \overline{\mathfrak{D}}_n$. Here in the diagrams $\bar{\mathfrak{D}}^2$ the vertices (except the root) are enumerated by the indices $\xi_{2m+1}, \dots, \xi_{2N}, \sigma_{2m+1}, \dots, \sigma_{2N}$, $N := m+n$. Since

[7] As we will see, structure of the set $\mathfrak{D}_m$ depends on $a^{(m)}_s(\tau_1)$ through the index m only. This will become more clear in Section 3, where we will construct $\mathfrak{D}_m$ carefully.

[8] Equivalently, the set $\overline{\mathfrak{D}}_n$ can be obtained by conjugating diagrams from the set $\mathfrak{D}_n$, then exchanging positions of conjugated and non-conjugated vertices in each block (we recall that, by convention, the non-conjugated vertices should always situate before the conjugated vertices), and exchanging the indices ξ_j with σ_j, so that again the non-conjugated vertices are enumerated by the indices ξ_j while the conjugated – by the indices σ_j.

the root of $\bar{\mathfrak{D}}^2$ is denoted σ_0, then all vertices of $\mathfrak{D}$ are enumerated by $4N+2$ indices ξ_i, σ_j, where $0 \le i, j \le 2N$ and always $\xi_0 = \sigma_0 = s$; see fig. 4. Then setting $I_s(\mathfrak{D}) = \tilde{I}_s(\mathfrak{D}^1)\tilde{I}_s(\bar{\mathfrak{D}}^2)$, where $\mathfrak{D} = \mathfrak{D}^1 \sqcup \bar{\mathfrak{D}}^2$, we find

$$\mathbb{E}a_s^{(m)}(\tau_1)\bar{a}_s^{(n)}(\tau_2) = \sum_{\mathfrak{D}\in\mathfrak{D}_m\times\overline{\mathfrak{D}}_n} \mathbb{E}I_s(\mathfrak{D}). \tag{2.18}$$

In fact, we do not write down the sums $\tilde{I}_s$ but directly write the products I_s, see Lemma 3.1 for their explicit form.

Step 2. To estimate the r.h.s. of (2.18) we study separately the expectation $\mathbb{E}I_s(\mathfrak{D})$ for each diagram $\mathfrak{D} = \mathfrak{D}^1 \sqcup \bar{\mathfrak{D}}^2 \in \mathfrak{D}_m \times \overline{\mathfrak{D}}_n$. Due to (2.16),

$$\mathbb{E}I_s(\mathfrak{D}) = \int \dots \int L^{-Nd} \sum_{\xi_1,\dots,\xi_{2N},\sigma_1,\dots,\sigma_{2N}} \mathbb{E}(\dots)\, dl_1 \dots dl_N, \tag{2.19}$$

where the term in the brackets is obtained by taking the product of the terms $(\dots)$ in the integrands of $\tilde{I}_s(\mathfrak{D}^1)$ and $\tilde{I}_s(\bar{\mathfrak{D}}^2)$, corresponding to the leaves of the diagram $\mathfrak{D}$. So, the brackets in (2.19) contain a product of $2N+2$ random variables of the form $[a_{s'}^{(0)}(t)]^*$, multiplied by a deterministic factor. Since $a_{s'}^{(0)}(t)$ are independent complex Gaussian random variables whose correlations are given by (2.15), then by the Wick theorem (see e.g. in [17]) the expectation $\mathbb{E}(\dots)$ is a sum over different Wick-pairings of the non-conjugated variables $a_{\xi_j}^{(0)}(l_k)$ with conjugated $\bar{a}_{\sigma_r}^{(0)}(l_q)$, where

$$\mathbb{E}a_{\xi_j}^{(0)}(l_k)\bar{a}_{\sigma_r}^{(0)}(l_q) \neq 0 \quad \text{only if} \quad \xi_j = \sigma_r. \tag{2.20}$$

The summands can be parametrised by Feynman diagrams $\mathfrak{F}$ (see e.g. in [17]), obtained from the diagram $\mathfrak{D}$ by connecting with an edge every pair of the Wick-coupled vertices $a_{\xi_j}^{(0)}(l_k)$ and $\bar{a}_{\sigma_r}^{(0)}(l_q)$, see fig. 5. So to every diagram $\mathfrak{D}$ correspond several Feynman diagrams $\mathfrak{F}$, one diagram for each Wick-pairing. By construction, every Feynman diagram decomposes to pairs of adjacent vertices, where in each pair either both vertices have zero (recall (2.7)) degree, or one vertex is virtual and another one has positive degree.

Therefore, by (2.18),

$$\mathbb{E}a_s^{(m)}(\tau_1)\bar{a}_s^{(n)}(\tau_2) = \sum_{\mathfrak{F}\in\mathfrak{F}_{m,n}} J_s(\mathfrak{F}),$$

where the sum is taken over the set $\mathfrak{F}_{m,n}$ of Feynman diagrams $\mathfrak{F}$ obtained via all possible Wick-couplings in various diagrams $\mathfrak{D} \in \mathfrak{D}_m \times \overline{\mathfrak{D}}_n$. Each (deterministic) sum $J_s(\mathfrak{F})$ has the form

$$J_s(\mathfrak{F}) = \int \dots \int L^{-Nd} \sum_{\xi_1,\dots,\xi_{2N},\sigma_1,\dots,\sigma_{2N}} (\dots)\, dl_1 \dots dl_N, \tag{2.21}$$

where the term in the brackets $(\dots)$ is obtained from those in (2.19) by a Wick-pairing associated to the diagram $\mathfrak{F}$. See Lemma 4.1 for an explicit form of the sums $J_s(\mathfrak{F})$.

Step 3. The indices ξ_i, σ_j in (2.21) are subject to the linear relations (2.17), where $1 \le j \le N$ and in (2.17)(2) the diagram $\mathfrak{D}$ is replaced by some $\mathfrak{F}$ (see (2.20)). By (2.17)(2) (with $\mathfrak{D}$ replaced by $\mathfrak{F}$), the multi-index $\sigma = (\sigma_1, \dots, \sigma_{2N})$ is a function of the multi-index $\xi = (\xi_1, \dots, \xi_{2N})$, so it remains to study eq. (2.17)(1), where we substitute $\sigma = \sigma(\xi)$. Analysing the diagrams $\mathfrak{F}$, forming the class $\mathfrak{F}_{m,n}$, we find an $\mathfrak{F}$-dependent affine parametrization of solutions ξ for this equation by poly-vectors $z = (z_1, \dots, z_N)$, $z_j \in \mathbb{Z}_L^d$. Then we write the normalized sum $L^{-Nd}\sum_{\xi,\sigma}$ in (2.21) as $L^{-Nd}\sum_z$, approximate it by an integral over $\mathbb{R}^{Nd}$ with accuracy $L^{-2}\nu^{-2}$, and accordingly find that

$$J_s(\mathfrak{F}) = \int_{\mathbb{R}^N} dl \int_{\mathbb{R}^{Nd}} dz\, F_s^{\mathfrak{F}}(l,z) e^{i\nu^{-1}\Omega^{\mathfrak{F}}(l,z)} + O(L^{-2}\nu^{-2}). \tag{2.22}$$

Here $l = (l_1, \dots, l_N) \in \mathbb{R}^N$, the density function $F_s^{\mathfrak{F}}$ (which we write down explicitly) is Schwartz in the variables (s, z), piecewise smooth and fast decaying in l, while the phase function $\Omega^{\mathfrak{F}}$ is quadratic in z and linear in l: $\Omega^{\mathfrak{F}}(l,z) = \sum_{1\le i,j\le N} \alpha_{ij}^{\mathfrak{F}} z_i \cdot z_j (l_i - l_j)$, where $z_i \cdot z_j$ is the scalar product in $\mathbb{R}^d$; see Theorem 5.9. The constant skew-symmetric matrix $\alpha^{\mathfrak{F}} = (\alpha_{ij}^{\mathfrak{F}})$, $\alpha_{ij}^{\mathfrak{F}} \in \{-1, 0, 1\}$, is given by an explicit formula. Usually it is degenerate, but all its rows and columns are non-zero vectors.

Step 4. The last step of the proof is to estimate integral (2.22) when $\nu \ll 1$. To this end, inspired by the stationary phase method, we apply to the latter the integral Parseval's identity, involving the fast oscillating Gaussian kernel $e^{i\nu^{-1}\Omega^{\mathfrak{F}}}$. The task is made complicated by the degeneracy of the quadratic form $\Omega^{\mathfrak{F}}(l,\cdot)$. To handle it, in Sections 6, 7 and 8.5 we prove abstract theorems which allow to estimate integrals of a more general forms than (2.22).

3. Diagrams $\mathfrak{D}$ and formula for products $a_s^{(m)}\bar{a}_s^{(n)}$

In this section we construct the set of diagrams $\mathfrak{D}_m \times \overline{\mathfrak{D}}_n$, associated to the product $a_s^{(m)}(\tau_1)\bar{a}_s^{(n)}(\tau_2)$, and express the latter through the random processes $a_k^{(0)}$ and $\bar{a}_k^{(0)}$ (see Lemma 3.1). The construction is rather tedious but the main difficulties are notational: once the diagrammatic language is developed and convenient notation are introduced, proof of Lemma 3.1 becomes a simple computation.

3.1. The set of diagrams $\mathfrak{D}_m \times \overline{\mathfrak{D}}_n$.

For integers $m, n \ge 0$ we define the set $\mathfrak{D}_m \times \overline{\mathfrak{D}}_n$ as a direct product of the sets of diagrams $\mathfrak{D}_m$ and $\overline{\mathfrak{D}}_n$, associated, correspondingly, with the variables $a_s^{(m)}$ and $\bar{a}_s^{(n)}$.

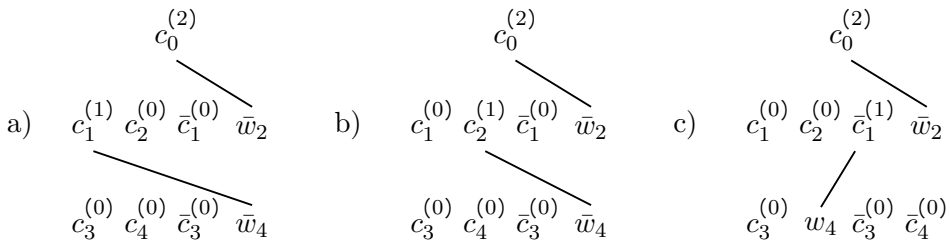

FIGURE 2. The set of diagrams $\mathfrak{D}_2$.

We start by constructing the set $\mathfrak{D}_m$, construction of $\overline{\mathfrak{D}}_n$ is similar. For example, the set $\mathfrak{D}_1$ consists of a unique diagram, which can be obtained from the diagram in fig. 3(a) by erasing the isolated vertex $\bar c_0^{(0)}$. Here each $c_j^{(p)}$ is regarded as an argument, which should be substituted by a variable $a_{\xi_j}^{(p)}(l)$, while $\bar c_j^{(p)}$ should be substituted by an $\bar a_{\sigma_j}^{(p)}(l')$. E.g., this substitution, applied to the diagram in fig. 2(c), gives the diagram in fig. 1(b). The set $\mathfrak{D}_2$ consists of three diagrams in fig. 2. The set $\mathfrak{D}_3$ consists of 12 diagrams, two of which are given in fig. 3(b,c). Construction of the diagrams seems to be quite clear from these examples. Nevertheless, below we explain it in detail.

3.1.1. *Agreements and first notation.* Each diagram $\mathfrak{D} \in \mathfrak{D}_m$ consists of a number of vertices, some of them are coupled by edges. Each vertex is either *non-conjugated* or *conjugated*. The non-conjugated vertices are denoted by c_j while the conjugated – by $\bar c_j$. If we do not want to indicate whether a vertex is conjugated or not, we write it as $\hat c_j$. Each vertex is characterised by its *degree* which we often write as an upper index:

$$\deg \hat c_j^{(k)} = k,$$

and each vertex is either *virtual* or *real*. If a vertex $c_j, \bar c_j$ or $\hat c_j$ is virtual, we may write it as $w_j, \bar w_j$ or $\hat w_j$. Each virtual vertex $\hat w_j$ has zero degree,

$$\deg \hat w_j = 0,$$

so we do not write for it the upper index. Real vertices of zero degree are called *leaves*.

The vertices of a diagram are organised in blocks of four, where the k-th block, $k \geq 1$, is $\{c_{2k-1}, c_{2k}, \bar c_{2k-1}, \bar c_{2k}\}$. Among these four vertices either the vertex c_{2k} or the vertex $\bar c_{2k}$ is virtual, while the other are real. If the vertex c_{2k} is virtual then we denote this block by B_k,

$$B_k = \{c_{2k-1}, w_{2k}, \bar c_{2k-1}, \bar c_{2k}\}, \tag{3.1}$$

and call it the *non-conjugated k-th block*. If the vertex $\bar c_{2k}$ is virtual, we denote

$$\bar B_k = \{c_{2k-1}, c_{2k}, \bar c_{2k-1}, \bar w_{2k}\}, \tag{3.2}$$

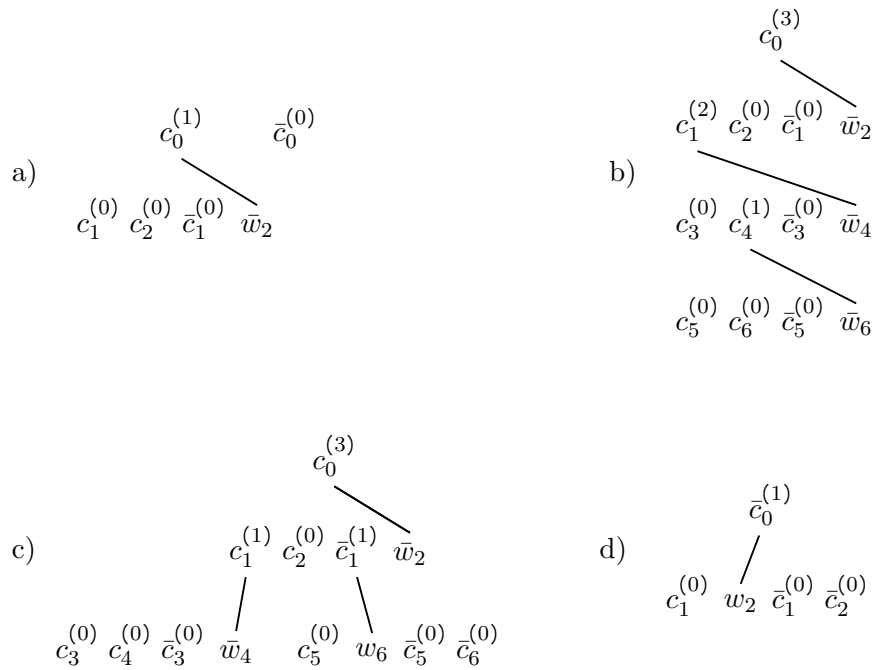

FIGURE 3. a) The unique diagram of the set $\mathfrak{D}_1 \times \overline{\mathfrak{D}}_0$. b,c) Some diagrams from the set $\mathfrak{D}_3$. d) The unique diagram of the set $\overline{\mathfrak{D}}_1$.

and call it the *conjugated k-th block*. If we do not want to emphasize whether a block is conjugated or not, we denote it by $\hat{B}_k$. We define the degree of each block as the sum of degrees of the forming it vertices. So

$$\deg B_k := \deg c_{2k-1} + \deg \bar{c}_{2k} + \deg \bar{c}_{2k-1}$$

(since $\deg \bar{w}_{2k} = 0$), and similarly for $\deg \bar{B}_k$. E.g., the diagram from fig. 2(a) has two conjugated blocks $\bar{B}_1$ and $\bar{B}_2$, where $\deg \bar{B}_1 = 1$ and $\deg \bar{B}_2 = 0$.

3.1.2. *Construction of the diagrams.* If a vertex c_j is coupled with a virtual vertex $\bar{w}_{2k}$ of a block $\bar{B}_k$ then we say that c_j *generates* the block $\bar{B}_k$ or that c_j is a *parent* of the block $\bar{B}_k$. The notion that a vertex $\bar{c}_j$ generates a block B_k is defined similarly.

The set of diagrams $\mathfrak{D}_m$ is constructed by the following inductive procedure. The set $\mathfrak{D}_0$ is formed by a unique diagram which consists of a unique (real) vertex $c_0^{(0)}$. If $m \geq 1$, the root of each diagram $\mathfrak{D} \in \mathfrak{D}_m$ is a real vertex $c_0^{(m)}$. Each vertex $\hat{c}_j$ of the diagram $\mathfrak{D}$ with $\deg \hat{c}_j \geq 1$ (i.e. which is real and is not a leave) generates a block $\hat{B}_k$ satisfying

$$\deg \hat{B}_k = \deg \hat{c}_j - 1. \tag{3.3}$$

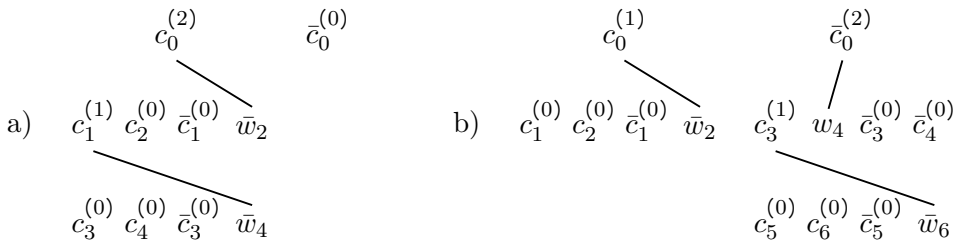

FIGURE 4. A diagram from the set a) $\mathfrak{D}_2 \times \overline{\mathfrak{D}}_0$ and b) $\mathfrak{D}_1 \times \overline{\mathfrak{D}}_2$.

Non-conjugated vertices c_j generate conjugated blocks $\bar{B}_k$ while conjugated vertices $\bar{c}_j$ generate non-conjugated blocks B_k. The numeration of the blocks (i.e. the dependence of k as a function of $(\hat{c}_j)$) is chosen in arbitrary way. E.g., from the left to the right and from the top to the bottom, as in fig. 2,3.

Rule (3.3) does not determine uniquely the degrees of vertices in the block $\hat{B}_k$, but fixes only their sum. So, for $m \neq 0, 1$ the diagram $\mathfrak{D}$ obtained by the inductive procedure above is not unique. We define $\mathfrak{D}_m$ as the set of all possible diagrams obtained by this procedure.

The set of diagrams $\overline{\mathfrak{D}}_n$ is constructed by a similar procedure. The only difference is that in diagrams $\bar{\mathfrak{D}} \in \overline{\mathfrak{D}}_n$ the root vertex is conjugated and has the form $\bar{c}_0^{(n)}$, see fig. 3(d). The set of diagrams $\overline{\mathfrak{D}}_n$ equals to the set $\mathfrak{D}_n$ in which we conjugate each diagram, and in every block of each diagram we exchange positions of conjugated and non-conjugated vertices, so that the non-conjugated vertices are situated first.

Finally, for a pair of diagrams $\mathfrak{D}^1 \in \mathfrak{D}_m$ and $\bar{\mathfrak{D}}^2 \in \overline{\mathfrak{D}}_n$ we consider the diagram $\mathfrak{D}^1 \sqcup \bar{\mathfrak{D}}^2$ (see fig. 3(a) and 4), obtained by drawing the diagrams $\mathfrak{D}^1$ and $\bar{\mathfrak{D}}^2$ side by side and enumerate the vertices of $\bar{\mathfrak{D}}^2$, except the root $\bar{c}_0$, not from 1 to $2n$ but from $2m+1$ to $2m+2n$. Thus, in $\mathfrak{D}^1 \sqcup \bar{\mathfrak{D}}^2$ we denote the two roots as c_0 and $\bar{c}_0$, then enumerate the remaining vertices of the diagram $\mathfrak{D}^1$, and finally those of the diagram $\bar{\mathfrak{D}}^2$. We define

$$\mathfrak{D}_m \times \overline{\mathfrak{D}}_n = \{\mathfrak{D}^1 \sqcup \bar{\mathfrak{D}}^2 : \mathfrak{D}^1 \in \mathfrak{D}_m, \bar{\mathfrak{D}}^2 \in \overline{\mathfrak{D}}_n\}.$$

3.1.3. *Properties of the set $\mathfrak{D}_m \times \overline{\mathfrak{D}}_n$ and some terminology.* Throughout the paper we use the notation

$$N := m + n.$$

The following properties of a diagram $\mathfrak{D} \in \mathfrak{D}_m \times \overline{\mathfrak{D}}_n$ are obvious.

• The number of blocks of $\mathfrak{D}$ equals to N.

• The diagram $\mathfrak{D}$ has $4N+2$ vertices, half of them are conjugated while another half are non-conjugated.

• If vertices $\hat{c}_i$ and $\hat{c}_j$ are coupled by an edge, we call them *adjacent* and write $\hat{c}_i \sim \hat{c}_j$. Each pair of adjacent vertices in $\mathfrak{D}$ consists of a conjugated

and non-conjugated vertex, and has either the form $\{c_i^{(p)}, \bar{w}_j\}$ or $\{\bar{c}_i^{(p)}, w_j\}$, where the degree $p \geq 1$.

• An edge which joins c_i with $\bar{c}_j$ is denoted $(c_i, \bar{c}_j)$ (either c_i or $\bar{c}_j$ is a virtual vertex). Each diagram $\mathfrak{D} \in \mathfrak{D}_m \times \overline{\mathfrak{D}}_n$ has N edges.

• Recall that a real vertex $\hat{c}_j^{(0)}$ of zero degree is called a leaf. There is no vertex in $\mathfrak{D}$, adjacent to a leaf. The number of conjugated leaves equals to the number of non-conjugated leaves and equals to $N + 1$.

3.1.4. *Values of vertices $\hat{c}_j$.* Consider a diagram $\mathfrak{D} \in \mathfrak{D}_m \times \overline{\mathfrak{D}}_n$. To every vertex $\hat{c}_i$ of a block $\hat{B}_k$, $1 \leq k \leq N$, we assign the same time variable $l_k \geq -T$. E.g., in the diagram from fig. 4(b) the times assigned to the vertices c_2 and $\bar{c}_6$ are l_1 and l_3, respectively. To the roots c_0 and $\bar{c}_0$ we assign the times τ_1 and τ_2. The vectors

$$l = (l_1, \dots, l_N) \qquad \text{and} \qquad \mathbf{l} = (\tau_1, \tau_2, l) \tag{3.4}$$

are called *the vectors of times*. Let

$$\xi = (\xi_0, \dots, \xi_{2N}), \quad \sigma = (\sigma_0, \dots, \sigma_{2N}),$$

where $\xi_i, \sigma_j \in \mathbb{Z}_L^d$, so $(\xi, \sigma) \in \mathbb{Z}_L^{(4N+2)d}$, and we recall that $4N + 2$ is the number of vertices in $\mathfrak{D}$. These will be the indices for summation in a formula for $a_s^{(m)}(\tau_1)\bar{a}_s^{(n)}(\tau_2)$, where ξ_j's are indices for the non-conjugated variables a_s and σ_j – for the conjugated variables $\bar{a}_s$ in the following sense.

We regard a real vertex $c_j^{(p)}$ of the diagram $\mathfrak{D}$ as a position to put there a component $a_{\xi_j}^{(p)}(t)$ of the random vector $a^{(p)}(t)$, and a real vertex $\bar{c}_j^{(p)}$ – as a position for a component $\bar{a}_{\sigma_j}^{(p)}(t')$ of the conjugate vector, with suitable $t = t(\mathbf{l})$, $t' = t'(\mathbf{l})$. More precisely, for $p \geq 0$ we view the real vertices $c_j^{(p)}$, $\bar{c}_j^{(p)}$ as random functions of the variables $\mathbf{l}, \xi, \sigma$, defined by

$$c_j^{(p)}(\mathbf{l}, \xi, \sigma) = a_{\xi_j}^{(p)}(t), \qquad \bar{c}_j^{(p)}(\mathbf{l}, \xi, \sigma) = \bar{a}_{\sigma_j}^{(p)}(t'), \tag{3.5}$$

where t and t' are components of the time-vector $\mathbf{l}$ in (3.4), assigned to the vertices $c_j^{(p)}$ and $\bar{c}_j^{(p)}$. We say that the vertices $c_j^{(p)}$ and $\bar{c}_j^{(p)}$ take values $a_{\xi_j}^{(p)}(t)$ and $\bar{a}_{\sigma_j}^{(p)}(t')$. For virtual vertices w_j and $\bar{w}_j$ we say that they take values w_{ξ_j} and $\bar{w}_{\sigma_j}$.

3.1.5. *Restrictions on the multi-indices (ξ, σ).* To calculate the value of a diagram $\mathfrak{D} \in \mathfrak{D}_m \times \overline{\mathfrak{D}}_n$ we will make substitutions (3.5), using multi-indices $(\xi, \sigma) \in \mathbb{Z}_L^{(4N+2)d}$. At some stages of our constructions it will be necessary to put restrictions on the (ξ, σ) and substitute in the diagram only the corresponding multi-indices which we call *admissible*. The set of all admissible multi-indices (ξ, σ) is denoted

$$\mathcal{A}_s(\mathfrak{D}) \subset \mathbb{R}^{(4N+2)d},$$

where we take s, ξ_i, σ_j in $\mathbb{R}^d$ rather than in $\mathbb{Z}^d_L$ to be able to extend the correlation from Theorem 2.2 to a function on $\mathbb{R}^d = \{s\}$. The set $\mathcal{A}_s(\mathfrak{D})$ is defined by the following three relations:

- We have

$$\xi_0 = \sigma_0 = s. \tag{3.6}$$

- For all $1 \le i \le N$,

$$\delta^{l\xi_{2i-1}\xi_{2i}}_{\sigma_{2i-1}\sigma_{2i}} = 1. \tag{3.7}$$

- If the vertices c_i and $\bar{c}_j$ are adjacent, the corresponding indices are equal:

$$c_i \sim \bar{c}_j \Rightarrow \xi_i = \sigma_j. \tag{3.8}$$

We will also use the corresponding discrete subset

$$\mathcal{A}^L_s(\mathfrak{D}) = \mathcal{A}_s(\mathfrak{D}) \cap \mathbb{Z}^{(4N+2)d}_L, \tag{3.9}$$

which is exactly the set of multi-indices in which we sum in (2.21). In view of condition (3.6) it may be not empty only if $s \in \mathbb{Z}^d_L$.

3.1.6. *More notation.* Let $\mathfrak{D} \in \mathfrak{D}_m \times \overline{\mathfrak{D}}_n$.

- We denote by $E(\mathfrak{D})$ the set of all edges of a diagram $\mathfrak{D}$, $E(\mathfrak{D}) = \{(c_i, \bar{c}_j)\}$.
- Let $(\xi, \sigma) \in \mathcal{A}_s(\mathfrak{D})$ be an admissible index–vector. Then $\xi_i = \sigma_j$ if $c_i \sim \bar{c}_j$. Denoting by φ an edge $\varphi = (c_i, \bar{c}_j)$, we set

$$\gamma^{\varphi}(\xi, \sigma) := \gamma_{\xi_i} = \gamma_{\sigma_j} \tag{3.10}$$

(γ_s's are defined in (1.4)).

- Recall that in each pair of adjacent vertices in $\mathfrak{D}$ always one vertex is real while another is virtual. Then, by l^{φ}_r and l^{φ}_w we denote the times assigned to the real and virtual vertices, connected by the edge φ. So each $l^{\varphi}_{r,w}$ is a mapping from $E(\mathfrak{D})$ to the set $\{\tau_1, \tau_2, l_1, \dots, l_N\}$.
- By $l^{\hat{c}_i}$ we denote the time, assigned to a vertex $\hat{c}_i$.
- By

$$L(\mathfrak{D}), \bar{L}(\mathfrak{D}) \subset \{0, \dots 2N\}$$

we denote the sets such that $\{c_i\}_{i \in L(\mathfrak{D})}$ and $\{\bar{c}_i\}_{i \in \bar{L}(\mathfrak{D})}$ form the sets of all non-conjugated leaves and all conjugated leaves, correspondingly. In other words, abusing notation and identifying the vertices with their values, we have $\{c_i\}_{i \in L(\mathfrak{D})} = \{a^{(0)}_{\xi_i}\}$ and $\{\bar{c}_i\}_{i \in \bar{L}(\mathfrak{D})} = \{\bar{a}^{(0)}_{\sigma_i}\}$. E.g., for the diagram from fig. 4(b) we have $L(\mathfrak{D}) = \{1, 2, 5, 6\}$ and $\bar{L}(\mathfrak{D}) = \{1, 3, 4, 5\}$.

3.2. **Formula for the product $a_s^{(m)}\bar{a}_s^{(n)}$.** Let us denote

$$\tau := (\tau_1, \tau_2), \tag{3.11}$$

where we recall that τ_1, τ_2 are the times for the roots $c_0, \bar{c}_0$. Recall also that the sets $E(\mathfrak{D})$, $L(\mathfrak{D})$, $\bar{L}(\mathfrak{D})$, the functions $\gamma^\varphi = \gamma^\varphi(\xi,\sigma)$ and the times $l^\varphi_{r,w}$ and $l^{c_j,\bar{c}_j}$ are defined in Section 3.1.6; in difference with the functions γ^φ, the time variables $l^\varphi_r, l^\varphi_w, l^{c_j}, l^{\bar{c}_j}$ are independent from (ξ,σ). To each diagram $\mathfrak{D} \in \mathfrak{D}_m \times \overline{\mathfrak{D}}_n$ we associate the (random) function

$$\begin{aligned} G_{\mathfrak{D}}(\tau, l, \xi, \sigma) := c_{\mathfrak{D}} \prod_{\varphi \in E(\mathfrak{D})} e^{-\gamma^\varphi(\xi,\sigma)(l^\varphi_r - l^\varphi_w)} \mathbb{I}_{\{-T \le l^\varphi_w \le l^\varphi_r\}}(\tau, l) \\ \prod_{j \in L(\mathfrak{D})} a^{(0)}_{\xi_j}(l^{c_j}) \prod_{j \in \bar{L}(\mathfrak{D})} \bar{a}^{(0)}_{\sigma_j}(l^{\bar{c}_j}), \end{aligned} \tag{3.12}$$

which is defined on the set of admissible multi-indices $(\xi,\sigma) \in \mathcal{A}^L_s(\mathfrak{D})$ (see (3.9)). See below for a discussion of this formula. Here $\mathbb{I}_{\{-T \le l^\varphi_w \le l^\varphi_r\}}$ denotes the indicator function of the set in which $-T \le l^\varphi_w \le l^\varphi_r$ and $c_{\mathfrak{D}}$ is the constant

$$c_{\mathfrak{D}} = (-1)^{\#w_{\mathfrak{D}}} i^N, \tag{3.13}$$

where $\#w_{\mathfrak{D}}$ is the number of non-conjugated virtual vertices of the diagram $\mathfrak{D}$, $N = m + n$ and $i = \sqrt{-1}$ is the imaginary unit. Set

$$\omega_j(\xi,\sigma) = \omega^{\xi_{2j-1}\xi_{2j}}_{\sigma_{2j-1}\sigma_{2j}} \quad \text{and} \quad \omega = (\omega_1, \dots, \omega_N), \tag{3.14}$$

we recall that the quadratic form $\omega^{s_1 s_2}_{s_3 s}$ is defined in (2.5). For vectors $\vec{x} = (x_1, \dots, x_N)$ and $\vec{t} = (t_1, \dots, t_N)$ set also

$$\Theta(\vec{x}, \vec{t}) = \prod_{j=1}^N \theta(x_j, t_j), \tag{3.15}$$

where θ is as in (2.14).

Lemma 3.1. *For any integers $m, n \ge 0$ satisfying $N := m + n \ge 1$, $s \in \mathbb{Z}^d_L$ and $\tau_1, \tau_2 \ge -T$, we have*

$$a^{(m)}_s(\tau_1)\bar{a}^{(n)}_s(\tau_2) = \sum_{\mathfrak{D} \in \mathfrak{D}_m \times \overline{\mathfrak{D}}_n} I_s(\mathfrak{D}), \tag{3.16}$$

where $I_s(\mathfrak{D}) = I_s(\tau; \mathfrak{D})$,

$$I_s(\tau; \mathfrak{D}) = \int_{\mathbb{R}^N} dl\, L^{-Nd} \sum_{(\xi,\sigma) \in \mathcal{A}^L_s(\mathfrak{D})} G_{\mathfrak{D}}(\tau, l, \xi, \sigma)\, \Theta(\omega(\xi,\sigma), \nu^{-1} l). \tag{3.17}$$

Components of the formula (3.17) are quite simple. Indeed, the factor $\prod_{j \in L(\mathfrak{D})} a^{(0)}_{\xi_j}(l^{c_j}) \prod_{j \in \bar{L}(\mathfrak{D})} \bar{a}^{(0)}_{\sigma_j}(l^{\bar{c}_j})$, entering $G_{\mathfrak{D}}(\tau, l, \xi, \sigma)$ (see (3.12)), is just the product of values of all leaves of the diagram $\mathfrak{D}$. The term $l^\varphi_r - l^\varphi_w$ is a difference of the times assigned to vertices, coupled by an edge φ, and the corresponding exponent in $G_{\mathfrak{D}}(\tau, l, \xi, \sigma)$ comes from the iteration of the

Duhamel formula (2.13) that generates φ. It appears together with the indicator function $\mathbb{I}_{\{-T\le l_w^\varphi\le l_r^\varphi\}}$, which is there due to the fact that in (2.13) the integration is performed not over $\mathbb{R}$ but only over the interval $[-T,\tau]$. The factor $\theta(\omega_j,\nu^{-1}l_j)$, entering the function $\Theta(\omega,\nu^{-1}l)$, comes from the iteration of the Duhamel formula, generating the j-th block of $\mathfrak{D}$.

The assertion of Lemma 3.1 easily (but clumsily) follows by iterating the formula (2.13). Instead of giving a complete proof we restrict ourselves to considering an example below.

3.2.1. *Example.* Here we establish formula (3.16) for the case $m=1$ and $n=0$. We get from (2.12) that

$$a_s^{(1)}(\tau_1)\bar a_s^{(0)}(\tau_2)=i\int_{-T}^{\tau_1}dl_1\,L^{-d}\sum_{\xi_1,\xi_2,\sigma_1,\sigma_2}\delta^{I^{\xi_1\xi_2}}_{\sigma_1\sigma_2}\delta^s_{\sigma_2}\,e^{-\gamma_{\sigma_2}(\tau_1-l_1)}$$
$$\big(a^{(0)}_{\xi_1}a^{(0)}_{\xi_2}\bar a^{(0)}_{\sigma_1}\big)(l_1)\bar a^{(0)}_s(\tau_2)\,\theta(\omega_1,\nu^{-1}l_1),\tag{3.18}$$

where we use notation (3.14). The set $\mathfrak{D}_1\times\overline{\mathfrak{D}}_0$ consists of a unique diagram, given in fig. 3(a), which we denote by $\mathfrak{D}^{1,0}$. Now the total number of vertices is $4N+2=6$, and the corresponding set of admissible multi-indices $\mathcal{A}_s^L(\mathfrak{D}^{1,0})$ has the form

$$\mathcal{A}_s^L(\mathfrak{D}^{1,0})=\big\{(\xi_0,\xi_1,\xi_2,\sigma_0,\sigma_1,\sigma_2)\in\mathbb{Z}_L^{6d}:\ \xi_0=\sigma_0=s,\ \delta^{I^{\xi_1\xi_2}}_{\sigma_1\sigma_2}=1,\ \xi_0=\sigma_2\big\}.$$

Then we rewrite the r.h.s. of (3.18) as

$$i\int_{-T}^{\tau_1}dl_1\,L^{-d}\sum_{(\xi,\sigma)\in\mathcal{A}_s^L(\mathfrak{D}^{1,0})}e^{-\gamma_{\sigma_2}(\tau_1-l_1)}\big(a^{(0)}_{\xi_1}a^{(0)}_{\xi_2}\bar a^{(0)}_{\sigma_1}\big)(l_1)\bar a^{(0)}_{\sigma_0}(\tau_2)\,\theta(\omega_1,\nu^{-1}l_1).$$

The unique edge of the diagram $\mathfrak{D}^{1,0}$ is $\varphi:=(c_0^{(1)},\bar w_2)$, so using the introduced in Section 3.1.6 notation we write $e^{-\gamma_{\sigma_2}(\tau_1-l_1)}=e^{-\gamma^\varphi(l_r^\varphi-l_w^\varphi)}$ and $\int_{-T}^{\tau_1}dl_1=\int_{\mathbb{R}}\mathbb{I}_{\{-T\le l_w^\varphi\le l_r^\varphi\}}\,dl_1$. Then, noting that the term $a^{(0)}_{\xi_1}a^{(0)}_{\xi_2}\bar a^{(0)}_{\sigma_1}\bar a^{(0)}_{\sigma_0}$ is a product of values of all leaves in the diagram $\mathfrak{D}^{1,0}$, we see that (3.18) takes the form $a_s^{(1)}(\tau_1)\bar a_s^{(0)}(\tau_2)=I_s(\mathfrak{D}^{1,0})$, where the sum $I_s(\mathfrak{D}^{1,0})$ is defined in (3.17). Thus, formula (3.16) holds for $m=1,n=0$.

4. Feynman diagrams and formula for expectations $\mathbb{E}a_s^{(m)}\bar a_s^{(n)}$

In this section we compute the expectation $\mathbb{E}a_s^{(m)}(\tau_1)\bar a_s^{(n)}(\tau_2)$. Due to (3.16), it suffices to find $\mathbb{E}I_s(\mathfrak{D})$ for every diagram $\mathfrak{D}\in\mathfrak{D}_m\times\overline{\mathfrak{D}}_n$. Note that the randomness enters $I_s(\mathfrak{D})$ only via the product of values of leaves $a^{(0)}_{\xi_i}$, $\bar a^{(0)}_{\sigma_j}$ in (3.12), which are Gaussian random variables. Since their correlations are given by (2.15), due to the Wick theorem $\mathbb{E}I_s(\mathfrak{D})$ is given by a sum over all Wick pairings of values $a^{(0)}_{\xi_i}(l^{c_i})$ of non-conjugated leaves in $\mathfrak{D}$ with values $\bar a^{(0)}_{\sigma_j}(l^{\bar c_j})$ of conjugated leaves. We parametrize this sum by *Feynman*

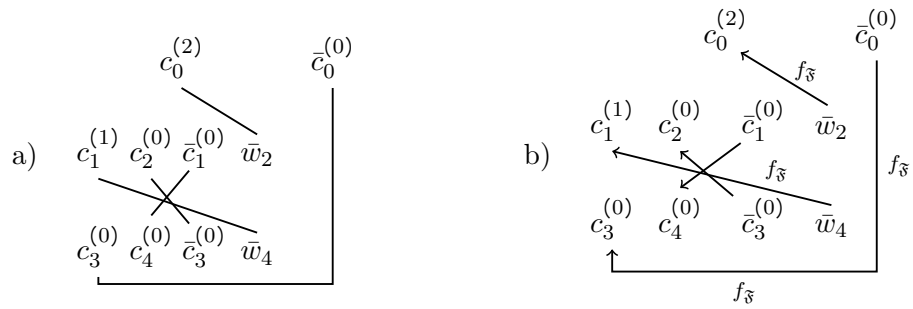

FIGURE 5. a) A Feynman diagram $\mathfrak{F}$ obtained from the diagram $\mathfrak{D}$ from fig. 4(a). b) Orientation of the Feynman diagram from item (a) and associated permutation $f_{\mathfrak{F}}$.

diagrams $\mathfrak{F}$ (see [17] for the notion), which are obtained from the diagram $\mathfrak{D}$ by connecting the Wick paired leaves by edges, see fig. 5.

Next we give a detailed definition of the Feynman diagrams and discuss related notions.

4.1. **Feynman diagrams.** Let us fix $m, n \geq 0$ satisfying $N = m + n \geq 1$, and consider a diagram $\mathfrak{D} \in \mathfrak{D}_m \times \overline{\mathfrak{D}}_n$. Its set of leaves consists of $N + 1$ conjugated leaves $\bar{c}_i^{(0)}$ and $N + 1$ non-conjugated leaves $c_j^{(0)}$. Consider any partition of this set to $N + 1$ non-intersecting pairs of the form $(c_i^{(0)}, \bar{c}_j^{(0)})$, where the leaves $c_i^{(0)}$ and $\bar{c}_j^{(0)}$ *do not belong to the same block* of $\mathfrak{D}$ (if such partition exists). We denote by $\mathfrak{F}$ a diagram obtained from $\mathfrak{D}$ by connecting the leaves in each pair by an edge (see fig. 5(a)) and call it a Feynman diagram. We never couple leaves belonging to the same block since for any j-th block the variables $a_{\xi_\alpha}^{(0)}$, $\bar{a}_{\sigma_\beta}^{(0)}$, assigned to the vertices in this block as in (3.5), satisfy $\mathbb{E} a_{\xi_\alpha}^{(0)} \bar{a}_{\sigma_\beta}^{(0)} = 0$ for $\alpha, \beta \in \{2j - 1, 2j\}$ and any admissible mutli-index $(\xi, \sigma) \in \mathcal{A}_s^L(\mathfrak{D})$, since $\xi_\alpha \neq \sigma_\beta$ in view of (1.10) and (3.7).

Let $\mathfrak{F}(\mathfrak{D})$ be the set of all Feynman diagrams, which may be obtained from $\mathfrak{D}$ using different partitions of the set of leaves to pairs. We define

$$\mathfrak{F}_{m,n} = \bigcup_{\mathfrak{D} \in \mathfrak{D}_m \times \overline{\mathfrak{D}}_n} \mathfrak{F}(\mathfrak{D}). \tag{4.1}$$

This is the set of all Feynman diagrams corresponding to the product $a_s^{(m)} \bar{a}_s^{(n)}$. By construction, the set of vertices of any Feynman diagram $\mathfrak{F} \in \mathfrak{F}_{m,n}$ is partitioned into pairs of adjacent vertices $(c_i, \bar{c}_j)$. Each pair has either the form $(c_i^{(p)}, \bar{w}_j)$ or $(w_i, \bar{c}_j^{(p)})$, where $p \geq 1$, or the form $(c_i^{(0)}, \bar{c}_j^{(0)})$. Moreover, adjacent vertices never belong to the same block. To any $\mathfrak{D} \in \mathfrak{D}_m \times \overline{\mathfrak{D}}_n$ the construction above corresponds many diagrams $\mathfrak{F} \in \mathfrak{F}_{m,n}$, but the inverse

mapping

$$\mathfrak{F}_{m,n} \to \mathfrak{D}_m \times \overline{\mathfrak{D}}_n, \quad \mathfrak{F} \mapsto \mathfrak{D}_{\mathfrak{F}}, \tag{4.2}$$

is well defined: the diagram $\mathfrak{D}_{\mathfrak{F}}$ is obtained from $\mathfrak{F}$ by erasing the edges that couple its leaves.

In fact, by demanding that in Feynman diagrams coupled leaves never belong to the same block, we have excluded only a part of vanishing Wick pairings. To exclude the remaining part, in Section 5.2 we will replace the set $\mathfrak{F}_{m,n}$ of Feynman diagrams by a smaller set $\mathfrak{F}^{true}_{m,n} \subset \mathfrak{F}_{m,n}$. For the Wick pairings corresponding to diagrams $\mathfrak{F} \notin \mathfrak{F}^{true}_{m,n}$ the assumption $\{s_1, s_2\} \neq \{s_3, s\}$, following from (1.9), again is violated due to (3.7), but in a less trivial way (see Proposition 5.6).

4.1.1. *Set of admissible multi-indices $\mathcal{A}'_s(\mathfrak{F})$.* Let us orient edges of a Feynman diagram $\mathfrak{F} \in \mathfrak{F}_{m,n}$ in the direction from conjugated vertices to non-conjugated, as in fig. 5(b), and consider a permutation $f_{\mathfrak{F}}$ of the set $\{0, \dots, 2N\}$ such that

$$f_{\mathfrak{F}}(j) = i \ \text{ if } \bar{c}_j \sim c_i, \text{ i.e. an edge of } \mathfrak{F} \text{ goes from } \bar{c}_j \text{ to } c_i, \tag{4.3}$$

see fig. 5(b). For a vector $\xi \in \mathbb{R}^{(2N+1)d}$ we denote

$$\xi_{f_{\mathfrak{F}}} = (\xi_{f_{\mathfrak{F}}(0)}, \dots, \xi_{f_{\mathfrak{F}}(2N)}). \tag{4.4}$$

Let $\mathcal{A}'_s(\mathfrak{F})$ be a set of all multi-indices $\xi \in \mathbb{R}^{(2N+1)d}$ satisfying the following relations: for $\sigma = \xi_{f_{\mathfrak{F}}}$,

- We have

$$\xi_0 = \sigma_0 = s. \tag{4.5}$$

- For any $1 \le i \le N$,

$$\delta^{I\xi_{2i-1}\xi_{2i}}_{\sigma_{2i-1}\sigma_{2i}} = 1. \tag{4.6}$$

In other words, we require (4.5) and (4.6) to be satisfied by the multi-indices ξ, σ, where σ is obtained from ξ accordingly to the edges of diagram $\mathfrak{F}$: $\sigma_j = \xi_i$ if a vertex $\bar{c}_j$ is coupled with the vertex c_i. In particular, if $\xi \in \mathcal{A}'_s(\mathfrak{F})$ then $(\xi, \xi_{f_{\mathfrak{F}}}) \in \mathcal{A}_s(\mathfrak{D}_{\mathfrak{F}})$, where $\mathcal{A}_s(\mathfrak{D})$ is defined in Section 3.1.5 and the diagram $\mathfrak{D}_{\mathfrak{F}}$ – in (4.2). Indeed, (3.8) holds by the definition (4.3) of the permutation $f_{\mathfrak{F}}$ while (3.6) and (3.7) are obvious.

We also consider the discrete set $\mathcal{A}'^L_s(\mathfrak{F}) = \mathcal{A}'_s(\mathfrak{F}) \cap \mathbb{Z}^{(2N+1)d}_L$.

Our motivation behind these definitions is as follows. As it is explained in the beginning of the section, each term $\mathbb{E}I(\mathfrak{D})$ can be written as a sum over the Feynman diagrams $\mathfrak{F}$,

$$\mathbb{E}I_s(\mathfrak{D}) = \sum_{\mathfrak{F} \in \mathfrak{F}(\mathfrak{D})} J_s(\mathfrak{F}).$$

Each summand $J_s(\mathfrak{F})$ has the form (3.17) where $G_{\mathfrak{D}}$ is replaced by a function obtained by the corresponding to $\mathfrak{F}$ Wick pairing of the values $a^{(0)}_{\xi_i}$, $\bar{a}^{(0)}_{\sigma_j}$ of leaves. In this formula for $J_s(\mathfrak{F})$ we should take summation only over those multi-indices $(\xi,\sigma)\in\mathcal{A}^L_s(\mathfrak{D})$ for which $\xi_i=\sigma_j$ once the vertices c_i and $\bar{c}_j$ are adjacent in $\mathfrak{F}$, that is over vectors of the form $(\xi,\xi_{f_{\mathfrak{F}}})$, where $\xi\in\mathcal{A}^{IL}_s(\mathfrak{F})$.

4.2. **Formula for the expectation $\mathbb{E}a^{(m)}_s\bar{a}^{(n)}_s$.** Below we introduce some more notation. Let $\mathfrak{F}\in\mathfrak{F}_{m,n}$ be a Feynman diagram.

• We denote by $E(\mathfrak{F})$ the set of edges of the diagram $\mathfrak{F}$, then $E(\mathfrak{D}_{\mathfrak{F}})\subset E(\mathfrak{F})$ (see (4.2)). We also set

$$E_D(\mathfrak{F}):=E(\mathfrak{D}_{\mathfrak{F}}),\quad E_L(\mathfrak{F}):=E(\mathfrak{F})\setminus E_D(\mathfrak{F})\quad\text{so}\quad E(\mathfrak{F})=E_D(\mathfrak{F})\cup E_L(\mathfrak{F}).$$

• For an edge $\vartheta\in E(\mathfrak{F})$ which couples some vertices c_i and $\bar{c}_j$, we set $c_\vartheta:=c_i$, $\bar{c}_\vartheta:=\bar{c}_j$, and denote by l^{c_ϑ} and $l^{\bar{c}_\vartheta}$ the times assigned to the vertices c_ϑ and $\bar{c}_\vartheta$. We also set

$$\gamma^\vartheta(\xi):=\gamma_{\xi_i}\quad\text{and}\quad b^\vartheta(\xi):=b(\xi_i)\quad\text{if}\ \ \vartheta\ \text{couples}\ c_i\ \text{and}\ c_j,$$

where γ_s is defined in (1.4) and $b(s)$ enters the definition of the random force η^ω.

• If $\varphi\in E_D(\mathfrak{F})$, we define the times l^φ_r and l^φ_w as in Section 3.1.6.

The main result of this section is Lemma 4.1, stated below. Recalling (3.11) and (3.12), (2.15), to each Feynman diagram $\mathfrak{F}$ we associate the density function $c_{\mathfrak{F}}F^{\mathfrak{F}}(\tau,l,\xi)$, where $c_{\mathfrak{F}}=c_{\mathfrak{D}_{\mathfrak{F}}}$ (see (3.13)) and $F^{\mathfrak{F}}(\tau,l,\xi)$ is the real function

$$F^{\mathfrak{F}}(\tau,l,\xi)=\prod_{\varphi\in E_D(\mathfrak{F})}e^{-\gamma^\varphi(l^\varphi_r-l^\varphi_w)}\mathbb{I}_{\{-T\le l^\varphi_w\le l^\varphi_r\}}(\tau,l)$$

$$\prod_{\psi\in E_L(\mathfrak{F})}\Big(e^{-\gamma^\psi|l^{c_\psi}-l^{\bar{c}_\psi}|}-e^{-\gamma^\psi(l^{c_\psi}+l^{\bar{c}_\psi}+2T)}\Big)\frac{(b^\psi)^2}{\gamma^\psi},\tag{4.7}$$

where $\gamma^\varphi=\gamma^\varphi(\xi)$, $\gamma^\psi=\gamma^\psi(\xi)$, $b^\psi=b^\psi(\xi)$. We also associate to $\mathfrak{F}$ the quadratic forms

$$\omega^{\mathfrak{F}}_j(\xi):=\omega_j(\xi,\xi_{f_{\mathfrak{F}}})=\omega^{\xi_{2j-1}\xi_{2j}}_{\xi_{f_{\mathfrak{F}}(2j-1)}\xi_{f_{\mathfrak{F}}(2j)}}\quad\text{and set}\quad\omega^{\mathfrak{F}}=(\omega^{\mathfrak{F}}_1,\dots,\omega^{\mathfrak{F}}_N),\tag{4.8}$$

see (3.14) and (4.4).

Lemma 4.1. *For any integers $m,n\ge 0$ satisfying $N=m+n\ge 1$, for $s\in\mathbb{Z}^d_L$ and $\tau_1,\tau_2\ge -T$, we have*

$$\mathbb{E}a^{(m)}_s(\tau_1)\bar{a}^{(n)}_s(\tau_2)=\sum_{\mathfrak{F}\in\mathfrak{F}_{m,n}}c_{\mathfrak{F}}J_s(\mathfrak{F}),\tag{4.9}$$

where $J_s(\mathfrak{F})=J_s(\tau;\mathfrak{F})$

$$J_s(\tau;\mathfrak{F})=\int_{\mathbb{R}^N}dl\,L^{-Nd}\sum_{\xi\in\mathcal{A}^{IL}_s(\mathfrak{F})}F^{\mathfrak{F}}(\tau,l,\xi)\,\Theta(\omega^{\mathfrak{F}}(\xi),\nu^{-1}l).\tag{4.10}$$

The density $F^{\mathfrak{F}}$, given by (4.7), is real, smooth in $\xi \in \mathbb{R}^{(2N+1)d}$, piecewise smooth in $l \in \mathbb{R}^N$ and such that for any vector $\kappa \in \mathbb{Z}_+^{d(2N+1)}$ we have:

$$|\partial_\xi^\kappa F^{\mathfrak{F}}(\tau, l, \xi)| \le C_\kappa^{\#}(\xi)\, e^{-\delta\left(\sum_{i=1}^m |\tau_1 - l_i| + \sum_{i=m+1}^N |\tau_2 - l_i|\right)}, \tag{4.11}$$

for any $\xi \in \mathcal{A}_s^I(\mathfrak{F})$, $l \in \mathbb{R}^N$ and $\tau_1, \tau_2 \ge -T$, where $\delta = \delta_N > 0$.

Below we do not use the explicit form (4.7) of the function $F^{\mathfrak{F}}$ but only the estimate (4.11).

Proof of (4.9) can be obtained by taking the expectation of both sides of (3.16) and is essentially explained in previous discussion of this section. Estimate (4.11) follows from the explicit form (4.7) of the density $F^{\mathfrak{F}}$. An accurate proof of Lemma 4.1 is straightforward but rather tedious in view of the complexity of notation, so we give it only in the arXiv version of this paper [10].

Let us note that

$$\text{if } \ m + n = 1, \ \text{ then } \ \mathbb{E} a_s^{(m)} \bar{a}_s^{(n)} = 0. \tag{4.12}$$

Indeed, in this case $\mathfrak{F}_{m,n} = \varnothing$, since every partition of the set of leaves into pairs couples vertices from the same block. Accordingly, we will often assume that $m + n \ge 2$.

5. Change of coordinates and final formula for expectations $\mathbb{E} a_s^{(m)} \bar{a}_s^{(n)}$

Lemma 4.1 gives an explicit formula for the expectation $\mathbb{E} a_s^{(m)} \bar{a}_s^{(n)}$, however, the structure of the set of multi-indices $\mathcal{A}_s^{IL}(\mathfrak{F})$, over which we take a summation in (4.10), is too complicated for further analysis. In this section we find new coordinates in which this set and the quadratic forms (4.8) take convenient form.

By (4.6) the vectors $\xi \in \mathcal{A}_s^{IL}$ satisfy the system of equations

$$\xi_{2k-1} + \xi_{2k} = \sigma_{2k-1} + \sigma_{2k}, \qquad \sigma := \xi_{f_{\mathfrak{F}}}, \qquad 1 \le k \le N. \tag{5.1}$$

We start with constructing coordinates $x = (x_0, \dots, x_{2N})$, where $x_0 = \xi_0$ and for every $1 \le k \le N$ we set either $x_{2k-1} = \xi_{2k-1} - \sigma_{2k-1}$ and $x_{2k} = \xi_{2k} - \sigma_{2k}$ or $x_{2k-1} = \xi_{2k-1} - \sigma_{2k}$ and $x_{2k} = \xi_{2k} - \sigma_{2k-1}$, so that equations (5.1) take trivial form $x_{2k} = -x_{2k-1}$, $\forall k$. We will make a choice (different for different k and dependent on the diagram $\mathfrak{F}$) between the two possibilities above in such a way that the transformation $\xi \mapsto x$ is invertible. Then, components $z_k := x_{2k-1}$, $1 \le k \le N$, form coordinates on the set $\mathcal{A}_s^{IL}$ while components x_{2k} are functions of $z = (z_k)_{1 \le k \le N}$.

To construct this transformation $\xi \mapsto x$ we add to the diagram $\mathfrak{F}$ dashed edges which couple vertices inside each block in such a way that the obtained diagram becomes a Hamilton cycle, see fig. 6(b) (we show that it is always possible to do). For every k we choose between the two possibilities above in accordance with the dashed edges. It turns out that in the coordinates z the

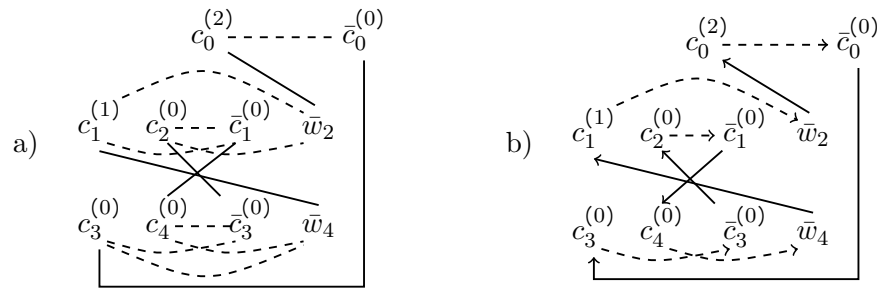

FIGURE 6. a) The graph $\mathfrak{U}_{\mathfrak{F}}$ obtained from the Feynman diagram $\mathfrak{F}$ from fig. 5. b) A Hamilton cycle $\mathfrak{C}_{\mathfrak{F}}$ in the graph $\mathfrak{U}_{\mathfrak{F}}$ from item (a).

function $\Theta(\omega^{\mathfrak{F}}, \nu^{-1} l)$ with $\theta(x,t) = e^{ixt}$ takes a simple form (5.21)-(5.22). This leads to an explicit formula for the sum $J_s(\mathfrak{F})$ and next in Sections 6, 7 allows to analyse its asymptotic behaviour as $\nu \to 0$, $L \to \infty$. Next we explain construction of the transformation $\xi \mapsto x$ in detail.

5.1. **Cycles $\mathfrak{C}_{\mathfrak{F}}, \hat{\mathfrak{C}}_{\mathfrak{F}}$ and permutation $\pi_{\mathfrak{F}}$.** Let us take a Feynman diagram $\mathfrak{F} \in \mathfrak{F}_{m,n}$ with $N = m + n \geq 2$. By construction, if two vertices of $\mathfrak{F}$ belong to the same block then they are not adjacent. Now inside each block $\hat{B}_k$, $k \geq 1$, (see (3.1)-(3.2)) we join by four dashed edges conjugated vertices with non-conjugated in all possible ways, see fig. 6(a). We also join by a dashed edge the two roots c_0, $\bar{c}_0$ and denote the resulting graph by $\mathfrak{U}_{\mathfrak{F}}$.

Lemma 5.1. *For each Feynman diagram $\mathfrak{F} \in \mathfrak{F}_{m,n}$ the graph $\mathfrak{U}_{\mathfrak{F}}$ has a Hamilton cycle in which solid and dashed edges alternate.*

Proof of Lemma 5.1, given in Appendix 8.2, follows from the fact that the diagram, made by the blocks of $\mathfrak{F}$, is connected. In general, the Hamilton cycle in Lemma 5.1 is not unique, so we fix one for each diagram $\mathfrak{U}_{\mathfrak{F}}$ and denote it as $\mathfrak{C}_{\mathfrak{F}}$. Examples of the Hamilton cycles $\mathfrak{C}_{\mathfrak{F}}$ are given in fig. 6(b). It is straightforward to see that, by construction of the diagrams $\mathfrak{U}_{\mathfrak{F}}$, the cycles $\mathfrak{C}_{\mathfrak{F}}$ contain all solid edges of $\mathfrak{U}_{\mathfrak{F}}$ (which are the edges of $\mathfrak{F}$), and a half of the dashed edges: for each block exactly two of four dashed edges that couple vertices inside the block enter the cycle $\mathfrak{C}_{\mathfrak{F}}$. On the cycle $\mathfrak{C}_{\mathfrak{F}}$ the dashes and solid edges alternate.

We orient the dashed edges of $\mathfrak{C}_{\mathfrak{F}}$ in the direction from non-conjugated vertices to conjugated and recall that the solid edges are oriented in the opposite direction, from conjugated vertices to non-conjugated. Then, $\mathfrak{C}_{\mathfrak{F}}$ becomes an oriented cycle, see fig. 6(b).

Let us denote by $\hat{\mathfrak{C}}_{\mathfrak{F}}$ the reduced oriented cycle $\mathfrak{C}_{\mathfrak{F}}$, that is the cycle $\mathfrak{C}_{\mathfrak{F}}$ in which we keep only non-conjugated vertices and "forget" all conjugated ones, see fig. 7. The cycle $\hat{\mathfrak{C}}_{\mathfrak{F}}$ defines a cyclic permutation $\pi_{\mathfrak{F}}$ of the set of

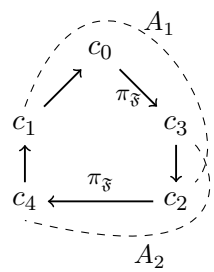

FIGURE 7. Cycle $\hat{\mathfrak{C}}_{\mathfrak{F}}$ corresponding to the cycle $\mathfrak{C}_{\mathfrak{F}}$ in fig. 6(b), the associated permutation $\pi_{\mathfrak{F}}$ and the arcs $A_1 := [c_1, c_2]$, $A_2 := [c_3, c_4]$. Here we have $\alpha_{12}^{\mathfrak{F}} = -\alpha_{21}^{\mathfrak{F}} = 1$.

parameters for the non-conjugated vertices $\{0, \dots, 2N\}$ as

$$\pi_{\mathfrak{F}}(j) = i \quad \text{if an edge of } \hat{\mathfrak{C}}_{\mathfrak{F}} \text{ goes from } c_j \text{ to } c_i.$$

The graph $\mathfrak{U}_{\mathfrak{F}}$ and the Hamilton cycle $\mathfrak{C}_{\mathfrak{F}}$ play a central role in our construction. Nevertheless, below we use only the "derivative" objects: the reduced cycle $\hat{\mathfrak{C}}_{\mathfrak{F}}$ and the permutation $\pi_{\mathfrak{F}}$.

5.2. **The change of coordinates.** For a fixed Feynman diagram $\mathfrak{F} \in \mathfrak{F}_{m,n}$ with $N = m + n \geq 1$, we set

$$x_0 := \xi_0, \quad x_j := \xi_j - \xi_{\pi_{\mathfrak{F}}(j)} \quad \forall 1 \leq j \leq 2N, \tag{5.2}$$

and define the $(2N+1)$–vector $x = (x_j)_{0 \leq j \leq 2N}$, $x_j \in \mathbb{R}^d$. Since $\pi_{\mathfrak{F}}$ is a cyclic permutation of the set $\{0, \dots, 2N\}$, then the transformation $\xi \mapsto x$ is a bijection of $(\mathbb{R}^d)^{2N+1}$. Indeed, it is invertible since iterating (5.2) we get

$$\xi_j = x_j + \xi_{\pi_{\mathfrak{F}}(j)} = x_j + x_{\pi_{\mathfrak{F}}(j)} + \xi_{\pi_{\mathfrak{F}}^2(j)} = \ldots = \sum_{i=0}^{k} x_{\pi_{\mathfrak{F}}^i(j)} + \xi_0,$$

where $k = k(j)$ denotes minimal positive integer satisfying $\pi_{\mathfrak{F}}^{k+1}(j) = 0$. In other words, since $\xi_0 = x_0$,

$$\xi_j = \sum_{i:\, c_i \in [c_j, c_0]} x_i, \qquad 0 \leq j \leq 2N, \tag{5.3}$$

where by $[c_a, c_b]$, $a \neq b$, we denote a set (an *arc*) of vertices of the cycle $\hat{\mathfrak{C}}_{\mathfrak{F}}$ that are situated between the vertices c_a and c_b according to the orientation of $\hat{\mathfrak{C}}_{\mathfrak{F}}$, including c_a and c_b. E.g., in fig. 7 $[c_4, c_3] = \{c_4, c_1, c_0, c_3\}$. For $a = b$ we set $[c_a, c_b] = \{c_a\}$.

Next we write the set of admissible multi-indices $\mathcal{A}_s'(\mathfrak{F}) \subset \mathbb{R}^{d(2N-1)}$ in the coordinates x. An important role below is played by the $N \times N$ *incidence*

matrix $\alpha^{\mathfrak{F}} = (\alpha^{\mathfrak{F}}_{ij})_{1\le i,j\le N}$ with components [9]

$$\alpha^{\mathfrak{F}}_{ij} = \begin{cases} 1 & \text{if} \quad c_{2j-1} \in [c_{2i-1}, c_{2i}], \quad c_{2j} \notin [c_{2i-1}, c_{2i}], \\ -1 & \text{if} \quad c_{2j-1} \notin [c_{2i-1}, c_{2i}], \quad c_{2j} \in [c_{2i-1}, c_{2i}], \\ 0 & \text{otherwise.} \end{cases} \tag{5.4}$$

It is straightforward to see that the matrix $\alpha^{\mathfrak{F}}$ is skew-symmetric, so it is S-regular in the following sense:

Definition 5.2. *A square matrix $\alpha = \{\alpha_{ij}\}$ is called S-regular if it is skew-symmetric and all its elements α_{ij} are such that $\alpha_{ij} \in \{-1, 0, 1\}$. If in addition all rows and columns of α are non-zero vectors, then it is called SS-regular.*

Given below Propositions 5.3 and 5.5 are in the hart of the proof of the main result of this work – Theorem 1.1. Previous constructions in Sections 3-5.1 essentially were made in order to create the notation, needed to state and prove these results.

Proposition 5.3. *For any diagram $\mathfrak{F} \in \mathfrak{F}_{m,n}$ and any $s \in \mathbb{R}^d$, the set $\mathcal{A}'_s(\mathfrak{F})$ consists of all vectors $\xi = \xi(x)$, where $\xi(x)$ is given by (5.3) and $x \in (\mathbb{R}^d)^{2N+1}$ satisfies the following relations, for every $1 \le k \le N$:*

$$(a)\ x_0 = s, \qquad (b)\ x_{2k} = -x_{2k-1}, \tag{5.5}$$

$$(a)\ x_{2k-1} \ne 0 \qquad (b)\ \sum_{i=1}^{N} \alpha^{\mathfrak{F}}_{ki} x_{2i-1} \ne 0. \tag{5.6}$$

Moreover, for vectors x satisfying (5.5)(b) we have

$$\xi_{2k-1}(x) - \xi_{\pi_{\mathfrak{F}}(2k)}(x) = \sum_{i=1}^{N} \alpha^{\mathfrak{F}}_{ki} x_{2i-1}. \tag{5.7}$$

Proof. In this proof we omit the lower index $\mathfrak{F}$ in the notation $\pi_{\mathfrak{F}}$ and $f_{\mathfrak{F}}$. We start by noting that for every $k \ge 1$

$$\{\pi(2k-1), \pi(2k)\} = \{f(2k-1), f(2k)\}. \tag{5.8}$$

Indeed, the permutation π encodes the change of indices of non-conjugated vertices of the cycle $\mathfrak{C}_{\mathfrak{F}}$ under the double shift, the first along dashed edges and the second along solid edges. The first shift preserves the set of indices $\{2k-1, 2k\}$ while the second is encoded by the permutation f.

Now let us recall that the set $\mathcal{A}'_s(\mathfrak{F})$ consists of all multi-indices ξ satisfying (4.5) and (4.6) with $\sigma = \xi_f$. In view of what is told above, we have

[9] Formula (5.4) admits the following interpretation. An element $\alpha^{\mathfrak{F}}_{ij}$ equals ± 1 if the arcs $[c_{2i-1}, c_{2i}]$ and $[c_{2j-1}, c_{2j}]$ are linked, and equals 0 otherwise. The sign "+" or "−" is determined by the order of linking; see fig. 7. So the incidence matrix $\alpha^{\mathfrak{F}}$ describes the linking of the arcs $[c_{2j-1}, c_{2j}]$ on the reduced cycle $\hat{\mathfrak{C}}_{\mathfrak{F}}$, or equivalently on the Hamiltonian cycle $\mathfrak{C}_{\mathfrak{F}}$ in the diagram $\mathfrak{F}$.

$\{\xi_{f(2k-1)}, \xi_{f(2k)}\} = \{\xi_{\pi(2k-1)}, \xi_{\pi(2k)}\}$, so $\delta^{I\xi_{2k-1}\xi_{2k}}_{\xi_{f(2k-1)}\xi_{f(2k)}} = \delta^{I\xi_{2k-1}\xi_{2k}}_{\xi_{\pi(2k-1)}\xi_{\pi(2k)}}$. Then eq. (4.6) is equivalent to $\delta^{I\xi_{2k-1}\xi_{2k}}_{\xi_{\pi(2k-1)}\xi_{\pi(2k)}} = 1$, that is to the equation

$$\xi_{2k-1} + \xi_{2k} = \xi_{\pi(2k-1)} + \xi_{\pi(2k)} \tag{5.9}$$

jointly with

$$\{\xi_{2k-1}, \xi_{2k}\} \cap \{\xi_{\pi(2k-1)}, \xi_{\pi(2k)}\} = \varnothing \tag{5.10}$$

(see (1.10)). Equation (5.9) is equivalent to (5.5)(b). The equality $\xi_0 = s$ from (4.5) is equivalent to (5.5)(a). The equality $\sigma_0 = s$, in view of (5.3), takes the form

$$s = \sigma_0 = \xi_{f(0)} = \sum_{i:\, c_i \in [c_{f(0)}, c_0]} x_i. \tag{5.11}$$

Since $f(0) = \pi(0)$, then $[c_{f(0)}, c_0] = [c_{\pi(0)}, c_0]$, and this interval contains all vertices of the cycle $\hat{\mathfrak{C}}_{\mathfrak{F}}$. Then, using (5.5)(b), we see that the r.h.s. of (5.11) equals to x_0, so (5.11) again takes the form (5.5)(a).

Now it remains to show that (5.10) is equivalent to (5.6). In view of (5.9), relation (5.10) is equivalent to

$$\xi_{2k-1} \neq \xi_{\pi(2k-1)} \quad \text{and} \quad \xi_{2k-1} \neq \xi_{\pi(2k)}. \tag{5.12}$$

The first inequality above is equivalent to (5.6)(a), while the second is equivalent to (5.6)(b) if (5.7) is established. So it remains to verify (5.7). Due to (5.3),

$$\xi_{2k-1} - \xi_{\pi(2k)} = \sum_{i:\, c_i \in [c_{2k-1}, c_0]} x_i - \sum_{i:\, c_i \in [c_{\pi(2k)}, c_0]} x_i.$$

Let us first assume that $c_{\pi(2k)} \in [c_{2k-1}, c_0]$ and $c_{\pi(2k)} \neq c_{2k-1}$. Then the identity above takes the form

$$\xi_{2k-1} - \xi_{\pi(2k)} = \sum_{i:\, c_i \in [c_{2k-1}, c_{2k}]} x_i,$$

where we have used that the vertex situating in $\hat{\mathfrak{C}}_{\mathfrak{F}}$ before $c_{\pi(2k)}$ is c_{2k}. In view of the assumption above, $c_0 \notin [c_{2k-1}, c_{2k}]$. Then, due to cancellations provided by (5.5)(b), we get (5.7).

The case $c_{\pi(2k)} \notin [c_{2k-1}, c_0]$ can be considered similarly. If $c_{\pi(2k)} = c_{2k-1}$, that is $\pi(2k) = 2k-1$, then the both sides of eq. (5.7) vanish. Indeed, $\alpha^{\mathfrak{F}}_{ki} = 0$ $\forall i$ since the arc $[c_{2k-1}, c_{2k}]$ contains all vertices $c_j \in \hat{\mathfrak{C}}_{\mathfrak{F}}$. □

Let us denote

$$z_j := x_{2j-1} \in \mathbb{R}^d \qquad \text{and} \qquad z = (z_j)_{1 \le j \le N} \in \mathbb{R}^{Nd}_L.$$

Due to Proposition 5.3, vector z forms coordinates on the set $\mathcal{A}'_s(\mathfrak{F})$ and, by (5.5)(b), $x_i = (-1)^{i+1} z_{\lceil i/2 \rceil}$. Then, using (5.5)(a), we see that for $\xi \in \mathcal{A}'_s(\mathfrak{F})$ relation (5.3) takes the form

$$\xi_j(z) = s + \sum_{i:\, c_i \in [c_j, c_0)} (-1)^{i+1} z_{\lceil i/2 \rceil}, \quad 1 \le j \le 2N, \tag{5.13}$$

where $[c_j, c_0) := [c_j, c_0] \setminus \{c_0\}$, and $\xi_0 = s$. Here we emphasize that the linear mapping $z \mapsto \xi$ depends on $\mathfrak{F}$ (through the interval $[c_j, c_0)$) and that ξ is an affine vector-function of the parameter $s \in \mathbb{R}^d$. Thus, in the z-coordinates

$$\mathcal{A}'_s(\mathfrak{F}) = \{\xi(z) : \ z \in \mathcal{Z}(\mathfrak{F})\}, \tag{5.14}$$

where, by (5.6),

$$\mathcal{Z}(\mathfrak{F}) = \{z \in \mathbb{R}^{dN} : \quad z_j \neq 0 \quad \text{and} \quad \sum_{i=1}^{N} \alpha^{\mathfrak{F}}_{ji} z_i \neq 0 \quad \forall 1 \leq j \leq N\}. \tag{5.15}$$

Remark 5.4. Since the choice of the Hamilton cycle $\mathfrak{C}_{\mathfrak{F}}$ in general is not unique, the obtained parametrization $z \mapsto \xi$ is not unique as well. However, one can show that if $z' \mapsto \xi$ is another parametrization, obtained by the procedure above, and $\alpha^{\mathfrak{F}'}$ is the associated incidence matrix, then for each j we have either $z'_j(\xi) = z_j(\xi)$ or $z'_j(\xi) = \sum_{i=1}^{N} \alpha^{\mathfrak{F}}_{ji} z_i$. In the latter case we also have the symmetric relation $z_j(\xi) = \sum_{i=1}^{N} \alpha^{\mathfrak{F}'}_{ji} z'_i$.

Next we write the quadratic forms $\omega^{\mathfrak{F}}_j(\xi)$, defined in (4.8), in the z-coordinates.

Proposition 5.5. *Functions $\omega^{\mathfrak{F}}_j(\xi)$, restricted to the set $\mathcal{A}'_s(\mathfrak{F}) \ni \xi$ and written in the z-coordinates, take the form*

$$\omega^{\mathfrak{F}}_j(z) = 2z_j \cdot \sum_{i=1}^{N} \alpha^{\mathfrak{F}}_{ji} z_i, \tag{5.16}$$

where the S-regular incidence matrix $\alpha^{\mathfrak{F}} = (\alpha^{\mathfrak{F}}_{ij})_{1 \leq i,j \leq N}$ is defined in (5.4).

Proof. In this proof we again skip the lower index $\mathfrak{F}$ in the notation $\pi_{\mathfrak{F}}$. Due to (5.8) and their definition (3.14),(4.8), the functions $\omega^{\mathfrak{F}}_j(\xi)$ have the form

$$\omega^{\mathfrak{F}}_j(\xi) = |\xi_{2j-1}|^2 - |\xi_{\pi(2j-1)}|^2 + |\xi_{2j}|^2 - |\xi_{\pi(2j)}|^2.$$

Using (5.5)(b), we obtain

$$\begin{aligned}
\omega^{\mathfrak{F}}_j(\xi) &= x_{2j-1} \cdot (\xi_{2j-1} + \xi_{\pi(2j-1)}) + x_{2j} \cdot (\xi_{2j} + \xi_{\pi(2j)}) \\
&= x_{2j-1} \cdot (\xi_{2j-1} + \xi_{\pi(2j-1)} - \xi_{2j} - \xi_{\pi(2j)}) \\
&= x_{2j-1} \cdot (2\xi_{2j-1} - x_{2j-1} - 2\xi_{\pi(2j)} - x_{2j}) \\
&= 2x_{2j-1} \cdot (\xi_{2j-1} - \xi_{\pi(2j)}).
\end{aligned}$$

Thus, by (5.7), $\omega^{\mathfrak{F}}_j(z) = 2z_j \cdot \sum_{i=1}^{N} \alpha^{\mathfrak{F}}_{ji} z_i$. □

Finally, we note that instead of considering the set of Feynman diagrams $\mathfrak{F}_{m,n}$ it suffices to study only its subset. Denote by

$$\mathfrak{F}^{true}_{m,n} \subset \mathfrak{F}_{m,n}$$

the subset of diagrams $\mathfrak{F}$ for which the corresponding incidence matrix $\alpha^{\mathfrak{F}}$ is not only S-regular, but is SS-regular (see Definition 5.2). In view of (5.15) we have:

Proposition 5.6. *If $\mathfrak{F} \notin \mathfrak{F}^{true}_{m,n}$, then $\mathcal{Z}(\mathfrak{F}) = \varnothing$. If $\mathfrak{F} \in \mathfrak{F}^{true}_{m,n}$, then $\mathcal{Z}(\mathfrak{F})$ is the complement to the union of a finite system of linear subspaces of positive codimension. In particular, it is is an open dense subset of $\mathbb{R}^{dN}$.*

Due to (5.14), Proposition 5.6 implies that the integrals $J_s(\mathfrak{F})$ from (4.10) with $\mathfrak{F} \notin \mathfrak{F}^{true}_{m,n}$ vanish.

5.3. **Final formula for expectations $\mathbb{E}a_s^{(m)}\bar{a}_s^{(n)}$.** Let $F_s^{\mathfrak{F}}(\tau, l, z)$ be the function $F^{\mathfrak{F}}(\tau, l, \xi)$, defined in (4.7), restricted to the set $\mathcal{A}'_s(\mathfrak{F})$ and written in the z-coordinates,

$$F_s^{\mathfrak{F}}(\tau, l, z) = F^{\mathfrak{F}}\big(\tau, l, \xi(z)\big). \tag{5.17}$$

The function $F_s^{\mathfrak{F}}$ depends on $s \in \mathbb{R}^d$ through the transformation $z \mapsto \xi$, see (5.13). Changing in the sums $J_s(\mathfrak{F})$ the coordinates ξ to z and using (1.28) we see that Lemma 4.1 joined with (5.14) and Propositions 5.6, 5.5 implies the following result (we recall (2.14)):

Theorem 5.7. *For any integers $m, n \geq 0$ satisfying $N = m + n \geq 1$, $s \in \mathbb{Z}^d_L$ and $\tau_1, \tau_2 \geq -T$, we have*

$$\mathbb{E}a_s^{(m)}(\tau_1)\bar{a}_s^{(n)}(\tau_2) = \sum_{\mathfrak{F} \in \mathfrak{F}^{true}_{m,n}} c_{\mathfrak{F}} J_s(\mathfrak{F}), \tag{5.18}$$

where $\mathfrak{F}^{true}_{m,n}$ is a finite set of diagrams, defined in Section 5.2, the constants $c_{\mathfrak{F}} \in \{\pm 1, \pm i\}$ are defined in Section 4.2 and

$$J_s(\tau; \mathfrak{F}) = \int_{\mathbb{R}^N} dl\, L^{-Nd} \sum_{z \in \mathcal{Z}(\mathfrak{F}) \cap \mathbb{Z}^{Nd}_L} F_s^{\mathfrak{F}}(\tau, l, z)\, \Theta(\omega^{\mathfrak{F}}(z), \nu^{-1} l). \tag{5.19}$$

Here $\Theta(\omega^{\mathfrak{F}}, \nu^{-1} l) = \prod_{j=1}^N \theta(\omega_j^{\mathfrak{F}}, \nu^{-1} l_j)$, while components $\omega_j^{\mathfrak{F}}$ of the vector $\omega^{\mathfrak{F}}$ are given by (5.16). The set $\mathcal{Z}(\mathfrak{F})$ is defined in (5.15) and the incidence matrix $\alpha^{\mathfrak{F}}$ from (5.15), (5.16) is SS-regular. The density function $F_s^{\mathfrak{F}}$ is real and satisfies

$$|\partial_s^\mu \partial_z^\kappa F_s^{\mathfrak{F}}(\tau, l, z)| \leq C^{\#}_{\mu,\kappa}(s) C^{\#}_{\mu,\kappa}(z)\, e^{-\delta\left(\sum_{i=1}^m |\tau_1 - l_i| + \sum_{i=m+1}^N |\tau_2 - l_i|\right)} \tag{5.20}$$

with a suitable $\delta = \delta_N > 0$, for any vectors $\mu \in \mathbb{Z}^d_+$, $\kappa \in \mathbb{Z}^{dN}_+$, and any $s \in \mathbb{R}^d$, $z \in \mathbb{R}^{Nd}$.

Since the density $F_s^{\mathfrak{F}}$ in (5.19) is an affine function of $s \in \mathbb{R}^d$ and satisfies (5.20), the sum $J_s(\mathfrak{F})$ is a Schwartz function of $s \in \mathbb{R}^d$. Thus, the correlations $\mathbb{E}a_s^{(m)}(\tau_1)\bar{a}_s^{(n)}(\tau_2)$ extends to a Schwartz function of $s \in \mathbb{R}^d$ via the

equality (5.18), as is affirmed in Theorem 2.2. In view of this result, below we always assume that $s \in \mathbb{R}^d$.

Subsequent analysis of the sums $J_s(\mathfrak{F})$ is based on the lemma below, where the function $\Theta(\omega^{\mathfrak{F}}(z), \nu^{-1}l)$ is defined as in (3.13) with $\theta(y,t) = e^{ity}$ (cf. (2.11)).

Lemma 5.8. *If $\theta(y,t) = e^{ity}$, then*

$$\Theta(\omega^{\mathfrak{F}}(z), \nu^{-1}l) = e^{i\nu^{-1}\Omega^{\mathfrak{F}}(l,z)}, \tag{5.21}$$

where the phase function $\Omega^{\mathfrak{F}}$ is given by

$$\Omega^{\mathfrak{F}}(l,z) = \sum_{1\le i,j\le N} \alpha^{\mathfrak{F}}_{ij}\,(l_i - l_j)\, z_i \cdot z_j. \tag{5.22}$$

Proof. Since $\theta(x,t) = e^{itx}$, the function Θ is given by (5.21) with $\Omega^{\mathfrak{F}}(l,z) = \sum_{j=1}^N l_j \omega^{\mathfrak{F}}_j(z)$. From (5.16) it follows that $\Omega^{\mathfrak{F}} = 2\sum_{1\le i,j\le N} l_j \alpha^{\mathfrak{F}}_{ji}\, z_j \cdot z_i$. Now (5.22) follows from the skew symmetry of the matrix $\alpha^{\mathfrak{F}}$. □

5.4. **Continuous approximation.** From now on we always assume that

$$\theta(y,t) = e^{ity}.$$

Using that by Proposition 5.6 $\mathcal{Z}(\mathfrak{F})$ with $\mathfrak{F} \in \mathfrak{F}^{true}_{m,n}$ is an open dense subset of $\mathbb{R}^{dN}$, we approximate the sum $L^{-Nd}\sum_{z\in\mathcal{Z}(\mathfrak{F})\cap\mathbb{Z}^{Nd}_L}$ from (5.19) by an integral over $\mathbb{R}^{dN}$ by applying Theorem 3.1 from [9]. For $d \ge 2$ we get

$$\left| J_s(\mathfrak{F}) - \int_{\mathbb{R}^N} dl \int_{\mathbb{R}^{dN}} dz\, F^{\mathfrak{F}}_s(\tau,l,z) e^{i\nu^{-1}\Omega^{\mathfrak{F}}(l,z)} \right| \le C^{\#}(s) L^{-2}\nu^{-2},$$

where we used (5.21). When $d = 1$ the r.h.s. of the inequality above should be modified by adding the term $C^{\#}(s)L^{-1}$, see Remark 3.2 in [9].

Together with (5.18) this estimate implies the main result of the section:

Theorem 5.9. *Assume that $d \ge 2$ and the function θ is as above. Then for any integers $m,n \ge 0$ satisfying $N = m+n \ge 1$, any $s \in \mathbb{R}^d$ and $\tau_1,\tau_2 \ge -T$, we have*

$$\left| \mathbb{E} a^{(m)}_s(\tau_1)\bar{a}^{(n)}_s(\tau_2) - \sum_{\mathfrak{F}\in\mathfrak{F}^{true}_{m,n}} c_{\mathfrak{F}} \tilde{J}_s(\mathfrak{F}) \right| \le C^{\#}(s) L^{-2}\nu^{-2}, \tag{5.23}$$

where

$$\tilde{J}_s(\tau;\mathfrak{F}) = \int_{\mathbb{R}^N} dl \int_{\mathbb{R}^{dN}} dz\, F^{\mathfrak{F}}_s(\tau,l,z) e^{i\nu^{-1}\Omega^{\mathfrak{F}}(l,z)}. \tag{5.24}$$

The constant $c_{\mathfrak{F}} = c_{\mathfrak{D}_{\mathfrak{F}}}$ is defined in (3.13) and the diagram $\mathfrak{D}_{\mathfrak{F}}$ – in (4.2). The real valued density $F^{\mathfrak{F}}_s$ is given by (5.17) together with (4.7) and satisfies the estimate (5.20). The phase function $\Omega^{\mathfrak{F}}$ is given by (5.22) where the incidence matrix $\alpha^{\mathfrak{F}}$ is SS-regular.

If $d = 1$ then the r.h.s. of (5.23) should be modified by adding to it the term $C^{\#}(s)L^{-1}$.

Theorem 5.9 provides an explicit formula for a correlation $\mathbb{E}a_s^{(m)}\bar{a}_s^{(n)}$ with arbitrary m and n, up to an error term of the size $O(L^{-2}\nu^{-2})$. In Section 8.4 of the arXiv version [10] of this paper we consider an example when $m = n = 2$.

Since the functions $F^{\mathfrak{F}}$ and $\Omega^{\mathfrak{F}}$ are given by explicit formulas (5.17) and (5.22), the integral over the variable l in (5.24) can be found explicitly. In Section 8.4 we give the result of this computation, relevant for further study of the energy spectrum of the quasisolutions.

6. Main estimate

6.1. **Estimation of integrals $\tilde{J}_s(\mathfrak{F})$.** Next we estimate the integrals $\tilde{J}_s(\mathfrak{F})$ from Theorem 5.9.

Theorem 6.1. *For any integers $m, n \geq 0$, satisfying $N = m + n \geq 2$, any $\mathfrak{F} \in \mathfrak{F}_{m,n}^{true}$ and $s \in \mathbb{R}^d$,*

$$|\tilde{J}_s(\mathfrak{F})| \leq C^{\#}(s)\nu^{\min(\lceil N/2\rceil, d)}\chi_d^N(\nu), \tag{6.1}$$

uniformly in $\tau_1, \tau_2 \geq -T$, wherethe function χ_d^N is defined in (1.15).

Together with Theorem 5.9, estimate (6.1) implies the desired inequality (2.8) and concludes the proof of Theorems 2.2 and 1.1. [10]

We deduce (6.1) from an abstract theorem below where we estimate integrals of the following more general form. Let

$$J_s = \int_{\mathbb{R}^K} dl \int_{\mathbb{R}^{dM}} dz\, F_s(l, z) e^{i\nu^{-1}Q(l,z)}, \tag{6.2}$$

where $M \geq 2$, $K \geq 1$, $d \geq 1$, $l = (l_1, \dots, l_K)$ and z is the polyvector $(z_1, \dots, z_M)$ with $z_i = (z_i^1, \dots, z_i^d) \in \mathbb{R}^d$. The phase function $Q(l, z)$ is assumed to be linear in l and quadratic in z,

$$Q(l, z) = \big(Q(l)z\big) \cdot z = \sum_{i,j=1}^{M} (q_{ij} \cdot l)\,(z_i \cdot z_j) = \sum_{i,j=1}^{M} q_{ij} \cdot l \sum_{m=1}^{d} z_i^m z_j^m .$$

Here $q_{ij} = q_{ji} = (q_{ij}^k)_{1\leq k\leq K}$ are K-vectors, so for $l \in \mathbb{R}^K$, $Q(l) = \big(q_{ij} \cdot l\big)_{1\leq i,j\leq M}$ is a real symmetric $M \times M$–matrix. Denoting by Q_k, $1 \leq k \leq K$, the real symmetric matrices

$$Q_k = (q_{ij}^k)_{1\leq i,j\leq M}\,,$$

we write $Q(l)$ as $\sum_k Q_k l_k$. We denote by $R \leq K$ the rank of the system of vectors $q_{ij} \in \mathbb{R}^K$, $1 \leq i, j \leq M$. It equals to the rank of the system of

[10] We recall that $\mathfrak{F}_{m,n}^{true} = \varnothing$ for $N = 1$, see (4.12), so we study only the case $N \geq 2$.

matrices $\{Q_k, 1 \le k \le K\}$ in the space of $M \times M$ matrices. Indeed, consider the row

$$(Q_1, \dots, Q_K),$$

interpreting it as a matrix with columns Q_j (regarded as vectors in $\mathbb{R}^{M^2}$). Then the two ranks we talk about are the column- and raw-ranks of this matrix, so they coincide.

Note that the phase function $\Omega^{\mathfrak{F}}$ in (5.22) may be written as $Q(l, z)$.

The complex density function $F_s(l, z)$ in (6.2) is assumed to be bounded and measurable in $l \in \mathbb{R}^K$, $z \in \mathbb{R}^{dN}$, smooth in z, depend on the parameter $s \in \mathbb{R}^d$, and satisfy

$$|\partial_z^\kappa F_s(l, z)| \le C_\kappa^\#(s) C_\kappa^\#(z) h_\kappa(|l - l_0|), \tag{6.3}$$

for any $\kappa \in \mathbb{Z}_+^{dM}$ and some $l_0 \in \mathbb{R}^K$. Each function h_κ is assumed to be monotone decreasing, bounded and such that the function $h_\kappa(|l|)$ is integrable over $\mathbb{R}^K$. Then

$$0 \le h_\kappa \le C_\kappa, \qquad \int_0^\infty h_\kappa(r) r^{K-1}\, dr \le C_\kappa, \tag{6.4}$$

for appropriate constants $C_\kappa > 0$. For $k \in \mathbb{N}$ we introduce the functions

$$\psi_d^k(\nu) = \begin{cases} -\ln \nu & \text{if} \quad k = d, \\ 1 & \text{otherwise.} \end{cases} \tag{6.5}$$

Theorem 6.2. *Assume that* $\operatorname{tr} Q_k = 0$ *for each* k*. Then, under assumption* (6.3)

$$|J_s| \le C^\#(s) \nu^{\min(R,d)} \psi_d^R(\nu). \tag{6.6}$$

The function $C^\#$ *depends on* F_s *only through the functions* $C_\kappa^\#$ *and the constants* C_κ *in* (6.3), (6.4)*. It also depends on the tensor* (q_{ij}^k)*.*

Theorem 6.2 is proven in the next subsection. In Appendix 8.5 we note that it implies an asymptotic estimate for parameter-depending integrals of some quotients with asymptotically degenerating divisors, including those which are obtained from the integrals $\tilde{J}_s(\mathfrak{F})$ by explicit integration over $\mathbb{R}_l^N$. That result is not needed for the present paper, however we believe that it is of independent interest. It is related to [18].

In view of (5.20), the integrals $\tilde{J}_s(\mathfrak{F})$ satisfy the assumptions of Theorem 6.2 with the functions $C_\kappa^\#$ and the constants C_κ independent from τ_1, τ_2. Here, in view of (5.22), we have $q_{ij} \cdot l = \alpha_{ij}^{\mathfrak{F}}(l_i - l_j)$ and it can be shown that the rank R satisfies $R \ge \lceil N/2 \rceil$ (this follows solely from the fact that the matrix $\alpha^{\mathfrak{F}}$ is skew-symmetric and does not have zero rows and columns since it is SS-regular). So Theorem 6.2 implies estimate (6.1), where the function χ_d^N is replaced by ψ_d^R. In the case $R = d = \lceil N/2 \rceil$ (which holds true for

certain matrices $\alpha^{\tilde{\mathfrak{F}}}$) this estimate is slightly weaker than (6.1). Indeed, it gives

$$|\tilde{J}_s(\mathfrak{F})| \le C^{\#}(s)\nu^{\lceil N/2\rceil}\ln\nu^{-1},$$

so compare to (6.1) we get an additional factor $\ln\nu^{-1}$, unless $N = 2, 3$. This is insufficient for the purposes of [9]. To overcome this difficulty, in Theorem 7.1 we prove that if the tensor (q_{ij}^k) has a block-diagonal form, the estimate (6.6) improves and implies (6.1). This is mostly a technical issue, and ideologically (6.1) follows from (6.6). So we postpone Theorem 7.1 till the next Section 7 and first establish estimate (6.6).

6.2. **Proof of Theorem 6.2.** In this proof the positive constants C, C_1, etc. and the functions $C^{\#}$ depend on F_s as in Theorem 6.2. To simplify presentation we assume that in (6.3) $l_0 = 0$, but emphasize that the estimates we obtain are uniform in l_0.

Let $\lambda_1(l), \dots, \lambda_M(l)$ be (real) continuous in l eigenvalues of the symmetric matrix $Q(l)$, enumerated in decreasing order:

$$|\lambda_1(l)| \ge |\lambda_2(l)| \ge \ldots \ge |\lambda_M(l)| \quad \text{for any} \quad l \in \mathbb{R}^K. \tag{6.7}$$

The corresponding normalized eigenvectors are vector–functions of l, forming an orthonormal basis in $\mathbb{R}^M$ for every l, and analytic outside an analytic subset of $\mathbb{R}^K_l$ of positive codimension (while as functions of $l \in \mathbb{R}^k$ they even are not continuous). Denoting by $w(l, z) = w = (w_1, \dots, w_M)$, $w_i \in \mathbb{R}^d$, the polyvector $z = (z_1, \dots, z_M)$, written in this basis, we write function Q as

$$Q(l, z) = \sum_{i=1}^{M} \lambda_i(l)|w_i|^2.$$

Since the transformation $z \mapsto w$ is orthogonal for every l, then $dz = dw$ and

$$J_s = \int_{\mathbb{R}^K} dl \int_{\mathbb{R}^{dM}} dw\, H_s(l, w) e^{i\nu^{-1}\sum_{i=1}^M \lambda_i(l)|w_i|^2}. \tag{6.8}$$

Here $H_s(l, w) = F_s(l, z(l, w))$, so that the function $H_s(l, w)$ again satisfies the estimate (6.3). The following observation plays the key role in the proof.

Lemma 6.3. *Let $A = (a_{ij})$ be a real symmetric $M \times M$-matrix, $M \ge 2$, such that $\operatorname{tr} A = 0$, and let $\lambda_1, \dots, \lambda_M$ be its eigenvalues. Then*

$$\sum_{i,j} a_{ij}^2 = -2\sum_{i<j} \lambda_i\lambda_j. \tag{6.9}$$

Proof. Let $\|A\|_{HS}$ be the Hilbert-Schmidt norm of the matrix A. From one hand, $\|A\|_{HS}^2$ equals to the l.h.s. of (6.9). From another hand,

$$\|A\|_{HS}^2 = \sum_{i=1}^{M} \lambda_i^2 = \Big(\sum_{i=1}^{M} \lambda_i\Big)^2 - 2\sum_{i<j} \lambda_i\lambda_j.$$

It remains to note that $\Big(\sum_{i=1}^M \lambda_i\Big)^2 = (\operatorname{tr} A)^2 = 0$. □

Applying Lemma 6.3 to the matrix $Q(l)$, we find

$$\frac{1}{2}\sum_{i,j}\left(q_{ij}\cdot l\right)^2 = -\sum_{i<j}\lambda_i(l)\lambda_j(l) =: \mathcal{K}(l) \geq 0.$$

Let us consider the sets

$$\mathcal{N}_\nu := \{l \in \mathbb{R}^K : \mathcal{K}(l) \geq \nu^2\} \quad \text{and} \quad \mathcal{N}_\nu^c := \mathbb{R}^K \setminus \mathcal{N}_\nu.$$

Below for a set $V \subset \mathbb{R}_l^K$ we denote by $\langle J_s, V\rangle$ the integral (6.8) with the domain of integration $\mathbb{R}_l^K$ replaced by V. Recall that R is the rank of the system of K–vectors (q_{ij}).

Lemma 6.4. *Under assumption* (6.3) *we have* $|\langle J_s, \mathcal{N}_\nu^c\rangle| \leq C^{\#}(s)\nu^R$.

Proof of Lemma 6.4 is given at the end of this section.
It remains to establish estimate (6.6) for the integral $\langle J_s, \mathcal{N}_\nu\rangle$. In view of (6.7),

$$|\lambda_1\lambda_2| \geq M^{-2}\sum_{i<j}\left|\lambda_i\lambda_j\right| \geq M^{-2}\mathcal{K}. \tag{6.10}$$

So for $l \in \mathcal{N}_\nu$ we have $|\lambda_1(l)\lambda_2(l)| \geq M^{-2}\nu^2$. It is known (see [7, 16]) that the (generalized) Fourier transform of the function

$$\mathbb{R}^{2d} \ni (w_1, w_2) \mapsto e^{-i\nu^{-1}(\lambda_1|w_1|^2+\lambda_2|w_2|^2)}$$

is

$$\zeta(\pi\nu)^d(|\lambda_1\lambda_2|)^{-d/2}e^{i\nu(\lambda_1^{-1}|\xi_1|^2+\lambda_2^{-1}|\xi_2|^2)/4},$$

where ζ is a complex constant of unit norm; here we define the Fourier transform of a regular function $u(x)$ as $\int e^{-ix\cdot\xi}u(x)\,dx$. Inspired by the stationary phase method [7, 16], we apply the Parseval's identity to the partial integral over w_1, w_2 and get

$$\begin{aligned}&\int_{\mathbb{R}^{2d}} H_s(l,w)e^{i\nu^{-1}(\lambda_1|w_1|^2+\lambda_2|w_2|^2)}\,dw_1dw_2\\ &= \zeta\frac{(\nu/4\pi)^d}{|\lambda_1\lambda_2|^{d/2}}\int_{\mathbb{R}^{2d}}\hat{H}_s^{1,2}(l,\xi_1,\xi_2,w_{\geq 3})e^{-i\frac{\nu}{4}(\lambda_1^{-1}|\xi_1|^2+\lambda_2^{-1}|\xi_2|^2)}\,d\xi_1d\xi_2,\end{aligned}$$

for any $w_{\geq 3} = (w_3, \ldots, w_M)$, where $\hat{H}_s^{1,2}$ stands for the Fourier transform of function H_s with respect to the variables w_1, w_2.[11] Since H_s satisfies assumption (6.3) (where $l_0 = 0$), the r.h.s. above is bounded in absolute value by

$$C^{\#}(s)C^{\#}(w_{\geq 3})\frac{\nu^d}{|\lambda_1\lambda_2|^{d/2}}h(|l|),$$

[11] See Section 12.2 in [9] for justification of this formula without using the generalized Fourier transform.

where the function $h \geq 0$ satisfies (6.4). So

$$|\langle J_s, \mathcal{N}_\nu\rangle| \leq \int_{\mathcal{N}_\nu} \frac{C^{\#}(s)\nu^d h(|l|)}{|\lambda_1(l)\lambda_2(l)|^{d/2}}\, dl \int_{\mathbb{R}^{(M-2)d}} C^{\#}(w_{\geq 3})\, dw_{\geq 3}$$
$$\leq C_1^{\#}(s)\nu^d \int_{\mathcal{N}_\nu} \frac{h(|l|)}{|\lambda_1(l)\lambda_2(l)|^{d/2}}\, dl.$$

Then, due to (6.10), we have

$$|\langle J_s, \mathcal{N}_\nu\rangle| \leq C^{\#}(s)\nu^d \int_{\mathcal{N}_\nu} \frac{h(|l|)}{(\mathcal{K}(l))^{d/2}}\, dl =: C^{\#}(s) I^\nu.$$

Lemma 6.5. *Assume that the function h satisfies assumption (6.4). Then $I^\nu \leq C\nu^{\min(R,d)}\psi_d^R(\nu)$.*

Lemma 6.5 together with Lemma 6.4 concludes the proof of the theorem. The lemmas are established below. □

Proof of Lemma 6.4. Since function H_s satisfies (6.3) with $l_0 = 0$, then

$$|\langle J_s, \mathcal{N}_\nu^c\rangle| \leq C^{\#}(s) \int_{\mathcal{N}_\nu^c} h_0(|l|)\, dl.$$

Let us denote by $v_1, \dots, v_R$ a maximal collection of linearly independent vectors from the set $(q_{ij})_{1\leq i,j\leq M}$. In the integral $\langle J_s, \mathcal{N}_\nu^c\rangle$ we make a linear (non-degenerate) change of variable from l to $t = (t_1, \dots, t_K)$ in such a way that $t_k = v_k \cdot l$ for $k \leq R$. Let $t^< = (t_1, \dots, t_R)$ and $t^> = (t_{R+1}, \dots, t_K)$. Due to the identity $\mathcal{K}(l) = \frac{1}{2}\sum (q_{ij}\cdot l)^2$,

$$\frac{|t^<|^2}{2} \leq \mathcal{K}(l(t)) \leq C|t^<|^2. \tag{6.11}$$

So denoting by $\mathcal{N}_{\nu,t}^c$ the set $\mathcal{N}_\nu^c$ written in the t-coordinates, we obtain $\mathcal{N}_{\nu,t}^c \subset \{t : |t^<| < \sqrt{2}\nu\}$. Since the function h_0 is decreasing, inequalities $|l| \geq C|t|$ and $|t| \geq |t^>|$ imply that $h_0(|l|) \leq h_0(C|t|) \leq h_0(C|t^>|)$. Then

$$|\langle J_s, \mathcal{N}_\nu^c\rangle| \leq C^{\#}(s) \int_{t^< \in \mathbb{R}^R:\, |t^<| < \sqrt{2}\nu} dt^< \int_{\mathbb{R}^{K-R}} h_0(C|t^>|)\, dt^>$$
$$\leq C_1^{\#}(s)\nu^R \int_{\mathbb{R}^{K-R}} h_0(C|t^>|)\, dt^>.$$

The last integral is bounded by $C\int_0^\infty h_0(r) r^{K-R-1}\, dr < C_1$, in view of assumption (6.4). □

Proof of Lemma 6.5. We make the change of coordinates $l \mapsto t$ as in the proof of Lemma 6.4 and use the notation $t^<$, $t^>$, introduced there. Denoting by $\mathcal{N}_{\nu,t}$ the set $\mathcal{N}_\nu$ written in the t-coordinates, in view of (6.11) we find $\mathcal{N}_{\nu,t} \subset \{t : |t^<| \geq C'\nu\}$. Then, using that $h(|l|) \leq h(C_1|t|)$, by (6.11) we get

$$I^\nu \leq 2^{d/2}\nu^d \int_{t:\, |t^<| \geq C\nu} \frac{h(C_1|t|)}{|t^<|^d}\, dt.$$

Since the function $h(|t|)$ is integrable, the r.h.s. above is bounded by $C\nu^d$, if we take the integral only over the set where $|t^<| \geq 1$. Then, it remains to get the desired estimate for the integral over the set where $C\nu \leq |t^<| \leq 1$. Since the function h is decreasing, we have $h(C_1|t|) \leq h(C_1|t^>|)$, so this integral is bounded by

$$2^{d/2}\,\nu^d \int\limits_{t\in\mathbb{R}^K:\, C\nu\leq|t^<|\leq 1} \frac{h(C_1|t^>|)}{|t^<|^d}\,dt \leq C_2\nu^d \int\limits_{t^<\in\mathbb{R}^R:\, C\nu\leq|t^<|\leq 1} \frac{1}{|t^<|^d}\,dt^<,$$

where we have integrated out the variable $t^>$. Passing to the spherical coordinates, we see that the latter integral is bounded by

$$C_3\nu^d \int_{C\nu}^1 \frac{r^{R-1}}{r^d}\,dr \leq \begin{cases} C_4\nu^d & \text{if} \quad d \leq R-1 \\ C_4\nu^d \ln\nu^{-1} & \text{if} \quad d = R \\ C_4\nu^R & \text{if} \quad d > R. \end{cases}$$

The r.h.s. of the last inequality can be rewritten as $C_4\nu^{\min(R,d)}\psi_d^R(\nu)$. □

7. Refinement of estimate (6.6) and proof of Theorem 6.1

In this section we establish Theorem 7.1, in which we improve estimate (6.6), and deduce from it Theorem 6.1.

7.1. **Refinement of (6.6).** We consider integral (6.2), where to simplify the formulation of result (and since it suffices for our purpose) we assume $M = K$ and denote $N := M = K \geq 2$. We consider a decomposition of the set $\{1,\dots N\}$ into $p \geq 1$ non-empty disjoint subsets

$$\{1,\dots N\} = \mathcal{I}_1 \cup \dots \cup \mathcal{I}_p, \tag{7.1}$$

and denote by l^k the vector $l^k = (l_i)_{i\in\mathcal{I}_k}$.

We assume that the phase function Q admits a block-decomposition

$$Q(l,z) = \sum_{k=1}^p U_k(l,z), \qquad U_k(l,z) = \sum_{i,j\in\mathcal{I}_k} (q_{ij}\cdot l^k)\,(z_i\cdot z_j),$$

where $q_{ij} = q_{ji} \in \mathbb{R}^{|\mathcal{I}_k|}$ for $i,j \in \mathcal{I}_k$. Concerning the density F_s we assume that it satisfies the estimate

$$|\partial_z^\kappa F_s(l,z)| \leq C_\kappa^\#(s)C_\kappa^\#(z)\prod_{i=1}^p h_\kappa^i(|l^i - l_0^i|), \tag{7.2}$$

for any κ and some vectors $l_0^i \in \mathbb{R}^{|\mathcal{I}_i|}$. The functions h_κ^i are assumed to satisfy the same assumption that we imposed on the functions h_κ in (6.3), where in (6.4) we replace K by $|\mathcal{I}_i|$. Denote by R_k the rank of the system of $|\mathcal{I}_k|$-vectors $(q_{ij})_{i,j\in\mathcal{I}_k}$.

Theorem 7.1. *Assume that* $\sum_{i\in\mathcal{I}_k} q_{ii} = 0 \in \mathbb{R}^{|\mathcal{I}_k|}$ *for any* $1 \le k \le p$. *Then, under assumption* (7.2) *we have*

$$|J_s| \le C^{\#}(s)\prod_{k=1}^{p}\nu^{\min(R_k,d)}\psi_d^{R_k}(\nu),$$

where the function $C^{\#}$ *depends on* F_s *as in Theorem 6.2.*

Proof. We argue by induction. In the case $p = 1$ the assertion follows from Theorem 6.2. Assume that theorem is proven for the case of $p-1$ subsets $\mathcal{I}_k$ in (7.1) and let us establish it for the case of p subsets. We recall the notation $l^p = (l_i)_{i\in\mathcal{I}_p}$ and set $l^< = (l_i)_{i\notin\mathcal{I}_p}$. Similarly, we denote $z^p = (z_i)_{i\in\mathcal{I}_p}$ and $z^< = (z_i)_{i\notin\mathcal{I}_p}$. Let $N_p := |\mathcal{I}_p|$ and

$$I_s(l^<, z^<) = \int_{\mathbb{R}^{N_p}} dl^p \int_{\mathbb{R}^{dN_p}} dz^p\, F_s(l,z)e^{i\nu^{-1}U_p(l,z)},$$

where we recall that the function $U_p(l,z)$ depends only on l^p and z^p. Then,

$$\partial^{\kappa}_{z^<} I_s(l^<, z^<) = \int_{\mathbb{R}^{N_p}} dl^p \int_{\mathbb{R}^{dN_p}} dz^p\, \left(\partial^{\kappa}_{z^<} F_s(l,z)\right) e^{i\nu^{-1}U_p(l,z)},$$

for any κ. Literally repeating the proof of Theorem 6.2, we get

$$|\partial^{\kappa}_{z^<} I_s(l^<, z^<)| \le C^{\#}(s)C^{\#}(z^<)\prod_{i=1}^{p-1} h^i(|l^i - l^i_0|)g(\nu),$$

where $g(\nu) = \nu^{\min(R_p,d)}\psi_d^{R_p}(\nu)$ and the functions h^i satisfy the same assumptions that the functions h^i_κ from (7.2). Thus, we see that the function $G_s(l^<, z^<) = (g(\nu))^{-1} I_s(l^<, z^<)$ satisfies assumption (7.2) with $p = p-1$. Note that

$$J_s = g(\nu)\int_{\mathbb{R}^{N-N_p}} dl^< \int_{\mathbb{R}^{d(N-N_p)}} dz^<\, G_s(l^<, z^<)e^{i\nu^{-1}\sum_{k=1}^{p-1}U_k(l,z)},$$

where $U_k(l,z)$ depend only on $l^<, z^<$. Then, by the induction hypothesis,

$$|J_s| \le C^{\#}(s)g(\nu)\prod_{k=1}^{p-1}\nu^{\min(R_k,d)}\psi_d^{R_k}(\nu) = C^{\#}(s)\prod_{k=1}^{p}\nu^{\min(R_k,d)}\psi_d^{R_k}(\nu).$$

□

7.2. **Proof of Theorem 6.1.** Recall that the phase function $\Omega^{\mathfrak{F}}$ from the definition (5.24) of the integral $\tilde{J}_s(\mathfrak{F})$ is given by (5.22) and that for $\mathfrak{F} \in \mathfrak{F}^{true}_{m,n}$ the matrix $\alpha^{\mathfrak{F}}$ is SS-regular (see Denition 5.2). Consider a partition (7.1) with $p \ge 1$ such that the matrices $\alpha_k = (\alpha^{\mathfrak{F}}_{ij})_{i,j\in\mathcal{I}_k}$ are irreducible and $\alpha^{\mathfrak{F}}_{ij} = 0$ once $i \in \mathcal{I}_k$ and $j \in \mathcal{I}_r$ with $k \ne r$. Then the integral $\tilde{J}_s(\mathfrak{F})$ satisfies assumptions of Theorem 7.1 with $q_{ij}\cdot l = \alpha^{\mathfrak{F}}_{ij}(l_i - l_j)$.

Lemma 7.2. *Let $A = (a_{ij})$ be an irreducible $n \times n$-matrix, $n \geq 2$, and the vectors $r_{ij} \in \mathbb{R}^n$ be given by $r_{ij} = a_{ij}(e_i - e_j)$, where $(e_i)_{1 \leq i \leq n}$ is a basis in $\mathbb{R}^n$. Then rank R of the system of n-vectors $(r_{ij})_{1 \leq i,j \leq n}$ equals $n - 1$.*

Proof of Lemma 7.2 is a simple exercise from linear algebra and we postpone it to Appendix 8.3. Set $N_k = |\mathcal{I}_k|$. Since the $N_k \times N_k$-matrices α_k are skew-symmetric and non-zero, we have $N_k \geq 2$. Applying Lemma 7.2 to the system of vectors $(q_{ij})_{i,j \in \mathcal{I}_k}$, we see that its rank R_k equals to $N_k - 1$. Then by Theorem 7.1 we get the estimate

$$|\tilde{J}_s(\mathfrak{F})| \leq C^{\#}(s) \prod_{i=1}^{p} \nu^{\min(N_i - 1, d)} \psi_d^{N_i - 1}(\nu). \tag{7.3}$$

Let us show that (7.3) implies the desired estimate (6.1). Assume first that $N_i - 1 < d$ for every $1 \leq i \leq p$. Then, in view of (7.3) and the identity $\sum_{i=1}^{p} N_i = N$, we have

$$|\tilde{J}_s(\mathfrak{F})| \leq C^{\#}(s) \nu^{N-p}.$$

Since $N_i \geq 2$ for any i, the number of blocks p satisfies $p \leq \lfloor N/2 \rfloor$, where $\lfloor \cdot \rfloor$ denotes the (lower) entire part. Then $N - p \geq \lceil N/2 \rceil$, so we get $\nu^{N-p} \leq \nu^{\lceil N/2 \rceil}$ which implies (6.1).

Now we assume that

$$N_j - 1 \geq d \quad \text{for some} \quad 1 \leq j \leq p, \tag{7.4}$$

so $\nu^{\min(N_j - 1, d)} = \nu^d$. Since for every i we have $\nu^{\min(N_i - 1, d)} \leq \nu$, by (7.3) we find

$$|\tilde{J}_s(\mathfrak{F})| \leq C^{\#}(s) \nu^{d+p-1} \prod_{i=1}^{p} \psi_d^{N_i - 1}(\nu) \leq C_1^{\#}(s) \nu^{d+p-2}.$$

In the case $p \geq 2$ this implies (6.1). Now it remains to consider the case when (7.4) holds with $p = 1$. Then $N - 1 \geq d$. If $N - 1 > d$ then (7.3) with $p = 1$ implies (6.1). Otherwise $N - 1 = d$, so

$$|\tilde{J}_s(\mathfrak{F})| \leq C^{\#}(s) \nu^{N-1} \ln \nu^{-1}. \tag{7.5}$$

If $N - 1 > \lceil N/2 \rceil$ then $\nu^{N-1} \ln \nu^{-1} < C \nu^{\lceil N/2 \rceil}$ and we are done. The opposite situation is possible only if $N \leq 3$. Since $d = N - 1$, it takes place if $d = 2$, $N = 3$ or $d = 1$, $N = 2$. In these cases we find $N - 1 = \lceil N/2 \rceil = d$, which is 2 or 1 correspondingly, so again (7.5) implies (6.1). □

8. Addenda

8.1. **Optimality of the estimate in Theorem 2.2.** Throughout this section we assume that $N = m + n \geq 2$ and $d \geq 2$. Accordingly to Theorem 5.9, we have

$$\mathbb{E} a_s^{(m)}(\tau_1) \bar{a}_s^{(n)}(\tau_2) = \sum_{\mathfrak{F} \in \mathfrak{F}_{m,n}^{true}} \tilde{J}_s(\mathfrak{F}) + O\big(C^{\#}(s) L^{-2} \nu^{-2}\big), \tag{8.1}$$

where we recall that the integrals $\tilde{J}_s(\mathfrak{F})$ are given by (5.24) and $\mathfrak{F}^{true}_{m,n}$ is a certain subset of the set of Feynman diagrams $\mathfrak{F}$ (which are defined in Section 4), associated with the product $a^{(m)}_s \bar{a}^{(n)}_s$. Each integral $\tilde{J}_s(\mathfrak{F})$ satisfies the estimate (6.1), and this implies the estimate in Theorem 2.2. Since $\nu \ln \nu^{-1} \le C$ these estimates admit the following slightly weaker forms:

$$
\begin{aligned}
&|\tilde{J}_s(\mathfrak{F})| \le C^{\#}(s)\nu^{\min(N/2,d)} \\
&\big|\mathbb{E} a^{(m)}_s(\tau_1)\bar{a}^{(n)}_s(\tau_2)\big| \le C^{\#}(s)\big(\nu^{-2}L^{-2} + \nu^{\min(N/2,d)}\big).
\end{aligned}
\tag{8.2}
$$

In this appendix we discuss the question if it is possible to get rid of the minimum with d in the estimates above, that is, if it is true that

$$
|\tilde{J}_s(\mathfrak{F})| \le C^{\#}(s)\nu^{N/2}, \qquad \big|\mathbb{E} a^{(m)}_s(\tau_1)\bar{a}^{(n)}_s(\tau_2)\big| \le C^{\#}(s)\big(\nu^{-2}L^{-2} + \nu^{N/2}\big),
\tag{8.3}
$$

for any $N \ge 2$. We discussed the motivation for this question in Section 1.4.

Let $\mathfrak{F}^2_{m,n} \subset \mathfrak{F}^{true}_{m,n}$ be a subset of Feynman diagrams $\mathfrak{F}$ for which there is $q = q(\mathfrak{F})$ such that the matrix $\alpha^{\mathfrak{F}}$ from (5.22) satisfies $\alpha^{\mathfrak{F}}_{ij} = 0$ if both $i \ne q$ and $j \ne q$ (so, all elements of the matrix $\alpha^{\mathfrak{F}}$ outside the q-th line and column are zero). It can be shown that $\mathfrak{F}^2_{m,n} \ne \varnothing$ if $m, n \ge 1$.

Proposition 8.1. *If $m, n \ge 1$ and $N > 2d$, then for any Feynman diagram $\mathfrak{F} \in \mathfrak{F}^2_{m,n}$ we have $C^{\#}_1(s)\nu^d \le |\tilde{J}_s(\mathfrak{F})| \le C^{\#}_2(s)\nu^d$.*

Proposition 8.1 shows that the first estimate from (8.3) is false for $\mathfrak{F} \in \mathfrak{F}^2_{m,n}$ when $N > 2d$, so in general we have only (8.2). Then, if the large integrals $\tilde{J}_s(\mathfrak{F})$ of the size ν^d did not cancel each other in the sum from (8.1), the second estimate from (8.3) would also fail. Moreover, if they did not cancel in the sums from (1.25) that give n^k_s, then as we have discussed in Section 1.4, this would imply that the energy spectrum corresponding to a truncation $\sum_{i=0}^m \rho^i v^{(i)}_s$ with $m > d$ of the decomposition (1.12) cannot be approximated by a solution of the WKE (1.23). However, we find some strong cancellations in the sum from (8.1):

Proposition 8.2. *We have $\big|\sum_{\mathfrak{F}\in\mathfrak{F}^2_{m,n}} \tilde{J}_s(\mathfrak{F})\big| \le C^{\#}(s)\nu^{N-1}$ for $N \ge 2$.*

Since $\nu^{N-1} \le \nu^{N/2}$, this estimate does not prevent the second estimate from (8.3) to be true. We expect that for $N \gg 2d$ the integrals $\tilde{J}_s(\mathfrak{F})$ with $\mathfrak{F} \in \mathfrak{F}^2_{m,n}$ are the biggest (see below) but still there are many other diagrams $\mathfrak{F} \notin \mathfrak{F}^2_{m,n}$ for which the corresponding integrals also are large (bigger than $\nu^{N/2}$). Nevertheless, we expect that similar cancellations take place for them as well, but do not know if they go till the order $\nu^{N/2}$. So, we state the following

Problem 8.3. *Prove that for any $d \geq 2$ and m, n satisfying $N = m + n \geq 2$,*

$$\Big| \sum_{\mathfrak{F} \in \mathfrak{F}^{true}_{m,n}} \tilde{J}_s(\mathfrak{F}) \Big| \leq C^{\#}(s) \nu^{N/2}. \tag{8.4}$$

If (8.4) is true, we get the second estimate from (8.3). In particular, under the "kinetic" scaling $\rho = \sqrt{\varepsilon}\nu^{-1/2}$ the energy spectrum corresponding to a truncation $\sum_{i=0}^{m} \rho^i v_s^{(i)}$ of series (1.12) with any $m \geq 2$ can be approximated by a solution of the WKE (1.23) at least with the accuracy ε^2, if $L^{-2}\nu^{-2} \leq \nu^m$.

Discussion of proofs of Propositions 8.1, 8.2 and of Problem 8.3. Below we do not give proofs but present only their rough ideas. Let us consider the matrix $\alpha_l^{\mathfrak{F}} = \big(\alpha_{ij}^{\mathfrak{F}}(l_i - l_j)\big)_{1 \leq i,j \leq N}$, which depends on $l \in \mathbb{R}^N$ (see (5.22)). Since it is symmetric, it has N real eigenvalues, which we enumerate in the decreasing order $|\lambda_1(l)| \geq |\lambda_2(l)| \geq \ldots \geq |\lambda_N(l)| \geq 0$, for any $l \in \mathbb{R}^N$. It is possible to show that the set $\mathfrak{F}^2_{m,n}$ can be equivalently defined as follows. For $k \geq 1$ we denote by $\mathfrak{F}^k_{m,n}$ a set of Feynman diagrams $\mathfrak{F}$ for which the eigenvalues $\lambda_1(l), \ldots, \lambda_k(l)$ are not identically zero while $\lambda_i(l) \equiv 0$ for any $i \geq k+1$. Since the matrix $\alpha_l^{\mathfrak{F}}$ is non-zero but has zero trace, we have $\mathfrak{F}^1_{m,n} = \varnothing$, so $\mathfrak{F}^{true}_{m,n} = \cup_{k=2}^{N} \mathfrak{F}^k_{m,n}$.

Let us take a diagram $\mathfrak{F} \in \mathfrak{F}^2_{m,n}$ and consider the integral over dz from the definition (5.24) of $\tilde{J}_s(\mathfrak{F})$. Applying to it the Parseval identity in the eigendirections corresponding to the eigenvalues λ_1 and λ_2, and using that for $\mathfrak{F} \in \mathfrak{F}^2_{m,n}$ we have $\lambda_3 = \ldots = \lambda_N = 0$, we find that $\tilde{J}_s(\mathfrak{F}) \sim \nu^d \int_{\mathbb{R}^N} \dfrac{h_s^{\mathfrak{F}}(l)\, dl}{|\lambda_1(l)\lambda_2(l)|^{d/2}}$ if $N > 2d$, for an appropriate integrable function $h_s^{\mathfrak{F}}$, which is fast decaying in s. Due to Lemma 6.3, we have $\lambda_1\lambda_2 = -\frac{1}{2} \sum_{1 \leq i,j \leq N} (\alpha_{ij}^{\mathfrak{F}})^2 (l_i - l_j)^2$. Using this formula, we show that the latter integral converges, which proves Proposition 8.1.

Below by a graph of a Feynman diagram we mean the diagram, where we do not specify which vertices are virtual, which have positive degree and which are leaves, so the graph is just a collection of vertices coupled by edges. To establish Proposition 8.2, we note that, up to a renumbering of vertices, all diagrams $\mathfrak{F} \in \mathfrak{F}^2_{m,n}$ have the same graph which we denote by $\mathfrak{G}^2$. To recover the set of diagrams $\mathfrak{F}^2_{m,n}$ from this graph, it suffices to look over all possible placings of virtual vertices (once positions of the virtual vertices are known, degrees of the other vertices are recovered uniquely).

The graph $\mathfrak{G}^2$ is such that in a k-th block with any $k \neq 1, m+1$, the virtual vertex can be placed in two different positions, c_{2k} or $\bar{c}_{2k}$. Now, using the stationary phase method [7,16], we develop each integral $\tilde{J}_s(\mathfrak{F})$ in ν. Change of position of a virtual vertex inside of a block, roughly speaking, changes only the sign in first p orders of this decomposition. This is where

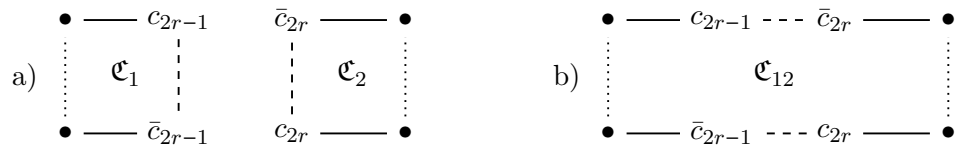

FIGURE 8. Construction of the cycle $\mathfrak{C}_{12}$ from the cycles $\mathfrak{C}_1$ and $\mathfrak{C}_2$. In this figure we do not make difference between solid and dashed edges.

the cancellations come from. The number p here equals to the number of blocks in which position of the virtual vertex is not determined uniquely, that is $p = N - 2$. After some work, we deduce from this Proposition 8.2.

Let us now discuss Problem 8.3. Take a diagram $\mathfrak{F}_0 \notin \mathfrak{F}^2_{m,n}$ and consider its graph $\mathfrak{G}_{\mathfrak{F}_0}$. We expect that in the sum

$$\sum_{\mathfrak{F}\in\mathfrak{F}^{true}_{m,n}:\, \mathfrak{G}_{\mathfrak{F}}=\mathfrak{G}_{\mathfrak{F}_0}} \tilde{J}_s(\mathfrak{F}) \tag{8.5}$$

there are similar cancellations. However, the graph $\mathfrak{G}^2$ is "very symmetric" (we are able to draw it explicitly), while for a general diagram $\mathfrak{F}_0$ there are less symmetries in the graph $\mathfrak{G}_{\mathfrak{F}_0}$. That is why we expect that in the sum (8.5) the number p of orders, in which the cancellations take place, is smaller than in the sum from Proposition 8.2, so we expect to get less "additional degrees" of ν than in Proposition 8.2. Then the estimate (8.4) can be true only if the integrals $\tilde{J}_s(\mathfrak{F})$ are smaller than those from Proposition 8.2, once $N > 2d$. Namely, if there exists $r > d$ sufficiently large such that for the diagrams $\mathfrak{F}$ satisfying $\mathfrak{G}_{\mathfrak{F}} = \mathfrak{G}_{\mathfrak{F}_0}$ we have $|\tilde{J}_s(\mathfrak{F})| \le C^{\#}(s)\nu^r$.

It is plausible that this is indeed the case and below we explain the reason. Let $k > 2$ be such that $\mathfrak{F}_0 \in \mathfrak{F}^k_{m,n}$. Then it is possible to show that $\mathfrak{F} \in \mathfrak{F}^k_{m,n}$ as well for any diagram $\mathfrak{F}$ satisfying $\mathfrak{G}_{\mathfrak{F}} = \mathfrak{G}_{\mathfrak{F}_0}$. Then, arguing as in the proof of Proposition 8.1 and applying the stationary phase method in the eigendirections corresponding to the eigenvalues $\lambda_1, \dots, \lambda_k$, we get $\tilde{J}_s(\mathfrak{F}) \sim \nu^{kd/2} \int_{\mathbb{R}^N} \frac{h^{\mathfrak{F}}_s(l)\, dl}{|\lambda_1(l)\cdots\lambda_k(l)|^{d/2}}$, if the latter integral converges. Perhaps, the factor $\nu^{kd/2}$ could lead to the desired estimate. However, we are not able to establish the convergence since we do not know a good estimate from below for the product $\lambda_1(l)\cdots\lambda_k(l)$ in the case $k > 2$, in difference with the situation when $k = 2$ considered in Proposition 8.1.

8.2. **Proof of Lemma 5.1.** Let us consider a graph $\mathfrak{C}^{\prime}_{\mathfrak{F}}$ which is obtained from $\mathfrak{U}_{\mathfrak{F}}$ by erasing in each block two (of four) dashed edges. We choose the edges we preserve arbitrarily, but in such a way that in the graph $\mathfrak{C}^{\prime}_{\mathfrak{F}}$ each vertex is incident to exactly two edges, one of which is solid and another one is dashed. We will modify the choice of the preserved dashed edges in

certain blocks of $\mathfrak{C}'_{\mathfrak{F}}$ in such a way that $\mathfrak{C}'_{\mathfrak{F}}$ will become a cycle. Since in the graph $\mathfrak{C}'_{\mathfrak{F}}$ every vertex is incident to two edges, $\mathfrak{C}'_{\mathfrak{F}}$ decouples to a union of $k \geq 1$ disjoint cycles $\mathfrak{C}_i$, $1 \leq i \leq k$, in which solid and dashed edges alternate. If $k = 1$ then we are done. Assume that $k > 1$. We claim that in this case there exists a block $\hat{B}_r = \{c_{2r-1}, c_{2r}, \bar{c}_{2r-1}, \bar{c}_{2r}\}$ such that some but not all of its vertices belongs to cycle $\mathfrak{C}_1$.

Indeed, assume that the claim is not true, so that each block either entirely belongs to the cycle $\mathfrak{C}_1$, or its intersection with $\mathfrak{C}_1$ is empty. Take a block $\hat{B}_p$ belonging to $\mathfrak{C}_1$. Then any vertex $\hat{c}_j \notin \hat{B}_p$ which is adjacent with a vertex from the block $\hat{B}_p$ (by construction of $\mathfrak{C}'_{\mathfrak{F}}$ such vertex exists), belongs to the cycle $\mathfrak{C}_1$ and, consequently, all other vertices from the block to which belongs $\hat{c}_j$ belong to $\mathfrak{C}_1$ as well. Arguing in this way, by induction we obtain that the cycle $\mathfrak{C}_1$ contains all blocks of the diagram $\mathfrak{C}'_{\mathfrak{F}}$, as well as the vertices c_0 and $\bar{c}_0$. Thus, $\mathfrak{C}_1 = \mathfrak{C}'_{\mathfrak{F}}$ which contradicts to the assumption $k > 1$, so the claim is true.

Let $\hat{B}_r$ be a block as in the claim. By construction of the diagram $\mathfrak{C}'_{\mathfrak{F}}$, its dashed edges either provide the couplings $c_{2r-1} \sim \bar{c}_{2r-1}$ and $c_{2r} \sim \bar{c}_{2r}$ or the couplings $c_{2r-1} \sim \bar{c}_{2r}$ and $c_{2r} \sim \bar{c}_{2r-1}$. For definiteness, we assume that $c_{2r-1} \sim \bar{c}_{2r-1} \in \mathfrak{C}_1$ and $c_{2r} \sim \bar{c}_{2r} \in \mathfrak{C}_2$, as in fig. 8(a). Let us erase the (dashed) edges that couple the vertices c_{2r-1} with $\bar{c}_{2r-1}$ and c_{2r} with $\bar{c}_{2r}$, and instead couple the vertices c_{2r-1} with $\bar{c}_{2r}$ and c_{2r} with $\bar{c}_{2r-1}$. Then from the diagram $\mathfrak{C}'_{\mathfrak{F}}$ we obtain another diagram $\mathfrak{C}''_{\mathfrak{F}}$, constructed from $\mathfrak{U}_{\mathfrak{F}}$ in the same way as the diagram $\mathfrak{C}'_{\mathfrak{F}}$, in which there is one cycle $\mathfrak{C}_{12}$ instead of the two disjoint cycles $\mathfrak{C}_1$ and $\mathfrak{C}_2$ in $\mathfrak{C}'_{\mathfrak{F}}$, see fig. 8(b). So the total number of cycles in the diagram $\mathfrak{C}''_{\mathfrak{F}}$ is $k - 1$, where we recall that by our assumption in $\mathfrak{C}'_{\mathfrak{F}}$ there are $k > 1$ cycles. Then, arguing by induction, we arrive at a diagram $\mathfrak{C}_{\mathfrak{F}}$ which is a cycle itself.

8.3. **Proof of Lemma 7.2.** Since the matrix A is irreducible, there exists k and indices $i_1, \dots, i_k$ satisfying $i_j \neq i_{j+1}$ $\forall j$, such that $a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_{k-1} i_k} \neq 0$ and the sequence $i_1, \dots, i_k$ contains every number from 1 to n. We construct a set B of linearly independent vectors r_{ij} by the following inductive procedure. For the base of induction we consider the indices i_1, i_2 and set $B = \{r_{i_1 i_2}\}$. For the inductive step, we assume that the indices $i_1, \dots, i_{m-1}$ are already considered, and we consider the index i_m. If the set B does not contain any vector r_{uv} for which $u = i_m$ or $v = i_m$ then we add the vector $r_{i_{m-1} i_m}$ to B and pass to the next step. Otherwise we just pass to the next step. We conclude the procedure when all indices $i_1, \dots, i_k$ are considered.

Since the sequence $i_1, \dots, i_k$ contains every $1 \leq j \leq n$, after the end of the procedure above we have $|B| = n - 1$. Due to the form of the vectors r_{ij}, vectors from the set B are linearly independent, so the rank R satisfies $R \geq n - 1$. Thus, it remains to show that $R < n$. If $R = n$ then the basis vectors e_m can be uniquely expressed through the vectors r_{ij}. However, this is not possible since the vectors r_{ij} are invariant with respect to translations

of the set (e_i): if we replace e_i by $e_i + e$ for every i and a vector e then the vectors r_{ij} remain unchanged. □

8.4. **Integrals $\tilde{J}_s(\mathfrak{F})$ in which the time variable l is integrated out.** In this appendix we integrate out the time variable l in the integrals $\tilde{J}_s(\mathfrak{F})$, which are defined in (5.24). We believe that it is relevant for further study of the energy spectrum (1.25) of quasisolutions since for the moment of writing we know asymptotical as $\nu \to 0$ behaviour of the integrals $\tilde{J}_s(\mathfrak{F})$ only in case $N = 2$, and to find it we have to write the integrals in such integrated in l form. See in [9], where we found this asymptotic and, using a particular case of Theorem 5.9, deduced from it the asymptotical as $\nu \to 0$, $L \to \infty$ behaviour of the moment $\mathbb{E}|a_s^{(1)}|^2$. Note, however, that in case of general N, even to get an upper bound for the integrals $\tilde{J}_s(\mathfrak{F})$ in Theorem 6.2 we write them in the form (5.24) and we do not know how to get the upper bound using their integrated in l form.

For simplicity of computation we assume $\tau_1 = \tau_2 =: t$ and $T = \infty$. We denote by $\mathcal{S}^N$ the set of all permutations q of the set $\{1, \dots, N\}$ and let

$$\mathcal{L}_q^N := \{l \in \mathbb{R}^N : -\infty \le l_{q(N)} \le \dots \le l_{q(1)} \le t\}, \quad \text{so that} \quad \mathbb{R}_l^N = \bigcup_{q \in \mathcal{S}^N} \mathcal{L}_q^N,$$

and the sets $\mathcal{L}_q^N$ with different q can intersect only over sets of zero Lebesgue measure. Then, in view of (4.7),

$$\tilde{J}_s(\mathfrak{F}) = \int_{\mathbb{R}^{dN}} \prod_{\psi \in E_L(\mathfrak{F})} \frac{(b^\psi(z))^2}{\gamma^\psi(z)} \sum_{q \in \mathcal{S}^N} I_s^q(\mathfrak{F}; z)\, dz\,, \tag{8.6}$$

where

$$I_s^q(\mathfrak{F}, z) = \int_{\mathcal{L}_q^N} e^{i\nu^{-1}\Omega^{\mathfrak{F}}(l,z)} \prod_{\varphi \in E_D(\mathfrak{F})} e^{-\gamma^\varphi (l_r^\varphi - l_w^\varphi)} \mathbb{I}_{\{-T \le l_w^\varphi \le l_r^\varphi\}}(t, l)$$
$$\prod_{\psi \in E_L(\mathfrak{F})} e^{-\gamma^\psi |l^{c_\psi} - l^{\bar{c}_\psi}|}\, dl, \tag{8.7}$$

where $b^\psi(z) = b^\psi(\xi(z))$, $\gamma^\psi(z) = \gamma^\psi(\xi(z))$ and $\gamma^\varphi(z) = \gamma^\varphi(\xi(z))$. It remains to compute the integrals I_s^q, and for this purpose we introduce the following notation. For $0 \le i < j \le N$ we set

$$\gamma_{ij}^{\mathfrak{F}} := \sum_{\vartheta : \hat{B}_i \sim \hat{B}_j} \gamma_\vartheta \ge 0,$$

where the sum is taken over all edges $\vartheta \in E(\mathfrak{F})$ of the diagram $\mathfrak{F}$ which couple a vertex from the block $\hat{B}_i$ with a vertex from the block $\hat{B}_j$. Here by $\hat{B}_0$ we denote the set of roots $\{c_0, \bar{c}_0\}$.

Let $\mathcal{S}_{\mathfrak{F}}^N \subset \mathcal{S}^N$ be the set of permutations q for which the relation $l \in \mathcal{L}_q^N$ implies

$$l_w^\varphi \le l_r^\varphi \quad \text{for all} \quad \varphi \in E_D(\mathfrak{F}),$$

cf. the indicator function in (8.7).

Below we denote $q(0) := 0$.

Lemma 8.4. *Assume $\tau_1 = \tau_2 = t$ and $T = \infty$. Then for any $m, n \geq 0$ satisfying $N := m + n \geq 2$ and $\mathfrak{F} \in \mathfrak{F}_{m,n}$ the integral $\tilde{J}_s(\mathfrak{F})$ is given by (8.6), where*

$$I_s^q(\mathfrak{F}; z) = \frac{1}{2\gamma_s} \prod_{k=2}^{N} \left(i\nu^{-1} \sum_{j=k}^{N} \omega_{q(j)}^{\mathfrak{F}}(z) + \sum_{0 \leq r < k \leq j \leq N} \gamma_{q(r)q(j)}^{\mathfrak{F}}(z) \right)^{-1} \tag{8.8}$$

if $q \in \mathcal{S}_{\mathfrak{F}}^N$ and $I_s^q(\mathfrak{F}; z) \equiv 0$ otherwise. Moreover, $\sum_{0 \leq r < k \leq j \leq N} \gamma_{q(r)q(j)}^{\mathfrak{F}} \geq 1$ for any $2 \leq k \leq N$, so the denominator in (8.8) is separated from zero.

The lemma can be proven by induction.

Note that outside the intersection of quadrics

$$Q := \{z : \omega_j^{\mathfrak{F}}(z) = 0 \quad \text{for } 1 \leq j \leq N\}$$

we have $I_s^q(\mathfrak{F}; z) \sim \nu^{N-1}$, while on Q we have $I_s^q(\mathfrak{F}; z) \sim 1$. So the integrand in (8.6) asymptotically degenerates on Q as $\nu \to 0$, and the integral in (8.6) is asymptotically singular.

8.5. **Estimation of asymptotically singular integrals of quotients.** In Lemma 8.4 we integrated out the time variable l in integrals $\tilde{J}_s(\mathfrak{F})$, defined in (5.24), and represented $\tilde{J}_s(\mathfrak{F})$ as sums of integrals of quotients with large quadratic forms in divisors, see (8.6),(8.8). In Theorem 6.2 we got an upper bound for a family of integrals of quotients that have more general form than the integrals $\tilde{J}_s(\mathfrak{F})$. As a corollary, in this appendix we obtain an upper bound for a natural family of integrals of more general form than those in (8.6), (8.8).

As in Section 6.1, we consider the family of quadratic forms

$$(Q_k z) \cdot z = \sum_{i,j=1}^{M} q_{ij}^k z_i \cdot z_j, \quad 1 \leq k \leq K,$$

where z is the polyvector $z = (z_1, \ldots, z_M), z_j \in \mathbb{R}^d$, and $Q_k = (q_{ij}^k)_{1 \leq i,j \leq M}$ are real symmetric matrices. As before, we denote by R the rank of the system of matrices $Q_1, \ldots, Q_K$ in the space of $M \times M$-matrices, equal to the rank of the system of vectors

$$q_{ij} \in \mathbb{R}^K, \quad 1 \leq i, j \leq M, \quad q_{ij} = (q_{ij}^1, \ldots, q_{ij}^K).$$

Let

$$J^\nu = \int_{\mathbb{R}^{dM}} \frac{G(z)}{\prod_{k=1}^{K} \left(i\nu^{-1}(Q_k z) \cdot z + \Gamma_k(z) \right)} \, dz, \tag{8.9}$$

where $0 < \nu \le 1/2$, G is a Schwartz function, the functions Γ_k are smooth, have at most polynomial growth at infinity together with their partial derivatives of all orders, and satisfy $\Gamma_k(z) \ge C_\Gamma$ for some constant $C_\Gamma > 0$ and any k and z.

Proposition 8.5. *Assume that* $\operatorname{tr} Q_k = 0$ *for every* $1 \le k \le K$. *Then*

$$|J^\nu| \le C\nu^{\min(R,d)}\psi_d^R(\nu), \tag{8.10}$$

where the function ψ_d^R *is defined in* (6.5).

To establish the proposition, we use representation

$$\frac{1}{i\nu^{-1}(Q_k z)\cdot z + \Gamma_k(z)} = \int_{-\infty}^{0} e^{l_k\left(i\nu^{-1}(Q_k z)\cdot z + \Gamma_k(z)\right)}\, dl_k,$$

for every k. Then the integral J^ν takes the form (6.2) and the desired estimate follows from Theorem 6.2. See [8] for a generalization of this result.

In [18] the asymptotic behaviour as $\nu \to 0$ of the integral J^ν was found in the case when $d \ge 2$ is a pair number, $K = 2$, $\Gamma_1 = \Gamma_2$ and $Q_1 = -Q_2$, where the quadratic form Q_1 was assumed to be non-degenerate and to have the index of the form (m, m), for some m. This result plays a crucial role in [9] when analysing the asymptotic behaviour of the term $n^{(2)}$ from the decomposition (1.21). In particular, it was shown that $J^\nu \sim \nu$ which is in accordance with the estimate (8.10). The family of integrals (8.9) is significantly more complicated than that in [18]. We get for them only the upper bound and for the moment do not now how to examine their asymptotic behaviour.

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