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Guillaume Garnier

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Guillaume Garnier. Decompounding with unknown noise through several independent channels and multiplicative decompounding. 2024. hal-04460693v1

**HAL Id: hal-04460693**

**<https://hal.science/hal-04460693v1>**

Preprint submitted on 16 Feb 2024 (v1), last revised 17 May 2024 (v2)

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# DECOMPOUNDING WITH UNKNOWN NOISE THROUGH SEVERAL INDEPENDENT CHANNELS AND MULTIPLICATIVE DECOMPOUNDING

GUILLAUME GARNIER

**ABSTRACT.** In this article, we consider two different statistical models. First, we focus on the estimation of the jump intensity of a compound Poisson process in the presence of unknown noise. This problem combines both the deconvolution problem and the decompounding problem. More specifically, we observe several independent compound Poisson processes but we assume that all these observations are noisy due to measurement noise. We construct an Fourier estimator of the jump density and we study its mean integrated squared error. Then, we propose an adaptive method to correctly select the cut-off of the estimator and we illustrate the efficiency of the method with numerical results.

Secondly, we introduce in this paper the multiplicative decompounding problem. We study this problem with Mellin density estimators. We develop an adaptive procedure to select the optimal cutoff parameter.

**Keywords:** adaptive density estimation, Fourier estimator, inverse problem, nonparametric statistical inference, deconvolution, decompounding, multiplicative decompounding, Mellin transform, empirical processes.

**MSC2020 subject classifications :** 62C12, 62C20, 62G05, 62G07.

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*Date:* February 8, 2024.

Sorbonne Université, Inria, Laboratoire Jacques-Louis Lions (LJLL), 75005 Paris, France.

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## 1. INTRODUCTION

1.1. **Motivation.** In the domain of non-parametric statistics, the deconvolution problem and the decompounding problem are classical problems that have been widely studied in the scientific literature.

Deconvolution has applications in many fields, such as image processing [25], microscopy [40], astronomy [38, 31], seismology [41, 35] and medicine [26, 32].

Let  $X$  and  $\varepsilon$  be two independent random variables with density  $f_X$  and  $f_\varepsilon$ . In its simplest form, the deconvolution problem consists in estimating  $f_X$  from a *i.i.d.* sample of  $Y = X + \varepsilon$ , *i.e.* from noisy observations

$$Y_i = X_i + \varepsilon_i, \quad i = 1, \dots, n,$$

where  $(X_i)_{i=1}^n$  is a family of *i.i.d.* random variables with density  $f_X$ ,  $(\varepsilon_i)_{i=1}^n$  is a family of random noise, *i.i.d.*, with density  $f_\varepsilon$  and independent of  $(X_i)_{i=1}^n$ .

This problem have been extensively studied in the literature. The most popular approach estimates  $f_X$  with Fourier estimators [8, 13, 39, 29]. Other approaches have also been developed, for instance by using spline-based methods [2], wavelet decomposition [24, 37] and penalization methods [12, 11]. In most studies, the author assume that the noise density  $f_\varepsilon$  is known, however this assumption is not necessary if an additional error sample is available. [23, 10]. Many adaptive methods have been developed [18].

In the literature, the term decompounding first appeared in the article of Buchmann and Grübel [6]. Decompounding has many applications in financial mathematics [19] and queuing theory [3, 22].

Let  $(X_k)_{k \in \mathbb{N}}$  be a family of *i.i.d.* random variable with density  $f_X$  and  $N$  be a homogeneous Poisson process with intensity  $\lambda \in (0, \infty)$ . Define  $Y = (Y_t)_{t \geq 0}$  as the compound Poisson process

$$Y_t = \sum_{k=1}^{N_t} X_k, \quad t \geq 0.$$

The decompounding problem consists in estimate  $f_X$  from observations  $(Y_{i\Delta})_{i=1}^n$  of the trajectory of  $Y$  over  $[0, T]$  at a sampling rate  $\Delta > 0$ , *i.e.* at time points  $(i\Delta)_{i=1}^n$ . The decompounding problem have been the subject of many articles, see e.g. [5, 7, 43]. Most of the time,  $f_X$  is estimated using its characteristic function and Fourier estimators []. The available data can be observed at high frequencies or at low frequencies [18, 9]. The decompounding problem is part of the more general framework of Levy processes that have been widely developed in recent years [33, 30, 1]

In this article, we study a model that combines both the deconvolution problem and the decompounding problem. In particular, we assume that we observe several compound Poisson processes but that all these observations are noisy due to measurement noise. More precisely, we consider a family of processes

$$Z_t^j = \left( \sum_{k=1}^{N_t^j} X_k^j \right) + \varepsilon_t^j, \quad t \geq 0, \quad j \in \{1, \dots, J\}.$$

We want to construct an estimator of the jump density of the process.

Our motivation to study this type of model comes from evolutionary biology. Many studies try to estimate the distribution of fitness effects (DFE) of cell. This probability density represents the effect of a new mutation on the fitness of a cell. An accurate determination of the DFE within a population would provide a better understanding of the evolutionary trajectory of the population [20].

In 2018, Robert et al. [34] developed new experimental methods and a probabilistic model to study the DFE in a population of Escherichia coli (E. coli).

The effect of each mutation on the fitness of a cell is assumed to be drawn according to a random variable  $X$  of density  $f_X$  where  $f_X$  represents the DFE of the cell. The authors conclude that the number of mutations in a cell follow a point Poisson process and they estimate the DFE using a method of moment.

More specifically, they observe 1476 cell lines using micro-fluidic methods [46]. For each lineage  $j \in \{1, \dots, 1476\}$ , it is assume that mutations are deleterious and appear according to a Poisson point process  $N^j = (N_t^j)_{t \geq 0}$  with intensity  $\lambda > 0$ . Each of these mutations modifies the selective value over time  $(W_t^j)_{t \in \mathbb{R}^+}$  of the  $j$ -lineage. Let denote  $t_i^j$  ( $i \in \mathbb{N}$ ) the

time of occurrence of the  $\{i\}$ -th mutation in lineage  $j$ . The quantity

$$s_i^j = \frac{W_{t_{i-1}}^j - W_{t_i}^j}{W_{t_{i-1}}^j}, \quad i > 0,$$

represent the relative effect of the  $\{i\}$ -th mutation on the fitness of the individual. Assuming further that these mutations have independent and identically distributed  $(s_i)$  effects, then

$$\frac{W_t^j}{W_0^j} = \prod_{i=1}^{N_t^j} (1 - s_i^j).$$

By composing by the logarithm, the evolution of each lineage is controlled by a compound Poisson process, *i.e.*

$$Y_t^j = \ln W_t^j = \sum_{i=1}^{N_t^j} \ln(1 - s_i^j).$$

However, the experimental realities in biology do not allow to really observe the process  $Y^j$  in continuous time. In reality, lineages can only be observed at a sampling rate  $\Delta > 0$  through discrete observations  $(Y_{i\Delta}^j)_{i=1}^n$ . Moreover, these observations are noisy due to measurement noise  $(\varepsilon_i^j)_{i=1}^n$  on the  $i$ -th observation. In fact, it is only possible to observe the different lineages

$$(1) \quad Z_t^j = \left( \sum_{k=1}^{N_t^j} X_k^j \right) + \varepsilon_t^j, \quad t \geq 0,$$

at sampling times  $(i\Delta)_{i=1}^n$ , with  $X_k^j = \ln(1 - s_k^j)$ . By abuse of notation, we consider that the noise  $\varepsilon_t^j$  exists for all  $t$  in each lineage  $j \in \{1, \dots, 1476\}$ .

The main goal of this article is to provide a statistical estimator that is adapted to estimate the probability density  $f_X$  from the sample of the noisy trajectories.

Such an approach has several advantages. First, it offers a way to study the jump density of a noisy compound Poisson process when we observe several independent processes, which has, to our knowledge, been little studied in the literature. Second this method has the advantage of studying a model very close to the biological situation.

However, using a logarithm on the model to transform it into an 'additive' model only gives us information on  $\log X$ . While it is true that the two models are theoretically equivalent, the additive form forces us to make assumptions about the logarithm of the density we want to reconstruct. These assumptions are not very natural.

To overcome this, we use a method similar to the previous one to study the multiplicative decomposing problem.

In this setting, we discretely observe one trajectory of a multiplicative compound Poisson process

$$W_t = \prod_{i=1}^{N_t} (1 - s_i).$$

where  $N$  is a Poisson process with intensity  $\lambda$  independent of the *i.i.d.* random variables  $(s_i)_i$  with density  $f$ .

The idea of studying ‘multiplicative’ models was developed in [45] and [44] where the authors studied a multiplicative deconvolution problem where the error  $U$  is uniformly distributed on  $[0, 1]$ . Such models can be studied using the Mellin transform, which can be seen as a multiplicative version of the Fourier transform. These estimators have been successfully used in [27, 28] to study the multiplicative deconvolution problem with unknown noise.

**1.2. Main results and organisation of the article.** In Section 2, we recall some basic definitions and we define the estimator of  $f_X$ . In Section 2.3, we establish a bound for the  $L^2$ -risk. Then we investigate optimal rates of convergence. In Section 2.5, we establish an adaptive procedure to automatically select the value of the threshold  $m$  according to the data.

In Section 3, we apply the same method to the multiplicative decompounding when there is no noise on the observations. In Section 3.1, we recall some properties of the Mellin transform. In Section 3.2, we define the empirical Mellin estimator and we give some of its properties. In Section 3.3, we introduce the multiplicative decompounding problem. In Section 3.4, we establish a risk bound. In Section 3.5, we introduce an adaptive procedure to select the optimal cutoff parameter for the Mellin estimator. We numerically illustrate our two methods on several examples in Section 4. The proofs are postponed to the Section 5.

We define the distinguished logarithm in Appendix A and describe some of its classical properties. In Appendix B, we recall some useful results about compound Poisson processes and we present some useful lemmas in Appendix C.

All of our computer code is available and documented on GitHub<sup>1</sup> with several examples, making it easy to use on biological experimental data.

## 2. DECOMPOUNDING WITH UNKNOWN NOISE

**2.1. Notations.** In this section, we define most of the notations that we use throughout this article.

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<sup>1</sup><https://github.com/guimgarnier/decompounding-with-noise>

2.1.1. *The empirical characteristic function.* Let  $\mu$  be a probability measure on the line. The Fourier transform of  $\mu$  is the complex-valued function defined for all  $u \in \mathbb{R}$  by

$$\mathcal{F}(\mu)(u) = \int_{\mathbb{R}} e^{iux} d\mu(x).$$

Let  $X$  be a real-valued random variable. The characteristic function of  $X$  is the Fourier transform of the measure  $\mathbb{P}_X$ . It is defined for all  $u \in \mathbb{R}$  by

$$\varphi_X(u) = \int_{\mathbb{R}} e^{iux} \mathbb{P}_X(dx) = \mathbb{E}[e^{iuX}].$$

If  $X$  has a density  $f_X \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$  w.r.t. the Lebesgue measure  $dx$ , then

$$\varphi_X(u) = \int_{\mathbb{R}} e^{iux} f_X(x) dx, \quad u \in \mathbb{R}.$$

We recall the inversion formula

$$f_X(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \varphi_X(u) e^{-iux} du, \quad x \in \mathbb{R},$$

and the Parseval's identity

$$\int_{\mathbb{R}} |f_X(x)|^2 dx = \frac{1}{2\pi} \int_{\mathbb{R}} |\varphi_X(u)|^2 du.$$

In all the following,  $\|\cdot\|_2$  represents the usual norm on  $L^2(\mathbb{R})$ ,  $\|f\|_2 = \left( \int_{\mathbb{R}} |f(x)|^2 dx \right)^{1/2}$ .

Let  $(X_1, \dots, X_N)$  be *i.i.d.* real-valued random variables with common characteristic function  $\varphi_X(t)$ . The complex-valued function

$$\widehat{\varphi}_X^N(u) = \frac{1}{N} \sum_{k=1}^N e^{iuX_k}, \quad u \in \mathbb{R},$$

is called the empirical characteristic function associated with the sample  $(X_1, \dots, X_N)$ .

It is well known in the statistical literature that the empirical characteristic function is an unbiased estimator of  $\varphi_X$  *i.e.*  $\mathbb{E}[\widehat{\varphi}_X^N(u)] = \varphi_X(u)$  and that

$$(2) \quad \mathbb{E}[|\widehat{\varphi}_X^N(u) - \varphi_X(u)|^2] = \frac{1}{N}(1 - |\varphi_X(u)|^2)$$

For more detailed information on the empirical characteristic function, we refer the reader to the book of Ushakov. [42]

2.1.2. *The distinguished logarithm.* In all the following, the notation  $\log$  represents the distinguished logarithm that we define in more detail in Appendix A.

**2.2. Statistical settings.** Let  $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ . Let  $(X_i^j)_{i,j \geq 0}$  be *i.i.d.* real-valued random variables with density  $f$  and for some  $J \in \mathbb{N}$ , let  $(N_t^j)_{j \in \llbracket 1; J \rrbracket}$  be a family of *i.i.d.* Poisson point process with intensity  $\lambda \in (0, \infty)$  independent of  $(X_i^j)_{i,j \geq 0}$ .

We consider a family  $(Y_t^j, t \in \mathbb{R}_+)_{j \in \llbracket 1; J \rrbracket}$  of compound Poisson processes with intensity  $\lambda$  and jump size density  $f$ ,

$$Y_t^j = \sum_{k=1}^{N_t^j} X_k^j, \quad t \in \mathbb{R}_+, \quad j \in \llbracket 1; J \rrbracket.$$

It is assumed that the observations are not precise and that they are disturbed by a random noise independent of the observed process, *i.e.* that we observe a process of the form

$$Z_t^j = Y_t^j + \varepsilon_t^j = \left( \sum_{k=1}^{N_t^j} X_k^j \right) + \varepsilon_t^j, \quad t \geq 0,$$

where  $\varepsilon_t^j$  is a measurement error. It is assume that for every  $i, j = 1, \dots, J$ , and for every reals  $t, s \geq 0$ , the random variables  $\varepsilon_t^j$  and  $\varepsilon_s^i$  are independents.

Also, we set the following assumptions:

- (H<sub>1</sub>)  $\varphi_{X_1}$  is integrable.
- (H<sub>2</sub>)  $\mathbb{E}(X_1^2) < \infty$ .
- (H<sub>3</sub>)  $\forall t \in \mathbb{R}_+, \mathbb{E}[\varepsilon_t] = 0$  and  $\mathbb{E}[\varepsilon_t^2] < \infty$ .
- (H<sub>4</sub>)  $\forall m \in \mathbb{R}_+, \exists c_m \in (0, \infty) : \forall u \in [0, m], |\varphi_\varepsilon(u)| \geq c_m$ .
- (H<sub>5</sub>)  $\forall u \in \mathbb{R}, |\varphi_\varepsilon(u)| > 0$ .

For the sake of simplicity, we write  $\varepsilon$  instead of  $\varepsilon_t^j$  in the rest of this document.

**2.2.1. Construction of the estimator.** Rather than trying to construct an estimator of  $f$  directly, we try to estimate the characteristic function  $\varphi_X$  of  $X$ . The main idea is to claim that if we have a good reconstruction of  $\varphi_X$ , by applying the inverse Fourier transform, we should have a good reconstruction of  $f$ .

For all  $t \in \mathbb{R}_+$ , the characteristic function of the process on a single channel  $Z_t^j$  is given by

$$(3) \quad \varphi_{Z_t}(u) = e^{-\lambda t + \lambda t \varphi_X(u)} \cdot \varphi_\varepsilon(u), \quad \forall u \in \mathbb{R}.$$

To simplify the notations, we consider in the rest of this article the particular case where  $\lambda = 1$ .

Consider two different times  $0 < t_1 < t_2$ , then

$$\frac{\varphi_{Z_{t_2}}}{\varphi_{Z_{t_1}}} = e^{-(t_2-t_1) + (t_2-t_1)\varphi_X(u)}, \quad \forall u \in \mathbb{R}.$$

Applying the distinguished logarithm, we obtain the explicit formula

$$\varphi_X(u) = 1 + \frac{1}{t_2 - t_1} \left[ \log \varphi_{Z_{t_2}}(u) - \log \varphi_{Z_{t_1}}(u) \right], \quad \forall u \in \mathbb{R}.$$



This formula leads us to consider the estimator

$$\forall u \in \mathbb{R}, \widehat{\varphi}_X^J(u) = 1 + \frac{1}{t_2 - t_1} \left[ \log \widehat{\varphi}_{Z_{t_2}}^J(u) - \log \widehat{\varphi}_{Z_{t_1}}^J(u) \right],$$

with for all  $u \in \mathbb{R}$ ,  $\tau \in \{t_1, t_2\}$

$$\widehat{\varphi}_{Z_\tau}^J(u) = \frac{1}{J} \sum_{j=1}^J iZ_\tau^j e^{iuZ_\tau^j}, \quad \widehat{\varphi}_{Z_\tau}^J(u) = \frac{1}{J} \sum_{j=1}^J e^{iuZ_\tau^j}, \quad \log \widehat{\varphi}_{Z_\tau}^J(u) = \int_0^u \frac{\widehat{\varphi}_{Z_\tau}^{\prime J}(z)}{\widehat{\varphi}_{Z_\tau}^J(z)} dz.$$

Since  $\varphi_X$  is a characteristic function, we know that its absolute value is bounded by 1. Nevertheless, this is not necessarily the case for its estimator  $\widehat{\varphi}_X^J$ . We avoid this explosion problem by cutting off the annoying frequencies, *i.e.* we consider the estimator

$$\widetilde{\varphi}_X^J(u) = 1 + \frac{1}{t_2 - t_1} \left[ \log \widehat{\varphi}_{Z_{t_2}}^J(u) \cdot \mathbf{1}_{|\log \widehat{\varphi}_{Z_{t_2}}^J(u)| \leq \ln(J)} - \log \widehat{\varphi}_{Z_{t_1}}^J(u) \cdot \mathbf{1}_{|\log \widehat{\varphi}_{Z_{t_1}}^J(u)| \leq \ln(J)} \right], \quad u \in \mathbb{R}.$$

To simplify the notations, we note in the following, for any positive real  $\tau > 0$  and for all  $u \in \mathbb{R}$

$$\log \widetilde{\varphi}_{Z_\tau}^J(u) = \log \widehat{\varphi}_{Z_\tau}^J(u) \cdot \mathbf{1}_{|\log \widehat{\varphi}_{Z_\tau}^J(u)| \leq \ln(J)}.$$

In particular, with this notation, we can rewrite

$$(4) \quad \widetilde{\varphi}_X^J(u) = 1 + \frac{1}{t_2 - t_1} \left[ \log \widetilde{\varphi}_{Z_{t_2}}^J(u) - \log \widetilde{\varphi}_{Z_{t_1}}^J(u) \right], \quad u \in \mathbb{R}.$$

Now, we find an estimator of  $f$  by performing an inverse Fourier transformation. However, there is no reason to believe that the quantity  $\widetilde{\varphi}_X^J$  is integrable. To get around this problem, we eliminate frequencies above a threshold  $m$  before applying an inverse Fourier transform.

$$(5) \quad \widehat{f}_{m,J}(x) = \frac{1}{2\pi} \int_{-m}^m e^{-iux} \widetilde{\varphi}_X^J(u) du, \quad x \in \mathbb{R}, \quad m \in (0, \infty).$$

In the same way, we perform an truncated inverse Fourier transformation from the true characteristic function of  $X$ .

$$f_m(x) = \frac{1}{2\pi} \int_{-m}^m e^{-iux} \varphi_X(u) du, \quad x \in \mathbb{R}, \quad m \in (0, \infty).$$

**2.3. Risk bounds.** In this section, we study the mean integrated squared error (MISE) of the estimator  $\widehat{f}_{m,J}$ .

For any  $0 < t_1 < t_2$ , we define

$$(6) \quad C_{t_1, t_2}^J = \min \left\{ m \geq 0 \mid 3t_2 - t_1 + \sup_{[-m, m]} |\log \varphi_\varepsilon(\cdot)| > \ln(J) \right\}.$$

**Theorem 1.** *Suppose that the assumptions  $(H_1)$  –  $(H_5)$  holds. Let  $0 < t_1 < t_2$  be two reals. We suppose that  $J$  is enough large, *i.e.* that for all  $i = 1, 2$  :*

- (1)  $t_i < \frac{1}{4} \log(Jt_i)$ ,
- (2)  $\sqrt{\log(Jt_i)}(Jt_i)^{2\delta_i - 1/2} < 1$  with  $\delta_i = t_i / \log(Jt_i)$ .

Then for any  $m < C_{t_1, t_2}^J$  with  $C_{t_1, t_2}^J$  defined by (6), the following inequality holds

$$\begin{aligned} \mathbb{E}\left(\|\widehat{f}_{m, J} - f\|_2^2\right) &\leq \|f_m - f\|_2^2 + \sum_{i=1}^2 \frac{4e^{4t_i}}{J(t_2 - t_1)^2} \int_{-m}^m \frac{du}{|\varphi_\varepsilon(u)|^2} \\ &\quad + \frac{4K_{J, t_1, t_2}}{(t_2 - t_1)^2} \cdot \left( \frac{\mathbb{E}[X_i^2]}{Jt_i} + \frac{\mathbb{E}[\varepsilon^2]}{Jt_i^2} + 4\frac{m}{(Jt_i)^2} \right). \end{aligned}$$

where  $K_{J, t_1, t_2} = m \ln^2(J) + 8mt_2^2 + \int_{-m}^m |\log \varphi_\varepsilon(u)|^2 du$ .

*Remark 2.* Theorem 1 is asymptotic and ensures that the variance term vanishes when  $J \rightarrow \infty$ . The upper bound obtained is both the sum of a bias term and of a variance term. In the variance, there is a term  $V \sim \frac{4e^{4t_i}}{J(t_2 - t_1)^2}$ . On one side, the presence of  $e^{4t_i}$  means that the estimator is more and more imprecise as we look the sample at a very large time  $t_2$ . On the other side, the presence of  $(t_2 - t_1)$  at the numerator means that we cannot take  $t_1$  and  $t_2$  too close to each other. We illustrate this trade-off in Section 4 with numerical simulations.

**2.4. Speed of convergence.** In this Section, we study the optimal choice of  $m$  based on the regularity of the jump density and regularity of the noise density, *i.e.* that when  $J \rightarrow \infty$ , we look at the asymptotic behavior of  $m$  when it minimizes the upper term in Theorem in 1. We organize the discussion according to different regularities of  $f$  and  $f_\varepsilon$ . More specifically, we consider two different type of regularity :

- Ordinary smooth densities : The characteristic function decays as  $|u|^{-2a}$  (as Gamma Law)
- Super smooth densities : The characteristic function decays as  $e^{-|u|^s}$  (as Cauchy Law)

2.4.1. *When  $f$  is ordinary smooth,  $f_\varepsilon$  is ordinary smooth.* In this case, we assume that  $f \in \mathcal{S}(\beta, L)$  for some reals  $\beta > 0$ ,  $L > 0$ , *i.e.*

$$f \in \mathcal{S}(\beta, L) = \left\{ f \in L^2(\mathbb{R}) \mid \int_{\mathbb{R}} (1 + |u|^2)^\beta |\varphi_X(u)|^2 du \leq L \right\}.$$

We also assume that  $f_\varepsilon$  is *ordinary smooth*, *i.e.* there exists two reals  $a > \frac{1}{2}$  and  $d > 0$ , such that

$$\forall u \in \mathbb{R}, \quad d \leq (1 + u^2)^a |\varphi_\varepsilon(u)|^2 \leq \frac{1}{d}.$$

It follows that  $\|f - f_m\|_2^2 \asymp m^{-2\beta}$  and that the variance term  $\frac{1}{J} \int_{-m}^m \frac{du}{|\varphi_\varepsilon(u)|^2}$  is of order  $\frac{1}{J} m^{2a+1}$ .

Moreover, the variance term  $\frac{4K_{J, t_1, t_2}}{(t_2 - t_1)^2} \cdot \left( \frac{\mathbb{E}[X_i^2]}{Jt_i} + \frac{\mathbb{E}[\varepsilon^2]}{(Jt_i)^3} + 4\frac{m}{(Jt_i)^2} \right)$  is of order  $\frac{4m}{(t_2 - t_1)^2} \cdot \ln^2(J) \cdot \frac{\mathbb{E}[X_i^2]}{Jt_1}$ .

Therefore, if the Assumptions of Theorem 1 holds, then there exist two constants  $A, B > 0$  that depends on  $t_1$  and  $t_2$  such that

$$(7) \quad \mathbb{E} \left[ \|\widehat{f}_{m,J} - f\|_2^2 \right] \leq m^{-2\beta} + \frac{A}{J} m^{2a+1} + Bm \cdot \frac{\ln^2(J)}{J}.$$

To obtain a value  $m^*$  that reaches the bias-variance compromise, we minimize the upper bound of Equation (7) by differentiating with respect to  $m$ . It follows that  $m^*$  is the solution of the equation

$$-2\beta m^{-2\beta-1} + (2a+1) \frac{A}{J} m^{2a} + B \cdot \frac{\ln^2(J)}{J} = 0.$$

Then,  $m^*$  is the solution of the equation

$$m^{2a+2\beta+1} + C \cdot \ln^2(J) m^{2\beta+1} = D \cdot J.$$

for some constants  $C, D > 0$  well defined.

It follows that  $m^* \sim J^{\frac{1}{2a+2\beta+1}}$  when  $J \rightarrow \infty$  and that

$$\mathbb{E} \left[ \|\widehat{f}_{m,J} - f\|_2^2 \right] = O \left( J^{\frac{-2\beta}{2\beta+2a+1}} \right), \quad J \rightarrow \infty.$$

$$m^* \sim J^{\frac{1}{2a+2\beta+1}} \\ \mathbb{E} \|\widehat{f}_{m,J} - f\|_2^2 = O \left( J^{\frac{-2\beta}{2\beta+2a+1}} \right)$$

2.4.2. *When  $f$  is ordinary smooth,  $f_\varepsilon$  is super smooth smooth.* In this case, we assume that  $f \in \mathcal{S}(\beta, L)$  for some reals  $\beta > 0, L > 0$ , i.e.

$$f \in \mathcal{S}(\beta, L) = \left\{ f \in L^2(\mathbb{R}) \mid \int_{\mathbb{R}} (1 + |u|^2)^\beta |\varphi_X(u)|^2 du \leq L \right\}.$$

We also assume that  $f_\varepsilon$  is *super smooth*, i.e. there exists  $a > 0, s > 0$  and  $d_1, d_2 > 0$  such that

$$\forall u \in \mathbb{R}, \quad d_1 \leq \exp(b \cdot |u|^s) \cdot |\varphi_\varepsilon(u)|^2 \leq d_2.$$

It follows that  $\|f - f_m\|_2^2 \asymp m^{-2\beta}$  and that the variance term  $\frac{1}{J} \int_{-m}^m \frac{du}{|\varphi_\varepsilon(u)|^2}$  is of order  $\frac{1}{J} m \cdot e^{b \cdot m^s}$ .

Moreover, the variance term  $\frac{4K_{J,t_1,t_2}}{(t_2-t_1)^2} \cdot \left( \frac{\mathbb{E}[X_i^2]}{Jt_i} + \frac{\mathbb{E}[\varepsilon^2]}{(Jt_i)^3} + 4 \frac{m}{(Jt_i)^2} \right)$  is of order  $\frac{4m}{(t_2-t_1)^2} \cdot \ln^2(J) \cdot \frac{\mathbb{E}[X_i^2]}{Jt_1}$ .

Therefore, if the Assumptions of Theorem 1 holds, then there exist two constants  $A, B > 0$  that depends on  $t_1$  and  $t_2$  such that

$$(8) \quad \mathbb{E} \|\widehat{f}_{m,J} - f\|_2^2 \leq m^{-2\beta} + \frac{A}{J} m \cdot e^{b \cdot m^s} + Bm \cdot \frac{\ln^2(J)}{J}.$$

To obtain a value  $m^*$  that reaches the bias-variance compromise, we minimize the upper bound of Equation (8) by differentiating with respect to  $m$ . It follows that  $m^*$  is the

$f/f_\varepsilon$	ordinary smooth : $f_\varepsilon \in \mathcal{S}(a, L)$	super smooth : $f_\varepsilon \in \mathcal{A}_{b,s}(L)$
ordinary smooth : $f \in \mathcal{S}(\beta, L)$	$m^* \sim J^{\frac{1}{2a+2\beta+1}}$ $\mathbb{E}\ \widehat{f}_{m,J} - f\ _2^2 = O\left(J^{\frac{-2\beta}{2\beta+2a+1}}\right)$	$m^* \sim \frac{\ln(J)}{b}$ $\mathbb{E}\ \widehat{f}_{m,J} - f\ _2^2 = O\left(\left(\frac{\ln(J)}{b}\right)^{\frac{-2\beta}{s}}\right)$
super smooth $f \in \mathcal{A}_{c,s}(L)$	$m^* \sim \left(\frac{\ln(J)}{c}\right)^{\frac{1}{s}}$ $\mathbb{E}\ \widehat{f}_{m,J} - f\ _2^2 = O(J^{-1})$	$m^* \sim \left(\frac{\ln(J)}{b+c}\right)^{\frac{1}{s}}$ $\mathbb{E}\ \widehat{f}_{m,J} - f\ _2^2 = O\left(J^{\frac{-c}{b+c}}\right)$

TABLE 1. Speed of the optimal upper bound with respect to the regularity of  $f_X$  and  $f_\varepsilon$ .

solution of the equation

$$-2\beta m^{-2\beta-1} + \frac{A}{J}(1 + sb \cdot m^s) \cdot e^{b \cdot m^s} + B \cdot \frac{\ln^2(J)}{J} = 0.$$

Then,  $m^*$  is the solution of the equation

$$(1 + sb \cdot m^s)m^{2\beta+1} \cdot e^{b \cdot m^s} + B \cdot \ln^2(J)m^{2\beta+1} = D \cdot J.$$

for some constants  $C, D > 0$  well defined.

It follows that  $m^* \sim \left(\frac{\ln(J)}{b}\right)^{1/s}$  when  $J \rightarrow \infty$  and that

$$\mathbb{E}\|\widehat{f}_{m,J} - f\|_2^2 = O\left(\left(\frac{\ln(J)}{b}\right)^{\frac{-2\beta}{s}}\right).$$

2.4.3. *When  $f$  is super smooth.* In this section, we assume that  $f$  belongs to the class of super smooth densities,

$$\mathcal{A}_{c,s}(L) = \left\{ f \in L^2(\mathbb{R}) \mid \int_{\mathbb{R}} \exp(c \cdot |u|^s) |\varphi_X(u)|^2 du \leq L \right\}.$$

It follows that  $\|f - f_m\|_2^2 \asymp \exp(-c \cdot |m|^s)$ .

We apply the same strategy that in Section 2.4.1 and Section 2.4.2 to compute  $m^*$  when  $f_\varepsilon$  is ordinary smooth or super smooth. Table 1 resumes the different results that we obtained with respect to the regularity of  $f_X$  and  $f_\varepsilon$ .

2.5. **The adaptive procedure.** In Section 2.3, we have constructed a statistical estimator  $\widehat{f}_{m,J}$  to estimate the jump size density  $f$ . However, as this estimator strongly depends on the choice of the parameter  $m$ , we would like to be able to select a value of  $m$  that depends only on the available data, without a priory knowledge on the regularity of the density  $f$ .

To do this, the main idea is to select the parameter  $m$  that minimizes the bound obtained in the theorem 1.

For this, we need to make some further regularity assumptions on the measurement noise, which is to say that there is a function  $g \in L^1(\mathbb{R})$  and a real  $d > 0$  such that

$$\forall u \in \mathbb{R}, \quad d \cdot g(u) < |\varphi_\varepsilon(u)|^2 < \frac{g(u)}{d}.$$

This assumption of regularity of the noise cannot be relaxed. In fact, it can even be shown that if one does not have the noise level, then it is impossible to obtain a convergent method [4].

In the rest of this section, we assume that the noise is ordinary smooth, *i.e.* there exists two reals  $a > \frac{1}{2}$  and  $d \in (0, 1)$ , such that

$$\forall u \in \mathbb{R}, \quad d \leq (1 + u^2)^a \cdot |\varphi_\varepsilon(u)|^2 \leq \frac{1}{d}.$$

Nevertheless, the methods used can be adapted for any function  $g$ .

In the following, we simplify the computations by considering the particular case  $a = 1$ , *i.e.* we assume that

$$\forall u \in \mathbb{R}, \quad d \leq (1 + u^2) \cdot |\varphi_\varepsilon(u)|^2 \leq \frac{1}{d}.$$

2.5.1. *How to select  $\widehat{m}_{J,t_2,t_1}$ .* As we said above, we want to select  $m$  so as to minimize the bound obtained in the theorem 1.

The dominant terms in this bound are the bias term  $\int_{u \in [-m, m]} |\varphi_X(u)|^2 du$  and the variance term  $\frac{4e^{4t_2}}{J(t_2 - t_1)^2} \int_{-m}^m \frac{du}{|\varphi_\varepsilon(u)|^2}$ . Through differentiation, the optimal  $\overline{m}_J$  satisfies

$$|\varphi_X(\overline{m}_J)|^2 = \frac{4ae^{4t_2}}{J(t_2 - t_1)^2} (1 + \overline{m}_J^2).$$

then

$$\left| \frac{\varphi_X(\overline{m}_J)}{\sqrt{1 + \overline{m}_J^2}} \right|^2 = \frac{4ae^{4t_2}}{J(t_2 - t_1)^2}.$$

However we do not know  $\varphi_X$ , so it is impossible to calculate directly  $\overline{m}_J$ . Following the strategy developed by Duval and Kappus [18], we consider

$$\overline{\varphi}_X^J(u) = \widetilde{\varphi}_X^J(u) \cdot \mathbf{1}_{\left| \frac{\widetilde{\varphi}_X^J(u)}{\sqrt{1+u^2}} \right| \geq \frac{\kappa_{J,t_1,t_2}}{\sqrt{J(t_2-t_1)}}}$$

where  $\kappa_{J,t_1,t_2} = 2e^{2t_2} + \kappa \sqrt{\ln(J(t_2 - t_1)^2)}$ ,  $\kappa > 0$ .

It leads us to define the empirical cutoff parameter

$$\widehat{m}_J = \max \left\{ u \geq 0 : \left| \frac{\overline{\varphi}_X(u)}{\sqrt{1+u^2}} \right| \geq \frac{\kappa_{J,t_1,t_2}}{\sqrt{J(t_2-t_1)}} \right\} \wedge (J(t_2 - t_1)^2)^\alpha, \quad \alpha \in (0, 1)$$

For simplicity, we note  $m_{\max} = (J(t_2 - t_1)^2)$ , keeping in mind that it depends on  $J, t_1$  and  $t_2$ .

We define a new estimator

$$\bar{f}_{m,J}(x) = \frac{1}{2\pi} \int_{-m}^m e^{-iux} \bar{\varphi}_X^J(u) \, du, \quad x \in \mathbb{R}.$$

**Theorem 3.** *Assume  $(H_1)$ – $(H_4)$ . For all reals  $0 < t_1 < t_2$  such that  $t_2 \leq \frac{1}{4} \log(Jt_2)$  and  $(m_{\max})^\alpha < C_{t_1, t_2}^J$ ,  $Jt_1 \rightarrow \infty$ ,  $Jt_2 \rightarrow \infty$  as  $J \rightarrow \infty$ . Then,*

$$\begin{aligned} \mathbb{E}[\|\bar{f}_{\hat{m}_J} - f\|_2^2] &\leq \inf_{m \in [0, (m_{\max})^\alpha]} \left\{ \|f_m - f\|_2^2 + C \frac{\ln(J(t_2 - t_1)^2) \cdot m \cdot (1 + m^2)}{J(t_2 - t_1)^2} + \tilde{C}A \right\} \\ &\quad + \left( 2 + \frac{2 \log(J)}{(t_2 - t_1)} \right)^2 \cdot T_J \end{aligned}$$

where

$$A = \sum_{i=1}^2 \frac{4e^{4t_i}}{J(t_2 - t_1)^2} \int_{-m}^m \frac{du}{|\varphi_\varepsilon(u)|^2} + \frac{4K_{J, t_1, t_2}}{(t_2 - t_1)^2} \cdot \left( \frac{\mathbb{E}[X_i^2]}{Jt_i} + \frac{\mathbb{E}[\varepsilon^2]}{rJt_i^2} + 4 \frac{m}{(Jt_i)^2} \right)$$

and

$$(9) \quad T_J \leq C_0(J(t_2 - t_1)^2)^{\alpha - c(\theta)^2} + \frac{C_1}{J(t_2 - t_1)^2} + \frac{C_2}{J(t_2 - t_1)^4}$$

and  $c(\theta) = \kappa(t_2 - t_1)e^{2t_2} \cdot \frac{d}{\sqrt{1 + (m_{\max})^2}}$  and where  $C_0, C_1$  and  $C_2$  depends on  $\mathbb{E}[X_1^2], \mathbb{E}[\varepsilon^2]$  and where  $C$  and  $\tilde{C}$  are two constants.

### 3. MULTIPLICATIVE DECOMPOUNDING

**3.1. Preliminaries on Mellin transform.** In this section, we first recall some classical results on Mellin transform. We define the multiplicative decomposing problem and we define a good non-parametric estimator.

Let  $\mu$  be a probability measure on  $(0, \infty)$ . Its Mellin transform is the complex-valued function defined by

$$(10) \quad \mathcal{M}[\mu](s) = \int_0^\infty x^{s-1} \mu(dx), \quad s \in \mathbb{C},$$

for those values of  $s$  for which this integral is well-defined.

*Remark 4.* If  $\mathcal{M}[\mu](u)$  is well-defined for some  $u \in \mathbb{R}$ , then  $\mathcal{M}[\mu]$  converges on the vertical line  $u + i\mathbb{R}$ . Furthermore, if the integral exists for  $u$  and  $v$  in  $\mathbb{R}$  ( $u < v$ ), then  $\mathcal{M}[\mu](w)$  exists for  $w \in (u, v)$ . It follows that the Mellin transform of a probability measure is well-defined on a vertical band  $\Xi_\mu$  of the complex plane. The region  $\Xi_\mu$  is called the fundamental strip of  $\mathcal{M}[\mu]$ .

Assume that  $\mu$  has a density  $f \in L_{\text{loc}}^1(\mathbb{R}_+)$  w.r.t. to the Lebesgue measure  $dx$ , then we denote

$$\mathcal{M}[f](s) := \mathcal{M}[f \, dx](s) = \int_0^\infty x^{s-1} f(x) \, dx, \quad s \in \mathbb{C},$$

for those values of  $s$  for which this integral is well-defined. We denote  $\Xi_f$  the fundamental strip of  $\mathcal{M}[f \, dx]$

Let  $f \in L^1_{\text{loc}}(\mathbb{R}_+)$  and  $c \in \mathbb{R}$  that belongs to  $\Xi_f$ . Then we define

$$\mathcal{M}_c[f](t) := \mathcal{M}[f](c + it) = \int_0^\infty x^{c-1+it} f(x) \, dx, \quad t \in \mathbb{R}.$$

We observe that the function  $x \mapsto x^{c-1}f(x)$  belongs to  $L^1(\mathbb{R}_+)$ .

We define the norm

$$\|f\|_{\omega_c} = \left( \int_0^\infty |f(x)|^2 x^{2c-1} \, dx \right)^{1/2}.$$

Let  $f, g \in L^1_{\text{loc}}(\mathbb{R}_+)$ . The multiplicative convolution of  $f$  and  $g$  is the real-valued function defined by

$$f \star g(x) = \int_0^\infty f(y)g\left(\frac{x}{y}\right) \, dy, \quad x \in [0, \infty).$$

**Proposition 5** (Multiplicative convolution and Mellin transform). *Let  $\mu$  and  $\nu$  be two probability measures on  $\mathbb{R}_+$ . We have*

$$\mathcal{M}[\mu \star \nu](s) = \mathcal{M}[\mu](s)\mathcal{M}[\nu](s), \quad s \in \mathbb{C},$$

whenever this integral is well-defined.

**Proposition 6** (Parseval's theorem for Mellin transform).

$$(11) \quad \int_0^\infty \frac{|f(x)|^2}{x} \, dx = \frac{1}{2\pi} \int_0^\infty \mathcal{M}[f](it) \, dt.$$

**Proposition 7.** (Inversion formula) *Let  $f \in L^1(\mathbb{R}_+)$  and  $c \in \Xi_f$ . Then*

$$f(x) = \mathcal{M}^{-1}\mathcal{M}[f](x) = \frac{1}{2\pi i} \int_{\nu-i\infty}^{\nu+i\infty} x^{-s} \mathcal{M}[f](s) \, ds, \quad x \in \mathbb{R}_+.$$

**3.2. The Mellin estimator.** Let  $f \in L^2_{\mathbb{R}_+}(\omega_c)$  be a probability distribution. Let us assume that we have a sample  $X_1, \dots, X_N$  independent and identically distributed according to  $f \, dx$ .

We define the Mellin estimator of  $f$

$$\widehat{\mathcal{M}}_c(t) = \frac{1}{n} \sum_{k=1}^N X_k^{c-1+it}, \quad t \in \mathbb{R}.$$

We use Proposition 7 to define the estimator.

$$\widehat{f}_m(x) = \frac{1}{2\pi} \int_{-m}^m x^{-c-it} \widehat{\mathcal{M}}_c(t) \, dt,$$

which is an estimator of

$$f_m(x) = \frac{1}{2\pi} \int_{-m}^m x^{-c-it} \mathcal{M}_1(t) dt .$$

The next proposition give a bound of the  $L^2_{\mathbb{R}_+}(\omega_c)$ -risk of  $\widehat{f}_k$ .

**Proposition 8** (Miguel et al. [27] Proposition 2.1). *If  $f \in L^2_{\mathbb{R}_+}(\omega_c)$  and  $\sigma_c^2 := \mathbb{E}_f(X^{2(c-1)}) < \infty$ , then for all  $k \in \mathbb{R}_+$ ,*

$$\mathbb{E}_f(\|f - \widehat{f}_k\|_{\omega_c}^2) \leq \|f - f_k\|_{\omega_c}^2 + \frac{\sigma_c^2 k}{\pi n} .$$

By choosing  $k = k_n$  such that  $n^{-1}k_n \rightarrow 0$  and  $k_n \rightarrow \infty$ ,  $\widehat{f}_{k_n}$  is a consistent estimator of  $f$ .

If  $X$  and  $Y$  are two independant random variables, then

$$\mathcal{M}[X \cdot Y](s) = \mathcal{M}[X](s) \cdot \mathcal{M}[Y](s) , s \in \mathbb{R} .$$

**3.3. Statistical setting: estimation procedure.** Consider the multiplicative compound process

$$(12) \quad Y_t = \prod_{i=1}^{N_t} X_i ,$$

where  $(N_t, t \in \mathbb{R}_+)$  is a Poisson process with constant intensity  $\lambda > 0$ , independent of the *i.i.d.* random variables  $(X_j)_{j \in \mathbb{N}}$  with common density  $f \in L^1(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$ .

We suppose that we observe one trajectory of  $(Y_t)$  over  $[0, T]$  at equidistant time  $\Delta, 2\Delta, \dots, n\Delta$ ,  $T = n\Delta$ . We denote  $Z_{k\Delta} = \frac{Y_{k\Delta}}{Y_{(k-1)\Delta}}$  the  $k$ -th increment of  $Y$ .

Let  $c \in \Xi_X \cap \Xi_\Delta$ . We have  $1 \in \Xi_X \cap \Xi_\Delta$ . Let  $\mathcal{M}_1[f]$  be the Mellin transform of  $X_1$  and  $\mathcal{M}_1[\Delta]$  be the Mellin transform of  $Z_\Delta$ .

**Lemma 9.** *For any  $s \in \mathbb{C}$ ,  $\mathcal{M}_1[\Delta](s) = \exp(\Delta\lambda(\mathcal{M}_1[f](s) - 1))$ .*

From Lemma 9, we have

$$\mathcal{M}_1[f](s) = 1 + \frac{1}{\lambda\Delta} \log \mathcal{M}_1[\Delta](s) ,$$

where  $\log \mathcal{M}_1[\Delta]$  is the distinguished logarithm of  $\mathcal{M}_1[\Delta]$  that we define in Appendix A. This last equation leads us to define the estimator

$$\widehat{\mathcal{M}}_1[f](s) = 1 + \frac{1}{\lambda\Delta} \log \widehat{\mathcal{M}}_1[\Delta](s) , s \in \mathbb{R} .$$

where

$$\widehat{\mathcal{M}}_1[\Delta](t) = \frac{1}{n} \sum_{k=1}^n Z_{k\Delta}^{c-1+it} , \quad \widehat{\mathcal{M}}'_c[\Delta](t) = \frac{1}{n} \sum_{k=1}^n i \log(Z_{k\Delta}) Z_{k\Delta}^{c-1+it} \quad \text{and} \quad \log \widehat{\mathcal{M}}_1[\Delta](s) = \int_0^u \frac{\widehat{\mathcal{M}}'_c[\Delta](t)}{\widehat{\mathcal{M}}_1[\Delta](t)} ds .$$



However, the quantity  $\log \widehat{\mathcal{M}}_1[\Delta](s)$  may explode. Then we define

$$\widetilde{\widehat{\mathcal{M}}_1[f]}(s) = \widehat{\mathcal{M}}_1[f](s) \cdot \mathbf{1}_{|\widehat{\mathcal{M}}_1[f](s)| \leq 4}, \quad s \in \mathbb{R}.$$

Finally, we apply a Mellin inversion

$$(13) \quad \widehat{f_{m,\Delta}}(x) = \frac{1}{2\pi} \int_{-m}^m x^{-c-it} \widetilde{\widehat{\mathcal{M}}_1[f]}(s) dt, \quad x \in (0, \infty),$$

### 3.4. Risk bounds.

**Theorem 10.** *Assume that  $\mathbb{E}[X_1^2] < \infty$ ,  $\delta \leq \frac{1}{4} \log(J\Delta)$  and  $J\Delta \rightarrow \infty$  as  $J \rightarrow \infty$ . Then, for any  $m \geq 0$ , it holds*

$$(14) \quad \mathbb{E}[\|\widehat{f_{m,\Delta}} - f\|_{\omega_1}^2] \leq \|f_m - f\|_{\omega_1}^2 + \frac{1}{2\pi n \Delta^2} \int_{-m}^m \frac{1}{|\widehat{\mathcal{M}}_1[\Delta](s)|^2} ds + \frac{25}{\pi} \left( \frac{\mathbb{E}[\ln(X_1)^2]}{\Delta n} + 4 \frac{m}{(n\Delta)^2} \right).$$

We prove this result in Appendix 5.3. In the demonstration, we need the two following lemmas.

**Lemma 11.** *Let  $m > 0$  and  $\zeta > 0$ . We consider the event*

$$\Omega_{\zeta,\Delta}(m) = \left\{ \forall u \in [-m, m], |\widehat{\mathcal{M}}_1[\Delta](u) - \mathcal{M}_1[\Delta](u)| \leq \zeta \sqrt{\frac{\log(n\Delta)}{n\Delta}} \right\}.$$

If  $\mathbb{E}[\ln(X_1)^2] < \infty$ , then

$$\mathbb{P}(\Omega_{\zeta,t}^c(m)) \leq \frac{\mathbb{E}[\ln(X_1)^2]}{\Delta n} + 4 \frac{m}{(n\Delta)^n}$$

*Proof.* Let  $c, h, \tau \in \mathbb{R}_+$  be some positive reals. We define the events

$$A(c) = \left\{ \left| \frac{1}{n} \sum_{k=1}^n |\ln(Z_{k\Delta})| - \mathbb{E}|\ln(Z_\Delta)| \right| \leq c \right\},$$

$$B_{h,\tau}(m) = \left\{ \forall |k| \leq \left\lceil \frac{m}{h} \right\rceil, |\widehat{\mathcal{M}}_1[\Delta](kh) - \mathcal{M}_1[\Delta](kh)| \leq \tau \sqrt{\frac{\log(n\Delta)}{n\Delta}} \right\}.$$

As we know that the function  $u \mapsto e^{iux}$  is 1-Lipschitz, then for all  $u \in \mathbb{R}$  and for all  $h \in \mathbb{R}_+$  we have

$$\begin{aligned}
 |\widehat{\mathcal{M}}_1[\Delta](u+h) - \widehat{\mathcal{M}}_1[\Delta](u)| &= \left| \frac{1}{n} \sum_{k=1}^n Z_{k\Delta}^{i(u+h)} - \frac{1}{n} \sum_{k=1}^n Z_{k\Delta}^{c-1+iu} \right| \\
 &= \frac{1}{n} \sum_{k=1}^n \left| Z_{k\Delta}^{i(u+h)} - Z_{k\Delta}^{iu} \right| \\
 &= \frac{1}{n} \sum_{k=1}^n \left| e^{i(u+h)\ln(Z_{k\Delta})} - e^{iu\ln(Z_{k\Delta})} \right| \\
 &\leq \frac{h}{n} \sum_{k=1}^n \left| \ln(Z_{k\Delta}) \right| \\
 &= h \left( \frac{1}{n} \sum_{k=1}^n \left[ |\ln(Z_{k\Delta})| - \mathbb{E}[|\ln(Z_\Delta)|] \right] + \mathbb{E}[|\ln(Z_\Delta)|] \right)
 \end{aligned}$$

The definition of  $A(c)$  ensures that  
 (16)

$$\forall u \in \mathbb{R}, \quad h > 0, \quad |\widehat{\mathcal{M}}_1[\Delta](u+h) - \widehat{\mathcal{M}}_1[\Delta](u)| \mathbf{1}_{A(c)} \leq h(c + \mathbb{E}[|\ln(Z_\Delta)|]) \leq h(c + \Delta \mathbb{E}[|\ln(X_1)|]).$$

If  $\mathbb{E}[\ln(X_1)^2]$  is finite, the Markov inequality leads to

$$(17) \quad \mathbb{P}(A(c)^c) \leq \frac{\Delta \mathbb{E}[\ln(X_1)^2]}{c^2 n}.$$

Moreover we have

$$\begin{aligned}
 \mathbb{P}(B_{h,\tau}^c) &= \mathbb{P} \left( \exists |k| \leq \left\lceil \frac{m}{h} \right\rceil, \left| \widehat{\mathcal{M}}_1[\Delta](kh) - \mathcal{M}_1[\Delta](kh) \right| > \tau \sqrt{\frac{\log(n\Delta)}{n\Delta}} \right) \\
 &\leq \sum_{k=-\left\lceil \frac{m}{h} \right\rceil}^{\left\lceil \frac{m}{h} \right\rceil} \mathbb{P} \left( \left| \widehat{\mathcal{M}}_1[\Delta](kh) - \mathcal{M}_1[\Delta](kh) \right| > \tau \sqrt{\frac{\log(n\Delta)}{n\Delta}} \right) \\
 &\leq \sum_{k=-\left\lceil \frac{m}{h} \right\rceil}^{\left\lceil \frac{m}{h} \right\rceil} \mathbb{P} \left( \left| \sum_{k=1}^n Z_{k\Delta}^{i(kh)} - \mathbb{E}[Z_\Delta^{i(kh)}] \right| > \tau \sqrt{\frac{n \log(n\Delta)}{\Delta}} \right)
 \end{aligned}$$

As  $|Z_\Delta^{i(kh)}| \leq 1$  almost surely, Hoeffding's inequality ensures that

$$\mathbb{P}(B_{h,\tau}^c) \leq \sum_{k=-\left\lceil \frac{m}{h} \right\rceil}^{\left\lceil \frac{m}{h} \right\rceil} 2 \exp \left( - \frac{\tau^2 \log(n\Delta)}{2t} \right) = 4 \left\lceil \frac{m}{h} \right\rceil (n\Delta)^{-\tau^2/(2\Delta)}$$

Let  $|u| \leq m$ . There exists  $k$  a positive integer such that  $u \in [kh - \frac{h}{2}, kh + \frac{h}{2}]$ . It follows that

$$\begin{aligned} \mathbb{1}_{A(c) \cap B_{h,\tau}(m)} |\widehat{\mathcal{M}}_1[\Delta](u) - \mathcal{M}_1[\Delta](u)| &\leq \mathbb{1}_{A(c) \cap B_{h,\tau}(m)} \left( |\widehat{\mathcal{M}}_1[\Delta](u) - |\widehat{\mathcal{M}}_1[\Delta](kh)| \right. \\ &\quad \left. + |\widehat{\mathcal{M}}_1[\Delta](kh) - \mathcal{M}_1[\Delta](kh)| + |\mathcal{M}_1[\Delta](kh) - \mathcal{M}_1[\Delta](u)| \right) \end{aligned}$$

We bound the three right terms by using respectively the Equation (30), the definition of  $B_{h,\tau}$  and the fact that  $u \rightarrow e^{iux}$  is 1-Lipschitz. It follows that

$$\mathbb{1}_{A(c) \cap B_{h,\tau}(m)} |\widehat{\mathcal{M}}_1[\Delta](u) - \mathcal{M}_1[\Delta](u)| = 2h\Delta \mathbb{E}[|\ln(X_1)|] + hc + \tau \sqrt{\frac{\log(n\Delta)}{n\Delta}}.$$

By fixing  $c = \Delta$ ,  $h = o\left(\sqrt{\frac{\log(n\Delta)}{n\Delta}}\right)$  such that  $h > \frac{1}{\sqrt{n\Delta}}$  and  $\zeta > \tau$ , it follows that

$$A(c) \cap B_{h,\tau}(m) \subset \Omega_{\zeta,\Delta}(m).$$

In addition,  $h > \frac{1}{\sqrt{n\Delta}}$ , (30), we prove that for all  $\eta > 0$ ,

$$\begin{aligned} \mathbb{P}(\Omega_{\zeta,t}^c(m)) &= \mathbb{P}(A(c)^c) + \mathbb{P}(B_{h,\tau}(m)^c) \leq \frac{\mathbb{E}[\ln(X_1)^2]}{\Delta n} + 4 \left\lceil \frac{m}{h} \right\rceil (n\Delta)^{-\tau^2/(2\Delta)} \\ &\leq \frac{\mathbb{E}[\ln(X_1)^2]}{\Delta n} + 4m(n\Delta)^{-\frac{t-\tau^2}{2t}} \end{aligned}$$

We obtain the result by taking  $\tau^2 = t(1 + 2\eta)$ . ■

**Lemma 12.** (Duval, Kappus [18] Lemma 5.2) *Let  $\gamma \in \mathbb{R}_+$  and consider*

$$M_{n,\Delta}^{(\gamma\Delta)} = \min \left\{ u \geq 0 : |\mathcal{M}_1[\Delta](u)| = \gamma c_m \sqrt{\log(Jt)/(Jt)} \right\},$$

with the convention  $\inf\{\emptyset\} = +\infty$ .

Let  $\zeta \in \mathbb{R}_+$  be a positive real s.t.  $0 < \zeta < \gamma$ . Then

$$\mathbb{1}_{|u| \leq M_{J,t}^{\gamma} \wedge m, \Omega_{\zeta,t}(m)} \cdot |\log \widehat{\mathcal{M}}_1[\Delta](s) - \log \mathcal{M}_1[\Delta](s)| \leq \frac{\gamma}{\zeta} \log \left( \frac{\gamma}{\gamma - \zeta} \right) \frac{|\widehat{\mathcal{M}}_1[\Delta](s) - \mathcal{M}_1[\Delta](s)|}{|\mathcal{M}_1[\Delta](s)|}$$

**3.5. The adaptive procedure.** We want to minimize the right side term in Theorem 13.

The optimal cutoff  $\overline{m}_n$  is defined by

$$\overline{m}_n \in \arg \min_{m>0} \left( \|f_m - f\|_{\omega_1}^2 + \frac{1}{2\pi n\Delta^2} \int_{-m}^m \frac{1}{|\mathcal{M}_1[\Delta](s)|^2} ds + \frac{25}{\pi} \left( \frac{\mathbb{E}[\ln(X_1)^2]}{\Delta n} + 4 \frac{m}{(n\Delta)^2} \right) \right)$$

In the variance term, the leading term is in  $\frac{me^{4\Delta}}{n\Delta}$ . Therefore, the optimal cutoff is such that

$$|\mathcal{M}_1[f](\overline{m}_n)|^2 = \frac{e^{4\Delta}}{n\Delta}$$

We can define the empirical optimal cutoff by

$$|\overline{\mathcal{M}}_1[f](\overline{m}_n)|^2 = \frac{e^{4\Delta}}{n\Delta}$$

where

$$\overline{\mathcal{M}_1[f]}(s) = \widetilde{\mathcal{M}_1[f]}(s) \mathbb{1}_{|\widetilde{\mathcal{M}_1[f]}(s)| \geq \kappa_{n,\Delta}/\sqrt{n\Delta}}.$$

with

$$\kappa_{n,\Delta} = (e^{2\Delta+\kappa} \sqrt{\log(n\Delta)}), \kappa > 0.$$

We consider the empirical cutoff

$$\widehat{m}_n = \max \{m \geq 0 : |\overline{\mathcal{M}_1[f]}(m)| = \kappa_{n,\Delta}/\sqrt{n\Delta}\} \wedge (n\Delta)^\alpha.$$

**Theorem 13** (Adaptative method). *Assume that  $\mathbb{E}[X_1^2] < \infty$ ,  $\delta \leq \frac{1}{4} \log(J\Delta)$  and  $J\Delta \rightarrow \infty$  as  $J \rightarrow \infty$ . Then, there exist two positive constants  $A$  and  $B$  depending on  $\kappa$  and  $\mathbb{E}[\ln(X_1)^2]$  such that*

$$\begin{aligned} & \mathbb{E}[\|\widehat{f}_{\widehat{m}_n,\Delta} - f\|_{\omega_1}^2] \\ & \leq A \left( \|f_m - f\|_{\omega_1}^2 + \frac{\log(n\Delta)m}{n\Delta} + \frac{1}{n\Delta^2} \int_{-m}^m \frac{1}{|\mathcal{M}_1[\Delta](s)|^2} ds + 4 \frac{m^2}{(n\Delta)^2} \right) \\ & \quad + B \left( (n\Delta)^{\alpha-c(\Delta)^2} + \frac{1}{n\Delta} \right). \end{aligned}$$

## 4. NUMERICAL EXPERIMENTS

### 4.1. Decomposing with noise.

4.1.1. *Implementation details.* For a given estimator  $\widehat{f}_{m,J}$  of  $f$ , we numerically evaluate the efficiency of  $\widehat{f}_{m,J}$ . More specifically, we compute the  $L^2$ -risks  $\mathbb{E}(\|\widehat{f}_{m,J} - f\|_2^2)$  by averaging the result obtained for 1000 independent simulations.

We test our method for three different jump distribution densities  $f_X$ :

- the mixture density  $0.3\mathcal{N}(-3.5, 1) + 0.7\mathcal{N}(3.5, 1)$ .
- the Gamma density  $\Gamma(2)$ .
- the Cauchy density  $\mathcal{C}(0, 1)$ .

and the noise density  $\mathcal{N}(0, 1)$ .

4.1.2. *Efficiency of the estimation.* The estimator is expected to be progressively more accurate as  $J$  increases. We compute the estimator for different values of  $J$  for mixed density (Figure 1) and Gamma density (Figure 2).

4.1.3. *Dependence of the  $L^2$ -risk to the cut-off  $m$  and efficiency of the adaptive procedure.* The value of  $m$  has a strong impact on the quality of the estimator  $\widehat{f}_{m,J}$ . When  $m$  is too low or too high, we expect that the estimator should deteriorate. The adaptive procedure gives an automatic way to select a "good"  $m$  based on observations.

We illustrate the efficiency of the adaptive procedure for the mixture density (Figure 3), Gamma density (Figure 4) and the Cauchy density (Figure 5).

The results of the simulations are satisfactory, as we can see that the value of  $m$  chosen seems to minimize our error.

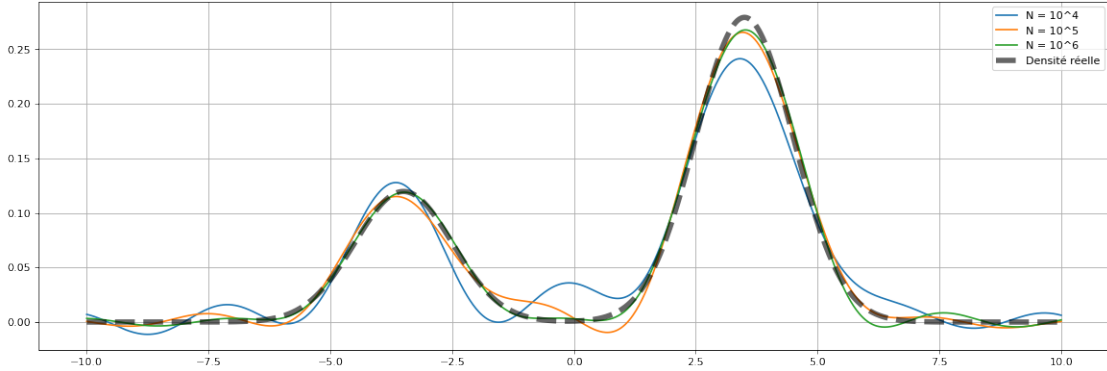


FIGURE 1. Illustration of the estimator.

*Input : Jump distribution  $X \sim 0.3\mathcal{N}(-3.5, 1) + 0.7\mathcal{N}(3.5, 1)$ . Observations are corrupted by a Gaussian noise  $\mathcal{N}(0, 1)$ .  $t_1 = 0.5, t_2 = 1, J = 10^5$ .*

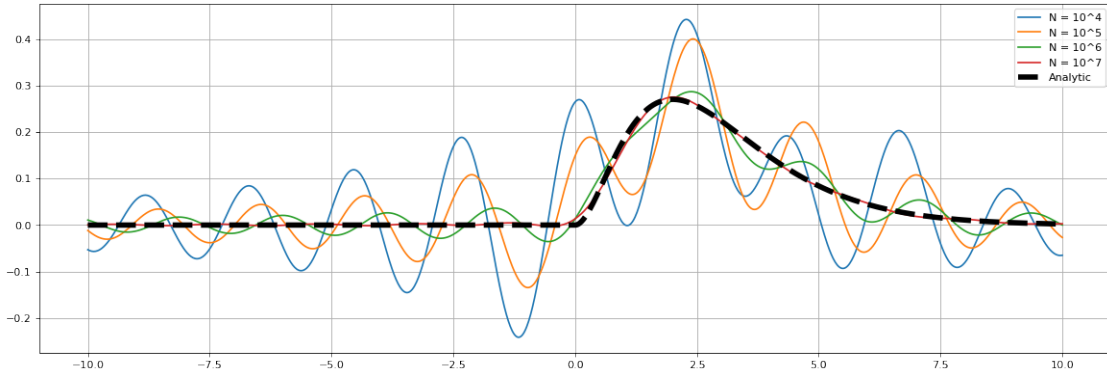


FIGURE 2. Illustration of the estimator.

*Input : Jump distribution  $X \sim \Gamma(3)$ . Observations are corrupted by a Gaussian noise  $\mathcal{N}(0, 1)$ .  $t_1 = 0.5, t_2 = 1, J = 10^5$ .*

4.1.4. *How to select an optimal time  $t_2$ .* To define our estimator, we have chosen two times  $t_1$  and  $t_2$  in a totally arbitrary way. Therefore, we would like to know how to select them in an optimal way, or at least have an intuition of the right way to choose them.

Theorem 1 states that  $t_2$  must be neither too large nor too close to  $t_1$ , which we can illustrate with numerical simulation.

Figure 6 shows the value of the  $L^2$ -risks  $\mathbb{E}(\|\widehat{f}_{m,J} - f\|_2^2)$  of the estimator for different values of  $t_2$ . Each of these values is the average of the results of 1000 different simulations.

Here, it seems that the estimate is better when  $t_2$  is between 1 and 1.5, which seems to correspond roughly to the intensity of the jump number.

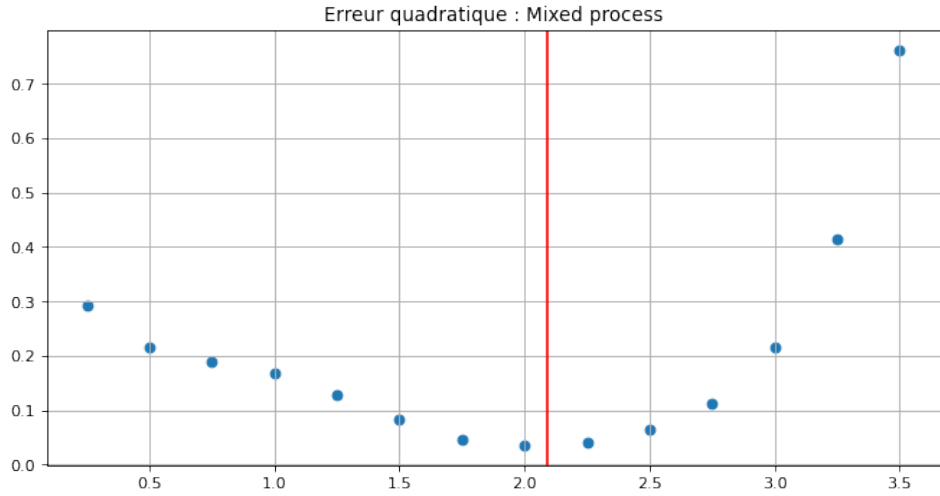


FIGURE 3. Computations of the  $L^2$ -risks  $\mathbb{E}(\|\widehat{f}_{m,J} - f\|_2^2)$  (y axis) for different values of  $m$  (x axis). The vertical line represents the value of  $\widehat{m}$  chosen automatically by the adaptive procedure.

*Input : Jump distribution  $X \sim 0.3\mathcal{N}(-2, 1) + 0.7\mathcal{N}(2, 1)$ . Observations are corrupted by a Gaussian noise  $\mathcal{N}(0, 1)$ .  $t_1 = 0.5, t_2 = 1, J = 10^5$ .*

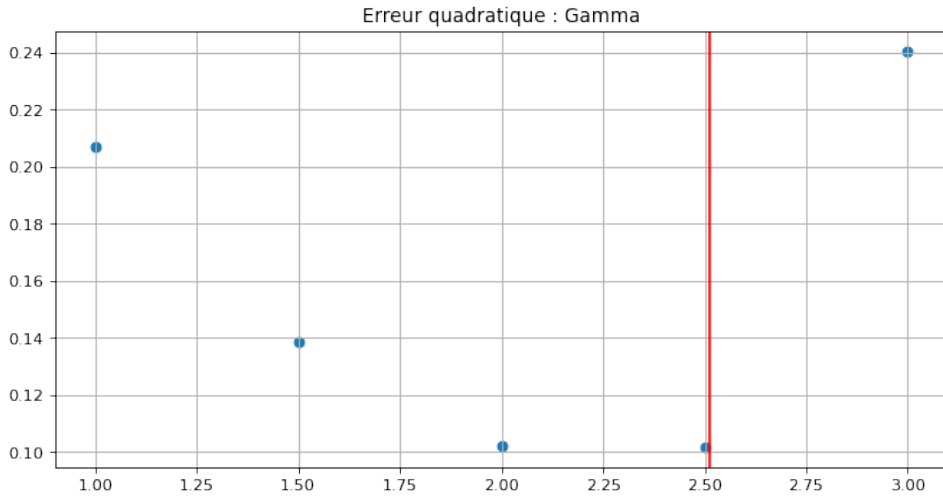


FIGURE 4. Computations of the  $L^2$ -risks  $\mathbb{E}(\|\widehat{f}_{m,J} - f\|_2^2)$  (y axis) for different values of  $m$  (x axis). The vertical line represents the value of  $\widehat{m}$  chosen automatically by the adaptive procedure.

*Input : Jump distribution  $X \sim \Gamma(2)$ . Observations are corrupted by a Gaussian noise  $\mathcal{N}(0, 1)$ .  $t_1 = 0.5, t_2 = 1, J = 10^5$ .*

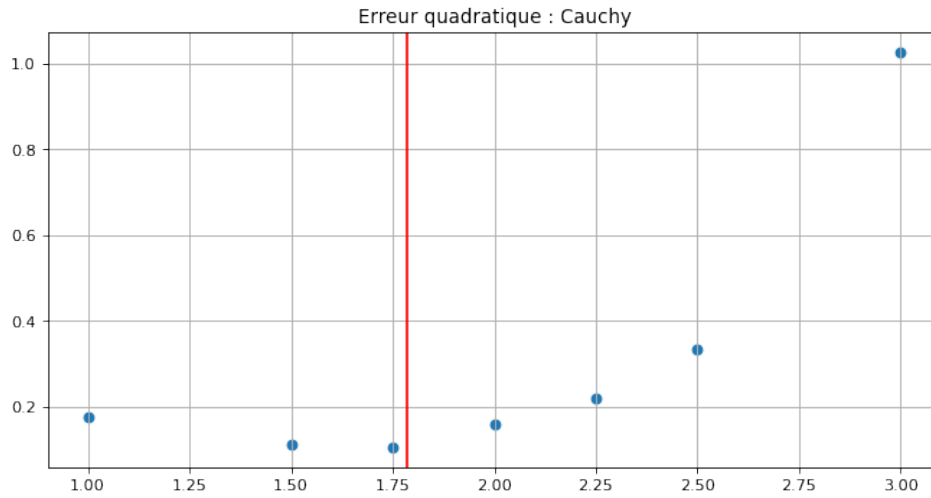


FIGURE 5. Computations of the  $L^2$ -risks  $\mathbb{E}(\|\hat{f}_{m,J} - f\|_2^2)$  (y axis) for different values of  $m$  (x axis). The vertical line represents the value of  $\hat{m}$  chosen automatically by the adaptive procedure.

*Input : Jump distribution  $X \sim \mathcal{C}(0, 1)$ . Observations are corrupted by a Gaussian noise  $\mathcal{N}(0, 1)$ .  $t_1 = 0.5, t_2 = 1, J = 10^5$ .*

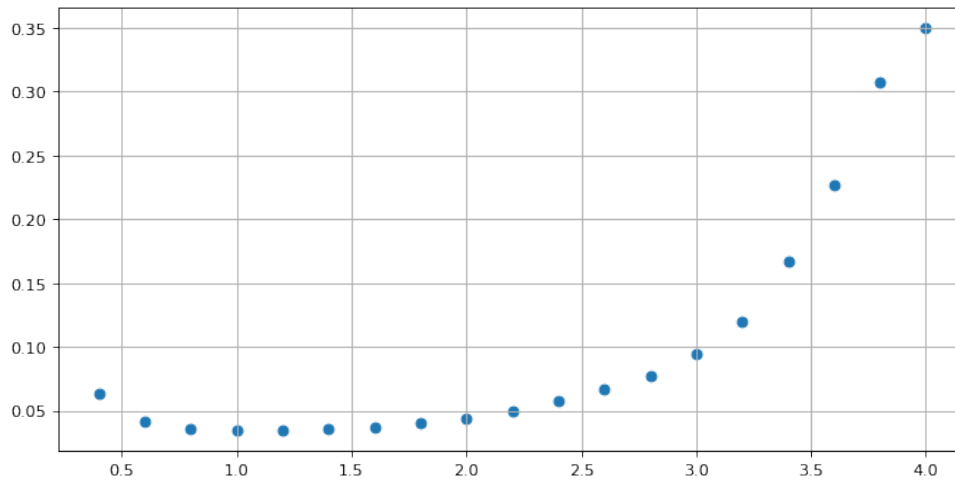


FIGURE 6. Computations of the  $L^2$ -risks  $\mathbb{E}(\|\hat{f}_{m,J} - f\|_2^2)$  (y axis) for different values of  $t_2$  (x axis).

*Input : Jump distribution  $X \sim 0.3\mathcal{N}(-2, 1) + 0.7\mathcal{N}(2, 1)$ . Observations are corrupted by a Gaussian noise  $\mathcal{N}(0, 1)$ .  $t_1 = 0.2, m = 2, J = 10^5$ .*

**4.2. Multiplicative decomposing.** In this section, we illustrate the efficiency of the Mellin estimator we built in Section 3.2 on a simulated sample of size  $n = 5000$ . Here, we assume that the jump distribution densities  $f_X$  is the Beta law  $\beta(200, 30)$ . The bandwidth  $m = 83.7$  was selected by the adaptive procedure described in Section 3.5. The resulting estimate is plotted in Fig. 7.

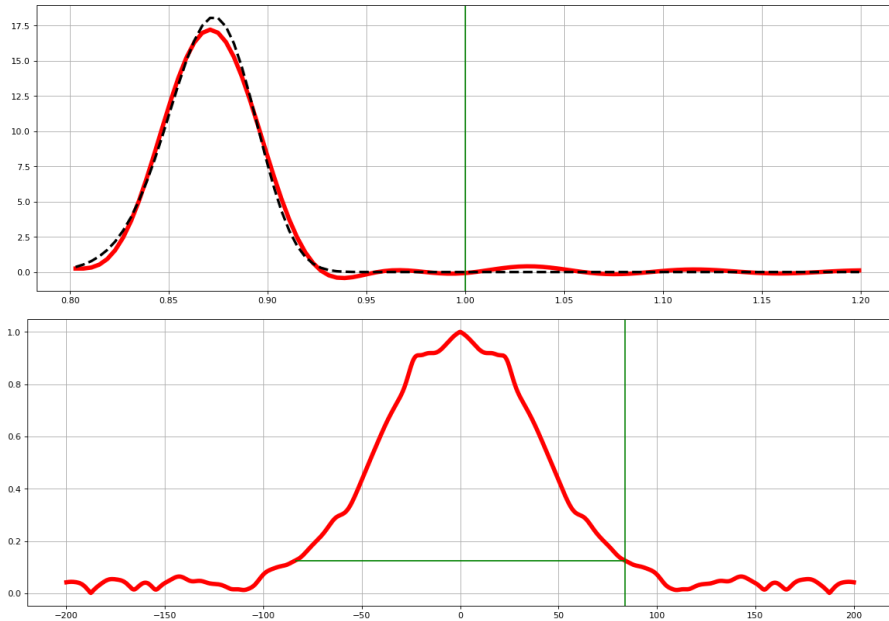


FIGURE 7. **Top:** Estimation of the Beta density  $\beta(200, 30)$  for 5000 observations. The real density is the black dotted-line and the estimator is plotted in red. **Bottom:** Illustration of the adaptive procedure

## 5. PROOFS

**5.1. Proof of Theorem 1.** This proof is an adaptation of the proof of Theorem 3.1 of Duval and Kappus [18]. Nevertheless, we have in addition a noise term that must be taken into account in our majorations.

From the triangular inequality and Parseval's equality we have

$$\begin{aligned} \|\widehat{f}_{m,J} - f\|_2^2 &\leq \|f_m - f\|_2^2 + \|\widehat{f}_{m,J} - f_m\|_2^2 \\ &= \|f_m - f\|_2^2 + \frac{1}{2\pi} \int_{-m}^m |\widetilde{\varphi}_X^J(u) - \varphi_X(u)|^2 du . \end{aligned}$$



By recalling that for any reals  $(a, b) \in \mathbb{R}^2$ ,  $|a + b|^2 \leq 2(a^2 + b^2)$ , and by using Definition 5 and Definition 4, then we have

$$\begin{aligned} & \int_{-m}^m |\tilde{\varphi}_X^J - \varphi_X(u)|^2 du \\ &= \frac{1}{(t_2 - t_1)^2} \int_{-m}^m |\log \tilde{\varphi}_{Z_{t_2}}^J(u) - \log \tilde{\varphi}_{Z_{t_1}}^J(u) - \log \varphi_{Z_{t_2}}(u) + \log \varphi_{Z_{t_1}}(u)|^2 du \\ &\leq \frac{2}{(t_2 - t_1)^2} \int_{-m}^m |\log \tilde{\varphi}_{Z_{t_1}}^J(u) - \log \varphi_{Z_{t_1}}(u)|^2 du + \frac{2}{(t_2 - t_1)^2} \int_{-m}^m |\log \tilde{\varphi}_{Z_{t_2}}^J(u) - \log \varphi_{Z_{t_2}}(u)|^2 du . \end{aligned}$$

In the rest of the proof we establish a majorization of the two terms on the right.

Let  $\tau \in \{t_1, t_2\}$  and  $I_{\tau, J, m}$  denote the quantity

$$I_{\tau, J, m} = \frac{1}{(t_2 - t_1)^2} \int_{-m}^m |\log \tilde{\varphi}_{Z_\tau}^J(u) - \log \varphi_{Z_\tau}(u)|^2 du .$$

Let  $(\gamma, \zeta) \in (0, \infty)^2$  be two positive reals such that  $\gamma c_m > \zeta$ .

In particular, we take  $\gamma$  such that  $\gamma c_m = \frac{2\zeta}{1 \wedge (t_2 - t_1)} > \zeta$ .

Consider the events

$$\Omega_{\zeta, \tau}(m) = \left\{ \forall u \in [-m, m], |\tilde{\varphi}_{Z_\tau}^J(u) - \varphi_{Z_\tau}(u)| \leq \zeta \sqrt{\frac{\log(J\tau)}{J\tau}} \right\} .$$

and

$$M_{J, t}^{(\gamma c_m)} = \min \left\{ u \geq 0 : |\varphi_{Z_t}(u)| = \gamma c_m \sqrt{\log(Jt)/(Jt)} \right\} ,$$

as defined in Lemma 22 and Lemma 23.

As the events

$$\{|u| \leq m \wedge M_{J, \tau}^{(\gamma c_m)}\} \cap \Omega_{\zeta, \tau}(m) , \quad \{|u| \in [M_{J, \tau}^{(\gamma c_m)}, m]\} \cap \Omega_{\zeta, \tau}(m) \quad \text{and} \quad \Omega_{\zeta, \tau}(m)^c$$

form a partition of the sample set  $\Omega$ , we only need to control  $I_{\tau,J,m}$  on each of them. This leads us to define the quantities

$$\begin{aligned} I_{\tau,J,m}^{(1)} &= \frac{1}{(t_2 - t_1)^2} \int_{-m \wedge M_{J,\tau}^{(\gamma c_m)}}^{m \wedge M_{J,\tau}^{(\gamma c_m)}} \mathbb{1}_{\Omega_{\zeta,\tau}(m)} |\log \tilde{\varphi}_{Z_\tau}^J(u) - \log \varphi_{Z_\tau}(u)|^2 du, \\ I_{\tau,J,m}^{(2)} &= \frac{1}{(t_2 - t_1)^2} \mathbb{1}_{m > M_{J,\tau}^{(\gamma c_m)}} \cdot \mathbb{1}_{\Omega_{\zeta,\tau}(m)} \int_{|u| \in [M_{J,\tau}^{(\gamma c_m)}, m]} |\log \tilde{\varphi}_{Z_\tau}^J(u) - \log \varphi_{Z_\tau}(u)|^2 du, \\ I_{\tau,J,m}^{(3)} &= \frac{1}{(t_2 - t_1)^2} \mathbb{1}_{\Omega_{\zeta,\tau}(m)^c} \int_{-m}^m |\log \tilde{\varphi}_{Z_\tau}^J(u) - \log \varphi_{Z_\tau}(u)|^2 du. \end{aligned}$$

Obviously, we have the decomposition

$$I_{\tau,J,m} = I_{\tau,J,m}^{(1)} + I_{\tau,J,m}^{(2)} + I_{\tau,J,m}^{(3)}.$$

Step 1 – Control  $I_{\tau,J,m}^{(1)}$ . Let us focus on the event  $A = \{|u| \leq m \wedge M_{J,\tau}^{(\gamma c_m)}\} \cap \Omega_{\zeta,\tau}(m)$ .

The triangular inequality ensures that

$$|\log \hat{\varphi}_{Z_\tau}^J(u)| \leq |\log \tilde{\varphi}_{Z_\tau}^J(u) - \log \varphi_{Z_\tau}(u)| + |\log \varphi_{Z_\tau}(u)|.$$

Then Lemma 23 applied to  $\zeta < \gamma c_m$  ensures that

$$|\log \hat{\varphi}_{Z_\tau}^J(u)| \leq \frac{\gamma c_m}{\zeta} \log \left( \frac{\gamma c_m}{\gamma c_m - \zeta} \right) \frac{|\hat{\varphi}_{Z_\tau}^J(u) - \varphi_{Z_\tau}(u)|}{|\varphi_{Z_\tau}(u)|} + |\log \varphi_{Z_\tau}(u)|.$$

Then we have by definition of  $\Omega_{\zeta,\tau}(m)$ ,

$$|\log \hat{\varphi}_{Z_\tau}^J(u)| \leq \frac{\gamma c_m}{\zeta} \log \left( \frac{\gamma c_m}{\gamma c_m - \zeta} \right) \frac{1}{|\varphi_{Z_\tau}(u)|} \zeta \sqrt{\frac{\log(J\tau)}{J\tau}} + |\log \varphi_{Z_\tau}(u)|.$$

For  $J\tau$  large enough, we have  $\gamma c_m \sqrt{\frac{\log(J\tau)}{J\tau}} < 1$ , hence on  $A$  the definition of  $M_{J,\tau}^{(\gamma c_m)}$  ensures that

$$\frac{1}{|\varphi_{Z_\tau}(u)|} \leq \frac{1}{\gamma c_m \sqrt{\frac{\log(J\tau)}{J\tau}}}.$$

It follows that

$$\begin{aligned} |\log \hat{\varphi}_{Z_\tau}^J(u)| &\leq \frac{\gamma c_m}{\zeta} \log \left( \frac{\gamma c_m}{\gamma c_m - \zeta} \right) \frac{1}{\gamma c_m \sqrt{\frac{\log(J\tau)}{J\tau}}} \zeta \sqrt{\frac{\log(J\tau)}{J\tau}} + |\log \varphi_{Z_\tau}(u)|, \\ &\leq \log \left( \frac{\gamma c_m}{\gamma c_m - \zeta} \right) + |\log \varphi_{Z_\tau}(u)|. \end{aligned}$$

By definition, this means (see also Proposition 17)

$$|\log \hat{\varphi}_{Z_\tau}^J(u)| \leq \log \left( \frac{\gamma c_m}{\gamma c_m - \zeta} \right) + |\log \varphi_{Y_\tau}(u) + \log \varphi_\varepsilon(u)|,$$

However, we know that for all  $u \in \mathbb{R}$

$$\varphi_X(u) = 1 + \frac{1}{\tau} \log \varphi_{Y_\tau}(u) .$$

It follows that

$$\begin{aligned} |\log \widehat{\varphi}_{Z_\tau}^J(u)| &\leq \log \left( \frac{\gamma c_m}{\gamma c_m - \zeta} \right) + |\tau(\varphi_X(u) + 1) + \log \varphi_\varepsilon(u)| \\ &\leq \log \left( \frac{\gamma c_m}{\gamma c_m - \zeta} \right) + 2\tau + |\log \varphi_\varepsilon(u)| , \\ &\leq \log \left( \frac{\gamma c_m}{\gamma c_m - \zeta} \right) + 2\tau + \sup_{[-m, m]} |\log \varphi_\varepsilon(\cdot)| . \end{aligned}$$

Therefore,

$$\left| \frac{\log \widehat{\varphi}_{Z_\tau}^J(u)}{t_2 - t_1} \right| \leq \frac{1}{t_2 - t_1} \log \left( \frac{\gamma c_m}{\gamma c_m - \zeta} \right) + \frac{2\tau}{t_2 - t_1} + \frac{\sup_{[-m, m]} |\log \varphi_\varepsilon(\cdot)|}{t_2 - t_1}$$

There are two different cases.

- If  $t_2 - t_1 \geq 1$ , then  $\gamma c_m = \frac{2\zeta}{1 \wedge (t_2 - t_1)} = 2\zeta$ . It follows that

$$\frac{1}{t_2 - t_1} \log \left( \frac{\gamma c_m}{\gamma c_m - \zeta} \right) = \frac{1}{t_2 - t_1} \log(2) < 1 .$$

- If  $t_2 - t_1 < 1$ , then  $\gamma c_m = \frac{2\zeta}{1 \wedge (t_2 - t_1)} = \frac{2\zeta}{t_2 - t_1}$ . It follows that

$$\begin{aligned} \frac{1}{t_2 - t_1} \log \left( \frac{\gamma c_m}{\gamma c_m - \zeta} \right) &= \frac{1}{t_2 - t_1} \log \left( \frac{\frac{2\zeta}{t_2 - t_1}}{\frac{2\zeta}{t_2 - t_1} - \frac{\zeta(t_2 - t_1)}{t_2 - t_1}} \right) \\ &= \frac{1}{t_2 - t_1} \log \left( \frac{2}{2 - (t_2 - t_1)} \right) \\ &\leq 1 , \end{aligned}$$

because for any real  $x \in (0, 1)$  we have

$$0 < \frac{1}{x} \log \left( \frac{2}{2 - x} \right) < 1 .$$

It leads to

$$\begin{aligned} \left| \frac{\log \widehat{\varphi}_{Z_\tau}^J(u)}{t_2 - t_1} \right| &\leq 1 + \frac{2\tau}{t_2 - t_1} + \frac{\sup_{[-m, m]} |\log \varphi_\varepsilon(\cdot)|}{t_2 - t_1} \\ (25) \quad &\leq 1 + \frac{2t_2}{t_2 - t_1} + \frac{\sup_{[-m, m]} |\log \varphi_\varepsilon(\cdot)|}{t_2 - t_1} . \end{aligned}$$

Since we consider  $0 \leq m \leq C_{t_1, t_2}^J$

$$\left| \log \widehat{\varphi}_{Z_\tau}^J(u) \right| \leq 3t_2 - t_1 + \sup_{[-m, m]} |\log \varphi_\varepsilon(\cdot)| \leq \ln(J)$$

Therefore, it comes from Equation (25) and Equation (4) that

$$\log \tilde{\varphi}_{Z_\tau}^J(u) = \log \hat{\varphi}_{Z_\tau}^J(u) .$$

By using Lemma 23 and the definition of  $\gamma$ , we have

$$\begin{aligned} \mathbb{E}[I_{\tau,J,m}^{(1)}] &= \frac{1}{(t_2 - t_1)^2} \int_{-m \wedge M_{J,\tau}^{(\gamma c_m)}}^{m \wedge M_{J,\tau}^{(\gamma c_m)}} \mathbb{E} \left[ \mathbb{1}_{\Omega_{\zeta,\tau}(m)} |\log \hat{\varphi}_{Z_\tau}^J(u) - \log \varphi_{Z_\tau}(u)|^2 \right] du \\ &\leq \frac{1}{(t_2 - t_1)^2} \int_{-m}^m \frac{\mathbb{E}[|\hat{\varphi}_{Z_\tau}^J(u) - \varphi_{Z_\tau}(u)|^2]}{|\varphi_{Z_\tau}(u)|^2} du , \end{aligned}$$

because  $\frac{\gamma c_m}{\zeta} \log \left( \frac{\gamma c_m}{\gamma c_m - \zeta} \right) \leq 1$  due to the fact that  $\gamma c_m > \zeta$ .

Direct computations lead to

$$\begin{aligned} \mathbb{E}[|\hat{\varphi}_{Z_\tau}^J(u) - \varphi_{Z_\tau}(u)|^2] &= \frac{1 - |\varphi_{Z_\tau}(u)|^2}{J} \\ &= \frac{1 - |\varphi_{Y_\tau}(u)|^2 |\varphi_\varepsilon(u)|^2}{J} \\ &= \frac{(1 - |\varphi_{Y_\tau}(u)|^2) |\varphi_\varepsilon(u)|^2 + 1 - |\varphi_\varepsilon(u)|^2}{J} \\ &\leq \frac{(1 - |\varphi_{Y_\tau}(u)|^2) |\varphi_\varepsilon(u)|^2 + 1 - |\varphi_\varepsilon(u)|^2}{J} \\ &\leq \frac{2}{J} . \end{aligned}$$

We deduce that

$$\mathbb{E}[I_{\tau,J,m}^{(1)}] \leq \frac{2}{J(t_2 - t_1)^2} \int_{-m}^m \frac{1}{|\varphi_{Z_\tau}(u)|^2} du .$$

Futhermore, we know that  $(Y_\tau)$  is a compound Poisson process with intensity  $\lambda = 1$ . It leads to

$$|\varphi_{Y_\tau}(u)| \geq e^{-2\tau} , \quad u \in \mathbb{R} .$$

Then,

$$|\varphi_{Z_\tau}(u)|^2 = |\varphi_{Y_\tau}(u) \varphi_\varepsilon(u)|^2 \geq e^{-4\tau} |\varphi_\varepsilon(u)|^2$$

It leads to

$$\mathbb{E}[I_{\tau,J,m}^{(1)}] \leq \frac{2e^{4\tau}}{J(t_2 - t_1)^2} \int_{-m}^m \frac{1}{|\varphi_\varepsilon(u)|^2} du .$$

Step 2 – Control  $I_{\tau,J,m}^{(3)}$ .. on the event  $\Omega_{\zeta,\tau}(m)^c$ , it is more complicated to control the empirical distinguished logarithm. Nevertheless, the cut-off on the estimator allows us to write that

$$\begin{aligned}
\int_{-m}^m |\log \tilde{\varphi}_{Z_\tau}^J(u) - \log \varphi_{Z_\tau}(u)|^2 du &\leq 2 \int_{-m}^m |\log \tilde{\varphi}_{Z_\tau}^J(u)|^2 du + 2 \int_{-m}^m |\log \varphi_{Z_\tau}(u)|^2 du \\
&\leq 4m \ln^2(J) + 2 \int_{-m}^m |\log \varphi_{Y_\tau}(u) + \log \varphi_\varepsilon(u)|^2 du \\
&\leq 4m \ln^2(J) + 4 \int_{-m}^m |\log \varphi_{Y_\tau}(u)|^2 du + 4 \int_{-m}^m |\log \varphi_\varepsilon(u)|^2 du \\
&\leq 4m \ln^2(J) + 32m\tau^2 du + 4 \int_{-m}^m |\log \varphi_\varepsilon(u)|^2 du .
\end{aligned}$$

Indeed, we have that  $\varphi_X = 1 + \frac{1}{\tau} \log \varphi_{Y_\tau}$ . It follows that  $|\log \varphi_{Y_\tau}| \leq 2\tau$ . Recall that we have

$$I_{\tau, J, m}^{(3)} = \frac{1}{(t_2 - t_1)^2} \mathbf{1}_{\Omega_{\zeta, \tau}(m)^c} \int_{-m}^m |\log \tilde{\varphi}_{Z_\tau}^J(u) - \log \varphi_{Z_\tau}(u)|^2 du ,$$

which leads to write

$$\mathbb{E}[I_{\tau, J, m}^{(3)}] \leq \frac{1}{(t_2 - t_1)^2} \cdot \left( 4m \ln^2(J) + 32m\tau^2 du + 4 \int_{-m}^m |\log \varphi_\varepsilon(u)|^2 du \right) \cdot \mathbb{P}(\Omega_{\zeta, \tau}(m)^c) .$$

We apply Lemma 22 with  $\eta = 2$  and  $\zeta > \sqrt{5\tau}$ . It ensures that

$$\mathbb{E}[I_{\tau, J, m}^{(3)}] \leq \frac{1}{(t_2 - t_1)^2} \cdot \left( 4m \ln^2(J) + 32m\tau^2 + 4 \int_{-m}^m |\log \varphi_\varepsilon(u)|^2 du \right) \cdot \left( \frac{\mathbb{E}[X_i^2]}{J\tau} + \frac{\mathbb{E}[\varepsilon^2]}{J\tau^2} + 4 \frac{m}{(J\tau)^2} \right)$$

Step 3 – Control  $I_{\tau, J, m}^{(2)}$ . By Assumption,  $\tau \leq \delta \log(J\tau)$  and  $\delta < 1/4$ . It follows that for all  $u \in [-m, m]$ ,

$$|\varphi_{Z_\tau}(u)| = |\varphi_{Y_\tau}| \cdot |\varphi_\varepsilon| > e^{-2\tau} c_m > (J\tau)^{-2\delta} c_m > \gamma c_m \sqrt{\log(J\tau)/(J\tau)} ,$$

when  $J\tau$  is enough high.

Finally,  $M_{J, \tau}^{(\gamma c_m)} \geq m$ , and  $I_{\tau, J, m}^{(2)} = 0$  a.s. .

**5.2. Proof of Theorem 3.** Let  $m \geq 0$ . The proof of the theorem is divided in two steps : in a first one, we control  $\mathbb{E} \|\widehat{f}_{\widehat{m}_n, J} - f\|_2^2$  on the event  $\mathcal{E} = \{\widehat{m}_J < m\}$ , then we control it on the complementary event  $\mathcal{E}^c = \{\widehat{m}_J \geq m\}$ .

As we know, it follows from the triangular inequality and Parseval's equality that

$$\begin{aligned}\|\bar{f}_{m,J} - f\|_2^2 &= \|f_m - f\|_2^2 + \frac{1}{2\pi} \int_{-m}^m |\tilde{\varphi}_X^J(u) - \varphi_X(u)|^2 du \\ &= \frac{1}{2\pi} \int_{|u| \in [-m,m]^c} |\varphi_X(u)|^2 du + \frac{1}{2\pi} \int_{-m}^m |\tilde{\varphi}_X^J(u) - \varphi_X(u)|^2 du.\end{aligned}$$

Therefore, by replacing the estimator  $\hat{f}_{m,J}$  by  $\hat{f}_{\hat{m}_n,J}$  produces a surplus of bias on  $\mathcal{E}$  and surplus of variance on  $\mathcal{E}^c$ . Then, we only need to control these surpluses.

Step 1 – Control on  $\mathcal{E}$ .

$$\begin{aligned}\mathbb{E}\left[\mathbf{1}_{\mathcal{E}} \int_{|u| \in [\hat{m}_J, m]} |\varphi_X(u)|^2 du\right] &\leq 2\mathbb{E}\left[\mathbf{1}_{\mathcal{E}} \int_{|u| \in [\hat{m}_J, m]} |\tilde{\varphi}_X^J(u)|^2 du\right] + 2\mathbb{E}\left[\mathbf{1}_{\mathcal{E}} \int_{|u| \in [\hat{m}_J, m]} |\tilde{\varphi}_X^J(u) - \varphi_X(u)|^2 du\right] \\ &\leq 2\mathbb{E}\left[\mathbf{1}_{\mathcal{E}} \int_{|u| \in [\hat{m}_J, m]} \frac{\kappa_{J,t_1,t_2}^2 \cdot (1+u^2)}{J(t_2-t_1)^2} du\right] + 2\mathbb{E}\left[\mathbf{1}_{\mathcal{E}} \int_{|u| \in [0, m]} |\tilde{\varphi}_X^J(u) - \varphi_X(u)|^2 du\right] \\ &\leq 4m \cdot (1+m^2) \cdot \frac{\kappa_{J,t_1,t_2}^2}{J(t_2-t_1)^2} + 2A,\end{aligned}$$

where  $A = \sum_{i=1}^2 \frac{4e^{4t_i}}{J(t_2-t_1)^2} \int_{-m}^m \frac{du}{|\varphi_\varepsilon(u)|^2} + \frac{4K_{J,t_1,t_2}}{(t_2-t_1)^2} \cdot \left( \frac{\mathbb{E}[X_i^2]}{Jt_i} + \frac{\mathbb{E}[\varepsilon^2]}{Jt_i^2} + 4\frac{m}{(Jt_i)^2} \right)$  is given by Theorem 1.

Given that  $\kappa_{J,t_1,t_2} = 2\sqrt{2}e^{2t_2} + \kappa\sqrt{\ln(J(t_2-t_1)^2)}$ , we have that

$$\mathbb{E}[\|\bar{f}_{\hat{m}_J} - f\|_2^2 \mathbf{1}_{\mathcal{E}}] \leq \|f_m - f\|_2^2 + C \frac{m \cdot (1+m^2) \cdot \ln(J(t_2-t_1)^2)}{J(t_2-t_1)^2} + \tilde{C}A$$

where  $C$  and  $\tilde{C}$  are two constants.

Step 2 – Control on  $\mathcal{E}^c$ . Now, we look at the case where where  $\hat{m}_J > m$ . By construction,  $\hat{m}_J$  is bounded by  $(m_{\max})^\alpha$ . It follows that

$$\mathbb{E}\left[\int_{m < |u| < \hat{m}_J} |\tilde{\varphi}_X^J(u) - \varphi(u)|^2 du \mathbf{1}_{\mathcal{E}^c}\right] \leq \mathbb{E}\left[\int_{m < |u| < (m_{\max})^\alpha} |\tilde{\varphi}_X^J(u) - \varphi(u)|^2 du\right]$$

Let  $\eta > 2$ , such that  $\alpha - \eta < -1$ . As for the proof of the theorem 1, Lemma 22 and Lemma 23 leads to

$$\begin{aligned}\mathbb{E}[|\tilde{\varphi}_X^J(u) - \varphi(u)|^2] &\leq |\varphi_X(u)|^2 + \mathbb{E}[|\tilde{\varphi}_X^J(u) - \varphi(u)|^2] \\ &\leq |\varphi_X(u)|^2 + \sum_{i=1}^2 \frac{4e^{4t_i}}{J(t_2-t_1)^2} \frac{du}{|\varphi_\varepsilon(u)|^2} + \frac{\mathbb{E}[X_1^2]}{Jt_1} + \frac{\mathbb{E}[\varepsilon^2]}{Jt_1^2} + 4\frac{m}{(Jt_1)^\eta}\end{aligned}$$

► On the event

$$\left| \frac{\varphi_X(u)}{\sqrt{1+u^2}} \right| \geq \frac{2e^{2t_2}}{\sqrt{J}(t_2 - t_1)}$$

we see that

$$\begin{aligned} \mathbb{E}[|\bar{\varphi}_X^J(u) - \varphi(u)|^2] &\leq |\varphi_X(u)|^2 + \frac{8}{J(t_2 - t_1)^2} \frac{e^{4t_2}}{|\varphi_\varepsilon(u)|^2} + \frac{\mathbb{E}[X_1^2]}{Jt_1} + \frac{\mathbb{E}[\varepsilon^2]}{Jt_1^2} + 4 \frac{m}{(Jt_1)^\eta} \\ &\leq |\varphi_X(u)|^2 + \frac{2|\varphi_X(u)|^2}{1+u^2} \frac{1}{|\varphi_\varepsilon(u)|^2} + \frac{\mathbb{E}[X_1^2]}{Jt_1} + \frac{\mathbb{E}[\varepsilon^2]}{Jt_1^2} + 4 \frac{m_{\max}}{(Jt_1)^\eta} \\ &\leq |\varphi_X(u)|^2 + \frac{2|\varphi_X(u)|^2}{(1+u^2)} \frac{(1+u^2)}{d} + \frac{\mathbb{E}[X_1^2]}{Jt_1} + \frac{\mathbb{E}[\varepsilon^2]}{Jt_1^2} + 4 \frac{(t_2 - t_1)^{2\alpha}}{J \cdot t_1^\eta} \\ &\leq C_0 |\varphi_X(u)|^2, \end{aligned}$$

as  $J \rightarrow \infty$ .

► From now on, all that remains to be done is to look at the event

$$\mathcal{A} = \left\{ \left| \frac{\varphi_X(u)}{\sqrt{1+u^2}} \right| \leq \frac{2e^{2t_2}}{\sqrt{J}(t_2 - t_1)} \right\}$$

By definition, we know that

$$\bar{\varphi}_X^J(u) \leq 1 + \frac{2 \log(J)}{(t_2 - t_1)}$$

Then, on the event

$$\left| \frac{\varphi_X(u)}{\sqrt{1+u^2}} \right| \leq \frac{2e^{2t_2}}{\sqrt{J}(t_2 - t_1)}$$

we see that

$$\begin{aligned}
 & \mathbb{E} \left[ \int_{[m, \widehat{m}_n]} |\widehat{\varphi}_X^J(u) - \varphi(u)|^2 \cdot \mathbb{1}_{\mathcal{A}} \right] \\
 & \leq \int_{|u| \in [m, (m_{\max})^\alpha]} |\varphi(u)|^2 du + \left( 2 + \frac{2 \log(J)}{t_2 - t_1} \right)^2 \int_{|u| \in [m, (m_{\max})^\alpha]} \mathbb{P} \left( \left| \frac{\widehat{\varphi}_X(u)}{\sqrt{1+u^2}} \right| \geq \frac{\kappa_{J,t_1,t_2}}{\sqrt{J}(t_2 - t_1)} \right) \mathbb{1}_{\mathcal{A}} du \\
 & \leq \int_{|u| \in [m, (m_{\max})^\alpha]} |\varphi(u)|^2 du + \left( 2 + \frac{2 \log(J)}{t_2 - t_1} \right)^2 \int_{|u| \in [m, (m_{\max})^\alpha]} \mathbb{P} \left( \left| \frac{\widehat{\varphi}_X(u)}{\sqrt{1+u^2}} \right| \geq \frac{\kappa_{J,t_1,t_2}}{\sqrt{J}(t_2 - t_1)} \right) \mathbb{1}_{\mathcal{A}} du \\
 & \leq \int_{|u| \in [m, (m_{\max})^\alpha]} |\varphi(u)|^2 du \\
 & \quad + \left( 2 + \frac{2 \log(J)}{t_2 - t_1} \right)^2 \int_{|u| \in [m, (m_{\max})^\alpha]} \mathbb{P} \left( \left| \frac{\widehat{\varphi}_X(u) - \varphi(u)}{\sqrt{1+u^2}} \right| \geq \frac{\kappa \sqrt{\ln(J(t_2 - t_1)^2)}}{\sqrt{J}(t_2 - t_1)} \right) du \\
 & \leq \int_{|u| \in [m, (m_{\max})^\alpha]} |\varphi(u)|^2 du \\
 & \quad + \left( 2 + \frac{2 \log(J)}{t_2 - t_1} \right)^2 \int_{|u| \in [m, (m_{\max})^\alpha]} \mathbb{P} \left( |\widehat{\varphi}_X(u) - \varphi(u)| \geq \frac{\kappa \sqrt{\ln(J(t_2 - t_1)^2)}}{\sqrt{J}(t_2 - t_1)} \right) du \\
 & \leq \int_{|u| \in [m, (m_{\max})^\alpha]} |\varphi(u)|^2 du + \left( 2 + \frac{2 \log(J)}{t_2 - t_1} \right)^2 \cdot T_J
 \end{aligned}$$

Finally, it only remains to bound  $T_J$  where

$$T_J = \int_{|u| \in [m, (m_{\max})^\alpha]} \mathbb{P} \left( |\widehat{\varphi}_X(u) - \varphi(u)| \geq \frac{\kappa \sqrt{\ln(J(t_2 - t_1)^2)}}{\sqrt{J}(t_2 - t_1)} \right) du$$

By Definition, we have



$$\begin{aligned}
& \mathbb{P}\left(\left|\widehat{\varphi}_X(u) - \varphi(u)\right| \geq \frac{\kappa\sqrt{\ln(J(t_2 - t_1)^2)}}{\sqrt{J}(t_2 - t_1)}\right) \\
&= \mathbb{P}\left(\left|\frac{\log \widehat{\varphi}_{Z_{t_2}}^J(u) - \log \widehat{\varphi}_{Z_{t_1}}^J(u) - \log \varphi_{Z_{t_2}}(u) + \log \varphi_{Z_{t_1}}(u)}{t_2 - t_1}\right| \geq \frac{\kappa\sqrt{\ln(J(t_2 - t_1)^2)}}{\sqrt{J}(t_2 - t_1)}\right) \\
&\leq \mathbb{P}\left(\left|\log \widehat{\varphi}_{Z_{t_1}}^J(u) - \log \varphi_{Z_{t_1}}(u)\right| + \left|\log \widehat{\varphi}_{Z_{t_2}}^J(u) - \log \varphi_{Z_{t_2}}(u)\right| \geq (t_2 - t_1) \frac{\kappa\sqrt{\ln(J(t_2 - t_1)^2)}}{\sqrt{J}(t_2 - t_1)}\right) \\
&\leq \mathbb{P}\left(\left|\log \widehat{\varphi}_{Z_{t_1}}^J(u) - \log \varphi_{Z_{t_1}}(u)\right| \geq (t_2 - t_1) \frac{1}{2} \cdot \frac{\kappa\sqrt{\ln(J(t_2 - t_1)^2)}}{\sqrt{J}(t_2 - t_1)}\right) \\
&\quad + \mathbb{P}\left(\left|\log \widehat{\varphi}_{Z_{t_2}}^J(u) - \log \varphi_{Z_{t_2}}(u)\right| \geq (t_2 - t_1) \frac{1}{2} \cdot \frac{\kappa\sqrt{\ln(J(t_2 - t_1)^2)}}{\sqrt{J}(t_2 - t_1)}\right)
\end{aligned}$$

Without loss of generality, we only consider the term

$$\mathbb{P}\left(\left|\log \widehat{\varphi}_{Z_{t_2}}^J(u) - \log \varphi_{Z_{t_2}}(u)\right| \geq (t_2 - t_1) \frac{\kappa\sqrt{\ln(J(t_2 - t_1)^2)}}{2\sqrt{J}(t_2 - t_1)}\right).$$

By taking  $\gamma_\Delta$  and  $\zeta > 0$ , like in the proof of Theorem 1, we have that

$$\begin{aligned}
& \mathbb{P}\left(\left|\log \widehat{\varphi}_{Z_{t_2}}^J(u) - \log \varphi_{Z_{t_2}}(u)\right| \geq (t_2 - t_1) \frac{\kappa\sqrt{\ln(J(t_2 - t_1)^2)}}{2\sqrt{J}(t_2 - t_1)}\right) \\
&\leq \mathbb{P}\left(\left|\widehat{\varphi}_{Z_{t_2}}^J(u) - \varphi_{Z_{t_2}}(u)\right| \geq (t_2 - t_1) |\varphi_{Z_{t_2}}(u)| \frac{\kappa\sqrt{\ln(J(t_2 - t_1)^2)}}{2\sqrt{J}(t_2 - t_1)}\right) + \mathbb{P}(\Omega_{\zeta, (t_2 - t_1)^2}^c).
\end{aligned}$$

Using hypothesis on the regularity of  $|\varphi_\varepsilon|$ , we have that

$$|\varphi_{Z_{t_2}}| = |\varphi_{Y_{t_2}}| \cdot |\varphi_\varepsilon| \geq |\varphi_{Y_{t_2}}| \cdot \frac{d}{\sqrt{1 + u^2}} \geq e^{2t_2} \cdot \frac{d}{\sqrt{1 + (m_{\max})^2}}$$

Let  $c(\theta) = \kappa(t_2 - t_1)e^{2t_2} \cdot \frac{d}{\sqrt{1 + (m_{\max})^2}}$

It follows from Hoeffding inequality and Lemma 22 that

$$\begin{aligned}
& \mathbb{P}\left(\left|\log \widehat{\varphi}_{Z_{t_2}}^J(u) - \log \varphi_{Z_{t_2}}(u)\right| \geq \frac{\kappa\sqrt{\ln(J(t_2 - t_1)^2)}}{2\sqrt{J}(t_2 - t_1)}\right) \\
&\leq 2(J(t_2 - t_1)^2)^{-c(\theta)^2} + \frac{\mathbb{E}[X_1^2]}{J(t_2 - t_1)^2} + \frac{\mathbb{E}[\varepsilon^2]}{J(t_2 - t_1)^4} + 4 \frac{m_{\max}}{(J(t_2 - t_1)^2)^\eta}
\end{aligned}$$

If we take  $\eta > 3$  such that  $2\alpha - \eta < -1$  and  $\zeta > \sqrt{(t_2 - t_1)^2(1 + 2\eta)}$ , then we have

$$\begin{aligned}
& \int_{|u| \in [m, (m_{\max})^\alpha]} \mathbb{P}\left(\left|\log \widehat{\varphi}_{Z_{t_2}}^J(u) - \log \varphi_{Z_{t_2}}(u)\right| \geq \frac{\kappa\sqrt{\ln(J(t_2 - t_1)^2)}}{2\sqrt{J}(t_2 - t_1)}\right) \\
&\leq 4(J(t_2 - t_1)^2)^{\alpha - c(\theta)^2} + \frac{C'}{J(t_2 - t_1)^2} + \frac{C'}{J(t_2 - t_1)^4}
\end{aligned}$$

where  $C'$  depends on  $\mathbb{E}[X_1^2], \mathbb{E}[\varepsilon^2]$ .

Similar reasoning on

$$\int_{|u| \in [m, (m_{\max})^\alpha]} \mathbb{P} \left( \left| \log \widehat{\varphi}_{Z_{t_1}}^J(u) - \log \varphi_{Z_{t_1}}(u) \right|, \right.$$

leads to

$$T_J \leq C_0 (J(t_2 - t_1)^2)^{\alpha - c(\theta)^2} + \frac{C_1}{J(t_2 - t_1)^2} + \frac{C_2}{J(t_2 - t_1)^4}$$

where  $C_0, C_1$  and  $C_2$  depends on  $\mathbb{E}[X_1^2], \mathbb{E}[\varepsilon^2]$ .

**5.3. Proof of Theorem 10.** This proof is an adaptation of the proof of Theorem 3.1 of Duval and Kappus [18]. Nevertheless, we are working with products of random variables and not with sums, which forces us to adapt the proof to use to take the Mellin transform in consideration.

We have the decomposition

$$\|\widehat{f}_m - f\|_{\omega_1}^2 \leq \|f_m - f\|_{\omega_1}^2 + \|\widehat{f}_m - f_m\|_{\omega_1}^2.$$

To prove the theorem, we have to bound the variance term  $\|\widehat{f}_m - f_m\|_{\omega_1}^2$ .

Using the isometry equation, we get

$$\|\widehat{f}_m - f_m\|_{\omega_1}^2 = \frac{1}{2\pi} \int_{-m}^m |\widetilde{\mathcal{M}}_1[f](s) - \mathcal{M}_1[f](s)|^2 ds.$$

To get a majorization of the right side term, we consider different events on which it can be control.

Fix  $\gamma > \zeta$  and set  $\gamma_\Delta = \frac{2\zeta}{1 \wedge \Delta}$ . We consider the events

$$\Omega_{\zeta, \Delta}(m) = \left\{ \forall u \in [-m, m], |\widetilde{\mathcal{M}}_1[\Delta](u) - \mathcal{M}_1[\Delta](u)| \leq \zeta \sqrt{\frac{\log(n\Delta)}{n\Delta}} \right\}$$

and

$$\{m \leq M_{n, \Delta}^{(\gamma_\Delta)}\} \quad \text{where} \quad M_{n, \Delta}^{(\gamma_\Delta)} = \min \left\{ u \geq 0 : |\mathcal{M}_1[\Delta](u)| = \gamma c_m \sqrt{\log(Jt)/(Jt)} \right\}.$$

We have

$$\begin{aligned} \frac{1}{2\pi} \int_{-m}^m |\widetilde{\mathcal{M}}_1[f](s) - \mathcal{M}_1[f](s)|^2 ds &= \frac{1}{2\pi} \int_{-m \wedge M_{n, \Delta}^{(\gamma_\Delta)}}^{m \wedge M_{n, \Delta}^{(\gamma_\Delta)}} \mathbb{1}_{\Omega_{\zeta, \Delta}(m)} |\widetilde{\mathcal{M}}_1[f](s) - \mathcal{M}_1[f](s)|^2 ds \\ &\quad + \frac{1}{2\pi} \mathbb{1}_{m > M_{n, \Delta}^{(\gamma_\Delta)}} \cdot \mathbb{1}_{\Omega_{\zeta, \Delta}(m)} \int_{|s| \in [M_{n, \Delta}^{(\gamma_\Delta)}, m]} |\widetilde{\mathcal{M}}_1[f](s) - \mathcal{M}_1[f](s)|^2 ds \\ &\quad + \mathbb{1}_{\Omega_{\zeta, \Delta}^c(m)} \frac{1}{2\pi} \int_{-m}^m |\widetilde{\mathcal{M}}_1[f](s) - \mathcal{M}_1[f](s)|^2 ds \end{aligned}$$

**Step 1.** Focus on the event  $A = \{|u| \leq m \wedge M_{n,\Delta}^{(\gamma_\Delta)}\} \cap \Omega_{\zeta,\Delta}^c(m)$ .

First, we prove that  $\widehat{\mathcal{M}}_1[f](s) = \widetilde{\mathcal{M}}_1[f](s)$

The triangular inequality ensures that

$$\widehat{\mathcal{M}}_1[f](s) \leq 1 + \frac{|\log \widehat{\mathcal{M}}_1[\Delta](s) - \log \mathcal{M}_1[\Delta](s)| + |\log \mathcal{M}_1[\Delta](s)|}{\Delta}.$$

and Lemma 23 ensures that

$$\widehat{\mathcal{M}}_1[f](s) \leq 1 + \frac{1}{\Delta} \log \left( \frac{\gamma}{\gamma - \zeta} \right) \frac{|\widehat{\mathcal{M}}_1[\Delta](s) - \mathcal{M}_1[\Delta](s)|}{|\mathcal{M}_1[\Delta](s)|} + \frac{|\log \mathcal{M}_1[\Delta](s)|}{\Delta}.$$

Furthermore, since

$$\mathcal{M}_1[f](s) = 1 + \frac{1}{\Delta} \log \mathcal{M}_1[\Delta](s),$$

it follows that

$$\widehat{\mathcal{M}}_1[f](s) \leq 3 + \frac{1}{\Delta} \log \left( \frac{\gamma}{\gamma - \zeta} \right) \leq 4.$$

Then  $\widehat{\mathcal{M}}_1[f](s) = \widetilde{\mathcal{M}}_1[f](s)$ .

Therefore,

$$\begin{aligned} \mathbb{E} \left[ \frac{1}{2\pi} \int_{-m \wedge M_{n,\Delta}^{(\gamma_\Delta)}}^{m \wedge M_{n,\Delta}^{(\gamma_\Delta)}} \mathbb{1}_{\Omega_{\zeta,\Delta}(m)} |\widetilde{\mathcal{M}}_1[f](s) - \mathcal{M}_1[f](s)|^2 ds \right] &= \frac{1}{2\pi} \int_{-m \wedge M_{n,\Delta}^{(\gamma_\Delta)}}^{m \wedge M_{n,\Delta}^{(\gamma_\Delta)}} \mathbb{E} \left[ \mathbb{1}_{\Omega_{\zeta,\Delta}(m)} |\widehat{\mathcal{M}}_1[f](s) - \mathcal{M}_1[f](s)|^2 \right] ds \\ &= \frac{1}{2\pi \Delta^2} \int_{-m \wedge M_{n,\Delta}^{(\gamma_\Delta)}}^{m \wedge M_{n,\Delta}^{(\gamma_\Delta)}} \mathbb{E} \left[ \mathbb{1}_{\Omega_{\zeta,\Delta}(m)} |\log \widehat{\mathcal{M}}_1[\Delta](s) - \log \mathcal{M}_1[\Delta](s)|^2 \right] ds \end{aligned}$$

Lemma 23 ensures that

$$\mathbb{E} \left[ \frac{1}{2\pi} \int_{-m \wedge M_{n,\Delta}^{(\gamma_\Delta)}}^{m \wedge M_{n,\Delta}^{(\gamma_\Delta)}} \mathbb{1}_{\Omega_{\zeta,\Delta}(m)} |\widetilde{\mathcal{M}}_1[f](s) - \mathcal{M}_1[f](s)|^2 ds \right] \leq \frac{1}{2\pi \Delta^2} \int_{-m \wedge M_{n,\Delta}^{(\gamma_\Delta)}}^{m \wedge M_{n,\Delta}^{(\gamma_\Delta)}} \frac{\mathbb{E} [|\widehat{\mathcal{M}}_1[\Delta](s) - \mathcal{M}_1[\Delta](s)|^2]}{|\mathcal{M}_1[\Delta](s)|^2} ds$$

We have

$$\mathbb{E} [|\widehat{\mathcal{M}}_1[\Delta](s) - \mathcal{M}_1[\Delta](s)|^2] = \text{Var}(\widehat{\mathcal{M}}_1[\Delta](s)) \leq \frac{1}{n} \mathbb{E} (|Z_\Delta^{c-1+it}|^2)$$

Then we have

$$\mathbb{E} \left[ \frac{1}{2\pi} \int_{-m \wedge M_{n,\Delta}^{(\gamma_\Delta)}}^{m \wedge M_{n,\Delta}^{(\gamma_\Delta)}} \mathbb{1}_{\Omega_{\zeta,\Delta}(m)} |\widetilde{\mathcal{M}}_1[f](s) - \mathcal{M}_1[f](s)|^2 ds \right] \leq \frac{1}{2\pi n \Delta^2} \int_{-m}^m \frac{1}{|\mathcal{M}_1[\Delta](s)|^2} ds$$

Then, set  $\zeta > \sqrt{5\Delta}$ . Using Lemma 22 with  $\eta = 2$  to get

$$\mathbb{E} \left[ \mathbb{1}_{\Omega_{\zeta,\Delta}^c(m)} \frac{1}{2\pi} \int_{-m}^m |\widetilde{\mathcal{M}}_1[f](s) - \mathcal{M}_1[f](s)|^2 ds \right] \leq \frac{50m}{2\pi} \left( \frac{\mathbb{E}[\ln(X_1)^2]}{\Delta n} + 4 \frac{m}{(n\Delta)^2} \right)$$

Finally, we observe that  $|\mathcal{M}_1(\Delta)(s)| > e^{-2\Delta}$ . We can use the same strategy that Duval Kappus [18] to show that  $M_{n,\Delta}^{\gamma\Delta} = +\infty$  and that

$$\mathbb{E}\left[\frac{1}{2\pi} \mathbf{1}_{m > M_{n,\Delta}^{\gamma\Delta}} \cdot \mathbf{1}_{\Omega_{\zeta,\Delta}(m)} \int_{|s| \in [M_{n,\Delta}^{\gamma\Delta}, m]} |\widetilde{\mathcal{M}}_1[f](s) - \mathcal{M}_1[f](s)|^2 ds\right] = 0.$$

**5.4. Proof of Theorem 13.** Let  $0 < m < (n\Delta)^\alpha$ . As for the Theorem 3, the proof is divided in two steps : in the first one, we control  $\mathbb{E}[\|\overline{f_{\widehat{m}_n, \Delta}} - f\|_{\omega_1}^2]$  on the event  $\mathcal{E} = \{\widehat{m}_n < m\}$ , then we control it on the complementary event  $\mathcal{E}^c = \{\widehat{m}_n \geq m\}$ .

*Step 1 - The first event  $\mathcal{E}$ .* The triangular inequality and the isometry equality leads to

$$\begin{aligned} \mathbb{E}\left[\mathbf{1}_{\mathcal{E}} \int_{|u| \in [\widehat{m}_n, m]} |\mathcal{M}_1[f](u)|^2 du\right] &\leq 2\mathbb{E}\left[\mathbf{1}_{\mathcal{E}} \int_{|u| \in [\widehat{m}_n, m]} |\widetilde{\mathcal{M}}_1[f](u)|^2 du\right] + 2\mathbb{E}\left[\mathbf{1}_{\mathcal{E}} \int_{|u| \in [\widehat{m}_n, m]} |\widetilde{\mathcal{M}}_1[f](u) - \mathcal{M}_1[f](u)|^2 du\right] \\ &\leq 2\mathbb{E}\left[\mathbf{1}_{\mathcal{E}} \int_{|u| \in [\widehat{m}_n, m]} \frac{\kappa_{n,\Delta}^2}{n\Delta} du\right] + 2\mathbb{E}\left[\mathbf{1}_{\mathcal{E}} \int_{|u| \in [0, m]} |\widetilde{\mathcal{M}}_1[f](u) - \mathcal{M}_1[f](u)|^2 du\right] \\ &\leq 4\frac{\kappa_{n,\Delta}^2 m}{n\Delta} + \frac{1}{\pi n\Delta^2} \int_{-m}^m \frac{1}{|\mathcal{M}_1[\Delta](s)|^2} ds + \frac{50}{\pi} m \left( \frac{\mathbb{E}[\ln(X_1)^2]}{\Delta n} + 4\frac{m}{(n\Delta)^2} \right). \end{aligned}$$

By definition,  $\kappa_{n,\Delta} = e^{2\Delta} + \kappa\sqrt{\log(n\Delta)}$ . It follows from Theorem 10 that

$$\begin{aligned} \mathbb{E}\left[\mathbf{1}_{\mathcal{E}} \|\overline{f_{\widehat{m}_n, \Delta}} - f\|_{\omega_1}^2\right] &\leq \|f_m - f\|_{\omega_1}^2 + C \frac{\log(n\Delta)m}{n\Delta} \\ &\quad + \frac{1}{\pi n\Delta^2} \int_{-m}^m \frac{1}{|\mathcal{M}_1[\Delta](s)|^2} ds + \frac{50}{\pi} m \left( \frac{\mathbb{E}[\ln(X_1)^2]}{\Delta n} + 4\frac{m}{(n\Delta)^2} \right) \end{aligned}$$

*Step 2 - The second event  $\mathcal{E}^c$ .* Henceforth, we consider the case where  $\widehat{m}_n \geq m$ . It remains to control the surplus in the variance of  $\widetilde{f_{\widehat{m}_n}}$ .

Since  $m \leq \widehat{m}_n \leq (n\Delta)^\alpha$ , it follows that

$$\mathbb{E}\left[\mathbf{1}_{\mathcal{E}^c} \int_{|u| \in [m, \widehat{m}_n]} |\overline{\mathcal{M}}_1[f](u) - \mathcal{M}[f](u)|^2\right] \leq \left[ \int_{|u| \in [m, (n\Delta)^\alpha]} \mathbb{E}[|\overline{\mathcal{M}}_1[f](u) - \mathcal{M}[f](u)|^2] \right]$$

Let  $\eta > 2$  such that  $\alpha - \eta < -1$ . Lemma 11 and Lemma 12 ensure that

$$\begin{aligned} \mathbb{E}[|\overline{\mathcal{M}}_1[f](u) - \mathcal{M}[f](u)|^2] &\leq |\mathcal{M}_1[f](s)|^2 + \mathbb{E}[|\widetilde{\mathcal{M}}_1[f](u) - \mathcal{M}[f](u)|^2] \\ &\leq |\mathcal{M}_1[f](s)|^2 + \frac{2}{n\Delta^2 |\mathcal{M}_1[\Delta](s)|^2} + \frac{\mathbb{E}[\ln(X_1)^2]}{\Delta n} + \frac{4}{n\Delta} \end{aligned}$$

First, we assume that  $\{|\mathcal{M}_1[f](s)|^2 > \frac{e^{2\Delta}}{\sqrt{n\Delta}}\}$ . Given that  $\mathcal{M}_1[\Delta](u) > e^{-2\Delta}$ , it follows in this case that

$$\mathbb{E}[|\overline{\mathcal{M}}_1[f](u) - \mathcal{M}_1[f](u)|^2] \leq |\mathcal{M}_1[f](u)|^2(6 + \mathbb{E}[\ln(X_1)^2]).$$

Therefore,

$$\begin{aligned} \mathbb{E}\left[\int_{|u| \in [m, \widehat{m}_n]} |\overline{\mathcal{M}}_1[f](u) - \mathcal{M}[f](u)|^2 \cdot \mathbf{1}_{\{|\mathcal{M}_1[f](s)|^2 > \frac{e^{2\Delta}}{\sqrt{n\Delta}}\}} \cdot \mathbf{1}_{\mathcal{E}^c}\right] du &\leq A \int_{|u| \in [m, \widehat{m}_n]} |\mathcal{M}_1[f](u)|^2 du \\ &\leq A \int_{[-m, m]^c} |\mathcal{M}_1[f](u)|^2 du \end{aligned}$$

where  $A = 6 + \mathbb{E}[\ln(X_1)^2]$ .

Secondly, we assume that  $\{|\mathcal{M}_1[f](s)|^2 \leq \frac{e^{2\Delta}}{\sqrt{n\Delta}}\}$ . By construction,  $|\overline{\mathcal{M}}_1[f](s)| \leq 4$ . It follows that

$$\begin{aligned} \mathbb{E}\left[\int_{|u| \in [m, \widehat{m}_n]} |\overline{\mathcal{M}}_1[f](u) - \mathcal{M}[f](u)|^2 \cdot \mathbf{1}_{\{|\mathcal{M}_1[f](s)|^2 \leq \frac{e^{2\Delta}}{\sqrt{n\Delta}}\}} \cdot \mathbf{1}_{\mathcal{E}^c}\right] du \\ \leq \int_{|u| \in [m, (n\Delta)^\alpha]} |\mathcal{M}_1[f](u)|^2 du + 25 \int_{|u| \in [m, (n\Delta)^\alpha]} \mathbb{P}\left(|\widehat{\mathcal{M}}_1[f]| \geq \frac{\kappa_{n, \Delta}}{\sqrt{n\Delta}}\right) \cdot \mathbf{1}_{\{|\mathcal{M}_1[f](s)|^2 \leq \frac{e^{2\Delta}}{\sqrt{n\Delta}}\}} du \\ \leq \int_{|u| \in [m, (n\Delta)^\alpha]} |\mathcal{M}_1[f](u)|^2 du + 25 \int_{|u| \in [m, (n\Delta)^\alpha]} \mathbb{P}\left(|\widehat{\mathcal{M}}_1[f] - \mathcal{M}_1[f]| \geq \kappa \sqrt{\log(n\Delta/(n\Delta))}\right) du \end{aligned}$$

It remains to bound the last term. For that, we use Lemma 11 and Lemma 12 by taking  $\gamma_\Delta$  and  $\zeta > 0$ , like in the proof of Theorem 10. By definition, we have that

$$\begin{aligned} &\mathbb{P}\left(|\widehat{\mathcal{M}}_1[f] - \mathcal{M}_1[f]| \geq \kappa \sqrt{\log(n\Delta/(n\Delta))}\right) \\ &= \mathbb{P}\left(|\log \widehat{\mathcal{M}}_1[\Delta] - \log \mathcal{M}_1[\Delta]| \geq \kappa \Delta \sqrt{\log(n\Delta/(n\Delta))}\right) \\ &\leq \mathbb{P}\left(|\widehat{\mathcal{M}}_1[\Delta] - \mathcal{M}_1[\Delta]| \geq |\mathcal{M}_1[\Delta]| \kappa \Delta \sqrt{\log(n\Delta/(n\Delta))}\right) + \mathbb{P}(\Omega_{\zeta, \Delta}^c((n\Delta)^\alpha)). \end{aligned}$$

Let  $c(\Delta) = k\Delta e^{-2\Delta}$ . The Hoeffding inequality and Lemma 11 ensure that

$$\mathbb{P}\left(|\widehat{\mathcal{M}}_1[f] - \mathcal{M}_1[f]| \geq \kappa \sqrt{\log(n\Delta/(n\Delta))}\right) \leq 2(n\Delta)^{\alpha - c(\Delta)^2} + \frac{\mathbb{E}[\ln(X_1)^2]}{n\Delta} + 4(n\Delta)^{\alpha - \eta}.$$

If we take  $\eta > 3$  such that  $2\alpha - \eta < -1$  and  $\zeta > \sqrt{\Delta(1 + 2\eta)}$ , then

$$\int_{|u| \in [m, (n\Delta)^\alpha]} \mathbb{P}\left(|\widehat{\mathcal{M}}_1[f] - \mathcal{M}_1[f]| \geq \kappa \sqrt{\log(n\Delta/(n\Delta))}\right) du \leq 4(n\Delta)^{\alpha - c(\Delta)^2} + \frac{B}{n\Delta}$$

where  $B$  depends on  $\mathbb{E}[\ln(X_1)^2]$ .

It follows that there exist two constant  $A, B$  such that

$$\begin{aligned} & \mathbb{E}[\|\widehat{f_{m_n, \Delta}} - f\|_{\omega_1}^2] \\ & \leq A \left( \|f_m - f\|_{\omega_1}^2 + \frac{\log(n\Delta)m}{n\Delta} + \frac{1}{n\Delta^2} \int_{-m}^m \frac{1}{|\mathcal{M}_1[\Delta](s)|^2} ds + 4 \frac{m^2}{(n\Delta)^2} \right) \\ & \quad + B \left( (n\Delta)^{\alpha - c(\Delta)^2} + \frac{1}{n\Delta} \right). \end{aligned}$$

We conclude by taking the infimum in  $m$ .

## 6. CONCLUSION AND PERSPECTIVES

In this article, we develop an adaptive procedure to estimate the jump density in a noisy compound Poisson process. In our case, we observe several noisy compound Poisson processes. We have shown that by looking at what happens at two different times, it is possible to reconstruct then density  $f_X$ .

In practice, the experimental data allow us to observe the process at different times. A discussion could be conducted later to understand if it is possible to combine estimators built with different  $t_1$  and  $t_2$  to improve our estimator.

In future work, we plan to apply our statistical method to biological data from the article by Robert et al. [34].

Furthermore, it is possible to link compound Poisson process to the fragmentation equation which has been widely studied in the literature [15, 14, 16]. We plan to develop and study this link, in future work.

## APPENDIX A. THE DISTINGUISHED LOGARITHM

In the section, we recall the definition and some properties of the distinguished logarithm. The reader who would like to have more information can refer to the articles of Duval and Kappus [17, 18] or to the article of Finkelstein et al. [21].

**Theorem 14.** *Let  $d \in \mathbb{N} - \{0\}$  and let  $\varphi \in \mathcal{C}^0(\mathbb{R}^d; \mathbb{C}^*)$  be a continuous application which never vanishes and such that  $\varphi(0) = 1$ .*

*Then there exists a unique continuous application  $\psi \in \mathcal{C}^0(\mathbb{R}^d; \mathbb{C})$  such that*

- (1)  $\psi(0) = 0$ ,
- (2) for all  $x \in \mathbb{R}^d$ ,  $\varphi(x) = e^{\psi(x)}$ .

*The application  $\psi$  is called the Distinguished Logarithm of  $\varphi$  and is denoted by  $\log \varphi$ .*

*Remark 15.* In general, the distinguished logarithm does not reduce to a composition by the principal determination of the logarithm. In his book [36], Sato remarks to his readers that one can have  $\varphi(z_1) = \varphi(z_2)$  and  $\log \varphi(z_1) \neq \log \varphi(z_2)$ .

Indeed, consider the application  $\varphi(t) = e^{it}$ , ( $t \in \mathbb{R}$ ). It verifies all the assumptions of Theorem 14. which ensures that

$$\log \varphi(t) = it, \quad (t \in \mathbb{R}),$$

It follows that

$$\varphi(0) = \varphi(2) = 1 \quad \text{and} \quad \log \varphi(0) = 0, \quad \log \varphi(2) = 2i\pi.$$

**Proposition 16.** (Theorem 2. of [21])

Let  $d > 0$  be a positive integer and  $f, (f_n)_{n \in \mathbb{N}} \in \mathcal{C}^0(\mathbb{R}^d; \mathbb{C}^*)$  be continuous functions which never vanishes and such that  $f(0) = f_n(0) = 1$  for all  $n \in \mathbb{N}$ . If  $(f_n)$  converges uniformly to  $f$  on compact subsets of  $\mathbb{R}^d$ , then the sequence  $(\log(f_n))_{n \in \mathbb{N}}$  converges uniformly to  $\log(f)$  on compact subsets of  $\mathbb{R}^d$ .

**Proposition 17.** (Lemma 3. of [17])

Let  $\varphi$  be a characteristic function without zeroes and assume that  $\varphi$  is differentiable. Then, it follows that

$$\log \varphi(u) = \int_0^u \frac{\varphi'(z)}{\varphi(z)} dz.$$

**Corollary 18.** Let  $d \in \mathbb{N} - \{0\}$  and  $f_1, f_2 \in \mathcal{C}^0(\mathbb{R}^d; \mathbb{C}^*)$  be two continuous functions which never vanishes and such that  $f_1(0) = f_2(0) = 1$ . Assume that  $f_1, f_2$  are both differentiable and let  $h : \mathbb{R}^d \rightarrow \mathbb{C}^*$  denotes the quotient  $h = \frac{f_2}{f_1}$ . Then,

$$\forall u \in \mathbb{R}^d, \quad \log h(u) = \log f_2(u) - \log f_1(u).$$

*Proof.*

$$\begin{aligned} \log h(u) &= \int_0^u \frac{h'(z)}{h(z)} dz = \int_0^u \frac{\left(\frac{f_2}{f_1}\right)'(z)}{\left(\frac{f_2}{f_1}\right)(z)} dz = \int_0^u \frac{\frac{f_2' f_1 - f_2 f_1'}{f_1^2}}{\left(\frac{f_2}{f_1}\right)} dz = \int_0^u \left(\frac{f_2' f_1 - f_2 f_1'}{f_1 f_2}\right) dz \\ &= \int_0^u \left(\frac{f_1'}{f_1}\right) dz - \int_0^u \left(\frac{f_2'}{f_2}\right) dz = \log f_2(u) - \log f_1(u) \end{aligned}$$

■

## APPENDIX B. COMPOUND POISSON PROCESSES

**Proposition 19.** Let  $(Y_t, t \in \mathbb{R}_+)$  be a compound Poisson process with intensity  $\lambda \in \mathbb{R}_+$  and jump size density  $f$ . Assume that  $X_1$  is integrable. Then for all  $t \in \mathbb{R}_+$ ,  $Y_t$  is integrable and

$$\mathbb{E}(Y_t) = \lambda t \mathbb{E}(X_1).$$

**Proposition 20.** Let  $(Y_t, t \in \mathbb{R}_+)$  be a compound Poisson process with intensity  $\lambda \in \mathbb{R}_+$  and jump size density  $f$ . Assume that  $X_1$  is square-integrable. Then for all  $t \in \mathbb{R}_+$ ,  $Z_t$  is square-integrable and

$$\text{Var}(Y_t) = \lambda t \mathbb{E}(X_1^2).$$

**Proposition 21.** *Let  $(Y_t, t \in \mathbb{R}_+)$  be a compound Poisson process with intensity  $\lambda \in \mathbb{R}_+$  and jump size density  $f$ . Then*

$$\varphi_{Y_t}(u) = e^{-\lambda t(1-\varphi_X(u))} .$$

*Proof.* We have

$$\begin{aligned} \varphi_{Y_t}(u) &= \mathbb{E}\left(e^{i\xi \sum_{i=1}^{N_t} X_i}\right) = \mathbb{E}\left[\mathbb{E}\left(e^{i\xi \sum_{i=1}^{N_t} X_i}\right) \middle| N_t\right] = \mathbb{E}\left[\mathbb{E}\left(e^{i\xi N_t X}\right) \middle| N_t\right] \\ &= \sum_{k=0}^{\infty} e^{-\lambda t} \frac{\lambda^k t^k}{k!} \times \mathbb{E}\left(e^{i\xi x}\right)^k = e^{-\lambda t} \sum_{k=0}^{\infty} \frac{\lambda^k t^k}{k!} \times \mathbb{E}\left(e^{i\xi x}\right)^k \\ &= e^{-\lambda t} \times e^{\lambda t \cdot \varphi_X(u)} . \end{aligned}$$

■

## APPENDIX C. USEFUL LEMMAS

### C.1. A first Lemma.

**Lemma 22.** *Let  $m, \zeta, t \in \mathbb{R}_+$  be positive reals. We consider the event*

$$\Omega_{\zeta,t}(m) = \left\{ \forall u \in [-m, m], |\widehat{\varphi}_{Z_t}^J(u) - \varphi_{Z_t}(u)| \leq \zeta \sqrt{\frac{\log(Jt)}{Jt}} \right\} .$$

If  $\mathbb{E}[X_1^2] < \infty$ , then

$$\forall \eta > 0, \forall \zeta > \sqrt{t(1+2\eta)}, \mathbb{P}(\Omega_{\zeta,t}(m)^c) \leq \frac{\mathbb{E}[X_1^2]}{Jt} + \frac{\mathbb{E}[\varepsilon^2]}{Jt^2} + 4 \frac{m}{(Jt)^\eta} .$$

*Proof.* Let  $c, h, \tau \in \mathbb{R}_+$  be some positive reals. We define the events

$$\begin{aligned} A(c) &= \left\{ \left| \frac{1}{J} \sum_{j=1}^J |Z_t^j| - \mathbb{E}[|Z_t|] \right| \leq c \right\} , \\ B_{h,\tau(m)} &= \left\{ \forall |k| \leq \left\lceil \frac{m}{h} \right\rceil, |\widehat{\varphi}_{Z_t}^J(kh) - \varphi_{Z_t}(kh)| \leq \tau \sqrt{\frac{\log(Jt)}{Jt}} \right\} . \end{aligned}$$



As we know that the function  $x \mapsto e^{iux}$  is 1-Lipschitz, then for all  $u \in \mathbb{R}$  and for all  $h \in \mathbb{R}_+$  we have

$$\begin{aligned} |\widehat{\varphi}_{Z_t}^J(u) - \widehat{\varphi}_{Z_t}^J(u+h)| &= \left| \frac{1}{J} \sum_{j=1}^J e^{iuZ_t^j} - \frac{1}{J} \sum_{j=1}^J e^{i(u+h)Z_t^j} \right| \\ &\leq \frac{1}{J} \sum_{j=1}^J \left| e^{iuZ_t^j} - e^{i(u+h)Z_t^j} \right| \\ &\leq \frac{h}{J} \sum_{j=1}^J |Z_t^j| \\ &= h \left( \frac{1}{J} \sum_{j=1}^J \left[ |Z_t^j| - \mathbb{E}[|Z_t|] \right] + \mathbb{E}[|Z_t|] \right) \end{aligned}$$

The definition of  $A(c)$  ensures that

$$(30) \quad \forall u \in \mathbb{R}, \quad h > 0, \quad |\widehat{\varphi}_{Z_t}^J(u) - \widehat{\varphi}_{Z_t}^J(u+h)| \mathbf{1}_{A(c)} \leq h(c + \mathbb{E}[|Z_t|]).$$

Moreover we have

$$(31) \quad \mathbb{E}[|Z_t|] \leq t\mathbb{E}[|X_1|] + \mathbb{E}[|\varepsilon|].$$

If  $\mathbb{E}[X_1^2] < \infty$ , by applying the Markov inequalities and Proposition 20,

$$\text{Var}[|Z_t|] \leq \text{Var}[Z_t] = t\mathbb{E}[X_1^2] + \mathbb{E}[\varepsilon^2]$$

we claim that

$$(32) \quad \mathbb{P}(A(c)^c) \leq \frac{t\mathbb{E}[X_1^2]}{c^2 J} + \frac{\mathbb{E}[\varepsilon^2]}{c^2 J}.$$

Moreover we have

$$(33a) \quad \mathbb{P}(B_{h,\tau}^c) = \mathbb{P} \left( \exists |k| \leq \left\lceil \frac{m}{h} \right\rceil, |\widehat{\varphi}_{Z_t}^J(kh) - \varphi_{Z_t}(kh)| > \tau \sqrt{\frac{\log(Jt)}{Jt}} \right)$$

$$(33b) \quad \leq \sum_{k=-\lceil \frac{m}{h} \rceil}^{\lceil \frac{m}{h} \rceil} \mathbb{P} \left( |\widehat{\varphi}_{Z_t}^J(kh) - \varphi_{Z_t}(kh)| > \tau \sqrt{\frac{\log(Jt)}{Jt}} \right)$$

$$(33c) \quad \leq \sum_{k=-\lceil \frac{m}{h} \rceil}^{\lceil \frac{m}{h} \rceil} \mathbb{P} \left( \left| \sum_{j=1}^J e^{ikhZ_t^j} - \mathbb{E}[e^{ikhZ_t^j}] \right| > \tau \sqrt{\frac{J \log(Jt)}{t}} \right)$$

As  $|e^{ikhZ_t^j}| \leq 1$  almost surely, Hoeffding's inequality ensures that

$$(33d) \quad \leq \sum_{k=-\lceil \frac{m}{h} \rceil}^{\lceil \frac{m}{h} \rceil} 2 \exp\left(-\frac{\tau^2 \log(Jt)}{2t}\right)$$

$$(33e) \quad = 4 \lceil \frac{m}{h} \rceil (Jt)^{-\tau^2/(2t)}$$

Let  $|u| \leq m$ . Then there exists  $k$  a positive integer such that  $u \in [kh - \frac{h}{2}, kh + \frac{h}{2}]$ . Then

$$(34) \quad \mathbb{1}_{A(c) \cap B_{h,\tau}(m)} |\widehat{\varphi}_{Z_t}^J(u) - \varphi_{Z_t}(u)| \leq \mathbb{1}_{A(c) \cap B_{h,\tau}(m)} \left( |\widehat{\varphi}_{Z_t}^J(u) - \widehat{\varphi}_{Z_t}^J(kh)| \right. \\ \left. + |\widehat{\varphi}_{Z_t}^J(kh) - \varphi_{Z_t}(kh)| + |\varphi_{Z_t}(kh) - \varphi_{Z_t}(u)| \right)$$

Since the function  $x \mapsto e^{ixt}$  is 1-Lipschitz, we have

$$|\varphi_{Z_t}(kh) - \varphi_{Z_t}(u)| = |\mathbb{E}[e^{ikhZ_t} - e^{iuZ_t}]| \\ \leq h \mathbb{E}[|Z_t|]$$

By applying this last result to Equation (34)

$$\mathbb{1}_{A(c) \cap B_{h,\tau}(m)} \sup_{|u| \leq m} |\widehat{\varphi}_{Z_t}^J(u) - \varphi_{Z_t}(u)| \leq \mathbb{1}_{A(c) \cap B_{h,\tau}(m)} \left( |\widehat{\varphi}_{Z_t}^J(u) - \widehat{\varphi}_{Z_t}^J(kh)| \right. \\ \left. + |\widehat{\varphi}_{Z_t}^J(kh) - \varphi_{Z_t}(kh)| + h \mathbb{E}[|Z_t|] \right)$$

It follows from Equation (30) that

$$\mathbb{1}_{A(c) \cap B_{h,\tau}(m)} \sup_{|u| \leq m} |\widehat{\varphi}_{Z_t}^J(u) - \varphi_{Z_t}(u)| \leq h(c + \mathbb{E}[|Z_t|]) + \mathbb{1}_{A(c) \cap B_{h,\tau}(m)} \left( |\widehat{\varphi}_{Z_t}^J(kh) - \varphi_{Z_t}(kh)| \right) + h \mathbb{E}[|Z_t|].$$

By using the definition of  $B_{h,\tau}(m)$ , we have

$$(36) \quad \mathbb{1}_{A(c) \cap B_{h,\tau}(m)} \sup_{|u| \leq m} |\widehat{\varphi}_{Z_t}^J(u) - \varphi_{Z_t}(u)| \leq h(c + \mathbb{E}[|Z_t|]) + \tau \sqrt{\frac{\log(Jt)}{Jt}} + h \mathbb{E}[|Z_t|] \\ \leq hc + 2h \mathbb{E}[|Z_t|] + \tau \sqrt{\frac{\log(Jt)}{Jt}} \\ \leq hc + 2h(t \mathbb{E}[|X_1|] + \mathbb{E}[|\varepsilon|]) + \tau \sqrt{\frac{\log(Jt)}{Jt}}.$$

In particular, if  $c = t$ ,  $h = o(\sqrt{\frac{\log(Jt)}{Jt}})$  such that  $h > \frac{1}{\sqrt{Jt}}$  and  $\zeta > \tau$ , Equation (36), allows us to say that

$$A(c) \cap B_{h,\tau(m)} \subset \Omega_{\zeta,t}(m).$$

In addition,  $h > \frac{1}{\sqrt{Jt}}$ , (30) et (32)), we prove that for all  $\eta > 0$ ,

$$\begin{aligned} \mathbb{P}(\Omega_{\zeta,t}^c(m)) &\leq \mathbb{P}(A^c(t)) + \mathbb{P}(B_{h,\tau}^c)(m) \\ &\leq \frac{\mathbb{E}[X_1^2]}{Jt} + \frac{\mathbb{E}[\varepsilon^2]}{Jt^2} + 4 \left\lceil \frac{m}{h} \right\rceil (Jt)^{-\frac{\tau^2}{2t}} \\ &\leq \frac{\mathbb{E}[X_1^2]}{Jt} + \frac{\mathbb{E}[\varepsilon^2]}{Jt^2} + 4m(Jt)^{-\frac{t-\tau^2}{2t}} \end{aligned}$$

We obtain the result by taking  $\tau^2 = t(1 + 2\eta)$ . ■

## C.2. A second lemma.

**Lemma 23.** (Lemma 5.2. of [18]) *Let  $\gamma \in \mathbb{R}_+$ ,  $t \in \mathbb{R}_+$  and consider*

$$M_{J,t}^{(\gamma)} = \min \left\{ u \geq 0 : |\varphi_{Z_t}(u)| = \gamma \sqrt{\log(Jt)/(Jt)} \right\},$$

with the convention  $\inf\{\emptyset\} = +\infty$ .

Let  $\zeta \in \mathbb{R}_+$  be a positive real s.t.  $0 < \zeta < \gamma$ . Then

$$\mathbb{1}_{|u| \leq M_{J,t}^{\gamma} \wedge m, \Omega_{\zeta,t}(m)} \cdot \left| \log(\widehat{\varphi}_{Z_t}^J(u)) - \log \varphi_{Z_t}(u) \right| \leq \frac{\gamma}{\zeta} \log \left( \frac{\gamma}{\gamma - \zeta} \right) \frac{|\widehat{\varphi}_{Z_t}^J(u) - \varphi_{Z_t}(u)|}{|\varphi_{Z_t}(u)|}$$

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