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Majdouline Borji^{a)}  and Christoph Kopper^{b)} 

AFFILIATIONS

Centre de Physique Théorique CPHT, CNRS, UMR 7644, Institut Polytechnique de Paris, 91128 Palaiseau, France

^{a)} Author to whom correspondence should be addressed: majdouline.borji@polytechnique.edu

^{b)} Electronic mail: christoph.kopper@polytechnique.edu

ABSTRACT

In this paper, we give a rigorous proof of the renormalizability of the massive ϕ_4^4 theory on a half-space using renormalization group flow equations. We find that five counterterms are needed to make the theory finite, namely, ϕ^2 , $\phi\partial_z\phi$, $\phi\partial_z^2\phi$, $\phi\Delta_x\phi$, and ϕ^4 for $(z, x) \in \mathbb{R}^+ \times \mathbb{R}^3$. The amputated correlation functions are distributions in position space. We consider a suitable class of test functions and prove inductive bounds for the correlation functions folded with these test functions. The bounds are uniform in the cutoff and, thus, directly lead to renormalizability.

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I. INTRODUCTION

The renormalization of quantum field theories that break translation invariance is of great importance since many interesting quantum field theories break this symmetry. One may ask if the renormalizability of a given theory only depends on the interaction introduced and the dimension of space-time or whether it also depends on the geometrical and topological properties of the space-time. In a previous work,¹ we studied the ϕ_4^4 -theory on a lattice, which is a regularization scheme that breaks translation invariance, and we found that the theory is renormalizable and the Euclidean symmetries are restored for renormalized correlation functions. In Ref. 2, the authors considered the breaking of translation invariance by studying the ϕ_4^4 interaction on a Riemannian manifold and found that it is renormalizable. Only one additional counterterm that renormalizes the curvature is needed, compared to the ϕ_4^4 -theory in the Euclidean space-time. In this work, we are interested in the renormalizability of the ϕ_4^4 scalar field theory on a space with a boundary, which is another manifestation of the breaking of translation invariance.

A simple model to study surface effects in quantum field theory is the semi-infinite scalar field model, which first appeared in 1971.³ It is defined starting from the massive ϕ_4^4 model in infinite space, with the difference that it is defined on a half space bounded by a plane. In this model, three types of boundary conditions are considered in the literature, namely, Dirichlet, Neumann, and Robin boundary conditions (b.c.'s.). From a mathematical point of view, each b.c. corresponds to a self-adjoint extension of the Laplacian in $\mathbb{R}^+ \times \mathbb{R}^3$. The self-adjointness of the Laplacian is normally required in order to be able to define the propagator of a quantum field theory. Each boundary condition defines a particular propagator.

Lubensky and Rubin^{4,5} studied a model of ferromagnetically coupled classical spins on a semi-infinite lattice. Using a mean-field approach, they provided a qualitatively correct understanding of the different phases undergone by the system, which are the ordinary, extraordinary, surface, and special transitions. The phenomenological theory of scaling⁶⁻⁸ was generalized to surfaces, and it implied relations between bulk critical exponents and the additional surface critical exponents, needed to describe the singular behavior of surface related properties. However, renormalization of the model is necessary⁹⁻¹³ when one wants to go beyond the mean-field approximation.

Diehl and Dietrich¹⁴⁻¹⁶ studied the critical behavior of the semi-infinite system using renormalization group methods. They considered the ordinary¹⁵ and special transitions,¹⁶ which correspond, respectively, to the Dirichlet and Robin boundary conditions and found that in addition to the usual two bulk counterterms, an additional surface counterterm is needed to make the two-point function finite in the case of

the Dirichlet boundary condition. For the Robin boundary condition, two surface counterterms are needed. The calculations were performed to two-loop order using dimensional regularization, and the surface counterterms were obtained by inserting the operators $\lim_{z \rightarrow 0} \partial_z \phi(z, x)$, where $x \in \mathbb{R}^3$ in the case of Dirichlet b.c. and $\phi(0, x)$, $\phi^2(0, x)$ in the case of the Robin b.c., where ϕ is the considered scalar field.

The semi-infinite model was also addressed by Symanzik in his study of the Schrödinger representation for renormalizable quantum fields¹⁷ in which he not only discussed the renormalization of surface operators in a different context but also found that surface counterterms are required to make the two-point function finite.

In Ref. 18, Albuquerque calculated the one-loop two-point function using a cut-off regularization in the case of the Robin boundary condition. In addition to the usual mass counterterm, two additional counterterms are needed to make the (non-amputated) tadpole finite. They depend on whether the external points are on the surface or not. If none of them is on the surface, then only one surface counterterm is required, and it diverges linearly in the cutoff. However, if at least one of the external points lies on the surface, then in addition to the linearly diverging counterterm, an extra surface counterterm is needed, and it diverges logarithmically with the cutoff. These findings suggest that the renormalization of the amputated and unamputated diagrams is different.⁹

In this paper, we give a rigorous proof of the renormalizability of the ϕ_4^4 massive semi-infinite model using the renormalization group flow equations (FEs).^{11,12} This paper is organized as follows. In Sec. II, we present the semi-infinite model scalar field theory with all the possible boundary conditions and their associated propagators. We also present the properties of the flowing propagator and the associated heat kernel together with the considered action and the system of perturbative flow equations satisfied by the connected amputated Schwinger (CAS) distributions. Section III will be devoted to prove some regularity properties of the support of the gaussian measure associated with the regularized propagator. To establish bounds on the CAS, which are distributions, they have to be folded first with test functions. In Sec. IV, a suitable class of test functions is introduced, together with tree structures that will be used in the bounds on the CAS to be derived. In Sec. V, we state the boundary and the renormalization conditions used to integrate the flow equations of the irrelevant and relevant terms, respectively. Section VI is the central one of this paper. We state and prove inductive bounds on the amputated Schwinger distributions folded with the introduced test functions, which, being uniform in the cutoff, directly lead to renormalizability.

II. THE ACTION AND THE FLOW EQUATIONS

A. The half space and the possible boundary conditions

We consider the half space or what we call also the semi-infinite space $\mathbb{R}^+ \times \mathbb{R}^3$, where $\mathbb{R}^+ := [0, \infty)$ and $\mathbb{R}^{++} := (0, \infty)$.

Let $\mathcal{C}_0^\infty(\mathbb{R}^+ \times \mathbb{R}^3)$ be the space of compactly supported smooth functions defined on the considered half space. For $k \in \mathbb{N}^*$, we consider the Hilbert Sobolev spaces $H^k := W^{k,2}$. We denote by $H_0^1(\mathbb{R}^+)$ the set of functions in $H^1(\mathbb{R}^+)$ that vanish at the boundary 0. The Laplacian Δ defined on the Hilbert space $L^2(\mathbb{R}^+ \times \mathbb{R}^3)$ has the following self-adjoint extensions:

- The Dirichlet Laplacian Δ_D defined by

$$\forall u \in \mathcal{C}_0^\infty(\mathbb{R}^+ \times \mathbb{R}^3), \quad \Delta_D u = \Delta u$$

with the domain

$$D(\Delta_D) := \left\{ u \in L^2(\mathbb{R}^+ \times \mathbb{R}^3) \mid u(z, \cdot) \in H^2(\mathbb{R}^3) \ \forall z \geq 0; \right. \\ \left. u(\cdot, x) \in H_0^1(\mathbb{R}^+) \cap H^2(\mathbb{R}^+) \ \forall x \in \mathbb{R}^3 \right\}.$$

- The Neumann Laplacian Δ_N defined by

$$\forall u \in \mathcal{C}_0^\infty(\mathbb{R}^+ \times \mathbb{R}^3), \quad \Delta_N u = \Delta u$$

with the domain

$$D(\Delta_N) := \left\{ u \in L^2(\mathbb{R}^+ \times \mathbb{R}^3) \mid u(z, \cdot) \in H^2(\mathbb{R}^3) \ \forall z \geq 0; \right. \\ \left. u(\cdot, x) \in H^2(\mathbb{R}^+), \partial_z u(z, x)|_{z=0} = 0 \ \forall x \in \mathbb{R}^3 \right\}.$$

- The Robin Laplacian Δ_R defined by

$$\forall u \in \mathcal{C}_0^\infty(\mathbb{R}^+ \times \mathbb{R}^3), \quad \Delta_R u = \Delta u$$

with the domain

$$D(\Delta_R) := \{u \in L^2(\mathbb{R}^+ \times \mathbb{R}^3) \mid u(z, \cdot) \in H^2(\mathbb{R}^3) \ \forall z \geq 0; u(\cdot, x) \in H^2(\mathbb{R}^+), \\ \partial_z u(z, x)|_{z=0} = cu(0, x) \ \forall x \in \mathbb{R}^3\}, \quad c > 0.$$

Each self-adjoint extension corresponds to a possible boundary condition. The massive propagators associated with these boundary conditions are defined from the functional calculus by

$$C_\bullet((z, x); (z', x')) = \int_0^\infty d\lambda \ e^{-\lambda(-\Delta_\bullet + m^2)}((z, x); (z', x')), \quad (1)$$

where $\bullet \in \{D, R, N\}$ for, respectively, Dirichlet, Robin, and Neumann boundary conditions. (1) can be written in terms of the heat kernels as

$$C_D((z, x); (z', x')) = \int_0^\infty d\lambda \ e^{-\lambda m^2} p_B(\lambda; x, x') \frac{1}{\sqrt{2\pi\lambda}} \left(e^{-\frac{(z-z')^2}{2\lambda}} - e^{-\frac{(z+z')^2}{2\lambda}} \right), \\ C_N((z, x); (z', x')) = \int_0^\infty d\lambda \ e^{-\lambda m^2} p_B(\lambda; x, x') \frac{1}{\sqrt{2\pi\lambda}} \left(e^{-\frac{(z-z')^2}{2\lambda}} + e^{-\frac{(z+z')^2}{2\lambda}} \right), \\ C_R((z, x); (z', x')) = \int_0^\infty d\lambda \ e^{-\lambda m^2} p_B(\lambda; x, x') p_R(\lambda; z, z'),$$

where

$$p_B(\lambda; x, x') := \frac{1}{(2\pi\lambda)^{\frac{3}{2}}} e^{-\frac{(x-x')^2}{2\lambda}} \quad (2)$$

and

$$p_R(\lambda; z, z') := p_N(\lambda; z, z') - 2 \int_0^\infty \frac{dw}{\sqrt{2\pi\lambda}} e^{-w} e^{-\frac{(z+z'+\frac{w}{c})^2}{2\lambda}}. \quad (3)$$

Here, p_N denotes the one-dimensional Neumann heat kernel,

$$p_N(\lambda; z, z') := \frac{1}{\sqrt{2\pi\lambda}} \left(\frac{e^{-\frac{(z-z')^2}{2\lambda}} + e^{-\frac{(z+z')^2}{2\lambda}}}{2} \right). \quad (4)$$

In the pz -representation, which corresponds to taking the partial Fourier transformation with respect to the variable $x \in \mathbb{R}^3$, the Dirichlet, Neumann, and Robin propagators simply read

$$C_D(p; z, z') = \frac{1}{2\sqrt{p^2 + m^2}} \left[e^{-\sqrt{p^2 + m^2}|z-z'|} - e^{-\sqrt{p^2 + m^2}|z+z'|} \right], \quad (5)$$

$$C_N(p; z, z') = \frac{1}{2\sqrt{p^2 + m^2}} \left[e^{-\sqrt{p^2 + m^2}|z-z'|} + e^{-\sqrt{p^2 + m^2}|z+z'|} \right], \quad (6)$$

$$C_R(p; z, z') = \frac{1}{2\sqrt{p^2 + m^2}} \left[e^{-\sqrt{p^2 + m^2}|z-z'|} + \frac{\sqrt{p^2 + m^2} - c}{\sqrt{p^2 + m^2} + c} e^{-\sqrt{p^2 + m^2}|z+z'|} \right]. \quad (7)$$

The Dirichlet boundary condition corresponds to $c \rightarrow \infty$, and the Neumann boundary condition corresponds to $c = 0$. We study the Robin boundary condition since the other two conditions are limit cases of the former. One can easily verify that we have

$$C_D(p; 0, z') = C_D(p; z, 0) = 0, \quad \lim_{z \rightarrow 0} \partial_z C_N(p; z, z') = \lim_{z' \rightarrow 0} \partial_{z'} C_N(p; z, z') = 0, \quad (8)$$

$$\lim_{z \rightarrow 0} \partial_z C_R(p; z, z') = c C_R(p; 0, z'), \quad \lim_{z' \rightarrow 0} \partial_{z'} C_R(p; z, z') = c C_R(p; z, 0), \quad (9)$$

where we used that the associated heat kernels verify, respectively, the Dirichlet, Neumann, and Robin boundary conditions.

B. ϕ_4^4 scalar field theory on the semi-infinite space

We will analyze the perturbative renormalizability of the semi-infinite ϕ_4^4 theory with Robin boundary conditions. It will be proved by analyzing the generating functional L^{Λ, Λ_0} of connected amputated Schwinger (CAS) distributions. The upper indices Λ_0 and Λ enter through the regularized propagator. We choose the following regularization:

$$C_R^{\Lambda, \Lambda_0}(p; z, z') = \int_{\frac{1}{\Lambda_0^2}}^{\frac{1}{\Lambda^2}} d\lambda e^{-\lambda(p^2+m^2)} p_R(\lambda; z, z'). \tag{10}$$

Clearly, C_R^{Λ, Λ_0} verifies the b.c. (9). For $\Lambda \rightarrow 0$ and $\Lambda_0 \rightarrow \infty$, we recover the unregularized propagator (7). We denote

$$\dot{C}_R^\Lambda(p; z, z') = \frac{\partial}{\partial \Lambda} C_R^{\Lambda, \Lambda_0}(p; z, z') = \dot{C}^\Lambda(p) p_R\left(\frac{1}{\Lambda^2}; z, z'\right), \tag{11}$$

where $\dot{C}^\Lambda(p) = -\frac{2}{\Lambda^3} e^{-\frac{p^2+m^2}{\Lambda^2}}$.

The starting point in writing an Euclidean quantum field theory is to define the associated path integral given by the corresponding gaussian measure. We assume that $0 \leq \Lambda \leq \Lambda_0 < \infty$ so that the flow parameter Λ takes the role of an infrared cutoff, whereas Λ_0 is a UV cutoff. The full propagator is recovered for $\Lambda = 0$ and $\Lambda_0 \rightarrow \infty$. For finite Λ_0 and infinite volume, the positivity and the regularity properties of C_R^{Λ, Λ_0} permit to define the theory rigorously from the functional integral,

$$e^{-\frac{1}{\hbar}(L^{\Lambda, \Lambda_0}(\phi) + I^{\Lambda, \Lambda_0})} := \int d\mu_{\Lambda, \Lambda_0, R}(\Phi) e^{-\frac{1}{\hbar} L^{\Lambda, \Lambda_0}(\Phi + \phi)}, \tag{12}$$

$$L^{\Lambda, \Lambda_0}(0) = 0,$$

where the factors of \hbar have been introduced to allow for a consistent loop expansion in the sequel. Here, $d\mu_{\Lambda, \Lambda_0, R}$ denotes the Gaussian measure with covariance $\hbar C_R^{\Lambda, \Lambda_0}$. The test functions ϕ and Φ are supposed to be in the support of the Gaussian measure $d\mu_{\Lambda, \Lambda_0, R}$, which, in particular, implies that they are in $\mathcal{C}^\infty(\mathbb{R}^+ \times \mathbb{R}^3)$, as we will prove in Sec. III. The normalization factor $e^{-\frac{1}{\hbar} I^{\Lambda, \Lambda_0}}$ is due to vacuum contributions. It diverges in infinite volume so that we can take the infinite volume limit only when it has been eliminated.¹⁹ We do not make the finite volume explicit here since it plays no role in the sequel.

The functional $L^{\Lambda, \Lambda_0}(\phi)$ is the bare interaction of a renormalizable theory, including counterterms, viewed as a formal power series in \hbar . For shortness, we will pose in the following, with $z \in \mathbb{R}^+$, $p \in \mathbb{R}^3$, and $x \in \mathbb{R}^3$,

$$\int_z := \int_0^\infty dz, \quad \int_p := \int_{\mathbb{R}^3} \frac{d^3 p}{(2\pi)^3}, \quad \int_S := \int_{\mathbb{R}^3} d^3 x, \quad \int_V := \int_0^\infty dz \int_{\mathbb{R}^3} d^3 x.$$

Since translation invariance is broken in the z -direction (the semi-line), all counterterms may be z -dependent. In general, the constraints on the bare action result from the symmetry properties of the theory, which are imposed on its field content and on the form of the propagator. It is, therefore, natural to consider the general bare interaction,

$$L^{\Lambda_0, \Lambda_0}(\phi) = \frac{\lambda}{4!} \int_V \phi^4(z, x) + \frac{1}{2} \int_V \left(a^{\Lambda_0}(z) \phi^2(z, x) - b^{\Lambda_0}(z) \phi(z, x) \Delta_x \phi(z, x) - d^{\Lambda_0}(z) \phi(z, x) \partial_z^2 \phi(z, x) + s^{\Lambda_0}(z) \phi(z, x) (\partial_z \phi)(z, x) + \frac{2}{4!} c^{\Lambda_0}(z) \phi^4(z, x) \right). \tag{13}$$

Here, we supposed the theory to be symmetric under $\phi \rightarrow -\phi$, and we included only relevant terms with respect to (12) in the sense of the renormalization group. The functions $a^{\Lambda_0}(z)$, $b^{\Lambda_0}(z)$, $c^{\Lambda_0}(z)$, $d^{\Lambda_0}(z)$, and $s^{\Lambda_0}(z)$ are supposed to be smooth.

The flow equation (FE) is obtained from (12) on differentiating with respect to Λ . For the steps of the computation, we refer the reader to Refs. 19 and 20. It is a differential equation for the functional L^{Λ, Λ_0} ,

$$\partial_\Lambda (L^{\Lambda, \Lambda_0} + I^{\Lambda, \Lambda_0}) = \frac{\hbar}{2} \left\langle \frac{\delta}{\delta \phi}, \dot{C}_R^\Lambda \frac{\delta}{\delta \phi} \right\rangle L^{\Lambda, \Lambda_0} - \frac{1}{2} \left\langle \frac{\delta}{\delta \phi} L^{\Lambda, \Lambda_0}, \dot{C}_R^\Lambda \frac{\delta}{\delta \phi} L^{\Lambda, \Lambda_0} \right\rangle. \tag{14}$$

By $\langle \cdot, \cdot \rangle$ we denote the standard inner product in $L^2(\mathbb{R}^+ \times \mathbb{R}^3)$.

We may expand the functional $L^{\Lambda, \Lambda_0}(\phi)$ in a formal power series with respect to \hbar ,

$$L^{\Lambda, \Lambda_0}(\phi) = \sum_{l=0}^\infty \hbar^l L_l^{\Lambda, \Lambda_0}(\phi).$$

Corresponding expansions for $a^{\Lambda_0}(z)$, $b^{\Lambda_0}(z)$, etc., are $a^{\Lambda_0}(z) = \sum_{l=0}^{\infty} \hbar^l a_l^{\Lambda_0}(z)$, etc. From $L_l^{\Lambda_0, \Lambda_0}(\phi)$, we obtain the CAS distributions of loop order l as

$$\mathcal{L}_{l,n}^{\Lambda_0, \Lambda_0}((z_1, x_1), \dots, (z_n, x_n)) := \delta_{\phi(z_1, x_1)} \cdots \delta_{\phi(z_n, x_n)} L_l^{\Lambda_0, \Lambda_0} |_{\phi=0},$$

where we used the notation $\delta_{\phi(z,x)} = \delta/\delta\phi(z,x)$.

Since translation invariance in the x -directions is preserved, we will use in all what follows a mixed representation, where the Fourier transform to p -space is performed only with respect to $x \in \mathbb{R}^3$. In this representation, we set

$$\mathcal{L}_{l,n}^{\Lambda_0, \Lambda_0}(z_1; \vec{p}_n; \Phi_n) = \int_0^\infty dz_2 \cdots dz_n \mathcal{L}_{l,n}^{\Lambda_0, \Lambda_0}((z_1, p_1), \dots, (z_n, p_n)) \phi_2(z_2) \cdots \phi_n(z_n). \quad (15)$$

Here, we denote

$$\Phi_n(z_2, \dots, z_n) := \prod_{i=2}^n \phi_i(z_i), \quad \vec{p}_n := (p_1, \dots, p_n) \in \mathbb{R}^{3n}, \quad \|\vec{p}_n\| := \sup_{1 \leq i \leq n} |p_i|$$

and

$$\delta^{(3)}(p_1 + \cdots + p_n) \mathcal{L}_{l,n}^{\Lambda_0, \Lambda_0}((z_1, p_1), \dots, (z_n, p_n)) = (2\pi)^{3(n-1)} \frac{\delta^n}{\delta\phi(z_1, p_1) \cdots \delta\phi(z_n, p_n)} L_l^{\Lambda_0, \Lambda_0}(\phi) |_{\phi=0}.$$

$\delta^{(3)}(p_1 + \cdots + p_n)$ appears because of the translation invariance in the x directions. The FEs for the CAS distributions derived from (14) are^{2,20}

$$\begin{aligned} \partial_\Lambda \partial^w \mathcal{L}_{l,n}^{\Lambda_0, \Lambda_0}((z_1, p_1), \dots, (z_n, p_n)) &= \frac{1}{2} \int_z \int_{z'} \int_k \partial^w \mathcal{L}_{l-1, n+2}^{\Lambda_0, \Lambda_0}((z_1, p_1), \dots, (z_n, p_n), (z, k), (z', -k)) \dot{C}_R^\Lambda(k; z, z') \\ &\quad - \frac{1}{2} \int_z \int_{z'} \sum_{l_1, l_2}^l \sum_{n_1, n_2} c_{w_i} \left[\partial^{w_1} \mathcal{L}_{l_1, n_1+1}^{\Lambda_0, \Lambda_0}((z_1, p_1), \dots, (z_{n_1}, p_{n_1}), (z, p)) \partial^{w_3} \dot{C}_R^\Lambda(p; z, z') \right. \\ &\quad \times \left. \partial^{w_2} \mathcal{L}_{l_2, n_2+1}^{\Lambda_0, \Lambda_0}((z', -p), \dots, (z_n, p_n)) \right]_{\text{rsym}}, \\ p &= -p_1 - \cdots - p_{n_1} = p_{n_1+1} + \cdots + p_n. \end{aligned} \quad (16)$$

Here, we wrote (16) directly in a form where a number $|w|$ of momentum derivatives, characterized by a multi-index, act on both sides and we used the shorthand notation,

$$\partial^w := \prod_{i=1}^n \prod_{\mu=0}^3 \left(\frac{\partial}{\partial p_{i,\mu}} \right)^{w_{i,\mu}} \quad \text{with } w = (w_{1,0}, \dots, w_{n,3}), \quad |w| = \sum w_{i,\mu}, \quad w_{i,\mu} \in \mathbb{N}^*. \quad (17)$$

The symbol “rsym” means summation over those permutations of the momenta $(z_1, p_1), \dots, (z_n, p_n)$, which do not leave invariant the (unordered) subsets $((z_1, p_1), \dots, (z_{n_1}, p_{n_1}))$ and $((z_{n_1+1}, p_{n_1+1}), \dots, (z_n, p_n))$ and, therefore, produce mutually different pairs of (unordered) image subsets, and the primes restrict the summations to $n_1 + n_2 = n$, $l_1 + l_2 = l$, and $w_1 + w_2 + w_3 = w$, respectively. The combinatorial factor $c_{\{w_i\}} = w!(w_1!w_2!w_3!)^{-1}$ stems from Leibniz’s rule. In the loop order $l = 0$, the first term on the RHS is absent.

III. REGULARITY OF THE SUPPORT OF THE REGULARIZED GAUSSIAN MEASURE

The bare interaction L^{Λ_0, Λ_0} is composed of powers of the field ϕ and of its derivatives. It cannot be given any mathematical meaning if the field ϕ is not sufficiently regular, e.g., in $\mathcal{C}^2(\mathbb{R}^+ \times \mathbb{R}^3)$. In this section, we prove that the field ϕ belongs to $\mathcal{C}^\infty(\mathbb{R}^+ \times \mathbb{R}^3)$. This result is due to the regularity properties of the support of the gaussian measure $\mu_{\Lambda_0, R}$.

Since the theory that we study is massive, we consider the UV-regularized propagator without an infrared cut-off

$$C_R^{\Lambda_0}(p; x, y) = \int_{\frac{1}{\Lambda_0^2}}^\infty d\lambda \ e^{-\lambda(p^2 + m^2)} p_R(\lambda; x, y),$$

assuming that $\Lambda_0 \geq 1$, $p \in \mathbb{R}^3$, and $x, y \in \mathbb{R}^+$.

The same arguments work for the Gaussian measure associated with the propagator $C_R^{\Lambda_0, \Lambda_0}(p; x, y)$. We prove the following.

Proposition 1. Let $\mu_{\Lambda_0, R}$ be the Gaussian measure associated with the propagator $C_R^{\Lambda_0}$. The support of $\mu_{\Lambda_0, R}$ satisfies by

$$\text{supp}\mu_{\Lambda_0,R} \subset \bigcap_{n \geq 1} \left\{ (-\Delta_R + m^2)^{-n} L^2(\mathbb{R}^+ \times \mathbb{R}^3) \right\}.$$

Before stating the Proof of Proposition 1, we recall the following corollary of the Minlos theorem.^{13,21,22}

Corollary 1. Given a nuclear space E , μ , a measure on E' , and C its characteristic function, we introduce a continuous inner product $(\cdot, \cdot)_0$ on E and let H_0 be the completion of E with respect to $(\cdot, \cdot)_0$. Suppose that C is continuous on H_0 . Let T be a Hilbert–Schmidt operator on H_0 satisfying the following:

- (a) T is one to one (injective map).
- (b) $E \subset \text{Im}T$ and $T^{-1}(E)$ is dense in H_0 .
- (c) The map $T^{-1} : E \rightarrow H_0$ is continuous.

Then, the support of μ is on $(T^{-1})^* H'_0 \subset E'$. The notations $(T^{-1})^*$ and H'_0 are used for the “adjoint” and the “dual space” in the pairing between E and E' .

For a proof of this corollary, see Refs. 21 and 22.

Proof. We apply Corollary 1 to $E = \mathcal{S}(\mathbb{R}^+ \times \mathbb{R}^3) \cong \mathcal{S}(\mathbb{R}^+) \otimes \mathcal{S}(\mathbb{R}^3)$. This is a nuclear space, which is a tensor product of two nuclear spaces. See Ref. 23 for the proof that $\mathcal{S}(\mathbb{R}^+)$ is a nuclear space. Theorem A.4.1 in Ref. 24 implies the existence of the Gaussian measure $\mu_{\Lambda_0,R}$ with covariance $C_R^{\Lambda_0}$ with support included in $\mathcal{S}'(\mathbb{R}^+ \times \mathbb{R}^3)$. We apply the corollary of Minlos’s theorem to the scalar product $\langle f, g \rangle_n := \langle f, P^{-2n} g \rangle$, where $\langle \cdot, \cdot \rangle$ is the usual scalar product in $L^2(\mathbb{R}^+ \times \mathbb{R}^3)$ and $P = -\Delta_R + m^2$. P^n is a unitary map from $L^2(\mathbb{R}^+ \times \mathbb{R}^3)$ into H_{-n} , the completion of $\mathcal{S}(\mathbb{R}^+ \times \mathbb{R}^3)$ with respect to $\langle \cdot, \cdot \rangle_n$. We verify first that the regularized covariance is continuous on H_{-n} for any $n \in \mathbb{N}$, that is,

$$\exists C > 0 \text{ such that } \forall f, g \in H_{-n} : \quad \left| \langle f, C_R^{\Lambda_0} g \rangle \right| \leq C \|f\|_{H_{-n}} \|g\|_{H_{-n}}.$$

One can verify that the operators $C_R^{\Lambda_0}$ and $(-\Delta_R + m^2)^{-n}$ commute. Since $(-\Delta_R + m^2)^{-n}$ is self-adjoint, we obtain

$$\langle f, C_R^{\Lambda_0} g \rangle = \left\langle (-\Delta_R + m^2)^{-n} f, (-\Delta_R + m^2)^{2n} C_R^{\Lambda_0} (-\Delta_R + m^2)^{-n} g \right\rangle. \quad (18)$$

By the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned} \left| \langle f, C_R^{\Lambda_0} g \rangle \right| &\leq \int \frac{d^3 p}{(2\pi)^3} \left\| (-\Delta_R + m^2)^{-n} f \right\|_{L^2(\mathbb{R}^+)}(p) \\ &\quad \times \left\| (-\Delta_R + m^2)^{2n} C_R^{\Lambda_0} (-\Delta_R + m^2)^{-n} g \right\|_{L^2(\mathbb{R}^+)}(p), \end{aligned} \quad (19)$$

where

$$\begin{aligned} &\left\| (-\Delta_R + m^2)^{2n} C_R^{\Lambda_0} (-\Delta_R + m^2)^{-n} g \right\|_{L^2(\mathbb{R}^+)}^2(p) \\ &= \int_0^\infty dx \left| \int_0^\infty dy (-\Delta_R + m^2)_x^{2n} C_R^{\Lambda_0}(p; x, y) \int_0^\infty dz (-\Delta_R + m^2)^{-n}(p; y, z) g(z, p) \right|^2. \end{aligned} \quad (20)$$

Using again the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned} &\left| \int_0^\infty dy (-\Delta_R + m^2)_x^{2n} C_R^{\Lambda_0}(p; x, y) \int_0^\infty dz (-\Delta_R + m^2)^{-n}(p; y, z) g(z, p) \right|^2 \\ &\leq \int_0^\infty dy \left| (-\Delta_R + m^2)_x^{2n} C_R^{\Lambda_0}(p; x, y) \right|^2 \left\| (-\Delta_R + m^2)^{-n} g \right\|_{L^2(\mathbb{R}^+)}^2. \end{aligned} \quad (21)$$

Therefore, (20) can be bounded by

$$\left(\int_0^\infty dx \int_0^\infty dy \left| (-\Delta_R + m^2)_x^{2n} C_R^{\Lambda_0}(p; x, y) \right|^2 \right) \left\| (-\Delta_R + m^2)^{-n} g \right\|_{L^2(\mathbb{R}^+)}^2. \quad (22)$$

We have

$$(-\Delta_R + m^2)^{2n}_x C_R^{\Lambda_0}(p; x, y) = \int_{\frac{1}{\Lambda_0}}^{\infty} d\lambda e^{-\lambda(p^2+m^2)} (-\partial_x^2 + p^2 + m^2)^{2n} p_R(\lambda; x, y). \tag{23}$$

Using the Leibniz formula, we get

$$(-\partial_x^2 + p^2 + m^2)^{2n} p_R(\lambda; x, y) = \sum_{k=0}^{2n} C_{2n}^k (-1)^k (p^2 + m^2)^{2n-k} \partial_x^{2k} p_R(\lambda; x, y). \tag{24}$$

One can prove by induction

$$\partial_x^{2k} \left(\frac{1}{\sqrt{2\pi\lambda}} e^{-\frac{(x-y)^2}{2\lambda}} \right) = \lambda^{-k} \sum_{i=0}^k c_i(k) \left(\frac{x-y}{\sqrt{\lambda}} \right)^i \left(\frac{1}{\sqrt{2\pi\lambda}} e^{-\frac{(x-y)^2}{2\lambda}} \right), \tag{25}$$

where $c_i(k) \in \mathbb{R}$, which implies that

$$|\partial_x^{2k} p_B(\lambda; x, y)| \leq c_k \lambda^{-k} p_B(2\lambda; x, y).$$

Here, c_k is a positive constant. Therefore, we find using (3) and (4) that

$$|\partial_x^{2k} p_R(\lambda; x, y)| \leq c_k \lambda^{-k} \left(p_B(2\lambda; x, y) + p_B(2\lambda; x, -y) + 2 \int_0^{\infty} dw e^{-w} p_B\left(2\lambda; x, -\frac{w}{c} - y\right) \right).$$

Using that

$$2 \int_0^{\infty} \frac{dw}{\sqrt{2\pi\lambda}} e^{-w} e^{-\frac{(x+y+\frac{w}{c})^2}{2\lambda}} = p_N(\lambda; x, y) - p_R(\lambda; x, y),$$

we obtain

$$\begin{aligned} |\partial_x^{2k} p_R(\lambda; x, y)| &\leq c_k \lambda^{-k} (p_B(2\lambda; x, y) + p_B(2\lambda; x, -y) + p_N(2\lambda; x, y) - p_R(2\lambda; x, y)) \\ &\leq 2c_k \lambda^{-k} p_N(2\lambda; x, y), \end{aligned}$$

and this implies that

$$\left| (-\partial_x^2 + p^2 + m^2)^{2n} p_R(\lambda; x, y) \right| \leq 2C_n \sum_{k=0}^{2n} \lambda^{-k} (p^2 + m^2)^{2n-k} p_N(2\lambda; x, y),$$

where $C_n := \sup_{1 \leq k \leq n} c_k$.

Therefore, we obtain

$$\left| (-\Delta_R + m^2)^{2n} C_R^{\Lambda_0}(p; x, y) \right| \leq \tilde{C}_n \Lambda_0^{4n} \sum_{k=0}^{2n} \left(\frac{p^2 + m^2}{\Lambda_0^2} \right)^{2n-k} \int_{\frac{1}{\Lambda_0}}^{\infty} d\lambda e^{-\lambda(p^2+m^2)} p_N(2\lambda; x, y),$$

where we used that

$$p_R(\tau; x, y) \leq p_N(\tau; x, y) \quad \forall \tau, x, y \in \mathbb{R}^+.$$

C_n and \tilde{C}_n are suitable positive constants that depend on n .

We have by the Cauchy-Schwarz inequality,

$$\begin{aligned} \left| \int_{\frac{1}{\Lambda_0}}^{\infty} d\lambda e^{-\lambda(p^2+m^2)} p_N(2\lambda; x, y) \right|^2 &\leq e^{-\frac{p^2+m^2}{\Lambda_0^2}} \int_{\frac{1}{\Lambda_0}}^{\infty} d\lambda e^{-\lambda(p^2+m^2)} |p_N(2\lambda; x, y)|^2 \\ &\leq \frac{4\Lambda_0}{\sqrt{2\pi}} e^{-\frac{p^2+m^2}{\Lambda_0^2}} \int_0^{\infty} d\lambda e^{-\lambda(p^2+m^2)} \frac{1}{\sqrt{2\pi\lambda}} e^{-\frac{(x-y)^2}{2\lambda}} \\ &\leq \frac{4\Lambda_0}{\sqrt{2\pi}} e^{-\frac{p^2+m^2}{\Lambda_0^2}} \frac{e^{-\sqrt{p^2+m^2}|x-y|}}{\sqrt{p^2+m^2}}, \end{aligned}$$

where we used that for $x, y \in \mathbb{R}^+$,

$$e^{-\frac{(x+y)^2}{2\lambda}} \leq e^{-\frac{(x-y)^2}{2\lambda}}.$$

Therefore, we have obtained the following bound for the first factor from (22):

$$\left(\int_0^\infty dx \int_0^\infty dy |(-\Delta_R + m^2)^{2n} C_R^{\Lambda_0}(p; x, y)|^2 \right) \leq \Lambda_0^{8n+1} \mathcal{P} \left(\frac{p^2 + m^2}{\Lambda_0^2} \right) \frac{e^{-\frac{p^2+m^2}{\Lambda_0^2}}}{(p^2 + m^2)^{\frac{3}{2}}}, \quad (26)$$

where \mathcal{P} is a suitable polynomial with positive coefficients. All constants were absorbed in the polynomial \mathcal{P} , and we obtain the final bound for (18) using again the Cauchy-Schwarz inequality with respect to the momenta p ,

$$\begin{aligned} \left| \langle f, C_R^{\Lambda_0} g \rangle \right| &\leq C_{\Lambda_0} \left\| (-\Delta_R + m^2)^{-n} f \right\|_{L^2(\mathbb{R}^+ \times \mathbb{R}^3)} \left\| (-\Delta_R + m^2)^{-n} g \right\|_{L^2(\mathbb{R}^+ \times \mathbb{R}^3)} \\ &= C_{\Lambda_0} \|f\|_{H_{-n}} \times \|g\|_{H_{-n}}. \end{aligned} \quad (27)$$

The constant C_{Λ_0} depends on Λ_0 . To obtain (27), we have performed the p -integral in (19) using bound (26).

We apply Corollary 1 with $T = P^{-2}$. The operator P^{-2} has the following kernel:

$$\begin{aligned} (-\Delta_R + m^2)^{-2}(p; x, y) &:= \int_0^\infty \frac{du}{4(p^2 + m^2)} \left(e^{-\sqrt{p^2+m^2}|x-u|} + \frac{\sqrt{p^2+m^2}-c}{\sqrt{p^2+m^2}+c} e^{-\sqrt{p^2+m^2}|x+u|} \right) \\ &\quad \times \left(e^{-\sqrt{p^2+m^2}|y-u|} + \frac{\sqrt{p^2+m^2}-c}{\sqrt{p^2+m^2}+c} e^{-\sqrt{p^2+m^2}|y+u|} \right). \end{aligned}$$

We can bound it as follows:

$$(-\Delta_R + m^2)^{-2}(p; x, y) \leq \int_0^\infty du e^{-\sqrt{p^2+m^2}|x-u|} e^{-\sqrt{p^2+m^2}|y-u|} \frac{1}{(\sqrt{p^2+m^2}+c)^2}.$$

T is a Hilbert-Schmidt operator on $L^2(\mathbb{R}^+ \times \mathbb{R}^3)$ since it is an integral operator with kernel in $L^2(\mathbb{R}^+ \times \mathbb{R}^3)$. This is a consequence of

$$\begin{aligned} \left\| (-\Delta_R + m^2)^{-2} \right\|_{L^2(\mathbb{R}^+ \times \mathbb{R}^3)}^2 &\leq \int_{\mathbb{R}^3} \frac{d^3p}{(2\pi)^3} \frac{1}{(\sqrt{p^2+m^2}+c)^4} \left(\int_0^\infty dx du e^{-2\sqrt{p^2+m^2}|x-u|} \right)^2 \\ &\leq C \int_{\mathbb{R}^3} \frac{d^3p}{(2\pi)^3} \frac{1}{(p^2+m^2)^4} < \infty \quad \text{for suitable } C > 0. \end{aligned}$$

T satisfies the hypotheses of Corollary 1. The dual of H_{-n} is the space H_n of functions whose image under P^n is in L^2 . Therefore, $\mu_{\Lambda_0, R}$ has support on the set $\{P^{2-n}f, f \in L^2(\mathbb{R}^+ \times \mathbb{R}^3)\}$. Since this is true for any $n \in \mathbb{N}^*$, we conclude that $\mu_{\Lambda_0, R}$ has its support within the set

$$\bigcap_{n \geq 1} \left\{ (-\Delta_R + m^2)^{-n} L^2(\mathbb{R}^+ \times \mathbb{R}^3) \right\}.$$

□

IV. TEST FUNCTIONS AND TREE STRUCTURES

A. Test functions

Standard proofs of perturbative renormalizability by flow equations use inductive bounds on the n -point correlation functions. These objects are no more functions if considered in the mixed position-momentum space, but rather tempered distributions. We introduce tempered distributions in $\mathcal{S}'(\mathbb{R}^{+n})$ with respect to the semi-norms,

$$\prod_{i=1}^n \mathcal{N}_2(\phi_i),$$

where $\mathcal{N}_2(\phi) := \sup_{0 \leq \alpha, \beta \leq 2} \left\| (1+z^\beta) \partial_z^\alpha \phi(z) \right\|_\infty$ and $\partial_z \phi|_{z=0} = \lim_{z \rightarrow 0^+} \partial_z \phi$. We refer the reader to Ref. 23 for additional information on the topological construction of $\mathcal{S}'(\mathbb{R}^+)$.

We now introduce test functions against which $\mathcal{L}_{l,n}^{\Lambda,\Lambda_0}$ will be integrated. In the sequel, we will bound the CAS folded with test functions of the following form:

Let $1 \leq s \leq n$, and we define

$$\tau := \inf \tau_{2,s} \text{ where } \tau_{2,s} = (\tau_2, \dots, \tau_s) \text{ with } \tau_i > 0,$$

and similarly $z_{2,s} = (z_2, \dots, z_s)$. Given $y_2, \dots, y_s \in \mathbb{R}^+$, we define

$$\phi_{\tau_{2,s}, y_{2,s}}(z_{2,s}) := \prod_{i=2}^s p_R(\tau_i; z_i, y_i) \prod_{i=s+1}^n \chi^+(z_i), \tag{28}$$

where $\chi^+(z_i)$ is the characteristic function of the semi-line \mathbb{R}^+ . This definition can be generalized by choosing any other subset of s coordinates among z_2, \dots, z_n . We also define for $2 \leq j \leq s$,

$$\phi_{\tau_{2,s}, y_{2,s}}^{(j)}(z_{2,n}) := p_R^{(1)}(\tau_j; z_j, z_1; y_j) \prod_{i=2, i \neq j}^s p_R(\tau_i; z_i, y_i) \prod_{i=s+1}^n \chi^+(z_i) \tag{29}$$

with

$$p_R^{(1)}(\tau_j; z_j, z_1; y_j) = p_R(\tau_j; z_j, y_j) - p_R(\tau_j; z_1, y_j). \tag{30}$$

B. Tree structures

We follow Ref. 2 and introduce the tree structures that will be used in establishing inductive bounds for the CAS.

- (i) We denote by \mathcal{T}^s the set of all trees that have a root vertex and $s - 1$ external vertices, where $s \geq 2$. Subsequently, we will identify for shortness the vertices of the trees with a set of points in \mathbb{R}^+ . For a tree $T^s \in \mathcal{T}^s$, we will call $z_1 \in \mathbb{R}^+$ its root vertex, and $Y = \{y_2, \dots, y_s\}$ the set of points in \mathbb{R}^+ to be identified with its external vertices. Likewise, we call $z = \{z_1, \dots, z_r\}$ with $r \geq 0$ the set of internal vertices of T^s .
- (ii) We call $c_1 = c(z_1)$ the incidence number of the root vertex, that is, the number of the lines of the tree that have the root vertex as an edge. The external vertices have incidence number 1; the internal vertices have incidence number > 1 . We call a line p an external line of the tree if one of its edges is in Y . The set of external lines is denoted as \mathcal{F} . The remaining lines are called internal lines of the tree and are denoted by \mathcal{I} .
- (iii) By T_l^s , we denote a tree $T^s \in \mathcal{T}^s$ satisfying $v_2 + \delta_{c_1,1} \leq 3l - 2 + s/2$ for $l \geq 1$ and satisfying $v_2 = 0$ for $l = 0$, where v_n is the number of vertices having incidence number n . Then, \mathcal{T}_l^s denotes the set of all trees T_l^s . We indicate the external vertices and internal vertices of the tree by writing $T_l^s(z_1, y_{2,s}, \vec{z})$ with $y_{2,s} = (y_2, \dots, y_s)$ and $\vec{z} = (z_2, \dots, z_{r+1})$.
- (iv) We also define for $i \leq s$ the set of twice rooted trees denoted as $\mathcal{T}_i^{s,(12)}$. The trees $T_i^{s,(12)} \in \mathcal{T}_i^{s,(12)}$ are defined exactly as the trees T_l^s apart from the fact that they have two root vertices z_1, z_2 with property (ii) and $s - 2$ external vertices.
- (v) For a tree $T_l^{s+2}(z_1, y_{2,s+2}, \vec{z})$, we define the reduced tree
 - $T_{l, y_i, y_j}^{s+2}(z_1, y_2, \dots, y_{i-1}, y_{i+1}, \dots, y_{j-1}, y_{j+1}, \dots, y_{s+2}, \vec{z}_{ij})$ to be the unique tree to be obtained from $T_l^{s+2}(z_1, y_{2,s}, \vec{z})$ through the following procedure:
 - By taking off the two external vertices y_i, y_j together with the external lines attached to them.
 - By taking off the internal vertices—if any—that have acquired incidence number $c = 1$ through the previous process and by also taking off the lines attached to them.
 - If a new vertex of incidence number 1 is created, the second step of the process is repeated.

C. Weight factors

Let $0 < \delta < 1$. Given a set $\tau_{2,s} := \{\tau_2, \dots, \tau_s\}$ with $\tau := \inf_{2 \leq i \leq s} \tau_i$, a set of external vertices $y_{2,s} = \{y_2, \dots, y_s\} \in (\mathbb{R}^+)^{s-1}$, and a set of internal vertices $\vec{z} = (z_2, \dots, z_{r+1}) \in (\mathbb{R}^+)^r$, and attributing positive parameters $\Lambda_{\mathcal{I}} = \{\Lambda_I | I \in \mathcal{I}\}$ to the internal lines, the weight factor $\mathcal{F}(\Lambda_{\mathcal{I}}, \tau; T_l^s(z_1, y_{2,s}, \vec{z}))$ of a tree $T_l^s(z_1, y_{2,s}, \vec{z})$ at scales Λ_I is defined as a product of heat kernels associated with the internal and external lines of the tree. We set

$$\mathcal{F}(\Lambda_{\mathcal{I}}, \tau; T_l^s(z_1, y_{2,s}, \vec{z})) := \prod_{I \in \mathcal{I}} p_B\left(\frac{1+\delta}{\Lambda_I^2}; I\right) \prod_{J \in \mathcal{F}} p_B(\tau_{J,\delta}; J), \tag{31}$$

where $\tau_{J,\delta}$ denotes the entry $\tau_{i,\delta}$ in τ carrying the index of the external coordinate y_i in which the external line J ends and $\tau_{i,\delta} := (1 + \delta)\tau_i$. For $I = \{a, b\}$, the notation $p_B(\frac{1+\delta}{\Lambda_I^2}; I)$ stands for $p_B(\frac{1+\delta}{\Lambda_I^2}; a, b)$. We also define the integrated weight factor

$$\mathcal{F}(\Lambda, \tau; T_l^s; z_1, y_{2,s}) := \sup_{\Lambda \leq \Lambda_l \leq \Lambda_0} \int_{\mathcal{Z}} \mathcal{F}(\Lambda, \tau; T_l^s(z_1, y_{2,s}, \vec{z})). \quad (32)$$

It depends on Λ_0 , but note that its limit for $\Lambda_0 \rightarrow \infty$ exists and that typically the sup is expected to be taken for the minimal values of Λ admitted. Therefore, we suppress the dependence on Λ_0 in the notation. Finally, we introduce the global weight factor $\mathcal{F}(\Lambda, \tau, z_1, y_{2,s})$ or more shortly $\mathcal{F}_{s,l}^\Lambda(\tau)$, which is defined through

$$\mathcal{F}_{s,l}^\Lambda(\Lambda, \tau, z_1, y_{2,s}) := \sum_{T_l^s \in \mathcal{T}_l^s} \mathcal{F}(\Lambda, \tau; T_l^s; z_1, y_{2,s}). \quad (33)$$

If this does not lead to ambiguity, we write shortly

$$\mathcal{F}_{s,l}^\Lambda(\tau) \equiv \mathcal{F}_{s,l}^\Lambda(\Lambda, \tau, z_1, y_{2,s}). \quad (34)$$

In complete analogy, we define the weight factors and global weight factors for twice rooted trees, which we denote as $\mathcal{F}(\Lambda, \tau; T_l^{s,(12)}; z_1, z_2, y_{2,s})$ or $\mathcal{F}_{s,l}^{(12)}(\Lambda, \tau, z_1, y_{2,s})$ or $\mathcal{F}_{s,l}^{12}(\Lambda, \tau)$.

For $s = 1$, we set $\mathcal{F}_{1,l}^\Lambda(\tau) \equiv 1$. This case corresponds to a tree T_l^s with no external vertices.

V. BOUNDARY AND RENORMALIZATION CONDITIONS

The relevant terms in the bare interaction are fixed by renormalization conditions at the value $\Lambda = 0$ of the flow parameter; all other boundary terms are fixed at $\Lambda = \Lambda_0$. To extract the relevant terms contained in

$$\mathcal{L}_{l,2}^{\Lambda,\Lambda_0}(z_1; 0, 0; \phi_2) := \int_{z_2} \mathcal{L}_{l,2}^{\Lambda,\Lambda_0}((z_1, 0), (z_2, 0)) \phi_2(z_2) \quad (35)$$

and

$$\mathcal{L}_{l,4}^{\Lambda,\Lambda_0}(z_1; 0, 0, 0, 0; \Phi_4) := \int_{z_{2,4}} \mathcal{L}_{l,4}^{\Lambda,\Lambda_0}((z_1, 0), \dots, (z_4, 0)) \prod_{i=2}^4 \phi_i(z_i), \quad (36)$$

we use a Taylor expansion of the test functions ϕ_2 and Φ_4 , which gives

$$\begin{aligned} \mathcal{L}_{l,2}^{\Lambda,\Lambda_0}(z_1; 0, 0; \phi_2) &= a_l^{\Lambda,\Lambda_0}(z_1) \phi_2(z_1) - s_l^{\Lambda,\Lambda_0}(z_1) (\partial_{z_1} \phi_2)(z_1) \\ &\quad + d_l^{\Lambda,\Lambda_0}(z_1) (\partial_{z_1}^2 \phi_2)(z_1) + l_{l,2}^{\Lambda,\Lambda_0}(z_1; \phi_2), \end{aligned} \quad (37)$$

$$(\partial_{p^2} \mathcal{L}_{l,2}^{\Lambda,\Lambda_0})(z_1; 0, 0; \phi_2) = b_l^{\Lambda,\Lambda_0}(z_1) \phi_2(z_1) + (\partial_{p^2} l_{l,2}^{\Lambda,\Lambda_0})(z_1; \phi_2), \quad (38)$$

$$\mathcal{L}_{l,4}^{\Lambda,\Lambda_0}(z_1; 0, \dots, 0; \Phi_4) = c_l^{\Lambda,\Lambda_0}(z_1) \phi_2(z_1) \phi_3(z_1) \phi_4(z_1) + l_{l,4}^{\Lambda,\Lambda_0}(z_1; \Phi_4), \quad (39)$$

where $\Phi_4(z_2, z_3, z_4) = \prod_{i=2}^4 \phi_i(z_i)$.

Then, the relevant terms appear as

$$a_l^{\Lambda,\Lambda_0}(z_1) = \int_0^\infty dz_2 \mathcal{L}_{l,2}^{\Lambda,\Lambda_0}((z_1, 0), (z_2, 0)), \quad (40)$$

$$s_l^{\Lambda,\Lambda_0}(z_1) = \int_0^\infty dz_2 (z_1 - z_2) \mathcal{L}_{l,2}^{\Lambda,\Lambda_0}((z_1, 0), (z_2, 0)), \quad (41)$$

$$d_l^{\Lambda,\Lambda_0}(z_1) = \int_0^\infty dz_2 (z_1 - z_2)^2 \mathcal{L}_{l,2}^{\Lambda,\Lambda_0}((z_1, 0), (z_2, 0)), \quad (42)$$

$$b_l^{\Lambda,\Lambda_0}(z_1) = \int_0^\infty dz_2 \partial_{p^2} \left(\mathcal{L}_{l,2}^{\Lambda,\Lambda_0}((z_1, p), (z_2, -p)) \right) \Big|_{p=0}, \quad (43)$$

$$c_l^{\Lambda,\Lambda_0}(z_1) = \int_0^\infty dz_2 dz_3 dz_4 \mathcal{L}_{l,4}^{\Lambda,\Lambda_0}((z_1, 0), \dots, (z_4, 0)), \quad (44)$$

and the reminders $l_{l,2}^{\Lambda,\Lambda_0}(z_1; \phi_2)$, $(\partial_{p^2} l_{l,2}^{\Lambda,\Lambda_0})(z_1; \phi_2)$, and $l_{l,4}^{\Lambda,\Lambda_0}(z_1; \Phi_4)$ have the form

$$l_{l,2}^{\Lambda,\Lambda_0}(z_1; \phi_2) = \int_0^\infty dz_2 \int_0^1 dt \frac{(1-t)^2}{2!} \partial_t^3 \phi_2(tz_2 + (1-t)z_1) \mathcal{L}_{l,2}^{\Lambda,\Lambda_0}((z_1; 0), (z_2; 0)), \quad (45)$$

$$\left(\partial_{p^2} l_{l,2}^{\Lambda,\Lambda_0}\right)(z_1; \phi_2) = \int_0^\infty dz_2 \int_0^1 dt \frac{(1-t)^2}{2!} \partial_t^3 \phi_2(tz_2 + (1-t)z_1) \partial_{p^2} \left(\mathcal{L}_{l,2}^{\Lambda,\Lambda_0}((z_1, p), (z_2, -p))\right) \Big|_{p=0},$$

and

$$l_{l,4}^{\Lambda,\Lambda_0}(z_1; \Phi_4) = \int_0^\infty dz_2 dz_3 dz_4 \mathcal{L}_{l,4}^{\Lambda,\Lambda_0}((z_1, 0), \dots, (z_4, 0)) \left[\int_0^1 dt \partial_t \phi_2(tz_2 + (1-t)z_1) \phi_3(z_3) \phi_4(z_4) + \phi_2(z_1) \int_0^1 dt \partial_t \phi_3(tz_3 + (1-t)z_1) \phi_4(z_4) + \phi_2(z_1) \phi_3(z_1) \int_0^1 dt \partial_t \phi_4(tz_4 + (1-t)z_1) \right]. \tag{46}$$

Boundary conditions at $\Lambda = \Lambda_0$

The bare interaction implies that at $\Lambda = \Lambda_0$,

$$\partial^w \mathcal{L}_{l,n}^{\Lambda_0,\Lambda_0}((z_1, p_1), \dots, (z_n, p_n)) = 0 \quad \forall n + |w| \geq 5, \quad \mathcal{L}_{0,2}^{\Lambda_0,\Lambda_0}((z_1, p), (z_2, -p)) = 0. \tag{47}$$

Renormalization conditions at $\Lambda = 0$

The renormalization conditions are fixed at $\Lambda = 0$ by imposing

$$a_l^{0,\Lambda_0}(z_1), \quad s_l^{0,\Lambda_0}(z_1), \quad d_l^{0,\Lambda_0}(z_1), \quad b_l^{0,\Lambda_0}(z_1), \quad c_l^{0,\Lambda_0}(z_1)$$

to be smooth functions in $\mathcal{E}^\infty(\mathbb{R}^+)$, uniformly bounded with respect to Λ_0 .

Typically, all the renormalization conditions are assumed to be cutoff-independent. The simplest renormalization conditions are BPHZ-renormalization conditions, where we set

$$a_l^{0,\Lambda_0}(z_1) \equiv 0, \quad s_l^{0,\Lambda_0}(z_1) \equiv 0, \quad d_l^{0,\Lambda_0}(z_1) \equiv 0, \quad b_l^{0,\Lambda_0}(z_1) \equiv 0, \quad c_l^{0,\Lambda_0}(z_1) \equiv 0. \tag{48}$$

These will be adopted in the following.

VI. PROOF OF RENORMALIZABILITY

We define for all $n \geq 2$ and $0 \leq r \leq 3$,

$$\mathcal{L}_{l,n}^{\Lambda,\Lambda_0}(z_1; \vec{p}_n; (z_1 - z_i)^r \Phi_s) := \int_0^\infty dz_{2,n} (z_1 - z_i)^r \mathcal{L}_{l,n}^{\Lambda,\Lambda_0}((z_1, p_1), \dots, (z_n, p_n)) \Phi_s(z_{2,s}), \tag{49}$$

and we define for all $n \geq 3$,

$$F_{12} \mathcal{L}_{l,n}^{\Lambda,\Lambda_0}(z_1, z_2; \vec{p}_n; \Phi_{s-1}) := (z_1 - z_2)^3 \int_0^\infty dz_{3,n} \mathcal{L}_{l,n}^{\Lambda,\Lambda_0}((z_1, p_1), \dots, (z_n, p_n)) \Phi_{s-1}(z_{3,s}). \tag{50}$$

For $n = 2$, we define

$$F_{12} \mathcal{L}_{l,2}^{\Lambda,\Lambda_0}(z_1, z_2; p) := (z_1 - z_2)^3 \mathcal{L}_{l,2}^{\Lambda,\Lambda_0}((z_1, p), (z_2, -p)). \tag{51}$$

Theorem 1 (boundedness). We consider $0 \leq \Lambda \leq \Lambda_0 < \infty$, $1 \leq s \leq n$, $2 \leq i \leq n$, $2 \leq j \leq s$, and $0 \leq r \leq 3$. We consider test functions either of the form $\phi_{\tau_2, s, y_{2,s}}(z_{2,n})$ or $\phi_{\tau_2, s, y_{2,s}}^{(j)}(z_{2,n})$, which are also denoted in shorthand as $\phi_{\tau, y_{2,s}}$ or $\phi_{\tau, y_{2,s}}^{(j)}$; see (28) and (29). Adopting (48), we claim

$$(A) \left| \partial^w \mathcal{L}_{l,n}^{\Lambda,\Lambda_0}(z_1; \vec{p}_n; (z_1 - z_i)^r \phi_{\tau, y_{2,s}}) \right| \leq (\Lambda + m)^{4-n-|w|-r} \mathcal{P}_1 \left(\log \frac{\Lambda + m}{m} \right) \mathcal{P}_2 \left(\frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathcal{Q}_1 \left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s,l}^\Lambda(\tau), \tag{52}$$

$$(B) \left| F_{12} \mathcal{L}_{l,n}^{\Lambda,\Lambda_0}(z_1, z_2; \vec{p}_n; \phi_{\tau, y_{2,s}}) \right| \leq (\Lambda + m)^{1-n} \mathcal{P}_3 \left(\log \frac{\Lambda + m}{m} \right) \mathcal{P}_4 \left(\frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathcal{F}_{s,l}^{12}(\Lambda, \tau), \tag{53}$$

$$(C) \left| \partial^w \mathcal{L}_{l,n}^{\Lambda,\Lambda_0}(z_1; \vec{p}_n; \phi_{\tau, y_{2,s}}^{(j)}) \right| \leq (\Lambda + m)^{3-n-|w|} \tau_j^{-\frac{1}{2}} \mathcal{P}_5 \left(\log \frac{\Lambda + m}{m} \right) \mathcal{P}_6 \left(\frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathcal{Q}_2 \left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s,l}^\Lambda(\tau). \tag{54}$$

(D) The functions defined in (49)–(51) and their momentum derivatives are in $\mathcal{E}^\infty(\mathbb{R}^+)$ with respect to z_1 .

Here and in the following, \mathcal{P}_i and \mathcal{Q}_i denote (each time they appear possibly new) polynomials with non-negative coefficients. The polynomials \mathcal{Q}_i are reduced to a constant if $s = 1$. The coefficients depend on $l, n, |w|, \delta$ but not on $\{p_i\}, \Lambda, \Lambda_0$, and z_1 . For $l = 0$, all polynomials \mathcal{P}_i reduce to constants. In the definition of \mathcal{F} (31), $\delta > 0$ may be chosen arbitrarily small.

Remarks 1.

- Bounds (53) and (54) are required to close the inductive argument in the subsequent proof. Bound (52) is the central result of the boundedness theorem (Theorem 1) needed later to prove the convergence of $\partial^w \mathcal{L}_{l,n}^{\Lambda, \Lambda_0}(z_1; p_1, \dots, p_n; (z_1 - z_i)^r \phi_{\tau, y_{2,s}})$ in the limits $\Lambda \rightarrow 0$ and $\Lambda_0 \rightarrow \infty$.
- The proof that we give in the following holds also for larger classes of test functions indexed by a strictly positive parameter τ such that

$$|\partial_z^\alpha \phi_\tau(z)| \leq \tau^{-\frac{\alpha}{2}} |\phi_\tau(z)| \quad \forall z \in \mathbb{R}^+, \quad \forall \alpha \geq 0.$$

The role of the parameter τ as it appears in the proof is to absorb negative powers of the flow parameter Λ by producing powers of $\frac{\tau}{\Lambda+m}$ that contribute to the polynomial \mathcal{Q} at each step of the induction. This preserves the power counting in terms of $\Lambda + m$. We choose a simple example of these functions, which are the Robin heat kernels. They can be proved to be dense in $\mathcal{A}(\mathbb{R}^{+n})$.

- The parameter δ depends on the loop order l , and it verifies $0 < \delta_l < \delta_{l+1} < \frac{1}{2}$. For simplicity, we omit the index l .
- The value of the integral

$$\int_0^\infty dw \delta_w \tag{55}$$

admits two possible choices, which are 1 and $\frac{1}{2}$. These two choices are called, respectively, the weak and strong definitions of the Dirac distribution.²⁵

The subsequent proof uses the strong definition of the Dirac distribution. For the weak definition, all the points of Theorem 1 hold except for (D). In the weak convention, the functions defined in (49)–(51) are in $\mathcal{C}^\infty(\mathbb{R}^{+*})$ with respect to z_1 and are not continuous at 0. One can verify that the proof of renormalizability is independent of the chosen convention. This comes from the fact that for a continuous function f , in both conventions, we have that

$$\int_0^\infty dz \left(\int_0^\infty dz' \delta(z - z') \right) f(z) = \int_0^\infty dz f(z).$$

Proof. The bounds are proven inductively using the standard inductive scheme, which proceeds upward in l , for given l upward in n and for given (n, l) downward in $|w|$ starting from some arbitrary $|w_{\max}| \geq 3$. The induction works because the terms on the right-hand side of the FE always are prior to the one of the left-hand side in the inductive order. Hence, bounds (52)–(54) may be used as an induction hypothesis on the right-hand side. Once verified in the first induction step, we integrate the FE, where the terms with $n + |w| + r \geq 5$ are integrated down from Λ_0 to Λ because of the boundary conditions (47) and the terms with $n + |w| + r \leq 4$ at the renormalization point are integrated upward from 0 to Λ using (48). We can write remembering (15),

$$\partial^w \mathcal{L}_{l,n}^{\Lambda, \Lambda_0}(z_1; \vec{0}; \phi_{\tau, y_{2,s}}) = \partial^w \mathcal{L}_{l,n}^{0, \Lambda_0}(z_1; \vec{0}; \phi_{\tau, y_{2,s}}) + \int_0^\Lambda d\lambda \partial_\lambda \partial^w \mathcal{L}_{l,n}^{\lambda, \Lambda_0}(z_1; \vec{0}; \phi_{\tau, y_{2,s}}). \tag{56}$$

Once a bound has been obtained at the renormalization point, it is possible to move away from the renormalization point using the integrated Taylor formula,

$$\begin{aligned} \partial^w \mathcal{L}_{l,n}^{\Lambda, \Lambda_0}(z_1; \vec{p}_n; \phi_{\tau, y_{2,s}}) &= \partial^w \mathcal{L}_{l,n}^{\Lambda, \Lambda_0}(z_1; \vec{0}; \phi_{\tau, y_{2,s}}) \\ &+ \sum_{i=1}^n \sum_{\mu=1}^4 p_{i,\mu} \int_0^1 dt \left(\partial_{p_{i,\mu}} \partial^w \mathcal{L}_{l,n}^{\Lambda, \Lambda_0} \right)(z_1; t \vec{p}_n; \phi_{\tau, y_{2,s}}). \end{aligned} \tag{57}$$

The induction starts with the pair (0, 4) for which the right-hand side of the FE vanishes so that

$$\mathcal{L}_{0,4}^{\Lambda, \Lambda_0}((z_1, p_1), \dots, (z_4, p_4)) = \lambda \prod_{i=2}^4 \delta(z_1 - z_i),$$

which implies for the test function $\phi_{\tau, y_{2,s}}$ defined in (28) that

$$\mathcal{L}_{0,4}^{\Lambda, \Lambda_0}(z_1; \vec{p}_4; \phi_{\tau, y_{2,s}}) = \lambda \prod_{i=2}^s p_R(\tau_i; z_1, y_i). \tag{58}$$

Using bounds (A4) and (A6) from the Appendix, we deduce that

$$\prod_{i=2}^s p_R(\tau_i; z_1, y_i) \leq 2^s \prod_{i=2}^s p_B(\tau_i; z_1, y_i) \leq 2^s (1 + \delta)^{\frac{s}{2}} \prod_{i=2}^s p_B(\tau_i, \delta; z_1, y_i),$$

which implies that (58) can be bounded by a tree with no internal vertices and with a root vertex z_1 linked to the external vertices $y_{2,s}$, which is in agreement with bound (52). The constants are absorbed in the polynomial \mathcal{P}_1 , which is of degree 0 at the tree order.

(I) Bounds on the right-hand side of the FE:

We want to establish the bounds

$$|\partial_\Lambda \partial^w \mathcal{L}_{l,n}^{\Lambda, \Lambda_0}(z_1; \vec{p}_n; (z_1 - z_i)^r \phi_{\tau, y_{2,s}})| \leq (\Lambda + m)^{3-n-|w|-r} \mathcal{P}_1 \left(\log \frac{\Lambda + m}{m} \right) \mathcal{P}_2 \left(\frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathcal{Q}_1 \left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s,l}^\Lambda(\tau). \tag{59}$$

(A) In this part, we consider the case $r = 0$.

(A₁) Let R_1 be the first term on the RHS of the FE,

$$R_1 := \int_p \int_{z_{2,n}, z, z'} \partial^w \mathcal{L}_{n+2, l-1}^{\Lambda, \Lambda_0}((z_n, \vec{p}_n), (z, p), (z', -p)) \hat{C}_R^\Lambda(p; z, z') \prod_{i=2}^s p_R(\tau_i; z_i, y_i),$$

which can be written as

$$-\frac{2e^{-\frac{p^2+m^2}{\Lambda^2}}}{\Lambda^3} \int_p \int_0^\infty du \partial^w \mathcal{L}_{n+2, l-1}^{\Lambda, \Lambda_0} \left(z_1; p, -p, \vec{p}_n; \phi_{\tau, y_{2,s}} \times p_R \left(\frac{1}{2\Lambda^2}; u, \cdot \right) \times p_R \left(\frac{1}{2\Lambda^2}; \cdot, u \right) \right), \tag{60}$$

where we used (11) and the semi-group property for p_R ; see (A3).

Applying the induction hypothesis gives the bound

$$|R_1| \leq (\Lambda + m)^{2-n-|w|} \mathcal{P}_1 \left(\log \frac{\Lambda + m}{m} \right) \mathcal{P}_2 \left(\frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathcal{Q}_1 \left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \times \int_0^\infty du \int_{\vec{z}} \sum_{T_{l-1}^{s+2}(z_1, y_{2,s}, u, u)} \mathcal{F} \left(\Lambda, \left\{ \tau, \frac{1}{2\Lambda^2}, \frac{1}{2\Lambda^2} \right\}; T_{l-1}^{s+2}(z_1, y_{2,s}, u, u, \vec{z}) \right). \tag{61}$$

In the sequel, we write the polynomials \mathcal{P}_1 , \mathcal{P}_2 , and \mathcal{Q}_1 without their lower indices. One should keep in mind that these polynomials may have, each time they appear, different positive coefficients, which depend on $l, n, |w|, \delta$ only and not on $\{p_i\}, \Lambda, \Lambda_0$, and z_1 .

For any contribution to (61), we denote by z', z'' the vertices in the tree $T_{l-1}^{s+2}(z_1, y_{2,s}, u, u)$ to which the test functions $p_B(\frac{1+\delta}{2\Lambda^2}; u, \cdot)$ and $p_B(\frac{1+\delta}{2\Lambda^2}; \cdot, u)$ are attached. Interchanging $\int_{\vec{z}}$ and \int_u and performing the integral over u using the semi-group property (A3), we obtain

$$\int_0^\infty du p_B \left(\frac{1+\delta}{2\Lambda^2}; z', u \right) p_B \left(\frac{1+\delta}{2\Lambda^2}; u, z'' \right) \leq p_B \left(\frac{1+\delta}{\Lambda^2}; z', z'' \right) \leq O(1)\Lambda$$

with a positive constant $O(1)$. Using this bound, we obtain

$$\int_0^\infty du \mathcal{F} \left(\Lambda, \left\{ \tau, \frac{1}{2\Lambda^2}, \frac{1}{2\Lambda^2} \right\}; T_{l-1}^{s+2}(z_1, y_{2,s}, u, u) \right) \leq O(1)\Lambda \mathcal{F} \left(\Lambda, \tau; T_l^s(z_1, y_{2,s}) \right),$$

where the tree T_l^s is the reduced tree obtained from T_{l-1}^{s+2} by taking away the two external lines ending in u . Note that v_2 , the number of vertices of incidence number 2, verifies $v_2 + \delta_{c,1} \leq 3l - 3 - 2 + s/2 + 1 \leq 3l - 2 + s/2$, which explains that the reduction of T_{l-1}^{s+2} is in \mathcal{T}_l^s . The elimination of vertices of incidence number 1 together with their adjacent line is justified by the fact that $\int_{z'} p_B(\frac{1+\delta}{\Lambda^2}; z', z'') \leq 1$.

The reduction process for each tree fixes uniquely the set of internal vertices of T_l^s in terms of those of T_{l-1}^{s+2} . Therefore, we obtain

$$|R_1| \leq (\Lambda + m)^{3-n-|w|} \mathcal{P} \left(\log \frac{\Lambda + m}{m} \right) \mathcal{P} \left(\frac{\|\tilde{p}_n\|}{\Lambda + m} \right) \times \mathcal{Q} \left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \sum_{T_i^s(z_1, y_{2,s})} \mathcal{F}(\Lambda, \tau; T_i^s; z_1, y_{2,s}). \tag{62}$$

(A₂) We now consider the second term on the RHS of the FE. It is enough to analyze the following term from the symmetrized sum:

$$R_2 := \int_{z_2, n, z, z'} \partial^{w_1} \mathcal{L}_{l_1, n_1+1}^{\Lambda, \Lambda_0}((z_1, p_1), \dots, (z_{n_1}, p_{n_1}), (z, p)) \partial^{w_3} \hat{C}_R^\Lambda(p; z, z') \times \partial^{w_2} \mathcal{L}_{l_2, n_2+1}^{\Lambda, \Lambda_0}((z_{n_1+1}, p_{n_1+1}), \dots, (z_n, p_n), (z', -p)) \phi_{\tau, y_{2,s}}(z_2, n),$$

in which the arguments (z_i, p_i) appear in $\mathcal{L}_{l_1, n_1+1}^{\Lambda, \Lambda_0}$ and $\mathcal{L}_{l_2, n_2+1}^{\Lambda, \Lambda_0}$ in an ordered way.

R_2 can be rewritten similarly as in (A₁),

$$R_2 := \int_u \int_{z_2, n, z, z'} \partial^{w_1} \mathcal{L}_{l_1, n_1+1}^{\Lambda, \Lambda_0}((z_1, p_1), \dots, (z_{n_1}, p_{n_1}), (z, p)) \partial^{w_3} \hat{C}^\Lambda(p) \times \partial^{w_2} \mathcal{L}_{l_2, n_2+1}^{\Lambda, \Lambda_0}((z_{n_1+1}, p_{n_1+1}), \dots, (z_n, p_n), (z', -p)) \phi_{\tau, y_{2,s}}(z_2, n) \times p_R \left(\frac{1}{2\Lambda^2}; z, u \right) p_R \left(\frac{1}{2\Lambda^2}; u, z' \right).$$

We define

$$\phi'_{s_1}(z_2, n_1) = \prod_{r=2}^{s_1} p_R(\tau_r; z_r, y_r), \quad \phi''_{s_2}(z_{n_1+1}, n-1) = \prod_{r=n_1+1}^{s_2+n_1} p_R(\tau_r; z_r, y_r),$$

where $s = s_1 + s_2$.

Therefore, R_2 can be rewritten as

$$R_2 = \int_{z_n} \int_u \partial^{w_1} \mathcal{L}_{l_1, n_1+1}^{\Lambda, \Lambda_0}(z_1; p_1, \dots, p_{n_1}, p; \phi'_{s_1} \times p_R \left(\frac{1}{2\Lambda^2}; \cdot, u \right)) \partial^{w_3} \hat{C}^\Lambda(p) \times \partial^{w_2} \mathcal{L}_{l_2, n_2+1}^{\Lambda, \Lambda_0}(z_n; -p, p_{n_1+1}, \dots, p_n; \phi''_{s_2} \times p_R \left(\frac{1}{2\Lambda^2}; u, \cdot \right)) \phi_n(z_n), \tag{63}$$

where $\phi_n(z_n) = p(\tau_n; z_n, y_n)$ if $s = n$ and $\phi_n(z_n) \equiv 1$ otherwise. Applying the induction hypothesis to both terms in (63), we obtain the bound

$$|R_2| \leq (\Lambda + m)^{8-n-|w|-2-3} \mathcal{P} \left(\log \frac{\Lambda + m}{m} \right) \mathcal{P} \left(\frac{\|\tilde{p}_n\|}{\Lambda + m} \right) \mathcal{Q} \left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \times \int_{z_n} \int_u \sum_{T_{l_1}^{s_1+1}, T_{l_2}^{s_2+1}} \mathcal{F} \left(\Lambda, \left\{ \tau', \frac{1}{2\Lambda^2} \right\}; T_{l_1}^{s_1+1}; z_1, y_{2, s_1}, u \right) \times \mathcal{F} \left(\Lambda, \left\{ \tau'', \frac{1}{2\Lambda^2} \right\}; T_{l_2}^{s_2+1}; z_n, u, y_{s_1+1}, \dots, y_{s(n)} \right) \phi_n(z_n), \tag{64}$$

where we used

$$|\partial^{w_3} \hat{C}^\Lambda(p)| \leq (\Lambda + m)^{-3-|w_3|} \mathcal{P} \left(\frac{|p|}{\Lambda + m} \right). \tag{65}$$

In (64), we set $s(n) = s$ if $s < n$ and $s(n) = s - 1$ if $s = n$.

Interchanging the integral over u with the sum over trees, we obtain

$$|R_2| \leq (\Lambda + m)^{3-n-|w|} \mathcal{P} \left(\log \frac{\Lambda + m}{m} \right) \mathcal{P} \left(\frac{\|\tilde{p}_n\|}{\Lambda + m} \right) \mathcal{Q} \left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \times \sum_{T_i^s(T_{l_1}^{s_1+1}, T_{l_2}^{s_2+1})} \int_{z_n} \mathcal{F}(\Lambda, \tau; T_i^s; z_1, y_{2,s}),$$

with the following explanations:

- Any contribution in the sum over trees $T_l^s(T_{l_1}^{s_1+1}, T_{l_2}^{s_2+1})(z_1, y_2, \dots, y_s, \bar{z})$ is obtained from $T_{l_1}^{s_1+1}(z_1, y_2, \dots, y_{s_1}, u, \bar{z}')$ and $T_{l_2}^{s_2+1}(z_n, y_2, \dots, y_{s_1+1}, u, \bar{z}'')$ by joining these two trees via the lines going from the vertices z' and z'' to u , where z' and z'' are the vertices attached to u in the two trees. These two lines have parameters $\frac{1+\delta}{2\Lambda^2}$. We use (A3) from the Appendix to obtain

$$\int_u p_B\left(\frac{1+\delta}{2\Lambda^2}; z', u\right) p_B\left(\frac{1+\delta}{2\Lambda^2}; u, z''\right) \leq p_B\left(\frac{1+\delta}{\Lambda^2}; z', z''\right)$$

so that the new internal line has a parameter in the interval $[\Lambda, \Lambda_0]$ over which the sup is taken in the definition of \mathcal{F} .

- When performing the integral over z_n , we note that z_n has been viewed as the root vertex of $T_{l_2}^{s_2+1}(z_n, u, y_{s_1+1}, \dots, y_s, \bar{z})$. We distinguish two cases:
 - If $s = n$, we set $\phi_n(z_n) = p_B(\tau_n; z_n, y_n)$, and z_n becomes an internal vertex and y_n becomes an external vertex of T_l^s .
 - If $s < n$, then $\phi_n(z_n) \equiv 1$ and the vertex z_n becomes an internal vertex of T_l^s unless $c(z_n) \equiv 1$. In this case, we use

$$\int_{z_n} p_B(t_{n,\delta}; z_n, z_j) \leq 1. \tag{66}$$

This integration permits to take away the vertex z_n and the internal line joining it to an internal vertex z_j of the tree $T_{l_2}^{s_2+1}$. If $c(z_j) = 2$, let z_l be the other vertex to which z_j is joined. The semi-group property (A3) implies

$$\int_{z_j} p_B(t_{n,\delta}; z_n, z_j) p_B(t_{j,\delta}; z_j, z_l) \leq p_B(t_{n,\delta} + t_{j,\delta}; z_n, z_l).$$

This elimination process continues until we reach $c(z_k) > 2$. Applying (66) for $z_j = z_k$ takes away the internal vertex z_n . If $c(z_k) = 3$ in $T_{l_2}^{s_2+1}$, then it becomes equal to 2 after integrating out z_n . Therefore, a new vertex of incidence number 2 is created in the new tree, and this is compatible with the definition of T_l^s . Namely, let $v_{2,i}$ be the number of vertices with incidence number 2 of the tree $T_{l_i}^{s_i+1}$. By definition of $T_{l_i}^{s_i+1}$, we have

$$v_{2,1} + \delta_{c_1,1} \leq 3l_1 - 2 + \frac{s_1 + 1}{2}, \quad v_{2,2} + \delta_{c_2,1} \leq 3l_2 - 2 + \frac{s_2 + 1}{2}.$$

Since $c_s := c(z_s) = 1$, we deduce that

$$v_{2,2} \leq 3l_2 - 3 + \frac{s_2 + 1}{2}.$$

The number of vertices v_2 with incidence number 2 in the new tree obtained from $T_l^s(T_{l_1}^{s_1+1}, T_{l_2}^{s_2+1})$ after integrating out z_n is equal to $v_{2,1} + v_{2,2} + 1$ so that

$$v_2 + \delta_{c_1,1} \leq 3l - 3 + \frac{s}{2}.$$

Therefore, we conclude that

$$|R_2| \leq (\Lambda + m)^{3-n-|w|} \mathcal{F}\left(\log \frac{\Lambda + m}{m}\right) \mathcal{F}\left(\frac{\|\bar{p}_n\|}{\Lambda + m}\right) \mathcal{F}\left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m}\right) \sum_{T_l^s(z_1, y_{2,s})} \mathcal{F}(\Lambda, \tau; T_l^s; z_1, y_{2,s}).$$

(B) We consider now the case $r \neq 0$:

For the first term on the right-hand side of the flow equation [Eq. (16)], the bounds are proven exactly as in (A₁). For the second term, we proceed similarly as in (A₂). We pick a generic term on the right-hand side,

$$\int_u \int_{z_{2,n}, z, z'} \phi_{\tau, y_{2,s}}(z_{2,n}) \partial^{w_3} \dot{C}^\Lambda(p) p_R\left(\frac{1}{2\Lambda^2}; z, u\right) p_R\left(\frac{1}{2\Lambda^2}; u, z'\right) (z_i - z_1)^r \\ \times \partial^{w_1} \mathcal{L}_{l_1, n_1+1}^{\Lambda, \Lambda_0}((z_1, p_1), \dots, (z_{n_1}, p_{n_1}), (z, p)) \partial^{w_2} \mathcal{L}_{l_2, n_2+1}^{\Lambda, \Lambda_0}((z_{n_1+1}, p_{n_1+1}), \dots, (z_n, p_n), (z', -p)).$$

In the case where $i \leq n_1$, the proof is the same as for $r = 0$, up to inserting the modified induction hypothesis for

$$\begin{aligned} & \partial^{w_1} \mathcal{L}_{l_1, n_1+1}^{\Lambda, \Lambda_0} \left(z_1; \vec{p}_n; (z_i - z_1)^r \phi'_{s_1} \times p_R \left(\frac{1}{2\Lambda^2}; \cdot, u \right) \right) \\ &= \int_{z_2, \dots, z_{n_1}, z} \partial^{w_1} \mathcal{L}_{l_1, n_1+1}^{\Lambda, \Lambda_0} \left((z_1, p_1), \dots, (z_{n_1}, p_{n_1}), (z, p) \right) (z_i - z_1)^r \phi'_{s_1}(z_{2, n_1}) p_R \left(\frac{1}{2\Lambda^2}; z, u \right). \end{aligned}$$

If $i > n_1$, we assume without restriction $i = n$ and proceed again as in (A₂) to obtain the bound

$$\begin{aligned} & (\Lambda + m)^{3-n-|w|} \mathcal{P} \left(\log \frac{\Lambda + m}{m} \right) \mathcal{P} \left(\frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathcal{Q} \left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \\ & \times \int_{z_n} \int_u |z_n - z_1|^r \sum_{T_{l_1}^{s_1+1}, T_{l_2}^{s_2+1}} \mathcal{F} \left(\Lambda, \left\{ \tau', \frac{1}{2\Lambda^2} \right\}; T_{l_1}^{s_1+1}; z_1, y_{2, s_1}, u \right) \\ & \times \mathcal{F} \left(\Lambda, \left\{ \tau'', \frac{1}{2\Lambda^2} \right\}; T_{l_2}^{s_2+1}; z_n, u, y_{s_1+1}, \dots, y_{s(n)} \right) \phi_n(z_n). \end{aligned}$$

We bound

$$|z_n - z_1| \leq \sum_{a=1}^q |v_a - v_{a-1}|, \tag{67}$$

where $\{v_a\}$ are the positions of the internal vertices in the tree $T_l^s(T_{l_1+1}^{s_1+1}, T_{l_2+1}^{s_2+1})$ defined as in (A₂) on the path joining $z_1 = v_0$ and $z_n = v_q$. Using inequality (A7) from the Appendix for $\tau = \frac{1}{\Lambda^2}$, we obtain for δ' such that $\delta < \delta' < \frac{1}{2}$

$$|v_a - v_{a-1}|^r p_B \left(\frac{1 + \delta}{\Lambda^2}; v_a, v_{a-1} \right) \leq O(1) \Lambda^{-r} p_B \left(\frac{1 + \delta'}{\Lambda^2}; v_a, v_{a-1} \right), \tag{68}$$

and this implies (59) with a new parameter $\delta' < \frac{1}{2}$. Note that the cases $s = n$ and $s < n$ are treated as in (A₂).

The previous reasoning also holds for $\partial_\Lambda F_{12} \mathcal{L}_{l, n}^{\Lambda, \Lambda_0}(z_1, z_2; \vec{p}_n; \phi_{\tau, y_{2, s}})$, where z_2 takes the role of z_n . After absorbing all constants in \mathcal{P} , we obtain

$$\left| \partial_\Lambda \partial^w \mathcal{L}_{l, n}^{\Lambda, \Lambda_0}(z_1; \vec{p}_n; (z_1 - z_i)^r \phi_{\tau, y_{2, s}}) \right| \leq (\Lambda + m)^{3-n-|w|-r} \mathcal{P} \left(\log \frac{\Lambda + m}{m} \right) \mathcal{P} \left(\frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathcal{Q} \left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s, l}^\Lambda(\tau), \tag{69}$$

$$\left| \partial_\Lambda F_{12} \mathcal{L}_{l, n}^{\Lambda, \Lambda_0}(z_1, z_2; \vec{p}_n; \phi_{\tau, y_{2, s}}) \right| \leq (\Lambda + m)^{-n} \mathcal{P} \left(\frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathcal{P} \left(\log \frac{\Lambda + m}{m} \right) \mathcal{F}_{s, l}^{12}(\Lambda, \tau). \tag{70}$$

The bounds for (40)–(44),

$$|\partial_\Lambda c_l^{\Lambda, \Lambda_0}(z_1)| \leq (\Lambda + m)^{-1} \mathcal{P} \left(\log \frac{\Lambda + m}{m} \right), \quad |\partial_\Lambda a_l^{\Lambda, \Lambda_0}(z_1)| \leq (\Lambda + m) \mathcal{P} \left(\log \frac{\Lambda + m}{m} \right), \tag{71}$$

$$|\partial_\Lambda b_l^{\Lambda, \Lambda_0}(z_1)| \leq (\Lambda + m)^{-1} \mathcal{P} \left(\log \frac{\Lambda + m}{m} \right), \quad |\partial_\Lambda d_l^{\Lambda, \Lambda_0}(z_1)| \leq (\Lambda + m)^{-1} \mathcal{P} \left(\log \frac{\Lambda + m}{m} \right), \tag{72}$$

$$|\partial_\Lambda s_l^{\Lambda, \Lambda_0}(z_1)| \leq \mathcal{P} \left(\log \frac{\Lambda + m}{m} \right), \tag{73}$$

are obtained on restricting the previous considerations to the case $s = 1$, in which all the coordinates $z_{2, n}$ are integrated over with $n = 2$ or $n = 4$ and the momenta \vec{p}_n set to $\vec{0}$

(C) We come to the bound on $\partial_\Lambda \partial^w \mathcal{L}_{l, n}^{\Lambda, \Lambda_0}(z_1; \vec{p}_n; \phi_{\tau, y_{2, s}}^{(j)})$ [i.e., remember (29) and (30)]. As compared to (A), the only case which requires new analysis is the bound on the second term from the right-hand side of the FE (16) in the case $j > s_1$. Then, we assume without restriction, similarly as in (A), that $j = s$. The term to be bounded corresponding to (64) is then

$$\begin{aligned}
 & (\Lambda + m)^{3-n-|w|} \mathcal{P}\left(\log \frac{\Lambda + m}{m}\right) \mathcal{P}\left(\frac{\|\vec{p}_n\|}{\Lambda + m}\right) \mathcal{Q}\left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m}\right) \int_{u_{T_{l_1}^{s_1+1}, T_{l_2}^{s_2+1}}} \sum_{T_{l_1}^{s_1+1}, T_{l_2}^{s_2+1}} \mathcal{F}\left(\Lambda, \left\{\tau', \frac{1}{2\Lambda^2}\right\}; T_{l_1}^{s_1+1}; z_1, y_{2s_1}, u\right) \\
 & \times e^{-\frac{m^2}{2\Lambda^2}} \int_{z_s} \mathcal{F}\left(\Lambda, \left\{\tau'', \frac{1}{2\Lambda^2}\right\}; T_{l_2}^{s_2+1}; z_s, u, y_{s_1+1}, \dots, y_{s(n)}\right) |p_R^{(1)}(\tau_s; z_s, z_1; y_s)|. \tag{74}
 \end{aligned}$$

We factorized $e^{-\frac{m^2}{2\Lambda^2}}$ from the derivative of the flowing propagator $\hat{C}^\Lambda(p)$, and we will use the bound

$$(\Lambda + m) e^{-\frac{m^2}{2\Lambda^2}} \leq C \Lambda, \quad C := \|(1 + x) e^{-\frac{x^2}{2}}\|_\infty. \tag{75}$$

To bound (74), we telescope the difference $p_R^{(1)}(\tau_s; z_s, z_1; y_s)$ along the tree $T_l^s(T_{l_1}^{s_1+1}, T_{l_2}^{s_2+1})$ similarly as in (67). We, then, have to bound expressions of the type

$$p_B\left(\frac{1 + \delta}{\Lambda_l^2}; v_{a-1}, v_a\right) |p_R(\tau_s; v_a, y_s) - p_R(\tau_s; v_{a-1}, y_s)|, \tag{76}$$

where v_{a-1}, v_a are adjacent internal vertices in $T_l^s(T_{l_1}^{s_1+1}, T_{l_2}^{s_2+1})$ on the unique path from z_1 to y_s . The Taylor expansion of $p_R(\tau_s; v_a, y_s)$ gives

$$p_R(\tau_s; v_a, y_s) = p_R(\tau_s; v_{a-1}, y_s) + \int_0^1 dt (\partial_t p_R)(\tau_s; tv_{a-1} + (1-t)v_a, y_s).$$

Lemma 1 from the Appendix implies that for all $0 < \delta' < 1$, we have

$$|p_R^{(1)}(\tau_s; v_a, v_{a-1}, y_s)| \leq C_{1,\delta} \frac{|v_a - v_{a-1}|}{\sqrt{\tau_s}} \int_0^1 dt p_B(\tau_{s,\delta'}; tv_{a-1} + (1-t)v_a, y_s).$$

Therefore, (76) is bounded by

$$C_{1,\delta} \tau_s^{-\frac{1}{2}} |v_a - v_{a-1}| p_B\left(\frac{1 + \delta}{\Lambda_l^2}; v_{a-1}, v_a\right) \int_0^1 dt p_B((1 + \delta')\tau_s; tv_{a-1} + (1-t)v_a, y_s). \tag{77}$$

Introducing for $2\delta < 1$,

$$b = 2 \frac{1 + 2\delta}{1 - 2\delta},$$

we distinguish between the two cases:

- Case (1): $\delta' \Lambda^2 \leq b \tau_s^{-1}$

Using (75), we obtain

$$(\Lambda + m)^{3-n-|w|} e^{-\frac{m^2}{2\Lambda^2}} \leq C \left(\frac{b}{\delta'}\right)^{\frac{1}{2}} (\Lambda + m)^{2-n-|w|} \tau_s^{-\frac{1}{2}}.$$

The tree $T_l^s(T_{l_1}^{s_1+1}, T_{l_2}^{s_2+1})$ is obtained from the two initial trees by joining them via u as in (A₂), and we bound

$$|p_R^{(1)}(\tau_s; z_s, z_1; y_s)| \leq |p_R(\tau_s; z_s, y_s)| + |p_R(\tau_s; z_1, y_s)|. \tag{78}$$

Here, $p_R(\tau_s, z_1, y_s)$ is associated with $\phi_s(z_s) \equiv 1$, and the integration over z_s in $T_l^s(T_{l_1}^{s_1+1}, T_{l_2}^{s_2+1})$ is performed similarly as in (A₂). This implies that for $\delta' \Lambda^2 \leq b \tau_s^{-1}$, (74) is bounded by

$$\left(\frac{b}{\delta'}\right)^{\frac{1}{2}} (\Lambda + m)^{2-n-|w|} \tau_s^{-\frac{1}{2}} \mathcal{P}\left(\log \frac{\Lambda + m}{m}\right) \mathcal{P}\left(\frac{\|\vec{p}_n\|}{\Lambda + m}\right) \mathcal{Q}\left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m}\right) \mathcal{F}_{s,l}^\Lambda(\tau). \tag{79}$$

- Case (2): $\delta' \Lambda^2 \geq b \tau_s^{-1}$

Using Lemma 2, we obtain the following bound:

$$|v_a - v_{a-1}| p_B \left(\frac{1 + \delta}{\Lambda_I^2}; v_{a-1}, v_a \right) \int_0^1 dt p_B(\tau_{s,\delta'}; tv_{a-1} + (1-t)v_a, y_s) \leq C_\delta \Lambda^{-1} p_B \left(\frac{1}{\Lambda_I^2}; v_{a-1}, v_a \right) p_B((1 + \delta')^3 \tau_s; v_{a-1}, y_s),$$

which implies that (77) can be bounded by

$$C_\delta C_{1,\delta} \Lambda^{-1} \tau_s^{-\frac{1}{2}} p_B \left(\frac{1}{\Lambda_I^2}; v_{a-1}, v_a \right) p_B((1 + \delta')^3 \tau_s; v_{a-1}, y_s).$$

Choosing δ' such that $(1 + \delta')^3 = 1 + \delta$, that is, $\delta' = \frac{\delta}{3} + O(\delta^2)$, and using bound (A6), the final bound obtained for (76), then, reads

$$C'_\delta \tau_s^{-\frac{1}{2}} \Lambda^{-1} p_B \left(\frac{1 + \delta}{\Lambda_I^2}; v_{a-1}, v_a \right) p_B(\tau_{s,\delta}; v_{a-1}, y_s), \tag{80}$$

where we used property (A5) from the Appendix and $C'_\delta = C_\delta C_{1,\delta}$.

Case (2) corresponds to a sum of two new trees of type \mathcal{T}_l^s , where in comparison to $T_l^s(T_l^{s_1+1}, T_l^{s_2+1})$, the incidence number of v_{a-1} has increased by one unit. In (74), an integral over z_s is performed. If in the new tree

- (a) z_s has $c(z_s) > 1$, then z_s takes the role of an internal vertex of the new tree and
- (b) z_s has $c(z_s) = 1$, we integrate over z_s using (A2) so that the vertex z_s disappears. As a consequence of bounds (79) and (80), on replacing again $s \rightarrow j$, we, thus, obtain for $n \geq 2$,

$$\left| \partial_\Lambda \partial^w \mathcal{L}_{l,n}^{\Lambda,\Lambda_0}(z_1; p_{1,n}; \phi_{\tau,y_{2,s}}^{(j)}) \right| \leq \left(\left(\frac{b}{\delta} \right)^{\frac{1}{2}} + C'_\delta \right) (\Lambda + m)^{2-n-|w|} \tau_j^{-\frac{1}{2}} \times \mathcal{P} \left(\log \frac{\Lambda + m}{m} \right) \mathcal{P} \left(\frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathcal{Q} \left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s,l}^\Lambda(\tau).$$

All the constants are absorbed in the polynomial \mathcal{Q} .

(D) To prove (D), we use (63) and (60) to show inductively that

$$\partial_\Lambda \partial^w \mathcal{L}_{l,n}^{\Lambda,\Lambda_0}(z_1; p_1, \dots, p_n; (z_1 - z_i)^r \phi_{\tau,y_{2,s}}), \tag{81}$$

$$\partial_\Lambda F_{12} \mathcal{L}_{l,n}^{\Lambda,\Lambda_0}(z_1, z_2; p_{1,n}; \phi_{\tau,y_{2,s}}) \tag{82}$$

are $\mathcal{E}^\infty(\mathbb{R}^+)$ with respect to z_1 . For (82), we will integrate from Λ to Λ_0 , and for (81), we integrate from 0 to Λ for $n + |w| + r \leq 4$ and from Λ to Λ_0 for $n + |w| + r \geq 5$. The details of these integrations can be deduced from (II,a) and (II,b).

(II) Integration of the FE:

From the bounds on the derivatives $\partial_\Lambda \partial^w \mathcal{L}_{n+2,l-1}^{\Lambda,\Lambda_0}(z_1; p_1, \dots, p_n; \phi_{\tau,y_{2,s}})$, we verify the induction hypothesis on integration over Λ . In all cases, we need the bound

$$\mathcal{F}_{s,l}^{\Lambda_2}(\tau) \leq \mathcal{F}_{s,l}^{\Lambda_1}(\tau) \quad \text{for } \Lambda_1 \leq \Lambda_2,$$

which follows directly from the definition of $\mathcal{F}_{s,l}^\Lambda(\tau)$.

(a) Irrelevant terms:

Since

$$\partial^w \mathcal{L}_{l,n}^{\Lambda_0,\Lambda_0}((z_1, p_1), \dots, (z_n, p_n)) = 0 \quad \forall n + |w| + r \geq 5,$$

integration from Λ to Λ_0 for $n + |w| + r \geq 5$ gives

$$\begin{aligned} & \left| \partial^w \mathcal{L}_{l,n}^{\Lambda,\Lambda_0}(z_1; \vec{p}_n; (z_1 - z_i)^r \phi_{\tau,y_{2,s}}) \right| \\ & \leq (\Lambda + m)^{4-n-|w|} \mathcal{P} \left(\log \frac{\Lambda + m}{m} \right) \mathcal{P} \left(\frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathcal{Q} \left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s,l}^\Lambda(\tau) \end{aligned}$$

and

$$\begin{aligned} & \left| \partial^w \mathcal{L}_{l,n}^{\Lambda,\Lambda_0}(z_1; \vec{p}_n; \phi_{\tau,y_{2,s}}^{(j)}) \right| \\ & \leq (\Lambda + m)^{3-n-|w|} \tau_j^{-\frac{1}{2}} \mathcal{P}_5 \left(\log \frac{\Lambda + m}{m} \right) \mathcal{P}_6 \left(\frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathcal{Q}_2 \left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s,l}^\Lambda(\tau) \end{aligned}$$

and

$$\left| F_{12} \mathcal{L}_{l,n}^{\Lambda,\Lambda_0}(z_1, z_2; \vec{p}_n; \phi_{\tau,y_{2,s}}) \right| \leq (\Lambda + m)^{1-n} \mathcal{P}_3 \left(\log \frac{\Lambda + m}{m} \right) \mathcal{P}_4 \left(\frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathcal{F}_{s,l}^{12}(\Lambda, \tau).$$

(b) Relevant terms:

(b₁) $n = 4, w = 0, r = 0$: We start from decomposition (39),

$$\mathcal{L}_{l,4}^{\Lambda,\Lambda_0}(z_1; \mathbf{0}, \dots, \mathbf{0}; \Phi_{4,s}) = c_l^{\Lambda,\Lambda_0}(z_1) \Phi_{4,s}(z_1, z_1, z_1) + l_{l,4}^{\Lambda,\Lambda_0}(z_1, \Phi_{4,s}), \quad s \leq 4, \quad (83)$$

where

$$c_l^{\Lambda,\Lambda_0}(z_1) := \int_0^\infty dz_2 dz_3 dz_4 \mathcal{L}_{l,4}^{\Lambda,\Lambda_0}((z_1, 0), \dots, (z_4, 0))$$

and

$$\Phi_{4,s}(z_2, z_3, z_4) = \prod_{i=2}^4 \phi_i(z_i), \quad \phi_i(z_i) = p_R(\tau_i; z_i, y_i) \quad \text{if } i \leq s, \quad \text{otherwise } \phi_i \equiv 1.$$

From the renormalization conditions, we have for all $l \geq 1$,

$$c_l^{0,\Lambda_0}(z_1) \equiv 0.$$

Integrating (59) from 0 to Λ at zero momenta, then, gives

$$|c_l^{\Lambda,\Lambda_0}(z_1)| \leq \mathcal{P} \left(\log \frac{\Lambda + m}{m} \right).$$

We decompose the test function

$$\Phi_{4,s}(z_2, z_3, z_4) = \prod_{i=2}^s p_R(\tau_i; z_i, y_i) = \Phi_{4,s}(z_1, z_1, z_1) + \psi(z_2, z_3, z_4), \quad (84)$$

where for $s = 4$,

$$\psi(z_2, z_3, z_4) := \sum_{j=2}^4 \prod_{f=2}^{j-1} p_R(\tau_f; z_1, y_f) p_R^{(1)}(\tau_j; z_j, z_1, y_j) \prod_{i=j+1}^4 p_R(\tau_i; z_i, y_i) = \sum_{j=2}^4 \phi_{\tau,y_{2,s}}^{(j)}(z_{2,4})$$

remembering definition (29). Note that if $\phi_i \equiv 1$ for some i , which corresponds to $s < 4$, then the corresponding contribution to the sum vanishes.

Using (84) and the linearity of $\mathcal{L}_{l,n}^{\Lambda,\Lambda_0}$ with respect to the test functions, we deduce that

$$\mathcal{L}_{l,4}^{\Lambda,\Lambda_0}(z_1; \mathbf{0}, \dots, \mathbf{0}; \Phi_{4,s}) = c_l^{\Lambda,\Lambda_0}(z_1) \Phi_{4,s}(z_1, z_1, z_1) + \mathcal{L}_{l,4}^{\Lambda,\Lambda_0}(z_1; \mathbf{0}, \dots, \mathbf{0}; \psi).$$

Therefore, we have $l_{l,4}^{\Lambda,\Lambda_0}(z_1; \Phi_{4,s}) = \mathcal{L}_{l,4}^{\Lambda,\Lambda_0}(z_1; \mathbf{0}, \dots, \mathbf{0}; \psi)$, and hence, FE (16) provides

$$\begin{aligned} \partial_\Lambda l_{l,4}^{\Lambda,\Lambda_0}(z_1; \Phi_{4,s}) &= \frac{1}{2} \int_{z_{2,4}, z, z'} \psi(z_2, z_3, z_4) \\ & \left[\int_k \mathcal{L}_{l-1,6}^{\Lambda,\Lambda_0}((z, k), (z_1, 0), \dots, (z_4, 0), (z', -k)) \dot{C}_R^\Lambda(k; z, z') \right. \\ & \left. - \frac{1}{2} \sum_{l_1+l_2=n_1+n_2=4} \left[\mathcal{L}_{l_1, n_1+1}^{\Lambda,\Lambda_0}((z_1, 0), \dots, (z_{n_1}, 0), (z, 0)) \dot{C}_R^\Lambda(0; z, z') \right. \right. \\ & \left. \left. \mathcal{L}_{l_2, n_2+1}^{\Lambda,\Lambda_0}((z', 0), \dots, (z_4, 0)) \right]_{rsym} \right]. \quad (85) \end{aligned}$$

The right-hand side is a sum over expressions of the same form as the one for $\partial_\Lambda \mathcal{L}_{l,4}^{\Lambda,\Lambda_0}(z_1, \phi_{\tau_2, y_2, s}^{(j)})$ in part (C). We obtain in the same way as there the bound

$$\left| \partial_\Lambda l_{l,4}^{\Lambda,\Lambda_0}(z_1; \Phi_{4,s}) \right| \leq (\Lambda + m)^{-2} \tau^{-\frac{1}{2}} \mathcal{P} \left(\log \frac{\Lambda + m}{m} \right) \mathcal{Q} \left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s,l}^\Lambda(\tau).$$

Integrating from Λ to Λ_0 and majorizing $(\lambda + m)^{-1}$ by $(\Lambda + m)^{-1}$, we obtain

$$\left| l_{l,4}^{\Lambda,\Lambda_0}(z_1; \Phi_{4,s}) \right| \leq \left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{P} \left(\log \frac{\Lambda + m}{m} \right) \mathcal{Q} \left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s,l}^\Lambda(\tau),$$

which gives the bound for $l_{l,4}^{\Lambda,\Lambda_0}(z_1, \Phi_{4,s})$. The bound is extended to general momenta using the Taylor formula (57).

(b₂) $n = 2, r = 0, w = 0$: We start from the decomposition (37),

$$\begin{aligned} \mathcal{L}_{l,2}^{\Lambda,\Lambda_0}(z_1; 0, 0; \phi_2) &= a_l^{\Lambda,\Lambda_0}(z_1) \phi_2(z_1) - s_l^{\Lambda,\Lambda_0}(z_1) (\partial_{z_1} \phi_2)(z_1) \\ &\quad + d_l^{\Lambda,\Lambda_0}(z_1) (\partial_{z_1}^2 \phi_2)(z_1) + l_{l,2}^{\Lambda,\Lambda_0}(z_1; \phi_2), \end{aligned} \tag{86}$$

where $\phi_2(z_2) := \phi_{\tau, y_2}(z_2) = p_R(\tau; z_2, y_2)$. Using $a_l^{0,\Lambda_0}(z_1), s_l^{0,\Lambda_0}(z_1), d_l^{0,\Lambda_0}(z_1) \equiv 0$, we obtain on integration from 0 to Λ ,

$$\left| a_l^{\Lambda,\Lambda_0}(z_1) \right| \leq (\Lambda + m)^2 \mathcal{P} \left(\log \frac{\Lambda + m}{m} \right), \quad \left| s_l^{\Lambda,\Lambda_0}(z_1) \right| \leq (\Lambda + m) \mathcal{P} \left(\log \frac{\Lambda + m}{m} \right), \tag{87}$$

$$\left| d_l^{\Lambda,\Lambda_0}(z_1) \right| \leq \mathcal{P} \left(\log \frac{\Lambda + m}{m} \right). \tag{88}$$

Since $(\partial_{z_1} \phi_2)(z_1) \leq \tau^{-\frac{1}{2}} \phi_2(z_1), (\partial_{z_1}^2 \phi_2)(z_1) \leq \tau^{-1} \phi_2(z_1)$, we obtain

$$\left| s_l^{\Lambda,\Lambda_0}(z_1) (\partial_{z_1} \phi_2)(z_1) \right| \leq (\Lambda + m)^2 \left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{P} \left(\log \frac{\Lambda + m}{m} \right) \phi_2(z_1),$$

$$\left| d_l^{\Lambda,\Lambda_0}(z_1) (\partial_{z_1}^2 \phi_2)(z_1) \right| \leq (\Lambda + m)^2 \left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right)^2 \mathcal{P} \left(\log \frac{\Lambda + m}{m} \right) \phi_2(z_1).$$

For the irrelevant part of the two-point function, we have

$$\partial_\Lambda l_{2,l}^{\Lambda,\Lambda_0}(z_1; \phi_2) = \int_0^\infty dz_2 \int_0^1 dt \frac{(1-t)^2}{2!} \partial_t^3 \phi_{\tau, y_1}(tz_2 + (1-t)z_1) \partial_\Lambda \mathcal{L}_{l,2}^{\Lambda,\Lambda_0}((z_1, 0), (z_2, 0)) \tag{89}$$

$$= \int_0^\infty dz_2 \int_0^1 dt \frac{(1-t)^2}{2!} \frac{\partial_t^3 \phi_2(tz_2 + (1-t)z_1)}{(z_2 - z_1)^3} \partial_\Lambda F_{12} \mathcal{L}_{l,2}^{\Lambda,\Lambda_0}((z_1, 0), (z_2, 0)). \tag{90}$$

Bound (70) for $n = 2, r = 0$ yields

$$\begin{aligned} \left| \partial_\Lambda l_{2,l}^{\Lambda,\Lambda_0}(z_1; \phi_2) \right| &\leq (\Lambda + m)^{-2} \mathcal{P} \left(\log \frac{\Lambda + m}{m} \right) \\ &\quad \times \int_0^\infty dz_2 \mathcal{F}_{s,l}^{12}(\Lambda, \tau) \int_0^1 dt \frac{(1-t)^2}{2!} \frac{\partial_t^3 p_R(\tau; tz_2 + (1-t)z_1, y_2)}{(z_2 - z_1)^3}. \end{aligned}$$

Using Lemma 1, we obtain

$$\begin{aligned} \left| \partial_\Lambda l_{2,l}^{\Lambda,\Lambda_0}(z_1; \phi_2) \right| &\leq O(1) (\Lambda + m)^{-2} \tau^{-\frac{3}{2}} \mathcal{P} \left(\log \frac{\Lambda + m}{m} \right) \\ &\quad \times \int_0^\infty dz_2 \mathcal{F}_{s,l}^{12}(\Lambda, \tau) \int_0^1 dt p_R(\tau_{\delta'}; tz_2 + (1-t)z_1, y_2). \end{aligned}$$

Remembering (33) and (34), we have

$$\begin{aligned} \mathcal{F}_{2,l}^{(12)}(\Lambda, \tau) &= \mathcal{F}_{2,l}(\Lambda; z_1, z_2) = \sum_{T^{2,(12)}} \mathcal{F}_{2,l}(\Lambda; T_l^{2,(12)}; z_1, z_2) \\ &= \sum_{n=1}^{3l-2} \sup_{\{\Lambda_{i_v} | \Lambda \leq \Lambda_{i_v} \leq \Lambda_0\}} \left[\prod_{1 \leq v \leq n} \int_{\tilde{z}_v} \right] p_B\left(\frac{1+\delta}{\Lambda_{l_1}^2}; z_1, \tilde{z}_1\right) \cdots p_B\left(\frac{1+\delta}{\Lambda_{l_n}^2}; \tilde{z}_n, z_2\right) \\ &\leq \sum_{n=1}^{3l-2} \sup_{\{\Lambda_{i_v} | \Lambda \leq \Lambda_{i_v} \leq \Lambda_0\}} p_B\left(\frac{1+\delta}{\Lambda_n^2}; z_1, z_2\right), \end{aligned} \tag{91}$$

where $\Lambda_n := (\sum_{v=1}^n \Lambda_{l_v}^{-2})^{-\frac{1}{2}}$. Sum (91) stems from the fact that the double rooted trees have all internal vertices with incidence number 2 only. This number v_2 is constrained by the relation $v_2 + \delta_{c_1,1} \leq 3l - 2 + 1/2$ from definitions (iii) and (iv) of Subsection IV B.

We use Lemma 2 together with

$$p_R(\tau\delta'; tz_2 + (1-t)z_1, y_2) \leq 2 p_B(\tau\delta'; tz_2 + (1-t)z_1, y_2).$$

For $\delta'\Lambda^2 \geq b\tau^{-1}$, we obtain

$$\left| \partial_{\Lambda} I_{2,l}^{\Lambda, \Lambda_0}(z_1; \phi_2) \right| \leq \tau^{-\frac{3}{2}} (\Lambda + m)^{-2} \mathcal{P}\left(\log \frac{\Lambda + m}{m}\right) p_B((1 + \delta')^3 \tau; z_1, y_2). \tag{92}$$

As before, we choose $\delta' = \frac{\delta}{3} + O(\delta^2)$ such that $(1 + \delta')^3 = 1 + \delta$, which implies that

$$\left| \partial_{\Lambda} I_{2,l}^{\Lambda, \Lambda_0}(z_1; \phi_2) \right| \leq \tau^{-\frac{3}{2}} (\Lambda + m)^{-2} \mathcal{P}\left(\log \frac{\Lambda + m}{m}\right) p_B(\tau\delta; z_1, y_2). \tag{93}$$

For $\delta'\Lambda^2 \leq b\tau^{-1}$, we use

$$\begin{aligned} \partial_{\Lambda} I_{l,2}^{\Lambda, \Lambda_0}(z_1; \phi_2) &= \partial_{\Lambda} \mathcal{L}_{l,2}^{\Lambda, \Lambda_0}(z_1; 0, 0; \phi_2) - \partial_{\Lambda} a_l^{\Lambda, \Lambda_0}(z_1) \phi_2(z_1) \\ &\quad - \partial_{\Lambda} s_l^{\Lambda, \Lambda_0}(z_1) (\partial_{z_1} \phi_2)(z_1) - \partial_{\Lambda} d_l^{\Lambda, \Lambda_0}(z_1) (\partial_{z_1}^2 \phi_2)(z_1) \end{aligned}$$

and bounds (69) and (71) to (73) to obtain

$$\left| \partial_{\Lambda} I_{2,l}^{\Lambda, \Lambda_0}(z_1; \phi_2) \right| \leq (\Lambda + m) \mathcal{P}\left(\log \frac{\Lambda + m}{m}\right) \mathcal{Q}\left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m}\right) p_B(\tau; z_1, y_2). \tag{94}$$

Since $\delta'\Lambda^2 \leq b\tau^{-1}$, we have

$$\begin{aligned} \left| \partial_{\Lambda} I_{l,2}^{\Lambda, \Lambda_0}(z_1; \phi_2) \right| &\leq \max\left(m^3, \tau^{-\frac{3}{2}} \left(\frac{b}{\delta'}\right)^{\frac{3}{2}}\right) (\Lambda + m)^{-2} \\ &\quad \times \mathcal{P}\left(\log \frac{\Lambda + m}{m}\right) \mathcal{Q}\left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m}\right) p_B(\tau; z_1, y_2). \end{aligned} \tag{95}$$

Combining (93) and (95) and using bound (A6), we obtain for all $\Lambda \geq 0$,

$$\begin{aligned} \left| \partial_{\Lambda} I_{l,2}^{\Lambda, \Lambda_0}(z_1; \phi_2) \right| &\leq \max\left(m^3, \tau^{-\frac{3}{2}} \left(\frac{b}{\delta'}\right)^{\frac{3}{2}}\right) (\Lambda + m)^{-2} \\ &\quad \times \mathcal{P}\left(\log \frac{\Lambda + m}{m}\right) \mathcal{Q}\left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m}\right) p_B(\tau\delta; z_1, y_2). \end{aligned} \tag{96}$$

Integrating from Λ to Λ_0 gives

$$\begin{aligned} |l_{2,l}^{\Lambda,\Lambda_0}(z_1, \phi_2)| &\leq \max\left(m^3, \tau^{-\frac{3}{2}}\left(\frac{b}{\delta^r}\right)^{\frac{3}{2}}\right)(\Lambda+m)^{-1}\mathcal{P}\left(\log\frac{\Lambda+m}{m}\right)\mathcal{Q}\left(\frac{\tau^{-\frac{1}{2}}}{\Lambda+m}\right)p_B(\tau_\delta; z_1, y_2) \\ &\leq \frac{\max\left(m^3, \tau^{-\frac{3}{2}}\left(\frac{b}{\delta^r}\right)^{\frac{3}{2}}\right)}{(\Lambda+m)^3}(\Lambda+m)^2\mathcal{P}\left(\log\frac{\Lambda+m}{m}\right)\mathcal{Q}\left(\frac{\tau^{-\frac{1}{2}}}{\Lambda+m}\right)p_B(\tau_\delta; z_1, y_2) \\ &\leq (\Lambda+m)^2\mathcal{P}\left(\log\frac{\Lambda+m}{m}\right)\tilde{\mathcal{Q}}\left(\frac{\tau^{-\frac{1}{2}}}{\Lambda+m}\right)p_B(\tau_\delta; z_1, y_2), \end{aligned}$$

where $\tilde{\mathcal{Q}}(x) = (1+x^3)\mathcal{Q}(x)$. Again, all the constants were absorbed in the coefficients of \mathcal{P} . This concludes the proof for $n=2, r=0$, and $w=0$.

To establish the bounds on $\partial^w \mathcal{L}_{l,2}^{\Lambda,\Lambda_0}(z_1; p_{1,m}; (z_1 - z_2)^r \phi_2)$ for $r=1, 2; w=0$ and $r=0; w=2$, we expand the respective test functions as follows:

$$\begin{aligned} \mathcal{L}_{l,2}^{\Lambda,\Lambda_0}(z_1; 0, 0; (z_1 - z_2)\phi_2) &= -s_l^{\Lambda,\Lambda_0}(z_1)\phi_2(z_1) + d_l^{\Lambda,\Lambda_0}(z_1)(\partial_{z_1}\phi_2)(z_1) \\ &\quad + \int_0^\infty dz_2 \int_0^1 dt (1-t)\partial_t^2 \phi_2(tz_2 + (1-t)z_1)(z_1 - z_2)\mathcal{L}_{l,2}^{\Lambda,\Lambda_0}((z_1, 0), (z_2, 0)), \end{aligned}$$

$$\begin{aligned} \mathcal{L}_{l,2}^{\Lambda,\Lambda_0}(z_1; 0, 0; (z_1 - z_2)^2 \phi_2) &= d_l^{\Lambda,\Lambda_0}(z_1)\phi_2(z_1) \\ &\quad + \int_0^\infty dz_2 \int_0^1 dt \partial_t \phi_2(tz_2 + (1-t)z_1)(z_1 - z_2)^2 \mathcal{L}_{l,2}^{\Lambda,\Lambda_0}((z_1, 0), (z_2, 0)), \end{aligned}$$

$$\begin{aligned} (\partial_{p_2} \mathcal{L}_{l,2}^{\Lambda,\Lambda_0})(z_1; 0, 0; \phi_2) &= b_l^{\Lambda,\Lambda_0}(z_1)\phi_2(z_1) \\ &\quad + \int_0^\infty dz_2 \int_0^1 dt \frac{(1-t)^2}{2!} \partial_t^3 \phi_2(tz_2 + (1-t)z_1) \partial_{p_2} \left(\mathcal{L}_{l,2}^{\Lambda,\Lambda_0}((z_1, p), (z_2, -p)) \right) \Big|_{p=0}. \end{aligned}$$

The relevant terms have been bounded in (87) and (88), and the reminders are treated as $l_{2,l}^{\Lambda,\Lambda_0}(z_1; \phi_2)$. We obtain

$$\begin{aligned} \left| \mathcal{L}_{l,2}^{\Lambda,\Lambda_0}(z_1; 0; (z_1 - z_2)\phi_2) \right| &\leq (\Lambda+m)\mathcal{P}\left(\log\frac{\Lambda+m}{m}\right)\mathcal{Q}\left(\frac{\tau^{-\frac{1}{2}}}{\Lambda+m}\right)\mathcal{F}_{s,l}^\Lambda(\tau), \\ \left| \mathcal{L}_{l,2}^{\Lambda,\Lambda_0}(z_1; 0; (z_1 - z_2)^2 \phi_2) \right| &\leq \mathcal{P}\left(\log\frac{\Lambda+m}{m}\right)\mathcal{Q}\left(\frac{\tau^{-\frac{1}{2}}}{\Lambda+m}\right)\mathcal{F}_{s,l}^\Lambda(\tau), \\ \left| \partial_p^2 \mathcal{L}_{l,2}^{\Lambda,\Lambda_0}(z_1; 0; \phi_2) \right| &\leq \mathcal{P}\left(\log\frac{\Lambda+m}{m}\right)\mathcal{Q}\left(\frac{\tau^{-\frac{1}{2}}}{\Lambda+m}\right)\mathcal{F}_{s,l}^\Lambda(\tau). \end{aligned}$$

The extension to general momenta is performed using the Taylor expansion of $\mathcal{L}_{l,2}^{\Lambda,\Lambda_0}(z_1; 0, 0; (z_1 - z_2)^r \phi_2)$ for $r=0, 1, 2$ with respect to the variable $p \in \mathbb{R}^3$.

Finally, note that for

$$\phi_2^{(2)}(z_2) = p_R(\tau; z_2, y_2) - p_R(\tau; z_1, y_2) = \phi_2(z_2) - \phi_2(z_1),$$

we have

$$\mathcal{L}_{l,2}^{\Lambda,\Lambda_0}(z_1; 0, 0; \phi_2^{(2)}) = s_l^{\Lambda,\Lambda_0}(z_1)(\partial_{z_1}\phi_2)(z_1) + d_l^{\Lambda,\Lambda_0}(z_1)(\partial_{z_1}^2 \phi_2)(z_1) + l_{l,2}^{\Lambda,\Lambda_0}(z_1; 0, 0; \phi_2). \tag{97}$$

Proceeding again similarly as before—see (90), (93), and (94)—provides

$$\left| \mathcal{L}_{l,2}^{\Lambda,\Lambda_0}(z_1; 0, 0; \phi_2^{(2)}) \right| \leq (\Lambda + m) \tau^{-\frac{1}{2}} \mathcal{P} \left(\log \frac{\Lambda + m}{m} \right) \mathcal{Q} \left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s,l}^{\Lambda}(\tau).$$

The extension to general momenta is done by Taylor expansion. This ends the Proof of Theorem 1. □

Theorem 2 (convergence). Let $0 \leq \Lambda \leq \Lambda_0 < \infty$. Using the same notations and conventions and adopting the same renormalization conditions (48) as in Theorem 1, we have the following bounds:

$$\begin{aligned} \left| \partial_{\Lambda_0} \partial^w \mathcal{L}_{l,n}^{\Lambda,\Lambda_0}(z_1; \vec{p}_n; (z_1 - z_i)^r \phi_{\tau,y_{2,s}}) \right| &\leq \frac{(\Lambda + m)^{5-n-|w|-r}}{(\Lambda_0 + m)^2} \tilde{\mathcal{P}}_1 \left(\log \frac{\Lambda_0 + m}{m} \right) \\ &\times \tilde{\mathcal{P}}_2 \left(\frac{\|\vec{p}_n\|}{\Lambda + m} \right) \tilde{\mathcal{Q}}_1 \left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s,l}^{\Lambda}(\tau) \quad \forall n + |w| + r \geq 4, \end{aligned} \tag{98}$$

$$\begin{aligned} \left| \partial_{\Lambda_0} \partial^w \mathcal{L}_{l,2}^{\Lambda,\Lambda_0}(z_1; p, -p; \phi_{\tau,y_2}) \right| &\leq \frac{(\Lambda + m)^{3-|w|}}{(\Lambda_0 + m)^2} \tilde{\mathcal{P}}_3 \left(\log \frac{\Lambda_0 + m}{m} \right) \tilde{\mathcal{P}}_4 \left(\frac{\|\vec{p}_n\|}{\Lambda + m} \right) \tilde{\mathcal{Q}}_2 \left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s,l}^{\Lambda}(\tau), \end{aligned} \tag{99}$$

$$\begin{aligned} \left| \partial_{\Lambda_0} F_{12} \mathcal{L}_{l,n}^{\Lambda,\Lambda_0}(z_1, z_2; \vec{p}_n; \phi_{\tau,y_{2,s}}) \right| &\leq \frac{(\Lambda + m)^{2-n}}{(\Lambda_0 + m)^2} \tilde{\mathcal{P}}_5 \left(\log \frac{\Lambda_0 + m}{m} \right) \tilde{\mathcal{P}}_6 \left(\frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathcal{F}_{s,l}^{12}(\Lambda, \tau) \quad \forall n \geq 2, \end{aligned} \tag{100}$$

$$\begin{aligned} \left| \partial_{\Lambda_0} \mathcal{L}_{l,n}^{\Lambda,\Lambda_0}(z_1; \vec{p}_n; \phi_{\tau,y_{2,s}}^{(j)}) \right| &\leq \frac{(\Lambda + m)^{4-n}}{(\Lambda_0 + m)^2} \tau_j^{-\frac{1}{2}} \tilde{\mathcal{P}}_7 \left(\log \frac{\Lambda_0 + m}{m} \right) \\ &\times \tilde{\mathcal{P}}_8 \left(\frac{\|\vec{p}_n\|}{\Lambda + m} \right) \tilde{\mathcal{Q}}_3 \left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s,l}^{\Lambda}(\tau) \quad \forall n \geq 4. \end{aligned} \tag{101}$$

Proof. We apply the method of the previous proof. The case $n = 4, l = 0$ evidently satisfies claim (98). We integrate the system of flow equation [Eq. (16)] and derive the individual n -point folded distributions (49) with respect to Λ_0 . We denote the right-hand side of (16) by $\partial^w \mathcal{R}_{l,n}^{\Lambda,\Lambda_0}((z_1, p_1), \dots, (z_n, p_n))$. We bound separately the relevant and the irrelevant terms.

- (I) $n \pm r + |w| > 4$:

In these cases, the boundary condition (46) implies that

$$-\partial^w \mathcal{L}_{l,n}^{\Lambda,\Lambda_0}(z_1; \vec{p}_n; (z_1 - z_i)^r \phi_{\tau,y_{2,s}}) = \int_{\Lambda}^{\Lambda_0} d\lambda \partial^w \mathcal{R}_{l,n}^{\lambda,\Lambda_0}(z_1; \vec{p}_n; (z_1 - z_i)^r \phi_{\tau,y_{2,s}}).$$

Therefore,

$$\begin{aligned} -\partial_{\Lambda_0} \partial^w \mathcal{L}_{l,n}^{\Lambda,\Lambda_0}(z_1; \vec{p}_n; (z_1 - z_i)^r \phi_{\tau,y_{2,s}}) &= \partial^w \mathcal{R}_{l,n}^{\Lambda_0,\Lambda_0}(z_1; \vec{p}_n; (z_1 - z_i)^r \phi_{\tau,y_{2,s}}) \\ &+ \int_{\Lambda}^{\Lambda_0} d\lambda \partial_{\Lambda_0} \partial^w \mathcal{R}_{l,n}^{\lambda,\Lambda_0}(z_1; \vec{p}_n; (z_1 - z_i)^r \phi_{\tau,y_{2,s}}). \end{aligned} \tag{102}$$

To the first term on the right-hand side, only the non-linear part on the right-hand side of (16) contributes because of the boundary condition (46). Using Theorem 1, we obtain as before the bound

$$\begin{aligned} \left| \partial^w \mathcal{R}_{l,n}^{\Lambda_0,\Lambda_0}(z_1; \vec{p}_n; (z_1 - z_i)^r \phi_{\tau,y_{2,s}}) \right| &\leq (\Lambda_0 + m)^{3-n-|w|-r} \mathcal{P} \left(\log \frac{\Lambda_0 + m}{m} \right) \mathcal{Q} \left(\frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathcal{Q} \left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s,l}^{\Lambda}(\tau). \end{aligned}$$

Since $n + |w| + r > 4$, we obtain for all $0 \leq \Lambda \leq \Lambda_0$,

$$\begin{aligned} & \left| \partial^w \mathcal{R}_{l,n}^{\Lambda_0, \Lambda_0} (z_1; \vec{p}_n; (z_1 - z_i)^r \phi_{\tau, y_{2,s}}) \right| \\ & \leq \frac{(\Lambda + m)^{5-n-|w|-r}}{(\Lambda_0 + m)^2} \mathcal{P} \left(\log \frac{\Lambda_0 + m}{m} \right) \mathcal{P} \left(\frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathcal{Q} \left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s,l}^\Lambda(\tau). \end{aligned}$$

For the second term on the right-hand side of (102), we obtain

$$\begin{aligned} \partial_{\Lambda_0} \partial^w \mathcal{R}_{l,n}^{\Lambda, \Lambda_0} ((z_1, p_1), \dots, (z_n, p_n)) &= \frac{1}{2} \int_z \int_{z'} \int_k \partial_{\Lambda_0} \partial^w \mathcal{L}_{l-1, n+2}^{\Lambda, \Lambda_0} ((z_1, p_1), \dots, (z_n, p_n), (z, k), (z', -k)) \dot{C}_R^\Lambda(p; z, z') \\ & - \frac{1}{2} \int_z \int_{z'} \sum_{l_1, l_2} \sum_{n_1, n_2} \sum_{w_i} c_{w_i} \left[\partial_{\Lambda_0} \partial^{w_1} \mathcal{L}_{l_1, n_1+1}^{\Lambda, \Lambda_0} ((z_1, p_1), \dots, (z_{n_1}, p_{n_1}), (z, p)) \partial^{w_3} \dot{C}_R^\Lambda(p; z, z') \right. \\ & \times \left. \partial^{w_2} \mathcal{L}_{l_2, n_2+1}^{\Lambda, \Lambda_0} ((z', -p), \dots, (z_n, p_n)) \right]_{rsym} \\ & - \frac{1}{2} \int_z \int_{z'} \sum_{l_1, l_2} \sum_{n_1, n_2} \sum_{w_i} c_{w_i} \left[\partial^{w_1} \mathcal{L}_{l_1, n_1+1}^{\Lambda, \Lambda_0} ((z_1, p_1), \dots, (z_{n_1}, p_{n_1}), (z, p)) \partial^{w_3} \dot{C}_R^\Lambda(p; z, z') \right. \\ & \times \left. \partial_{\Lambda_0} \partial^{w_2} \mathcal{L}_{l_2, n_2+1}^{\Lambda, \Lambda_0} ((z', -p), \dots, (z_n, p_n)) \right]_{rsym}, \\ & p = -p_1 - \dots - p_{n_1} = p_{n_1+1} + \dots + p_n, \end{aligned}$$

where we used that $\partial_{\Lambda_0} \dot{C}_R^\Lambda(k; z, z') = 0$. Using Theorem 1 and induction hypothesis (98) and following the same steps as in the Proof of Theorem 1, we get

$$\begin{aligned} & \left| \partial_{\Lambda_0} \partial^w \mathcal{R}_{l,n}^{\Lambda, \Lambda_0} (z_1; \vec{p}_n; (z_1 - z_i)^r \phi_{\tau, y_{2,s}}) \right| \\ & \leq \frac{(\Lambda + m)^{4-n-|w|-r}}{(\Lambda_0 + m)^2} \mathcal{P} \left(\log \frac{\Lambda_0 + m}{m} \right) \mathcal{P} \left(\frac{\|\vec{p}_n\|}{\Lambda + m} \right) \mathcal{Q} \left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s,l}^\Lambda(\tau). \end{aligned}$$

Integrating from Λ to Λ_0 and using that

$$\frac{\|\vec{p}_n\|}{\lambda + m} \leq \frac{\|\vec{p}_n\|}{\Lambda + m}, \quad \frac{\tau^{-\frac{1}{2}}}{\lambda + m} \leq \frac{\tau^{-\frac{1}{2}}}{\Lambda + m}, \quad \mathcal{F}_{s,l}^\lambda(\tau) \leq \mathcal{F}_{s,l}^\Lambda(\tau) \quad \forall \Lambda \leq \lambda \quad (103)$$

give a bound on the second term on the right-hand side of (102), which is of type (98).

(100) and (101) are proved following the same steps.

- (II) $(n, r, |w|) = (4, 0, 0)$, $(n, r, |w|) = (2, 0, 0)$, and $(n, r, |w|) = (2, 0, 2)$:

The FE provides inductive bounds on the relevant parts in these cases, and since the renormalization conditions do not depend on Λ_0 , we obtain

$$|\partial_{\Lambda_0} c_l^{\Lambda, \Lambda_0}(z_1)| \leq \frac{(\Lambda + m)}{(\Lambda_0 + m)^2} \mathcal{P} \left(\log \frac{\Lambda_0 + m}{m} \right), \quad (104)$$

$$|\partial_{\Lambda_0} a_l^{\Lambda, \Lambda_0}(z_1)| \leq \frac{(\Lambda + m)^3}{(\Lambda_0 + m)^2} \mathcal{P} \left(\log \frac{\Lambda_0 + m}{m} \right), \quad (105)$$

$$|\partial_{\Lambda_0} b_l^{\Lambda, \Lambda_0}(z_1)| \leq \frac{(\Lambda + m)}{(\Lambda_0 + m)^2} \mathcal{P} \left(\log \frac{\Lambda_0 + m}{m} \right), \quad (106)$$

$$|\partial_{\Lambda_0} d_l^{\Lambda, \Lambda_0}(z_1)| \leq \frac{(\Lambda + m)}{(\Lambda_0 + m)^2} \mathcal{P} \left(\log \frac{\Lambda_0 + m}{m} \right), \quad (107)$$

$$|\partial_{\Lambda_0} s_l^{\Lambda, \Lambda_0}(z_1)| \leq \frac{(\Lambda + m)^2}{(\Lambda_0 + m)^2} \mathcal{P} \left(\log \frac{\Lambda_0 + m}{m} \right). \quad (108)$$

In the case $n = 4$, we use decomposition (83) together with (29), (101), and (104) to obtain the bound

$$\begin{aligned} \left| \partial_{\Lambda_0} \partial^w \mathcal{L}_{l,4}^{\Lambda, \Lambda_0}(z_1; \vec{0}; \phi_{\tau, y_2, s}) \right| &\leq \frac{(\Lambda + m)}{(\Lambda_0 + m)^2} \tilde{\mathcal{P}}_1 \left(\log \frac{\Lambda_0 + m}{m} \right) \\ &\times \tilde{\mathcal{P}}_2 \left(\frac{\|\tilde{P}_n\|}{\Lambda + m} \right) \tilde{\mathcal{Q}}_1 \left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) \mathcal{F}_{s,l}^{\Lambda}(\tau). \end{aligned} \quad (109)$$

For $n = 2$, we use decomposition (86) and follow the same steps as in part (b2) of the Proof of Theorem 1. Using (89) and bound (100), we obtain for all $0 < \delta' < 1$,

$$\begin{aligned} \partial_{\Lambda_0} I_{2,l}^{\Lambda, \Lambda_0}(z_1; \phi_2) &= \int_0^\infty dz_2 \int_0^1 dt \frac{(1-t)^2}{2!} \frac{\partial_t^3 \phi_2(tz_2 + (1-t)z_1)}{(z_2 - z_1)^3} \partial_{\Lambda_0} F_{12} \mathcal{L}_{l,2}^{\Lambda, \Lambda_0}((z_1, 0), (z_2, 0)) \\ &\leq (\Lambda_0 + m)^{-2} \mathcal{P} \left(\log \frac{\Lambda_0 + m}{m} \right) \int_0^\infty dz_2 \mathcal{F}_{s,l}^{12}(\Lambda, \tau) \int_0^1 dt p_R(\tau \delta'; tz_2 + (1-t)z_1, y_2). \end{aligned}$$

Following the same steps as used before, we obtain

$$\left| \partial_{\Lambda_0} I_{2,l}^{\Lambda, \Lambda_0}(z_1; \phi) \right| \leq \tau^{-\frac{3}{2}} (\Lambda_0 + m)^{-2} \mathcal{P} \left(\log \frac{\Lambda_0 + m}{m} \right) p_B(\tau \delta; z_1, y_2) \quad \forall \delta' \Lambda^2 \geq b\tau^{-1}, \quad (110)$$

where we choose as before $\delta' = \delta/3 + O(\delta^2)$.

For the case $\delta' \Lambda^2 \leq b\tau^{-1}$, the bound is obtained as in the Proof of Theorem 1 by using decomposition (86), which yields

$$\begin{aligned} \partial_{\Lambda} \partial_{\Lambda_0} I_{l,2}^{\Lambda, \Lambda_0}(z_1; \phi_2) &= \partial_{\Lambda} \partial_{\Lambda_0} \mathcal{L}_{l,2}^{\Lambda, \Lambda_0}(z_1; \mathbf{0}, \mathbf{0}; \phi_2) - \partial_{\Lambda} \partial_{\Lambda_0} a_l^{\Lambda, \Lambda_0}(z_1) \phi_2(z_1) \\ &\quad + \partial_{\Lambda} \partial_{\Lambda_0} s_l^{\Lambda, \Lambda_0}(z_1) (\partial_{z_1} \phi_2)(z_1) - \partial_{\Lambda} \partial_{\Lambda_0} d_l^{\Lambda, \Lambda_0}(z_1) (\partial_{z_1}^2 \phi_2)(z_1). \end{aligned}$$

Using induction hypothesis (98), we obtain as in (94),

$$\left| \partial_{\Lambda} \partial_{\Lambda_0} I_{l,2}^{\Lambda, \Lambda_0}(z_1; \phi_2) \right| \leq \frac{(\Lambda + m)^2}{(\Lambda_0 + m)^2} \mathcal{P} \left(\log \frac{\Lambda_0 + m}{m} \right) \mathcal{Q} \left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) p_B(\tau \delta; z_1, y_2).$$

Since $\delta' \Lambda^2 \leq b\tau^{-1}$, we find

$$\left| \partial_{\Lambda} \partial_{\Lambda_0} I_{2,l}^{\Lambda, \Lambda_0}(z_1, \phi_2) \right| \leq \max \left(m^4, \tau^{-2} \left(\frac{b}{\delta'} \right)^2 \right) \frac{(\Lambda + m)^{-2}}{(\Lambda_0 + m)^2} \mathcal{P} \left(\log \frac{\Lambda_0 + m}{m} \right) \mathcal{Q} \left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) p_B(\tau \delta; z_1, y_2). \quad (111)$$

Integrating from Λ to Λ_0 (with Λ_0 being large enough), we obtain

$$\partial_{\Lambda_0} I_{2,l}^{\Lambda, \Lambda_0}(z_1, \phi_2) = \int_{\Lambda}^{\frac{b}{\delta'} \tau^{-\frac{1}{2}}} d\lambda \partial_{\lambda} \partial_{\Lambda_0} I_{2,l}^{\Lambda, \Lambda_0}(z_1, \phi_2) + \int_{\frac{b}{\delta'} \tau^{-\frac{1}{2}}}^{\Lambda_0} d\lambda \partial_{\lambda} \partial_{\Lambda_0} I_{2,l}^{\Lambda, \Lambda_0}(z_1, \phi_2).$$

Using (111), we obtain

$$\begin{aligned} &\left| \int_{\Lambda}^{\frac{b}{\delta'} \tau^{-\frac{1}{2}}} d\lambda \partial_{\lambda} \partial_{\Lambda_0} I_{2,l}^{\Lambda, \Lambda_0}(z_1, \phi_2) \right| \\ &\leq \max \left(m^4, \tau^{-2} \left(\frac{b}{\delta'} \right)^2 \right) \frac{(\Lambda + m)^{-1}}{(\Lambda_0 + m)^2} \mathcal{P} \left(\log \frac{\Lambda_0 + m}{m} \right) \mathcal{Q} \left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) p_B(\tau \delta; z_1, y_2), \end{aligned}$$

which can be bounded by

$$\frac{(\Lambda + m)^3}{(\Lambda_0 + m)^2} \mathcal{P} \left(\log \frac{\Lambda_0 + m}{m} \right) \mathcal{Q} \left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m} \right) p_B(\tau \delta; z_1, y_2). \quad (112)$$

We have

$$\int_{\frac{b}{\delta^r} \tau^{-\frac{1}{2}}}^{\Lambda_0} d\lambda \partial_\lambda \partial_{\Lambda_0} I_{2,l}^{\Lambda, \Lambda_0}(z_1, \phi_2) = \partial_{\Lambda_0} I_{2,l}^{\frac{b}{\delta^r} \tau^{-\frac{1}{2}}, \Lambda_0}(z_1, \phi_2). \tag{113}$$

Using (110) for $\delta^r \Lambda^2 \geq b\tau^{-1}$, we deduce that

$$\left| \int_{\tau^{-\frac{1}{2}}}^{\Lambda_0} d\lambda \partial_\lambda \partial_{\Lambda_0} I_{2,l}^{\Lambda, \Lambda_0}(z_1, \phi_2) \right| \leq \frac{(\Lambda + m)^3}{(\Lambda_0 + m)^2} \mathcal{P}\left(\log \frac{\Lambda_0 + m}{m}\right) \left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m}\right)^3 p_B(\tau_\delta; z_1, y_2). \tag{114}$$

(112) together with (114) implies that for all $0 \leq \Lambda \leq \Lambda_0$, we have

$$\left| \partial_{\Lambda_0} I_{2,l}^{\Lambda, \Lambda_0}(z_1, \phi_2) \right| \leq \frac{(\Lambda + m)^3}{(\Lambda_0 + m)^2} \mathcal{P}\left(\log \frac{\Lambda_0 + m}{m}\right) \mathcal{Q}\left(\frac{\tau^{-\frac{1}{2}}}{\Lambda + m}\right) p_B(\tau_\delta; z_1, y_2).$$

This concludes the proof for $n = 2, r = 0$, and $w = 0$. Cases $n = 2, r = 0$, and $w = 2$ are treated similarly. Extension to general momenta is again achieved via the Taylor formula (57). Note that compared to the Proof of Theorem 1, we do not need extra bounds for the cases $n = 2, r = 2$, and $n = 2, r = 1$ since they are not required to close the inductive scheme. Bounds (105)–(108) leading to convergence are obtained using only the FE together with inductive hypotheses (98) and (99) in addition to bound (52).

Thus, the Proof of Theorem 2 is complete. □

Integration of bounds (98) and (99) over Λ_0 immediately proves the convergence of all $\mathcal{L}_{l,n}^{\Lambda, \Lambda_0}(z_1, \vec{p}_n; \phi_{\tau, y_{2,s}})$ for fixed Λ to finite limits when $\Lambda_0 \rightarrow \infty$. In particular, one obtains for all $\Lambda'_0 > \Lambda_0$ and $\vec{p}_n \in \mathbb{R}^{3n}$,

$$\begin{aligned} & \left| \mathcal{L}_{l,n}^{0, \Lambda_0}(z_1; \vec{p}_n; \phi_{\tau, y_{2,s}}) - \mathcal{L}_{l,n}^{0, \Lambda'_0}(z_1; \vec{p}_n; \phi_{\tau, y_{2,s}}) \right| \\ & < \frac{m^{5-n}}{\Lambda_0} \left(\log \frac{\Lambda_0 + m}{m}\right)^v \tilde{\mathcal{P}}_2\left(\frac{\|\vec{p}_n\|}{m}\right) \tilde{\mathcal{Q}}_1\left(\frac{\tau^{-\frac{1}{2}}}{m}\right) \mathcal{F}_{s,l}^0(\tau). \end{aligned}$$

Then, the Cauchy criterion in $\mathcal{C}^\infty(\mathbb{R}^+)$ with respect to Λ_0 implies the existence of finite limits to all loop orders l .

AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Majdouline Borji: Conceptualization (equal); Formal analysis (lead); Investigation (lead); Methodology (lead); Project administration (supporting); Resources (equal); Writing – original draft (lead); Writing – review & editing (supporting). **Christoph Kopper:** Conceptualization (equal); Funding acquisition (lead); Methodology (equal); Project administration (lead); Supervision (lead); Validation (lead); Writing – review & editing (supporting).

DATA AVAILABILITY

The data that support the findings of this study are available within the article.

APPENDIX: THE HEAT KERNEL p_B : PROPERTIES AND BOUNDS

Here, we collect inequalities verified by the one-dimensional heat kernel defined by

$$p_B(\tau; z_1, z_2) = \frac{1}{\sqrt{2\pi\tau}} e^{-\frac{(z_1 - z_2)^2}{2\tau}}, \quad \tau > 0. \tag{A1}$$

Clearly, we have the following properties of (A1):

- Normalization:

$$\int_{\mathbb{R}} du p_B(\tau; z, u) = 1, \quad \forall z \in \mathbb{R}. \tag{A2}$$

- The semi-group property:

$$\int_{\mathbb{R}} du p_B(\tau_1; z_1, u) p_B(\tau_2; u, z_2) = p_B(\tau_1 + \tau_2; z_1, z_2), \quad \forall z_1, z_2 \in \mathbb{R}. \tag{A3}$$

- Let p_R be the Robin heat kernel given by

$$p_R(\tau; z, z') = p_B(\tau; z, z') + p_B(\tau; z, -z') - 2 \int_0^\infty dw e^{-w} p_B\left(\tau; z, -\frac{w}{c} - z'\right) \quad \forall z, z' \in \mathbb{R}^+.$$

Then,

$$p_R(\tau; z, z') \leq 2 p_B(\tau; z, z') \quad \forall z, z' \in \mathbb{R}^+. \tag{A4}$$

- A simple computation gives that for all z_1 and z_2 in \mathbb{R}^+ , we have

$$\int_{\mathbb{R}} du p_B(\tau_1; z_1, u) p_B(\tau_2; u, z_2) \leq 2 \int_{\mathbb{R}^+} du p_B(\tau_1; z_1, u) p_B(\tau_2; u, z_2). \tag{A5}$$

- For $0 < \delta < 1$, $\tau_\delta = (1 + \delta)\tau$, and $z_1, z_2 \in \mathbb{R}^+$, we have

$$p_B(\tau; z_1, z_2) \leq \sqrt{1 + \delta} p_B(\tau_\delta; z_1, z_2), \tag{A6}$$

and for $\delta' > \delta$, we have

$$|z_1 - z_2|^r p_B(\tau_\delta; z_1, z_2) \leq C_{\delta, \delta'} \tau^{\frac{r}{2}} p_B(\tau_{\delta'}; z_1, z_2), \tag{A7}$$

where $C_{\delta, \delta'} = \sqrt{\frac{1+\delta'}{1+\delta}} \|x^r e^{-\frac{x^2}{2} \frac{\delta' - \delta}{(1+\delta)(1+\delta')}}\|_\infty$.

The following lemmas are used repeatedly in the inductive proof of bounds (52)–(54).

Lemma 1. For all t, u, v , and y in \mathbb{R}^+ , $\tau > 0$, and some constant $C_{k, \delta} > 0$,

$$\left| \partial_t^k p_B(\tau; tu + (1-t)v, y) \right| \leq C_{k, \delta} \frac{|u-v|^k}{\tau^{\frac{k}{2}}} p_B(\tau_\delta; tu + (1-t)v, y).$$

Proof. One can prove by induction that

$$\partial_t^k p_B(\tau; tu + (1-t)v, y) = \frac{(u-v)^k}{\tau^{\frac{k}{2}}} \mathcal{P}_k\left(\frac{tu + (1-t)v - y}{\sqrt{\tau}}\right) p_B(\tau; tu + (1-t)v, y).$$

\mathcal{P}_k is a polynomial of degree k and has at least one root if k is odd. Therefore,

$$\begin{aligned} \left| \partial_t^k p_B(\tau; tu + (1-t)v, y) \right| &\leq \left| \mathcal{P}_k\left(\frac{tu + (1-t)v - y}{\sqrt{\tau}}\right) \right| e^{-\frac{(tu+(1-t)v-y)^2}{2(1+\delta)\tau} \cdot \frac{\delta}{1+2\delta}} \\ &\quad \times \frac{|u-v|^k}{\tau^{\frac{k}{2}}} p_B(\tau_\delta; tu + (1-t)v, y). \end{aligned}$$

The lemma follows directly with $C_{k, \delta} := \sup_{x \in \mathbb{R}} \left| \mathcal{P}_k(x) e^{-\frac{x^2}{2} \cdot \frac{\delta}{1+2\delta}} \right|$. □

Corollary 2. For all t, u, v , and y in \mathbb{R}^+ , $\tau > 0$, and some constant $C'_{k, \delta} > 0$,

$$\left| \partial_t^k p_R(\tau; tu + (1-t)v, y) \right| \leq C'_{k, \delta} \frac{|u-v|^k}{\tau^{\frac{k}{2}}} p_B(\tau_\delta; tu + (1-t)v, y).$$

Proof. We have

$$p_R(\tau; z, z') = p_B(\tau; z, z') + p_B(\tau; z, -z') - 2 \int_0^\infty dw e^{-w} p_B\left(\tau; z, -\frac{w}{c} - z'\right).$$

Using Lemma 1, we obtain

$$\begin{aligned} \left| \partial_t^k p_R(\tau; tu + (1-t)v, y) \right| &\leq C_{k,\delta} \frac{|u-v|^k}{\tau^{\frac{k}{2}}} (p_B(\tau_\delta; tu + (1-t)v, y) \\ &+ p_B(\tau_\delta; tu + (1-t)v, -y) + 2 \int_0^\infty dw e^{-w} p_B\left(\tau_\delta; tu + (1-t)v, -y - \frac{w}{c}\right)). \end{aligned}$$

Using that for $z, z', w \in \mathbb{R}^+$,

$$e^{-\frac{(z+z'+w/c)^2}{2}} \leq e^{-\frac{(z+z')^2}{2}} \leq e^{-\frac{(z-z')^2}{2}},$$

we obtain

$$\partial_t^k p_R(\tau; tu + (1-t)v, y) \leq 4 C_{k,\delta} \frac{|u-v|^k}{\tau^{\frac{k}{2}}} p_B(\tau_\delta; tu + (1-t)v, y).$$

We may set $C'_{k,\delta} := 4C_{k,\delta}$. □

Lemma 2. For $0 < \delta < 1$, we define $b = 2 \frac{1+2\delta}{1-2\delta}$ and let $0 < \delta' < 1$ and $\Lambda_I \geq \Lambda$ such that $\delta' \Lambda^2 \geq b\tau^{-1}$. For z_1, z_2 , and y_1 in \mathbb{R}^+ and $\tau > 0$, we have

$$\begin{aligned} |z_1 - z_2| p_B\left(\frac{1+\delta}{\Lambda_I^2}; z_1, z_2\right) &\int_0^1 dt p_B(\tau_{\delta'}; tz_2 + (1-t)z_1, y_1) \\ &\leq C_\delta \Lambda^{-1} p_B\left(\frac{1}{\Lambda_I^2}; z_1, z_2\right) p_B((1+\delta')^3 \tau; z_1, y_1). \end{aligned} \tag{A8}$$

Proof. We have

$$p_B\left(\frac{1+\delta}{\Lambda_I^2}; z_1, z_2\right) = \sqrt{\frac{1+2\delta}{1+\delta}} p_B\left(\frac{1+2\delta}{\Lambda_I^2}; z_1, z_2\right) e^{-\frac{\Lambda_I^2(z_1-z_2)^2}{2(1+\delta)} \cdot \frac{\delta}{1+2\delta}},$$

which implies that

$$|z_1 - z_2| p_B\left(\frac{1+\delta}{\Lambda_I^2}; z_1, z_2\right) \leq C_\delta \Lambda^{-1} p_B\left(\frac{1+2\delta}{\Lambda_I^2}; z_1, z_2\right), \tag{A9}$$

where $C_\delta := \sqrt{\frac{1+2\delta}{1+\delta}} \left\| x e^{-\frac{x^2}{2(1+\delta)} \cdot \frac{\delta}{1+2\delta}} \right\|_\infty$, and we used that $\Lambda \leq \Lambda_I$.

Now, we bound

$$p_B\left(\frac{1+2\delta}{\Lambda_I^2}; z_1, z_2\right) \int_0^1 dt p_B(\tau_{\delta'}; tz_2 + (1-t)z_1, y_1). \tag{A10}$$

For $0 < \delta' < 1$ and $\delta' \Lambda_I^2 \geq b\tau^{-1}$, we have

$$\begin{aligned} &\frac{\Lambda_I^2}{1+2\delta} (z_1 - z_2)^2 + \frac{1}{\tau(1+\delta')} (tz_2 + (1-t)z_1 - y_1)^2 \\ &= \frac{\Lambda_I^2 |z_1 - z_2|^2}{b} + \frac{\Lambda_I^2 |z_1 - z_2|^2}{2} + \frac{1}{\tau(1+\delta')} (tz_2 + (1-t)z_1 - y_1)^2 \\ &\geq \frac{|z_1 - z_2|^2}{\tau \delta'} + \frac{\Lambda_I^2 |z_1 - z_2|^2}{2} + \frac{1}{\tau(1+\delta')^2} (tz_2 + (1-t)z_1 - y_1)^2. \end{aligned} \tag{A11}$$

Let $0 \leq t \leq 1$, and we have

$$\frac{|z_1 - z_2|^2}{\delta'} + \frac{|tz_2 + (1-t)z_1 - y_1|^2}{(1+\delta')^2} \geq \frac{1}{(1+\delta')^2} \left[\frac{1}{\delta'} |z_1 - z_2|^2 + |tz_2 + (1-t)z_1 - y_1|^2 \right].$$

Using that

$$|z_1 - z_2| = |z_1 - tz_2 - (1-t)z_1| + |tz_2 + (1-t)z_1 - z_2|,$$

we obtain

$$\begin{aligned} & \frac{1}{\delta'} |z_1 - z_2|^2 + |tz_2 + (1-t)z_1 - y_1|^2 \\ & \geq \frac{1}{\delta'} |z_1 - tz_2 - (1-t)z_1|^2 + |tz_2 + (1-t)z_1 - y_1|^2 \\ & \geq \frac{1}{1+\delta'} (|z_1 - tz_2 - (1-t)z_1| + |tz_2 + (1-t)z_1 - y_1|)^2 \\ & \geq \frac{|z_1 - y_1|^2}{1+\delta'}. \end{aligned}$$

Therefore, we obtain that

$$\frac{\Lambda_I^2}{1+2\delta} (z_1 - z_2)^2 + \frac{1}{\tau(1+\delta')} (tz_2 + (1-t)z_1 - y_1)^2 \geq \frac{|z_1 - y_1|^2}{\tau(1+\delta')^3} + \frac{\Lambda_I^2 |z_1 - z_2|^2}{2},$$

which implies that (A10) can be bounded by

$$C_\delta p_B \left(\frac{1}{\Lambda_I^2}; z_1, z_2 \right) p_B \left((1+\delta')^3 \tau; z_1, y_1 \right),$$

which together with (A9) gives final bound (A8). □

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