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INCREASING THE SECOND UNIFORM INDISCERNIBLE BY STRONGLY SSP FORCING

BEN DE BOND T AND BOBAN VELIČKOVIĆ

ABSTRACT. We introduce a new and natural stationary set preserving forcing $\mathbb{P}^{c-c}(\lambda, \mu)$ that (under NS_{ω_1} precipitous + existence of $H_\theta^\#$ for a sufficiently large regular θ) increases the second uniform indiscernible \mathbf{u}_2 beyond some given ordinal λ . The forcing \mathbb{P}^{c-c} shares this property with forcings defined in [2] and [9]. As a main tool we use certain natural open two player games which are of independent interest, viz. the capturing games $\mathbf{G}_M^{cap}(X)$ and the catching-capturing games $\mathbf{G}_M^{c-c}(X)$. In particular, these games are used to isolate a special family of countable elementary submodels $M \prec H_\theta$ that occur as side conditions in \mathbb{P}^{c-c} and thus allow to control the forcing in a strong way.

1. INTRODUCTION

To date, the study of forcing axioms is one of the foremost lines in set theoretic research. Reflecting upon the nature of this phenomenon, one could argue that the true reason for the tremendous success of a forcing axiom as PFA is largely twofold. On the one hand, proper forcings are sufficiently well-behaved to allow a very fruitful iteration theory to be developed. Secondly, and just as crucially, the class of proper forcings is sufficiently rich to allow for a variety of very different proper forcings, giving in turn rise to many applications. A versatile technique for constructing such proper forcings, allowing them to be tailored to perform various different tasks, was introduced by Todorcevic ([17] and [18], see also [19], [20], ...) in the 1980s and proceeds by including in a forcing condition finitely many working parts, together with a finite chain of countable elementary substructures of H_θ . The behaviour of such forcing is determined by carefully fixing the allowed configurations of working parts and models occurring in the conditions. One can contrast this with the situation for stationary set preserving (from now on: ssp) forcings. The satisfactory iteration theory of semiproper forcings makes up for the lack of a general iteration theory for ssp forcings and allows to prove the consistency relative to a supercompact cardinal of the ssp-forcing axiom MM. Furthermore, the power of Woodin's \mathbb{P}_{\max} -forcing technique, in conjunction with the recent results by Asperó and Schindler ([1]) rephrasing (under suitable conditions) Woodin's axiom (*) as a bounded forcing axiom, suggest many possible applications of ssp forcing strictly beyond proper forcing. However, a technique for constructing ssp forcings just as versatile as Todorcevic's side condition technique has largely remained absent, even though the \mathcal{L} -forcing technique developed by Jensen ([8]) and further developed in [2],[5] and ultimately in [1], may be considered a candidate. In this article we exhibit one rudimentary instance of an alternative approach towards constructing specific ssp forcings that stays much closer to the side condition method for proper forcings. More precisely, we consider a reasonably simple forcing \mathbb{P}^{c-c} that is almost, but not entirely, a pure side condition forcing. The twists are the following: the side conditions are (morally) allowed to grow through the

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forcing procedure and working parts are included in the forcing conditions with the sole purpose of guiding this growth of the models. To guarantee that these models still fulfill their task as side conditions (and thus make the resulting forcing ssp) they can however not be allowed to absorb any new countable ordinals. Consequently, the crucial idea in this prima facie quite flexible adaptation of the proper-style side condition method is that only models $M \prec H_\theta$ exhibiting the right kind of growth behaviour are admissible as side conditions. The gain for the particular forcing $\mathbb{P}^{c-c}(\lambda, \mu)$ is that instead of just adding any sequence $(M_\alpha^G : \alpha < \omega_1)$ of countable models with union equal to H_θ (as a trivial pure side condition forcing would do), it will add a special such sequence that transitive collapses to a generic iteration with ω_1 -th iterate of the form H_λ . Moreover under the existence of $H_\lambda^\#$, it can be arranged that the first model in this generic iteration is iterable. Recall that under the presence of sharps for reals, all ω_1 -th iterates \mathcal{M}_{ω_1} occurring in some generic iteration $\mathbf{It} = (\mathcal{M}_\beta, j_{\alpha\beta}, G_\alpha : \alpha < \beta \leq \omega_1)$ of a countable transitive iterable model \mathcal{M}_0 are, by standard boundedness arguments, bounded in rank by the second uniform indiscernible ordinal \mathbf{u}_2 . Thus, it makes sense to consider the forcing $\mathbb{P}^{c-c}(\lambda, \mu)$ within the light of the still poorly understood problem of controlling the size of \mathbf{u}_2 by forcing. This problem has a significant history, starting with [16] where Steele and Van Wesep force over suitable determinacy models to obtain the consistency of $\text{ZFC} + \text{NS}_{\omega_1}$ saturated $+ \mathbf{u}_2 = \omega_2$. Later, Woodin showed [21, Theorem 3.17] that in fact $\mathbf{u}_2 = \omega_2$ follows from NS_{ω_1} saturated $+ \mathcal{P}(\omega_1)^\#$ exists. Thus, $\mathbf{u}_2 = \omega_2$ can be forced over a ZFC-model from a Woodin cardinal with a measurable above it using the forcing introduced by Shelah to make NS_{ω_1} saturated [21, Theorem 2.64]. In [15], Steel and Welch showed that $(\forall x \in \mathbb{R}) x^\#$ exists $+ \mathbf{u}_2 = \omega_2$ implies the existence of an inner model with a strong cardinal. The above premise also follows from the theory $\text{BMM} + \text{NS}_{\omega_1}$ precipitous, whose precise consistency strength is itself unclear. That $\text{BMM} + \text{NS}_{\omega_1}$ precipitous implies $\mathbf{u}_2 = \omega_2$ is in turn a consequence of the existence of ssp forcings such as \mathbb{P}^{c-c} , which, working under the theory $\text{ZFC} + \text{NS}_{\omega_1}$ precipitous $+ \text{existence of sufficient sharps}$, increase the value of \mathbf{u}_2 beyond some arbitrarily high ordinal. So far, two different such ssp forcings had been described in the literature, namely a Namba-like forcing in [9] and an \mathcal{L} -forcing in [2]. It is worth to note that both these two forcings, as well as the forcing \mathbb{P}^{c-c} described here, use the same argument to affect the value of \mathbf{u}_2 which goes back to Woodin's argument [21, Theorem 3.17], namely they all proceed through adding a countable transitive iterable model \mathcal{M} together with a suitable length- ω_1 generic iteration that blows it up to arbitrarily large rank. Also in some other aspects all three forcings behave very much alike. In particular, all three share with Namba forcing the property that they give some large regular cardinal cofinality ω . We believe that a main advantage of the construction as presented here is that the inclusion of suitable elementary substructures into the forcing conditions elucidates the proof of preservation of stationarity of ground-model stationary subsets of ω_1 . Indeed, the argument for proving that \mathbb{P}^{c-c} is ssp becomes entirely standard once the right class of models to be used as side conditions has been identified and the existence of projective stationary many such models has been proved. This additionally allows us to abstractly define a property of forcings that we call *strongly ssp* and is reminiscent of the similar notion of strong properness that is commonly satisfied by proper side condition forcings. Strongly ssp forcings enjoy additional strong properties beyond being merely ssp such as not adding any fresh functions $f : \omega_1 \rightarrow V$. The forcing \mathbb{P}^{c-c} is strongly ssp and therefore must add the many new reals that thin out the uniform indiscernibles, without adding any fresh subsets of ω_1 .

2. PRELIMINARIES

We gather here several notions and results that will be needed below.

2.1. Iterable structures. NS_{ω_1} denotes the non-stationary ideal on ω_1 , $\text{NS}_{\omega_1}^+$ is the set of stationary subsets of ω_1 . We also use $\text{NS}_{\omega_1}^+$ as a shorthand for the partially ordered set $(\text{NS}_{\omega_1}^+, \subseteq)$, that is, whenever $\text{NS}_{\omega_1}^+$ is without further indication treated as a poset, we have silently endowed it with the partial order $S \leq T \Leftrightarrow S \subseteq T$.

ZFC^- is the theory obtained from ZF by removing the powerset axiom and adding the collection axiom and the axiom that every set can be wellordered. Let ZFC^\bullet be the theory

$$\text{ZFC}^- + \mathcal{P}(\mathcal{P}(\omega_1)) \text{ exists} + \text{NS}_{\omega_1} \text{ is precipitous.}$$

Whenever \mathcal{H} is a transitive model of ZFC^- and $M \prec \mathcal{H}$ is a countable elementary substructure, then $\delta(M)$ denotes the countable ordinal $M \cap \omega_1^{\mathcal{H}}$, and for every $x \in \bigcup M$, we define

$$\text{Hull}(M, x) = \{f(x) : f \in M, x \in \text{dom}(f)\}.$$

Note that under these conditions, $\text{Hull}(M, x)$ is the minimal countable elementary submodel of \mathcal{H} that \subseteq -extends M and contains x as an element. The meaning of the statement

$$x \parallel_{\omega_1} M,$$

is by definition that

$$\text{Hull}(M, x) \cap \omega_1^{\mathcal{H}} = M \cap \omega_1^{\mathcal{H}}.$$

Definition 2.1. A countable model $M \prec \mathcal{H} \models \text{ZFC}^\bullet$ is called *selfgeneric* if for every $\mathcal{A} \in M$ such that

$$\mathcal{H} \models \mathcal{A} \subseteq \text{NS}_{\omega_1}^+ \text{ is a maximal antichain,}$$

there exists $S \in \mathcal{A} \cap M$ such that $\delta(M) \in S$.

Definition 2.2. Let \mathcal{M} be a countable transitive model of the theory ZFC^\bullet .

A *generic iteration* of $\mathcal{M} = \mathcal{M}_0$ of length γ (with $\gamma \leq \omega_1 + 1$) is a commuting system $\mathbf{It} = (\mathcal{M}_\beta, j_{\alpha\beta}, G_\alpha : \alpha < \beta < \gamma)$ satisfying

- (a) for every $\alpha + 1 < \gamma$, $G_\alpha \subset (\text{NS}_{\omega_1}^+)^{\mathcal{M}_\alpha}$ is a filter that is $(\text{NS}_{\omega_1}^+)^{\mathcal{M}_\alpha}$ -generic over \mathcal{M}_α ,
- (b) for every $\alpha + 1 < \gamma$, $\mathcal{M}_{\alpha+1}$ is the (transitivised) generic ultrapower of \mathcal{M}_α induced by G_α , $j_{\alpha, \alpha+1}$ is the corresponding ultrapower map,
- (c) for every limit ordinal $\beta < \gamma$, $(\mathcal{M}_\beta, (j_{\alpha\beta})_{\alpha < \beta})$ is the (transitivised) direct limit of the system $(\mathcal{M}_{\beta'}, j_{\alpha\beta'} : \alpha < \beta' < \beta)$.

The ZFC^\bullet -model \mathcal{M} is called *iterable* if for every limit ordinal $\gamma \leq \omega_1$ and every generic iteration $\mathbf{It} = (\mathcal{M}_\beta, j_{\alpha\beta}, G_\alpha : \alpha < \beta < \gamma)$ of length γ of \mathcal{M} , the direct limit of the system $(\mathcal{M}_\beta, j_{\alpha\beta} : \alpha < \beta < \gamma)$ is well-founded.

We will make use of the following standard way of constructing iterable models from sharps.

Lemma 2.3. *Suppose \mathcal{H} is a transitive ZFC^\bullet -model with $\omega_1 \subseteq \mathcal{H}$ and $\theta \in \mathcal{H}$ is an ordinal such that*

- (a) $\mathcal{H} \models \theta$ is a regular cardinal and $\theta > |\mathcal{P}(\mathcal{P}(\omega_1))|$,
- (b) $\mathcal{H} \models H_\theta$ and $H_\theta^\#$ both exist.

If $M \prec \mathcal{H}$ is countable and contains θ , then the transitive collapse of $M \cap H_\theta^{\mathcal{H}}$ is an iterable ZFC^\bullet -model.

A proof of Lemma 2.3 can be found by applying standard modifications to the argument given in [21] to prove [21, Theorem 3.14].

Definition 2.4. Let \mathcal{H} be a transitive ZFC^\bullet -model and let $(M_\alpha : \alpha < \omega_1)$ be a continuous \subseteq -increasing sequence of countable elementary submodels of \mathcal{H} . We call $(M_\alpha : \alpha < \omega_1)$ an *iteration-inducing filtration* of \mathcal{H} if

$$\bigcup_{\alpha < \omega_1} M_\alpha = \mathcal{H},$$

and additionally the following two conditions hold for every $\alpha < \omega_1$:

- M_α is selfgeneric,
- $M_{\alpha+1} = \text{Hull}(M_\alpha, \delta(M_\alpha))$.

There is a straightforward way to translate between generic iterations of length ω_1 and iteration-inducing filtrations. Indeed, the following is easily checked.

Lemma 2.5. *Let \mathcal{H} be a transitive ZFC^\bullet -model.*

- (1) *If $\mathbf{It} = (\mathcal{M}_\beta, j_{\alpha\beta}, G_\alpha : \alpha < \beta < \omega_1)$ is a generic iteration of length ω_1 with (transitivised) direct limit equal to \mathcal{H} , then $(j_{\alpha\omega_1}[\mathcal{M}_\alpha] : \alpha < \omega_1)$ is an iteration-inducing filtration of \mathcal{H} ,*
- (2) *if $(M_\alpha : \alpha < \omega_1)$ is an iteration-inducing filtration of \mathcal{H} , then there exists a unique generic iteration $\mathbf{It} = (\mathcal{M}_\beta, j_{\alpha\beta}, G_\alpha : \alpha < \beta < \omega_1)$ of length ω_1 such that the following diagrams $(\alpha < \beta < \omega_1)$ commute:*

$$\begin{array}{ccc} M_\alpha & \subseteq & M_\beta \\ \downarrow \pi_\alpha & & \downarrow \pi_\beta \\ \mathcal{M}_\alpha & \xrightarrow{j_{\alpha\beta}} & \mathcal{M}_\beta \end{array}$$

with $\pi_\alpha : M_\alpha \rightarrow \mathcal{M}_\alpha$ for each $\alpha < \omega_1$ being the transitivising isomorphism. Moreover, the direct limit of the system \mathbf{It} is equal to \mathcal{H} .

2.2. The second uniform indiscernible. If $x^\#$ exists for every real, then there exists a proper class of ordinals which are x -indiscernible for every real x , these are called the uniform indiscernibles. Clearly, every uncountable cardinal is a uniform indiscernible and ω_1 is the minimal uniform indiscernible ordinal. The second uniform indiscernible $\mathbf{u}_2 \leq \omega_2$ deserves special interest because it is simultaneously the second projective ordinal δ_2^1 . These projective ordinals

$$\delta_n^1 = \sup\{\alpha : \alpha \text{ is the length of a } \Delta_n^1 \text{ prewellordering of } \mathbb{R}\}$$

were introduced by Moschovakis ([12]) to gauge the extent to which CH can fail in a projectively definable way. One can check that if the sharp of every real exists, additionally

$$\mathbf{u}_2 = \sup\{\omega_1^{+L[x]} : x \in \mathbb{R}\}.$$

Results of respectively Martin ([10]) and Woodin ([21, Theorem 3.17]), give that under both $(\text{ZF}+)\text{AD}$ and under NS_{ω_1} saturated $+ \mathcal{P}(\omega_1)^\#$, the second uniform indiscernible \mathbf{u}_2 has the maximal value it could have, namely $\mathbf{u}_2 = \omega_2$. We need the following lemma which is also essentially used in Woodin's argument ([21, Theorem 3.17]) for

$$\text{NS}_{\omega_1} \text{ saturated } + \mathcal{P}(\omega_1)^\# \text{ exists } \Rightarrow \mathbf{u}_2 = \omega_2.$$

Lemma 2.6 ([21, Lemma 3.15]). *Let \mathcal{M} be a countable transitive model of ZFC^\bullet that is iterable. Let x be a real that codes both \mathcal{M} and some countable ordinal α . Then for every generic iteration $\mathbf{It} = (\mathcal{M}_\beta, j_{\alpha\beta}, G_\alpha : \alpha < \beta < \alpha + 1)$ of \mathcal{M} of length $\alpha + 1$,*

$$\text{rank}(\mathcal{M}_\alpha) < \omega_1^{+L[x]}.$$

Finally, we will also make use of the well known fact that existence of sharps of all (transitive) sets in a fragment of the universe is not destroyed by small forcing. See e.g. [14, Lemma 3.4] for a much finer result of this type, for the present purposes the following lemma suffices.

Lemma 2.7. *Let κ be a cardinal and suppose that $H_{\kappa^+}^\#$ exists. Then for every forcing \mathbb{P} of size at most κ and every filter G that is \mathbb{P} -generic over V ,*

$$V[G] \models X^\# \text{ exists for every transitive set } X \in H_{\kappa^+}.$$

3. THE STRONGLY SSP PROPERTY

Notation 3.1. For a set X and a cardinal θ , we write $X \ll \theta$ to mean that

$$\mathcal{P}(\mathcal{P}(X)) \in H_\theta.$$

Definition 3.2. Given a \mathbb{P} -condition p^* , the condition p^* is called (M, \mathbb{P}) -semigeneric if for every $q \leq p^*$ and every maximal antichain $\mathcal{A} \subseteq \mathbb{P}$ with $\mathcal{A} \in M$, there exists some $r \in \mathcal{A}$ such that

$$r \parallel q \text{ and } r \parallel_{\omega_1} M.$$

Equivalently, p^* is (M, \mathbb{P}) -semigeneric iff for every \mathbb{P} -name $\tau \in M$ for a countable ordinal,

$$p^* \Vdash \tau < \delta(M).$$

The condition p^* is called *strongly* (M, \mathbb{P}) -semigeneric if for every $q \leq p^*$ there exists $\text{tr}(q|M) \in \mathbb{P}$ such that $\text{tr}(q|M) \parallel_{\omega_1} M$ and for every $r \in \text{Hull}(M, \text{tr}(q|M)) \cap \mathbb{P}$,

$$r \leq \text{tr}(q|M) \Rightarrow r \parallel q.$$

\mathbb{P} is said to be *(strongly) semiproper* with respect to $M \prec H_\theta$ if for every $p \in \mathbb{P} \cap M$, there exists $p^* \leq p$ such that p^* is (strongly) (M, \mathbb{P}) -semigeneric.

Recall that a set $X \subseteq [H_\theta]^\omega$ is *projective stationary* if for every stationary subset $S \subseteq \omega_1$ the set $\{M \in X : M \cap \omega_1 \in S\}$ is stationary in $[H_\theta]^\omega$.

Lemma 3.3. *If \mathbb{P} is strongly semiproper with respect to $M \prec H_\theta$, then \mathbb{P} is semiproper with respect to $M \prec H_\theta$. If \mathbb{P} is semiproper with respect to projective stationary many $M \prec H_\theta$, then \mathbb{P} is ssp.*

Proof. The first statement is clear because every condition that is strongly (M, \mathbb{P}) -semigeneric is also (M, \mathbb{P}) -semigeneric. To prove the second statement, let be given a stationary set $S \subseteq \omega_1$, a name for a club $\dot{C} \in H_\theta$ and an arbitrary $p \in \mathbb{P}$. By assumption there exists $M \prec H_\theta$ such that \mathbb{P} is semiproper with respect to M and such that $p, \dot{C} \in M$ and $M \cap \omega_1 \in S$. By semiproperness there exists $p^* \leq p$ that is (M, \mathbb{P}) -semigeneric. Then p^* forces that $M[G] \cap \omega_1 = M \cap \omega_1$ and hence also that $M \cap \omega_1 \in \dot{C}$. \square

In fact (see [6, Lemma 4.8]), a forcing is ssp if and only if for every sufficiently large regular cardinal θ , there exist projective stationary many countable $M \prec H_\theta$ such that \mathbb{P} is semiproper with respect to M . Replacing here semiproper by *strongly* semiproper results in the following stronger notion.

Definition 3.4. A forcing \mathbb{P} is *strongly ssp* if for every regular cardinal θ which is sufficiently large ($\theta \gg \mathbb{P}$), there exist projective stationary many countable $M \prec H_\theta$ such that \mathbb{P} is strongly semiproper with respect to M .

The following lemma makes apparent that even ccc forcings and countably closed forcings can fail to be strongly ssp.

Lemma 3.5. *If \mathbb{P} is strongly ssp, then \mathbb{P} does not add any fresh functions $\omega_1 \rightarrow V$.*

Proof. Let \dot{f} be a \mathbb{P} -name and suppose $p \in \mathbb{P}$ such that

$$p \Vdash \dot{f} : \omega_1 \rightarrow V \quad \text{and} \quad (\forall \alpha < \omega_1) \dot{f} \upharpoonright \alpha \in V.$$

Choose a sufficiently large regular θ and let $M \prec H_\theta$ countable such that $\mathbb{P}, \dot{f}, p \in M$ and such that \mathbb{P} is strongly semiproper with respect to M . Then pick $p^* \leq p$ which is strongly (M, \mathbb{P}) -semigeneric and also pick $q \leq p^*$ and $g : M \cap \omega_1 \rightarrow V$ such that

$$q \Vdash \dot{f} \upharpoonright M \cap \omega_1 = g.$$

We claim that $\text{tr}(q|M)$ decides \dot{f} .

Suppose not, then there exist $r_1, r_2 \in \mathbb{P} \cap \text{Hull}(M, \text{tr}(q|M))$, $\alpha \in M \cap \omega_1$ and $x \neq y$ such that

$$r_1, r_2 \leq \text{tr}(q|M), \quad r_1 \Vdash \dot{f}(\alpha) = x, \quad r_2 \Vdash \dot{f}(\alpha) = y.$$

This contradicts $r_1 \parallel q$, $r_2 \parallel q$ and $q \Vdash \dot{f}(\alpha) = g(\alpha)$. □

As a consequence, a strongly ssp forcing adds no new uncountable branches to a ground-model tree of height ω_1 .

Remark 3.6. Of course, the strongly ssp property is inspired by Mitchell's ([11]) notion of strong properness. A forcing $\mathbb{P} \in H_\theta$ is strongly proper with respect to $M \prec H_\theta$ if for every $p \in M \cap \mathbb{P}$, there exists $p^* \leq p$ that is strongly (M, \mathbb{P}) -generic. p^* being strongly (M, \mathbb{P}) -generic means that for every $q \leq p^*$, there exists $\text{tr}(q|M) \in \mathbb{P} \cap M$ such that

$$(\forall r \in \mathbb{P} \cap M) r \leq \text{tr}(q|M) \Rightarrow r \parallel q.$$

\mathbb{P} is strongly proper if for every regular $\theta \gg \mathbb{P}$ there exist club many countable $M \prec H_\theta$ such that \mathbb{P} is strongly proper with respect to M .

Unsurprisingly, it is easy to check that every strongly proper forcing is strongly ssp.

4. GAMES: CATCHING AND CAPTURING

In this section three types of games will be introduced,

- the antichain catching games $\mathbf{G}_M^{\text{cat}}$,
- the capturing games $\mathbf{G}_M^{\text{cap}}(X)$,
- and the catching-capturing games $\mathbf{G}_M^{\text{c-c}}(X)$.

All three types of games will be played by two players $P1$ (*the Challenger*) and $P2$ (*the Constructor*). $P1$ and $P2$ thank their nicknames to the fact that the moves of $P2$ are used to construct a growing sequence $(M_n : n < \omega)$ of countable elementary submodels of some H_θ , while the moves of $P1$ can be thought of as challenges for $P2$ to make the models M_n satisfy certain conditions. All games considered have length ω . If \mathbf{G} is one of these games, we write $P1 \nearrow \mathbf{G}$ for the statement that $P1$ has a winning strategy for the game \mathbf{G} (and likewise for $P2 \nearrow \mathbf{G}$).

We proceed straight to the definitions.

Definition 4.1 (The antichain catching game). Let θ be a regular cardinal and suppose $\mathcal{P}(\omega_1) \in H_\theta$. Let also $M \prec H_\theta$ be countable. The *antichain catching game* $\mathbf{G}_M^{\text{cat}}$ for M is defined as follows. Define $M_0 = M$. In round n of the game,

- $P1$ plays some maximal antichain $\mathcal{A}_n \subseteq \text{NS}_{\omega_1}^+$ with $\mathcal{A}_n \in M_n$,
- $P2$ has to play some $S_n \in \mathcal{A}_n$ such that $M_n \cap \omega_1 \in S_n$ and $S_n \parallel_{\omega_1} M_n$,
- define $M_{n+1} = \text{Hull}(M_n, S_n)$.

Thus, a play of the game $\mathbf{G}_M^{\text{cat}}$ can be pictured as follows.

	M_0	$\overset{+S_0}{\rightsquigarrow}$	M_1	$\overset{+S_1}{\rightsquigarrow}$	M_2	\dots
	\cup		\cup		\cup	
$P1$	\mathcal{A}_0		\mathcal{A}_1		\mathcal{A}_2	\dots
$P2$	S_0		S_1		\dots	

A winner is assigned as decided by the following rules.

If there is a player who first disobeys the rules, this player loses, else, $P2$ wins.

The catching game is clearly just one of the many variations on the classic theme of antichain catching and it is easy to observe that $P1$ has a winning strategy for the catching game \mathbf{G}_M^{cat} if and only if there exists some countable selfgeneric $N \prec H_\theta$ such that $M \subseteq N$ and $M \cap \omega_1 = N \cap \omega_1$.

Definition 4.2 (The capturing game). Let θ be a regular cardinal and $X \in H_\theta$. Suppose $M \prec H_\theta$ is countable and $X \in M$. The X -capturing game $\mathbf{G}_M^{cap}(X)$ for M is defined as follows. Define $M_0 = M$. In round n of the game,

- $P1$ plays some $x_n \in X$ (x_n not necessarily belonging to M_n),
- $P2$ then has to play some function $f_n : \omega_1 \rightarrow X$, with $f_n(M_n \cap \omega_1) = x_n$ and $f_n \upharpoonright_{\omega_1} M_n$,
- define $M_{n+1} = \text{Hull}(M_n, f_n)$.

Thus, a play of the game $\mathbf{G}_M^{cap}(X)$ can be pictured as follows.

	M_0	$\overset{+f_0}{\rightsquigarrow}$	M_1	$\overset{+f_1}{\rightsquigarrow}$	M_2	\dots
$P1$	x_0		x_1		x_2	\dots
$P2$	f_0		f_1		\dots	

The winner is again assigned in accordance with the following same rules.

If there is a player who first disobeys the rules, this player loses, else, $P2$ wins.

Finally, the catching-capturing game $\mathbf{G}_M^{c-c}(X)$ is simply a mix of the two above games \mathbf{G}_M^{cat} and $\mathbf{G}_M^{cap}(X)$.

Definition 4.3 (The catching - capturing game). Let θ be a regular cardinal with $\mathcal{P}(\omega_1) \in H_\theta$. Suppose $M \prec H_\theta$ is countable and $X \in M$. The catching-capturing game $\mathbf{G}_M^{c-c}(X)$ for M is defined as follows. In round n , player $P1$ now has two options:

- either $P1$ makes a move $\mathcal{A}_n \in M_n$ as in the game \mathbf{G}_M^{cat} ,
 - the player $P2$ then has to answer with some S_n , respecting the same restrictions as in the game \mathbf{G}_M^{cat} ,
 - define $M_{n+1} = \text{Hull}(M_n, S_n)$.
- or $P1$ decides to play some $x_n \in X$, as in the game $\mathbf{G}_M^{cap}(X)$.
 - $P2$ then has to answer with some function $f_n : \omega_1 \rightarrow X$, respecting the same restrictions as in the game $\mathbf{G}_M^{cap}(X)$,
 - define $M_{n+1} = \text{Hull}(M_n, f_n)$.

A play of the game $\mathbf{G}_M^{c-c}(X)$ can be pictured as follows.

	M_0	$\overset{+S_0}{\rightsquigarrow}$	M_1	$\overset{+f_1}{\rightsquigarrow}$	M_2	$\overset{+S_2}{\rightsquigarrow}$	M_3
	\cup		\cup		\cup		
$P1$	\mathcal{A}_0		x_1		\mathcal{A}_2		$x_3 \dots$
$P2$	S_0		f_1		S_2		\dots

The winning conditions remain unaltered:

If there is a player who first disobeys the rules, this player loses,
else, $P2$ wins.

Note that the games \mathbf{G}_M^{cat} , $\mathbf{G}_M^{cap}(X)$, $\mathbf{G}_M^{c-c}(X)$ are all determined, the winning conditions for $P2$ being closed. We will allow countable models $M \prec H_\theta$ to occur as side conditions in the forcing \mathbb{P}^{c-c} only if $P2 \nearrow \mathbf{G}_M^{c-c}(X)$ for a sufficiently large set $X \in H_\theta$. Thus it will be crucial for this approach that sufficiently many models exist for which $P2$ has a winning strategy in $\mathbf{G}_M^{c-c}(X)$. If the non-stationary ideal on ω_1 is precipitous, this is warranted by the following proposition.

Proposition 4.4. *Suppose $\theta \gg \omega_1$ is a regular cardinal and $X \in H_\theta$. The following are all equivalent.*

- (a) *there exist projective stationary many countable $M \prec H_\theta$ for which $P2 \nearrow \mathbf{G}_M^{c-c}(X)$,*
- (b) *there exist projective stationary many countable $M \prec H_\theta$ for which $P2 \nearrow \mathbf{G}_M^{cat}$,*
- (c) *there exist projective stationary many countable selfgeneric $M \prec H_\theta$,*
- (d) *\mathbf{NS}_{ω_1} is precipitous.*

Proof.

(d) \Rightarrow (a) We first prove that (d) implies (a). The argument is a relative of those in [4, Theorem 3.8] and [9, Proposition 2.4].

Let $A \subseteq \omega_1$ be stationary and let $F : H_\theta \rightarrow H_\theta$ be arbitrary. We use an elementarity-argument to prove that there exists a countable model $M \prec H_\theta$ which is closed under F and satisfies $M \cap \omega_1 \in A$ and $P2 \nearrow \mathbf{G}_M^{c-c}(X)$. Let $G \subseteq \mathbf{NS}_{\omega_1}^+$ be a filter generic over V and containing A . Using precipitousness of \mathbf{NS}_{ω_1} , let W be the corresponding transi-tivised ultrapower of V and let $j : V \rightarrow W$ be the corresponding elementary embedding. Select (in V) an arbitrary countable elementary submodel $N \prec H_\theta$ that contains X and has the property that for every function $f : \omega_1 \rightarrow H_\theta$ which is in N , also the function $F \circ f : \omega_1 \rightarrow H_\theta$ is in N .

Next, we continue working in W . Note that, in W , $j(N)$ is a countable elementary sub-model of $H_{j(\theta)}$. Hence, the model

$$\tilde{N} := \{f(\xi) : \xi \in \omega_1^V, f \in j(N) \text{ with } \xi \in \text{dom}(f)\},$$

is, in W , a countable elementary submodel of $H_{j(\theta)}$.

By construction we have that $\tilde{N} \cap \omega_1^W = \omega_1^V \in j(A)$ and it clearly follows from the choice of N that \tilde{N} is closed under the function $j(F) : H_{j(\theta)} \rightarrow H_{j(\theta)}$. Using elementarity of the embedding j , the proof will be completed if we can establish that

$$(\text{ the model } \tilde{N} \prec H_{j(\theta)} \text{ satisfies } P2 \nearrow \mathbf{G}_{\tilde{N}}^{c-c}(j(X)))^W.$$

Arguing by contradiction, suppose that this is not the case. Then there exists in W some winning strategy Σ for $P1$ (the Challenger) in the game $\mathbf{G}_{\tilde{N}}^{c-c}(j(X))$. Let T be the tree (as defined in W) of all partial plays of the game $\mathbf{G}_{\tilde{N}}^{c-c}(j(X))$ in which $P1$ plays by the strategy Σ and in which $P2$ has not lost yet, then

$$(\text{ the tree } T \text{ is well-founded })^W.$$

We now leave W and describe in $V[G]$ an infinite branch B in the tree T , thus arriving at the desired contradiction (since W is transitive, well-foundedness is absolute between W and $V[G]$). The branch B can be defined as follows:

- $P1$ makes a $\mathbf{G}_{\tilde{N}}^{cap}(j(X))$ -move in round n :
 Suppose $P1$ plays the set $x \in j(X)$. There exists $f : \omega_1^V \rightarrow X$ such that $x = j(f)(\omega_1^V)$.
 $P2$ replies by playing the function $j(f)$.
- $P1$ makes a $\mathbf{G}_{\tilde{N}}^{cat}$ -move in round n :
 Suppose $P1$ plays the maximal antichain \mathcal{A}_n , then it is of the form $j(f)(\xi, j(\bar{y}))$, where $j(\bar{y})$ is the sequence of moves that have been played by $P2$ in the first n moves, $\xi \in \omega_1^V$, and where
 $f : \text{dom}(f) \rightarrow \{\mathcal{A} \subseteq \text{NS}_{\omega_1}^+ : \mathcal{A} \text{ is a maximal antichain in } \text{NS}_{\omega_1}^+\}^V$
 is a function which belongs to N . Then
 $\mathcal{B} := f(\xi, \bar{y}) \in \{\mathcal{A} \subseteq \text{NS}_{\omega_1}^+ : \mathcal{A} \text{ is a maximal antichain in } \text{NS}_{\omega_1}^+\}^V$,
 so there exists some $S \in \mathcal{B} \cap G$.
 $P2$ replies by playing the set $j(S)$.

To check that $P2$ does not lose by playing this way, we note that

$$M_n \subseteq \{f(\xi, j(w)) : \xi \in \omega_1, w \in H_\theta^{<\omega}, f \in j(N) = j[N]\} \subseteq j[H_\theta]$$

and

$$j[H_\theta] \cap \omega_1^W = \omega_1^V = \tilde{N} \cap \omega_1^W.$$

$(a) \Rightarrow (b)$ The implication $(a) \Rightarrow (b)$ is trivial.

$(b) \Rightarrow (c)$ It was already noted above that

$$P2 \nearrow \mathbf{G}_M^{cat}$$

is equivalent to the statement that there exists an extension of M to a countable selfgeneric $N \prec H_\theta$ containing the same countable ordinals. It is easily seen to follow from this that (b) and (c) are equivalent.

$(c) \Rightarrow (d)$ Finally, the direction $(c) \Rightarrow (d)$ is well known and can be found in [4, Lemma 3.6].

We repeat this short argument for the sake of completeness. Suppose NS_{ω_1} were not precipitous. There would exist $S \in \text{NS}_{\omega_1}^+$ such that

$S \Vdash$ there exists an infinite $<_G$ -decreasing sequence
 of ground-model functions $f_n : \omega_1 \rightarrow \text{Ord}$.

Let $M \prec H_\theta$ be countable and selfgeneric with $S \in M$ and $\delta := M \cap \omega_1 \in S$. Let $\pi : M \rightarrow \mathcal{M}$ be the transitive collapse of M . We can assume θ was sufficiently large so that inside the transitive ZFC^\bullet -model \mathcal{M} , $\pi(S)$ is a stationary subset of ω_1 and

$\pi(S) \Vdash$ there exists an infinite $<_G$ -decreasing sequence
 of ground-model functions $f_n : \omega_1 \rightarrow \text{Ord}$.

Moreover, by selfgenericity,

$$G_\delta = \{\pi(T) : T \in \text{NS}_{\omega_1}^+ \cap M \text{ with } \delta \in T\}$$

is a $\pi(\text{NS}_{\omega_1}^+)$ -filter that is generic over \mathcal{M} . This contradicts the prior statement about $\pi(S)$ forcing ill-foundedness of $\text{Ult}(V, G)$ because $\pi(S) \in G_\delta$ and $\text{Ult}(\mathcal{M}, G_\delta)$ is well-founded as it is \in -isomorphic to $\text{Hull}(M, \delta)$. □

Although not needed for the discussion in the next section of the forcing \mathbb{P}^{c-c} , we end this section with some results that show that the equivalences in Proposition 4.4 fall apart when one replaces *projective stationary many* by *club many*.

It is well known that the existence of club many selfgeneric models is equivalent to saturation of the non-stationary ideal (see e.g. [7, Vol. 2, Lemma 13.3.46]). Therefore, if NS_{ω_1} is saturated there exist club many countable $M \prec H_\theta$ for which $P2 \nearrow \mathbf{G}_M^{\text{cat}}$. In contrast, the existence of club many countable models $M \prec H_\theta$ for which $P2 \nearrow \mathbf{G}_M^{\text{cap}}(X)$ can be translated into strong versions of Chang's conjecture.

Definition 4.5. (cof-Strong Chang's Conjecture, [3, Definition 4.3]) SCC^{cof} is the statement that for all regular cardinals $\theta \gg \omega_2$ and every countable $M \prec H_\theta$ there exist cofinally many $\xi \in \omega_2$ such that $\xi \parallel_{\omega_1} M$.

Definition 4.6. (Global Strong Chang's Conjecture, [5, Definition 5.6], [3, Definition 4.9]) $\text{Glob-SCC}^{\text{cof}}$ is the statement that for all regular cardinals $\theta \gg \lambda \geq \omega_2$, for every $\lambda \in M \prec H_\theta$ and every $X \in [H_\lambda]^{\omega_1}$, there exists $Y \in [H_\lambda]^{\omega_1}$ such that

$$X \subseteq Y \text{ and } Y \parallel_{\omega_1} M.$$

Lemma 4.7. *The following two statements are equivalent.*

- (a) SCC^{cof} ,
- (b) for every regular $\theta \gg \omega_2$ and every $M \prec H_\theta$ countable, $P2$ has as a winning strategy in the ω_2 -capturing game $\mathbf{G}_M^{\text{cap}}(\omega_2)$ for M .

Proof.

$(a) \Rightarrow (b)$ Suppose first $P1$ plays $\rho \in \omega_1$. Use SCC^{cof} to find $M \subseteq M^* \prec H_\theta$ such that $M \cap \omega_1 = M^* \cap \omega_1$ and such that there exists $\xi^* \in M^* \cap \omega_2$ satisfying $\text{otp}(M^* \cap \xi^*) = \rho$. Choose inside M^* a bijection g from ω_1 onto ξ^* . Let $f : \omega_1 \rightarrow \omega_1$ be defined by $f(\alpha) = \text{otp}(g[\alpha])$. Then $f \parallel_{\omega_1} M$ and $f(\delta(M)) = \rho$.

Suppose then that $P1$ plays $\rho \in \omega_2 \setminus \omega_1$. By SCC^{cof} , we can assume that there exists $\xi \in M$ such that $\rho < \xi$. Choose in M a bijection g from ω_1 onto ξ . Using the argument in the foregoing case, one finds $f : \omega_1 \rightarrow \omega_1$ such that $f \parallel_{\omega_1} M$ and $f(\delta(M)) = g^{-1}(\rho)$.

$(b) \Rightarrow (a)$ Given $\rho \in \omega_2$, it follows from $P2 \nearrow \mathbf{G}_M^{\text{cap}}(\omega_2)$ that there exists $f : \omega_1 \rightarrow \omega_2$ such that $f \parallel_{\omega_1} M$ and $f(\delta(M)) = \rho$. Then $\xi = \sup(f)$ satisfies $\xi > \rho$ and $\xi \parallel_{\omega_1} M$. \square

Proposition 4.8. *The following two statements are equivalent.*

- (a) $\text{Glob-SCC}^{\text{cof}}$,
- (b) for all regular $\theta \gg \lambda \geq \omega_2$, and every $M \prec H_\theta$ with $\lambda \in M$, $P2$ has a winning strategy in the H_λ -capturing game $\mathbf{G}_M^{\text{cap}}(H_\lambda)$ for M .

Proof.

$(a) \Rightarrow (b)$ $\text{Glob-SCC}^{\text{cof}}$ implies that for every $x \in H_\lambda$ there exists some $\rho < \omega_1$ and $f : \omega_1 \rightarrow H_\lambda$ such that $f(\rho) = x$ and $f \parallel_{\omega_1} M$. $\text{Glob-SCC}^{\text{cof}}$ also implies SCC^{cof} , which by Lemma 4.7 implies that there exists a function $g : \omega_1 \rightarrow \omega_1$ such that $g(\delta(M)) = \rho$ and $\{g, f\} \parallel_{\omega_1} M$.

$(b) \Rightarrow (a)$ Given $\lambda \in M \prec H_\theta$ and $X \in [H_\lambda]^{\omega_1}$, find a function $g : \omega_1 \rightarrow [H_\lambda]^{\omega_1}$ such that $g(\delta(M)) = X$ and $g \parallel_{\omega_1} M$. Then $Y = \bigcup_{\alpha \in \omega_1} g(\alpha)$ satisfies

$$|Y| = \aleph_1, \quad X \subseteq Y, \quad Y \parallel_{\omega_1} M.$$

\square

5. THE FORCING $\mathbb{P}^{c-c}(\lambda, \mu)$

Throughout this section, fix two regular cardinals $\omega_1 \ll \lambda \ll \mu$. We further write \mathbf{G}_M^{c-c} for $\mathbf{G}_M^{c-c}(H_\lambda)$. If z is a finite run of the game \mathbf{G}_M^{c-c} , we write $P2(z)$ for the finite set of all moves that have been made by $P2$ in z . The conditions in the forcing $\mathbb{P}^{c-c}(\lambda, \mu)$ will be finite sets of special triples, we call those triples *building blocks*.

Definition 5.1. A set $B \in H_\mu$ is called a *building block* if it is a three-tuple $B = (M, z, \Sigma)$, with:

- (a) $M \prec H_\mu$ countable with $\lambda \in M$ and such that $P2 \nearrow \mathbf{G}_M^{c-c}$,
- (b) Σ a winning strategy for $P2$ in \mathbf{G}_M^{c-c} ,
- (c) z a finite play of the game \mathbf{G}_M^{c-c} in which $P2$ follows Σ .

Notation 5.2. Let $B = (M, z, \Sigma)$ and $B' = (M', z', \Sigma')$ be building blocks. We define

- $\text{Ext}(B) := \text{Hull}(M, P2(z)) \prec H_\mu$,
- $B \sqsubseteq_{\text{game}} B'$ holds if and only if $M = M', \Sigma = \Sigma'$ and $z \subseteq z'$.

We are now ready to define the main forcing $\mathbb{P}^{c-c}(\lambda, \mu)$.

Definition 5.3. p is a $\mathbb{P}^{c-c}(\lambda, \mu)$ -condition if

- (a) p is a finite set of building blocks,
- (b) for every two building blocks $B_1, B_2 \in p$,

$$\text{Ext}(B_1) \ni B_2 \text{ or } B_1 = B_2 \text{ or } B_1 \in \text{Ext}(B_2).$$

The order on $\mathbb{P}^{c-c}(\lambda, \mu)$ is defined as follows. If $p, q \in \mathbb{P}^{c-c}(\lambda, \mu)$, then

$$q \leq p \iff (\forall B \in p)(\exists B' \in q) B \sqsubseteq_{\text{game}} B'.$$

In analysing the forcing $\mathbb{P}^{c-c}(\lambda, \mu)$ it will be convenient to use the following additional notation.

Notation 5.4.

- Call a building block B an M -block if and only if $B = (M, z, \Sigma)$ for certain z, Σ ,
- let $<$ be the strict partial order defined on building blocks by

$$B_1 < B_2 \iff B_1 \in \text{Ext}(B_2).$$

Lemma 5.5. Suppose $B = (M, z, \Sigma)$ is a building block and p^* is a $\mathbb{P}^{c-c}(\lambda, \mu)$ -condition that contains B as an element. If $\mu \ll \theta$ regular and $N \prec H_\theta$ is countable with $\lambda, \mu \in N$ and satisfies

$$N \cap H_\mu = M,$$

then p^* is strongly $(N, \mathbb{P}^{c-c}(\lambda, \mu))$ -semigeneric.

Proof. To check that p^* is strongly $(N, \mathbb{P}^{c-c}(\lambda, \mu))$ -semigeneric, consider a $\mathbb{P}^{c-c}(\lambda, \mu)$ -condition $q \leq p^*$ and define

$$\text{tr}(q|N) = q \cap \text{Ext}(B_M^q) \in \mathbb{P}^{c-c}(\lambda, \mu),$$

where B_M^q is the unique M -building block that belongs to q . From $\text{tr}(q|N) \in \text{Ext}(B_M^q)$ follows

$$\text{tr}(q|N) \parallel_{\omega_1} N.$$

Suppose then that $r \in \mathbb{P}^{c-c}(\lambda, \mu) \cap \text{Hull}(N, \text{tr}(q|N))$ and $r \leq \text{tr}(q|N)$. To prove that $r \parallel q$, one checks that $r \cup (q \setminus \text{tr}(q|N))$ is a $\mathbb{P}^{c-c}(\lambda, \mu)$ -condition. This follows because for every two building blocks $B_1 \in r$ and $B_2 \in q \setminus \text{tr}(q|N)$, either $B_1 < B_M^q = B_2$ or $B_1 < B_M^q \leq B_2$. \square

The foregoing lemma allows to follow the well known side condition pattern for proving (semi)properness: to find $p^* \leq p$ that is a (semi)generic condition for M , simply add M on top of p .

Proposition 5.6. *If NS_{ω_1} is precipitous, then $\mathbb{P}^{c-c}(\lambda, \mu)$ is strongly ssp.*

Proof. Let $\theta \gg \mu$ be regular. Using Proposition 4.4, it suffices to check that $\mathbb{P}^{c-c}(\lambda, \mu)$ is strongly ssp with respect to every countable model $M \prec H_\theta$ that contains λ, μ and satisfies $P2 \nearrow \mathbf{G}_M^{c-c}$. Note that in this case also $P2 \nearrow \mathbf{G}_{M \cap H_\mu}^{c-c}$. Thus, if $p \in M \cap \mathbb{P}^{c-c}(\lambda, \mu)$, choose Σ_M such that $B_M = (M \cap H_\mu, \emptyset, \Sigma_M)$ is a building block, define $p^* = p \cup \{B_M\}$ and note that $p^* \in \mathbb{P}^{c-c}(\lambda, \mu)$. By Lemma 5.5, p^* is strongly $(M, \mathbb{P}^{c-c}(\lambda, \mu))$ -semigeneric. \square

Throughout the rest of this section, we assume that NS_{ω_1} is precipitous and thus that $\mathbb{P}^{c-c}(\lambda, \mu)$ is strongly ssp. Under this assumption, we analyse the generic object.

Definition 5.7. Let G be a $\mathbb{P}^{c-c}(\lambda, \mu)$ -filter generic over V .

- Define

$$C^G = \{\delta(M) : (\exists p \in G)(\exists B \in p) B = (M, z, \Sigma) \text{ for certain } z, \Sigma\}.$$

Clearly, C^G is unbounded in ω_1 , so

- let $(\delta_\alpha^G : \alpha < \omega_1)$ be the increasing enumeration of C^G .
- Define for every $\alpha < \omega_1$,

$$M_\alpha^G = \bigcup_{p \in G} \{\text{Ext}(B) \cap H_\lambda^V : B \in p \text{ is an } M\text{-block and } M \cap \omega_1 = \delta_\alpha^G\}.$$

Note that inside $V[G]$, M_α^G is a countable elementary submodel of H_λ^V . Countability of M_α^G follows because for every two $p, q \in G$, and every $(M_1, z_1, \Sigma_1) = B_1 \in p$, $(M_2, z_2, \Sigma_2) = B_2 \in q$:

$$M_1 \cap \omega_1 = M_2 \cap \omega_1 \quad \Rightarrow \quad B_1 \sqsubseteq_{\text{game}} B_2 \text{ or } B_2 \sqsubseteq_{\text{game}} B_1.$$

Proposition 5.8. *The sequence $(M_\alpha^G : \alpha < \omega_1)$ is an iteration-inducing filtration of H_λ^V .*

Proof. This is checked through the following series of density arguments.

Claim 5.8.1. *$(M_\alpha^G : \alpha < \omega_1)$ is a continuous sequence of countable elementary substructures of H_λ^V that together cover H_λ^V .*

Proof. First note that for every $x \in H_\lambda^V$ there are densely many $p \in \mathbb{P}^{c-c}(\lambda, \mu)$ that contain some block $B = (M, z, \Sigma)$ with $x \in M$.

Suppose next that $\beta < \omega_1$ is limit, then

$$\bigcup_{\alpha < \beta} M_\alpha^G = M_\beta^G$$

because

\sqsubseteq if $\alpha < \beta$ and $x \in M_\alpha^G$, then there exists $p \in G$ with $(M_1, z_1, \Sigma_1), (M_2, z_2, \Sigma_2) \in p$ such that $M_1 \cap \omega_1 = \delta_\alpha^G$, $M_2 \cap \omega_1 = \delta_\beta^G$ and $x \in \text{Hull}(M_1, P2(z_1)) \cap H_\lambda^V$.

Then $(M_1, z_1, \Sigma_1) \triangleleft (M_2, z_2, \Sigma_2)$, so $x \in \text{Hull}(M_2, P2(z_2)) \cap H_\lambda^V$.

\supseteq if $x \in M_\beta^G$, then there exists $p \in G$ and $B_1, B_2 \in p$ such that

$$\begin{aligned} B_1 &= \max\{B \in p : B \triangleleft B_2\}, \\ B_2 &= (M_2, z_2, \Sigma_2), p \Vdash M_2 \cap \omega_1 = \delta_\beta^G, x \in \text{Ext}(B_2), \\ B_1 &= (M_1, z_1, \Sigma_1). \end{aligned}$$

Consider the play z'_1 which is an extension of z_1 with one round in which $P1$ plays x as a \mathbf{G}^{cap} -move and $P2$ answers following Σ_1 . Then $p' = p \setminus \{B_1\} \cup \{(M_1, z'_1, \Sigma_1)\} \in \mathbb{P}^{c-c}(\lambda, \mu)$ forces that $x \in \bigcup_{\alpha < \beta} M_\alpha^G$. \blacksquare

Claim 5.8.2. *Every M_α^G is selfgeneric.*

Proof. Suppose that $p \in \mathbb{P}^{c-c}(\lambda, \mu)$ with $(M, z, \Sigma) \in p$, $p \Vdash M \cap \omega_1 = \delta_\alpha^G$ and \mathcal{A} is a maximal antichain of $\text{NS}_{\omega_1}^+$ such that $\mathcal{A} \in \text{Hull}(M, P2(z))$. Then consider the play z' which is an extension of z with one round in which $P1$ plays \mathcal{A} as a \mathbf{G}^{cat} -move and $P2$ answers following Σ , say the answer is $S \in \mathcal{A}$. Then $p' = p \setminus \{(M, z, \Sigma)\} \cup \{(M, z', \Sigma)\} \in \mathbb{P}^{c-c}(\lambda, \mu)$ forces that $S \in M_\alpha^G \cap \mathcal{A}$ and $M_\alpha^G \cap \omega_1 = \delta_\alpha^G \in S$. \blacksquare

Claim 5.8.3. $M_{\alpha+1}^G = \text{Hull}(M_\alpha^G, \delta_\alpha^G)$.

Proof. \supseteq is already clear.

\subseteq Suppose that $p \in \mathbb{P}^{c-c}(\lambda, \mu)$ with $(M_1, z_1, \Sigma_1), (M_2, z_2, \Sigma_2) \in p$, $p \Vdash M_1 \cap \omega_1 = \delta_\alpha^G$, $p \Vdash M_2 \cap \omega_1 = \delta_{\alpha+1}^G$ and $x \in \text{Hull}(M_2, P2(z_2)) \cap H_\lambda^V$ then consider the play \tilde{z}_1 which is an extension of z_1 with one round in which $P1$ plays x as a \mathbf{G}^{cap} -move and $P2$ answers following Σ_1 . Say this answer of $P2$ is the function f . Then $\tilde{p} = p \setminus \{(M_1, z_1, \Sigma_1)\} \cup \{(M_1, \tilde{z}_1, \Sigma_1)\} \in \mathbb{P}^{c-c}(\lambda, \mu)$ forces that $x = f(\delta_\alpha^G) \in \text{Hull}(M_\alpha^G, \delta_\alpha^G)$. \blacksquare

□

Definition 5.9. For every $\alpha < \omega_1$, define \mathcal{M}_α^G to be the transitive collapse of M_α^G .

We now put the foregoing together.

Theorem 5.10. *Suppose NS_{ω_1} is precipitous. If G is a $\mathbb{P}^{c-c}(\lambda, \mu)$ -filter generic over V , then in $V[G]$ there exists a generic iteration $\mathbf{It} = (\mathcal{M}_\beta, j_{\alpha\beta}, G_\alpha : \alpha < \beta < \omega_1)$ of length ω_1 of the model \mathcal{M}_0^G with direct limit equal to H_λ^V . Moreover, if $H_\lambda^\#$ exists in V , then the model \mathcal{M}_0^G is iterable.*

Proof. This follows from Proposition 5.8 together with Lemma 2.5. Iterability of \mathcal{M}_0^G follows from Lemma 2.3. \square

By combining Theorem 5.10 and Woodin's Lemma 2.6, one derives the following conclusions. Note in the first point of Corollary 5.11 that the $\mathbb{P}^{c-c}(\lambda, \mu)$ -extension is closed under sharps for reals because of Lemma 2.7. In the third point of Corollary 5.11, we reobtain a result of Claverie and Schindler ([2, Corollary 21]), it is implied by the second point of Corollary 5.11 together with Schindler's theorem that BMM implies the existence of sharps for all sets (see [13, Theorem 1.3]).

Corollary 5.11. *Suppose NS_{ω_1} is precipitous.*

(1) *If $H_\theta^\#$ exists for $\theta = (2^{<\mu})^+$, then*

$$\mathbb{P}^{c-c}(\lambda, \mu) \Vdash \mathbf{u}_2 > \lambda.$$

(2) *If $H_\lambda^\#$ exists and the bounded forcing axiom for the forcing $\mathbb{P}^{c-c}(\lambda, \mu)$ holds,*


$$\text{then } \mathbf{u}_2 = \omega_2.$$

(3) [2, Corollary 21] *If Bounded Martin's Maximum BMM holds, then $\mathbf{u}_2 = \omega_2$.*

Remark 5.12.

(1) By [5, Theorem 5.7], the forcing $\mathbb{P}^{c-c}(\lambda, \mu)$ is semiproper if and only if the principle (\dagger) holds, that is, $\mathbb{P}^{c-c}(\lambda, \mu)$ is semiproper if and only if every ssp forcing is semiproper. Also by [5, Theorem 5.7], (\dagger) is equivalent to $\text{Glob-SCC}^{\text{cof}}$ which by Proposition 4.8 is equivalent to the existence of club many countable $M \prec H_\theta$ for all $\theta \gg \lambda$ such that $P2 \nearrow \mathbf{G}_M^{c-c}(H_\lambda)$.

- (2) It is interesting to compare the forcing $\mathbb{P}^{c-c}(\lambda, \mu)$ with the forcings from [2] and [9]; and although all three are very similar, the exact relation between these forcings remains largely open. We note that the model M_0^G , added by $\mathbb{P}^{c-c}(\lambda, \mu)$, has the same strong properties as the countable family of functions that occurs on the generic branch of the Namba forcing in [9]. Namely:
- every countable elementary submodel of H_λ^V that contains M_0^G as a subset is selfgeneric,
 - M_0^G contains as a subset a countable set of functions $\{g_n : n < \omega\}$ from $\omega_1^{<\omega}$ to H_λ^V such that $H_\lambda^V = \bigcup_{n < \omega} \text{ran}(g_n)$.
- The second of these two points is already clear from the foregoing, the first one is slightly more surprising and can be checked through a similar density argument.
- (3) The proof of [5, Lemma 2.1] can directly be adapted to show that if NS_{ω_1} is precipitous, $H_\theta^\#$ exists for $\theta = (2^{<\mu})^+$, and the bounded forcing axiom for the forcing $\mathbb{P}^{c-c}(\lambda, \mu)$ holds, then also the admissible club guessing principle ACG holds. ACG states that the club-filter on ω_1 is generated by sets of the form $C_x = \{\alpha < \omega_1 : L_\alpha[x] \models \text{KP}\}$ with $x \in \mathbb{R}$.

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