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The equality between the Stochastic CRB and its semiparametric version for elliptical data

Stefano Fortunati and Esa Ollila, *Senior, IEEE*

Abstract—In this letter, the proof of the following quite surprising result is provided: if the data are elliptically distributed, the “parametric” Stochastic Cramér-Rao Bound (derived by assuming the a priori knowledge of the density generator) and its semiparametric counterpart (obtained by considering the density generator as an additional nuisance parameter) are equal. In other words, this means that, in the presence of finite-dimensional nuisance parameters (the source correlation matrix and noise power in the decomposition of the array covariance/scatter matrix), not knowing the data density generator does not lead to an additional loss of efficiency.

Index Terms—DOA estimation, Stochastic Cramér-Rao Bound, semiparametric models.

I. INTRODUCTION

The celebrated Stochastic Cramér-Rao Bound (SCRB) is a well-known tool in array processing since it represent a lower bound for the Mean Square Error (MSE) of any Direction of Arrival (DOA) estimation algorithm. While the SCRB is generally exploited in Gaussian-based scenarios, recent advances have allowed for a relaxation of this rather stringent assumption in favor of the more general hypothesis of elliptically distributed data. This letter shows a surprising equality between the parametric version of the elliptical SCRB and its semiparametric counterpart [2].

Notation: In this letter, italics indicates scalar quantities (a), lower case and upper case boldface indicate column vectors (\mathbf{a}) and matrices (\mathbf{A}), respectively. The superscripts T , $*$ and H indicate the transpose, the complex conjugate and the Hermitian operators, respectively. Each entry of a vector \mathbf{a} and of a matrix \mathbf{A} is indicated as $a_i \triangleq [\mathbf{A}]_i$ and $a_{i,j} \triangleq [\mathbf{A}]_{i,j}$, respectively. Let $\mathbf{A}(\boldsymbol{\theta})$ be a matrix (or possibly a vector or even a scalar) function of the *real* vector $\boldsymbol{\theta} \in \Theta \subseteq \mathbb{R}^d$, then $\mathbf{A}_0 \triangleq \mathbf{A}(\boldsymbol{\theta}_0)$ while $\mathbf{A}_i^0 \triangleq \frac{\partial \mathbf{A}(\boldsymbol{\theta})}{\partial \theta_i} |_{\boldsymbol{\theta}=\boldsymbol{\theta}_0}$, where $\boldsymbol{\theta}_0$ is a particular (or *true*) value of $\boldsymbol{\theta} \in \Theta$. For random variables or vectors, $=_d$ stands for “has the same distribution as”.

II. DOA ESTIMATION IN ELLIPTICAL DATA

Let $\mathbf{x} \in \mathbb{C}^m$ be the observation vector collected by an array of m sensors and let $\boldsymbol{\nu}_0 = (\nu_{0,1}, \dots, \nu_{0,q})^T \in \Gamma \subseteq \mathbb{R}^q$ be the vector of the location parameters (in some reference systems) of q narrowband sources. As firstly proposed in [1], we suppose that the observation vector is CES-distributed,

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i.e. $\mathbf{x} \sim CES_m(\mathbf{x}; \mathbf{0}, \boldsymbol{\Sigma}_0, g_0)$, and its scatter matrix has the following peculiar structure:

$$\boldsymbol{\Sigma}_0 \equiv \boldsymbol{\Sigma}(\boldsymbol{\nu}_0, \boldsymbol{\Gamma}_0, \sigma_0^2) = \mathbf{A}_0 \boldsymbol{\Gamma}_0 \mathbf{A}_0^H + \sigma_0^2 \mathbf{I}, \quad (1)$$

- $\mathbf{A}_0 \triangleq \mathbf{A}(\boldsymbol{\nu}_0) = [\mathbf{a}(\nu_{0,1}) \cdots \mathbf{a}(\nu_{0,q})] \in \mathbb{C}^{m \times q}$ such that (s.t.) $\mathbf{a}(\nu_{0,k})$ is the steering vector for the k -th source,
- $\boldsymbol{\Gamma}_0 \in \mathcal{M}_q \subset \mathbb{C}^{q \times q}$ is the source covariance matrix and \mathcal{M}_q stands for the set of the Hermitian, positive definite matrices of dimension $q \times q$,
- $\sigma_0^2 \in \mathbb{R}^+$ is the noise power,
- $g_0 \in \mathcal{G}$ is the density generator.

To avoid the scale ambiguity between the density generator and the data covariance matrix, we define the set \mathcal{G} as [2]:

$$\mathcal{G} \triangleq \left\{ g : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+ \mid 2^{-1} s_m \int_0^{+\infty} q^m g(q) dq = m \right\}, \quad (2)$$

where $s_m \triangleq 2\pi^m \Gamma(m)^{-1}$ and $\Gamma(\cdot)$ is the Gamma function. Note that, if $\mathbf{x} \sim CES_m(\mathbf{x}; \mathbf{0}, \boldsymbol{\Sigma}_0, g_0)$, then it admits the following stochastic representation [3]:

$$\mathbf{x} =_d \sqrt{Q} \boldsymbol{\Sigma}_0^{1/2} \mathbf{v}, \quad (3)$$

where $\mathbf{v} \sim U(\mathbb{C}S^m)$ is uniformly distributed on the unit $m-1$ sphere and $Q \sim p_Q(q) = 2^{-1} s_m q^{m-1} g_0(q)$ is called the *2nd-order modular variate*.

In the rest of this letter, instead of $\boldsymbol{\Gamma}_0$, we will make use of $\boldsymbol{\zeta}_0 \in \Psi \subseteq \mathbb{R}^{q^2}$ defined as:

$$\boldsymbol{\zeta}_0 \triangleq (\text{diag}(\boldsymbol{\Gamma}_0)^T, \text{vec}_l(\text{Re}(\boldsymbol{\Gamma}_0))^T, \text{vec}_l(\text{Im}(\boldsymbol{\Gamma}_0))^T)^T, \quad (4)$$

where the operator $\text{vec}_l(\cdot)$ selects all the entries strictly below the main diagonal of $\boldsymbol{\Gamma}_0$ taken in the column-wise order [4, Sec. 2.4] while $\text{diag}(\boldsymbol{\Gamma}_0)$ is a column vector collecting the diagonal elements of $\boldsymbol{\Gamma}_0$. Finally, let us define $\boldsymbol{\xi}_0$ as:

$$\boldsymbol{\xi}_0 \triangleq (\boldsymbol{\zeta}_0^T, \sigma_0^2)^T \in \Psi \times \mathbb{R}^+ \subseteq \mathbb{R}^{q^2+1}. \quad (5)$$

The celebrated “Stochastic” Cramér-Rao Bound (SCRB) has been introduced as the CRB on the estimation of the sources location vector $\boldsymbol{\nu}_0$ in the presence of the nuisance vector $\boldsymbol{\xi}_0$ (see e.g. [5]). According to the degree of a priori knowledge on $g_0 \in \mathcal{G}$, we may have the following three results:

- C1 g_0 is assumed to be equal to the density generator of the Gaussian distribution. This leads to the *Gaussian SCRB* $\text{GSCRb}(\boldsymbol{\nu}_0 | \boldsymbol{\xi}_0, \sigma_0^2)$ derived in, among others, [5].
- C2 g_0 is assumed to be equal to a specific density generator. This leads to the *Elliptical SCRb* $\text{ESCRb}(\boldsymbol{\nu}_0 | \boldsymbol{\xi}_0, \sigma_0^2)$ that can be obtained by inverting the Fisher Information Matrix introduced in [6].
- C3 g_0 is assumed to belong to the CES class but, unlike in C1 and C2, it is left fully unspecified, i.e. it is not assumed

to be a priori known. This leads to the *Semiparametric SCRB* $\text{SSCRB}(\nu_0|\xi_0, \sigma_0^2, g_0)$ derived in [2].

In this letter, we show that the Elliptical SCRB is equal to the Semiparametric CRB for each possible $g_0 \in \mathcal{G}$, i.e.

$$\text{ESCRB}(\nu_0|\xi_0, \sigma_0^2) = \text{SSCRB}(\nu_0|\xi_0, \sigma_0^2, g_0), \quad \forall g_0 \in \mathcal{G}. \quad (6)$$

In word, this means that *if we do not know ξ_0 , not knowing the density generator g_0 does not lead to an additional loss in estimation efficiency.*

To provide a proof of this (quite surprising) result, let us start by framing this estimation problem in the geometric, ‘‘Hilbert-spaces based’’ context of semiparametric models.

III. MODELS, TANGENT SPACES AND SCORE VECTORS

Let us consider the general semiparametric model:

$$\mathcal{P} = \{p_X(\mathbf{x}|\nu, \xi, g) : \nu \in \Gamma, \xi \in \Psi, g \in \mathcal{G}\}, \quad (7)$$

where $\Gamma \subseteq \mathbb{R}^q$ is the set of the (finite-dimensional) parameter vectors ν of interests, $\Psi \subseteq \mathbb{R}^r$ is the set of (finite-dimensional) nuisance parameter vectors ξ and \mathcal{G} is the space of the (infinite-dimensional) nuisance functions g . Note that, in the scenario described in Sect. II, $r = q^2 + 1$. We indicate as $\nu_0 \in \Gamma$, $\xi_0 \in \Psi$ and $g_0 \in \mathcal{G}$ the true, but unknown, related quantities and with $E_0\{f\} \triangleq \int f(\mathbf{x})dP_X(\mathbf{x}|\nu_0, \xi_0, g_0)$ the expectation of a measurable function f with respect to (w.r.t.) the true distribution $P_X(\mathbf{x}|\nu_0, \xi_0, g_0)$. Before moving on, let us introduce an Hilbert space that holds fundamental importance in our subsequent derivations. Specifically, we indicate with $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ the (infinite-dimensional) Hilbert space of the measurable functions with *i*) zero-mean $E_0\{h\} = 0$ and *ii*) finite variance $E_0\{h^2\} = \sigma_h^2 < +\infty$ ([7, Sect. 2.2] and [8, Ch. 2]). The related inner product is given by $\langle h_1, h_2 \rangle = E_0\{h_1 h_2^*\}$.

By using a self-explanatory notation, we now introduce three *parametric* submodels of \mathcal{P} as:

$$\mathcal{P}_1 = \{p_X(\mathbf{x}|\nu, \xi_0, g_0) : \nu \in \Gamma\}, \quad (8)$$

$$\mathcal{P}_2 = \{p_X(\mathbf{x}|\nu_0, \xi, g_0) : \xi \in \Psi\}, \quad (9)$$

$$\mathcal{P}_{1,2} = \{p_X(\mathbf{x}|\nu, \xi, g_0) : \nu \in \Gamma, \xi \in \Psi\}, \quad (10)$$

along with the *non-parametric* model

$$\mathcal{P}_3 = \{p_X(\mathbf{x}|\nu_0, \xi_0, g) : g \in \mathcal{G}\}. \quad (11)$$

Let us define the score vector \mathbf{s}_{ν_0} in \mathcal{P}_1 as:

$$[\mathbf{s}_{\nu_0}]_i \triangleq [\mathbf{s}_{\nu_0}(\mathbf{x})]_i = \partial \ln p_X(\mathbf{x}|\nu, \xi_0, g_0) / \partial \nu_i |_{\nu=\nu_0}, \quad (12)$$

for $i = 1, \dots, q$, that represents the score vector of the parameters of interest. Similarly, the score vector \mathbf{s}_{ξ_0} of the finite-dimensional nuisance parameters ξ_0 in \mathcal{P}_2 is given by:

$$[\mathbf{s}_{\xi_0}]_j \triangleq [\mathbf{s}_{\xi_0}(\mathbf{x})]_j = \partial \ln p_X(\mathbf{x}|\nu_0, \xi, g_0) / \partial \xi_j |_{\xi=\xi_0}, \quad (13)$$

for $j = 1, \dots, r$. Under the standard regularity conditions discussed in [9, Sect. 6.2 and 6.3], it is immediate to verify that $[\mathbf{s}_{\nu_0}]_i, [\mathbf{s}_{\xi_0}]_j \in \mathcal{H}, \forall i, j$.

We can now introduce the (finite-dimensional) tangent space of the parametric submodels \mathcal{P}_2 as the linear span of \mathbf{s}_{ξ_0} in \mathcal{H} (see e.g. [8, Sect. 2.3] or [7, eq. (24)]):

$$\mathcal{H} \supseteq \mathcal{T}_2 \triangleq \text{Span}\{[\mathbf{s}_{\xi_0}]_1, \dots, [\mathbf{s}_{\xi_0}]_r\}. \quad (14)$$

Remark: It is worth noticing that in this letter we prefer to work only with the Hilbert space \mathcal{H} instead of introducing the related ‘‘ q -replicating space’’ $\mathcal{H}_q \triangleq \mathcal{H} \times \dots \times \mathcal{H}$ as in [7, Sect. 2.2]. The two approaches are fully equivalent. In particular, if we work on (subspaces of) \mathcal{H} , instead of \mathcal{H}_q , all the operators, as expectations and projections, has to be interpreted componentwise (see e.g. [8, Sect. 2.4, Remark 2]). Specifically, for each $\mathbf{h} = (h_1, \dots, h_q)^T$ s.t. $h_i \in \mathcal{H}$ and $u \in \mathcal{U} \subseteq \mathcal{H}$, we have, for $i = 1, \dots, q$:

$$[\Pi(\mathbf{h}|\mathcal{U})]_i \triangleq \Pi([\mathbf{h}]_i|\mathcal{U}) = \Pi(h_i|\mathcal{U}), \quad (15)$$

$$[E_0\{\mathbf{h}u^*\}]_i \triangleq E_0\{[\mathbf{h}]_i u^*\} = \langle h_i, u \rangle, \quad (16)$$

Finally, the (infinite-dimensional) nuisance tangent space $\mathcal{T}_3 \subseteq \mathcal{H}$ of the non-parametric model \mathcal{P}_3 is defined as in [7, eq. (44)]. Note that, by construction, \mathcal{T}_2 and \mathcal{T}_3 are finite- and infinite-dimensional closed subspaces of \mathcal{H} .

According to the previous definition, let us define the *efficient score vector* $\bar{\mathbf{t}}_{\nu_0}$ for the vector ν_0 of the parameter of interest in the parametric submodel $\mathcal{P}_{1,2}$ in (10) as [7, Sect. 2.3, Def. 4]:

$$\bar{\mathbf{t}}_{\nu_0} \triangleq \mathbf{s}_{\nu_0} - \Pi(\mathbf{s}_{\nu_0}|\mathcal{T}_2), \quad (17)$$

where the projection operator has to be interpreted componentwise as defined in (15). Since \mathcal{T}_2 is a finite-dimensional subspace of \mathcal{H} , the projection operator $\Pi(\cdot|\mathcal{T}_2)$ can be derived in closed form as [7, eq. (16)], [8, Sect. 2.4, Example 1]:

$$[\Pi(\mathbf{h}|\mathcal{T}_2)]_i = E_0\{h_i \mathbf{s}_{\xi_0}^H\} \mathbf{I}_{\xi_0}^{-1} \mathbf{s}_{\xi_0}, \quad h_i \in \mathcal{H} \forall i, \quad (18)$$

where:

$$\mathbf{I}_{\xi_0} \triangleq E_0\{\mathbf{s}_{\xi_0} \mathbf{s}_{\xi_0}^H\}, \quad (19)$$

is the FIM for ξ_0 in the parametric submodel \mathcal{P}_2 in (9). Then, we have that $\bar{\mathbf{t}}_{\nu_0}$ can be explicitly expressed as:

$$\bar{\mathbf{t}}_{\nu_0} = \mathbf{s}_{\nu_0} - E_0\{\mathbf{s}_{\nu_0} \mathbf{s}_{\xi_0}^H\} \mathbf{I}_{\xi_0}^{-1} \mathbf{s}_{\xi_0} = \mathbf{s}_{\nu_0} - \mathbf{I}_{\nu_0 \xi_0} \mathbf{I}_{\xi_0}^{-1} \mathbf{s}_{\xi_0}, \quad (20)$$

where

$$\mathbf{I}_{\nu_0 \xi_0} \triangleq E_0\{\mathbf{s}_{\nu_0} \mathbf{s}_{\xi_0}^H\} \quad (21)$$

is the matrix of the cross-information terms in the parametric submodel $\mathcal{P}_{1,2}$ in (10).

On the same line, the *semiparametric efficient score vector* $\bar{\mathbf{s}}_{\nu_0}$ for ν_0 in the semiparametric model \mathcal{P} in (7) is given by [10, Sect. 3.4, eq. (18)]:

$$\bar{\mathbf{s}}_{\nu_0} \triangleq \mathbf{s}_{\nu_0} - \Pi(\mathbf{s}_{\nu_0}|\mathcal{T}_2 + \mathcal{T}_3),$$

where, since \mathcal{T}_2 is a (closed) finite-dimensional subspace and \mathcal{T}_3 is closed, then $\mathcal{T}_2 + \mathcal{T}_3$ is closed. In general, since $\mathcal{T}_2 + \mathcal{T}_3$ is infinite-dimensional, a closed form for the projection operator $\Pi(\cdot|\mathcal{T}_2 + \mathcal{T}_3)$ does not exist in general. Consequently, an explicit expression for $\bar{\mathbf{s}}_{\nu_0}$ cannot be obtained. Fortunately, some further manipulation is still possible. Since $\mathcal{T}_2 \subseteq \mathcal{T}_2 + \mathcal{T}_3$,

by using [10, A2, eq. (11)], $\bar{\mathbf{s}}_{\nu_0}$ can be expressed as:

$$\begin{aligned}\bar{\mathbf{s}}_{\nu_0} &\triangleq \mathbf{s}_{\nu_0} - \Pi(\mathbf{s}_{\nu_0} | \mathcal{T}_2 + \mathcal{T}_3) \\ &= \mathbf{s}_{\nu_0} - \Pi(\mathbf{s}_{\nu_0} | \mathcal{T}_2) - \Pi(\mathbf{s}_{\nu_0} | (\mathcal{T}_2 + \mathcal{T}_3) \cap \mathcal{T}_2^\perp) \\ &= \bar{\mathbf{t}}_{\nu_0} - \Pi(\bar{\mathbf{t}}_{\nu_0} | (\mathcal{T}_2 + \mathcal{T}_3) \cap \mathcal{T}_2^\perp),\end{aligned}\quad (22)$$

where the last equality holds true since, by definition, $(\mathcal{T}_2 + \mathcal{T}_3) \cap \mathcal{T}_2^\perp \subseteq \mathcal{T}_2^\perp$ and $\mathcal{T}_2 \ni \Pi([\mathbf{s}_{\nu_0}]_i | \mathcal{T}_2) \perp \mathcal{T}_2^\perp, \forall i$.

The two following Lemmas are given, without proof, in [10, Sect. 3.4, Prop. 3]. Here, to make the presentation as self-contained as possible, we provide a complete proof of these two key results.

Lemma 1. *The efficient Semiparametric FIM (SFIM) for ν_0 in the presence of the finite- and infinite-dimensional nuisance terms ξ_0 and g_0 , can be evaluated as:*

$$\bar{\mathbf{I}}(\nu_0 | \xi_0, g_0) = \bar{\mathbf{I}}(\nu_0 | \xi_0) - E_0 \{ \mathbf{w} \mathbf{w}^H \}, \quad (23)$$

where

$$\mathbf{w} \triangleq \Pi(\bar{\mathbf{t}}_{\nu_0} | (\mathcal{T}_2 + \mathcal{T}_3) \cap \mathcal{T}_2^\perp), \quad (24)$$

and

$$\bar{\mathbf{I}}(\nu_0 | \xi_0) \triangleq E_0 \{ \bar{\mathbf{t}}_{\nu_0} \bar{\mathbf{t}}_{\nu_0}^H \}. \quad (25)$$

Proof. We start by recalling that $E_0 \{ (\cdot)(\cdot)^* \} = \langle \cdot, \cdot \rangle$ in \mathcal{H} (see [7, Sect. 2.2]). From (22), it follows that $\bar{\mathbf{t}}_{\nu_0} = \bar{\mathbf{s}}_{\nu_0} + \mathbf{w}$. Then, through direct calculation, we get:

$$\begin{aligned}E_0 \{ \bar{\mathbf{s}}_{\nu_0} \bar{\mathbf{s}}_{\nu_0}^H \} &= \bar{\mathbf{I}}(\nu_0 | \xi_0) - E_0 \{ \bar{\mathbf{t}}_{\nu_0} \mathbf{w}^H \} - \\ &\quad + E_0 \{ \mathbf{w} \bar{\mathbf{t}}_{\nu_0}^H \} + E_0 \{ \mathbf{w} \mathbf{w}^H \} \\ &= \bar{\mathbf{I}}(\nu_0 | \xi_0) - 2\text{Re} [E_0 \{ \bar{\mathbf{s}}_{\nu_0} \mathbf{w}^H \}] - E_0 \{ \mathbf{w} \mathbf{w}^H \}.\end{aligned}\quad (26)$$

Then the equality in (23) follows immediately iff $E_0 \{ \bar{\mathbf{s}}_{\nu_0} \mathbf{w}^H \} = \mathbf{0}$. From the componentwise application of the inner product, we have that:

$$E_0 \{ \bar{\mathbf{s}}_{\nu_0} \mathbf{w}^H \} = \mathbf{0} \Leftrightarrow \langle [\bar{\mathbf{s}}_{\nu_0}]_i, w_j \rangle = 0, \quad (27)$$

$\forall i, j \in \{1, \dots, q\}$. To show that $[\bar{\mathbf{s}}_{\nu_0}]_i$ is orthogonal to $w_j, \forall i, j$, we note that according to its definition given in (22), $[\bar{\mathbf{s}}_{\nu_0}]_i \in (\mathcal{T}_2 + \mathcal{T}_3)^\perp$. Moreover, again by definition in (24), $w_j \in (\mathcal{T}_2 + \mathcal{T}_3) \cap \mathcal{T}_2^\perp \subseteq \mathcal{T}_2 + \mathcal{T}_3$. Consequently, $\langle [\bar{\mathbf{s}}_{\nu_0}]_i, w_j \rangle = 0, \forall i, j$ since $(\mathcal{T}_2 + \mathcal{T}_3) \perp (\mathcal{T}_2 + \mathcal{T}_3)^\perp$. \square

It is worth noticing that the matrix $\bar{\mathbf{I}}(\nu_0 | \xi_0)$ is the efficient FIM for ν_0 in the presence of the finite-dimensional nuisance vector ξ_0 in the parametric submodel $\mathcal{P}_{1,2}$ in (10). Consequently the CRB for ν_0 in the presence of ξ_0 is given by:

$$\text{CRB}(\nu_0 | \xi_0) = \bar{\mathbf{I}}(\nu_0 | \xi_0)^{-1} = \left[\mathbf{I}_{\nu_0} - \mathbf{I}_{\nu_0 \xi_0} \mathbf{I}_{\xi_0}^{-1} \mathbf{I}_{\nu_0 \xi_0}^H \right]^{-1}, \quad (28)$$

where:

$$\mathbf{I}_{\nu_0} \triangleq E_0 \{ \mathbf{s}_{\nu_0} \mathbf{s}_{\nu_0}^H \}, \quad (29)$$

is the FIM for ν_0 in the parametric submodel \mathcal{P}_1 in (8).

Lemma 2. *The efficient SFIM $\bar{\mathbf{I}}(\nu_0 | \xi_0, g_0)$ for the model \mathcal{P} in (7) is equal to the parametric efficient FIM $\bar{\mathbf{I}}(\nu_0 | \xi_0)$ in $\mathcal{P}_{1,2}$ iff $E_0 \{ \mathbf{w} \mathbf{w}^H \} = 0$ and consequently iff $\mathbf{w} = \mathbf{0}$. Moreover,*

$$\mathbf{w} = \mathbf{0} \Leftrightarrow [\bar{\mathbf{t}}_{\nu_0}]_i = [\mathbf{s}_{\nu_0}]_i - \Pi([\mathbf{s}_{\nu_0}]_i | \mathcal{T}_2) \perp \mathcal{T}_3, \quad (30)$$

for $i = 1, \dots, q$.

Proof. To prove the double implication in (30), let us start with the implication \Leftarrow . From [10, A2, eq. (4)] and by recalling that $\mathcal{T}_2 + \mathcal{T}_3$ is closed, we have that $(\mathcal{T}_2 + \mathcal{T}_3) \cap \mathcal{T}_2^\perp = (\mathcal{T}_2^\perp \cap \mathcal{T}_3^\perp)^\perp \cap \mathcal{T}_2^\perp$. Then, since $(\mathcal{T}_2^\perp \cap \mathcal{T}_3^\perp) \subseteq \mathcal{T}_2^\perp$, from [10, A2, Prop. 4], we have that:

$$\Pi(\bar{\mathbf{t}}_{\nu_0} | (\mathcal{T}_2 + \mathcal{T}_3) \cap \mathcal{T}_2^\perp) = \Pi(\bar{\mathbf{t}}_{\nu_0} | \mathcal{T}_2^\perp) - \Pi(\bar{\mathbf{t}}_{\nu_0} | \mathcal{T}_2^\perp \cap \mathcal{T}_3^\perp). \quad (31)$$

By definition, $[\bar{\mathbf{t}}_{\nu_0}]_i \in \mathcal{T}_2^\perp$. Moreover, if $[\bar{\mathbf{t}}_{\nu_0}]_i \perp \mathcal{T}_3$, then $[\bar{\mathbf{t}}_{\nu_0}]_i \in \mathcal{T}_3^\perp$. Then, $[\bar{\mathbf{t}}_{\nu_0}]_i \in \mathcal{T}_2^\perp \cap \mathcal{T}_3^\perp$ for $i = 1, \dots, q$. This implies that, from (31):

$$\Pi(\bar{\mathbf{t}}_{\nu_0} | (\mathcal{T}_2 + \mathcal{T}_3) \cap \mathcal{T}_2^\perp) = \bar{\mathbf{t}}_{\nu_0} - \bar{\mathbf{t}}_{\nu_0} = \mathbf{0}. \quad (32)$$

Let us now prove the second implication \Rightarrow . If the projection $\Pi(\bar{\mathbf{t}}_{\nu_0} | (\mathcal{T}_2 + \mathcal{T}_3) \cap \mathcal{T}_2^\perp) = \mathbf{0}$, then, from [10, A.2, eq. (4)]:

$$[\bar{\mathbf{t}}_{\nu_0}]_i \in [(\mathcal{T}_2 + \mathcal{T}_3) \cap \mathcal{T}_2^\perp]^\perp = (\mathcal{T}_2^\perp \cap \mathcal{T}_3^\perp) + \mathcal{T}_2, \quad (33)$$

since \mathcal{T}_2 and $(\mathcal{T}_2 + \mathcal{T}_3)^\perp$ are closed subspaces. Moreover, since $(\mathcal{T}_2^\perp \cap \mathcal{T}_3^\perp) \subseteq \mathcal{T}_2^\perp$, we have that:

$$[\bar{\mathbf{t}}_{\nu_0}]_i \in (\mathcal{T}_2^\perp \cap \mathcal{T}_3^\perp) + \mathcal{T}_2 = (\mathcal{T}_2^\perp \cap \mathcal{T}_3^\perp) \oplus \mathcal{T}_2, \quad (34)$$

where \oplus indicates the direct sum of two subspaces. Then, since by definition $[\bar{\mathbf{t}}_{\nu_0}]_i \in \mathcal{T}_2^\perp$, as a consequence of (34), we need to have that $[\bar{\mathbf{t}}_{\nu_0}]_i \in \mathcal{T}_3^\perp$ as well, or equivalently, $[\bar{\mathbf{t}}_{\nu_0}]_i \perp \mathcal{T}_3$ for $i = 1, \dots, q$. \square

IV. THE EQUALITY BETWEEN THE ESCRB AND SSCRB

We can now use the results of Lemma 2 to show the equality in (6). We start by specializing the semiparametric model \mathcal{P} in (7) for the DOA estimation described in Sect. II as:

$$\begin{aligned}\mathcal{P} &= \{ p_X(\mathbf{x} | \nu, \xi, g) = CES_m(\mathbf{x}; \mathbf{0}, \Sigma(\nu, \xi), g) : \\ &\quad \nu \in \Gamma, \xi = (\zeta^T, \sigma^2)^T \in \Psi, g \in \mathcal{G} \},\end{aligned}\quad (35)$$

then, its parametric submodel $\mathcal{P}_{1,2}$ in (10) becomes:

$$\begin{aligned}\mathcal{P}_{1,2} &= \{ p_X(\mathbf{x} | \nu, \xi, g_0) = CES_m(\mathbf{x}; \mathbf{0}, \Sigma(\nu, \xi), g_0) : \\ &\quad \nu \in \Gamma, \xi = (\zeta^T, \sigma^2)^T \in \Psi, \}, g_0 \in \mathcal{G}.\end{aligned}\quad (36)$$

The Elliptical Stochastic CRB ESCRB($\nu_0 | \xi_0, \sigma_0^2$) for a specific density generator $g_0 \in \mathcal{G}$ can then be obtained as the inverse of the efficient FIM $\bar{\mathbf{I}}(\nu_0 | \xi_0)$ for the model $\mathcal{P}_{1,2}$ in (36). From (28), this is equivalent to calculating the first top-left block submatrix of the inverse of the joint FIM for ν_0 and ξ_0 in the model $\mathcal{P}_{1,2}$ in (36), i.e.

$$\mathbf{I}(\nu_0, \xi_0) = \begin{pmatrix} \mathbf{I}_{\nu_0} & \mathbf{I}_{\nu_0 \xi_0} \\ \mathbf{I}_{\nu_0 \xi_0}^H & \mathbf{I}_{\xi_0} \end{pmatrix}, \quad (37)$$

that can be evaluated by means of the elliptical Slepian-Bangs formula provided in [6]. On the other hand, the Semiparametric SCRB for the model \mathcal{P} in (35) can be expressed as SSCRB($\nu_0 | \xi_0, \sigma_0^2, g_0$) = $[\bar{\mathbf{I}}(\nu_0 | \xi_0, g_0)]^{-1}$, where $\bar{\mathbf{I}}(\nu_0 | \xi_0, g_0)$ is given in (23) and it has been evaluated in [2].

Consequently, the equality (6) is verified iff:

$$\bar{\mathbf{I}}(\nu_0 | \xi_0) = \bar{\mathbf{I}}(\nu_0 | \xi_0, g_0) \Leftrightarrow [\bar{\mathbf{t}}_{\nu_0}]_i \perp \mathcal{T}_3, \quad (38)$$

for $i = 1, \dots, q$, where the double implication comes from a direct application of Lemma 2. Moreover, from (20), confirm-

ing the orthogonality condition on the RHS of equation (38) is equivalent to verifying the following equality:

$$\Pi(\bar{\mathbf{t}}_{\nu_0}|\mathcal{T}_3) = \Pi(\mathbf{s}_{\nu_0}|\mathcal{T}_3) - \Pi(\mathbf{I}_{\nu_0}\boldsymbol{\xi}_0\mathbf{I}_{\xi_0}^{-1}\mathbf{s}_{\xi_0}|\mathcal{T}_3) = \mathbf{0}. \quad (39)$$

Let us now prove that (39) actually holds. To this end, we rely on the results already derived in [2]. By adopting the same notation of [2], let us introduce the quantities:

$$\psi_0(t) \triangleq d \ln g_0(t)/dt, \quad (40)$$

$$\mathbf{P}_i^0 \triangleq \boldsymbol{\Sigma}_0^{-1/2} \boldsymbol{\Sigma}_i^0 \boldsymbol{\Sigma}_0^{-1/2}, \quad (41)$$

$$\text{tr}(\mathbf{P}_i^0) = \text{tr}(\boldsymbol{\Sigma}_i^0 \boldsymbol{\Sigma}_0^{-1}) = \text{vec}(\boldsymbol{\Sigma}_0^{-1})^H \text{vec}(\boldsymbol{\Sigma}_i^0), \quad (42)$$

for $i = 1, \dots, q + r$ where $r = q^2 + 1$.

As discussed in [7, Sect. 3.2.2] (see also [10, Sect 4.2]) the projection operator $\Pi(\cdot|\mathcal{T}_3)$ onto the nuisance tangent space \mathcal{T}_3 , defined in [7, Sect. 3.2.2, Lemma 1], can be evaluated as the following conditional expectation:

$$[\Pi(\mathbf{h}|\mathcal{T}_3)]_i = E_0\{h_i|\sqrt{\mathcal{Q}}\}, \quad \forall h_i \in \mathcal{H}. \quad (43)$$

Consequently, from [2, eq. (42)], we have:¹

$$\Pi([\mathbf{s}_{\nu_0}]_i|\mathcal{T}_3) = d - (1 + m^{-1} \mathcal{Q} \psi_0(\mathcal{Q})) \text{tr}(\mathbf{P}_i^0), \quad (44)$$

$$\Pi([\mathbf{s}_{\xi_0}]_j|\mathcal{T}_3) = d - (1 + m^{-1} \mathcal{Q} \psi_0(\mathcal{Q})) \text{tr}(\mathbf{P}_j^0), \quad (45)$$

for $i = 1, \dots, q$, $j = 1, \dots, r$ and

$$\varphi_0(\mathcal{Q}) \triangleq -(1 + m^{-1} \mathcal{Q} \psi_0(\mathcal{Q})). \quad (46)$$

Moreover, let us introduce \mathbf{W}_{ν_0} and \mathbf{W}_{ξ_0} as matrices whose i -th column is given by $\text{vec}(\boldsymbol{\Sigma}_i^0)$, i.e. :

$$\mathbb{C}^{m^2 \times q} \ni \mathbf{W}_{\nu_0} \triangleq \nabla_{\nu}^T \text{vec}(\boldsymbol{\Sigma}_0), \quad (47)$$

$$\mathbb{C}^{m^2 \times r} \ni \mathbf{W}_{\xi_0} = [\mathbf{W}_{\zeta_0} | \text{vec}(\mathbf{I})] \triangleq \nabla_{\xi}^T \text{vec}(\boldsymbol{\Sigma}_0). \quad (48)$$

Then, from (42), it is immediate to verify that eqs. (44) and (45) can be rewritten in vectorial form as:

$$\Pi(\mathbf{s}_{\nu_0}|\mathcal{T}_3) =_d \varphi_0(\mathcal{Q}) \mathbf{W}_{\nu_0}^T \text{vec}(\boldsymbol{\Sigma}_0^{-1})^*, \quad (49)$$

$$\Pi(\mathbf{s}_{\xi_0}|\mathcal{T}_3) =_d \varphi_0(\mathcal{Q}) \mathbf{W}_{\xi_0}^T \text{vec}(\boldsymbol{\Sigma}_0^{-1})^*. \quad (50)$$

From [6], the matrices \mathbf{I}_{ξ_0} and $\mathbf{I}_{\nu_0}\boldsymbol{\xi}_0$ can be expressed as:

$$\mathbf{I}_{\xi_0} = \frac{E\{\mathcal{Q}^2 \psi_0(\mathcal{Q})^2\}}{m(m+1)} (\mathbf{K} \mathbf{W}_{\xi_0})^H (\mathbf{K} \mathbf{W}_{\xi_0}), \quad (51)$$

$$\mathbf{I}_{\nu_0}\boldsymbol{\xi}_0 = \frac{E\{\mathcal{Q}^2 \psi_0(\mathcal{Q})^2\}}{m(m+1)} (\mathbf{K} \mathbf{W}_{\nu_0})^H (\mathbf{K} \mathbf{W}_{\xi_0}), \quad (52)$$

where $\mathbf{K} \triangleq (\boldsymbol{\Sigma}_0^{-T/2} \otimes \boldsymbol{\Sigma}_0^{-1/2})$. Following the notation of [5], let us introduce the matrices $\mathbf{G} \triangleq \mathbf{K} \mathbf{W}_{\nu_0}$ and $\boldsymbol{\Delta} \triangleq \mathbf{K} \mathbf{W}_{\xi_0} = [\mathbf{V} | \mathbf{u}]$, where $\mathbf{u} \triangleq \mathbf{K} \text{vec}(\mathbf{I}) = \text{vec}(\boldsymbol{\Sigma}_0^{-1})$.

Then, by collecting the previous results, we have:

$$\begin{aligned} \Pi(\mathbf{I}_{\nu_0}\boldsymbol{\xi}_0\mathbf{I}_{\xi_0}^{-1}\mathbf{s}_{\xi_0}|\mathcal{T}_3) &= \mathbf{I}_{\nu_0}\boldsymbol{\xi}_0\mathbf{I}_{\xi_0}^{-1} E_0\{\mathbf{s}_{\xi_0}|\sqrt{\mathcal{Q}}\} \\ &= _d \varphi_0(\mathcal{Q}) \mathbf{G}^H \boldsymbol{\Delta} [\boldsymbol{\Delta}^H \boldsymbol{\Delta}]^{-1} \mathbf{W}_{\xi_0}^T \mathbf{u}^*. \end{aligned} \quad (53)$$

By substituting (49) and (53) in (39), and by using the

properties of the Kronecker product, we can easily verify that:

$$\begin{aligned} \Pi(\bar{\mathbf{t}}_{\nu_0}|\mathcal{T}_3) &= _d \varphi_0(\mathcal{Q}) \left[\mathbf{W}_{\nu_0}^T - \mathbf{G}^H \boldsymbol{\Delta} [\boldsymbol{\Delta}^H \boldsymbol{\Delta}]^{-1} \mathbf{W}_{\xi_0}^T \right] \mathbf{u}^* \\ &= \varphi_0(\mathcal{Q}) \left[\mathbf{G}^H - \mathbf{G}^H \boldsymbol{\Pi}_{\boldsymbol{\Delta}} \right] \text{vec}(\mathbf{I}) = \varphi_0(\mathcal{Q}) \mathbf{G}^H \boldsymbol{\Pi}_{\boldsymbol{\Delta}}^{\perp} \text{vec}(\mathbf{I}), \end{aligned} \quad (54)$$

where, for notational simplicity, we introduced the orthogonal projection matrices:

$$\boldsymbol{\Pi}_{\boldsymbol{\Delta}} \triangleq \boldsymbol{\Delta} [\boldsymbol{\Delta}^H \boldsymbol{\Delta}]^{-1} \boldsymbol{\Delta}^H, \quad (55)$$

$$\boldsymbol{\Pi}_{\boldsymbol{\Delta}}^{\perp} \triangleq \mathbf{I} - \boldsymbol{\Pi}_{\boldsymbol{\Delta}} = \boldsymbol{\Pi}_{\mathbf{V}}^{\perp} - \frac{\boldsymbol{\Pi}_{\mathbf{V}}^{\perp} \mathbf{u} \mathbf{u}^H \boldsymbol{\Pi}_{\mathbf{V}}^{\perp}}{\mathbf{u}^H \boldsymbol{\Pi}_{\mathbf{V}}^{\perp} \mathbf{u}}, \quad (56)$$

where the last equality is defined by [5, eq. (14)].

Thus the condition (30) of Lemma 2 is equivalent to:

$$\Pi(\bar{\mathbf{t}}_{\nu_0}|\mathcal{T}_3) = \mathbf{0} \Leftrightarrow \text{vec}(\mathbf{I})^H \boldsymbol{\Pi}_{\boldsymbol{\Delta}}^{\perp} \mathbf{G} = \mathbf{0}. \quad (57)$$

To conclude, we need to prove that $\text{vec}(\mathbf{I})^H \boldsymbol{\Pi}_{\boldsymbol{\Delta}}^{\perp} \mathbf{G} = \mathbf{0}$.

Following again [5], let us indicate as \mathbf{g}_i the i -th column of \mathbf{G} . Then, by using the equality in (56), the condition (57) can be written as:

$$\text{vec}(\mathbf{I})^H \boldsymbol{\Pi}_{\mathbf{V}}^{\perp} \mathbf{g}_i - \frac{(\text{vec}(\mathbf{I})^H \boldsymbol{\Pi}_{\mathbf{V}}^{\perp} \mathbf{u})(\mathbf{u}^H \boldsymbol{\Pi}_{\mathbf{V}}^{\perp} \mathbf{g}_i)}{\mathbf{u}^H \boldsymbol{\Pi}_{\mathbf{V}}^{\perp} \mathbf{u}} = 0, \quad (58)$$

$i = 1, \dots, q$. Now, the second term in (58) is equal to 0, since from [5, eq. (28)], $\mathbf{u}^H \boldsymbol{\Pi}_{\mathbf{V}}^{\perp} \mathbf{g}_i = 0$, $i = 1, \dots, q$. Then, the last step consists in proving that:

$$\text{vec}(\mathbf{I})^H \boldsymbol{\Pi}_{\mathbf{V}}^{\perp} \mathbf{g}_i = 0, \quad i = 1, \dots, q. \quad (59)$$

To this end, by exploiting the relation in [5, eq. (23)] and the properties of the vec operator, we have that the following chain of equality holds true:

$$\text{vec}(\mathbf{I})^H \boldsymbol{\Pi}_{\mathbf{V}}^{\perp} \mathbf{g}_i = 2 \text{Real} \left(\text{tr} \left(\boldsymbol{\Pi}_{\boldsymbol{\Sigma}_0^{-1/2} \mathbf{A}_0}^{\perp} \mathbf{Z}_i \right) \right) = 0, \quad (60)$$

$i = 1, \dots, q$, where the matrix \mathbf{Z}_i is given in [5, eq. (18)] as:

$$\mathbf{Z}_i \triangleq \boldsymbol{\Sigma}_0^{-1/2} \mathbf{A}_0 \mathbf{c}_{0,i} \mathbf{d}_{0,i}^H \boldsymbol{\Sigma}_0^{-1/2}, \quad (61)$$

where $\mathbf{c}_{0,i}$ is the i -th column of the matrix $\boldsymbol{\Gamma}_0$ in (1) while

$$\mathbf{d}_{0,i} \triangleq d\mathbf{a}(\nu_i)/d\nu_i|_{\nu_k=\nu_{0,i}}. \quad (62)$$

We note that the last equality in (60) follows directly from the fact that $\boldsymbol{\Pi}_{\boldsymbol{\Sigma}_0^{-1/2} \mathbf{A}_0}^{\perp} \mathbf{Z}_i = \mathbf{0}$, $i = 1, \dots, q$.

This proves the identity (6).

V. CONCLUSION

We showed that the *Elliptical SCR*B $\text{ESCRB}(\nu_0|\xi_0, \sigma_0^2)$, evaluated at a specific and *a-priori known* density generator $g_0 \in \mathcal{G}$, is actually equal to the *Semiparametric SCR*B $\text{SSCRB}(\nu_0|\xi_0, \sigma_0^2, g_0)$ where, on the contrary, g_0 is assumed to be an additional unknown nuisance function:

$$\begin{aligned} \text{ESCRB}(\nu_0|\zeta_0, \sigma_0^2) &= \text{SSCRB}(\nu_0|\zeta_0, \sigma_0^2, g_0) \\ &= \frac{m(m+1)\sigma_0^2}{2E\{\mathcal{Q}^2\psi_0(\mathcal{Q})^2\}} [\mathbf{C}(\nu_0, \zeta_0, \cdot)]^{-1}, \quad \forall g_0 \in \mathcal{G}, \end{aligned} \quad (63)$$

$$\mathbf{C}(\nu_0, \zeta_0, \cdot) \triangleq \text{Re}(\mathbf{D}_0^H \boldsymbol{\Pi}_{\mathbf{A}_0}^{\perp} \mathbf{D}_0) \odot (\boldsymbol{\Gamma}_0 \mathbf{A}_0^H \boldsymbol{\Sigma}_0^{-1} \mathbf{A}_0 \boldsymbol{\Gamma}_0)^T, \quad (64)$$

where \mathbf{D}_0 is a matrix whose i -th column is $\mathbf{d}_{0,i}$ in (62).

¹In [2, eq. (42)] there were some typos that have been fixed in [11].

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