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On a q -analogue of the Zeta polynomial of posets

F. Chapoton

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Abstract

We introduce a q -analogue of the classical Zeta polynomial of finite partially ordered sets, as a polynomial in one variable x with coefficients depending on the indeterminate q . We prove some properties of this polynomial invariant, including its behaviour with respect to duality, product and disjoint union. The leading term is a q -analogue of the number of maximal chains, but not always with non-negative coefficients. The value at $q = 0$ turns out to be essentially the characteristic polynomial.

In the study of finite partially ordered sets (posets), one uses frequently polynomial invariants. They are useful to distinguish the posets or recognize them under different disguises, but also for testing the solidity of our understanding. Computing these invariants may require a good handle on the combinatorial mechanism behind the scene. This can lead to structural results. For example, sometimes these polynomials factor nicely and one would like to understand the reason for this.

One of the most classical such polynomial is the Zeta polynomial, whose values at positive integers count chains of elements, introduced in [20, §3]. This polynomial is related quite closely to the order polynomial, introduced in [18], which describes integer points in a polytope naturally attached to the poset. Yet another polynomial is the characteristic polynomial, recording the values of the Möbius function, introduced in [17]. An important but less well-known polynomial is the Coxeter polynomial, which contains information about the derived category of modules over the incidence algebra. For more information on this last topic, see the survey article [10]

The aim of the present article is to introduce and study a q -analogue of the Zeta polynomial, defined for posets endowed with a height function. This is a polynomial in the variable x , whose values at q -integers count chains according to the sum of heights of their elements. We will show that it shares many properties of the Zeta polynomial, to which it reduces when setting $q = 1$.

The main motivation for this definition comes from the q -analogue of Ehrhart theory introduced in [9], where the objects of study are lattice polytopes together with a linear form. It is well-known that the usual Zeta polynomial of the distributive lattice $J(P)$ of lower ideals in a poset P is equal to the order polynomial of the partial order P , and therefore to the Ehrhart polynomial of the order polytope of P introduced in [21]. The present article started with the aim to understand what happens to this story when one replaces the Ehrhart polynomial of P by the q -Ehrhart polynomial. The conclusion is that everything works nicely, and a rather clean theory can be established.

The construction is mostly as expected, with the appropriate changes and dependencies on the choice of the height function on the poset. There is one unexpected property, namely the value of the q -Zeta polynomial at $q = 0$ can under some conditions be identified with the characteristic polynomial.

The reader should be warned that, in some sense, a large part of the surrounding context and proofs are not new and indeed very classical since the pioneering works of Stanley on P -partitions. The precise relation of q -Ehrhart polynomials with the theory of P -partitions has been described in [9, §4]. A nice reference on the history of the subject of P -partitions can be found in [12], in which the connection between P -partitions and Zeta polynomials appears in §8.1.

The study of Ehrhart generating series is the main focus of the theory of P -partitions. From this perspective, the q -Zeta polynomials are just describing in another compact way the coefficients of these Ehrhart series. This is a slightly different viewpoint, which offers other insights.

The article is organized as follows. After recalling briefly the classical Zeta polynomial, the q -Zeta polynomial is defined and some examples are given in Section 1. In Section 2, some basic properties are proved. In Section 3, the relationship, in the case of distributive lattices, with the q -order polynomial is spelled out. In Section 4, a property relating the leading coefficient of the q -Zeta polynomial and the numerator of the Ehrhart series is proved. In Section 5, a known criterion is recalled for the positivity of the coefficients in the numerator of the Ehrhart series. Section 6 describes the unexpected relationship, at $q = 0$, with the characteristic polynomial. Section 7 proves a q -analogue of a reciprocity theorem for Zeta polynomials of Eulerian posets. The article ends with three appendices : **A** on classical results on flag vectors, positivity and R -labellings, **B** on q -analogues of polynomials with integer values and **C** on a q -analogue of the incidence algebra.

I would like to humbly dedicate this article to Richard Stanley, with my sincere admiration.

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Introduction

Let us first recall briefly the classical Zeta polynomial of posets.

Let P be a finite poset. For every integer $n \geq 2$, consider the set $C_n(P)$ of chains $e_1 \leq e_2 \leq \dots \leq e_{n-1}$ in P . Then the cardinality of $C_n(P)$ is given by the value at n of a polynomial $Z_P(x)$, called the Zeta polynomial of P .

One can easily prove this fact by the following computation.

For $k \geq 1$, let $\text{Ch}_k(P)$ be the set of strict chains with k elements in P , *i.e.* sequences $c_1 < \dots < c_k$.

By gathering chains according to their underlying strict chain and multiplic-

ities, one finds that

$$Z_P(n) = \sum_{e_1 \leq \dots \leq e_{n-1}} 1 = \sum_{k \geq 1} \left(\sum_{\substack{m_1, \dots, m_k \geq 1 \\ \sum_i m_i = n-1}} 1 \right) \# \text{Ch}_k. \quad (1)$$

But the inner sum is just the binomial coefficient $\binom{n-2}{k-1}$, so that

$$Z_P(n) = \sum_{k \geq 1} \binom{n-2}{k-1} \# \text{Ch}_k, \quad (2)$$

which is obviously a polynomial evaluated at n .

The Zeta polynomial is a useful invariant of posets, with nice general properties. By Formula (2), it belongs to the ring of polynomials with rational coefficients taking integer values on \mathbb{N} . The Zeta polynomial is multiplicative with respect to the Cartesian product of posets, additive with respect to disjoint union and invariant under duality. For more on this classical subject, see for instance [24, §3.11].

1 q -analogue

Let us now turn to our proposal for a q -analogue of the Zeta polynomial.

The letter q stands for the indeterminate in $\mathbb{Q}(q)$. We use the standard notation $[n]_q = (q^n - 1)/(q - 1)$ for the q -analogue of the integer $n \in \mathbb{Z}$, which is a polynomial in q if $n \geq 0$ and a Laurent polynomial otherwise. We will denote by $[n]!_q$ the q -factorial of n when $n \geq 0$. Note that $[0]_q = 0$ and $[1]_q = 1$.

Let P be a finite poset. We need the additional data of a height function $h : P \rightarrow \mathbb{N}$ such that $h(x) < h(y)$ for every cover relation $x < y$ in P .

A poset P is *graded* if there exists a height function that increase by 1 along every cover relation. Then there is a preferred choice for such a height function h , by assuming further that it has minimal value 0 on every connected component of P . This specific height function will be denoted by rk .

Every poset P can be endowed with a height function by choosing an arbitrary linear extension and using it as a height function.

Note that some posets that are not graded are nevertheless naturally endowed with natural height functions, for example the Tamari lattices using their description as posets of tilting modules [25].

For a strict chain $c = (c_1, \dots, c_k)$ in $\text{Ch}_k(P)$, let $h(c)$ denote the sequence of heights $(h(c_1), \dots, h(c_k))$ and let $\sum h(c)$ denote the sum of this sequence. The hypothesis on h ensures that the sequence $h(c)$ is strictly increasing.

For every integer $n \geq 2$, let us define

$$Z_{P,h}([n]_q) = \sum_{e_1 \leq \dots \leq e_{n-1}} q^{\sum_j h(e_j)} = \sum_{k \geq 1} \sum_{c \in \text{Ch}_k(P)} \sum_{\substack{m_1, \dots, m_k \geq 1 \\ \sum_i m_i = n-1}} q^{\sum_i m_i h(c_i)}. \quad (3)$$

For the moment, the left hand side is an abuse of notation, as we will only prove later that the right hand side is indeed the evaluation of a polynomial at $[n]_q$.

The reader can safely but temporarily assume that the left-hand side just means some function of n and q depending of P and h .

This can be rewritten as

$$Z_{P,h}([n]_q) = \sum_{k \geq 1} \sum_{c \in \text{Ch}_k(P)} q^{\sum h(c)} \sum_{\substack{m'_1, \dots, m'_k \geq 0 \\ \sum_i m'_i = n - k - 1}} q^{\sum_i m'_i h(c_i)}. \quad (4)$$

At this point, one can recognize the innermost sum as being essentially the q -Ehrhart polynomial of a simplex. Before proceeding further, let us recall the theory of q -Ehrhart polynomials, whose details can be found in [9].

1.1 q -Ehrhart polynomials and reciprocity

Fix a lattice polytope Q and an integral linear form ℓ on the ambient lattice. Assume that ℓ is not constant on any edge of Q and takes values in \mathbb{N} on Q . Then there exists a unique polynomial $\text{Ehr}_{Q,\ell}(x) \in \mathbb{Q}(q)[x]$ such that

$$\text{Ehr}_{Q,\ell}([n]_q) = \sum_{z \in nQ} q^{\ell(z)}, \quad (5)$$

for every integer $n \geq 0$. In words, this is counting lattice points in the dilates of Q according to the value of ℓ on each lattice point. By the formula above, the polynomial $\text{Ehr}_{Q,\ell}$ belongs to the ring of polynomials whose value at every q -integer $[n]_q$ with $n \geq 0$ is a polynomial in q with positive integer coefficients. This is called the q -Ehrhart polynomial of the lattice polytope Q with respect to the linear form ℓ . Note that setting $q = 1$ recovers the classical Ehrhart polynomial of lattice polytopes, not depending on the linear form ℓ . The degree of $\text{Ehr}_{Q,\ell}$ is the maximum value of ℓ on Q .

Let us also recall Ehrhart reciprocity in this setting. Let $n \geq 1$ be an integer. Then the evaluation of $\text{Ehr}_{Q,\ell}$ at $[-n]_q$ is given up to sign by the similar sum over interior points in the dilates of Q :

$$\text{Ehr}_{Q,\ell}([-n]_q) = (-1)^d \sum_{z \in \text{Int}(nQ)} q^{-\ell(z)}, \quad (6)$$

where d is the dimension of the polytope Q and Int denotes the interior of a polytope. Note that the interior of a 0-dimensional polytope (a point) is just itself.

For $k \geq 1$, the lattice polytope in \mathbb{N}^k whose vertices are the basis vectors in \mathbb{N}^k will be called the standard basic simplex.

For every k -tuple a of distinct elements of \mathbb{N} , let us denote by \mathbf{E}_a the q -Ehrhart polynomial of the standard basic simplex in \mathbb{N}^k with respect to the linear form given by the standard scalar product with a . The degree of \mathbf{E}_a is the maximal element of a . At $q = 1$, \mathbf{E}_a becomes the Ehrhart polynomial of the standard basic simplex, namely $\binom{x+k-1}{k-1}$, no longer depending on a .

For example, for the tuple $a = (1, 2, 3)$, one gets

$$\mathbf{E}_a = ((q-1)x+1) \cdot (qx+1) \cdot \frac{(q^2x+q+1)}{q+1}.$$

1.2 Definition of q -Zeta polynomial

Let us go back to our proposed definition (4) for $Z_{P,h}$. One therefore finds that

$$Z_{P,h}([n]_q) = \sum_{k \geq 1} \sum_{c \in \text{Ch}_k(P)} q^{\sum h(c)} E_{h(c)}([n-k-1]_q). \quad (7)$$

Here one has to be a little cautious about the substitution of the inner summation in (4) by the q -Ehrhart polynomial $E_{h(c)}$. This is a priori allowed only if $n \geq k+1$. On the one hand, if $2 \leq n \leq k$, then the inner summation in (4) vanishes because it runs over an empty set. On the other hand, by Ehrhart reciprocity, the value $E_{h(c)}([-d]_q)$ vanishes when $1 \leq d \leq k-1$, because in this case the d -th dilate of the standard basic simplex with k vertices has empty interior.

By the appropriate shift of variables relating $[n]_q$ and $[n-k-1]_q$, one finally reaches the following definition, consistent with all previous formulas.

Definition 1. *The q -Zeta polynomial $Z_{P,h}$ of a finite poset P with respect to the height function h is given by*

$$Z_{P,h}(x) = \sum_{k \geq 1} \sum_{c \in \text{Ch}_k(P)} q^{\sum h(c)} E_{h(c)} \left(\frac{x - [k+1]_q}{q^{k+1}} \right). \quad (8)$$

This polynomial is an element of the ring $\mathbb{Q}(q)[x]$. Its degree is the maximal value of the height function h on P . It belongs to the sub-ring of polynomials whose values at every q -integer $[n]_q$ with $n \geq 2$ is a polynomial in q with positive integer coefficients.

The first and second properties are clear from Formula (8) and the general properties of the q -Ehrhart polynomials. The third property follows directly from (4), which holds by construction, as well as (3).

In other words, by (3), one has the following statement:

Lemma 1.1. *The values of $Z_{P,h}$ at q -integers $[n]_q$ for $n \geq 2$ are q -analogues of the numbers of chains $e_1 \leq \dots \leq e_{n-1}$ in P , where the power of q is the sum of the heights of elements in the chain.*

As expected, the polynomial $Z_{P,h}$ deserves the name of q -analogue of the Zeta polynomial.

Lemma 1.2. *For any height function h , the specialisation of $Z_{P,h}$ at $q = 1$ is the usual Zeta polynomial Z_P of the poset P .*

Proof. This follows by comparing (1) and (3). □

1.3 Examples

Let us now give a few examples, using the height function rk coming from the grading of the posets, unless stated otherwise.

Example 1.3. *For the unique poset \circ with one element with respect to rk , the q -Zeta polynomial is 1. More generally, for the height function on \circ with value $H \in \mathbb{N}$, the value of the q -Zeta polynomial at $[n]_q$ is $q^{(n-1)H}$ and therefore*

$$Z_{\circ,H}(x) = \left(\frac{1 + (q-1)x}{q} \right)^H. \quad (9)$$

Example 1.4. For the total order on $d \geq 2$ elements, one finds

$$\frac{\prod_{j=0}^{d-2} ([j]_q + q^j x)}{[d-1]_q!}. \quad (10)$$

This follows from the identification of the value at $[n]_q$, using the Formula (3) as a weighted sum over all chains, with the standard q -binomial coefficient counting lattice paths in a $(n-1) \times (d-1)$ rectangle according to the area below.

Example 1.5. For the graded poset on 5 elements with one minimum $\hat{0}$, one maximum $\hat{1}$ and 3 pairwise incomparable elements in between, one finds

$$\frac{x \cdot ((q+2)x - 1)}{q+1}. \quad (11)$$

This was found using a computer. By hand, it can be computed by interpolation, as the degree is known to be 2. This is made easier by the statements about the values at $[0]_q$ and $[1]_q$ given below in Lemma 2.3 and Lemma 2.2.

Example 1.6. Let us also consider the poset on $\{a, b, c, d\}$ where a and b are both smaller than both c and d . One obtains

$$\frac{2(q+1)}{q}(x-1), \quad (12)$$

whose value at $[1]_q$ is 0.

Example 1.7. For the poset on $\{a, b, c\}$ where a is smaller than b and c , one obtains

$$2x - 1, \quad (13)$$

whereas for the dual poset one gets

$$\frac{((q+1)x - 1)}{q}. \quad (14)$$

Example 1.8. For the poset on $\{a, b, c, d, e\}$ where $a \leq b, a \leq c, b \leq d, c \leq e$, one obtains

$$\frac{2qx^2 + 2x - q - 1}{q+1}. \quad (15)$$

2 Basic properties

Let us present in this section a few basic properties of $Z_{P,h}$.

Let P and Q be two posets with height functions g and h . Consider $P \times Q$ with the height function $g+h$ sending (a, b) to $g(a) + h(b)$. Consider also $P \sqcup Q$ with the height function $g \sqcup h$ defined by g on P and h on Q .

Lemma 2.1. With the notations above, $Z_{P \times Q, g+h} = Z_{P,g} Z_{Q,h}$ and $Z_{P \sqcup Q, g \sqcup h} = Z_{P,g} + Z_{Q,h}$.

Proof. This is most easily seen using Formula (3).

A chain e in the Cartesian product $P \times Q$ is the same as a pair of chains (e', e'') in P and Q . The sum over e of the heights in $P \times Q$ is that of e' in P plus that of e'' in Q .

For the disjoint union $P \sqcup Q$, chains are either entirely in P or entirely in Q and the result follows. \square

For a poset P , let χ_P denote the Euler characteristic of the order complex of P , which is the simplicial complex made of strict chains. Recall that if P has a unique minimum or maximum, the order complex is contractible.

Lemma 2.2. *The value of $Z_{P,h}$ at $[1]_q$ is χ_P . In particular, if P has a unique minimum or maximum, this is 1.*

Proof. Let us first compute $E_{h(c)}([-k]_q)$ for $k \geq 1$ and $c \in \text{Ch}_k(P)$. By Ehrhart reciprocity, $E_{h(c)}([-k]_q) = (-1)^{k-1} q^{-\sum h(c)}$ because the only interior point in the k -th dilate of the standard basic simplex is $(1, \dots, 1)$. Then using Formula (7), one finds

$$Z_{P,h}([1]_q) = \sum_{k \geq 1} (-1)^{k-1} \# \text{Ch}_k(P).$$

This is exactly the expected Euler characteristic. \square

Lemma 2.3. *Assuming that P has a unique minimum $\hat{0}$, the value of $Z_{P,h}$ at $[0]_q$ is $q^{-h(\hat{0})}(1 - \chi_{P \setminus \hat{0}})$. In particular, if P also has a unique maximum $\hat{1}$ distinct from $\hat{0}$, this is 0.*

Proof. The special case when P has only one element is clear, see Example 1.3.

Assume now that P has at least 2 elements. First one can show that

$$E_{h(c)}([-k-1]_q) = (-1)^{k-1} q^{-\sum h(c)} \sum_i q^{-h(c_i)},$$

using Ehrhart reciprocity, for $k \geq 1$ and $c \in \text{Ch}_k(P)$. Therefore one finds

$$Z_{P,h}([0]_q) = \sum_{k \geq 1} (-1)^{k-1} \sum_{c \in \text{Ch}_k(P)} \sum_i q^{-h(c_i)}.$$

In this sum, the strict chain reduced to $\hat{0}$ contributes the term $q^{-h(\hat{0})}$.

On the remaining chains, removing or adding $\hat{0}$ define bijections exchanging strict chains with $k+1$ elements containing $\hat{0}$ and strict chains with k elements not containing $\hat{0}$. The contributions of such a pair of chains to the previous sum almost cancel mutually, except one summand $q^{-h(\hat{0})}$ in the inner sum, contributing

$$\sum_{k \geq 1} (-1)^k \sum_{c \in \text{Ch}_k(P \setminus \hat{0})} q^{-h(\hat{0})},$$

in which one can recognize $-q^{-h(\hat{0})} \chi_{P \setminus \hat{0}}$. \square

Recall that a poset is **bounded** if it has unique minimum $\hat{0}$ and unique maximum $\hat{1}$.

Lemma 2.4. *If the poset P is bounded, then*

$$Z_{P,h}([-1]_q) = q^{-h(\hat{0})-h(\hat{1})} \mu_P(\hat{0}, \hat{1}), \quad (16)$$

where μ_P is the usual Möbius function of the poset.

Proof. The proof is very similar to that of the previous lemmas.

The special case when P has only one element is clear, see Example 1.3. One can therefore assume that $\hat{0} \neq \hat{1}$.

Using Ehrhart reciprocity to evaluate $E_{h(c)}$ at $[-k-2]_q$ for $c \in \text{Ch}_k(P)$, one gets the formula

$$\sum_{k \geq 1} (-1)^{k-1} \sum_{c \in \text{Ch}_k(P)} \sum_i \left(q^{-2h(c_i)} + \sum_{i \neq j} q^{-h(c_i) - h(c_j)} \right).$$

Then one first separates strict chains according to whether they start by $\hat{0}$ or not. Contributions of pairs of chains almost cancel. In the remaining sum, one separates strict chains according to whether they end by $\hat{1}$ or not. Once again, there are cancellations by pairs. There remains only

$$q^{-h(\hat{0}) - h(\hat{1})} \left(-1 + \sum_{k \geq 1} (-1)^{k-1} \# \text{Ch}_k(P') \right),$$

where $\# \text{Ch}_k(P')$ is the number of strict chains $c_1 < c_2 < \dots < c_k$ in the poset $P' = P \setminus \{\hat{0}, \hat{1}\}$. By a classical result, the reduced Euler characteristic between the parentheses is the Möbius number $\mu_P(\hat{0}, \hat{1})$. \square

Let us consider a poset P and its dual poset \bar{P} . To any height function h on P and for any integer H at least equal to the maximal value of h , the function $H - h$ is a height function on \bar{P} . In that case, the q -Zeta polynomial of P w.r.t. h and that of \bar{P} w.r.t. $H - h$ are not directly related in a simple way, but their values are related as follows.

Lemma 2.5. *One has the following relation:*

$$Z_{\bar{P}, H-h}([n]_q) = q^{(n-1)H} (Z_{P,h}([n]_q))|_{q=1/q}, \quad (17)$$

for all $n \in \mathbb{Z}$. Note that the replacement of q by $1/q$ also affects the coefficients of the polynomial.

Proof. For $n \geq 2$, this follows directly from (3). The case $n = 1$ also follows from Lemma 2.2. It is therefore enough to check that the right hand side is the value of a polynomial at the q -integer $[n]_q$. Indeed, let us write

$$Z_{P,h}(x) = \sum_{j=0}^H \kappa_j x^j,$$

as the degree of this polynomial is at most H . Then the right hand side in the expected relation is

$$\sum_{j=0}^H \kappa_j|_{q=1/q} (q^{n-1})^{H-j} [n]_q^j,$$

which is the value at $[n]_q$ of the polynomial

$$\sum_{j=0}^H \kappa_j|_{q=1/q} \left(\frac{1 + (q-1)x}{q} \right)^{H-j} x^j.$$

\square

See Example 1.7 for a simple case of this relationship between q -Zeta polynomials of dual posets.

Lemma 2.6. *Let P be a poset with height function h . Let $Z_{P,h}(x)$ be the corresponding q -Zeta polynomial. Then the q -Zeta polynomial $Z_{P,h+1}$ for P with the shifted height function $h + 1$ is given by*

$$Z_{P,h+1}(x) = \frac{1}{q}(1 + (q - 1)x) \cdot Z_{P,h}(x). \quad (18)$$

Proof. Using Formula (3) for the height function $h + 1$ gives directly

$$Z_{P,h+1}([n]_q) = q^{n-1}Z_{P,h}([n]_q),$$

from which the result follows. \square

Lemma 2.7. *Let P be a poset with height function h . Let $Z_{P,h}(x)$ be the corresponding q -Zeta polynomial. Let $D \geq 1$ be an integer. Then the q -Zeta polynomial $Z_{P,Dh}$ for P with the scaled height function Dh is given by*

$$Z_{P,Dh}(x) = Z_{P,h} \Big|_{q=q^D} \left(\frac{(1 + (q - 1)x)^D - 1}{q^D - 1} \right). \quad (19)$$

Proof. From (3), it follows that $Z_{P,Dh}([n]_q) = Z_{P,h}|_{q=q^D}([n]_{q^D})$ for $n \geq 2$. Then expressing the argument $[n]_{q^D}$ using $[n]_q$ gives the formula. \square

This property can be seen in Example 1.3.

Recall that the coefficients of the flag f -vector of a graded poset P are the numbers of strict chains c with a fixed sequence of ranks $\text{rk}(c)$. For more on this, see [5] and [24, §3.12].

Lemma 2.8. *Let P be a graded poset and consider the height function rk . The polynomial $Z_{P,\text{rk}}$ is entirely determined by the flag f -vector of P .*

Proof. It follows from Formula (7) that $Z_{P,\text{rk}}$ is a linear combination of q -Ehrhart polynomials E_a whose coefficients are exactly coefficients of the flag f -vector. \square

Note: one can wonder about the converse implication. Most probably, the flag f -vector should be a finer invariant than the q -Zeta polynomial.

3 Relation with the q -order polynomial

The order polytope Q_P of a poset P [21] is the lattice polytope in \mathbb{N}^P defined by inequalities $0 \leq z_p \leq 1$ for all $p \in P$ and $z_p \leq z_q$ if $p \leq q$ in P . The application $(z_p)_p \mapsto (1 - z_p)_p$ is a bijection between Q_P and $Q_{\overline{P}}$ for the dual poset \overline{P} .

Let us recall the q -order polynomial L_P of a poset P as introduced in [9]. This is the q -Ehrhart polynomial of the order polytope Q_P , with respect to the linear form $(1, 1, \dots, 1)$, *i.e.* the sum of coordinates. The value of L_P at $[n]_q$ for $n \geq 0$ is therefore

$$L_P([n]_q) = \sum_{z \in nQ_P} q^{\sum_p z_p}. \quad (20)$$

The degree of L_P is $\#P$.

Let P be a finite poset. Let $J(P)$ be the distributive lattice of lower ideals in P under the order of containment. This lattice is graded by the cardinality of the lower ideal. Fix $n \geq 2$, and consider a chain $e_1 \leq \dots \leq e_{n-1}$ of elements of $J(P)$. This is an increasing chain of lower ideals of P . As such, it is characterised by the following data : for each element p of P , let z_p be the smallest integer between 0 and $n - 2$ such that $p \in e_{z_p+1}$ if it exists and $n - 1$ otherwise. Then these vectors $(z_p)_{p \in P}$ satisfy that $z_p \leq z_q$ if $p \leq q$ in P . This map defines a bijection between chains $e_1 \leq \dots \leq e_{n-1}$ in $J(P)$ and lattice points in the $(n - 1)$ -dilate of the order polytope Q_P .

Proposition 3.1. *For every poset P , one has $Z_{J(P), \text{rk}}(1 + qx) = L_{\overline{P}}(x)$.*

Proof. Let us compare the values at $x = [n - 1]_q$ for $n \geq 2$. Let us compute the monomial in q attached in $Z_{J(P), \text{rk}}(1 + qx) = Z_{J(P), \text{rk}}([n]_q)$ to one chain

$$e_1 \leq \dots \leq e_{n-1}$$

in terms of the corresponding lattice point $(z_p)_{p \in P}$ in $(n - 1)Q_P$. For the chain, the monomial is q to the power $\sum_i \#e_i = \sum_i \sum_{p \in e_i} 1$. By the bijection between chains and lattice points, the exponent of q becomes $\sum_{p \in P} (n - 1 - z_p)$ where $z \in (n - 1)Q_P$. Using the bijection $(z_p)_p \mapsto (1 - z_p)_p$ between Q_P and $Q_{\overline{P}}$, the exponent of q becomes $\sum_{p \in \overline{P}} z_p$ where $z \in (n - 1)Q_{\overline{P}}$. Summing these monomials over z gives exactly the value at $[n - 1]_q$ of the q -order polynomial of \overline{P} . \square

For example, let P be the poset with three elements a, b and c , such that a is less than b and c . Then the q -Ehrhart polynomial of \overline{P} is

$$\frac{1}{[2]_q [3]_q} \cdot (qx + 1) \cdot (q^2x + q + 1) \cdot ((q^3 + q^2)x + q^2 + q + 1),$$

and the q -Zeta polynomial of $J(P)$ is

$$\frac{1}{[2]_q [3]_q} \cdot x \cdot (qx + 1) \cdot ((q^2 + q)x + 1).$$

4 Ehrhart series and volumes

4.1 P -partitions and q -Ehrhart series for posets

Let us first recall the following classical setting, part of the more general theory of P -partitions, due to Stanley [19, §8].

The q -Ehrhart series of any poset P is the formal power series defined by

$$\mathcal{L}_P = \sum_{n \geq 0} L_P([n]_q) t^n, \quad (21)$$

and can be expressed as a rational fraction

$$\mathcal{L}_P = \frac{\mathbb{H}_P(q, t)}{(1 - t)(1 - qt) \dots (1 - q^{\#P}t)}, \quad (22)$$

where \mathbb{H}_P is a polynomial in q and t with non-negative integer coefficients. This polynomial has a known combinatorial interpretation, using descents for t and major indices for q , as a sum over all linear extensions of the poset P .

Let us also introduce the q -volume of a poset, as defined in [9, §4.2].

Definition 2. The q -volume of a poset P is the leading coefficient of the q -Ehrhart polynomial \mathbb{L}_P times the q -factorial $[\#P]!_q$.

Proposition 4.9 in [9] gives the following relationship between the polynomial $\mathbb{H}_{\overline{P}}$ and the q -volume of P .

Proposition 4.1. The q -volume of P is equal to $q^{\binom{\#P+1}{2}}$ times $\mathbb{H}_{\overline{P}}(1/q, 1)$.

4.2 q -Ehrhart series for q -Zeta polynomials

Inspired by Proposition 3.1 which identifies the q -Zeta polynomials of distributive lattices with q -order polynomials of their posets of join-irreducibles, it is natural to extend the constructions of the previous paragraph to the general case, for the q -Zeta polynomials of arbitrary posets.

Let (P, h) be a poset endowed with a height function h . In this section, H will denote the maximal value of h .

Let us consider the generating series of values of $Z_{P,h}$:

$$\mathcal{Z}_{P,h} = \sum_{n \geq 0} Z_{P,h}([n+1]_q) t^n, \quad (23)$$

similar to the q -Ehrhart series (21).

Proposition 4.2. This series can be expressed as a rational fraction:

$$\mathcal{Z}_{P,h} = \frac{\mathbb{H}_{P,h}(q, t)}{(1-t)(1-qt) \dots (1-q^H t)}, \quad (24)$$

where \mathbb{H}_P is a polynomial in q, q^{-1} and t with integer coefficients. The degree of $\mathbb{H}_{P,h}$ with respect to t is at most H .

Proof. Let us write Z for $Z_{P,h}$. Consider the polynomial $Z(1+qx)$. By Equation (3) and Lemma 2.2, its value at every $[n]_q$ for $n \geq 0$ is a polynomial in q with integer coefficients. By Proposition B.2, it can therefore be expressed as a sum

$$Z(1+qx) = \sum_{0 \leq j \leq H} c_j B_j(x), \quad (25)$$

where $c_j \in \mathbb{Z}[q, q^{-1}]$ and B_j are polynomials defined in Appendix B by (34). Then one concludes using the generating series from Lemma B.5. \square

By analogy with Definition 2, let us introduce the volume associated with the q -Zeta polynomial.

Definition 3. The q -Zeta volume of a poset P with respect to the height function h is the leading coefficient of the q -Zeta polynomial $Z_{P,h}$ times the q -factorial $[H]!_q$.

In this context, one has the following general property, for arbitrary posets. Let us choose $H-h$ as the height function on the dual poset \overline{P} .

Proposition 4.3. The q -Zeta volume of P w.r.t. h is $q^{\binom{H}{2}}$ times $\mathbb{H}_{\overline{P}, H-h}(1/q, 1)$.

Proof. The proof is similar to the proof of [9, Prop. 4.9]. Let us write

$$\mathcal{Z}_{P,h} = \frac{\sum_{j=0}^H h_j t^j}{\prod_{\ell=0}^H 1 - q^\ell t}, \quad (26)$$

for some coefficients h_j in $\mathbb{Z}[q, q^{-1}]$.

By Lemma 2.5 translated into generating series using (23), one has the equality

$$\mathbb{H}_{\overline{P}, H-h}(q, t) = \mathbb{H}_{P,h}(1/q, q^H t).$$

After replacing q by $1/q$ in the evaluation at $t = 1$, one gets

$$\mathbb{H}_{\overline{P}, H-h}(1/q, 1) = \sum_{j=0}^H h_j q^{-jH}.$$

On the other hand, one can compute directly the coefficient of x^H in $Z_{P,h}(1+qx)$ from (26). For the term $j = 0$, one can use Lemma B.5 and (34) that gives the corresponding coefficient of x^H explicitly as

$$\frac{1}{[H]!_q} \prod_{j=1}^H q^j = \frac{q^{\binom{H+1}{2}}}{[H]!_q}.$$

For the terms of index $j \geq 1$, one notes that the product by t in the series amounts to replace x by $(x-1)/q$ in the polynomial coefficient, which multiplies the leading coefficient of x^H by q^{-H} . The total leading coefficient of $Z_{P,h}(1+x)$ is therefore given by

$$\left(\sum_{j=0}^H h_j q^{-jH} \right) \frac{q^{\binom{H+1}{2}}}{[H]!_q}.$$

For the leading coefficient of $Z_{P,h}(x)$, one has moreover to divide by q^H .

One concludes by comparing the obtained expressions. \square

For example, for the self-dual poset of Example 1.5 with $H = 2$, the series is

$$\mathcal{Z}_{P,\text{rk}} = \frac{1 + 2qt}{(1-t)(1-qt)(1-q^2t)}$$

and the q -volume is $q + 2$.

5 Positivity properties

By Proposition 3.1 and the discussion in §4.1, when P is a distributive lattice, then $\mathbb{H}_{P,\text{rk}}$ is a polynomial in q and t with non-negative integer coefficients.

This positivity property of $\mathbb{H}_{P,\text{rk}}$ is not true for all posets. Small counterexamples are the posets of Example 1.7 and Example 1.8.

In the case of Example 1.8 with $H = 2$, one gets the numerator

$$-q^3 t^2 + (2 + q + q^2) t - 1.$$

In this section, we will give a sufficient criterion for positivity of $\mathbb{H}_{P,\text{rk}}$, in terms of the existence of an R -labelling.

5.1 R -labellings

Let L be a set of labels, endowed with an arbitrary relation denoted by $\overset{\heartsuit}{\prec}$. For $p < q$ in P , a maximal chain $p = e_0 < e_1 < \dots < e_{n+1} = q$ is *increasing*¹ if $\lambda(e_{i-1}, e_i) \overset{\heartsuit}{\prec} \lambda(e_i, e_{i+1})$ for all $1 \leq i \leq n$.

An R -labelling of a poset P by $(L, \overset{\heartsuit}{\prec})$ is a function λ from the set of edges of the Hasse diagram of P to L such that:

- for every pair of comparable elements $p \leq q$ in P , there is exactly one increasing maximal chain from p to q .

For example, the weak order on the symmetric group S_3 has no R -labelling, as both maximal chains from the minimum to the maximum are necessarily increasing.

Remark 5.1. *Every EL -labelling, as defined in [6], is also an R -labelling, as it satisfies a stronger condition. Supersolvable lattices and upper-semimodular lattices always have an R -labelling. This is proved in [24, Examples 3.13.4 and 3.13.5].*

5.2 Positivity criterion

Proposition 5.2. *If P is a bounded and graded poset that admits an R -labelling, then $\mathbb{H}_{P, \text{rk}}$ is a polynomial with positive integer coefficients.*

Proof. The proof follows from Theorem A.1 that gives positivity of the flag h -vector under the given hypothesis and the fact that the coefficient of $\mathbb{H}_{P, \text{rk}}$ are non-negative linear combinations of the flag h -vector elements, as proved in Theorem A.2. \square

Lemma 5.3. *If P is a bounded and graded poset that admits an R -labelling, then the q -Zeta volume of $Z_{P, \text{rk}}$ is a non-negative q -analogue of the number of maximal chains in P .*

Proof. This follows from Proposition 5.2 and Proposition 4.3. The relationship between the q -Zeta volume and $\mathbb{H}_{P, \text{rk}}$ involves the dual poset P , but the set of maximal chains is preserved by duality. \square

Proposition 5.2 above applies to several classical posets attached to finite Coxeter groups. The first ones are non-crossing partitions lattices issued from the absolute order [8, 7, 4, 13] and shard-intersection orders [15, 16]. These two families are known to be EL -shellable by results of [1, 14].

The intersection lattice of an essential central hyperplane arrangement is a geometric lattice and is therefore EL -shellable [6, Th. 3.1]. This applies in particular to posets of generalized set partitions defined as intersection lattices of reflection hyperplane arrangements.

Hence, all these posets have an R -labelling and non-negative $\mathbb{H}_{P, \text{rk}}$.

¹This terminology comes from the case where $\overset{\heartsuit}{\prec}$ is a partial order relation, which is not assumed here. One may say concatenable or friendly for a better intuition.

There is a less well-known family of posets attached to finite Coxeter groups, the parabolic-support posets [3, 2]. Every interval in this family is shellable and upper-semimodular, hence has an R -labelling [3]. But these posets are not bounded, hence the criterion does not apply to the full posets. Positivity of $\mathbb{H}_{P,\text{rk}}$ seems nevertheless to hold for the parabolic-support posets. This remains to be proved and explained.

Another interesting example is given by the root posets of type B , which are not bounded posets either. One can endow these posets with an R -labelling according to the direction (North-East or North-West) of the cover relations. It seems that the coefficients of powers t^k in $\mathbb{H}_{P,\text{rk}}$ are non-negative q -analogues of $\binom{n}{k}^2$, the Narayana numbers of type B .

One can wonder what would be a necessary and sufficient condition on posets for positivity of the coefficients of $\mathbb{H}_{P,\text{rk}}$.

6 Value at $q = 0$ and characteristic polynomial

Assume in this section that P is bounded and graded with a unique minimal element $\hat{0}$ and a unique maximal element $\hat{1}$. Let H be the maximal value of the rank function rk on P .

The characteristic polynomial of P is defined as

$$X_P(y) = \sum_{p \in P} \mu(\hat{0}, p) y^{H - \text{rk}(p)}. \quad (27)$$

We will use the notions of flag f -vector and flag h -vector recalled in Appendix A and the notations introduced there.

The characteristic polynomial can be expressed using the flag h -vector as follows.

Lemma 6.1. *For every graded and bounded poset P ,*

$$y^H X_P(1/y) = 1 + \sum_{j=1}^H (\beta_P([1, \dots, j-1]) + \beta_P([1, \dots, j])) (-y)^j. \quad (28)$$

Proof. The polynomial on the left is

$$1 + \sum_{j=1}^H \left(\sum_{\substack{p \in P \\ \text{rk}(p)=j}} \mu(\hat{0}, p) \right) y^j.$$

Then, for $j \geq 1$, the coefficient of y^j is the difference

$$\sum_{\substack{p \in P \\ \text{rk}(p) \leq j}} \mu(\hat{0}, p) - \sum_{\substack{p \in P \\ \text{rk}(p) \leq j-1}} \mu(\hat{0}, p),$$

in which each sum is minus the Möbius number μ_S of a rank-selected sub-poset, for the sets $S = [1, \dots, j]$ and $S = [1, \dots, j-1]$. The last step is then to use the relationship $\mu_S = (-1)^{\#S-1} \beta_P(S)$ ([24, 3.12]). \square

Theorem 6.2. *The q -Zeta polynomial $Z_{P,\text{rk}}$ has no pole at $q = 0$ and its value at $q = 0$ is related to the characteristic polynomial by*

$$Z_{P,\text{rk}}|_{q=0}(1-y) = y^H X_P(1/y). \quad (29)$$

Remark 6.3. *It can well happen, when the poset P has no unique minimum, that $Z_{P,h}$ has poles at $q = 0$, for instance for the poset of Example 1.6 and the dual poset in Example 1.7.*

Proof. Let us denote $Z_{P,\text{rk}}$ by Z . Let us start with Theorem A.2. One deduces that

$$t\mathbb{H}_{P,\text{rk}}(q,t) = \sum_{S \subseteq \{1,\dots,H-1\}} \beta_P(S) t^{\#S+1} q^{\Sigma S}.$$

As P is bounded, $Z(0) = 0$ by Lemma 2.3. Therefore the previous expression is the numerator of the generating series of values of Z for $n \geq 0$. By the correspondence stated in appendix B, this means that

$$Z_{P,\text{rk}}(x) = \sum_{S \subseteq \{1,\dots,H-1\}} \beta_P(S) q^{\Sigma S} \left[\begin{matrix} H-\#S-1 \\ H \end{matrix} ; x \right]_q.$$

One can then use Lemma B.8 to let q be 0 and obtain

$$Z|_{q=0}(x) = \sum_{j=0}^{H-1} \beta_P([1,\dots,j]) x(x-1)^j.$$

Indeed, all sets S that are not formed of consecutive integers starting at 1 appear with a strictly positive power of q , hence vanish when $q = 0$. One deduces

$$Z|_{q=0}(1-y) = \sum_{j=0}^{H-1} \beta_P([1,\dots,j]) (1-y)(-y)^j.$$

One can then conclude by an easy comparison with Lemma 6.1. \square

7 Reciprocity for Eulerian posets

Recall from [24, §3.14] that a finite graded poset P with $\hat{0}$ and $\hat{1}$ is Eulerian if its Möbius function μ is given for all $x \leq y$ in P by $\mu(x,y) = (-1)^{\text{rk}(x,y)}$ where rk is the rank function of P .

The following proposition is a q -analogue of a classical reciprocity property for Zeta polynomials of Eulerian posets, see [23, §2] and [24, Prop. 3.14.1].

We will use in the proof the notations of Appendix C and work in $\text{Inc}_q(P, \text{rk})$.

Proposition 7.1. *Let P be an Eulerian poset endowed with its rank function rk and let n be the rank of $\hat{1}$. Then*

$$Z(x) = (-q)^n Z|_{q=1/q}(-qx).$$

Proof. It is enough to prove the equality when evaluated at any negative q -integer $[-m]_q$ for $m \geq 2$. This becomes

$$Z([-m]_q) = (-q)^n (Z([m]_q))|_{q=1/q}. \quad (30)$$

Recall from Appendix C the standard Zeta matrix Z of the poset and the diagonal matrix D_{rk} whose coefficients are $q^{\text{rk}(e)}$ for $e \in P$. Let M be the standard Möbius matrix defined by $M = Z^{-1}$.

By the description given in Appendix C, the left hand side of (30) is the corner coefficient in the inverse in $\text{Inc}_q(P, \text{rk})$ of the matrix

$$Z(D_{\text{rk}}Z)^{m-1},$$

which is

$$D_{\text{rk}}^{-1}(MD_{\text{rk}}^{-1})^m.$$

The corner coefficient is therefore the sum over all paths $\hat{0} = e_0 \leq e_1 \leq e_2 \leq \dots \leq e_m = \hat{1}$ of

$$q^0 \mu(\hat{0}, e_1) q^{-\text{rk}(e_1)} \mu(e_1, e_2) q^{-\text{rk}(e_2)} \dots q^{-\text{rk}(e_{m-1})} \mu(e_{m-1}, \hat{1}) q^{-n}.$$

Using that the poset is Eulerian and cancelling signs by pairs except the right-most one, this is the same as $(-q)^{-n}$ times the sum over the same set of paths of

$$q^{-\text{rk}(e_1)} q^{-\text{rk}(e_2)} \dots q^{-\text{rk}(e_{m-1})}.$$

This last sum is exactly the value $Z([m]_q$ in which q was replaced by $1/q$. \square

This proposition easily translates into properties of every individual coefficient. For example, for the the face lattice of the icosahedron, where $n = 4$, the q -Zeta polynomial is the quotient of

$$(q^4 + 17q^3 - 6q^2 + 17q + 1)x^4 - 16(q^3 - 2q^2 + 2q - 1)x^3 - 24(q^2 - q + 1)x^2 - 8(q - 1)x$$

by $(q^2 + 1)(q^2 + q + 1)$.

Similarly, for the associahedron of dimension 3 with 14 vertices, the q -Zeta polynomial is the quotient of

$$(q^4 + 6q^3 + 7q^2 + 6q + 1)x^4 - 5(q^3 + q^2 - q - 1)x^3 - 15qx^2 + 5(q - 1)x$$

by $(q^2 + 1)(q^2 + q + 1)$.

A Flag f -vectors and h -vectors

Let us recall the standard definitions of flag f -vectors and flag h -vectors and then state theorems about their relationship with maximal chains. A standard reference on the subject is [24, §3.12].

Let P be a graded and bounded poset with unique minimum $\hat{0}$ and unique maximum $\hat{1}$. Let rk be the rank function on P , with minimal value 0 on $\hat{0}$ and maximal value H on $\hat{1}$.

For a subset $S \subseteq \{1, 2, \dots, H - 1\}$ of cardinality k , let $\alpha_P(S)$ be the number of chains $\hat{0} < p_1 < p_2 < \dots < p_k < \hat{1}$ in P such that $S = \{\text{rk}(p_1), \dots, \text{rk}(p_k)\}$.

The **flag f -vector** of P is the collection of numbers $\alpha_P(S)$, indexed by subsets S of $\{1, 2, \dots, H - 1\}$.

The **flag h -vector** of P is the collection of numbers $\beta_P(S)$, also indexed by subsets of $\{1, 2, \dots, H - 1\}$, defined by Möbius inversion as follows:

$$\beta_P(S) = \sum_{T \subseteq S} (-1)^{\#S - \#T} \alpha_P(T) \quad \text{and conversely} \quad \alpha_P(S) = \sum_{T \subseteq S} \beta_P(T). \quad (31)$$

Note that the numbers $\beta_P(S)$ are not obviously non-negative, as their definition by Möbius inversion involves signs.

Assume now that P admits an R -labelling λ for the relation $\overset{\circ}{\leftrightarrow}$, as defined in §5.1. To each maximal chain $M : \hat{0} = p_0 < p_1 < p_2 < \dots < p_H = \hat{1}$ in P , one can associate its descent set²

$$D(M) = \{i \in \{1, \dots, H-1\} \mid \text{not } \lambda(p_{i-1}, p_i) \overset{\circ}{\leftrightarrow} \lambda(p_i, p_{i+1})\}. \quad (32)$$

Theorem A.1 (Björner and Stanley). *Let P be a bounded and graded poset with an R -labelling. Let $S \subseteq \{1, 2, \dots, H-1\}$ be any subset the set of heights. The number $\alpha_P(S)$ counts maximal chains in P with descent set contained in S . The number $\beta_P(S)$ counts maximal chains in P with descent set S and is therefore non-negative.*

For the proof, see [24, §3.13] or [6, Th. 2.7]. One can check that the proof of [24, Th. 3.13.2] works *verbatim* without the hypothesis that $\overset{\circ}{\leftrightarrow}$ is a partial order relation.

Some interesting information about this statement can be found in [12, §8.1].

There is a simple relationship between the flag h -vector and the numerator of the q -Ehrhart series as defined in 4.2.

Theorem A.2. *The polynomial $\mathbb{H}_{P, \text{rk}}$ is the sum*

$$\sum_{S \subseteq \{1, \dots, H-1\}} \beta_P(S) t^{\#S} q^{\Sigma(S)} \quad (33)$$

and has therefore non-negative coefficients.

Proof. Introduce formal variables u_0, u_1, \dots, u_H . For a chain K in $P \setminus \{\hat{0}, \hat{1}\}$, let u_K be the product of $u_{\text{rk}(p)}$ over elements p in the chain. For a subset $S \subseteq \{1, 2, \dots, H-1\}$, let u_S be the product of u_i over elements of S .

According to [24, Ex. 3.67(b)], one has the formula

$$\sum_K u_K = \frac{\sum_S \beta_P(S) u_S}{\prod_{\ell=1}^{H-1} 1 - u_\ell},$$

where the sum over K is running over all chains in $P \setminus \{\hat{0}, \hat{1}\}$.

By extending the sum over K to all chains in P , one gets

$$\sum_K u_K = \frac{\sum_S \beta_P(S) u_S}{\prod_{\ell=0}^H 1 - u_\ell},$$

with just two additional factors in the denominator accounting for the multiplicities of $\hat{0}$ and $\hat{1}$ in the chain.

By specializing every u_ℓ to $q^\ell t$, one gets the equality

$$\sum_{n \geq 0} Z_{P, \text{rk}}([n+1]_q) t^n = \frac{\sum_S \beta_P(S) t^{\#S} q^{\Sigma(S)}}{\prod_{\ell=0}^H 1 - q^\ell t}.$$

Comparing with the definition of $\mathbb{H}_{P, \text{rk}}$ as a numerator in Proposition 4.2, one obtains the expected formula. \square

²Again, the terminology comes from the case where $\overset{\circ}{\leftrightarrow}$ is a partial order.

B Rings of polynomials with integer Laurent values

Let A_q be the sub-ring of $\mathbb{Q}(q)[x]$ made of polynomials P such that $P([n]_q) \in \mathbb{Z}[q, q^{-1}]$ for all $n \geq 0$. This is an analogue of the ring of integer-valued polynomials.

For $k \geq 0$, let B_k be the polynomial defined by

$$B_k(x) = \frac{\prod_{j=1}^k [j]_q + q^j x}{[k]!_q}. \quad (34)$$

Because the values $B_k([n]_q)$ for $n \geq 0$ are standard q -binomial coefficients, every B_k belongs to A_q . We will use the convention that $B_{-1} = 0$.

Let Δ_q be the operator acting on polynomials in x defined by

$$\Delta_q(P)(x) = \frac{P(x) - P((x-1)/q)}{1 + (q-1)x}. \quad (35)$$

The right hand side is well defined as a polynomial because the numerator has a root at $x = 1/(1-q)$. Assuming moreover that $P \in A_q$, this formula implies that the values $\Delta_q(P)([n]_q)$ for $n \geq 1$ are Laurent polynomials in q with integer coefficients.

Lemma B.1. *The operator Δ_q maps B_k to B_{k-1} for all $k \geq 0$.*

Proof. It is enough to prove that it holds for the value at every q -integer $[n]_q$ for $n \geq 1$. This in turn follows from the classical formula

$$\left(\binom{[n+k]_q}{k}_q - \binom{[n-1+k]_q}{k}_q \right) / q^n = \binom{[n+k-1]_q}{k-1}_q$$

for q -binomial coefficients. \square

Proposition B.2. *The polynomials B_k form a basis over $\mathbb{Z}[q, q^{-1}]$ of the ring A_q .*

Proof. These polynomials are linearly independent, as B_k has degree k . It remains to prove that they span A_q over $\mathbb{Z}[q, q^{-1}]$. The proof is by induction on the degree. This is clear in degree 0 as $B_0 = 1$.

So let P be a polynomial of degree $d > 0$ in A_q . Then $Q = \Delta_q(P)$ is a polynomial with values in $\mathbb{Z}[q, q^{-1}]$ for $n \geq 1$. Moreover, if the leading coefficient of P is $cx^d q^{\binom{d+1}{2}} / [d]!_q$, then the leading coefficient of Q is $cx^{d-1} q^{\binom{d}{2}} / [d-1]!_q$. Iterating d times the operator Δ_q , one reaches a constant polynomial, with values in $\mathbb{Z}[q, q^{-1}]$ for $n \geq d$ and with leading term c . Therefore $c \in \mathbb{Z}[q, q^{-1}]$ and the polynomial $P - cB_d$ belongs to A_q , with degree at most $d-1$. By induction, this difference is a linear combination over $\mathbb{Z}[q, q^{-1}]$ of B_k for $0 \leq k \leq d-1$. Hence P is a linear combination over $\mathbb{Z}[q, q^{-1}]$ of B_k for $0 \leq k \leq d$. \square

Corollary B.3. *Let P be an element of A_q . Then $P([n]_q) \in \mathbb{Z}[q, q^{-1}]$ for all $n \in \mathbb{Z}$. The operator Δ_q acts on A_q .*

Proof. The first statement holds because this is true for all basis elements B_k by an easy computation. The second one follows from the first and (35). \square

Lemma B.4. For $k \geq 0$, there holds $B_k((x-1)/q) = q^{-k}(B_k(x) - B_{k-1}(x))$.

Proof. The special case $k = 0$ is immediate. It is enough to prove this equality when evaluated at $[n]_q$ for $n \geq 1$. This reduces to a standard property of q -binomial coefficients. \square

The following statement can be found for instance in [9, Lem. 4.7].

Lemma B.5. For $k \geq 0$,

$$\sum_{n \geq 0} B_k([n]_q)t^n = \frac{1}{\prod_{\ell=0}^k (1 - q^\ell t)}. \quad (36)$$

The action of Δ_q can be translated into an action on sequences of Laurent polynomials as follows.

Abusing notation, let Δ_q be the linear operator acting on sequences $(a_n)_{n \in \mathbb{Z}}$ of Laurent polynomials by the formula

$$(\Delta_q(a))_n = (a_n - a_{n-1})/q^n. \quad (37)$$

This is the same as (35) when $a_n = P([n]_q)$.

Lemma B.6. A sequence of Laurent polynomials $(a_n)_{n \geq 0}$ is the sequence of values of an element of A_q at q -integers $[n]_q$ with $n \geq 0$ if and only if it is annihilated by some power of Δ_q .

Proof. Assume first that $a_n = P([n]_q)$ for some element P of A_q and for all $n \geq 0$. Then because the operator Δ_q acts by sending the basis element B_k to B_{k-1} , iterating Δ_q sufficiently gives the constant sequence 0.

Conversely, one proceeds by induction on d where the sequence a is annihilated by the power $d + 1$ of Δ_q . If the sequence a is identically zero, the statement is obvious.

Otherwise, the sequence $\Delta_q^d(a)$ is a constant Laurent polynomial c . The sequence $a'_n = a_n - cB_d([n]_q)$ is then annihilated by Δ_q^d , hence by induction $a'(n) = P([n]_q)$ for some element P of A_q . The statement follows. \square

For integers a and b , let us introduce the following polynomials

$$\left[\begin{matrix} a; x \\ b \end{matrix} \right]_q = \frac{\prod_{j=a-b+1}^a [j]_q + q^j x}{[b]_q!}. \quad (38)$$

Their evaluation at $x = [n]_q$ is the standard q -binomial coefficient $\left[\begin{matrix} a+n \\ b \end{matrix} \right]_q$. Note that $B_k(x) = \left[\begin{matrix} k; x \\ k \end{matrix} \right]_q$ for $k \geq 0$.

These polynomials are useful in the following correspondence.

Proposition B.7. Consider the generating series

$$\sum_{n \geq 0} Z_n t^n = \frac{\sum_{k=0}^d h_k t^k}{\prod_{\ell=0}^d (1 - q^\ell t)}, \quad (39)$$

where h_k are arbitrary coefficients. Then the Z_n are the values at $[n]_q$ of the polynomial

$$\sum_{k=0}^d h_k \left[\begin{matrix} d-k; x \\ d \end{matrix} \right]_q. \quad (40)$$

Proof. Starting from Lemma B.5 corresponding to $k = 0$, one can obtain by induction on k the equality

$$\sum_{n \geq 0} \begin{bmatrix} d-k+n \\ d \end{bmatrix}_q t^n = \frac{t^k}{\prod_{\ell=0}^d (1 - q^\ell t)}.$$

The statement readily follows. \square

These polynomials $\begin{bmatrix} a;x \\ b \end{bmatrix}_q$ also have a very simple limit at $q = 0$, once normalized appropriately.

Lemma B.8. *For $0 \leq i \leq d$, the polynomial $\begin{bmatrix} d-i;x \\ d \end{bmatrix}_q q^{\binom{i}{2}}$ has a limit when $q = 0$ and this limit is 1 if $i = 0$ and $x(x-1)^i$ otherwise.*

Proof. One uses the explicit product Formula (38) for these polynomials. The statement is clear if $i = 0$. When $i > 0$, this is a simple computation term-by-term in the product, by distributing the factor $q^{\binom{i}{2}}$ in the appropriate factors to compensate for poles in q . \square

C q -analogue of the incidence algebra

The aim of this section is to explain an interpretation of the values of q -Zeta polynomials as corner coefficients of powers of matrix-like objects, similar to the classical case taking place in the incidence algebra.

The idea of a possible q -analogue of the incidence algebra has been alluded to in the last paragraph of [22].

Let us consider a poset P endowed with the height function h .

Let $\text{Inc}_q(P, h)$ be the vector space of square matrices A with rows and columns indexed by P , with coefficients in the ring of Laurent polynomials in q , and such that $A_{x,y} = 0$ if x is not smaller than or equal to y in P . Let D_h be the diagonal matrix with coefficients $q^{h(x)}$ for $x \in P$.

The space $\text{Inc}_q(P, h)$ is endowed with the following bilinear product

$$(A \times_q B)_{x,z} = AD_h B. \quad (41)$$

The product \times_q is associative, with the diagonal matrix D_h^{-1} as unit.

This ring is a twisted version of the usual incidence algebra, to which it reduces when $q = 1$. It can be identified with a sub-ring of the usual ring of square matrices by multiplying all matrices on the left by D_h .

Let $Z \in \text{Inc}_q(P, h)$ be the zeta matrix of P , defined by $Z_{x,y} = 1$ for all relations $x \leq y$ in P and 0 otherwise. The matrix Z is invertible in $\text{Inc}_q(P, h)$, because it is upper triangular with invertible diagonal.

Let now P be a bounded poset. Let H be the maximal value of h on P . For $n \in \mathbb{Z}$, let $Z^{\times_q n}$ be the n -th power of Z in $\text{Inc}_q(P, h)$.

We will use the ring A_q and the operator Δ_q as defined in §B.

Proposition C.1. *7 The sequence of matrices $(Z^{\times_q n})_{n \in \mathbb{Z}}$ is annihilated by Δ_q^{H+1} . For all $n \in \mathbb{Z}$, the value of the q -Zeta polynomial $Z_{P,h}([n]_q)$ is the coefficient of index $(\hat{0}, \hat{1})$ in the matrix $Z^{\times_q n}$.*

Proof. For $n \geq 2$, the definition of the matrix $Z^{\times q^n}$ as a product $ZD_h ZD_h \cdots Z$ implies directly that its corner coefficient is the weighted sum over all chains $\hat{0} \leq e_1 \leq \cdots \leq e_{n-1} \leq \hat{1}$, where the weight is $q^{\sum_j h(e_j)}$. This is exactly (3). This implies that the sequence $(Z_{\hat{0}, \hat{1}}^{\times q^n})_{n \geq 2}$ is annihilated by Δ_q^{H+1} .

The same proof applies, for every relation $x \leq y$, to the sequence of coefficients $(Z_{x,y}^{\times q^n})_{n \geq 2}$, which is therefore also annihilated by Δ_q^{H+1} . The sequence of matrices $(Z^{\times q^n})_{n \geq 2}$ is annihilated by Δ_q^{H+1} acting by (37).

Now consider the sequence $\Delta^{H+1}(Z^{\times q^n})_{n \geq N}$ for some $N \in \mathbb{Z}$. This is

$$\Delta^{H+1}(Z^{\times q^n})_{n \geq N} = Z^{\times q^{(N-2)}} \times_q \Delta^{H+1}(Z^{\times q^n})_{n \geq 2} = 0. \quad (42)$$

It follows that the whole sequence $(Z^{\times q^n})_{n \in \mathbb{Z}}$ is annihilated by Δ_q^{H+1} . Hence every sequence of coefficients of fixed index (x, y) is a polynomial of degree at most H evaluated at $[n]_q$, and in particular the corner coefficient coincides with the q -Zeta polynomial of P . \square

Note that this statement also implies Lemma 2.4.

References

- [1] Christos A. Athanasiadis, Thomas Brady, and Colum Watt. Shellability of noncrossing partition lattices. *Proc. Amer. Math. Soc.*, 135(4):939–949, 2007.
- [2] Pierre Baumann, Frédéric Chapoton, Christophe Hohlweg, and Hugh Thomas. Chains in shard intersection lattices and parabolic support posets. *J. Comb.*, 9(2):309–325, 2018.
- [3] Nantel Bergeron, Christophe Hohlweg, and Mike Zabrocki. Posets related to the connectivity set of Coxeter groups. *J. Algebra*, 303(2):831–846, 2006.
- [4] David Bessis. The dual braid monoid. *Ann. Sci. École Norm. Sup. (4)*, 36(5):647–683, 2003.
- [5] Louis J. Billera and Gábor Hetyei. Linear inequalities for flags in graded partially ordered sets. *J. Combin. Theory Ser. A*, 89(1):77–104, 2000.
- [6] Anders Björner. Shellable and Cohen-Macaulay partially ordered sets. *Trans. Amer. Math. Soc.*, 260(1):159–183, 1980.
- [7] Thomas Brady and Colum Watt. $K(\pi, 1)$'s for Artin groups of finite type. In *Proceedings of the Conference on Geometric and Combinatorial Group Theory, Part I (Haifa, 2000)*, volume 94, pages 225–250, 2002.
- [8] Thomas Brady and Colum Watt. Non-crossing partition lattices in finite real reflection groups. *Trans. Amer. Math. Soc.*, 360(4):1983–2005, 2008.
- [9] Frédéric Chapoton. q -analogues of Ehrhart polynomials. *Proc. Edinb. Math. Soc. (2)*, 59(2):339–358, 2016.

- [10] José Antonio de la Peña. Coxeter transformations and the representation theory of algebras. In *Finite dimensional algebras and related topics. Proceedings of the NATO Advanced Research Workshop on Representations of algebras and related topics. Ottawa, Canada, August 10-18, 1992*, pages 223–253. Dordrecht: Kluwer Academic Publishers, 1994.
- [11] The SageMath Developers. sagemath/sage: 10.0, 2023.
- [12] Ira M. Gessel. A historical survey of P -partitions. In *The mathematical legacy of Richard P. Stanley*, pages 169–188. Amer. Math. Soc., Providence, RI, 2016.
- [13] Jon McCammond. Noncrossing partitions in surprising locations. *Amer. Math. Monthly*, 113(7):598–610, 2006.
- [14] T. Kyle Petersen. On the shard intersection order of a Coxeter group. *SIAM J. Discrete Math.*, 27(4):1880–1912, 2013.
- [15] Nathan Reading. Noncrossing partitions and the shard intersection order. In *21st International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2009)*, volume AK of *Discrete Math. Theor. Comput. Sci. Proc.*, pages 745–756. Assoc. Discrete Math. Theor. Comput. Sci., Nancy, 2009.
- [16] Nathan Reading. Noncrossing partitions and the shard intersection order. *J. Algebraic Combin.*, 33(4):483–530, 2011.
- [17] Gian-Carlo Rota. On the foundations of combinatorial theory. I: Theory of Möbius functions. *Z. Wahrscheinlichkeitstheor. Verw. Geb.*, 2:340–368, 1964.
- [18] Richard P. Stanley. A chromatic-like polynomial for ordered sets. Proc. 2nd Chapel Hill Conf. Combin. Math. Appl., Univ. North Carolina 1970, 421-427 (1970)., 1970.
- [19] Richard P. Stanley. *Ordered structures and partitions*. American Mathematical Society, Providence, R.I., 1972. Memoirs of the American Mathematical Society, No. 119.
- [20] Richard P. Stanley. Combinatorial reciprocity theorems. *Adv. Math.*, 14:194–253, 1974.
- [21] Richard P. Stanley. Two poset polytopes. *Discrete Comput. Geom.*, 1:9–23, 1986.
- [22] Richard P. Stanley. Subdivisions and local h -vectors. *J. Am. Math. Soc.*, 5(4):805–851, 1992.
- [23] Richard P. Stanley. A survey of Eulerian posets. In *Polytopes: abstract, convex and computational. Proceedings of the NATO Advanced Study Institute, Scarborough, Ontario, Canada, August 20 - September 3, 1993*, pages 301–333. Dordrecht: Kluwer Academic Publishers, 1994.

- [24] Richard P. Stanley. *Enumerative combinatorics. Volume 1*, volume 49 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 2012.
- [25] Hugh Thomas. The Tamari lattice as it arises in quiver representations. In *Associahedra, Tamari lattices and related structures. Tamari memorial Festschrift*, pages 281–291. Basel: Birkhäuser, 2012.