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# INDEX, PRIME IDEAL FACTORIZATION IN SIMPLEST QUARTIC FIELDS AND COUNTING THEIR DISCRIMINANTS.

MOHAMMED SEDDIK

ABSTRACT. We consider the simplest quartic number fields  $\mathbb{K}_m$  defined by the irreducible quartic polynomials

$$x^4 - mx^3 - 6x^2 + mx + 1,$$

where  $m$  runs over the positive rational integers such that the odd part of  $m^2 + 16$  is squarefree. In this paper, we study the common index divisor  $I(\mathbb{K}_m)$  and determine explicitly the prime ideal decomposition for any prime number in any simplest quartic number fields  $\mathbb{K}_m$ . On the other hand, we establish an asymptotic formula for the number of simplest quartic fields with discriminant  $\leq x$  and given index.

## 1. INTRODUCTION AND PRELIMINARIES

Let  $\mathbb{K}$  be a number field. We start with given most important known results on the common factor of indices  $I(\mathbb{K})$ , the prime ideal factorization and the asymptotic formula for number fields with discriminant  $\leq x$ .

1.1. **Index  $I(\mathbb{K})$  for number fields  $\mathbb{K}$ .** Let  $\mathbb{K}$  be a number field of degree  $n$  over  $\mathbb{Q}$  and let  $\mathbb{O}_{\mathbb{K}}$  be its ring of integers. Denote by  $\widehat{\mathbb{O}}_{\mathbb{K}}$  the set of primitive elements of  $\mathbb{O}_{\mathbb{K}}$ . For any  $\theta \in \mathbb{O}_{\mathbb{K}}$  we denote  $F_{\theta}(x)$  the characteristic polynomial of  $\theta$  over  $\mathbb{Q}$ . Let  $D_{\mathbb{K}}$  be the discriminant of  $\mathbb{K}$ . It is well known that if  $\theta \in \widehat{\mathbb{O}}_{\mathbb{K}}$ , the discriminant of  $F_{\theta}(x)$  has the form

$$(1.1) \quad D(\theta) = I(\theta)^2 D_{\mathbb{K}},$$

where  $I(\theta) = (\mathbb{O}_{\mathbb{K}} : \mathbb{Z}[\theta])$  is called the index of  $\theta$ . Let

$$(1.2) \quad I(\mathbb{K}) = \gcd_{\theta \in \widehat{\mathbb{O}}_{\mathbb{K}}} I(\theta).$$

A prime number  $p$  is called a common factor of indices if  $p \mid I(\mathbb{K})$ .

The knowledge of the index  $I(\mathbb{K})$  makes it possible to find the explicit decomposition of the prime numbers in the fields  $\mathbb{K}$ : If the prime number  $p \nmid I(\mathbb{K})$ , by equation (1.2) there exist a primitive integer  $\theta$  where  $p \nmid I(\theta)$  and by Dedekind's theorem [5, §18] we explicitly have the factorization of  $p$  using  $\theta$ . However, if  $p$  is a common factor divisor, the prime ideal decomposition in  $\mathbb{O}_{\mathbb{K}}$  is more difficult.

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Let us recall the statement of Dedekind's theorem. Let  $\mathbb{K} = \mathbb{Q}(\theta)$  be an algebraic number field with  $\theta \in \mathbb{O}_{\mathbb{K}}$ . Let  $p$  be a rational prime. Let

$$f(x) = \text{Irr}_{\mathbb{Q}}(x, \theta) \in \mathbb{Z}[x].$$

We consider the canonical surjection map  $\mathbb{Z}[x] \rightarrow \mathbb{Z}/p\mathbb{Z}[x]$ . We write

$$\bar{f}(x) = g_1(x)^{e_1} \cdots g_r(x)^{e_r},$$

where  $g_1(x), \dots, g_r(x)$  are distinct monic irreducible polynomials in  $\mathbb{Z}/p\mathbb{Z}[x]$  and  $e_1, \dots, e_r$  are positive integers.

For  $i = 1, 2, \dots, r$  denote by  $f_i(x)$  any monic polynomial of  $\mathbb{Z}[x]$  such that  $\bar{f}_i = g_i$ . We then set

$$P_i = \langle p, f_i(\theta) \rangle.$$

If  $I(\theta) \not\equiv 0 \pmod{p}$  then we have  $P_1, \dots, P_r$  are distinct prime ideals of  $\mathbb{O}_{\mathbb{K}}$  with

$$p\mathbb{O}_{\mathbb{K}} = P_1^{e_1} \cdots P_r^{e_r},$$

and

$$N(P_i) = p^{\deg(f_i)}.$$

In the following we review some known results on the computation of the index  $I(\mathbb{K})$ .

- If  $\mathbb{K}$  is a quadratic field ( $n = 2$ ), by classical number theory on quadratic fields, one can show that  $I(\mathbb{K}) = 1$ .
- In case  $\mathbb{K}$  is a cubic field ( $n = 3$ ), Engstrom [6] showed that  $I(\mathbb{K}) = 1$  or 2. Llorente and Nart in [14, Theorem 1] determine the *type of decomposition* of the rational primes and in [14, Theorem 4] give a necessary and sufficient condition for the index of  $\mathbb{K}$  to be 2. Moreover, in the paper [19] Spearman and Williams give the *explicit prime ideal factorization* of 2 in cubic fields with index 2.
- In the case  $\mathbb{K}$  is a quartic field ( $n = 4$ ), Engstrom [6] showed that

$$(1.3) \quad I(\mathbb{K}) = 1, 2, 3, 4, 6 \text{ or } 12.$$

- If the field  $\mathbb{K}$  is a cyclic quartic field, Spearman and Williams [20] showed that  $I(\mathbb{K})$  assumes all of these values and they give necessary and sufficient conditions for each to occur and find an asymptotic formula for the number of cyclic quartic fields with discriminant  $\leq x$  and  $I(\mathbb{K}) = i$  for each  $i \in \{1, 2, 3, 4, 6, 12\}$ ,

$$N(x; i) = \alpha_i x^{\frac{1}{2}} + O(x^{\frac{1}{3}} \log^3 x),$$

where

$$\begin{aligned} \alpha_1 &\approx 0.0970153, & \alpha_2 &\approx 0.0067627, & \alpha_3 &\approx 0.0101764, \\ \alpha_4 &\approx 0.0067627, & \alpha_6 &\approx 0.0006321, & \alpha_{12} &\approx 0.0006321. \end{aligned}$$

- Funakura [7, Theorem 5] showed that in the case of a pure quartic field, we have  $I(\mathbb{K}) = 1, 2$ .
- In the pure quartic field, Spearman and Williams [21] gives the explicit prime ideal factorization of 2 when the index is equal to 2.

1.2. **Counting discriminants with given index.** The general problem in algebraic number theory is that of counting the number of fields by discriminant. If we let  $N_n(x)$  denote the number of fields of degree  $n$  over  $\mathbb{Q}$  whose discriminants do not exceed  $x$  in absolute value. Then there is a conjecture that

$$\lim_{x \rightarrow \infty} \frac{N_n(x)}{x}$$

exist and is non-zero. This conjecture is proved in many special cases:

- For quadratic field, the problem is simple we have

$$N_2(x) = \sum_{\substack{|d| \leq x \\ d \equiv 1 \pmod{4}}} 1 + \sum_{\substack{|d| \leq \frac{x}{4} \\ d \equiv 2,3 \pmod{4}}} 1 \sim \frac{6}{\pi^2} x.$$

- For cyclic cubic fields H. Cohn [4] and Cohen-Diaz y Diaz in [3, p.577, §3] showed that

$$N_3(x) \sim C_3 x^{1/2}$$

where

$$C_3 = \frac{11\sqrt{3}}{36\pi} \prod_{\substack{p \equiv 1 \pmod{6} \\ p \text{ prime}}} \left(1 - \frac{2}{p(p+1)}\right).$$

- For cyclic quartic fields Baily [2, p. 209, Theorem 9] and its revised form by many authors see [13], [18] and [3, p.580, §5] by different methods, as follows

$$N_4(x) \sim C_4 x^{1/2}$$

where

$$C_4 = \frac{3}{\pi^2} \left( \left(1 + \frac{\sqrt{2}}{24}\right) \prod_{\substack{p \equiv 1 \pmod{4} \\ p \text{ prime}}} \left(1 + \frac{2}{p^{3/2} + p^{1/2}}\right) - 1 \right).$$

1.3. **Summary of the paper.** In this paper, we investigate the family of the simplest quartic fields which are defined by adjunction to  $\mathbb{Q}$  of a root of the polynomial

$$P_m = x^4 - mx^3 - 6x^2 + mx + 1, \quad m \in \mathbb{Z}^+$$

where  $m^2 + 16$  is not divisible by an odd square. It is easy to see that  $m$  may be specified greater than zero as  $P_m$  and  $P_{-m}$  generate the same extension.

M.N. Gras in [9] proved that those polynomials are reducibles precisely when  $m^2 + 16$  is a square, which occurs only for excluded cases  $m = 0, 3$ , and show that the form  $m^2 + 16$  represents infinitely many square-free integers. Olajos [17] proved that  $\mathbb{K}_m$  admits power integral bases only for  $m = 2, 4$  and he gave all generators of power integral bases. Recently, Gaál and Pentrányi [8] compute the minimal index of the simplest quartic fields.

The purpose of this paper :

- We study the common indices  $I(\mathbb{K}_m)$  and the prime ideal decomposition is determined explicitly,
- We establish an asymptotic formula for the number of the simplest quartic fields with given index and the discriminants less than  $x$ .

## 2. STATEMENT OF MAIN RESULTS

We state our main results.

### 2.1. Computation of the index $I(\mathbb{K}_m)$ and prime decomposition.

**Theorem 2.1.** Let  $m \in \mathbb{Z}^+$ ,  $m \neq 0, 3$  and  $\mathbb{K}_m = \mathbb{Q}(\theta)$  where  $\theta$  be a root of  $P_m$ . Then we have

(1)

$$I(\mathbb{K}_m) = \begin{cases} 2 & \text{if } m \text{ odd,} \\ 1 & \text{if } m \text{ even.} \end{cases}$$

(2) The prime ideal factorization of 2 in  $\mathbb{K}_m$  with index 2 is

$$2\mathbb{O}_{\mathbb{K}_m} = \left\langle 2, \frac{1 + \sqrt{m^2 + 16}}{2} \right\rangle \left\langle 2, \frac{1 - \sqrt{m^2 + 16}}{2} \right\rangle.$$

(3) The prime ideal factorization of 2 in  $\mathbb{K}_m$  with index 1 is as follows:

1) If  $v_2(m) = 1$ , then,

$$2\mathbb{O}_{\mathbb{K}_m} = \left\langle 2, \frac{m\theta^3 + 10\theta^2 - m\theta + 6}{4} \right\rangle^2.$$

2) If  $v_2(m) = 2$ , then,

$$2\mathbb{O}_{\mathbb{K}_m} = \left\langle 2, \frac{1 + \theta + \theta^2 + \theta^3}{4} \right\rangle^4.$$

3) If  $v_2(m) = 3$ , then,

$$2\mathbb{O}_{\mathbb{K}_m} = \left\langle 2, \frac{(m^3 + 25m + 4)\theta^3 + (5m^2 + 168)\theta^2 - (m^3 + 21m - 56)\theta - m^2 + 8}{16} \right\rangle^2.$$

4) If  $v_2(m) \geq 4$ , then,

$$2\mathbb{O}_{\mathbb{K}_m} = \left\langle 2, \frac{2 + 7\theta + \theta^3}{4} \right\rangle^2 \left\langle 2, \frac{5 + 7\theta + \theta^3}{4} \right\rangle^2.$$

**Proposition 2.1.** Let  $m \in \mathbb{Z}^+$ ,  $m \neq 0, 3$  and  $\mathbb{K}_m = \mathbb{Q}(\theta)$  where  $\theta$  be a root of  $P_m$ . Then

$$I(\theta) = \begin{cases} 2 & \text{if } v_2(m) = 0, \\ 2^2 & \text{if } v_2(m) = 1, \\ 2^3 & \text{if } v_2(m) = 2, \\ 2^4 & \text{if } v_2(m) \geq 3. \end{cases}$$

**Remark.** Note that we can show that for any  $p$  odd prime, we can give explicitly the factorization of  $p$  using  $\theta$ . To do it, we need to factorize  $P_m \pmod{p}$  and we use Dedekind's theorem.

**2.2. Asymptotic number of simplest quartic fields with discriminant  $\leq x$  and given index.** Let  $\mathbb{K}_m = \mathbb{Q}(\theta)$  where  $\theta$  be a root of  $P_m$  (simplest quartic fields).

We define for a positive integer  $i$

$$(2.1) \quad N(x, i) = \text{number of } \mathbb{K}_m \text{ with } D(\mathbb{K}_m) \leq x \text{ and } I(\mathbb{K}_m) = i.$$

We state our second main result.

**Theorem 2.2.** For  $i = 1, 2, 3, 4, 6, 12$  we have

(1)  $N(x, i) = 0$  for  $i \neq 1, 2$ .

(2) For  $i = 1, 2$ , we have the asymptotic formulas

$$N(x, i) \sim C_i x^{\frac{1}{6}},$$

where

$$C_1 = \left( \frac{1}{4\sqrt[3]{4}} + \frac{1}{4\sqrt[3]{2}} + \frac{1}{4} \right) \prod_{\substack{p \equiv 1 \pmod{4} \\ p \text{ prime}}} \left( 1 - \frac{2}{p^2} \right)$$

and

$$C_2 = \frac{1}{4} \prod_{\substack{p \equiv 1 \pmod{4} \\ p \text{ prime}}} \left( 1 - \frac{2}{p^2} \right)$$

As corollary, from Theorem 2.2, the number  $N(x)$  of simplest quartic fields  $\mathbb{K}_m$  with discriminant  $D(\mathbb{K}_m) \leq x$  is given by

$$N(x) = N(x, 1) + N(x, 2) \sim \left( \frac{1}{4\sqrt[3]{4}} + \frac{1}{4\sqrt[3]{2}} + \frac{1}{2} \right) \prod_{\substack{p \equiv 1 \pmod{4} \\ p \text{ prime}}} \left( 1 - \frac{2}{p^2} \right) x^{\frac{1}{6}}, \text{ as } x \rightarrow \infty$$

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