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Iterating Semi-proper Forcing using Virtual Models

Obrad Kasum*, Boban Veličković

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Abstract

By a virtual model, we mean a model of set theory which is elementary in its transitive closure. Virtual models are first used by Neeman [Nee14] to iterate forcing. That paper is concerned with proper forcing. The method was then adjusted by Veličković to iterate the case of semi-proper forcing and this was drafted in [Vel14]. We here straighten the details and futher elaborate on Veličković's method. The first section collects facts about virtual model, the second section describes the iteration, and the third one illustrates the method in the case of getting saturation of NS_{ω_1} (loosely relying on [Sch16]).

Contents

1 Virtual Models **2**
1.1 Admissible Structures . . . 2
1.2 Virtual Models: Definition . . . 2
1.3 Elementary Rank-initial Segments . . . 3
1.4 Reduction of a Virtual Model . . . 4
1.5 α-isomorphism . . . 5
1.6 Comparing Virtual Models . . . 5
1.7 Forcing Extensions of Virtual Models . . . 6
1.8 Semi-proper Forcing . . . 9
1.9 Iterated Forcing Extensions of Virtual Models . . . 10

2 Semi-proper Iteration **13**
2.1 Setup for the Iteration . . . 13
2.2 Recursive Step of the Definition . . . 14
2.3 Basic Properties . . . 15
2.4 Statement of Transfer Theorem . . . 17
2.5 Some Lemmas for Transfer Theorem . . . 18
2.6 Proof of Transfer Theorem . . . 24
2.6.1 Successor Case . . . 24
2.6.2 Limit Case . . . 26
2.7 Semi-properness and Chain Condition . . . 33

3 Saturating NS_{ω_1} **36**
3.1 Careful Collapse . . . 36
3.2 Sealed Predense Collections . . . 39
3.3 Sealing Iteration . . . 40

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1 Virtual Models

1.1 Admissible Structures

[1.1.1] **Summary.** We consider a basic notion of an admissible structure. It will depend on parameter $\mathcal{L}$, where $\mathcal{L}$ is a recursively enumerable first order language containing $\in$. The language $\mathcal{L}$ will most often be kept implicit.

[1.1.2] **Definition.** An *admissible structure* $\mathbb{A} := (A, \in, \dots)$ in language $\mathcal{L}$ is a transitive structure which satisfies ZFC^- in the extended language.

[1.1.3] **Remark.** We will often say "A is admissible" when we really mean "$\mathbb{A}$ is admissible".

[1.1.4] **Example.** If A is transitive model for ZFC^-, then $(A, \in)$ is admissible.

[1.1.5] **Example.** If A is admissible and if $U \subseteq A$ is definable with parameters over $(A, \in)$, then $(A, \in, U)$ is admissible.

[1.1.6] **Example.** If $\theta > \omega$ is regular, then every structure $(H_\theta, \in, \dots)$ is admissible.

1.2 Virtual Models: Definition

[1.2.1] **Summary.** We introduce now a generalization of an admissible structure which we call a virtual model. This notion will play a crucial role in the main construction.

[1.2.2] **Definition.** A *virtual model (in language $\mathcal{L}$)* is a structure $(M, \in, \dots)$ such that there exists an admissible $\mathbb{A}$ with

$$(M, \in, \dots) \prec \mathbb{A}.$$

[1.2.3] **Example.** Every admissible structure is also a virtual model.

[1.2.4] **Remark.** If $M \prec \mathbb{A}$, where $\mathbb{A}$ is admissible, then the structure on M is uniquely determined by structure $\mathbb{A}$.

[1.2.5] **Goal.** We will now show that given virtual model M, there exists unique minimal structure $\widehat{M}$ witnessing the fact that M is a virtual model.

[1.2.6] **Lemma.** Let A be admissible and let $B \subseteq A$ be transitive. Then $B \prec_0 A$.

[1.2.7] **Lemma.** Let A be admissible and let $M \prec_0 A$ be such that $\mathsf{trcl}(M) = A$. Then $M \prec A$.

[1.2.8] **Remark.** We treat function symbols as relation symbols. In particular, if A is admissible and $M \subseteq A$ contains all the constants, then M inherits a structure from A. In that case, the inherited structure is the default structure on M.

[1.2.9] **Lemma.** Let A be admissible and let $M \prec A$. Then $\mathsf{trcl}(M)$ is admissible and

$$M \prec \mathsf{trcl}(M) \prec A.$$

[1.2.10] **Proposition.** Let M be a virtual model. Then there exists unique structure $\widehat{M}$ on $\mathsf{trcl}(M)$ satisfying $M \prec \widehat{M}$. Furthermore, for every admissible A, it holds that $M \prec A$ if and only if $\widehat{M} \prec A$.

Proof. Existence follows from the previous lemma. To verify uniqueness, consider a relation R of structure M and note that

$$R^{\widehat{M}} = \bigcup_{\xi < \mathsf{Ord} \cap \widehat{M}} (R^{\widehat{M}} \cap (\widehat{M} \restriction \xi)) = \bigcup_{\xi < \mathsf{Ord} \cap M} (R^{\widehat{M}} \cap (\widehat{M} \restriction \xi)) = \bigcup_{\xi < \mathsf{Ord} \cap M} (R \cap V_\xi)^M$$

which depends only on M. $\square$

1.3 Elementary Rank-initial Segments

[1.3.1] **Definition.** Let A be admissible. Then $\mathscr{E}_A := \{\alpha < \mathsf{Ord}^A : A \restriction \alpha \prec A\}$.

[1.3.2] **Remark.** Set $\mathscr{E}_A$ is closed in Ord^A. If $A = V_\kappa$ where κ is inaccessible, then $\mathscr{E}_A$ is club in κ.

[1.3.3] **Proposition.** Let A, B be admissible and let $\alpha \in \mathscr{E}_A \cap \mathscr{E}_B$. Suppose that $A \restriction \alpha = B \restriction \alpha$. Then

$$\mathscr{E}_A \cap [0, \alpha] = \mathscr{E}_B \cap [0, \alpha] \in A \cap B.$$

[1.3.4] **Definition.** Let M be a virtual model. Then $\mathscr{E}_M := \{\alpha \in \mathsf{Ord} \cap M : M \cap V_\alpha \prec M\}$.

[1.3.5] **Remark.** Every admissible structure is also a virtual model. In that case, the two definitions coincide.

[1.3.6] **Proposition.** Let M be a virtual model. Then $\mathscr{E}_M = \mathscr{E}_{\widehat{M}} \cap M$.

[1.3.7] **Corollary.** Let M be a virtual model and let γ be a limit point of $\mathscr{E}_M$. Then $\gamma \in \mathscr{E}_{\widehat{M}}$.

Proof. Follows from the fact that $\mathscr{E}_M \subseteq \mathscr{E}_{\widehat{M}}$ and the fact that $\mathscr{E}_{\widehat{M}}$ is closed. □

[1.3.8] **Lemma.** Let M be a virtual model and let $\alpha \in \mathscr{E}_{\widehat{M}}$. Suppose that $\beta := \min(M \cap [\alpha, \infty))$. Then $\beta \in \mathscr{E}_M$.

Proof.

1° We apply Tarski-Vaught test. Suppose $a \in M \cap V_\beta$ and $M \models \exists y \phi(a, y)$. We want to find $b \in M \cap V_\beta$ such that $M \models \phi[a, b]$.

2° By elementarity, $\widehat{M} \models \exists y \phi(a, y)$. We also have $a \in M \cap V_\beta \subseteq M \cap V_\alpha \subseteq \widehat{M} \restriction \alpha$.

3° Since $\alpha \in \mathscr{E}_{\widehat{M}}$, there exists $y \in \widehat{M} \restriction \alpha$ such that $\widehat{M} \models \phi[a, y]$.

4° Since $\alpha \leq \beta$, we have

$$\widehat{M} \models (\exists y \in V_\beta)\phi(a, y).$$

By elementarity,

$$M \models (\exists y \in V_\beta)\phi(a, y).$$

5° Thus, there is $b \in M \cap V_\beta$ such that $M \models \phi[a, b]$.

□

[1.3.9] **Proposition.** Let M be a virtual model. Suppose that γ is a limit point of $\mathscr{E}_{\widehat{M}}$ and a limit point of $M \cap \mathsf{Ord}$. Then γ is a limit point of $\mathscr{E}_M$.

Proof.

1° Let $\alpha < \gamma$ be arbitrary. We want to find $\beta \in \mathscr{E}_M \cap \gamma$ such that $\alpha < \beta$.

2° There is $\alpha' \in \mathscr{E}_{\widehat{M}} \cap \gamma$ such that $\alpha < \alpha'$.

3° There is $\alpha'' \in M \cap \gamma$ such that $\alpha' < \alpha''$.

4° By [1.3.8], we have $\beta := \min(M \cap [\alpha', \infty)) \in \mathscr{E}_M$.

5° By 3°, we have $\beta \in [\alpha', \alpha''] \subseteq (\alpha, \gamma)$.

□

1.4 Reduction of a Virtual Model

[1.4.1] **Summary.** We introduce a basic construction of reducing the amount of information captured by a virtual model.

[1.4.2] **Definition.** Let M be a virtual model and let X be an arbitrary set. We define

$$\mathsf{Hull}(M, X) := \{f(x) : f : d \to r,\ f \in M,\ x \in X^{<\omega} \cap d\}.$$

[1.4.3] **Proposition.** M be a virtual model, let $\alpha := \sup(\mathsf{Ord} \cap M)$, and let X be an arbitrary set. Then the following holds.

a. $M \prec \mathsf{Hull}(M, X) \prec \widehat{M}$

b. $\widehat{\mathsf{Hull}(M, X)} = \widehat{M}$

c. $X \cap \widehat{M} \subseteq \mathsf{Hull}(M, X)$

d. $\mathsf{Hull}(M, X) = \mathsf{Hull}(M, X \cap \widehat{M})$

e. For every virtual model N satisfying $M \prec N$ and $X \cap \widehat{M} \subseteq N$, we have $\mathsf{Hull}(M, X) \prec N$.

f. $|\mathsf{Hull}(M, X)| \leq |M| + |X|$

□

[1.4.4] **Definition.** *The α-reduction* $M \downarrow \alpha$ of virtual model M is defined as follows: let $\pi : \mathsf{Hull}(M, V_\alpha) \to N$ be the transitive collapse and set $M \downarrow \alpha := \pi[M]$.

[1.4.5] **Remark.** We do not require $V_\alpha \subseteq \widehat{M}$ or even $\delta := \sup(\mathsf{Ord} \cap M) \geq \alpha$. What we do is simply collapsing M while "freezing" its part below rank α. In particular, if $\delta \leq \alpha$, this means that $M \subseteq V_\alpha$ and $\mathsf{Hull}(M, V_\alpha) = \widehat{M}$, i.e. $M \downarrow \alpha = M$.

[1.4.6] **Exercise.** Let M be a virtual model and let α be an ordinal. Then $\widehat{M \downarrow \alpha}$ is the transitive collapse of $\mathsf{Hull}(M, V_\alpha)$.

[1.4.7] **Definition.** A virtual model M is said to be *α-generated* if $\widehat{M} = \mathsf{Hull}(M, V_\alpha)$.

[1.4.8] **Lemma.** Let N be a virtual model and let $\pi : N \to \overline{N}$ be the transitive collapse. Then for every Σ_0 formula $\phi(\overline{x})$, we have

$$(\forall \overline{x} \in N^{<\omega})(\phi(\overline{x}) \iff \phi(\pi(\overline{x}))).$$

Proof.

$$\begin{aligned}
\phi(\overline{x}) &\iff \widehat{N} \models \phi(\overline{x}) && (1)\\
&\iff N \models \phi(\overline{x}) && (2)\\
&\iff \overline{N} \models \phi(\pi(\overline{x})) && (3)\\
&\iff \phi(\pi(\overline{x})), && (4)
\end{aligned}$$

where:

(1) follows by Σ_0-absoluteness;

(2) follows by elementarity;

(3) follows since π is an isomorphism;

(4) follows by Σ_0-absoluteness.

□

[1.4.9] **Proposition.** Let M be a virtual model. Then $M \downarrow \alpha$ is α-generated.

[1.4.10] **Proposition.** Let M be a virtual model and let $\beta \leq \alpha$. Then $(M \downarrow \alpha) \downarrow \beta = M \downarrow \beta$.

Proof. This is a straight forward computation; use [1.4.8] to show that collapses agree with the computations of Hull's. □

1.5 α-isomorphism

[1.5.1] **Summary.** The relation introduced in this part has for the goal to capture the idea that two virtual models carry the same information up to α.

[1.5.2] **Definition.** Let M, N be virtual models. We define $M \cong_\alpha N$ to hold if there is an isomorphism $f : \mathsf{Hull}(M, V_\alpha) \cong \mathsf{Hull}(N, V_\alpha)$ satisfying $f[M] = N$.

[1.5.3] **Proposition.** Let M be a virtual model. Then $M \cong_\alpha M \downarrow \alpha$.

[1.5.4] **Proposition.** Let M, N be α-generated virtual models. If $M \cong_\alpha N$, then $M = N$.

[1.5.5] **Proposition.** Let M, N be virtual models. Then $M \cong_\alpha N$ if and only if $M \downarrow \alpha = N \downarrow \alpha$.

[1.5.6] **Corollary.** $\cong_\alpha$ is an equivalence relation between virtual models.

1.6 Comparing Virtual Models

[1.6.1] **Summary.**

[1.6.2] **Definition.** Let M, N be countable virtual models. Then we define $M \lhd_\alpha N$ to hold if there exists $M' \in N$ such that M' with the inherited structure is a virtual model and $M \cong_\alpha M'$.

[1.6.3] **Definition.** A virtual model M is said to be *ξ-strong* if $V_\xi \subseteq \widehat{M}$.

[1.6.4] **Lemma.** Let M be an $(\omega+1)$-strong virtual model and let $N \in M$ be a countable virtual model. Then $|N|^M = \omega$.

Proof.

1° Let $\pi : M \to \overline{M}$ be the transitive collapse of $(M, \in)$. We have $\overline{M} \in H_{\omega_1}$ and there is a surjection $f : \omega \to Y$.

2° We have $f \in H_{\omega_1} \subseteq \widehat{N}$.

3° Thus, $\widehat{N} \models (\overline{M}$ is countable$)$.

4° By absoluteness of transitive collapse, $\widehat{N} \models (\overline{M}, \in) \cong (M, \in)$.

5° Thus, $\widehat{N} \models |M| = |\overline{M}| = \omega$ and consequently $N \models |M| = \omega$.

□

[1.6.5] **Proposition.** Suppose that $\alpha > \omega$. Then relation $\lhd_\alpha$ is a partial order between countable $(\omega+1)$-strong virtual models.

Proof.

1° Suppose that M, N, P are countable $(\omega+1)$-strong virtual models satisfying $M \lhd_\alpha N \lhd_\alpha P$.

2° There exist a virtual model $M' \in N$, an isomorphism

$$f : \mathsf{Hull}(M, V_\alpha) \cong \mathsf{Hull}(M', V_\alpha)$$

with $f[M] = M'$, a virtual model $N' \in P$, and an isomorphism

$$g : \mathsf{Hull}(N, V_\alpha) \cong \mathsf{Hull}(N', V_\alpha)$$

with $g[N] = N'$.

3° By [1.6.4], we have $M' \subseteq N$ and $g(M') = g[M']$.

4° For $h \in N'$ and $x \in \mathsf{dom}(f) \cap V_\alpha^{<\omega}$, we have

$$g(h(x)) = g(h)(g(x)) = g(h)(x).$$

We conclude

$$g[\mathsf{Hull}(M', V_\alpha)] = \mathsf{Hull}(g[M'], V_\alpha) = \mathsf{Hull}(g(M'), V_\alpha).$$

5° Thus, $g(M') \in P$ and
$$g \circ f : \mathsf{Hull}(M, V_\alpha) \cong \mathsf{Hull}(g(M'), V_\alpha).$$

6° We also verify
$$(g \circ f)[M] = g[f[M]] = g[M'] = g(M'),$$
which yields the conclusion $M \lhd_\alpha P$.

□

[1.6.6] **Proposition.** Suppose that M, M', N, N' are countable virtual models such that $M \cong_\alpha M'$ and $N \cong_\alpha N'$. Then $M \lhd_\alpha N$ if and only if $M' \lhd_\alpha N'$. □

[1.6.7] **Proposition.** Suppose that M, N are countable virtual models and $M \in N$. Then $M \lhd_\alpha N$. □

[1.6.8] **Proposition.** Let α be an ordinal and let M, N be countable virtual models. Suppose that $M, \alpha \in N$, that M is α-generated, and $M \lhd_\alpha N$. Then $M \in N$.

Proof.

1° There is a virtual model $M' \in N$ such that $M \cong_\alpha M'$.

2° Since M is α-generated, we have $M = M' \downarrow \alpha \in N$.

□

[1.6.9] **Proposition.** Let α be a beth-fixed-point and let M, N be countable α-strong virtual models such that $M \lhd_\alpha N$. Let $\theta \in M \cap V_\alpha$ be regular and uncountable. Then $M \cap H_\theta \in N \cap H_\theta$.

Proof. Since N is α-strong and $\theta < \alpha$, we have that $H_\theta = H_\theta^N \in N$. Let $M' \in N$ be such that $M \cong_\alpha M'$. We have $M \cap H_\theta = M' \cap H_\theta \in N$. On the other hand, since $M \cap H_\theta \in [H_\theta]^\omega$, we conclude $M \cap H_\theta \in H_\theta$. □

1.7 Forcing Extensions of Virtual Models

[1.7.1] **Definition.** Let A be admissible, let $\mathbb{P} \in A$ be a poset, and let $M \prec A$ with $\mathbb{P} \in M$. For $G \rightsquigarrow A^{\mathbb{P}}$, we define
$$M[G] := \{\tau^G : \tau \in M^{\mathbb{P}} = A^{\mathbb{P}} \cap M\}, \quad M^G := M[G] \cap A.$$

[1.7.2] **Proposition.** Let A be admissible, let $\mathbb{P} \in A$ be a poset, let $M \prec A$ with $\mathbb{P} \in M$, and let $G \rightsquigarrow A^{\mathbb{P}}$. Then the following holds.

a. $(M[G], \in, M^G, \mathbb{P}, G) \prec (A[G], \in, A, \mathbb{P}, G)$

b. $\widehat{M[G]} = \widehat{M}[G]$

c. $M \prec M^G \prec A$

d. $M^G[G] = M[G]$

e. $(M^G)^G = M^G$.

□

[1.7.3] **Proposition.** Let M be an α-strong virtual model, let α be such that $\widehat{M} \restriction \alpha \models \mathsf{ZFC}^-$, let $\mathbb{P} \in M \cap V_\alpha$, and let $G \rightsquigarrow V^{\mathbb{P}}$. Then $(\widehat{M} \restriction \alpha)[G] \subseteq \widehat{M[G]}$. □

[1.7.4] **Lemma.** Let M be a virtual model, let $\alpha \in \widehat{M}$ satisfy $\widehat{M} \restriction \alpha \models \mathsf{ZFC}^-$, let $\mathbb{P} \in M \cap V_\alpha$, and let $G \in V$ satisfy $G \rightsquigarrow \widehat{M}^{\mathbb{P}}$. Then $\mathsf{Hull}(M[G], V_\alpha) = \mathsf{Hull}(M, V_\alpha)[G]$.

Proof.

1° We first consider inclusion ($\supseteq$).

1′ Let $\dot{y} \in \mathsf{Hull}(M, V_\alpha)$ and let us verify $\dot{y}^G \in \mathsf{Hull}(M[G], V_\alpha)$.

2′ We have $\dot{y} = f(x)$ for a function $f \in M$ and an $x \in V_\alpha \cap \mathsf{dom}(f)$. We may choose f so that $\mathsf{im}(f) \subseteq \widehat{M}^{\mathbb{P}}$.

3′ Let $g \in M[G]$ be the function satisfying $\mathsf{dom}(g) = \mathsf{dom}(f)$ and $g(u) := f(u)^G$. We have $g \in M[G]$ and $x \in \mathsf{dom}(g) \cap V_\alpha$.

4′ Hence, $\dot{y}^G = f(x)^G = g(x) \in \mathsf{Hull}(M[G], V_\alpha)$.

2° We consider now inclusion ($\subseteq$)

1′ Let us consider $x_0 \in \mathsf{Hull}(M[G], V_\alpha)$ and let us show $x_0 \in \mathsf{Hull}(M, V_\alpha)[G]$.

2′ There exist a function $f \in M[G]$ and $x \in V_\alpha \cap \mathsf{dom}(f)$ such that $x_0 = f(x)$.

3′ There exist $\mathbb{P}$-names $\dot{f}, \dot{d}$ such that

$$\widehat{M}^{\mathbb{P}} \models (\dot{f} \text{ is a function on } \dot{d})$$

and $\dot{f}^G = f$.

4′ Note that

$$x \in \widehat{M[G]} \restriction \alpha = \widehat{M}[G] \restriction \alpha = (\widehat{M} \restriction \alpha)[G],$$

so there exists a $\mathbb{P}$-name $\dot{x} \in \widehat{M} \restriction \alpha$ such that $\dot{x}^G = x$.

5′ Let $\xi \in M$ satisfy $\xi > \alpha$ and let $e := \{\dot{y} \in V_\xi^M : \dot{y} \text{ is a } \mathbb{P}\text{-name}\} \in M$.

6′ There exists a function g such that $\mathsf{dom}(g) = e$ and

$$(\forall \dot{y} \in e)(\forall p \in \mathbb{P})(p \Vdash_{\mathbb{P}}^{\widehat{M}} \dot{y} \in \dot{d} \implies p \Vdash_{\mathbb{P}}^{\widehat{M}} \dot{f}(\dot{y}) = \ulcorner g(\dot{y}) \urcorner).$$

By elementarity, function g can be chosen in M.

7′ We have $\dot{z} := g(\dot{x}) \in \mathsf{Hull}(M, V_\alpha)$ and $\dot{z}^G \in \mathsf{Hull}(M, V_\alpha)[G]$.

8′ Let $p \in G$ be such that $p \Vdash_{\mathbb{P}}^{\widehat{M}} \dot{x} \in \dot{d}$. Since $\dot{x} \in e$, we have $p \Vdash_{\mathbb{P}}^{\widehat{M}} \dot{f}(\dot{x}) = \dot{z}$ and consequently $\dot{f}^G(\dot{x}^G) = \dot{z}^G$.

9′ Hence, $x_0 = f(x) = \dot{f}^G(\dot{x}^G) = \dot{z}^G \in \mathsf{Hull}(M, V_\alpha)[G]$.

□

[1.7.5] **Proposition.** Let M, N be virtual models and let $\alpha \in \widehat{M} \cap \widehat{N}$. Suppose that

$$\widehat{M} \restriction \alpha = \widehat{N} \restriction \alpha \models \mathsf{ZFC}^-$$

and that $M \cong_\alpha N$. Let $\mathbb{P} \in M \cap V_\alpha$ and let $G \in V$ satisfy $G \rightsquigarrow \widehat{M}^{\mathbb{P}}$. Then

$$\mathbb{P} \in N \cap V_\alpha, \quad G \rightsquigarrow \widehat{N}^{\mathbb{P}}, \quad M[G] \cong_\alpha N[G], \quad M^G \cong_\alpha N^G.$$

Furthermore, the isomorphism witnessing $M[G] \cong_\alpha N[G]$ extends the isomorphism witnessing $M^G \cong_\alpha N^G$, which in turn extends the isomorphism witnessing $M \cong_\alpha N$.

Proof.

1° $M \cong_\alpha N$ implies $M \cap V_\alpha = N \cap V_\alpha$, so $\mathbb{P} \in N \cap V_\alpha$. The fact that $G \rightsquigarrow \widehat{N}^{\mathbb{P}}$ is obvious.

2° We verify below that $M[G] \cong_\alpha N[G]$ and that the isomorphism witnessing this fact extends the isomorphism witnessing $M \cong_\alpha N$. Since M^G and N^G are computed as the appropriate grounds, we will immediately have $M^G \cong_\alpha N^G$, via the restricted isomorphism.

3° Let $F : \mathsf{Hull}(M, V_\alpha) \cong \mathsf{Hull}(N, V_\alpha)$ be such that $F[M] = N$.

4° By [1.7.4], every element of $\mathsf{Hull}(M[G], V_\alpha)$ is of the form $\dot{x}^G$ for some $\dot{x} \in \mathsf{Hull}(M, V_\alpha)^{\mathbb{P}}$. Also, $\dot{y}^G \in \mathsf{Hull}(N[G], V_\alpha)$ for every $\dot{y} \in \mathsf{Hull}(N, V_\alpha)^{\mathbb{P}}$.

5° Hence, we can define $F[G] : \mathsf{Hull}(M[G], V_\alpha) \to \mathsf{Hull}(N[G], V_\alpha)$ by setting $F[G](\dot{x}^G) := F(\dot{x})^G$ for $\dot{x} \in \mathsf{Hull}(M, V_\alpha)^{\mathbb{P}}$. We want to show that $F[G]$ is an isomorphism, that $(F[G])[M[G]] = N[G]$, and that $F[G] \restriction \mathsf{Hull}(M, V_\alpha) = F$.

6° **Claim.** For $* \in \{\in, =\}$ and $\dot{x}_0, \dot{x}_1 \in \mathsf{Hull}(M, V_\alpha)^{\mathbb{P}}$, we have

$$\dot{x}_0^G * \dot{x}_1^G \iff F(\dot{x}_0)^G * F(\dot{x}_1)^G.$$

Proof.

$$\begin{aligned} \dot{x}_0^G * \dot{x}_1^G \quad &\iff \quad (\exists p \in G)\mathsf{Hull}(M, V_\alpha) \models (p \Vdash \dot{x}_0 * \dot{x}_1) && (5)\\ &\iff \quad (\exists p \in G)\mathsf{Hull}(N, V_\alpha) \models (p \Vdash F(\dot{x}_0) * F(\dot{x}_1)) && (6)\\ &\iff \quad F(\dot{x}_0)^G * F(\dot{x}_1)^G, && (7) \end{aligned}$$

where (6) follows from $F \restriction V_\alpha = \mathsf{id}$ and $\mathbb{P} \in V_\alpha$. □

7° Hence, $F[G]$ is correctly defined injection agreeing with $\in$.

8° Analogously, we have injection

$$F^{-1}[G] : \mathsf{Hull}(N[G], V_\alpha) \to \mathsf{Hull}(M[G], V_\alpha)$$

agreeing with $\in$.

9° Clearly, $F[G] \circ F^{-1}[G] = \mathsf{id}$ and $F^{-1}[G] \circ F[G] = \mathsf{id}$, leading to the conclusion that

$$F[G] : \mathsf{Hull}(M[G], V_\alpha) \cong \mathsf{Hull}(N[G], V_\alpha).$$

10° For $x \in \mathsf{Hull}(M, V_\alpha)$, we have

$$F[G](x) = F[G](\check{x}^G) = F(\check{x})^G = ((F(x))\check{\ })^G = F(x).$$

11° Hence, it remains to establish $(F[G])[M[G]] = N[G]$, i.e. $(F[G])[M[G]] \subseteq N[G]$ and $(F^{-1}[G])[N[G]] \subseteq M[G]$.

12° We consider here only the first conjuct. Let $y \in M[G]$ be arbitrary.

13° By definition of $M[G]$, we have $y = \dot{x}^G$ for some $x \in M^{\mathbb{P}} \subseteq \mathsf{Hull}(M, V_\alpha)^{\mathbb{P}}$.

14° By definition of $F[G]$, we have $F[G](y) = F(\dot{x})^G$.

15° Since $\dot{x} \in M$ and $F[M] = N$, we have $F(\dot{x}) \in N^{\mathbb{P}}$ and consequently $F[G](y) = F(\dot{x})^G \in N[G]$.

□

[1.7.6] **Corollary.** Let M, N be countable virtual models and let $\alpha \in \widehat{M} \cap \widehat{N}$. Suppose that

$$\widehat{M} \restriction \alpha = \widehat{N} \restriction \alpha \models \mathsf{ZFC}^-$$

and that $M \lhd_\alpha N$. Let $\mathbb{P} \in M \cap V_\alpha$ and let $G \in V$ satisfy $G \rightsquigarrow \widehat{M}^{\mathbb{P}}$. Then $M[G]$ and $N[G]$ are countable and $M[G] \lhd_\alpha N[G]$.

Proof.

1° M, N surject onto $M[G], N[G]$, respectively, so $M[G], N[G]$ are countable.

2° Let $M' \in N$ be a virtual model in V satisfying $M \cong_\alpha M'$.

3° By [1.7.5], we have $M[G] \cong_\alpha M'[G] \in N[G]$.

4° Thus, $M[G] \lhd_\alpha N[G]$.

□

[1.7.7] **Lemma.** Let M be a virtual model and let $\alpha \in \widehat{M}$ be such that $\widehat{M} \restriction \alpha \models \mathsf{ZFC}^-$. Suppose that M is α-generated. Let $\mathbb{P} \in M \cap V_\alpha$ and let $G \in V$ satisfy $G \rightsquigarrow \widehat{M}^{\mathbb{P}}$. Then $M[G]$ and M^G are α-generated.

Proof. To see that $M[G]$ is α-generated, compute as follows:

$$\widehat{M[G]} = \widehat{M}[G] = \mathsf{Hull}(M, V_\alpha)[G] = \mathsf{Hull}(M[G], V_\alpha).$$

To see that M^G is α-generated, recall that $M \prec M^G \prec \widehat{M}$. □

[1.7.8] **Proposition.** Let M be a virtual model and let $\alpha \in \widehat{M}$ be such that $\widehat{M} \restriction \alpha \models \mathsf{ZFC}^-$, let $\mathbb{P} \in M \cap V_\alpha$, and let $G \in V$ satisfy $G \rightsquigarrow \widehat{M}^{\mathbb{P}}$. Then $(M \downarrow \alpha)[G] = M[G] \downarrow \alpha$ and $(M \downarrow \alpha)^G = M^G \downarrow \alpha$.

Proof. Since $M \downarrow \alpha \cong_\alpha M$, we have $(M \downarrow \alpha)[G] \cong_\alpha M[G]$. Since $M \downarrow \alpha$ is α-generated, so is $(M \downarrow \alpha)[G]$ and we conclude

$$(M \downarrow \alpha)[G] = M[G] \downarrow \alpha.$$

The other part is completely analogous. □

1.8 Semi-proper Forcing

[1.8.1] **Definition.** Let $\mathbb{P}$ be a poset, let $\theta > 2^{2^{|\mathsf{trcl}(\mathbb{P})|}}$ be regular, and let $M \prec (H_\theta, \in, \mathbb{P})$ be countable.

a. For $G \rightsquigarrow V^{\mathbb{P}}$, we say that G is *semi-$M^{\mathbb{P}}$-generic* if $M[G] \cap \omega_1^V = M \cap \omega_1^V$.

b. For $p \in \mathbb{P}$, we say that p is *semi-$M^{\mathbb{P}}$-generic* if $p \Vdash M[\dot{G}] \cap \omega_1^V = M \cap \omega_1^V$.

[1.8.2] **Lemma.** Let $\mathbb{P}$ be a poset, let $\theta > 2^{2^{|\mathsf{trcl}(\mathbb{P})|}}$ be regular, and let $M \prec (H_\theta, \in, \mathbb{P})$ be countable. If $p \in \mathbb{P}$ is semi-$M^{\mathbb{P}}$-generic, then $p \Vdash \omega_1 = \omega_1^V$.

Proof.

1° Let $g \rightsquigarrow V^{\mathbb{P}}$ with $p \in g$.

2° Since $M[g] \prec H_\theta^{V[g]}$, it holds that "$\omega_1$ is the smallest ordinal ξ with the property that $\sup(M[g] \cap \xi) < \xi$".

3° Since $M[g] \cap \omega_1^V = M \cap \omega_1^V$ and the last set is an ordinal, we see that the solution for ξ is exactly $\xi = \omega_1^V$.

□

[1.8.3] **Definition.** Let $\mathbb{P}$ be a poset and let $\theta > 2^{2^{|\mathsf{trcl}(\mathbb{P})|}}$ be regular. Poset $\mathbb{P}$ is *semi-proper* if for every countable $M \prec (H_\theta, \in, \mathbb{P})$ and every $p \in \mathbb{P} \cap M$ there exists $q \leq p$ which is semi-$M^{\mathbb{P}}$-generic.

[1.8.4] **Proposition.** Suppose that $\mathbb{P}$ is semi-proper. Then:

a. $\mathbb{P} \Vdash \omega_1 = \omega_1^V$

b. for every stationary $S \subseteq \omega_1$, it holds that $\mathbb{P} \Vdash (S \text{ is stationary})$.

Proof.

1° Let θ be sufficiently large regular.

2° By [1.8.2], it suffices for a to show that

$$D := \{p \in \mathbb{P} : (\exists M \prec (H_\theta, \in, \mathbb{P}))(|M| = \omega \wedge p \text{ is semi-}M^{\mathbb{P}}\text{-generic})\}$$

is dense in $\mathbb{P}$.

3° Let $p_0 \in \mathbb{P}$ be arbitrary and let $M \prec (H_\theta, \in, \mathbb{P}, p_0)$ be countable.

4° Since $\mathbb{P}$ is semi-proper, there is $p \leq p_0$ which is semi-$M^{\mathbb{P}}$-generic. Clearly, $p \in D$.

5° For b, suppose that S is stationary in ω_1, that $\Vdash \dot{f} : \omega_1 \to \omega_1$, and let $p \in \mathbb{P}$ be arbitrary. We want to find $q \leq p$ and $\xi \in S$ such that $q \Vdash \dot{f}[\xi] \subseteq \xi$.

6° Set of all countable $M \prec (H_\theta, \in, \mathbb{P}, p, \dot{f})$ is club in $[H_\theta]^\omega$.

7° Since $S \subseteq \omega_1$ is stationary in ω_1, there exists a countable $M \prec (H_\theta, \in, \mathbb{P}, p, \dot{f})$ satisfying $\xi := M \cap \omega_1 \in S$.

8° There exists $q \leq p$ which is semi-$M^{\mathbb{P}}$-generic.

9° For every $g \rightsquigarrow M^{\mathbb{P}}$ with $q \in g$, we have $\xi = M[g] \cap \omega_1$ and $M[g] \models \dot{f}^g : \omega_1 \to \omega_1$. We conclude $\dot{f}^g[\xi] \subseteq \xi$.

□

[1.8.5] **Definition.** Let α be a beth-fixed point, let M be a countable α-strong virtual model, and let $\mathbb{P} \in V_\alpha \cap M$ be a poset.

a. For $G \rightsquigarrow V^{\mathbb{P}}$, we say that G is *semi-$M^{\mathbb{P}}$-generic* if $M[G] \cap \omega_1^V = M \cap \omega_1^V$.

b. For $p \in \mathbb{P}$, we say that p is *semi-$M^{\mathbb{P}}$-generic* if $p \Vdash M[\dot{G}] \cap \omega_1^V = M \cap \omega_1^V$.

[1.8.6] **Proposition.** Let α be a beth-fixed point, let M be a countable α-strong virtual model, let $\mathbb{P} \in V_\alpha \cap M$ be a poset, and let $\theta > 2^{2^{|\mathsf{trcl}(\mathbb{P})|}}$ be a regular cardinal satisfying $\theta < \alpha$. For every $G \rightsquigarrow V^{\mathbb{P}}$, we have that G is semi-$M^{\mathbb{P}}$-generic in the sense of [1.8.5] if and only if it is semi-$(H_\theta \cap M)^{\mathbb{P}}$-generic in the sense of [1.8.1]. □

[1.8.7] **Proposition.** Let γ satisfy $V_\gamma \models \mathsf{ZFC}^-$, let M, N be countable γ-strong virtual models, let $\mathbb{P} \in M \cap V_\gamma$, and let $G \rightsquigarrow V^{\mathbb{P}}$. Suppose that $M \cong_\gamma N$. Then G is semi-$M^{\mathbb{P}}$-generic if and only if it is semi-$N^{\mathbb{P}}$-generic.

Proof. We have $M[G] \cong_\gamma N[G]$ and consequently

$$M[G] \cap \omega_1^V = N[G] \cap \omega_1^V.$$

□

[1.8.8] **Corollary.** Let γ satisfy $V_\gamma \models \mathsf{ZFC}^-$, let M be a countable γ-strong virtual model, let $\mathbb{P} \in M \cap V_\gamma$, and let $p \in \mathbb{P}$. Then p is semi-$M^{\mathbb{P}}$-generic if and only if it is semi-$(M \downarrow \gamma)^{\mathbb{P}}$-generic. □

1.9 Iterated Forcing Extensions of Virtual Models

[1.9.1] **Definition.** A *forcing iteration (with support E)* is a family $\vec{\mathbb{P}} = (\mathbb{P}_\alpha : \alpha \in E)$ where:

a. $E \subseteq \mathsf{Ord}$ and $E \neq \emptyset$;

b. $\mathbb{P}_\alpha$ is a poset for all α;

c. $\mathbb{P}_\alpha$ is a complete sub-poset of $\mathbb{P}_\beta$ for all $\alpha \leq \beta$ in E.

[1.9.2] **Proposition.** Let $\vec{\mathbb{P}}$ be an iteration with support E and let $\mathbb{Q} := \cup\{\mathbb{P}_\alpha : \alpha \in E\}$. Then $\vec{\mathbb{P}}^\frown\mathbb{Q}$ is an iteration with support $E \cup \{\sup\{\xi + 1 : \xi \in E\}\}$.

[1.9.3] **Definition.** Let $\vec{\mathbb{P}}$ be a forcing iteration with support E, let $\delta \in E$, let $\alpha \in E \cap \delta$, and let $G_\delta \rightsquigarrow V^{\mathbb{P}_\delta}$. Then the *$\alpha$-restriction of the generic* G_δ is defined as $G_\alpha := G_\delta \cap \mathbb{P}_\alpha$.

[1.9.4] **Remark.** It holds that $G_\alpha \rightsquigarrow V^{\mathbb{P}_\alpha}$.

[1.9.5] **Definition.** Let A be admissible and let $\vec{\mathbb{P}}$ be a forcing iteration with support E. Iteration $\vec{\mathbb{P}}$ is said to be *point-wise definable over A* if $\vec{\mathbb{P}} \subseteq A$ and for every $\alpha \in E$, constant $\mathbb{P}_\alpha$ is definable over A from the parameter α.

[1.9.6] **Definition.** Let A be admissible and let $\vec{\mathbb{P}}$ be an iteration with support E. Suppose that $\vec{\mathbb{P}}$ is point-wise definable over A. Let $M \prec A$, let $\delta \in E$, and let $G_\delta \rightsquigarrow V^{\mathbb{P}_\delta}$. Elementary submodels $M_{<\alpha}^{G_\delta}$ and $M_\alpha^{G_\delta}$ of A and elementary submodels $M_\alpha[G_\delta]$ of $A[G_\alpha]$ (inside $V[G_\alpha]$) are defined by recursion on $\alpha \in E \cap [0, \delta]$, as follows.

a. $M_{<\alpha}^{G_\delta} := M \cup \bigcup_{\xi \in E \cap \alpha} M_\xi^{G_\delta}$

b. if $\alpha \in M_{<\alpha}^{G_\delta}$, we define $M_\alpha[G_\delta] := M_{<\alpha}^{G_\delta}[G_\alpha]$ and $M_\alpha^{G_\delta} := (M_{<\alpha}^{G_\delta})^{G_\alpha} = M_\alpha[G_\delta] \cap A$;

c. if $\alpha \notin M_{<\alpha}^{G_\delta}$, we define $M_\alpha^{G_\delta} := M_{<\alpha}^{G_\delta}$ and leave $M_\alpha[G_\delta]$ undefined.

[1.9.7] **Proposition.** The previous definition is correct and for all $\alpha, \beta \in E$, we have:

a. $M_{<\alpha}^{G_\delta} \prec M_\alpha^{G_\delta} \prec A$

b. $M_\alpha^{G_\delta} \prec M_{<\beta}^{G_\delta}$ whenever $\alpha < \beta$;

c. $M_\alpha[G_\delta] \prec A[G_\alpha]$ whenever $\alpha \in M_{<\alpha}^{G_\delta}$.

□

[1.9.8] **Proposition.** Let A be admissible, let $\vec{\mathbb{P}}$ be an iteration with support E that is point-wise definable over A, let $M \prec A$, let $\delta_1, \delta_2 \in E$ with $\delta_1 \leq \delta_2$, let $G_{\delta_2} \rightsquigarrow V^{\mathbb{P}_{\delta_2}}$. Then for all $\alpha \in E \cap [0, \delta_1]$, it holds that:

a. $M_{<\alpha}^{G_{\delta_1}} = M_{<\alpha}^{G_{\delta_2}}$

b. $M_{\alpha}^{G_{\delta_1}} = M_{\alpha}^{G_{\delta_2}}$

c. $M_\alpha[G_{\delta_1}] = M_\alpha[G_{\delta_2}]$ whenever is either of them defined.

□

[1.9.9] **Proposition.** Let A be admissible, let $\vec{\mathbb{P}}$ be an iteration with support E that is point-wise definable over A, let $M \prec A$, let $\delta \in E$, let $G_\delta \rightsquigarrow V^{\mathbb{P}_\delta}$, and let $\alpha \in E \cap [0, \delta]$. Suppose that $\alpha \in M$. Then

$$M_\alpha^{G_\delta} = M^{G_\alpha}, \quad M_\alpha[G_\delta] = M[G_\alpha].$$

Proof.

1° It suffices to prove $M_\alpha[G_\delta] = M[G_\alpha]$.

2° We have $M \subseteq M_{<\alpha}^{G_\delta}$, so we conclude

$$M[G_\alpha] \subseteq M_{<\alpha}^{G_\delta}[G_\alpha] = M_\alpha[G_\delta].$$

3° We consider now the other inclusion. Let us first verify that

$$M_{<\alpha}^{G_\delta} \subseteq M[G_\alpha].$$

4° This is done by inductively showing

$$M_\xi^{G_\delta} \subseteq M[G_\alpha]$$

for all $\xi \in E \cap \alpha$.

5° Assume that the statement is true all $\eta \in E \cap \xi$ and let us verify it for ξ.

6° By the assumption,

$$M_{<\xi}^{G_\delta} = \bigcup_{\eta \in E \cap \xi} M_\eta^{G_\delta} \subseteq M[G_\alpha].$$

7° If $\xi \notin M_{<\xi}^{G_\delta}$, then

$$M_\xi^{G_\delta} = M_{<\xi}^{G_\delta} \subseteq M[G_\alpha].$$

Hence, let us assume $\xi \in M_{<\xi}^{G_\delta}$

8° Let $x \in M_\xi^{G_\delta}$ be arbitrary. Then there exists $\dot{x} \in M_{<\xi}^{G_\delta} \cap V^{\mathbb{P}_\xi}$ such that $x = \dot{x}^{G_\xi}$.

9° By 6°, we have $\dot{x}, \xi \in M[G_\alpha]$.

10° Since also $G_\alpha \in M[G_\alpha]$, we conclude $G_\xi \in M[G_\alpha]$.

11° Hence, $x = \dot{x}^{G_\xi} \in M[G_\alpha]$.

12° This concludes the induction. Let us now go back to the main point, i.e. $M_\alpha[G_\delta] \subseteq M[G_\alpha]$.

13° Let $x \in M_\alpha[G_\delta]$ be arbitrary. Then there exists $\dot{x} \in M_{<\alpha}^{G_\delta} \cap V^{\mathbb{P}_\alpha}$ such that $x = \dot{x}^{G_\alpha}$.

14° By what we have just proved, we have $\dot{x} \in M[G_\alpha]$.

15° Since $G_\alpha \in M[G_\alpha]$ as well, we conclude $x = \dot{x}^{G_\alpha} \in M[G_\alpha]$.

□

[1.9.10] **Notation.** Let A be admissible, let $\vec{\mathbb{P}}$ be an iteration with support E that is point-wise definable over A, let $M \prec A$, let $\delta \in E$, let $G_\delta \rightsquigarrow V^{\mathbb{P}_\delta}$, and let $\alpha \in E \cap [0, \delta]$. We shall henceforth write

$$M^{G_\alpha} := M_\alpha^{G_\delta}, \quad M[G_\alpha] := M_\alpha[G_\delta].$$

By the previous two propositions, this notation introduces no ambiguity.

[1.9.11] **Proposition.** Let A be admissible, let $\vec{\mathbb{P}}$ be an iteration with the support E which is point-wise definable over A, let $M \prec A$, let $\alpha, \delta \in E$ with $\alpha \leq \delta$, and let $G_\delta \rightsquigarrow V^{\mathbb{P}_\delta}$. Suppose that $\delta \in M^{G_\alpha}$. Then

$$M[G_\delta] = M^{G_\alpha}[G_\delta] \text{ and } M^{G_\delta} = (M^{G_\alpha})^{G_\delta}.$$

Proof. This is in fact obvious. Namely, the second equality simply states that extending M to M^{G_δ} according to the iteration $(\mathbb{P}_\xi : \xi \in E \cap [0, \delta])$ is the same as extending M to M^{G_α} according to $(\mathbb{P}_\xi : \xi \in E \cap [0, \alpha])$ and then extending M^{G_α} to $(M^{G_\alpha})^{G_\delta}$ according to $(\mathbb{P}_\xi : \xi \in E \cap (\alpha, \delta])$. An additional note is only required in the case $\alpha = \delta \in M^{G_\delta}$, where we apply e of [1.7.2]. For the first equality, we now have

$$M^{G_\alpha}[G_\delta] = (M^{G_\alpha})^{G_\delta}[G_\delta] = M^{G_\delta}[G_\delta] = (M^{G_\delta}_{<\delta})^{G_\delta}[G_\delta] = M^{G_\delta}_{<\delta}[G_\delta] = M[G_\delta],$$

where the first equality follows from [1.7.2], the second follows from what we have just prove, the third by definition, the forth again by [1.7.2], and the fifth by definition. □

[1.9.12] **Proposition.** Let γ be such that $V_\gamma \models \mathsf{ZFC}^-$, let A be admissible, let $M \prec A$ be a γ-strong virtual model, let $\vec{\mathbb{P}} \in V_\gamma$ be an iteration with support E which is point-wise definable over A, let $\alpha \in E$, and let $G_\alpha \rightsquigarrow V^{\mathbb{P}_\alpha}$. Suppose that $M[G_\alpha]$ is defined. Then $M[G_\alpha]$ is γ-strong in $V[G_\alpha]$.

Proof. This follows immediately from [1.7.3]. □

[1.9.13] **Proposition.** Let γ be such that $V_\gamma \models \mathsf{ZFC}^-$, let A be admissible, let $M, N \prec A$ be γ-strong virtual models, let $\vec{\mathbb{P}} \in V_\gamma$ be an iteration with support E which is point-wise definable over A, let $\alpha \in E$, and let $G_\alpha \rightsquigarrow V^{\mathbb{P}_\alpha}$. Suppose that $M \cong_\gamma N$. Then $M^{G_\alpha} \cong_\gamma N^{G_\alpha}$ and the isomorphism witnessing this fact extends the isomorphism witnessing $M \cong_\gamma N$.

Proof. This follows by induction on α. To handle the successor step, we simply apply [1.7.5]. Since the isomorphisms extend each other, the limit step is handled by simply taking the union of the previous isomorphisms. □

[1.9.14] **Proposition.** Let γ be such that $V_\gamma \models \mathsf{ZFC}^-$, let A be admissible, let $M, N \prec A$ be γ-strong virtual models, let $\vec{\mathbb{P}} \in V_\gamma$ be an iteration with support E which is point-wise definable over A, let $\alpha \in E$, and let $G_\alpha \rightsquigarrow V^{\mathbb{P}_\alpha}$. Suppose that $M[G_\alpha]$ is defined. Then $N[G_\alpha]$ is also defined and $M[G_\alpha] \cong_\gamma N[G_\alpha]$.

Proof. This follows from the previous proposition and [1.7.5]. □

2 Semi-proper Iteration

2.1 Setup for the Iteration

[2.1.1] **Definition.** A *blueprint for a semi-proper iteration* is a pair $(\mathbb{V}, \mathbf{Q})$ satisfying

a. $\mathbb{V} = (V_\kappa, \in, U)$,

b. κ is inaccessible,

c. $U \subseteq V_\kappa$,

d. $\mathbf{Q} : \kappa \times V_\kappa \to V_\kappa$ is definable without parameters over $\mathbb{V}$,

e. for all $\alpha < \kappa$ and for all $\mathbb{P} \in V_\kappa$, if $\mathbb{P}$ is a poset, then $\mathbf{Q}(\alpha, \mathbb{P}) \in V^{\mathbb{P}}$ and $\Vdash_{\mathbb{P}}$"$\mathbf{Q}(\alpha, \mathbb{Q})$ is semi-proper".

[2.1.2] **Declaration.** We fix a blueprint $(\mathbb{V}, \mathbf{Q})$ for a semi-proper iteration, where $\mathbb{V} = (V_\kappa, \in, U)$. We will assume for the rest of the notes that the default language for virtual models is $\{\in, \dot{U}\}$, where $\dot{U}$ is a unary predicate.

[2.1.3] **Notation.**

a. For $\alpha < \kappa$, we denote $\mathbb{V}_\alpha := (V_\alpha, \in, U \cap V_\alpha)$.

b. We denote $\mathscr{E} := \mathscr{E}_{\mathbb{V}} = \{\alpha < \kappa : \mathbb{V}_\alpha \prec \mathbb{V}\}$.

c. $\mathscr{E}^+ := \{\alpha + 1 : \alpha \in \mathscr{E}\}$

d. $\mathscr{E}^* := \mathscr{E} \cup \mathscr{E}^+$.

[2.1.4] **Definition.**

a. For $\alpha \in \mathscr{E}$, we say that a virtual model M is *α-correct* if $\mathbb{V}_\alpha \prec \widehat{M}$.

b. For $\alpha \in \mathscr{E}$, we define set

$$\mathcal{C}_\alpha := \{M : M \text{ is countable, } \alpha\text{-correct, and } \alpha\text{-generated virtual model}\}.$$

c. For $S \subseteq \mathsf{Ord}$, we define

$$\mathcal{C}_S := \bigcup_{\alpha \in S \cap \mathscr{E}} \mathcal{C}_\alpha.$$

[2.1.5] **Remark.** Traditionally, forcing iterations are constructed as follows:

$$\begin{aligned} \mathbb{P}_0 &:= \{1\} \\ \mathbb{P}_{\alpha+1} &:= \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha \\ \mathbb{P}_\lambda &\subseteq \{p : (\forall \alpha < \lambda)(p \restriction \alpha \in \mathbb{P}_\alpha)\} \end{aligned}$$

(λ limit). In the limit step, threads are required to satisfy some sort of a "support condition". In our iteration, we will double the steps: after adding a poset $\dot{\mathbb{Q}}_\alpha$, we will have an additional step of adding a "scaffolding". At the limit stages, the threading will now be controlled by that scaffolding.

[2.1.6] **Notation.**

a. For convenience, the iteration will be indexed by $\mathscr{E}^*$ and the initial poset $\mathbb{P}_{\min \mathscr{E}}$ will not be equal to $\{1\}$. Nevertheless, poset $\mathbb{P}_{\min \mathscr{E}}$ will stil be trivial.

b. Stages $\alpha \in \mathscr{E}^+$ correspond to adding a poset.

c. Stages α where α is a successor point of $\mathscr{E}$ correspond to the operation of adding a scaffolding.

d. Limit points of $\mathscr{E}$ are limit stages of the iteration.

e. The canonical name for a $\mathbb{P}_\alpha$-generic will be denoted by $\dot{G}_\alpha$ (for $\alpha \in \mathscr{E}^*$). Accordingly, a $\mathbb{P}_\alpha$-generic will have denotation G_α and for $\beta \in \mathscr{E}^* \cap \alpha$, we will write $G_\beta := G_\alpha \cap \mathbb{P}_\beta$.

[2.1.7] **Remark.** The iteration $(\mathbb{P}_\alpha : \alpha \in \mathscr{E}^*)$ is defined by recursion. The recursive step of the definition is described in the following subsection.

2.2 Recursive Step of the Definition

[2.2.1] **Declaration.**

a. Let $\delta \in \mathscr{E}$ and let $\vec{\mathbb{P}} = (\mathbb{P}_\alpha : \alpha \in \mathscr{E}^* \cap \delta)$ be a forcing iteration.

b. Suppose that $\vec{\mathbb{P}}$ is point-wise definable over $\mathbb{V}$.

c. Suppose that for every $\alpha \in \mathscr{E} \cap \delta$ and every $p \in \mathbb{P}_\alpha$, it holds that that $p = (w_p, \mathcal{M}_p)$ for some finite partial function $w_p :: \mathscr{E} \cap \alpha \to V_\kappa$ and some finite $\mathcal{M}_p \subseteq \mathcal{C}_{\leq\alpha}$.

d. Let $(\dot{\mathbb{Q}}_\alpha : \alpha \in \mathscr{E} \cap \delta)$ be a sequence of names satisfying $\dot{\mathbb{Q}}_\alpha \in V^{\mathbb{P}_\alpha}$ and

$$\mathbb{P}_\alpha \Vdash \dot{\mathbb{Q}}_\alpha \text{ is a semi-proper poset}$$

for every α.

[2.2.2] **Definition.** Suppose that $\delta = \min \mathscr{E}$. Then *poset* $\mathbb{P}_\delta$ consists of all pairs $p = (w_p, \mathcal{M}_p)$ where $w_p = \emptyset$ and $\mathcal{M}_p$ is a finite subset of $\mathcal{C}_\delta$. *The order* $q \leq p$ *in* $\mathbb{P}_\delta$ holds if and only if $\mathcal{M}_q \supseteq \mathcal{M}_p$.

[2.2.3] **Definition.** Let $M \in \mathcal{C}_{\geq\delta}$, let $\alpha \in \mathscr{E} \cap \delta$, and let $G_\alpha \rightsquigarrow V^{\mathbb{P}_\alpha}$. We say that M^{G_α} *is active at* δ in either of the following cases:

a. there exists a predecessor γ of ordinal δ inside $\mathscr{E}$ and $\gamma \in M^{G_\alpha}$;

b. ordinal δ is a limit point of $\mathscr{E}$ and $\sup(M^{G_\alpha} \cap \mathscr{E} \cap \delta) = \delta$.

[2.2.4] **Definition.** Let $\mathcal{M} \in \mathcal{C}_{[0,\kappa]}$, let $\alpha \in \mathscr{E} \cap \delta$, and let $G_\alpha \rightsquigarrow V^{\mathbb{P}_\alpha}$. We define

$$\mathcal{M}^\delta[G_\alpha] := \{M \downarrow \delta : M \in \mathcal{M},\ M^{G_\alpha} \text{ is active at } \delta\}.$$

[2.2.5] **Definition.** Let $\mathcal{M} \subseteq \mathcal{C}_\delta$, let $\alpha < \delta$, and let $G_\alpha \rightsquigarrow V^{\mathbb{P}_\alpha}$. The set $\mathcal{M}$ is *weak* $\lhd_\delta$*-chain w.r.t.* G_α if for every $M, N \in \mathcal{M}$, it holds that:

a. $\omega_1 \cap M = \omega_1 \cap N \implies M = N$;

b. $\omega_1 \cap M < \omega_1 \cap N \implies M \lhd_\delta N_\alpha^{G_\alpha}$.

[2.2.6] **Remark.** If in the second point the model $N_\alpha^{G_\alpha}$ is replaced by N, we get the usual notion of a $\lhd_\delta$-chain.

[2.2.7] **Remark.** Suppose that $\alpha \leq \beta < \delta$, that $G_\beta \rightsquigarrow V^{\mathbb{P}_\beta}$, and that $\mathcal{M}$ is a weak $\lhd_\delta$-chain w.r.t. G_α. Then $\mathcal{M}$ is a weak $\lhd_\delta$-chain w.r.t. G_β.

[2.2.8] **Notation.** Let $p = (w_p, \mathcal{M}_p)$ where w_p is a partial function on $\mathscr{E}$ and $\mathcal{M}_p \subseteq \mathcal{C}_{[0,\kappa]}$ and let $\alpha \in \mathscr{E}$. Then

$$\mathcal{M}_p \downarrow \alpha := \{M \downarrow \alpha : M \in \mathcal{M}_p\},$$
$$p \restriction \alpha := (w_p \restriction \alpha, \mathcal{M}_p \downarrow \alpha),$$
$$p \restriction (\alpha + 1) := (w_p \restriction (\alpha + 1), \mathcal{M}_p \downarrow \alpha).$$

[2.2.9] **Definition.** Suppose that δ has a predecessor γ inside $\mathscr{E}$. The *the poset* $\mathbb{P}_\delta$ consists of all pairs $p = (w_p, \mathcal{M}_p)$ satisfying:

a. object w_p is a finite partial function $\mathscr{E} \cap \delta \to V_\kappa$;

b. object $\mathcal{M}_p$ is a finite subset of $\mathcal{C}_{\leq\delta}$;

c. $p \restriction \alpha \in \mathbb{P}_\alpha$ for every $\alpha \in \mathscr{E}^* \cap \delta$;

d. $p \restriction (\gamma + 1) \Vdash_{\mathbb{P}_{\gamma+1}} (\mathcal{M}_p^\delta[\dot{G}_\gamma]$ is a weak $\lhd_\delta$ -chain w.r.t. $\dot{G}_{\gamma+1})$

e. if $\gamma \in \mathsf{dom}(w_p)$, then

$$p \restriction \gamma \Vdash_{\mathbb{P}_\gamma} (\forall M \in \mathcal{M}_p^\delta[\dot{G}_\gamma])(w_p(\gamma) \text{ is semi-} M[\dot{G}_\gamma]^{\dot{\mathbb{Q}}_\gamma}\text{-generic}).$$

The order $q \leq p$ *in* $\mathbb{P}_\delta$ holds if and only if:

f. $q \restriction \alpha \leq_{\mathbb{P}_\alpha} p \restriction \alpha$ for every $\alpha \in \mathscr{E}^* \cap \delta$;

g. $\mathcal{M}_q \cap \mathcal{C}_\delta \supseteq \mathcal{M}_p \cap \mathcal{C}_\delta$.

[2.2.10] **Definition.** Suppose that δ is a limit point of $\mathscr{E}$. Then *the poset* $\mathbb{P}_\delta$ consists of all pairs $p = (w_p, \mathcal{M}_p)$ satisfying:

a. object w_p is a finite partial function $\mathscr{E} \cap \delta \to V_\kappa$;
b. object $\mathcal{M}_p$ is a finite subset of $\mathcal{C}_{\leq\delta}$;
c. $p \restriction \alpha \in \mathbb{P}_\alpha$ for every $\alpha \in \mathscr{E}^* \cap \delta$;
d. there exists $\delta_0 < \delta$ such that for every $\alpha \in \mathscr{E} \cap (\delta_0, \delta)$ we have that

$$p \restriction (\alpha+1) \Vdash_{\mathbb{P}_{\alpha+1}} (\mathcal{M}_p^\delta[\dot{G}_\alpha] \text{ is a weak } \lhd_\delta\text{-chain w.r.t. } \dot{G}_{\alpha+1}).$$

The order $q \leq p$ *in* $\mathbb{P}_\delta$ holds if and only if:

e. $q \restriction \alpha \leq_{\mathbb{P}_\alpha} p \restriction \alpha$ for every $\alpha \in \mathscr{E}^* \cap \delta$;
f. $\mathcal{M}_q \cap \mathcal{C}_\delta \supseteq \mathcal{M}_p \cap \mathcal{C}_\delta$.

[2.2.11] **Definition.** $\dot{\mathbb{Q}}_\delta := \mathbf{Q}(\delta, \mathbb{P}_\delta)$

[2.2.12] **Definition.** *Poset* $\mathbb{P}_{\delta+1}$ consists of all pairs $(w_p, \mathcal{M}_p)$ satisfying:

a. object w_p is a finite partial function of the type $\mathscr{E} \cap (\delta+1) \to V_\kappa$;
b. object $\mathcal{M}_p$ is a finite subset of $\mathcal{C}_{\leq\delta}$;
c. $p \restriction \delta \in \mathbb{P}_\delta$
d. if $\delta \in \mathsf{dom}(w_p)$, then $w_p(\delta)$ is a canonical $\mathbb{P}_\delta$-name for an element of $\dot{\mathbb{Q}}_\delta$.

The order $q \leq p$ *in* $\mathbb{P}_{\delta+1}$ holds if and only if:

e. $q \restriction \delta \leq_{\mathbb{P}_\delta} p \restriction \delta$
f. if $\delta \in \mathsf{dom}(w_p)$, then $\delta \in \mathsf{dom}(w_q)$ and $q \restriction \delta \Vdash_{\mathbb{P}_\delta} w_q(\delta) \leq w_p(\delta)$.

2.3 Basic Properties

[2.3.1] **Definition.** *The semi-proper iteration given by the blueprint* $(\mathbb{V}, \mathbf{Q})$ is obtained by recursively iterating the construction of Subsection 2.2.

[2.3.2] **Definition.** Let $\mathbb{P}$ and $\mathbb{Q}$ be posets and let $\pi : \mathbb{Q} \to \mathbb{P}$. Suppose that $\mathbb{P}$ is a suborder of $\mathbb{Q}$. Then π is said to be *restriction of conditions* if

a. $(\forall p \in \mathbb{P})(\pi(p) = p)$,
b. $(\forall p, q \in \mathbb{Q})(p \leq q \implies \pi(p) \leq \pi(q))$,
c. $(\forall p \in \mathbb{P})(\forall q \in \mathbb{Q})(p \leq \pi(q) \implies p \parallel q)$.

[2.3.3] **Proposition.** $(\mathbb{P}_\alpha : \alpha \in \mathscr{E}^*)$ is a correctly defined forcing iteration. For elements $\alpha \leq \delta$ of $\mathscr{E}^*$, the mapping $\mathbb{P}_\delta \to \mathbb{P}_\alpha : p \mapsto p \restriction \alpha$ is a restriction of conditions.

Proof.

1° Let $\delta \in \mathscr{E}$ and let us assume recursively that $(\mathbb{P}_\alpha : \alpha \in \mathscr{E}^* \cap \delta)$ has been defined correctly and that it is an iteration.

2° We can now define the poset $\mathbb{P}_\delta$. We need to verify $\mathbb{P}_\alpha$ is a complete subposet of $\mathbb{P}_\delta$ for every $\alpha \in \mathscr{E}^* \cap \delta$.

3° It is immediate from the definitions that $\mathbb{P}_\alpha$ is a suborder of $\mathbb{P}_\delta$. It suffices for the conclusion to verify that the mapping $\mathbb{P}_\delta \to \mathbb{P}_\alpha : p \mapsto p \restriction \alpha$ is a restriction of conditions, which is also obvious.

4° We can now define the poset $\mathbb{P}_{\delta+1}$.

5° It is clear that $\mathbb{P}_\delta$ is a suborder of $\mathbb{P}_{\delta+1}$ and it is a matter of routine to verify that the mapping $\mathbb{P}_{\delta+1} \to \mathbb{P}_\delta : p \mapsto p \restriction \delta$ is a restriction of conditions. Hence, $\mathbb{P}_\delta$ is a complete subposet of $\mathbb{P}_{\delta+1}$.

□

[2.3.4] **Definition.** $\mathbb{P}_\kappa := \bigcup_{\alpha\in\mathscr{E}^*} \mathbb{P}_\alpha$

[2.3.5] **Proposition.** $(\mathbb{P}_\alpha : \alpha \in \mathscr{E}^* \cup \{\kappa\})$ is a forcing iteration. Mapping $\mathbb{P}_\kappa \to \mathbb{P}_\alpha : p \mapsto p \restriction \alpha$ is a restriction of conditions for all $\alpha \in \mathscr{E}^*$. □

[2.3.6] **Proposition.** Let $\delta \in \mathscr{E}$.

- Let $p = (w_p, \mathcal{M}_p)$. Then $p \in \mathbb{P}_\delta$ if and only if the following holds:
  a. w_p is a finite partial function from $\mathscr{E} \cap \delta$ into V_κ;
  b. $\mathcal{M}_p \in [\mathcal{C}_{\leq\delta}]^{<\omega}$
  c. for all $\gamma \in (\min \mathscr{E}, \delta]$, there exists $\beta \in \mathscr{E} \cap \gamma$ such that for all $\alpha \in \mathscr{E} \cap [\beta, \gamma)$,
  $$p \restriction (\alpha+1) \Vdash_{\mathbb{P}_{\alpha+1}} (\mathcal{M}_p^\gamma[\dot{G}_\alpha] \text{ is a weak } \lhd_\gamma\text{-chain w.r.t. } \dot{G}_{\alpha+1});$$
  d. for all $\beta \in \mathscr{E} \cap \delta$ and for γ the successor of β inside $\mathscr{E}$,
  $$\beta \in \mathsf{dom}(w_p) \implies p \restriction \beta \Vdash_{\mathbb{P}_\beta} (\forall M \in \mathcal{M}_p^\gamma[\dot{G}_\beta])(w_p(\beta) \text{ is semi-}M[\dot{G}_\beta]^{\dot{\mathbb{Q}}_\beta}\text{-generic}).$$
- Let $p, q \in \mathbb{P}_\delta$. Then $p \leq_{\mathbb{P}_\delta} q$ if and only if:
  e. $\mathsf{dom}(w_p) \supseteq \mathsf{dom}(w_q)$
  f. for all $\alpha \in \mathsf{dom}(w_q)$,
  $$p \restriction \alpha \Vdash_{\mathbb{P}_\alpha} w_p(\alpha) \leq_{\mathbb{Q}_\alpha} w_q(\alpha);$$
  g. for all $N \in \mathcal{M}_q$ and for $\gamma := \sup\{\beta : V_\beta \subseteq \widehat{N}\}$, there exists $M \in \mathcal{M}_q$ such that $N = M \downarrow \gamma$.

[2.3.7] **Proposition.** Iteration $(\mathbb{P}_\alpha : \alpha \in \mathscr{E}^*)$ is point-wise definable over $\mathbb{V}$. □

[2.3.8] **Proposition.** Caridinality of $\mathbb{P}_\alpha$ is strictly less then β for $\alpha \in \mathscr{E}^*$ and $\beta = \min(\mathscr{E} - (\alpha+1))$. □

[2.3.9] **Proposition.** $\mathbb{P}_{\delta+1}$ is forcing equivalent to $\mathbb{P}_\delta * \dot{\mathbb{Q}}_\delta$ for every $\delta \in \mathscr{E}$. □

[2.3.10] **Remark.** The previous equivalence is canonical.

[2.3.11] **Remark.** Suppose that $\beta < \gamma$ are elements of $\mathscr{E}^*$. Recall that we have a canonical name $\mathbb{P}_\gamma/\mathbb{P}_\beta \in V^{\mathbb{P}_\beta}$ that satisfies
$$\Vdash_{\mathbb{P}_\beta} \mathbb{P}_\gamma/\mathbb{P}_\beta = \{q \in \mathbb{P}_\gamma : q \restriction \beta \in \dot{G}_\beta\}.$$
This is a name for a poset and the following is verified.

a. $\mathbb{P}_\beta * (\mathbb{P}_\gamma/\mathbb{P}_\beta)$ is forcing equivalent to $\mathbb{P}_\gamma$.
b. For every dense subset D of the poset $\mathbb{P}_\gamma$, the set $\{q \in D : q \restriction \beta \in \dot{G}_{\mathbb{P}_\beta}\}$ is dense in the poset $\mathbb{P}_\gamma/\mathbb{P}_\beta$ inside the universe $V^{\mathbb{P}_\beta}$.
c. If $G_\beta \rightsquigarrow V^{\mathbb{P}_\beta}$ and $H \rightsquigarrow V[G_\beta]^{(\mathbb{P}_\gamma/\mathbb{P}_\beta)^{G_\beta}}$, then H is also a filter in $\mathbb{P}_\gamma$. If we want to think of it as such, we denote it by $G_\beta \cdot H$. Note that $G_\beta \cdot H$ corresponds to $G_\beta * H$ under the equivalence $\mathbb{P}_\gamma \simeq \mathbb{P}_\beta * (\mathbb{P}_\gamma/\mathbb{P}_\beta)$. In particular, $G_\beta \cdot H \rightsquigarrow V^{\mathbb{P}_\gamma}$ and $G_\beta \subseteq G_\beta \cdot H$.

[2.3.12] **Definition.** Let $\alpha \leq \beta$ be elements of $\mathscr{E}^*$, let $p \in \mathbb{P}_\alpha$, and let $q \in \mathbb{P}_\beta$. Suppose that $p \leq q \restriction \alpha$. Then we define $pq = (w, \mathcal{M})$ as follows:
$$w := w_p \cup (w_q \restriction (\alpha, \beta]), \quad \mathcal{M} := \mathcal{M}_p \cup \mathcal{M}_q.$$

[2.3.13] **Proposition.** Let $\alpha \leq \beta$ be elements of $\mathscr{E}^*$, let $p \in \mathbb{P}_\alpha$, and let $q \in \mathbb{P}_\beta$. Suppose that $p \leq q \restriction \alpha$. Then $pq \in \mathbb{P}_\beta$, it satisfies $pq \leq p, q$ and $pq \restriction \alpha = p$, and
$$(\forall r \in \mathbb{P}_\beta)(r \leq p, q \iff r \leq pq).$$

□

2.4 Statement of Transfer Theorem

[2.4.1] **Definition.** Let

a. M : virtual model,

b. $(\mathbb{S}_\alpha : \alpha \in E)$: forcing iteration point-wise definable over $\widehat{M}$,

c. $\gamma \in E$,

d. $p \in \mathbb{S}_\gamma$.

Suppose that there exists a beth-fixed point α such that $\vec{\mathbb{S}} \in V_\alpha \subseteq \widehat{M}$.
Then we say that p *is locally* $M^{\mathbb{S}_\gamma}$*-generic* if

$$p \Vdash^V_{\mathbb{S}_\gamma} M^{\dot{G}_\gamma} = M^{\dot{G}_\gamma}_{<\gamma}.$$

[2.4.2] **Remark.** If $E = \{0\}$, then p is locally $M^{\mathbb{S}_0}$-generic if and only if p is $M^{\mathbb{S}_0}$-generic[1]. In general, "$M^{\mathbb{S}_\gamma}$-generic" implies "locally $M^{\mathbb{S}_\gamma}$-generic", but we will use this notion here to propagate semi-genericity through our iteration.

[2.4.3] **Transfer Theorem.** Let $\gamma \in \mathscr{E}$. Suppose that for all $\alpha \in \mathscr{E} \cap \gamma$, for all $M \in \mathcal{C}_{>\alpha}$, for all $p \in \mathbb{P}_\alpha$ satisfying $M \downarrow \alpha \in \mathcal{M}_p$, $p \Vdash_{\mathbb{P}_\alpha} M^{\dot{G}_\alpha} \cap \omega_1^V = M \cap \omega_1^V$. Then for all $M \in \mathcal{C}_{>\gamma}$, every condition $p \in \mathbb{P}_\gamma$ satisfying $M \downarrow \gamma \in \mathcal{M}_p$ is locally $M^{\mathbb{P}_\gamma}$-generic.

Proof. The case $\gamma = \min \mathscr{E}$ is obvious since the poset $\mathbb{P}_\gamma$ is trivial. The case where γ is a successor point of $\mathscr{E}$ is proved in Subsubsection 2.6.1 and the case where γ is a limit point of $\mathscr{E}$ is proved in Subsubsection 2.6.2. □

[2.4.4] **Local Genericity Criterion.**

Let

- $(\mathbb{S}_\alpha : \alpha \in E)$: forcing iteration,
- M : virtual model,
- $\gamma \in E$,
- $p \in \mathbb{S}_\gamma$.

Suppose that

1. there exists a beth-fixed point α such that $\vec{\mathbb{S}} \in V_\alpha \subseteq \widehat{M}$,
2. for all $q \leq_{\mathbb{S}_\gamma} p$, for all dense open subsets D of $\mathbb{S}_\gamma$ satisfying $q \Vdash_{\mathbb{S}_\gamma} D \in M^{\dot{G}_\gamma}_{<\gamma}$, there exist $r \in D$ and $s \leq_{\mathbb{S}_\gamma} q, r$ such that

$$s \Vdash_{\mathbb{S}_\gamma} \gamma \notin M^{\dot{G}_\gamma}_{<\gamma} \vee r \in M^{\dot{G}_\gamma}_{<\gamma}.$$

Then p is locally $M^{\mathbb{S}_\gamma}$-generic.

Proof.

1° Let $p_0 \leq p$ be arbitrary and let us show that there exists $p_1 \leq p_0$ such that

$$q \Vdash M^{\dot{G}_\gamma} = M^{\dot{G}_\gamma}_{<\gamma}.$$

2° If $p_0 \Vdash \gamma \notin M^{\dot{G}_\gamma}_{<\gamma}$, then the conclusion follows by definition. Let us assume that $p_0 \nVdash \gamma \notin M^{\dot{G}_\gamma}_{<\gamma}$.

3° Then there exists $p_1 \leq p_0$ such that $p_1 \Vdash \gamma \in M^{\dot{G}_\gamma}_{<\gamma}$. We claim that p_1 is as required.

4° Let $\dot{\tau} \in V^{\mathbb{S}_\gamma}$ be such that $p_1 \Vdash \dot{\tau} \in (M^{\dot{G}_\gamma}_{<\gamma})^{\mathbb{S}_\gamma}$. We want to show that $p_1 \Vdash \dot{\tau}^{\dot{G}_\gamma} \in M^{\dot{G}_\gamma}_{<\gamma}$.

5° Let $p_2 \leq p_1$ be arbitrary. We want to find $s \leq p_2$ such that $s \Vdash \dot{\tau}^{\dot{G}_\gamma} \in M^{\dot{G}_\gamma}_{<\gamma}$.

[1] in the sense of Remark 1.2-XI.f. of [Kas22a]

6° Note that $p_2 \Vdash \dot{\tau} \in V^{\mathbb{S}_\gamma}$. This means that there exists $p_3 \leq p_2$ and some $\sigma \in V^{\mathbb{S}_\gamma}$ such that $p_3 \Vdash \dot{\tau} = \check{\sigma}$.

7° Let D be the set of all conditions in $\mathbb{S}_\gamma$ that decide the value of σ. Set D is dense open in $\mathbb{S}_\gamma$. For $r \in D$, let x_r be the value of σ as decided by r. We have

$$p_3 \Vdash D, (x_r : r \in D) \in M^{G_\gamma}_{<\gamma}.$$

8° There exist $r \in D$ and $s \leq p_3, r$ such that

$$s \Vdash r \in M^{G_\gamma}_{<\gamma}.$$

9° Then

$$s \Vdash \dot{\tau}^{\dot{G}_\gamma} = \check{\sigma}^{\dot{G}_\gamma} = x_s = x_r \in M^{G_\gamma}_{<\gamma},$$

the last fact being due to $s \Vdash (x_r)_{r \in D}, r \in M^{G_\gamma}_{<\gamma}$.

□

[2.4.5] **Remark.** In the case $E = \{\emptyset\}$, the above criterion reduces to the usual genericity criterion (see for example Lemma 1.2-IV of [Kas22b]).

2.5 Some Lemmas for Transfer Theorem

[2.5.1] **Lemma.** Let $\beta \in \mathscr{E}$, let $N \in \mathcal{C}_{\geq\beta}$, and let $p \in \mathbb{P}_\beta \cap N$. Then there exists $q \leq_{\mathbb{P}_\beta} p$ such that $\mathsf{dom}(w_q) = \mathsf{dom}(w_p)$ and $\mathcal{M}_q = \mathcal{M}_p \cup \{N \downarrow \beta\}$.

Proof.

1° Let $\mathcal{M}_q := \mathcal{M}_p \cup \{N \downarrow \beta\}$. We need to define w_q in such a way as to ensure $q := (w_q, \mathcal{M}_q) \in \mathbb{P}_\beta$ and $q \leq p$.

2° Note that for all $P \in \mathcal{M}_p$, we have $P \cap \omega_1^V < N \cap \omega_1^V$ and $P \lhd_\beta N$. Hence, we will be done with the argument if we construct w_q satisfying:

a. $\mathsf{dom}(w_q) = \mathsf{dom}(w_p) =: d$

b. $q \restriction \alpha \Vdash_{\mathbb{P}_\alpha} w_q(\alpha) \leq w_p(\alpha)$ for all $\alpha \in d$;

c. $q \restriction \alpha \Vdash_{\mathbb{P}_\alpha} \left(\alpha \in N^{\dot{G}_\alpha} \implies (w_p(\alpha) \text{ is semi-}N[\dot{G}_\alpha]^{\dot{\mathbb{Q}}_\alpha}\text{-generic})\right)$ for all $\alpha \in d$.

(Cf. Proposition [2.3.6].)

3° $w_q(\alpha)$ is defined by recursion on $\alpha \in d$. Suppose that $w_q \restriction \alpha$ has been defined and let us define $w_q(\alpha)$.

4° Let $G_\alpha \rightsquigarrow V^{\mathbb{P}_\alpha}$ be an arbitrary generic containing $q \restriction \alpha$.

5° Suppose that $\alpha \in N^{G_\alpha}$. Then $N[G_\alpha]$ is defined.

6° Since $\mathbb{Q}_\alpha := \dot{\mathbb{Q}}_\alpha^{G_\alpha} \in N[G_\alpha]$ is semi-proper and $w_p(\alpha)^{G_\alpha} \in N[G_\alpha]$, there exists $w \leq_{\mathbb{Q}_\alpha} w_p(\alpha)^{G_\alpha}$ which is semi-$N[G_\alpha]^{\mathbb{Q}_\alpha}$-generic.

7° Since G_α was arbitrary containing $q \restriction \alpha$, we can find a name $\dot{w} \in V^{\mathbb{P}_\alpha}$ such that

$$q \restriction \alpha \Vdash_{\mathbb{P}_\alpha} \dot{w} \leq_{\dot{\mathbb{Q}}_\alpha} w_p(\alpha) \wedge \left(\alpha \in N^{\dot{G}_\alpha} \implies (\dot{w} \text{ is semi-}N^{G_\alpha}\text{-generic})\right).$$

8° We can now set $w_q(\alpha) := \dot{w}$.

□

[2.5.2] **Lemma.** Let

- $\mu, \nu \in \mathscr{E}$: ν is the successor of μ in $\mathscr{E}$,
- $p \in \mathbb{P}_\nu$,
- $G_\mu \rightsquigarrow V^{\mathbb{P}_\mu}$: $p \restriction \mu \in G_\mu$,

- $M \in \mathcal{M}_p^\nu[G_\mu]$,
- $\mathbb{Q}_\mu := \dot{\mathbb{Q}}_\mu^{G_\mu}$,
- $u \in \mathbb{Q}_\mu \cap M[G_\mu]$.

<u>Then</u> there exists $v \in \mathbb{Q}_\mu$ such that $v \leq u$ and that for all $N \in \mathcal{M}_p^\nu[G_\mu]$ satisfying $N \cap \omega_1^V \geq M \cap \omega_1^V$, it holds that v is semi-$N[G_\mu]^{\mathbb{Q}_\mu}$-generic.

Proof.

1° Let
   - $\{M_i : i < n\} := \mathcal{M}_p^\nu[G_\mu]$: for all $i < j < n$, $M_i \cap \omega_1^V < M_j \cap \omega_1^V$,
   - $k < n$: $M_k = M$,
   - $\lambda := |\mathbb{Q}_\mu|$.

2° For all $i < n$, we have $\mathbb{Q}_\mu \in M_i[G_\mu]$ and $\mathbb{Q}_\mu$ is semi-proper (this follows from activity).

3° We may assume w.l.o.g. that $\mathbb{Q}_\mu = \lambda$.

4° Consider $i < n$. We let $N_i := M_i[G_\mu] \cap H((2^\lambda)^+) \prec H((2^\lambda)^+)$. Note that $w \in \mathbb{Q}_\mu$ is semi-$M_i[G_\mu]^{\mathbb{Q}_\mu}$-generic if and only if it is semi-$N_i^{\mathbb{Q}_\mu}$-generic.

5° Hence, it suffices to find $v \in \mathbb{Q}_\mu$ such that $v \leq u$ and which is semi-$N_i^{\mathbb{Q}_\mu}$-generic for all $i \in [k, n)$.

6° Since $M_i \lhd_\nu M_j[G_\mu]$ for all $i < j < n$, Proposition [1.6.9] implies that $N_i \in N_j$ for all $i < j < n$.

7° By recursion on $i \in [k, n)$, we construct a sequence $(u_i : k \leq i < n)$ of $\mathbb{Q}_\mu$-conditions satisfying $u_k := u \in N_k$ and satisfying for all $i \in (k, n)$ that $u_i \in N_i$, $u_i \leq_{\mathbb{Q}_\mu} u_{i-1}$, and u_i is semi-$N_{i-1}^{\mathbb{Q}_\mu}$-generic.

8° There exists $v \leq u_{n-1}$ such that v is semi-$N_{n-1}^{\mathbb{Q}_\mu}$-generic.

□

[2.5.3] **Lemma.** <u>Let</u>

- $\mu, \nu \in \mathscr{E}$: ν is the successor of μ in $\mathscr{E}$,
- $p \in \mathbb{P}_\nu$,
- $M \in \mathcal{M}_p$: $p \restriction \mu \Vdash_{\mathbb{P}_\mu} (M^{\dot{G}_\mu}$ is active at $\nu)$,
- $\dot{u} \in V^{\mathbb{P}_\mu}$: $p \restriction \mu \Vdash_{\mathbb{P}_\mu} \dot{u} \in \dot{\mathbb{Q}}_\mu \cap M[\dot{G}_\mu]$.

<u>Then</u> there exists a canonical $\mathbb{P}_\mu$-name $\dot{v}$ for an element of $\dot{\mathbb{Q}}_\mu$ such that $p \restriction \mu$ forces that "$\dot{v} \leq \dot{u}$ and that for all $N \in \mathcal{M}_p^\nu[\dot{G}_\mu]$ satisfying $N \cap \omega_1^V \geq M \cap \omega_1^V$, it holds that $\dot{v}$ is semi-$N[\dot{G}_\mu]^{\dot{\mathbb{Q}}_\mu}$-generic.

Proof. This is immediate from the previous lemma. □

[2.5.4] **Lemma.** Let $\gamma \in \mathscr{E}$, let $p \in \mathbb{P}_\gamma$, and let $\beta \in \mathscr{E} \cap \gamma$. Then there exists $p_\beta \leq_{\mathbb{P}_\gamma} p$ such that $\beta \in \mathsf{dom}(w_{p_\beta})$.

Proof.

1° We may assume w.l.o.g. that $(\beta, \gamma) \cap \mathscr{E} = \emptyset$.

2° Let $G_\beta \rightsquigarrow V^{\mathbb{P}_\beta}$ be arbitrary containing $p \restriction \beta$, let $M \in \mathcal{M}_p^\gamma[G_\beta]$ be such that $M \cap \omega_1^V$ is minimal, and let $\mathbb{Q}_\beta := \dot{\mathbb{Q}}_\beta^{G_\beta}$.

3° We may apply Lemma [2.5.2] to
   a. $\underset{\sim}{\mu} := \beta$, $\underset{\sim}{\nu} := \gamma$,
   b. $\underset{\sim}{p} := p \restriction \beta$,
   c. $\underset{\sim}{G}_\mu := G_\beta$,
   d. $\underset{\sim}{M} := M$,

e. $\underset{\sim}{u} := 1_{\mathbb{Q}_\beta}$,

which yields

$$(\exists v \in \mathbb{Q}_\beta)(\forall N \in \mathcal{M}_p^\gamma[G_\beta])(v \text{ is semi-}N[G_\beta]^{\mathbb{Q}_\beta}\text{-generic}).$$

4° Since G_β was arbitrary, we can find a canonical $\mathbb{P}_\beta$ name $\dot{v}$ for an element of $\dot{\mathbb{Q}}_\beta$ such that

$$p \restriction \beta \Vdash^V_{\mathbb{P}_\beta} (\forall N \in \mathcal{M}_p^\gamma[\dot{G}_\beta])(\dot{v} \text{ is semi-}N[\dot{G}_\beta]^{\dot{\mathbb{Q}}_\beta}\text{-generic}).$$

5° Let $w_{p_\beta} := w_p \cup \{(\beta, \dot{v})\}$, $\mathcal{M}_{p_\beta} := \mathcal{M}_p$, and $p_\beta := (w_{p_\beta}, \mathcal{M}_{p_\beta})$. We see that p_β is as required.

□

[2.5.5] **Lemma.** Let $\beta \in \mathscr{E} - \{\min \mathscr{E}\}$ and let $\xi < \beta$. Then $\Vdash_{\mathbb{P}_\beta} |\xi| \leq \omega_1$.

Proof.

1° It suffices to consider the case when β has a predecessor α inside $\mathscr{E}$.

2° Let $\dot{\mathcal{M}}$ be a $\mathbb{P}_\beta$-name satisfying

$$\Vdash_{\mathbb{P}_\beta} \dot{\mathcal{M}} = \bigcup_{p \in \dot{G}_\beta} \mathcal{M}_p^\beta[\dot{G}_\alpha].$$

3° We want to show that:

a. $\Vdash_{\mathbb{P}_\beta} \xi \subseteq \bigcup \dot{\mathcal{M}}$

b. $\Vdash_{\mathbb{P}_\beta} |\dot{\mathcal{M}}| \leq \omega_1$.

This suffices for the conclusion.

4° Let us first verify 3°a. Let $\eta < \xi$ and let $D := \{p \in \mathbb{P}_\beta : (\exists M \in \mathcal{M}_p)(\eta, \alpha \in M)\}$. We are done if we show that D is dense in $\mathbb{P}_\beta$.

5° Let $p_0 \in \mathbb{P}_\beta$ be arbitrary. Then there exists $N \prec \mathbb{V}$ such that $\alpha, \beta, \eta, p_0 \in N$.

6° By Lemma [2.5.1], there exists $p \leq_{\mathbb{P}_\beta} p_0$ such that $N \downarrow \beta \in \mathcal{M}_p$.

7° Since $\eta, \alpha \in N \downarrow \beta$, we conclude $p \in D$.

8° Let us now verify 3°b. Let $G_\beta \rightsquigarrow V^{\mathbb{P}_\beta}$ be arbitrary and let us work inside $V[G_\beta]$.

9° Let $\mathcal{M} := \dot{\mathcal{M}}^{G_\beta} \subseteq \mathcal{C}_\beta$. It suffices to show that the mapping

$$\mathcal{M} \to \omega_1^V : M \mapsto M \cap \omega_1^V$$

is an injection.

10° Let $M, N \in \mathcal{M}$ be arbitrary satisfying $M \cap \omega_1^V = N \cap \omega_1^V$. We want to show that $M = N$.

11° We have that $\alpha \in M \cap N$ and there exist $p, q \in G_\beta$ such that $M \in \mathcal{M}_p^\beta[G_\alpha]$ and $N \in \mathcal{M}_q^\beta[G_\alpha]$.

12° Since $p, q \in G_\beta$, there exists $r \in G_\beta$ such that $r \leq_{\mathbb{P}_\beta} p, q$. We have that $M, N \in \mathcal{M}_r^\beta[G_\alpha]$.

13° Since $r \in G_\beta$, we have that $\mathcal{M}_r^\beta[G_\alpha]$ is a weak $\lhd_\alpha$-chain w.r.t. $G_{\alpha+1}$. In particular, the fact that $M \cap \omega_1^V = N \cap \omega_1^V$ implies that $M = N$.

□

[2.5.6] **Lemma.** Let M be a virtual model, let δ be an inaccessible of M, let $\mathbb{P} \in M \cap V_\delta$ be a poset, and let $G \rightsquigarrow \widehat{M}^{\mathbb{P}}$. Then

$$\sup(M[G] \cap \delta) = \sup(M \cap \delta).$$

Proof.

1° Let $\tau \in M^{\mathbb{P}}$ be such that $\tau_G < \delta$. We want to show that there exists some $\beta \in M \cap \delta$ such that $\tau_G < \beta$.

2° We may assume w.l.o.g. that $\Vdash^V_{\mathbb{P}} \tau < \delta$.

3° There exists a maximal antichain $A \in M$ of $\mathbb{P}$ such that for all $p \in A$, there exists $\alpha_p < \delta$ such that $p \Vdash^V_{\mathbb{P}} \tau = \alpha_p$.

4° Note that $(\alpha_p : p \in A) \in M$. Since δ is inaccessible in M, we conclude that $\beta := \sup_{p \in A} \alpha_p \in M \cap \delta$.

5° It is now clear that β is as required.

□

[2.5.7] **Lemma.**

Let

- γ : limit point of $\mathscr{E}$,
- $M \in \mathcal{C}_{[0,\kappa]}$,
- $p \in \mathbb{P}_\gamma : M \downarrow \gamma \in \mathcal{M}_p$.

Suppose that for all $\alpha \in \mathscr{E} \cap \gamma$, condition $p \restriction \alpha \Vdash_{\mathbb{P}_\alpha} M^{\dot{G}_\alpha} \cap \omega_1^V = M \cap \omega_1^V$.

Then there exist ordinals $\beta < \gamma^* \leq \gamma$ and a condition $q \leq_{\mathbb{P}_\gamma} p$ such that $\mathcal{M}_q \cap \mathcal{C}_\gamma = \mathcal{M}_p \cap \mathcal{C}_\gamma$ and

$$(\forall \alpha \in \mathscr{E} \cap (\beta, \gamma))(q \restriction \alpha \Vdash_{\mathbb{P}_\alpha} \sup(M^{\dot{G}_\alpha} \cap \gamma) = \gamma^*).$$

Proof.

1° If $M \cap \mathsf{Ord} \subseteq \gamma$, then we may set $\gamma^* := \sup(M \cap \mathsf{Ord})$, $\beta := 0$, and $q := p$. Hence, let us assume that there exists $\xi \in M$ with $\xi \geq \gamma$.

2° Let η be the least ordinal $\eta \geq \gamma$ satisfying

$$(\exists \beta \in \mathscr{E}^* \cap \gamma)(\exists r \leq_{\mathbb{P}_\beta} p \restriction \beta)(r \Vdash_{\mathbb{P}_\beta} \eta \in M^{\dot{G}_\beta}).$$

3° Let $\beta_0 \in \mathscr{E}^* \cap \gamma$ be the least ordinal satisfying

$$(\exists r \leq_{\mathbb{P}_{\beta_0}} p \restriction \beta_0)(r \Vdash_{\mathbb{P}_{\beta_0}} \eta \in M^{\dot{G}_{\beta_0}}).$$

Let r be a witness for the last formula.

4° By lemma [1.3.8], we have that $\eta \in \mathscr{E}$.

5° **Claim.** $r \Vdash_{\mathbb{P}_{\beta_0}} \beta_0 \in M^{\dot{G}_{\beta_0}}_{<\beta_0}$

Proof.

1′ Assume otherwise. Then there exists $r' \leq_{\mathbb{P}_{\beta_0}} r$ such that $r' \Vdash \beta_0 \notin M^{\dot{G}_{\beta_0}}_{<\beta_0}$.

2′ This means that $r' \Vdash \eta \in M^{\dot{G}_{\beta_0}} = M^{\dot{G}_{\beta_0}}_{<\beta_0}$.

3′ Hence, there are $r'' \leq_{\mathbb{P}_{\beta_0}} r'$ and $\beta_1 \in \mathscr{E}^* \cap \beta_0$ such that $r'' \Vdash \eta \in M^{\dot{G}_{\beta_1}}$. This contradicts the minimality of β_0.

□

6° Hence, $r \Vdash_{\mathbb{P}_{\beta_0}} (M[\dot{G}_{\beta_0}]$ is defined).

7° Let $s \leq_{\mathbb{P}_{\beta_0}} r$ and let $\lambda \in \mathsf{Ord}$ be such that $s \Vdash_{\mathbb{P}_{\beta_0}} \mathsf{cof}^{M[\dot{G}_{\beta_0}]}(\eta) = \lambda$.

8° **Claim.** There exists $\beta \in \mathscr{E} \cap (\beta_0, \gamma)$ and $t \leq_{\mathbb{P}_\beta} s$ such that

$$t \Vdash_{\mathbb{P}_\beta} \beta \in M^{\dot{G}_\beta}_{<\beta} \wedge \mathsf{cof}^{M[G_\beta]}(\eta) \in \{\omega, \omega_1, \eta\}.$$

Proof.

1′ If $\lambda = \eta$, it suffices to take $\beta := \min(\mathscr{E} - \beta_0)$ and $t := s$. Hence, let us assume that $\lambda < \eta$.

2′ Note that $s \Vdash_{\mathbb{P}_{\beta_0}} \lambda \in M^{\dot{G}_{\beta_0}}$. By the choice of η, we then have that $\lambda < \gamma$.

3′ Let $G_{\beta_0} \rightsquigarrow V^{\mathbb{P}_{\beta_0}}$ be an arbitrary generic containing s and let us work in $V[G_{\beta_0}]$.

4′ Since $\eta \in \mathscr{E}$, the fact that $\mathscr{E} \cap (\lambda, \eta) \neq \emptyset$ can be expressed as saying that there exists $\beta \in (\lambda, \eta)$ such that $\widehat{M} \restriction \beta \prec \widehat{M} \restriction \eta$. Since this is a first order fact of $\widehat{M}$ with parameters from $M^{G_{\beta_0}} \prec \widehat{M}$, there exists such an ordinal β in $M^{G_{\beta_0}}$.

5′ By the choice of η, we have $\beta \in \mathscr{E} \cap (\lambda, \gamma)$.

6′ Let $G_\beta \rightsquigarrow V^{\mathbb{P}_\beta}$ extend G_{β_0} and let us work inside $V[G_\beta]$.

7′ We have $\beta \in M^{G_{\beta_0}} \subseteq M^{G_\beta}_{<\beta}$.

8′ By Lemma [2.5.5], we have $|\lambda| \leq \omega_1$ and consequently $\mathsf{cof}^{M[G_\beta]}(\eta) \leq \omega_1$.

9′ Condition t is now obtained by Forcing Theorem.

□

9° Up to strengthening t, we may assume that there exists γ^* such that $t \Vdash_{\mathbb{P}_\beta} \gamma^* = \sup(M[\dot{G}_\beta] \cap \gamma)$. Let $q := tp \in \mathbb{P}_\gamma$ and let us show that β, γ^*, and q are as required. It is immediate that $\mathcal{M}_q \cap \mathcal{C}_\gamma = \mathcal{M}_p \cap \mathcal{C}_\gamma$.

10° Since $t \Vdash_{\mathbb{P}_\beta} \beta \in M[\dot{G}_\beta] \cap \gamma \wedge \gamma^* = \sup(M[\dot{G}_\beta] \cap \gamma)$, we conclude $\beta < \gamma^* \leq \gamma$.

11° **Claim.** For all $\alpha \in \mathscr{E} \cap (\beta, \gamma)$, we have that $q \restriction \alpha \Vdash_{\mathbb{P}_\alpha} \sup(M^{\dot{G}_\alpha} \cap \gamma) = \gamma^*$.

Proof.

1′ Let $G_\alpha \rightsquigarrow V^{\mathbb{P}_\alpha}$ be an arbitrary generic containing $q \restriction \alpha$ and let us work inside $V[G_\alpha]$. We will distinguish between two cases, depending on whether $\mathsf{cof}^{M[G_\beta]}(\eta) \leq \omega_1^{V[G_\beta]}$ or $\mathsf{cof}^{M[G_\beta]}(\eta) = \eta$.

2′ *Suppose first that* $\mathsf{cof}^{M[G_\beta]}(\eta) \leq \omega_1^{V[G_\beta]}$. Then there exists $f \in M[G_\beta]$ such that

$$f : \omega_1^{V[G_\beta]} \to \eta$$

and

$$M[G_\beta] \models (f \text{ is cofinal in } \eta).$$

3′ By the assumption of the lemma, we have that

$$M[G_\beta] \cap \omega_1^V = M \cap \omega_1^V.$$

In particular, $\omega_1^V \not\subseteq M[G_\beta]$, which means that ω_1^V is not countable in $V[G_\beta]$.

4′ Hence, $\omega_1^V = \omega_1^{V[G_\beta]}$ and $f : \omega_1^V \to \eta$ is such that

$$\sup(f[M \cap \omega_1^V]) = \sup(M^{G_\beta} \cap \eta).$$

5′ This further implies that

$$\sup(f[M \cap \omega_1^V]) = \sup(M^{G_\beta} \cap \gamma) = \gamma^*.$$

6′ Observe that $M[G_\beta] \prec \widehat{M}[G_\beta]$, that $\widehat{M}[G_\alpha]$ is a generic extension of $\widehat{M}[G_\beta]$, and that $M[G_\alpha] \prec \widehat{M}[G_\alpha]$. This implies that

$$M[G_\alpha] \models (f : \omega_1^V \to \eta \text{ cofinally}).$$

7′ We can now compute as follows:

$$\sup(M^{G_\alpha} \cap \gamma) = \sup(M^{G_\alpha} \cap \eta) = \sup(f[M^{G_\alpha} \cap \omega_1^V]) = \sup(f[M \cap \omega_1^V]) = \gamma^*,$$

where the first equality follows by the choice of η, the second one from the previous point, the third one from the hypothesis that

$$M^{G_\alpha} \cap \omega_1^V = M \cap \omega_1^V,$$

and the fourth one from 5′.

8′ *Suppose now that* $\mathsf{cof}^{M[G_\beta]}(\eta) = \eta$.

9′ Since $M[G_\beta] \cap \eta = M^{G_\beta} \cap \eta = M^{G_\beta} \cap \gamma$, we see that η is a strong limit in $M[G_\beta]$. Hence,

$$M[G_\beta] \models (\eta \text{ is inaccessible}).$$

10′ Since posets $(\mathbb{P}_\xi : \xi \in \mathscr{E}^* \cap \gamma)$ are of rank $< \gamma \leq \eta$, the iterative application of Lemma [2.5.6] yields that

$$\sup(M^{G_\xi} \cap \eta) = \sup(M^{G_\beta} \cap \eta),$$

for all $\xi \in \mathscr{E}^* \cap [\beta, \alpha]$.

11′ In particular, we have that $\sup(M^{G_\alpha} \cap \eta) = \gamma^*$, which further implies

$$\sup(M^{G_\alpha} \cap \gamma) = \gamma^*.$$

□

12° This completes the proof of the lemma.

□

[2.5.8] **Lemma.** Let

a. M : virtual model,
b. $\gamma \in \mathscr{E}_M$: limit point of $\mathscr{E}_{\widehat{M}}$,
c. $\gamma^* := \sup(M \cap \gamma)$,
d. N : virtual model satisfying $N \lhd_{\gamma^*} M$.

Then there exists a unique γ-generated model $N^+ \in M$ such that $N \cong_{\gamma^*} N^+$. Moreover, if $\widehat{M} \restriction \gamma^* \prec \widehat{N}$, then $\widehat{M} \restriction \gamma \prec \widehat{N^+}$.

Proof.

1° To establish existence of such a N^+, note that by definition there exists $P \in M$ such that $N \cong_{\gamma^*} P$. Then $N^+ := P \downarrow \gamma \in M$ is as required.

2° Let us verify uniqueness. Let $N' \in M$ be a γ-generated virtual model such that $N \cong_{\gamma^*} N'$. It suffices to show $N' \cong_\gamma N^+$.

3° Let $\alpha \in \mathscr{E}_M \cap \gamma$ be arbitrary. Then $\alpha < \gamma^*$ and consequently $N' \cong_\alpha N \cong_\alpha N^+$.

4° Hence,

$$M \models (\forall \alpha < \gamma)(V_\alpha \prec (V_\gamma, \in, U \cap V_\gamma) \implies N' \cong_\alpha N^+).$$

5° By elementarity, the same statement is true in $\widehat{M}$. This means that for all $\alpha \in \mathscr{E}_{\widehat{M}} \cap \gamma$, we have that $N' \cong_\alpha N^+$.

6° For $\alpha \in \mathscr{E}_{\widehat{M}} \cap \gamma$, let $f_\alpha : \mathsf{Hull}(N', V_\alpha) \cong \mathsf{Hull}(N^+, V_\alpha)$ be the unique isomorphism witnessing $N' \cong_\alpha N^+$. For $\alpha_0 < \alpha_1$, we have $f_{\alpha_0} \subseteq f_{\alpha_1}$.

7° It is now easily seen that

$$f_\gamma := \bigcup_{\alpha \in \mathscr{E}_{\widehat{M}} \cap \gamma} f_\alpha : \mathsf{Hull}(N', V_\gamma) \to \mathsf{Hull}(N^+, V_\gamma)$$

witnesses $\mathsf{Hull}(N', V_\gamma) \cong_\gamma \mathsf{Hull}(N^+, V_\gamma)$.

8° Let us establish moreover part. Let $\alpha \in \mathscr{E}_M \cap \gamma$ be arbitrary. We have that $\alpha < \gamma^*$.

9° This implies that $\widehat{M} \restriction \alpha \prec \widehat{N}$ and consequently $\widehat{M} \restriction \alpha = \mathsf{Hull}(N, V_{\gamma^*}) \cap V_\alpha \prec \mathsf{Hull}(N, V_{\gamma^*})$.

10° Since $N \cong_{\gamma^*} N^+$, we conclude that

$$\widehat{M} \restriction \alpha = \mathsf{Hull}(N^+, V_{\gamma^*}) \cap V_\alpha \prec \mathsf{Hull}(N^+, V_{\gamma^*}) \prec \widehat{N^+}$$

and consequently $M \models V_\alpha \prec \widehat{N^+}$.

11° Hence,

$$M \models (\forall \alpha < \gamma)(V_\alpha \prec (V_\gamma, \in, U \cap V_\gamma) \implies V_\alpha \prec \widehat{N^+}).$$

12° Since $M \prec \widehat{M}$, we conclude that for all $\alpha \in \mathscr{E}_{\widehat{M}} \cap \gamma$, we have $\widehat{M} \restriction \alpha \prec \widehat{N^+}$. This suffices for the conclusion.

□

2.6 Proof of Transfer Theorem

2.6.1 Successor Case

1° Let $\gamma \in \mathscr{E}$ be the successor of some β inside $\mathscr{E}$. We are assuming that

$$(\forall N \in \mathcal{C}_{>\beta})(\forall q \in \mathbb{P}_\beta)(N \downarrow \beta \in \mathcal{M}_q \implies (q \text{ is semi-}N^{\mathbb{P}_\beta}\text{-generic})). \tag{8}$$

Let $M \in \mathcal{C}_{>\gamma}$ and let $p \in \mathbb{P}_\gamma$ satisfying $M \downarrow \gamma \in \mathcal{M}_p$. We want to show that p is locally $M^{\mathbb{P}_\gamma}$-generic.

2° We will use Local Genericity Criterion [2.4.4]. Let D be a dense open subset of $\mathbb{P}_\gamma$ and let $q \leq_{\mathbb{P}_\gamma} p$ be arbitrary satisfying

$$q \Vdash_{\mathbb{P}_\gamma} D \in M^{\dot{G}_\gamma}_{<\gamma}.$$

We want to find $r \in D$ and $s \leq_{\mathbb{P}_\gamma} q, r$ such that

$$s \Vdash_{\mathbb{P}_\gamma} \gamma \notin M^{\dot{G}_\gamma}_{<\gamma} \vee r \in M^{\dot{G}_\gamma}_{<\gamma}.$$

3° Up to strengthening q, we may assume that $q \in D$.

4° If $q \Vdash_{\mathbb{P}_\gamma} \gamma \notin M^{\dot{G}_\gamma}_{<\gamma}$, we can take $s := r := q$. Hence, let us assume that $q \nVdash_{\mathbb{P}_\gamma} \gamma \notin M^{\dot{G}_\gamma}_{<\gamma}$.

5° Up to strengthening q, we may assume that $q \Vdash_{\mathbb{P}_\gamma} \gamma \in M^{\dot{G}_\gamma}_{<\gamma}$. Since β is definable from γ over $\widehat{M}$, we have that $q \Vdash_{\mathbb{P}_\gamma} \beta \in M^{\dot{G}_\gamma}_{<\gamma}$. This implies that $q \Vdash_{\mathbb{P}_\beta} \beta \in M^{\dot{G}_\beta}$ and $q \Vdash_{\mathbb{P}_\beta} M \downarrow \gamma \in \mathcal{M}^\gamma_q[\dot{G}_\beta]$.

6° By Lemma [2.5.4], we may strengthen q to ensure $\beta \in \mathsf{dom}(w_q)$.

7° **Claim.** $q \restriction (\beta+1) \Vdash_{\mathbb{P}_{\beta+1}} M^{\dot{G}_{\beta+1}} \cap \omega_1^V = M \cap \omega_1^V$

Proof.

1′ By (8), we have that

$$q \restriction \beta \Vdash_{\mathbb{P}_\beta} M^{\dot{G}_\beta} \cap \omega_1^V = M \cap \omega_1^V.$$

2′ Let $G_{\beta+1}$ be a generic containing $q \restriction (\beta+1)$ and let us verify that $M^{G_{\beta+1}} \cap \omega_1^V = M \cap \omega_1^V$. Let $\mathbb{Q}_\beta := \dot{\mathbb{Q}}_\beta^{G_\beta}$.

3′ Since $\beta \in \mathsf{dom}(w_q)$ and $M \downarrow \gamma \in \mathcal{M}^\gamma_q[G_\beta]$, we have that $w_q(\beta)^{G_\beta}$ is semi-$(M \downarrow \gamma)[G_\beta]^{\mathbb{Q}_\beta}$-generic in $V[G_\beta]$.

4′ Hence,

$$(M \downarrow \gamma)[G_{\beta+1}] \cap \omega_1^{V[G_\beta]} = (M \downarrow \gamma)[G_\beta] \cap \omega_1^{V[G_\beta]}.$$

5′ This shows that

$$M^{G_{\beta+1}} \cap \omega_1^V = M^{G_\beta} \cap \omega_1^V = M \cap \omega_1^V.$$

□

8° We may assume that there exists $\mathcal{M} \in [\mathcal{C}_\gamma]^{<\omega}$ such that

$$q \restriction \beta \Vdash_{\mathbb{P}_\beta} \mathcal{M}^\gamma_q[\dot{G}_\beta] = \mathcal{M}.$$

Namely, there exists $\bar{q} \leq_{\mathbb{P}_\beta} q \restriction \beta$ such that $\bar{q}$ decides $\mathcal{M}^\gamma_q[\dot{G}_\beta]$. Since $\mathcal{M}_{\bar{q}q} \cap \mathcal{C}_\gamma = \mathcal{M}_q \cap \mathcal{C}_\gamma$, we may replace q by $\bar{q}q$.

9° Note that $M \downarrow \gamma \in \mathcal{M}$ and

$$q \restriction (\beta+1) \Vdash_{\mathbb{P}_{\beta+1}} \mathcal{M} \text{ is a weak } \lhd_\gamma\text{-chain w.r.t. } \dot{G}_{\beta+1}.$$

10° **Claim.** There exist $r \in D$ and $s_0 \in \mathbb{P}_{\beta+1}$ such that

a. $\beta \in \mathsf{dom}(w_r)$,

b. $\{N \in \mathcal{M} : N \cap \omega_1^V < M \cap \omega_1^V\} \subseteq \mathcal{M}_r$,

c. $s_0 \leq_{\mathbb{P}_{\beta+1}} q \restriction (\beta+1), r \restriction (\beta+1)$,

d. $s_0 \Vdash_{\mathbb{P}_{\beta+1}} r \in M^{\dot{G}_{\beta+1}}$.

Proof.

1′ Let $\mathcal{N} := \{N \in \mathcal{M} : N \cap \omega_1^V < M \cap \omega_1^V\}$ and let $G_{\beta+1} \rightsquigarrow V^{\mathbb{P}_{\beta+1}}$ arbitrary containing $q \restriction (\beta+1)$.

2′ For all $N \in \mathcal{N}$, we have

$$N \lhd_\gamma (M \downarrow \gamma)^{G_{\beta+1}} \cong_\gamma M^{G_{\beta+1}},$$

which yields $P \in M^{G_{\beta+1}}$ such that $N = P \downarrow \gamma$; since $\gamma \in M^{G_{\beta+1}}$, we conclude $N \in M^{G_{\beta+1}}$. Hence, $\mathcal{N} \in M^{G_{\beta+1}}$.

3′ Let $D^* := \{r \in D : r \restriction (\beta+1) \in G_{\beta+1},\ \beta \in \mathsf{dom}(w_r),\ \mathcal{N} \subseteq \mathcal{M}_r\} \in M^{G_{\beta+1}}$.

4′ Note that $q \in D^*$. By elementarity, $M^{G_{\beta+1}} \models D^* \neq \emptyset$. Hence, there exists $r \in D \cap M^{G_{\beta+1}}$.

5′ By Forcing Theorem, there exists $s_0 \in G_{\beta+1}$ such that

a. $s_0 \leq q \restriction (\beta+1), r \restriction (\beta+1)$,

b. $s_0 \Vdash_{\mathbb{P}_{\beta+1}} r \in M^{\dot{G}_{\beta+1}}$.

Conditions r, s_0 are as required.

□

11° **Claim.** There exists $s \in \mathbb{P}_\gamma$ such that $s \leq_{\mathbb{P}_\gamma} q, r, s_0$.

Proof.

1′ Let $w_s := w_{s_0}$, $\mathcal{M}_s := \mathcal{M}_{s_0} \cup \mathcal{M}_q \cup \mathcal{M}_r$, $s := (w_s, \mathcal{M}_s)$. We will be done once we show that $s \in \mathbb{P}_\gamma$.

2′ We have that $s \restriction (\beta+1) = s_0 \in \mathbb{P}_{\beta+1}$, so it remains to verify that

$$s \restriction (\beta+1) \quad \Vdash_{\mathbb{P}_{\beta+1}} \quad (\mathcal{M}_s^\gamma[\dot{G}_\beta] \text{ is a weak } \lhd_\gamma\text{-chain w.r.t. } \dot{G}_{\beta+1}), \tag{9}$$

$$s \restriction \beta \quad \Vdash_{\mathbb{P}_\beta} \quad (\forall N \in \mathcal{M}_s^\gamma[\dot{G}_\beta])(w_s(\beta) \text{ is semi-}N[\dot{G}_\beta]^{\dot{\mathbb{Q}}_\beta}\text{-generic}). \tag{10}$$

3′ Let $G_{\beta+1} \rightsquigarrow V^{\mathbb{P}_\beta}$ be arbitrary containing $s \restriction (\beta+1)$. Let us first verify that $\mathcal{M}_s^\gamma[G_\beta]$ is a weak $\lhd_\gamma$-chain w.r.t. $G_{\beta+1}$.

4′ Let $\mathcal{M}^* := \{N \in \mathcal{M} : N \cap \omega_1^V \geq M \cap \omega_1^V\}$. Since $\{N \in \mathcal{M}_q^\gamma[G_\beta] : N \cap \omega_1^V < M \cap \omega_1^V\} \subseteq \mathcal{M}_r$, we have that

$$\mathcal{M}_s^\gamma[G_\beta] = \mathcal{M}_q^\gamma[G_\beta] \cup \mathcal{M}_r^\gamma[G_\beta] = \mathcal{M}_r^\gamma[G_\beta] \cup \mathcal{M}^*.$$

5′ Both $\mathcal{M}_r^\gamma[G_\beta]$ and $\mathcal{M}^*$ are weak $\lhd_\gamma$-chains w.r.t. $G_{\beta+1}$ and for all $N \in \mathcal{M}_r^\gamma[G_\beta]$ and all $P \in \mathcal{M}^*$, we have that

$$N \cap \omega_1^V < M^{G_{\beta+1}} \cap \omega_1^V = M \cap \omega_1^V \leq P \cap \omega_1^V$$

("<" follows from $r \in M^{G_{\beta+1}}$ and "=" from Claim 7°). Hence, it suffices to verify that for all $N \in \mathcal{M}_r^\gamma[G_\beta]$ and all $P \in \mathcal{M}^*$, we have that $N \lhd_\gamma P^{G_{\beta+1}}$.

6′ Since $r \in M^{G_{\beta+1}}$, we have $N \lhd_\gamma M^{G_{\beta+1}}$.

7′ Since $M \downarrow \gamma, P \in \mathcal{M}$ and $M \cap \omega_1^V \leq P \cap \omega_1^V$, we have $M \downarrow \gamma = P$ or $M \downarrow \gamma \lhd_\gamma P^{G_{\beta+1}}$.

8′ Hence,

$$N \lhd_\gamma (M \downarrow \gamma)^{G_\beta+1} \subseteq P^{G_{\beta+1}},$$

as required.

9′ Let us now verify that for every $N \in \mathcal{M}_s^\gamma[G_\beta]$, we have that $w_s(\beta)^{G_\beta}$ is semi-$N[G_\beta]^{\mathbb{Q}_\beta}$-generic.

10′ Either $N \in \mathcal{M}_q^\gamma[G_\beta]$ or $N \in \mathcal{M}_r^\gamma[G_\beta]$.

11′ If $N \in \mathcal{M}_q^\gamma[G_\beta]$, the conclusion follows from

$$w_s(\beta)^{G_\beta} = w_{s_0}(\beta)^{G_\beta} \leq w_q(\beta)^{G_\beta};$$

if $N \in \mathcal{M}_r^\gamma[G_\beta]$, the conclusion follows from

$$w_s(\beta)^{G_\beta} = w_{s_0}(\beta)^{G_\beta} \leq w_r(\beta)^{G_\beta}.$$

12′ This concludes the proof of Claim 11°.

□

12° It is clear that r and s are as required in 2°. □

2.6.2 Limit Case

1° We are assuming that γ is a limit point of $\mathscr{E}$ and that

$$(\forall \alpha \in \mathscr{E} \cap \gamma)(\forall N \in \mathcal{C}_{>\alpha})(\forall q \in \mathbb{P}_\alpha)(N \downarrow \alpha \in \mathcal{M}_q \implies q \Vdash_{\mathbb{P}_\alpha} N^{\dot{G}_\alpha} \cap \omega_1^V = N \cap \omega_1^V).$$

Let us fix some $M \in \mathcal{C}_{>\gamma}$ and some $p \in \mathbb{P}_\gamma$ such that $M \downarrow \gamma \in \mathcal{M}_p$. We want to show that p is locally $M^{\mathbb{P}_\gamma}$-generic.

2° We use Local Genericity Criterion [2.4.4]. Let $q \leq_{\mathbb{P}_\gamma} p$ be an arbitrary condition and let D be an arbitrary dense open subset of $\mathbb{P}_\gamma$ such that

$$q \Vdash_{\mathbb{P}_\gamma} D \in M_{<\gamma}^{\dot{G}_\gamma}.$$

We want to find $r \in D$ and $s \leq_{\mathbb{P}_\gamma} q, r$ such that

$$s \Vdash_{\mathbb{P}_\gamma} \gamma \notin M_{<\gamma}^{\dot{G}_\gamma} \vee r \in M_{<\gamma}^{\dot{G}_\gamma}.$$

3° By Lemma [2.5.7], we may assume, up to strengthening q, that there exist $\beta \in \mathscr{E} \cap \gamma$ and $\gamma^* \in (\beta, \gamma]$ such that

$$(\forall \xi \in \mathscr{E} \cap [\beta, \gamma))(q \restriction \xi \Vdash_{\mathbb{P}_\xi} \sup(M^{\dot{G}_\xi} \cap \gamma) = \gamma^*).$$

4° If $q \Vdash_{\mathbb{P}_\gamma} \gamma \notin M_{<\gamma}^{\dot{G}_\gamma}$, we can set $r = s$ to be an arbitrary element of D which is $\leq q$. Hence, we will assume that $q \nVdash_{\mathbb{P}_\gamma} \gamma \notin M_{<\gamma}^{\dot{G}_\gamma}$. Up to strengthening q, we may assume that $q \Vdash_{\mathbb{P}_\gamma} \gamma \in M_{<\gamma}^{\dot{G}_\gamma}$. Up to further strengthening and increasing of β, we may assume that $q \Vdash_{\mathbb{P}_\gamma} \gamma \in M_{<\beta}^{\dot{G}_\gamma}$, or equivalently

$$q \restriction \beta \Vdash_{\mathbb{P}_\beta} \gamma \in M_{<\beta}^{\dot{G}_\beta}.$$

5° Up to strengthening q and increasing β, we may also assume that

$$q \restriction \beta \Vdash_{\mathbb{P}_\beta} D \in M_{<\beta}^{\dot{G}_\beta}.$$

6° **Claim.** We may assume that for all $N \in \mathcal{M}_q$, either

$$q \restriction \beta \Vdash_{\mathbb{P}_\beta} \sup(N^{\dot{G}_\beta} \cap \gamma^*) = \gamma^*$$

or

$$(\forall \xi \in \mathscr{E} \cap [\beta, \gamma^*))(q \restriction \xi \Vdash_{\mathbb{P}_\xi} \sup(N^{\dot{G}_\xi} \cap \gamma^*) \leq \beta).$$

Proof.

1′ Let $\mathcal{M}_q = \{N_i : i < k\}$, let $\eta_{-1} := \beta$, and let $t_{-1} := q \restriction \gamma^*$.

2′ Let $i < k$. Suppose recursively that an ordinal $\eta_{i-1} \in \mathscr{E} \cap [\beta, \gamma^*)$ and a condition $t_{i-1} \leq_{\mathbb{P}_{\gamma^*}} q \restriction \gamma^*$ satisfying

$$\mathcal{M}_{t_{i-1}} \cap \mathcal{C}_{\gamma^*} = \mathcal{M}_{q \restriction \gamma^*} \cap \mathcal{C}_{\gamma^*}$$

have been defined and that for all $j < i$, either

$$t_{i-1} \restriction \eta_{i-1} \Vdash_{\mathbb{P}_{\eta_{i-1}}} \sup(N_j^{\dot{G}_{\eta_{i-1}}} \cap \gamma^*) = \gamma^*$$

or

$$(\forall \xi \in \mathscr{E} \cap [\eta_{i-1}, \gamma^*))(t_{i-1} \restriction \xi \Vdash_{\mathbb{P}_\xi} \sup(N_j^{\dot{G}_\xi} \cap \gamma^*) \leq \eta_{i-1})).$$

3′ We want to define an ordinal $\eta_i \in \mathscr{E} \cap [\eta_{i-1}, \gamma^*)$ and a condition $t_i \leq_{\mathbb{P}_{\gamma^*}} t_{i-1}$ satisfying

$$\mathcal{M}_{t_i} \cap \mathcal{C}_{\gamma^*} = \mathcal{M}_{q \restriction \gamma^*} \cap \mathcal{C}_{\gamma^*}$$

such that for all $j \leq i$, either

$$t_i \restriction \eta_i \Vdash_{\mathbb{P}_{\eta_i}} \sup(N_j^{\dot{G}_{\eta_i}} \cap \gamma^*) = \gamma^*$$

or

$$(\forall \xi \in \mathscr{E} \cap [\eta_i, \gamma^*))(t_i \restriction \xi \Vdash_{\mathbb{P}_\xi} \sup(N_j^{\dot{G}_\xi} \cap \gamma^*) \leq \eta_i)).$$

4′ If $N_i \in \mathcal{C}_{<\gamma^*}$, then let η_i be the minimum of the set

$$\mathscr{E} \cap [\max\{\eta_{i-1}, \sup(N_i \cap \mathsf{Ord})\}, \gamma^*)$$

and let $t_i := t_{i-1}$.

5′ Otherwise, we have $N_i \in \mathcal{C}_{\geq \gamma^*}$. Then we can apply Lemma [2.5.7] to

a. $\underset{\sim}{\gamma} := \gamma^*$,
b. $\underset{\sim}{M} := N_i$,
c. $\underset{\sim}{p} := t_{i-1}$,

and obtain ordinals $\mu < \nu \leq \gamma^*$ and a condition $t_i \leq_{\mathbb{P}_{\gamma^*}} t_{i-1}$ satisfying

$$\mathcal{M}_{t_i} \cap \mathcal{C}_{\gamma^*} = \mathcal{M}_{t_{i-1}} \cap \mathcal{C}_{\gamma^*},$$

$$(\forall \xi \in \mathscr{E} \cap [\mu, \gamma^*))(t_i \restriction \xi \Vdash_{\mathbb{P}_\xi} \sup(N_i^{\dot{G}_\xi} \cap \gamma^*) = \nu).$$

6′ If $\nu = \gamma^*$, then set

$$\eta_i := \max\{\eta_{i-1}, \min(\mathscr{E} \cap [\mu, \gamma^*))\}.$$

If $\nu < \gamma^*$, then set

$$\eta_i := \max\{\eta_{i-1}, \min(\mathscr{E} \cap [\mu, \gamma^*)), \nu\}.$$

7′ By Lemma [1.3.9], we have $\nu \in \mathscr{E}$ and consequently $\eta_i \in \mathscr{E}$.

8′ This concludes the recursion. Note that $\mathcal{M}_{t_{k-1}} - \mathcal{M}_q \subseteq \mathcal{C}_{<\gamma^*}$, so there exists $\eta_k \in \mathscr{E} \cap [\eta_{k-1}, \gamma^*)$ such that for all $N \in \mathcal{M}_{t_{k-1}} - \mathcal{M}_q$ and for all $\xi \in \mathscr{E} \cap [\eta_k, \gamma^*)$,

$$\Vdash_{\mathbb{P}_\xi} \sup(N^{G_\xi} \cap \gamma^*) \leq \sup(N \cap \mathsf{Ord}) \leq \eta_k.$$

9′ The claim now follows by replacing q with $t_{k-1}q$ and by replacing β with η_k.

□

7° Up to increasing β, we may assume that $\mathsf{dom}(w_q) \cap \gamma^* \subseteq \beta$.

8° For all $\xi \in \mathscr{E} \cap [\beta, \gamma^*)$, condition $q \restriction \xi$ decides the value of $\mathcal{M}_q^{\gamma^*}[\dot{G}_\xi]$. Up to increasing β, we may assume that all these values are equal to some fixed $\mathcal{M} \in [\mathcal{C}_{\gamma^*}]^{<\omega}$. Hence, we can ensure the following:

a. $(\forall \xi \in \mathscr{E} \cap [\beta, \gamma^*))(q \restriction \xi \Vdash_{\mathbb{P}_\xi} \mathcal{M}_q^{\gamma^*}[\dot{G}_\xi] = \mathcal{M})$,

b. $q \restriction \beta \Vdash_{\mathbb{P}_\beta}$ ($\mathcal{M}$ is a weak $\lhd_{\gamma^*}$-chain w.r.t. $\dot{G}_\beta$),

c. $M \downarrow \gamma^* \in \mathcal{M}$.

9° By 3° and Lemma [1.3.9], we have that

$$q \restriction \beta \Vdash_{\mathbb{P}_\beta} \sup(\mathscr{E} \cap M^{\dot{G}_\beta} \cap \gamma) = \gamma^* > \beta.$$

This means that we may strengthen q and increase β as to ensure

$$q \restriction \beta \Vdash_{\mathbb{P}_\beta} \beta \in M^{\dot{G}_\beta}_{<\beta}.$$

10° **Claim.** There exist $r \in D$ and $s_0 \in \mathbb{P}_\beta$ such that

a. $(\forall N \in \mathcal{M} : N \cap \omega_1^V < M \cap \omega_1^V)(\exists N^+ \in \mathcal{M}_r \cap \mathcal{C}_\gamma)(N = N^+ \downarrow \gamma)$,

b. $s_0 \leq q \restriction \beta, r \restriction \beta$,

c. $s_0 \Vdash_{\mathbb{P}_\beta} r \in M^{\dot{G}_\beta}$.

Proof.

1′ Let G_β be a $V^{\mathbb{P}_\beta}$-generic containing $q \restriction \beta$ and let us work inside $V[G_\beta]$.

2′ For $N \in \mathcal{M}$ with $N \cap \omega_1^V < M \cap \omega_1^V$, we may apply Lemma [2.5.8] to

a. $\underset{\sim}{M} := M^{G_\beta}$,

b. $\underset{\sim}{\gamma} := \gamma$,

c. $\underset{\sim}{N} := N$,

and obtain the unique $N^+ \in M^{G_\beta} \cap \mathcal{C}_\gamma$ satisfying $N^+ \downarrow \gamma^* = N$. Let

$$\mathcal{N} := \{N^+ : N \in \mathcal{M},\ N \cap \omega_1^V < M \cap \omega_1^V\} \in M^{G_\beta}.$$

3′ Let $D^* := \{r \in D : r \restriction \beta \in G_\beta,\ \mathcal{N} \subseteq \mathcal{M}_r\}$. We want to find $r \in D^* \cap M^{G_\beta}$.

4′ Since $\beta \in M^{G_\beta}_{<\beta}$, model $M[G_\beta]$ is defined and $D^* \in M[G_\beta]$. By elementarity, it suffices to show that $D^* \neq \emptyset$.

5′ **Subclaim.** $\mathcal{N}$ is a weak $\lhd_\gamma$-chain w.r.t. G_β.

Proof.

1″ Let $N, P \in \mathcal{M}$ be arbitrary satisfying $N \cap \omega_1^V \leq P \cap \omega_1^V < M \cap \omega_1^V$. We want to show that

- if $N \cap \omega_1^V = P \cap \omega_1^V$, then $N^+ = P^+$,
- if $N \cap \omega_1^V < P \cap \omega_1^V$, then $N^+ \lhd_\gamma (P^+)^{G_\beta}$.

The first case is obvious, since we have $N = P$. Let us consider the case when $N \cap \omega_1^V < P \cap \omega_1^V$.

2″ We have that $N \lhd_{\gamma^*} P^{G_\beta}$. Since $P \cong_{\gamma^*} P^+$, we have that $N \lhd_{\gamma^*} (P^+)^{G_\beta}$.

3″ Hence, there exists $N' \in (P^+)^{G_\beta}$ such that $N \cong_{\gamma^*} N'$.

4″ Since $P^+ \in (M \downarrow \gamma)^{G_\beta}$, we conclude that

$$(P^+)^{G_\beta} \subseteq (M \downarrow \gamma)^{G_\beta}.$$

5″ This means that $N' \in (M \downarrow \gamma)^{G_\beta}$. Then there exists $N'' \in M^{G_\beta}$ such that $N' \cong_\gamma N''$.

6″ Hence, $N' \downarrow \gamma = N'' \downarrow \gamma \in M^{G_\beta}$, while $(N' \downarrow \gamma) \downarrow \gamma^* = N$. We conclude that

$$N^+ = N' \downarrow \gamma \cong_\gamma N' \in (P^+)^{G_\beta},$$

which means $N^+ \lhd_\gamma (P^+)^{G_\beta}$.

□

6′ There exists $t \in G_\beta$ such that

a. $t \leq q \restriction \beta$,

b. $t \Vdash^V_{\mathbb{P}_\beta}$ ($\check{\mathcal{N}}$ is a weak $\lhd_\gamma$-chain w.r.t. $\dot{G}_\beta$),

c. $t \Vdash^V_{\mathbb{P}_\beta}$ $(\forall P \in \check{\mathcal{N}})(P^{\dot{G}_\beta}$ is active at $\gamma^*)$.

7′ Let $w_u := w_t$, let $\mathcal{M}_u := \mathcal{M}_t \cup \mathcal{N}$, and let $u := (w_u, \mathcal{M}_u)$. The previous point implies that $u \in \mathbb{P}_\gamma$. We also have that $u \restriction \beta = t \in G_\beta$, which means that $u \in (\mathbb{P}_\gamma/\mathbb{P}_\beta)^{G_\beta}$.

8′ Since $\{v \in D : v \restriction \beta \in G_\beta\}$ is dense in $(\mathbb{P}_\gamma/\mathbb{P}_\beta)^{G_\beta}$, we conclude that there exists $v \in D$ such that $v \restriction \beta \in G_\beta$ and $v \leq_{\mathbb{P}_\gamma} u$; in particular, $\mathcal{N} \subseteq \mathcal{M}_v$ and consequently $v \in D^*$.

9′ Hence, there exists $r \in D^* \cap M[G_\beta]$. This means that

a. $r \in D$,

b. $r \restriction \beta \in G_\beta$,

c. $\mathcal{N} \subseteq \mathcal{M}_r$.

10′ By Forcing Theorem, there exists $s_0 \in G_\beta$ such that

a. $s_0 \leq q \restriction \beta, r \restriction \beta$,

b. $s_0 \Vdash_{\mathbb{P}_\beta} r \in M^{\dot{G}_\beta}$.

□

11° **Claim.** There exists $s \in \mathbb{P}_\gamma$ such that $s \leq_{\mathbb{P}_\gamma} q, r, s_0$.

Proof.

1′ Let $t := s_0 q \in \mathbb{P}_\gamma$. Note that

$$\mathsf{dom}(w_t) \cap [\beta, \gamma^*) = \mathsf{dom}(w_q) \cap [\beta, \gamma^*) = \emptyset.$$

2′ Let $\mathsf{dom}(w_s) := \mathsf{dom}(w_t) \cup \mathsf{dom}(w_r)$. For $\xi \in \mathsf{dom}(w_s) - [\beta, \gamma^*)$, let $w_s(\xi) := w_t(\xi)$.

3′ For $\xi \in \mathsf{dom}(w_s) \cap [\beta, \gamma^*) = \mathsf{dom}(w_r) - \beta$, let us denote by η its successor in $\mathscr{E}$.

4′ Since $s_0 \Vdash_{\mathbb{P}_\beta} r \in M^{\dot{G}_\beta}$ and $t \restriction \xi \leq s_0$, we have that $t \restriction \xi \Vdash_{\mathbb{P}_\xi}$ ($M^{\dot{G}_\xi}$ is active at η).

5′ Hence, we may apply Lemma [2.5.3] to

a. $\underset{\sim}{\mu} := \xi$, $\underset{\sim}{\nu} := \eta$,

b. $\underset{\sim}{p} := t \restriction \eta$,

c. $\underset{\sim}{M} := M \downarrow \eta$,

d. $\underset{\sim}{\dot{u}} := w_r(\xi)$

and obtain a canonical $\mathbb{P}_\xi$-name $w_s(\xi)$ for an element of $\dot{\mathbb{Q}}_\xi$ such that

$$t \restriction \xi \Vdash_{\mathbb{P}_\xi} w_s(\xi) \leq w_r(\xi),$$

$$t \restriction \xi \Vdash_{\mathbb{P}_\xi} (\forall N \in \mathcal{M}^\eta_t[\dot{G}_\xi])(N \cap \omega_1^V \geq M \cap \omega_1^V \implies (w_s(\xi) \text{ is semi-}N[\dot{G}_\xi]^{\dot{\mathbb{Q}}_\xi}\text{-generic})).$$

6′ Let $\mathcal{M}_s := \mathcal{M}_t \cup \mathcal{M}_r$ and let $s := (w_s, \mathcal{M}_s)$. We will be done if we show $s \in \mathbb{P}_\gamma$. We show by induction on $\eta \in \mathscr{E} \cap [\beta, \gamma]$ that $s \restriction \eta \in \mathbb{P}_\eta$.

7′ **Case.** $\eta = \beta$

Proof. This case is obvious since $s \restriction \beta = s_0$. □

8′ **Case.** Let us assume that $\eta \in (\beta, \gamma^*]$ and that η has a predecessor ξ inside $\mathscr{E}$.

Proof.

1″ We need to verify that

$$\xi \in \mathsf{dom}(w_s) \implies s \restriction \xi \Vdash_{\mathbb{P}_\xi} (\forall N \in \mathcal{M}^\eta_s[\dot{G}_\xi])(w_s(\xi) \text{ is semi-}N[\dot{G}_\xi]^{\dot{\mathbb{Q}}_\xi}\text{-generic})$$

and that

$$s \restriction (\xi + 1) \Vdash_{\mathbb{P}_{\xi+1}} (\mathcal{M}^\eta_s[\dot{G}_\xi] \text{ is a weak } \lhd_\eta\text{-chain w.r.t. } \dot{G}_{\xi+1}).$$

2″ Let $G_\xi \rightsquigarrow V^{\mathbb{P}_\xi}$ be an arbitrary generic containing $s \restriction \xi$ and let $\mathbb{Q}_\xi := \dot{\mathbb{Q}}_\xi^{G_\xi}$. Note that

$$\mathcal{M}_s^\eta[G_\xi] = \mathcal{M}_t^\eta[G_\xi] \cup \mathcal{M}_r^\eta[G_\xi].$$

3″ Since $\mathcal{M}_t^\eta[G_\xi] \subseteq \mathcal{M}_t \cap \mathcal{C}_{\geq\eta} = \mathcal{M}_q$, we have that $\mathcal{M}_t^\eta[G_\xi] = \mathcal{M}_q^\eta[G_\xi]$ and consequently

$$\mathcal{M}_s^\eta[G_\xi] = \mathcal{M}_q^\eta[G_\xi] \cup \mathcal{M}_r^\eta[G_\xi].$$

4″ **Subclaim.** $\mathcal{M}_q^\eta[G_\xi] \subseteq \mathcal{M} \downarrow \eta$

Proof.

1‴ Let $N \in \mathcal{M}_q^\eta[G_\xi]$ be arbitrary. Then there exists $P \in \mathcal{M}_q \cap \mathcal{C}_{\geq\eta}$ such that $\xi \in P^{G_\xi}$ and $N = P \downarrow \eta$.

2‴ We now have $\sup(P^{G_\xi} \cap \gamma) > \xi \geq \beta$ and it follows from Claim 6° that

$$\sup(P^{G_\xi} \cap \gamma) = \gamma^*.$$

3‴ This means that $P \downarrow \gamma^* \in \mathcal{M}_q^{\gamma^*}[G_\xi] = \mathcal{M}$, while $N = (P \downarrow \gamma^*) \downarrow \eta$.

□

5″ **Subclaim.** $\{N \in \mathcal{M}_q^\eta[G_\xi] : N \cap \omega_1^V < M \cap \omega_1^V\} \subseteq \mathcal{M}_r^\eta[G_\xi]$

Proof.

1‴ Let $N \in \mathcal{M}_q^\eta[G_\xi]$ be such that $N \cap \omega_1^V < M \cap \omega_1^V$.

2‴ By Subclaim 4″, there exists $P \in \mathcal{M}$ such that $N = P \downarrow \eta$.

3‴ We have $P \cap \omega_1^V < M \cap \omega_1^V$, so Claim 10°a implies that there exists $Q \in \mathcal{M}_r \cap \mathcal{C}_\gamma$ such that $P = Q \downarrow \gamma^*$.

4‴ Note that

$$\xi \in N^{G_\xi} \cong_\eta P^{G_\xi} \cong_{\gamma^*} Q^{G_\xi},$$

so $Q \downarrow \eta \in \mathcal{M}_r^\eta[G_\xi]$, while $N = Q \downarrow \eta$.

□

6″ **Subclaim.** For all $\xi \in \mathsf{dom}(w_s)$ and for all $N \in \mathcal{M}_s^\eta[G_\xi]$, we have that $w_s(\xi)^{G_\xi}$ is semi-$N[G_\xi]^{\mathbb{Q}_\xi}$-generic.

Proof.

1‴ Suppose first that $N \in \mathcal{M}_r^\eta[G_\xi]$. Since $\xi \in [\beta, \gamma^*)$, we have $\xi \in \mathsf{dom}(w_r) - \beta$ and consequently that $w_r(\xi)^{G_\xi}$ is semi-$N[G_\xi]^{\mathbb{Q}_\xi}$-generic. The conclusion now follows from the fact that $w_s(\xi)^{G_\xi} \leq w_r(\xi)^{G_\xi}$.

2‴ Suppose next that $N \in \mathcal{M}_q^\eta[G_\xi]$ and that $N \cap \omega_1^V \geq M \cap \omega_1^V$. Then $w_s(\xi)^{G_\xi}$ is semi-$N[G_\xi]^{\mathbb{Q}_\xi}$-generic by the choice of $w_s(\xi)$.

3‴ Suppose finally that $N \in \mathcal{M}_q^\eta[G_\xi]$ and that $N \cap \omega_1^V < M \cap \omega_1^V$. By Subclaim 5″, we have that $N \in \mathcal{M}_r^\eta[G_\xi]$, which means that $w_r(\xi)^{G_\xi}$ is semi-$N[G_\xi]^{\mathbb{Q}_\xi}$-generic. The desired conclusion now follows from the fact that $w_s(\xi)^{G_\xi} \leq w_r(\xi)^{G_\xi}$.

□

7″ **Subclaim.** Let $G_{\xi+1} \rightsquigarrow V^{\mathbb{P}_{\xi+1}}$ be an arbitrary generic extending G_ξ and containing $s \restriction (\xi+1)$. Then $\mathcal{M}_s^\eta[G_\xi]$ is a weak $\lhd_\eta$-chain w.r.t. $G_{\xi+1}$.

Proof.

1‴ Let $\mathcal{M}^* := \{P \in \mathcal{M}_q^\eta[G_\xi] : P \cap \omega_1^V \geq M \cap \omega_1^V\}$. By Subclaim 5″, set $\mathcal{M}_s^\eta[G_\xi]$ is the union of sets $\mathcal{M}_r^\eta[G_\xi]$ and $\mathcal{M}^*$.

2‴ Each of sets $\mathcal{M}_r^\eta[G_\xi]$ and $\mathcal{M}^*$ is a weak $\lhd_\eta$-chain w.r.t. $G_{\xi+1}$. Furthermore, for every $N \in \mathcal{M}_r^\eta[G_\xi]$ and every $P \in \mathcal{M}^*$, we have

$$N \cap \omega_1^V < M \cap \omega_1^V \leq P \cap \omega_1^V.$$

3‴ Hence, it remains to verify that for every $N \in \mathcal{M}_r^\eta[G_\xi]$ and every $P \in \mathcal{M}^*$, we have $N \lhd_\eta P^{G_{\xi+1}}$.

4‴ Since $N \in \mathcal{M}_r^\eta[G_\xi]$ and $r \in M^{G_\xi}$, we have $N \lhd_\eta M^{G_\xi}$.

5‴ By Subclaim 4″, there exists $Q \in \mathcal{M}$ such that $P = Q \downarrow \eta$. Since $Q \cap \omega_1^V \geq M \cap \omega_1^V$, we conclude that $M \downarrow \gamma^* = Q$ or $M \downarrow \gamma^* \lhd_{\gamma^*} Q^{G_\xi}$.

6‴ Hence, there exists $R \in Q^{G_\xi} \cup \{Q\}$ such that $M \cong_{\gamma^*} R$.

7‴ We conclude $N \lhd_\eta R^{G_\xi} \subseteq Q^{G_\xi}$.

8‴ Thus,

$$N \lhd_\eta Q^{G_\xi} \cong_\eta P^{G_\xi} \subseteq P^{G_{\xi+1}},$$

as required.

□

8″ This concludes the proof of Case 8′

□

9′ **Case.** Let us assume that $\eta \in (\beta, \gamma^*]$ and that η is a limit point of $\mathscr{E}$.

Proof.

1″ We need to show that there exists $\beta_0 < \eta$ such that

$$(\forall \xi \in \mathscr{E} \cap (\beta_0, \eta))(s \restriction (\xi+1) \Vdash_{\mathbb{P}_{\xi+1}} (\mathcal{M}_s^\eta[\dot{G}_\xi] \text{ is a weak } \lhd_\eta\text{-chain w.r.t. } \dot{G}_{\xi+1})).$$

We will show that $\beta_0 := \beta$ works.

2″ Let $\xi \in \mathscr{E} \cap (\beta, \eta)$ and let $G_\xi \rightsquigarrow V^{\mathbb{P}_\xi}$ satisfying $s \restriction \xi \in G_\xi$.

3″ We have $\mathcal{M}_s^\eta[G_\xi] = \mathcal{M}_t^\eta[G_\xi] \cup \mathcal{M}_r^\eta[G_\xi] = \mathcal{M}_q^\eta[G_\xi] \cup \mathcal{M}_r^\eta[G_\xi]$.

4″ **Subclaim.** $\mathcal{M}_q^\eta[G_\xi] \subseteq \mathcal{M} \downarrow \eta$

Proof.

1‴ Let $N \in \mathcal{M}_q^\eta[G_\xi]$ be arbitrary. Then there exists $P \in \mathcal{M}_q \cap \mathcal{C}_{\geq \eta}$ such that

$$\sup(P^{G_\xi} \cap \eta) = \eta$$

and $N = P \downarrow \eta$.

2‴ We now have $\sup(P^{G_\xi} \cap \gamma) > \beta$ and it follows from Claim 6° that

$$\sup(P^{G_\xi} \cap \gamma) = \gamma^*.$$

3‴ This means that $P \downarrow \gamma^* \in \mathcal{M}_q^{\gamma^*}[G_\xi] = \mathcal{M}$, while $N = (P \downarrow \gamma^*) \downarrow \eta$.

□

5″ **Subclaim.** $\{N \in \mathcal{M}_q^\eta[G_\xi] : N \cap \omega_1^V < M \cap \omega_1^V\} \subseteq \mathcal{M}_r^\eta[G_\xi]$

Proof.

1‴ Let $N \in \mathcal{M}_q^\eta[G_\xi]$ be such that $N \cap \omega_1^V < M \cap \omega_1^V$.

2‴ By Subclaim 4″, there exists $P \in \mathcal{M}$ such that $N = P \downarrow \eta$.

3‴ We have $P \cap \omega_1^V < M \cap \omega_1^V$, so Claim 10°a implies that there exists $Q \in \mathcal{M}_r \cap \mathcal{C}_\gamma$ such that $P = Q \downarrow \gamma^*$.

4‴ Since

$$N^{G_\xi} \cong_\eta P^{G_\xi} \cong_{\gamma^*} Q^{G_\xi},$$

we have that

$$\sup(Q^{G_\xi} \cap \eta) = \sup(N^{G_\xi} \cap \eta) = \eta,$$

i.e. $Q \downarrow \eta \in \mathcal{M}_r^\eta[G_\xi]$.

5‴ The conclusion now follows from the fact that $N = Q \downarrow \eta$.

□

6″ **Subclaim.** Let $G_{\xi+1} \rightsquigarrow V^{\mathbb{P}_{\xi+1}}$ be an arbitrary generic extending G_ξ and containing $s \restriction (\xi+1)$. Then $\mathcal{M}_s^\eta[G_\xi]$ is a weak $\lhd_\eta$-chain w.r.t. $G_{\xi+1}$.

Proof.

1‴ Let $\mathcal{M}^* := \{P \in \mathcal{M}_q^\eta[G_\xi] : P \cap \omega_1^V \geq M \cap \omega_1^V\}$. By Subclaim 5″, set $\mathcal{M}_s^\eta[G_\xi]$ is the union of sets $\mathcal{M}_r^\eta[G_\xi]$ and $\mathcal{M}^*$.

2‴ Each of sets $\mathcal{M}_r^\eta[G_\xi]$ and $\mathcal{M}^*$ is a weak $\lhd_\eta$-chain w.r.t. $G_{\xi+1}$. Furthermore, for every $N \in \mathcal{M}_r^\eta[G_\xi]$ and every $P \in \mathcal{M}^*$, we have

$$N \cap \omega_1^V < M \cap \omega_1^V \leq P \cap \omega_1^V.$$

3‴ Hence, it remains to verify that for every $N \in \mathcal{M}_r^\eta[G_\xi]$ and every $P \in \mathcal{M}^*$, we have $N \lhd_\eta P^{G_{\xi+1}}$.

4‴ Since $N \in \mathcal{M}_r^\eta[G_\xi]$ and $r \in M^{G_\xi}$, we have $N \lhd_\eta M^{G_\xi}$.

5‴ By Subclaim 4″, there exists $Q \in \mathcal{M}$ such that $P = Q \downarrow \eta$. Since $Q \cap \omega_1^V \geq M \cap \omega_1^V$, we conclude that $M \downarrow \gamma^* = Q$ or $M \downarrow \gamma^* \lhd_{\gamma^*} Q^{G_\xi}$.

6‴ Hence, there exists $R \in Q^{G_\xi} \cup \{Q\}$ such that $M \cong_{\gamma^*} R$.

7‴ We conclude $N \lhd_\eta R^{G_\xi} \subseteq Q^{G_\xi}$.

8‴ Thus,

$$N \lhd_\eta Q^{G_\xi} \cong_\eta P^{G_\xi} \subseteq P^{G_{\xi+1}},$$

as required.

□

7″ This concludes the proof of Case 9′.

□

10′ **Case.** Let us assume that $\eta \in (\gamma^*, \gamma]$.

Proof.

1″ Since $s_0 \Vdash_{\mathbb{P}_\beta} r \in M^{\dot{G}_\beta} \cap \mathbb{P}_\gamma$, we have that $\mathsf{dom}(w_r) \subseteq \gamma^*$ and $\mathcal{M}_r \subseteq \mathcal{C}_{<\gamma^*}$. Hence,

$$w_{s\restriction\eta} = (w_s \restriction \gamma^*) \cup (w_t \restriction [\gamma^*, \eta)) = (w_s \restriction \gamma^*) \cup (w_q \restriction [\gamma^*, \eta)),$$

$$\mathcal{M}_{s\restriction\eta} = (\mathcal{M}_s \cap \mathcal{C}_{\leq\gamma^*}) \cup (\mathcal{M}_t \downarrow \eta) = (\mathcal{M}_s \cap \mathcal{C}_{\leq\gamma^*}) \cup (\mathcal{M}_q \downarrow \eta).$$

2″ Suppose first that η is the successor of some β inside $\mathscr{E}$. We have to show that

$$\xi \in \mathsf{dom}(w_s) \implies s \restriction \xi \Vdash_{\mathbb{P}_\xi} (\forall N \in \mathcal{M}_s^\eta[\dot{G}_\xi])(w_s(\xi) \text{ is semi-}N[\dot{G}_\xi]^{\dot{\mathbb{Q}}_\xi}\text{-generic}) \tag{11}$$

and that

$$s \restriction (\xi+1) \Vdash_{\mathbb{P}_{\xi+1}} (\mathcal{M}_s^\eta[\dot{G}_\xi] \text{ is a weak } \lhd_\eta\text{-chain w.r.t. } \dot{G}_{\xi+1}). \tag{12}$$

3″ By 1″, we have that $\Vdash_{\mathbb{P}_\xi} \mathcal{M}_s^\eta[\dot{G}_\xi] = \mathcal{M}_q^\eta[\dot{G}_\xi]$ and $w_s(\xi) = w_q(\xi)$. Properties (11) and (12) now follow from the fact that q is a condition and that $s \restriction (\xi+1) \leq_{\mathbb{P}_{\xi+1}} q \restriction (\xi+1)$.

4″ Suppose now that η is a limit point of $\mathscr{E}$. We have to show that there exists $\bar{\eta} < \eta$ such that for all $\xi \in \mathscr{E} \cap (\bar{\eta}, \eta)$,

$$s \restriction (\xi+1) \Vdash_{\mathbb{P}_{\xi+1}} (\mathcal{M}_s^\eta[\dot{G}_\xi] \text{ is a weak } \lhd_\eta\text{-chain w.r.t. } \dot{G}_{\xi+1}). \tag{13}$$

5″ We claim that $\bar{\eta} := \gamma^*$ works. Namely, for all $\xi \in \mathscr{E} \cap (\gamma^*, \eta)$, we have that

$$\Vdash_{\mathbb{P}_\xi} \mathcal{M}_s^\eta[\dot{G}_\xi] = \mathcal{M}_q^\eta[\dot{G}_\xi],$$

$$s \restriction (\xi+1) \leq_{\mathbb{P}_{\xi+1}} q \restriction (\xi+1),$$

so the conclusion again follows from the fact that q is a condition.

□

11′ This concludes the proof of Claim 11°.

□

12° It is now clear that s and r as required in 2°, concluding the argument. □

2.7 Semi-properness and Chain Condition

[2.7.1] **Lemma.** Let $\gamma \in \mathscr{E}$, let $M \in \mathcal{C}_{>\gamma}$, and let $p \in \mathbb{P}_\gamma$ satisfy $M \downarrow \gamma \in \mathcal{M}_p$. Then

$$p \Vdash_{\mathbb{P}_\gamma} M^{\dot{G}_\gamma} \cap \omega_1^V = M \cap \omega_1^V.$$

Proof.

1° This is shown by induction on γ.

2° **Case.** $\gamma = \min \mathscr{E}$

Proof. By Transfer Theorem, condition p is locally $M^{\mathbb{P}_\gamma}$-generic, which in this case translates to $M^{\mathbb{P}_\gamma}$-generic. □

3° **Case.** Suppose that γ is a successor of some β inside $\mathscr{E}$.

Proof.

1′ Let $G_\gamma \rightsquigarrow V^{\mathbb{P}_\gamma}$ be arbitrary containing p and let us verify that

$$M^{G_\gamma} \cap \omega_1^V = M \cap \omega_1^V.$$

2′ By the IH and Transfer Theorem, we have that

$$M^{G_\beta} \cap \omega_1^V = M \cap \omega_1^V, \quad M^{G_\gamma} = M^{G_{\beta+1}}.$$

Hence, it suffices to verify that

$$M^{G_{\beta+1}} \cap \omega_1^V = M^{G_\beta} \cap \omega_1^V.$$

3′ If $\beta \notin M_{<\beta}^{G_\beta}$, then

$$M_{<\beta}^{G_\beta} = M^{G_\beta} = M^{G_\beta+1}.$$

Let us then assume that $\beta \in M_{<\beta}^{G_\beta}$.

4′ Since conditions $q \in \mathbb{P}_\gamma$ such that $\beta \in \mathsf{dom}(w_q)$ are dense below p, there exists $q \in G_\gamma$ such that $q \leq p$ and $\beta \in \mathsf{dom}(w_q)$.

5′ We have $M \downarrow \gamma \in \mathcal{M}_q^\gamma[G_\beta]$. Since $\beta \in \mathsf{dom}(w_q)$, we conclude that $w_q(\beta)^{G_\beta}$ is semi-$(M \downarrow \gamma)[G_\beta]^{\mathbb{Q}_\beta}$-generic.

6′ Since $q \in G_\beta$, we conclude that

$$(M \downarrow \gamma)[G_{\beta+1}] \cap \omega_1^{V[G_\beta]} = (M \downarrow \gamma)[G_\beta] \cap \omega_1^{V[G_\beta]}.$$

This suffices for the conclusion.

□

4° **Case.** γ is a limit point of $\mathscr{E}$.

Proof. By the IH and Transfer Theorem, we have that

$$p \Vdash_{\mathbb{P}_\gamma} M^{\dot{G}_\gamma} = M_{<\gamma}^{\dot{G}_\gamma}.$$

Hence,

$$p \Vdash_{\mathbb{P}_\gamma} M^{\dot{G}_\gamma} \cap \omega_1^V = \bigcup_{\alpha \in \mathscr{E} \cap \gamma} (M^{G_\alpha} \cap \omega_1^V) = M \cap \omega_1^V$$

by another application of the IH. □

5° This concludes the induction.

□

[2.7.2] **Proposition.** For all $\gamma \in \mathscr{E}^*$, poset $\mathbb{P}_\gamma$ is semi-proper.

Proof.

1° Suppose first that $\gamma \in \mathscr{E}$. It suffices to show that for every $M \in \mathcal{C}_{>\gamma}$ and every $p \in M$, there exists $q \leq p$ which is semi-$M^{\mathbb{P}_\gamma}$-generic.

2° By Lemma [2.5.1], there exists $q \leq p$ such that $M \downarrow \gamma \in \mathcal{M}_q$. By the previous lemma, condition q is semi-$M^{\mathbb{P}_\gamma}$-generic.

3° Suppose now that $\gamma \in \mathscr{E}^+$. Then

$$\mathbb{P}_\gamma \simeq \mathbb{P}_{\gamma-1} * \dot{\mathbb{Q}}_{\gamma-1},$$

which suffices to conclude semi-properness.

□

[2.7.3] **Lemma.** Suppose that

1. $\gamma \in \mathscr{E}$,
2. $\mathsf{cof}(\gamma) = \omega_1$,
3. $p \in \mathbb{P}_\gamma$.

Then there exist q and $\bar{\gamma}$ such that

a. $q \leq_{\mathbb{P}_\gamma} p$,
b. $\bar{\gamma} \in \mathscr{E} \cap \gamma$,
c. $\mathsf{dom}(w_q) \subseteq \bar{\gamma}$,
d. for all $\alpha \in \mathscr{E} \cap [0, \gamma]$, for all $N \in \mathcal{M}_q$, $q \restriction \alpha \Vdash_{\mathbb{P}_\alpha} \sup(N^{\dot{G}_\alpha} \cap \gamma) \leq \bar{\gamma}$.

Proof.

1° We note that γ is necessarily a limit point of $\mathscr{E}$.

2° Let $M \in \mathcal{C}_{>\gamma}$ be such that $\gamma, p \in M$ and let $q \leq p$ be such that $\mathsf{dom}(w_q) = \mathsf{dom}(w_p)$ and $\mathcal{M}_q = \mathcal{M}_p \cup \{M \downarrow \gamma\}$ (cf. Lemma [2.5.1]).

3° Let $\bar{\gamma} := \sup(\gamma \cap M) < \gamma$. We have that $\mathsf{cof}(\bar{\gamma}) = \omega$, that $\bar{\gamma}$ is a limit point of $\mathscr{E}$, and that $\mathsf{dom}(w_q) = \mathsf{dom}(w_p) \subseteq \bar{\gamma}$.

4° **Claim.** $q \Vdash_{\mathbb{P}_\gamma} \sup(M^{\dot{G}_\gamma} \cap \gamma) = \bar{\gamma}$

Proof.

1′ Let $G_\gamma \rightsquigarrow V^{\mathbb{P}_\gamma}$ be an arbitrary generic containing q and let us show that

$$\sup(M^{G_\gamma} \cap \gamma) = \bar{\gamma}.$$

2′ Let $f \in M$ map cofinally ω_1^V into γ.

3′ By Lemma [2.7.1], we have that $M^{G_\gamma} \cap \omega_1^V = M \cap \omega_1^V$.

4′ Hence,

$$\sup(M^{G_\gamma} \cap \gamma) = \sup(f[M^{G_\gamma} \cap \omega_1^V]) = \sup(f[M \cap \omega_1^V]) = \sup(M \cap \gamma) = \bar{\gamma}.$$

□

5° Consequently, for all $\alpha \in \mathscr{E} \cap [0, \gamma]$ and for all $N \in \mathcal{M}_q$, we have

$$q \restriction \alpha \Vdash_{\mathbb{P}_\alpha} \sup(N^{\dot{G}_\alpha} \cap \gamma) \leq \bar{\gamma}.$$

□

[2.7.4] **Theorem.** $\mathbb{P}_\kappa$ has κ-c.c.

Proof.

1° Assume otherwise.

2° Let $S := \{\alpha \in \mathscr{E} : \mathsf{cof}(\alpha) = \omega_1\}$. Then there exists an anti-chain $(p_\alpha : \alpha \in S)$ in $\mathbb{P}_\kappa$.

3° By the previous lemma, for all $\alpha \in S$, there exist q_α and $\bar{\alpha}$ such that

a. $q_\alpha \leq_{\mathbb{P}_\alpha} p_\alpha \restriction \alpha$,

b. $\bar{\alpha} \in \mathscr{E} \cap \alpha$,

c. $\mathsf{dom}(w_{q_\alpha}) \subseteq \bar{\alpha}$,

d. for all $\xi \in \mathscr{E} \cap [0, \alpha]$, for all $N \in \mathcal{M}_{q_\alpha}$, $q_\alpha \restriction \xi \Vdash_{\mathbb{P}_\xi} \sup(N^{\dot{G}_\xi} \cap \alpha) \leq \bar{\alpha}$.

4° By pressing down, there exist a stationary $S_1 \subseteq S$ and $\gamma \in \mathscr{E}$ such that for all $\alpha \in S_1$, we have $\bar{\alpha} = \gamma$.

5° There exist $\alpha, \beta \in S_1$ such that

a. $\gamma < \alpha < \beta$,

b. $q_\alpha \in \mathbb{P}_\beta$,

c. $q_\alpha \restriction \gamma = q_\beta \restriction \gamma$.

6° Let $q := q_\alpha \restriction \gamma = q_\beta \restriction \gamma$ and let

$$\begin{aligned} w_r &:= w_q \cup (w_{p_\alpha} \restriction [\gamma, \beta)) \cup (w_{p_\beta} \restriction [\beta, \kappa)), \\ \mathcal{M}_r &:= \mathcal{M}_q \cup \mathcal{M}_{p_\alpha} \cup \mathcal{M}_{p_\beta}, \\ r &:= (w_r, \mathcal{M}_r). \end{aligned}$$

It is now easily seen that $r \in \mathbb{P}_\kappa$ and $r \leq p_\alpha$, p_β, which contradicts $p_\alpha \perp_{\mathbb{P}_\kappa} p_\beta$.

□

[2.7.5] **Theorem.** $\mathbb{P}_\kappa$ is semi-proper.

Proof.

1° Let θ be large enough regular, let $M \prec (H_\theta, \in, \kappa, U)$ be countable, let $p \in \mathbb{P}_\kappa \cap M$, and let us find $q \leq_{\mathbb{P}_\kappa} p$ which is semi-$M^{\mathbb{P}_\kappa}$-generic.

2° Let $\bar{\kappa} := \sup(\kappa \cap M) \in \mathscr{E} \cap \kappa$, let $N := M \cap V_\kappa \in \mathcal{C}_{\bar{\kappa}}$, and let $q \in \mathbb{P}_{\bar{\kappa}}$ be such that

a. $q \leq p$,

b. $\mathsf{dom}(w_q) = \mathsf{dom}(w_p)$,

c. $\mathcal{M}_q = \mathcal{M}_p \cup \{N\}$

(cf. Lemma [2.5.1]; note that $p \in \mathbb{P}_{\bar{\kappa}}$). We want to show that q is semi-$M^{\mathbb{P}_\kappa}$-generic.

3° Let $\tau \in M^{\mathbb{P}_\kappa}$ with $\Vdash_{\mathbb{P}_\kappa} \tau < \omega_1^V$ be arbitrary. We want to show that $q \Vdash_{\mathbb{P}_\kappa} \tau \in M$.

4° There exists $A \in M$ such that

a. A is a maximal anti-chain of $\mathbb{P}_\kappa$,

b. for all $r \in A$, there exists unique $\alpha_r < \omega_1^V$ such that $r \Vdash_{\mathbb{P}_\kappa} \tau = \alpha_r$.

5° By the previous theorem, there exists $\gamma \in \mathscr{E} \cap M$ such that $(w_q, \mathcal{M}_p) \in \mathbb{P}_\gamma$ and $A \subseteq \mathbb{P}_\gamma$.

6° We can now find $\sigma \in N^{\mathbb{P}_\gamma}$ such that for all $r \in A$, we have $r \Vdash_{\mathbb{P}_\gamma} \sigma = \alpha_r$. Note that then $\Vdash_{\mathbb{P}_\kappa} \tau = \sigma$ and consequently $\Vdash_{\mathbb{P}_\gamma} \sigma < \omega_1^V$.

7° Let $q_\tau := (w_q, \mathcal{M}_p \cup \{N \downarrow \gamma\})$. It is clear that $q_\tau \in \mathbb{P}_\gamma$ and $q \leq_{\mathbb{P}_\kappa} q_\tau$.

8° By Lemma [2.7.1], we have that q_τ is semi-$N^{\mathbb{P}_\gamma}$-generic. In particular, $q_\tau \Vdash_{\mathbb{P}_\gamma} \sigma \in N$.

9° Hence,

$$q \leq_{\mathbb{P}_\kappa} q_\tau \Vdash_{\mathbb{P}_\kappa} \tau = \sigma \in N \subseteq M,$$

as required.

□

3 Saturating NS_{ω_1}

3.1 Careful Collapse

[3.1.1] **Notation.** For M a virtual model, we denote $\delta_M := \min(\mathsf{Ord} - M)$.

[3.1.2] **Definition.** Suppose that

1. $\theta > \omega$ is regular,
2. U is a stationary subset of $[H_\theta]^\omega$.

The collapsing poset Col_U *guided by* U, is defined as follows.

a. $p \in \mathsf{Col}_U$ if and only if $p = (\mathcal{M}_p, d_p)$ where
   i. for all $M \in \mathcal{M}_p$, either $M \in U$ or there does not exist $N \in U$ satisfying $M \subseteq N$ and $\delta_N = \delta_M$,
   ii. $\mathcal{M}_p$ is a finite $\in$-chain,
   iii. $d_p : \mathcal{M}_p \to [H_\theta]^{<\omega}$,
   iv. for all $M, N \in \mathcal{M}_p$, if $M \in N$, then $d_p(M) \subseteq N$.

b. $p \leq q$ in Col_U if and only if $\mathcal{M}_p \supseteq \mathcal{M}_q$ and for all $M \in \mathcal{M}_q$, $d_p(M) \supseteq d_q(M)$.

[3.1.3] **Lemma.** Suppose that

1. $\theta > \omega$ is regular,
2. U is a stationary subset of $[H_\theta]^\omega$,
3. $x \in H_\theta$,
4. $D := \{p \in \mathsf{Col}_U : \exists M \in \mathcal{M}_p, x \in M\}$.

Then D is dense in Col_U.

Proof.

1° Let $p_0 \in \mathsf{Col}_U$ be arbitrary and let us find $p \in D$ satisfying $p \leq p_0$.

2° Since U is stationary, there exists $M \in U$ such that $x, p_0 \in M$.

3° Then $p := (\mathcal{M}_{p_0} \cup \{M\}, d_{p_0} \cup \{(M, \emptyset)\})$ is as required.

□

[3.1.4] **Lemma.** Suppose that

1. $\theta > \omega$ is regular,
2. U is a stationary subset of $[H_\theta]^\omega$,
3. $G \rightsquigarrow V^{\mathsf{Col}_U}$,
4. $\mathcal{M}_G := \cup\{\mathcal{M}_p : p \in G\}$.

Then

a. $\mathcal{M}$ is an $\in$-chain,

b. $\mathsf{otp}(\mathcal{M}, \in) \leq \omega_1^V$,

c. $\cup\mathcal{M} = H_\theta$.

Proof.

1° Part a. is obvious.

2° Let $(M_\xi : \xi < \alpha)$ be the $\in$-increasing enumeration of $\mathcal{M}$. The following mapping is strictly increasing:

$$\alpha \to \omega_1^V : \xi \mapsto \delta_{M_\xi}.$$

This shows part b.

3° Lemma [3.1.3]. immediately implies part c.

□

[3.1.5] **Definition.** Suppose that

1. $\theta > \omega$ is regular,
2. U is a stationary subset of $[H_\theta]^\omega$,
3. $g \rightsquigarrow V^{\mathsf{Col}_U}$.

Then we define

a. $\mathcal{M}_g := \cup\{\mathcal{M}_p : p \in g\}$,
b. $\alpha_g := \mathsf{otp}(\mathcal{M}_g, \in)$,
c. for all $\xi < \alpha_g$, $\mathcal{M}_g(\xi)$ is the ξth element of $(\mathcal{M}_g, \in)$.

[3.1.6] **Proposition.** Suppose that

1. $\theta > \omega$ is regular,
2. U is a stationary subset of $[H_\theta]^\omega$.

Then in V^{Col_U}, for all limit $\eta < \alpha_g$, $\mathcal{M}_g(\eta) = \bigcup_{\xi<\eta} \mathcal{M}_g(\xi)$.

Proof.

1° Assume otherwise. Then there exist p, α, η, x, M such that

a. $p \Vdash \alpha_g = \alpha$,
b. η is a limit ordinal strictly less than α,
c. $x \in H_\theta$,
d. $p \Vdash x \in \mathcal{M}_g(\eta) - \bigcup_{\xi<\eta} \mathcal{M}_g(\xi)$,
e. $p \Vdash \mathcal{M}_g(\eta) = M$,
f. $M \in \mathcal{M}_p$.

2° There exist $q \leq p$ and $N \in \mathcal{M}_q \cap M$ such that $x \in d_q(N)$.

3° Let $r \leq q$ and $\xi < \eta$ be such that $r \Vdash \mathcal{M}_g(\xi) = N$. We have

$$r \Vdash x \in d_r(N) \subseteq \mathcal{M}_g(\xi + 1),$$

which is a contradiction.

□

[3.1.7] **Proposition.** Suppose that

1. $\theta > \omega$ is regular,
2. U is stationary in $[H_\theta]^\omega$.

Then Col_U is strongly semiproper.

Proof.

1° Let us consider arbitrary

a. $\chi \gg \theta$,
b. $M \prec (H_\chi, \in, U)$ countable,
c. $p \in \mathsf{Col}_U \cap M$.

We want to find $q \leq p$ which is strongly semigeneric for (M, Col_U).

2° **Case I.** There exists $N \in U$ such that $M \cap H_\theta \subseteq N$ and $\delta_N = \delta_M$.

Proof.

1′ Let $q := (\mathcal{M}_p \cup \{N\}, d_p \cup \{(N, \emptyset)\})$. We want to show that q is strongly semigeneric for (M, Col_U).

2′ Let $r \leq q$ be arbitrary and let us find some $r_M \in \mathsf{Col}_U$ such that for $P := \mathsf{Hull}(M, r_M)$,

a. $\delta_P = \delta_M$,

b. for all $s \in \mathsf{Col}_U \cap P$, if $s \leq r_M$, then $s \parallel r$.

3′ Let $r_M := (\mathcal{M}_r \cap N, d_r \restriction N) \in N$. We have that

$$\mathsf{Hull}(M, r_M) \cap H_\theta = \mathsf{Hull}(M \cap H_\theta, r_M) \subseteq N.$$

4′ In particular, for $P := \mathsf{Hull}(M, r_M)$, we have $\delta_P = \delta_M$.

5′ Property 2′b. is now routinely verified.

□

3° **Case II.** There does not exist $N \in U$ such that $M \cap H_\theta \subseteq N$ and $\delta_N = \delta_M$.

Proof.

1′ In this case, we have that $q := (\mathcal{M}_p \cup \{M \cap H_\theta\}, d_p \cup \{(M \cap H_\theta, \emptyset)\} \in \mathsf{Col}_U$ and $q \leq p$. We want to verify that q is strongly generic for (M, Col_U).

2′ Let $r \leq q$ be arbitrary and let us find some $r_M \in \mathsf{Col}_U \cap M$ such that for all $s \in \mathsf{Col}_U \cap M$, if $s \leq r_M$, then $s \parallel r$.

3′ It is routinely verified that $r_M := (\mathcal{M}_r \cap M, d_r \restriction M)$ is as required.

□

4° Having verified the two cases, we reach the conclusion.

□

[3.1.8] **Corollary.** Suppose that

1. $\theta > \omega$ is regular,
2. U is stationary in $[H_\theta]^\omega$,
3. $g \rightsquigarrow V^{\mathsf{Col}_U}$.

Then $\mathsf{otp}(\mathcal{M}_g, \in) = \alpha_g = \omega_1$.

Proof. This follows from the facts that $\alpha_g \leq \omega_1^V = \omega_1$ and $H_\theta^V = \cup \mathcal{M}_g$. □

[3.1.9] **Corollary.** Suppose that

1. $\theta > \omega$ is regular,
2. U is stationary in $[H_\theta]^\omega$.

Then in V^{Col_U}, $|H_\theta^V| = \omega_1$. □

[3.1.10] **Lemma.** Suppose that

1. $\theta > \omega$ is regular,
2. U is stationary in $[H_\theta]^\omega$,
3. $\chi \gg H_\theta$,
4. $M \prec (H_\chi, \in, U)$ is countable,
5. $p \in \mathsf{Col}_U$ is such that $M \cap H_\theta \in \mathcal{M}_p$.

Then p is strongly generic for (M, Col_U).

Proof.

1° Let $q \leq p$ be arbitrary. We want to find $q_M \in \mathsf{Col}_U \cap M$ such that for all $r \in \mathsf{Col}_U \cap M$, if $r \leq q_M$, then $r \parallel q$.

2° Let $q_M := (\mathcal{M}_q \cap M, d_q \restriction M)$. It is easily seen that q_M is as required.

□

[3.1.11] **Proposition.** Suppose that

1. $\theta > \omega$ is regular,
2. U is stationary in $[H_\theta]^\omega$.

Then in V^{Col_U}, U is stationary in $[H_\theta^V]^\omega$.

Proof.

1° Assume otherwise. Then there exist $C \in V^{\mathsf{Col}_U}$ and $p \in \mathsf{Col}_U$ such that

a. $\Vdash_{\mathsf{Col}_U}$ "C is a club in $[H_\theta^V]^\omega$",

b. $p \Vdash_{\mathsf{Col}_U} C \cap U = \emptyset$.

2° Let $\chi \gg H_\theta, C$. Then there exists countable $M \prec (H_\chi, \in, \theta, C, p)$ such that $M \cap H_\theta \in U$.

3° Let $N := M \cap H_\theta \in U$ and let $q := (\mathcal{M}_p \cup \{N\}, d_p \cup \{(N, \emptyset)\}) \leq p$ in Col_U. By Lemma [3.1.10], we have that q is generic for (M, Col_U).

4° Let $g \rightsquigarrow V^{\mathsf{Col}_U}$ contain q. We have that $C \in M[g]$, so $M[g] \cap H_\theta^V \in C$.

5° Since q is generic for (M, Col_U) in V, we have that $M[g] \cap H_\theta^V = N$.

6° Thus, $N \in C \cap U$, which is a contradiction.

□

3.2 Sealed Predense Collections

[3.2.1] **Definition.** Let $D \subseteq \mathbb{P}_{\mathsf{NS}_{\omega_1}}$. We say that D *is sealed* if there exists a mapping $f : \omega_1 \to D$ and a club C in ω_1 such that $C \subseteq \nabla f$.

[3.2.2] **Proposition.** If $D \subseteq \mathbb{P}_{\mathsf{NS}_{\omega_1}}$ is sealed, then D is predense in $\mathbb{P}_{\mathsf{NS}_{\omega_1}}$. □

[3.2.3] **Proposition.** The following are equivalent.

a. NS_{ω_1} is saturated.

b. Every predense subset of $\mathbb{P}_{\mathsf{NS}_{\omega_1}}$ is sealed. □

[3.2.4] **Proposition.** Suppose that $D \subseteq \mathbb{P}_{\mathsf{NS}_{\omega_1}}$ is sealed and that $\mathbb{P}$ is a stationary set preserving poset. Then $V^{\mathbb{P}} \models$ "D is sealed in $\mathbb{P}_{\mathsf{NS}_{\omega_1}}$".

[3.2.5] **Definition.** Suppose that

1. D is predense in $\mathbb{P}_{\mathsf{NS}_{\omega_1}}$,
2. $\theta \geq \omega_2$ is regular,
3. $M \prec H_\theta$ is countable.

Then we say that M *captures* D iff there exists $S \in D \cap M$ such that $\delta_M \in S$.

[3.2.6] **Lemma.** Suppose that

1. D is predense in $\mathbb{P}_{\mathsf{NS}_{\omega_1}}$,
2. $\theta \geq \omega_2$ is regular,
3. $U := \{M \prec H_\theta : |M| = \omega, \text{“}M \text{ captures } D\text{”}\}$.

Then U is projectively stationary.

Proof.

1° Let $S \subseteq \omega_1$ be stationary and let $F : H_\theta^{<\omega} \to H_\theta$. There exists $T \in D$ such that $T \cap S$ is stationary.

2° There exists $M \prec H_\theta$ such that $T \in M$, $F[M^\omega] \subseteq M$, and $\delta_M \in T \cap S$.

3° Hence, $M \in U$, $\delta_M \in S$, and M is closed for F, as required.

□

[3.2.7] **Proposition.** Suppose that

1. D is a predense set in $\mathbb{P}_{\mathsf{NS}_{\omega_1}}$,
2. $U := \{M \prec H_{\omega_2} : |M| = \omega, \text{“}M \text{ captures } D\text{”}\}$,
3. g is V-generic for Col_U,
4. $U^* := \{\xi < \omega_1 : \mathcal{M}_g(\xi) \in U\}$,
5. h is generic over $V[g]$ such that $\omega_1^{V[g][h]} = \omega_1$,
6. in $V[g][h]$, U^* contains a club.

Then in $V[g][h]$, D is sealed.

Proof.

1° For all $\xi < \omega_1$, we enumerate $D \cap \mathcal{M}_g(\xi)$ as $(S_n^\xi : n < \omega)$.

2° We define $f : \omega_1 \to D$ as follows: for all $\alpha < \omega_1$, for the unique $\xi < \omega_1$ and $n < \omega$ such that $\alpha = \omega\xi + n$, we let $f(\alpha) := S_n^\xi$.

3° Let C be a club contained in U^*. By Proposition [3.1.6], the set $E := \{\delta(\mathcal{M}_g(\alpha)) : \alpha \in C, \omega\alpha = \alpha\}$ is a club.

4° We will be done if we show that $E \subseteq \nabla f$. To that end, let $\delta \in E$ be arbitrary and let us find $\gamma < \delta$ such that $\delta \in f(\gamma)$.

5° There exists $\alpha \in C$ such that $\omega\alpha = \alpha$ and $\delta = \delta(\mathcal{M}_g(\alpha))$.

6° Since $\alpha \in U^*$, we have that $\mathcal{M}_g(\alpha) \in U$, so there exists $S \in D \cap \mathcal{M}_g(\alpha)$ such that $\delta \in S$.

7° Again by Proposition [3.1.6], there exist $\xi < \alpha$ and $n < \omega$ such that $S = S_n^\xi$.

8° Let $\gamma := \alpha\xi + n < \alpha$. Then $\delta \in S_n^\xi = f(\gamma)$, as required.

□

3.3 Sealing Iteration

[3.3.1] **Definition.** Let $\delta > \omega$ be a cardinal and let $A \subseteq V_\delta$. Then $\kappa < \delta$ is *A-reflecting below δ* if for all multiplicatively closed $\lambda \in (\kappa, \delta)$, there exists a strong (κ, λ)-extender E satisfying $i_E(A \cap V_\kappa) \cap V_\lambda = A \cap V_\lambda$.

[3.3.2] **Definition.** Let $\delta > \omega$ be a cardinal. A *Woodin diamond for δ* is a function $\mathbf{U} : \kappa \to V_\kappa$ such that

a. for all $\xi < \delta$, $\mathbf{U}(\xi) \subseteq V_\xi$,

b. for all $A \subseteq V_\delta$, the set

$$\{\kappa < \delta : A \cap V_\kappa = \mathbf{U}(\kappa) \text{ and } \kappa \text{ is } A\text{-reflecting below } \delta\}$$

is stationary in δ.

[3.3.3] **Proposition.** Suppose that there exists a Woodin diamond for δ. Then δ is a Woodin cardinal. □

[3.3.4] **Proposition.** Suppose that δ is a Woodin cardinal. Then in $V^{\mathsf{Col}(\delta,\delta)}$, there exists a Woodin diamond for δ.

Proof. Cf. [Sch16, Lemma 1.3] □

[3.3.5] **Definition.** Let $\mathbf{U}$ be a Woodin diamond for δ. *The blueprint for the* NS_{ω_1}*-saturating iteration given by* $\mathbf{U}$ is the pair $((V_\delta, \in, \mathbf{U}), \mathbf{Q})$ where $\mathbf{Q} : \delta \times V_\delta \to V_\delta$ is defined over $(V_\delta, \in, \mathbf{U})$ as follows: for all $\kappa < \delta$ and for all $\mathbb{P} \in V_\delta$, if it is the case that

a. κ is inaccessible,
b. $\mathbb{P} \subseteq V_\kappa$ is a poset,
c. $\mathbf{U}(\kappa) = \mathbb{P} \oplus (\dot{S}_i : i < \kappa)$,
d. $V^{\mathbb{P}} \models$ "$\{\dot{S}_i : i < \kappa\}$ is predense in $\mathbb{P}_{\mathsf{NS}_{\omega_1}}$",
e. $\dot{U}$ is the canonical name such that

$$V^{\mathbb{P}} \models \dot{U} = \{M \prec H_{\omega_2} : |M| = \omega, \text{“}M \text{ captures } \{\dot{S}_i : i < \kappa\}\text{”}\},$$

then we set $\mathbb{Q}_\kappa$ to be the canonical $\mathbb{P}$-name for $\mathsf{Col}_{\dot{U}}$, while otherwise we set $\mathbb{Q}_\kappa$ to be the canonical name for the trivial poset.

[3.3.6] **Proposition.** Suppose that

1. $\mathbf{U}$ is a Woodin diamond for δ,
2. $(\mathbb{V}, \mathbf{Q})$ is the blueprint for the NS_{ω_1}-saturating iteration given by $\mathbf{U}$,
3. $(\mathbb{P}_\alpha : \alpha \in \mathscr{E}^*)$ is the semi-proper forcing iteration given by the blueprint $(\mathbb{V}, \mathbf{Q})$,
4. $\mathbb{P} := \bigcup_{\alpha \in \mathscr{E}^*} \mathbb{P}_\alpha$.

Then it holds that

a. $\mathbb{P}$ is semi-proper,
b. $\mathbb{P}$ is δ-c.c,
c. $\omega_2^{V^{\mathbb{P}}} = \delta$,
d. in $V^{\mathbb{P}}$, NS_{ω_1} is saturated.

Proof.

1° Parts a-c. are general facts about the iteration. Let us verify that in $V^{\mathbb{P}}$, NS_{ω_1} is saturated.

2° Let $(\dot{S}_i : i < \delta)$ be a name for a predense collection in $\mathbb{P}_{\mathsf{NS}_{\omega_1}}$ in $V^{\mathbb{P}}$. We want to show that the corresponding predense collection in $V^{\mathbb{P}}$ is sealed.

3° The set

$$\mathscr{E}_0 := \Big\{\alpha < \delta : V_\alpha \prec (V_\delta, \in, \mathbf{U}, \mathbb{P}, (\dot{S}_i : i < \delta))\Big\}$$

is a club in δ. This means that there exists $\kappa \in \mathscr{E}_0$ such that

a. $\mathbf{U}(\kappa) = \mathbb{P}_{<\kappa} \oplus (\dot{S}_i : i < \kappa)$,
b. κ is $\mathbb{P} \oplus (S_i : i < \delta)$-reflecting below δ.

We will show that $(\dot{S}_i : i < \kappa)$ is sealed inside $V^{\mathbb{P}_\kappa}$.

4° Let $\lambda, E, j, \mathcal{M}$ be such that

a. $\lambda \in \mathscr{E}_0$ is inaccessible $> \kappa$,
b. E is a strong (κ, λ)-extender,
c. $j := i_E : V \prec \mathsf{Ult}(V, E) =: \mathcal{M}$,
d. $j(\mathbb{P}_{<\kappa}) \cap V_\lambda = \mathbb{P}_{<\lambda}$,
e. $j((\dot{S}_i : i < \kappa)) \restriction \lambda = (\dot{S}_i : i < \lambda)$.

5° Let G_κ be V-generic for $\mathbb{P}_\kappa$ and let us work in $V[G_\kappa]$. The family $(S_i : i < \kappa)$ is predense in $\mathbb{P}_{\mathsf{NS}_{\omega_1}}$, so $\mathbb{Q}_\kappa = \mathsf{Col}_U$, where

$$U := \{M \prec H_{\omega_2} : |M| = \omega, \text{“}M \text{ captures } \{S_i : i < \kappa\}\text{”}\}.$$

6° Let g be $V[G_\kappa]$-generic for Col_U, let $G_{\kappa+1}$ be the V-generic for $\mathbb{P}_{\kappa+1}$ corresponding to $G_\kappa * g$, and let us work in $V[G_{\kappa+1}]$. We denote by U^* the set

$$\{\xi < \omega_1 : \mathcal{M}_g(\xi) \in U\}.$$

7° Let G^* be V-generic for $j(\mathbb{P}_\kappa)$ extending $G_{\kappa+1}$ and let $k : V[G_\kappa] \prec \mathcal{M}[G^*]$ be such that $k \restriction V = j$ and $k(G_\kappa) = G^*$.

8° **Claim.** In $\mathcal{M}[G^*]$, $\omega_1 - U^*$ is not stationary.

Proof.

1′ Let us assume otherwise. Then there exists $i_0 < j(\kappa)$ such that the set

$$k((\dot{S}_i : i < \kappa))(i_0) \cap (\omega_1 - U^*)$$

is stationary in ω_1.

2′ In $\mathcal{M}$, let $(\dot{T}_i : i < j(\kappa)) = k((\dot{S}_i : i < \kappa))$ and let $(M_\xi : \xi < \omega_1)$ be a continuous $\in$-chain of countable elementary submodels of

$$(\mathcal{M} \restriction j(\lambda), \in, \kappa, \lambda, \dot{T}_i).$$

3′ Back in $\mathcal{M}[G^*]$, the set

$$\{\delta(M_\xi[G^*]) : \xi < \omega_1\}$$

is a club in ω_1, so there exists $\xi_0 < \omega_1$ such that

$$\alpha := \delta(M_{\xi_0}[G^*]) \in T_{i_0} \cap (\omega_1 - U^*).$$

4′ Since $\alpha \notin U^*$, we have that $\mathcal{M}_g(\alpha) \notin U$. In other words, inside $\mathcal{M}[G_\kappa]$, there does not exists $N \in U$ such that $N \supseteq \mathcal{M}_g(\alpha)$ and $\delta_N = \delta(\mathcal{M}_g(\alpha))$.

5′ Since $\mathcal{M}_g(\alpha)$ is a countable subset of H_{ω_2} inside $\mathcal{M}[G_\kappa]$ and since $\omega_2^{\mathcal{M}[G_\kappa]} = \kappa$, we have that $k(\mathcal{M}_g(\alpha)) = \mathcal{M}_g(\alpha)$.

6′ Consequently, in $\mathcal{M}[G^*]$, there does not exist $N \in k(U)$ such that $N \supseteq \mathcal{M}_g(\alpha)$ and $\delta_N = \delta(\mathcal{M}_g(\alpha))$.

7′ Since $g \in M_{\xi_0}[G^*]$, we get that

$$M_{\xi_0}[G^*] \cap H_{\omega_2}^{V[G_\kappa]} = \bigcup_{\xi<\alpha} \mathcal{M}_g(\xi) = \mathcal{M}_g(\alpha).$$

Hence, $M_{\xi_0}[G^*] \supseteq \mathcal{M}_g(\alpha)$ and $\delta(M_{\xi_0}[G^*]) = \delta(\mathcal{M}_g(\alpha))$.

8′ On the other hand, $M_{\xi_0}[G^*]$ captures $k(\{S_i : i < \kappa\})$, as witnessed by T_{i_0}. This means that $M_{\xi_0}[G^*] \in k(U)$.

9′ The previous three points together yield a contradiction.

□

9° Thus, in $\mathcal{M}[G^*]$, U^* contains a club.

10° By Proposition [3.2.7] applied in $\mathcal{M}[G_\kappa]$, we have that $\{S_i : i < \kappa\}$ is sealed in $\mathcal{M}[G^*]$.

11° This means that $k(\{S_i : i < \kappa\})$ is sealed in $\mathcal{M}[G^*]$, which by elementarity of k yields that $\{S_i : i < \kappa\}$ is sealed in $V[G_\kappa]$, as required.

□

[3.3.7] **Corollary.** Suppose that δ is a Woodin cardinal. Then there exists a poset $\mathbb{P}$ such that

a. $\mathbb{P}$ is semi-proper,

b. $\mathbb{P}$ is δ-c.c,

c. $\omega_2^{V^{\mathbb{P}}} = \delta$,

d. in $V^{\mathbb{P}}$, NS_{ω_1} is saturated. □

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