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Abstract. If Δ and Γ are two derivations of a commutative algebra A such that $\Delta\Gamma - \Gamma\Delta = \Delta$ is locally nilpotent, one can endow A with a new product $*$ whose filtered semiclassical limit is the Poisson structure $\Delta \wedge \Gamma$. In this article we first study theses (Poisson) algebras from an algebraic point of view, and when A is a polynomial algebra, we investigate their homological properties. In particular, if the derivations Δ and Γ are linear, we prove that the algebras $(A, *)$ are Artin-Schelter regular algebras. Assuming furthermore a technical condition on Γ, we show that the algebra $(A, *)$ is Calabi-Yau if and only if the trace of Γ is equal to 1 if and only if the Poisson algebra $(A, \Delta \wedge \Gamma)$ is unimodular.

Contents

Introduction

The philosophy behind noncommutative projective algebraic geometry [Rog14] is to study noncommutative version of the projective spaces $\mathbb{P}^n$ by describing suitable noncommutative algebras that would be thought of as their homogeneous coordinates rings: the so-called Artin-Schlelter regular algebras (AS regular algebras for shorts) [AS87]. Numerous example of AS regular algebras are known but a compete classification is known only up to projective dimension 2. The work of Pym [Pym15] provides a partial answer in dimension 3 by focusing on a classification of graded Calabi-Yau algebras [Gin06] that are flat deformations of $\mathbb{C}[x_0, x_1, x_2, x_3]$. In [LS19], Sierra and the second author introduced a family of AS regular algebras $R(n, a)$ of Gel'fand-Kirillov dimension $n + 1$ indexed by a scalar a. The authors proved that for any integer $n \geqslant 0$ there exists a unique scalar a such that the algebra $R(n, a)$ is a Calabi-Yau. This is of particular interest since it explains a specific choice of parameter made in the exceptional component $E(3)$ of Pym's classification.

Back to the algebra $R(n, a)$, the parameter n can be interpreted as the index of nilpotency of a maximal Jordan block seen as the restriction to degree 1 polynomials of a locally nilpotent derivation Δ

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of a polynomial algebra in $n+1$ variables. The construction of $R(n,a)$ then relies on a second derivation Γ such that $\Delta\Gamma - \Gamma\Delta = \Delta$ is nilpotent together with a deformation formula of Coll, Gerstenhaber, and Giaquinto [CGG89]. The aim of this article is to investigate the following natural generalization: what are the possible algebras arising from this construction by relaxing the conditions on Δ? For instance, the case of homogeneous locally nilpotent derivations Δ of a polynomial rings whose canonical Jordan normal form acting on the set of degree 1 polynomial is not a maximal block will be of particular interest. In this situation we prove that we obtain AS regular algebras and, among them, we describe a family of Calabi-Yau algebras indexed by $r-1$ scalar parameters, where r is the number of blocks in the canonical Jordan normal form of Δ (the algebras studied in [LS19] correspond to the case $r=1$).

This article is divided into eight sections. In Section 1 we present the general construction of a Poisson algebra $A_{(\Delta,\Gamma)}$ and of an associative algebra $R_{(\Delta,\Gamma)}$ associated to a pair of derivations (Δ,Γ) of a commutative algebra A such that $\Delta\Gamma - \Gamma\Delta = \Delta$ is nilpotent. Such a pair of derivation will be called a *solvable pair*. Section 2 is devoted to the study of general algebraic properties of $A_{(\Delta,\Gamma)}$ and $R_{(\Delta,\Gamma)}$. A lots of example of solvable pairs are given. Moreover we compute the (Poisson) center in many cases (Lemma 2.9) and exhibit examples of important Poisson derivations of $A_{(\Delta,\Gamma)}$ and automorphisms of $R_{(\Delta,\Gamma)}$ (see Lemma 2.15 and Theorem 2.17). Furthermore we remark that the natural filtration associated to the locally nilpotent derivation Δ is compatible with both the Poisson structure of $A_{(\Delta,\Gamma)}$ and the associative structure of $R_{(\Delta,\Gamma)}$. This filtration is presented and studied in Section 2.4, it will be one of our main tool to prove a lot of results in this article. In the general situation not much can be said about normal elements. However, among them, we identify and study a significant subset consisting of the so-called strongly normal elements (Definition 2.29) which will be of particular interest latter on in this article. To end Section 2 we observe that up to localization the Poisson algebra $A_{(\Delta,\Gamma)}$ and the algebra $R_{(\Delta,\Gamma)}$ are particularly simple: they are isomorphic to (Poisson) Ore extensions over (a localization of) the kernel of Δ thanks to the local slice construction.

The rest of the article is devoted to the case where the solvable pair is defined on polynomial algebra A. In Section 3 the rank of a non abelian Poisson algebra $A_{(\Delta,\Gamma)}$ is shown to be equal to 2. This implies that a Poisson algebra $A_{(\Delta,\Gamma)}$ cannot be isomorphic non trivially to tensor products of Poisson algebras of the same type. In that sense we obtain new Poisson algebras with our construction. Section 4 deals with the case of homogeneous derivations Δ and Γ. In this situation both the Poisson center of $A_{(\Delta,\Gamma)}$ and the center of $R_{(\Delta,\Gamma)}$ are completely determined (Corollary 4.9). Moreover we construct a finer filtration than the previous one, having the benefit that the associated graded algebra is a polynomial algebra. In Section 5 we show that the algebra $R_{(\Delta,\Gamma)}$ is AS regular and we completely determine the (Poisson) normal elements: they are precisely the strongly normal elements introduced in Section 2. In Sections 6 and 7 we focus on the case where Γ is diagonalizable. In this situation, we are able to determine the Poisson derivation when Γ is generic, as well as we are able to give a presentation by generators and relations of $R_{(\Delta,\Gamma)}$. Our last section is devoted to characterize when our algebras (resp. Poisson algebras) are Calabi-Yau (resp. unimodular), see Corollary 8.12. Finally the article ends with Appendices which recap some of the results frequently used in the article, in particular it explains the structure of the graded ring for the finer of the filtrations we study. In this article we always assume that $\Bbbk$ is a field of characteristic zero.

1. A deformation formula for non commuting derivations

Let A be a commutative $\Bbbk$-algebra and Δ,Γ be two derivations of A such that $[\Delta,\Gamma] = \Delta\Gamma - \Gamma\Delta = \Delta$. It is easily verified that the biderivation $\{-,-\} = \Delta \wedge \Gamma$ satisfies the Jacobi identity, hence is a Poisson bracket on A. More precisely we have

$$\{f,g\} = \Delta(f)\Gamma(g) - \Delta(g)\Gamma(f) \tag{1.1}$$

for any $f, g \in A$. Moreover, if Δ is locally nilpotent, by setting for any $f, g \in A$

$$f * g = \sum_{i \geqslant 0} \Delta^i(f) \binom{\Gamma}{i}(g) = fg + \Delta(f)\Gamma(g) + \sum_{i \geqslant 2} \Delta^i(f) \binom{\Gamma}{i}(g), \tag{1.2}$$

where $\binom{\Gamma}{i} = \frac{1}{i!}\Gamma \circ (\Gamma - \mathrm{id}) \circ \cdots \circ (\Gamma - (i-1)\mathrm{id})$, we define an associative product on A, see [CGG89] for the formal version and [Pym15, Lemma 3.3] for the algebraic version. We set

- $A = A_{(\Delta,\Gamma)} = (A, \{-,-\})$,
- $R = R_{(\Delta,\Gamma)} = (A, *)$.

The noncommutative algebra R is a deformation of the Poisson algebra A in the following sense. Consider A as an algebra over $\Bbbk[t]$ and define

$$f *_t g = \sum_{i \geqslant 0} t^i \Delta^i(f) \binom{\Gamma}{i}(g).$$

Then $A_t = (A, *_t)$ is an associative and noncommutative algebra over $\Bbbk[t]$ such that

- $A_t/(t-1)A_t \cong R$,
- A_t/tA_t is commutative and can be endowed with a Poisson structure given by

$$\{f + tA_t, g + tA_t\} = \frac{f *_t g - g *_t f}{t} + tA_t$$

- A_t/tA_t and A are isomorphic as Poisson algebras.

Example 1.3. For $A = \Bbbk[X_0, \ldots, X_n]$ and the derivations $\Delta(X_i) = X_{i-1}$ ($X_{-1} = 0$) and $\Gamma(X_i) = (a+i)X_i$ ($a \in \Bbbk$), the (Poisson) algebras obtained by the formulae (1.1) and (1.2) have been studied in [LS19]. The Poisson algebra is denoted by $A(n, a)$ and the noncommutative algebra $R(n, a)$. In this article, our aim is to generalize results of [LS19] to other pairs of derivations.

2. Solvable pairs

2.1. **Generalities.** A pair (Δ, Γ) of derivations of A such that $[\Delta, \Gamma] = \Delta$ is locally nilpotent will be called a *solvable pair*. Since $[\Delta, \Gamma] = \Delta$ we observe thanks to equations (1.1) and (1.2) that

$$\Gamma = 0 \implies \Delta = 0 \implies \begin{cases} A_{(\Delta,\Gamma)} \text{ is Poisson commutative,} \\ R_{(\Delta,\Gamma)} \text{ is commutative.} \end{cases}$$

The aim of this section is to study general facts about solvable pairs. The following result study the influence of a change of generators of A on the algebras $A_{(\Delta,\Gamma)}$ and $R_{(\Delta,\Gamma)}$.

Lemma 2.1. *Let (Δ, Γ) be a solvable pair on A and σ and automorphism of the algebra A. Then*

(i) $(\widetilde{\Delta}, \widetilde{\Gamma}) := (\sigma\Delta\sigma^{-1}, \sigma\Gamma\sigma^{-1})$ *is a solvable pair on A,*
(ii) *the Poisson algebras $A_{(\Delta,\Gamma)}$ and $A_{(\widetilde{\Delta},\widetilde{\Gamma})}$ are isomorphic,*
(iii) *the algebras $R_{(\Delta,\Gamma)}$ and $R_{(\widetilde{\Delta},\widetilde{\Gamma})}$ are isomorphic.*

Proof. (i) is a straightforward computation.

(ii) and (iii). In both case we show that σ is an isomorphism between the appropriate algebras. Since $\sigma\Delta = \widetilde{\Delta}\sigma$ and $\sigma\Gamma = \widetilde{\Gamma}\sigma$ we have for all $f, g \in A$

$$\begin{aligned} \sigma(\{f,g\}_{(\Delta,\Gamma)}) &= \sigma\Delta(f)\sigma\Gamma(g) - \sigma\Delta(g)\sigma\Gamma(f) \\ &= \widetilde{\Delta}\sigma(f)\widetilde{\Gamma}\sigma(g) - \widetilde{\Delta}\sigma(g)\widetilde{\Gamma}\sigma(f) = \{\sigma(f), \sigma(g)\}_{(\widetilde{\Delta},\widetilde{\Gamma})}. \end{aligned}$$

By induction we obtain easily that $\sigma\Delta^\ell = \widetilde{\Delta}^\ell\sigma$ and $\sigma\binom{\Gamma}{\ell} = \binom{\widetilde{\Gamma}}{\ell}\sigma$ for all $\ell \in \mathbb{N}$. The equality $\sigma(f *_{(\Delta,\Gamma)} g) = \sigma(f) *_{(\widetilde{\Delta},\widetilde{\Gamma})} \sigma(g)$ follows from the definition of the product $*$ given by (1.2). □

Remark 2.2. When $\Delta = 0$ the Poisson algebra A is Poisson commutative and the algebra R is commutative for every choice of derivation Γ. Therefore there is no hope in general that a Poisson isomorphism between $A_{(\Delta,\Gamma)}$ and $A_{(\Delta',\Gamma')}$ implies that there exists $\sigma \in \mathrm{Aut}(A)$ such that $(\Delta', \Gamma') = (\sigma\Delta\sigma^{-1}, \sigma\Gamma\sigma^{-1})$ (the same remark also applies to $R_{(\Delta,\Gamma)}$). In other words, the Poisson isomorphism class of $A_{(\Delta,\Gamma)}$ or the $\Bbbk$-algebra isomorphism class of $R_{(\Delta,\Gamma)}$ does not determine the isomorphism class of the representation of the 2-dimensional solvable Lie algebra.

For a less trivial example showing that even the conjugacy class of Δ is not determined, see Remark 2.13.

We conclude this section with some examples.

Proposition 2.3. *Let $A = \Bbbk[X_0, X_1]$. Up to isomorphism any solvable pair (Δ, Γ) on A with $\Delta \neq 0$ is of the form*

$$\Delta = P\partial_{X_1}, \quad \Gamma = Q\partial_{X_0} + R\partial_{X_1}$$

where $P, Q \in \Bbbk[X_0]$, $P \neq 0$ and $R = R_0 + R_1X_1$ for some $R_0, R_1 \in \Bbbk[X_0]$ are such that

(1) if $P \in \Bbbk^\times$ or $Q = 0$, then $R_1 = 1$

(2) if $\deg_{X_0}(P) > 0$ and $Q \neq 0$, then $R_1 \neq 1$ and we have $P(R_1 - 1) = Q\partial_{X_0}(P)$.

In particular if $\deg_{X_0}(P) > 0$, $Q \neq 0$ and $R_1 \in \Bbbk \setminus \{1\}$, then there exists $\Lambda \in \Bbbk^\times$, $\alpha \in \Bbbk$ and an integer $n > 0$ such that $Q = \frac{R_1 - 1}{n}(X_0 - \alpha)$ and $P = \Lambda(X_0 - \alpha)^n$.

Proof. Let (δ, γ) be a solvable pair on A. Thanks to Rentschler's theorem, see [Fr06, Theorem 4.1], there exists $\alpha \in \mathrm{Aut}(A)$ such that $\Delta := \alpha\gamma\alpha^{-1} = P\partial_{X_1}$ for some nonzero polynomial $P \in \Bbbk[X_0]$. Set $\Gamma := \alpha\delta\alpha^{-1} = Q\partial_{X_0} + R\partial_{X_1}$ for some $Q, R \in A$. Then $[\Delta, \Gamma] = \Delta$ implies that

$$0 = \Delta(X_0) = \Delta\Gamma(X_0) - \Gamma\Delta(X_0) = \Delta(Q)$$

so that $Q \in \ker\Delta = \Bbbk[X_0]$ since $P \neq 0$. Moreover we have

$$P = \Delta(X_1) = \Delta\Gamma(X_1) - \Gamma\Delta(X_1) = \Delta(R) - \Gamma(P) = P\partial_{X_1}(R) - Q\partial_{X_0}(P)$$

so that $P(\partial_{X_1}(R) - 1) = Q\partial_{X_0}(P)$. Since $P \neq 0$ two cases can happened

(1) $(Q = 0 \text{ or } P \in \Bbbk^\times) \iff \partial_{X_1}(R) = 1 \iff R = X_1 + R_0$ with $R_0 \in \Bbbk[X_0]$

(2) $(Q \neq 0 \text{ and } \deg_{X_0} P > 0) \iff \partial_{X_1}(R) \neq 1$.

In the second case, every factors in $P(\partial_{X_1}(R) - 1) = Q\partial_{X_0}(P)$ is nonzero so that the degree in X_1 of $\partial_{X_1}(R) - 1$ is equal to 0. Hence $R = R_0 + R_1X_1$ with $R_0, R_1 \in \Bbbk[X_0]$ and $R_1 \neq 1$. When $R_1 = \mu \in \Bbbk \setminus \{1\}$ we have $(\mu - 1)P = (-Q)P'$ which classically implies that $\deg_{X_0}(Q) = 1$ and thus $Q = \lambda(X_0 - \alpha)$ for some $\lambda \in \Bbbk^\times$ and some $\alpha \in \Bbbk$ and $P = \Lambda(X_0 - \alpha)^n$ for some $\Lambda \in \Bbbk^\times$ and some integer n and the relation $(\mu - 1)P = (-Q)P'$ gives $\lambda = (\mu - 1)/n$. □

Corollary 2.4. *Every Poisson structure on $A = \Bbbk[X_0, X_1]$ arising from a solvable pair is, up to isomorphism, of the form $\{X_0, X_1\} = S$, where $S \in \Bbbk[X_0]$. The deformed algebra $(R, *)$ is given by two generators X_0, X_1 and one relation $X_0 * X_1 - X_1 * X_0 = S$. In particular, the Poisson field $\mathrm{Frac}(A)$ is either Poisson commutative or a Poisson Weyl field. Similarly, the skewfield $\mathrm{Frac}(R)$ is either commutative or is a Weyl skewfield.*

Proof. We have the result for A with $S = -PQ$ thanks to Proposition 2.3 and by setting, if $S \neq 0$, $X_1' = X_1 * S^{-1}$ so that $\{X_0, X_1'\} = 1$. Note that the powers X_1^{*i} for $i \geqslant 0$ form a basis of $R = (A, *)$ over $\Bbbk[X_0]$ so that R is isomorphic to the Ore extension $\Bbbk[X_0][X_1; S\partial_{X_0}]$ (see Proposition 2.32 for a proof in a more general situation). In particular R is a Noetherian domain, hence admits a skewfield of fraction $\mathrm{Frac}(R)$. If $S \neq 0$, set $X_1' = X_1 * S^{-1}$ so that $X_0 * X_1' - X_1' * X_0 = 1$ and R is isomorphic to a Weyl skewfield. □

Example 2.5. The Poisson structure on $A = \Bbbk[X_0, X_1]$ given by $\{X_0, X_1\} = X_0X_1$ cannot be obtained by a solvable pair. It follows from Corollary 2.4 since $\mathrm{Frac}(A)$ (resp. $\mathrm{Frac}(R)$) cannot contains any Poisson bracket (resp. commutator) equal to 1, see [GL11] (resp. [AD94]).

Example 2.6. Let $A = \Bbbk[X_0, X_1]$ and set $(\Delta, \Gamma) = (\partial_{X_0}, X_0\partial_{X_0} + X_1^i\partial_{X_1})$ for some integer $i \in \mathbb{N}$. One easily verify that (Δ, Γ) is a solvable pair on A and that

$$\{X_0, X_1\} = X_1^i \qquad \text{and} \qquad X_0 * X_1 - X_1 * X_0 = X_1^i = X_1^{*i}$$

where X_1^{*i} denote the i^{th} power of X_1 with respect to the product $*$. We retrieve the following classical (Poisson) algebras:

$i = 0$ the (Poisson) Weyl algebra,
$i = 1$ the (symmetric) enveloping algebra of the two dimensional non abelian Lie algebra,
$i = 2$ the (Poisson) Jordan plane.

Note that the (Poisson) Weyl algebra is classically obtained via the commuting pair of derivations $(\partial_{X_0}, \partial_{X_1})$.

Example 2.7. Let $A = \Bbbk[X, Y_1, \dots, Y_n]$ and set $\Delta = \partial_X$ and $\Gamma = X\partial_X + \lambda_1 Y_1\partial_{Y_1} + \cdots + \lambda_n Y_n\partial_{Y_n}$ for some scalars $\lambda_1, \dots, \lambda_n$. Then (Δ, Γ) is a solvable pair on A and $A_{(\Delta,\Gamma)}$ (resp. $R_{(\Delta,\Gamma)}$) is isomorphic to the symmetric (resp. enveloping) algebra of the Lie algebra $\mathfrak{g}$ with basis $\{x, y_1, \dots, y_n\}$ and non zero Lie bracket

$$[x, y_i] = \lambda_i y_i, \quad i = 1, \dots, n.$$

2.2. **Center.** In this section, we study the elementary and general properties of solvable pairs related to the center of $A_{(\Delta,\Gamma)}$ and $R_{(\Delta,\Gamma)}$.

Definition 2.8. Let A be a Poisson algebra. The Poisson center of A is the set

$$Z_P(A) = \{z \in A \mid \{z, a\} = 0 \text{ for all } a \in A\}$$

of Poisson central elements. It is a Poisson-commutative subalgebra of A.

Lemma 2.9. *Assume that $\Delta \neq 0$ and that $A = A_{(\Delta,\Gamma)}$ is a domain. Set $R = R_{(\Delta,\Gamma)}$.*

(1) Then $\ker\Delta \cap \ker\Gamma \subseteq Z_P(A)$ with equality if $\ker\Delta \neq \ker\Gamma$.
(2) We have $\ker\Delta \cap \ker\Gamma \subseteq Z(R)$. Moreover, $Z(R) \cap \ker\Delta \subseteq \ker\Gamma$.
(3) Assume that $\ker\Delta \neq \ker\Gamma$. Then $Z(R) = \ker\Delta \cap \ker\Gamma$.

Proof. (1) The inclusion $\ker\Delta \cap \ker\Gamma \subseteq Z_P(A)$ is clear. Let $z \in Z_P(A)$. First assume that there exists $x \in \ker\Delta \setminus \ker\Gamma$. Then $0 = \{z, x\} = \Delta(z)\Gamma(x)$, which implies $z \in \ker\Delta$. For $y \notin \ker\Delta$ we have $0 = \{y, z\} = \Delta(y)\Gamma(z)$ and so $z \in \ker\Gamma$. Now assume that there exists $x \in \ker\Gamma \setminus \ker\Delta$. Then $0 = \{x, z\} = \Delta(x)\Gamma(z)$ so that $z \in \ker\Gamma$. Since $\ker\Gamma \neq A$ (otherwise $\Gamma = 0$ which implies $\Delta = 0$) we have for any $y \notin \ker\Gamma$ that $0 = \{z, y\} = \Delta(z)\Gamma(y)$ and so $z \in \ker\Delta$.

(2) The inclusion $\ker\Delta \cap \ker\Gamma \subseteq Z(R)$ is clear. Let us now consider $z \in Z(R) \cap \ker\Delta$. Since $\Delta \neq 0$ and Δ is locally nilpotent, there exists $x \in \ker\Delta^2 \setminus \ker\Delta$. We then have $z * x = zx$ since $z \in \ker\Delta$ and $x * z = zx + \Delta(x)\Gamma(z)$ since $x \in \ker\Delta^2$. But $z \in Z(R)$ hence $\Delta(x)\Gamma(z) = 0$ and $\Delta(x) \neq 0$. Thus $\Gamma(z) = 0$.

(3) We first show that $\ker\Delta \not\subseteq \ker\Gamma$. If $\ker\Delta \subseteq \ker\Gamma$ then $\ker\Delta \subsetneq \ker\Gamma$ since $\ker\Delta \neq \ker\Gamma$. Consider $x \in \ker\Gamma \setminus \ker\Delta$ and $n \geq 0$ such that $\Delta^n(x) \neq 0$ and $\Delta^{n+1}(x) = 0$. In particular $\Delta^n(x) \in \ker\Delta \subseteq \ker\Gamma$. Thus $[\Delta^n, \Gamma](x) = n\Delta^n(x)$ (see Lemma 2.20) implies $\Delta^n(\Gamma(x)) = n\Delta^n(x)$. But $x \in \ker\Gamma$ thus $n\Delta^n(x) = 0$ thus $n = 0$ and $\Delta(x) = 0$ which is absurd. Hence we can consider $x \in \ker\Delta$ such that $\Gamma(x) \neq 0$ and choose $z \in Z(R)$. Then $x * z = xz$ since $x \in \ker\Delta$. Moreover

$$0 = z * x - x * z = \sum_{i \geq 1} \Delta^i(z) \binom{\Gamma}{i}(x) \tag{2.10}$$

But $[\Delta, \Gamma] = \Delta$ implies that $\ker \Delta$ is stable by Γ, hence stable by $\binom{\Gamma}{i}$ for all $i \geq 1$ (see Lemma A.6). Therefore we have $\binom{\Gamma}{i}(x) \in \ker \Delta$ for all $i \geq 1$. Let $N \geq 1$ be the smallest positive integer such that $\Delta^N(z) = 0$. Assume that $N \geq 2$. By applying Δ^{N-2} to equation (2.10) we obtain that $\Gamma(x)\Delta^{N-1}(z) = 0$. Since $\Gamma(x) \neq 0$, we get a contradiction. Hence $N = 1$ i.e. $z \in \ker \Delta$ and assertion (2) provides us with $z \in \ker \Gamma$. □

Example 2.11. Recall the derivations $\Delta = \partial_{X_0}$ and $\Gamma = X_0\partial_{X_0} + X_1^i\partial_{X_1}$ from Example 2.6. One easily verify that $\ker \Delta = \Bbbk[X_1]$ and $\ker \Gamma = \Bbbk$ so that $Z_P(A) = Z(R) = \Bbbk$.

Example 2.12. Recall the derivations $\Delta = \partial_X$ and $\Gamma = X\partial_X + \lambda_1 Y_1\partial_{Y_1} + \cdots + \lambda_n Y_n\partial_{Y_n}$ from Example 2.7. Then $\ker \Delta = \Bbbk[Y_1, \ldots, Y_n] \neq \ker \Gamma$ (as long as not all the λ_i are zeros) so that $Z_P(A) = Z(R) = \ker \Delta \cap \ker \Gamma$ (take $x = Y_i$ for $\lambda_i \neq 0$ in the proposition). Note that to understand

$$\ker \Delta \cap \ker \Gamma = \ker \Gamma|_{\Bbbk[Y_1,\ldots,Y_n]} = \mathrm{Vect}\{Y_1^{\alpha_1} \cdots Y_n^{\alpha_n} \mid \lambda_1\alpha_1 + \cdots + \lambda_n\alpha_n = 0,\ \alpha_i \in \mathbb{N}\}$$

one needs to study the structure of the submonoid $\sum_{i=1}^n \lambda_i\mathbb{N}$ of $(\Bbbk, +)$ (or the $\mathbb{Z}$-module $\sum_{i=1}^n \lambda_i\mathbb{Z}$ if we allow for localization of the Y_i's). For instance if the λ_i are $\mathbb{N}$-linearly independent then $Z_P(A) = Z(R) = \Bbbk$.

The following example illustrate the fact that when $\ker \Delta = \ker \Gamma$ the inclusion $\ker \Delta \cap \ker \Gamma \subseteq Z_P(A)$ can be strict.

Example 2.13. Let $A = \Bbbk[X_0, \ldots, X_n]$ and consider the solvable pair $(\Delta, \Gamma) = (X_0\partial_{X_1}, X_1\partial_{X_1})$. We have $A = Z_P(A)$ but $\ker \Delta = \ker \Gamma = \Bbbk[X_0, X_2, X_3, \ldots, X_n] \subsetneq A$ (see also Corollary 4.9). Similarly we have $Z(R) = R$ so $Z(R) \not\subseteq \ker \Delta = \ker \Gamma$.

2.3. **Derivation and automorphism.**

Definition 2.14. Let A be a Poisson algebra.

(1) A *Poisson derivation* is a derivation δ of A such that for all $a, b \in A$

$$\delta(\{a, b\}) = \{\delta(a), b\} + \{a, \delta(b)\}.$$

(2) The set $\mathrm{P.Der}(A)$ of Poisson derivations of A is a Lie subalgebra of the Lie algebra of derivations of the associative commutative algebra A.

Let $A = A_{(\Delta,\Gamma)}$. When $\Delta \neq 0$, the Lie algebra $\mathrm{P.Der}(A)$ is not reduced to 0 thanks to the following easy lemma.

Lemma 2.15. *The derivation Δ is a Poisson derivation of A.*

Remark 2.16. Poisson center and Poisson derivation are compatible in the following sense. Let A be a Poisson algebra. The Poisson center of A is stable by any Poisson derivation of A: for $\delta \in \mathrm{P.Der}(A)$ and $a \in Z_P(A)$, an easy computation shows that $\delta(a) \in Z_P(A)$.

The following proposition can be seen as a deformation formula for the Poisson derivation Δ. Note that the map Δ is not a derivation of $R_{(\Delta,\Gamma)}$ in general.

Theorem 2.17. *Let (Δ, Γ) be a solvable pair. For $a \in \Bbbk$, the map $\phi_a := (\mathrm{id} + \Delta)^a = \sum_{k\geq 0} \binom{a}{k}\Delta^k$ is a (graded if Δ is graded) $\Bbbk$-algebra automorphism of $R = R_{(\Delta,\Gamma)}$. The linear map*

$$\delta = \sum_{r\geqslant 1} \frac{(-1)^{r-1}}{r}\Delta^r$$

is a (graded if Δ is graded) derivation of R.

Strictly speaking δ is an element of the formal power series ring $\Bbbk[\![\Delta]\!]$ but note that since Δ is locally nilpotent $\delta(f)$ is a well-defined element of R for any $f \in R$.

Proof. We first prove that $\mathrm{id}+\Delta$ is an automorphism of R.

$$\begin{aligned}(\mathrm{id}+\Delta)(f)*(\mathrm{id}+\Delta)(g) &= \textstyle\sum_{i\geq 0}(\Delta^i(f)+\Delta^{i+1}(f))(\binom{\Gamma}{i}(g)+\binom{\Gamma}{i}\Delta(g))\\ &= f*g+\textstyle\sum_{i\geq 0}\Delta^{i+1}(f)\binom{\Gamma}{i}(g)+\Delta^i(f)\binom{\Gamma}{i}\Delta(g)+\Delta^{i+1}(f)\binom{\Gamma}{i}\Delta(g) \qquad (A.6)\\ &= f*g+\textstyle\sum_{i\geq 0}\Delta^{i+1}(f)\binom{\Gamma}{i}(g)+f*\Delta(g)+\sum_{i\geq 0}\Delta^{i+1}(f)\left[\Delta,\binom{\Gamma}{i+1}\right](g)\\ &= f*g+\textstyle\sum_{i\geq 0}\Delta^{i+1}(f)\binom{\Gamma}{i}(g)+f*\Delta(g)+\sum_{i\geq 0}\Delta^{i}(f)\left[\Delta,\binom{\Gamma}{i}\right](g)\\ &= f*g+\textstyle\sum_{i\geq 0}\Delta^{i+1}(f)\binom{\Gamma}{i}(g)+\sum_{i\geq 0}\Delta^{i}(f)\Delta\binom{\Gamma}{i}(g)\\ &= f*g+\Delta(f*g)\end{aligned}$$

Proposition 2 of [Vid82] allows us to conclude that δ is a derivation of R. Hence $a\delta$ is a locally nilpotent derivation of R. So $\exp(a\delta)=(\mathrm{id}+\Delta)^a$ is an automorphism of R.

This last property can be shown by a direct computation using the combinatorial relation

$$\begin{aligned}\textstyle\sum_{i=0}^{u}\binom{a}{v+i}\binom{v+i}{i}\binom{v}{u-i} = \sum_{i=0}^{u}\frac{a(a-1)\cdots(a-v-i+1)}{i!(v-u+i)!(u-i)!} &= \binom{a}{v}\sum_{i=0}^{u}\frac{(a-v)\cdots(a-v-i+1)v!}{i!(v-u+i)!(u-i)!}\\ &= \binom{a}{v}\textstyle\sum_{i=0}^{u}\binom{a-v}{i}\binom{v}{u-i} = \binom{a}{v}\binom{a}{u}\end{aligned}$$

where the last equality is the Chu-Vandermonde identity (see Appendix A). □

Lemma 2.18. *For any $a,b\in\Bbbk$ we have $\phi_a\circ\phi_b=\phi_{a+b}$. Moreover if $\Delta\neq 0$, $\phi_a=\mathrm{id}$ if and only if $a=0$.*

Proof. Let $f\in R$. By using equation (A.4) we have

$$\phi_a\circ\phi_b(f)=\sum_{k,\ell\geq 0}\binom{a}{k}\binom{b}{\ell}\Delta^{k+\ell}(f)=\sum_{u\geq 0}\left(\sum_{k+\ell=u}\binom{a}{k}\binom{b}{\ell}\right)\Delta^u(f)=\sum_{u\geq 0}\binom{a+b}{u}\Delta^u(f)=\phi_{a+b}(f).$$

When $\Delta\neq 0$, there exists $f\in R$ such that $\Delta^2(f)=0$ and $\Delta(f)\neq 0$. Then $\phi_a(f)=f+a\Delta(f)$. □

2.4. **A filtration on $R_{(\Delta,\Gamma)}$.** It is a classical fact that locally nilpotent derivations induce filtrations on commutative algebras. In this section we show that the sequence $(\ker\Delta^{i+1})_{i\in\mathbb{N}}$ provides us with a filtration of both the Poisson algebra $A_{(\Delta,\Gamma)}$ and the noncommutative algebra $R_{(\Delta,\Gamma)}$.

Definition 2.19. Let Δ be a locally nilpotent derivation on A. Set $\varepsilon(0)=-\infty$ and for any nonzero element $f\in A$ define $\varepsilon(f)=\min\{i\in\mathbb{N},\ \Delta^{i+1}(f)=0\}$ and set $A^{\leqslant i}=\{f\in A,\ \varepsilon(f)\leqslant i\}=\ker\Delta^{i+1}$. By convention set $A^{\leqslant -1}=\{0\}$.

Note that $\varepsilon=\deg_\Delta$ in the notation of [Fr06, Section 1.1.8] which is a degree function thanks to [No94, Proposition 6.1.1] since Δ is locally nilpotent.

Lemma 2.20. *For any polynomial $P\in\Bbbk[T]$ we have $[P(\Delta),\Gamma]=Q(\Delta)$ where $Q=XP'$. In particular, for any integer $i\geqslant 0$ we have $\Delta^i\Gamma=\Gamma\Delta^i+i\Delta^i$.*

Proof. By linearity, it is enough to prove the relation for $P=X^i$ where $i\in\mathbb{N}$ which is easily obtained by induction. □

Lemma 2.21. *Let (Δ,Γ) be a solvable pair on A and fix integers $i,j\geqslant 0$.*

(1) We have $\Delta(A^{\leqslant i})\subseteq A^{\leqslant i-1}$ and more precisely $\varepsilon(\Delta(f))=\varepsilon(f)-1$ for every $f\in A$.
(2) We have $\Gamma(A^{\leqslant i})\subseteq A^{\leqslant i}$. Hence $\varepsilon(P(\Gamma)(f))\leqslant\varepsilon(f)$ for every $f\in A$ and $P\in\Bbbk[T]$.
(3) If $f,g\in A$ are such that $\varepsilon(f)=i$ and $\varepsilon(g)=j$, then $\varepsilon(fg)=i+j$.
(4) For any $f,g\in A$ we have

$$f*g-g*f=\Delta(f)\Gamma(g)-\Delta(g)\Gamma(f)+\sum_{i\geqslant 2}\left(\Delta^i(f)\binom{\Gamma}{i}(g)-\Delta^i(g)\binom{\Gamma}{i}(f)\right)$$

In particular, if $\varepsilon(f)=i$ and $\varepsilon(g)=j$ then $\varepsilon(f*g-g*f)\leqslant i+j-1$.

Proof. Assertion (1) is obvious since $\Delta(f) \in \ker \Delta^{i-1}$ if and only if $f \in \ker \Delta^i$. Assertion (2) follows from Lemma 2.20 showing that $\ker \Delta^i$ is stable by Γ and hence by every polynomial in Γ. Assertion (3) is true since from Leibniz formula we get $\Delta^{i+j}(fg) = \binom{i+j}{i}\Delta^i(f)\Delta^j(g) \neq 0$ because A is a domain and $\operatorname{char} \Bbbk = 0$. Assertion (4) follows from (1.2) and then assertions (1) and (2). □

Recall that if $(T^{\leqslant i})_{i\in\mathbb{N}}$ is a filtration of a ring T, its associated graded ring $\operatorname{gr}(T) = \bigoplus T^{\leqslant i}/T^{\leqslant i-1}$ is an $\mathbb{N}$-graded ring whose homogeneous elements of degree i are denoted by $x + T^{\leqslant i-1}$ for an element $x \in T^{\leqslant i} \setminus T^{\leqslant i-1}$.

Proposition 2.22. *Assume that A is a domain.*

1. *The family $(A^{\leqslant i})_{i\geqslant 0}$ is a Poisson algebra filtration of $A_{(\Delta,\Gamma)}$ of degree -1, meaning that it is an algebra filtration of A together with $\{A^{\leqslant i}, A^{\leqslant j}\} \subseteq A^{\leqslant i+j-1}$ for all integers $i, j \geqslant 0$. Moreover, the associated graded algebra $\operatorname{gr}(A)$ is a domain.*
2. *The family $(A^{\leqslant i})_{i\geqslant 0}$ is an algebra filtration of $R_{(\Delta,\Gamma)}$. Moreover, the associated graded algebra $\operatorname{gr}(R)$ is equal to $\operatorname{gr}(A)$.*
3. *The commutative algebra $\operatorname{gr}(R) = \operatorname{gr}(A)$ can be endowed with the following three Poisson brackets*
   (a) $\{\overline{f}, \overline{g}\}' := (f * g - g * f) + A^{\leqslant i+j-2}$
   (b) $\{\overline{f}, \overline{g}\}'' := \{f, g\} + A^{\leqslant i+j-2}$
   (c) $\{\overline{f}, \overline{g}\}''' := \overline{\Delta}(\overline{f})\overline{\Gamma}(\overline{g}) - \overline{\Gamma}(\overline{f})\overline{\Delta}(\overline{g}) \in A^{\leqslant i+j-1}/A^{\leqslant i+j-2}$
   for homogeneous elements $\overline{f}$ and $\overline{g}$ of respective ε-degree i and j. In (c) the pair of maps $(\overline{\Delta}, \overline{\Gamma})$ is the solvable pair of homogeneous derivations of $\operatorname{gr}(A)$ of respective degree -1 and 0 which is induced by the solvable pair of filtered derivations Δ and Γ of A.
4. *The Poisson structures defined in (3) are all equal and make $\operatorname{gr}(A) = \operatorname{gr}(R)$ into a graded Poisson algebra of degree -1.*

Proof. (1). The fact that $(A^{\leqslant i})_{i\geqslant 0}$ is an algebra filtration of A is an easy consequence of Leibniz formula, see [Fr06, Section 1.1.8] for instance. It is a Poisson filtration of degree -1 thanks to equation (1.1) and assertions (1) and (2) of Lemma 2.21. Moreover $\operatorname{gr}(A)$ is a domain thanks to assertion (3) of Lemma 2.21.

(2). The fact that $(A^{\leqslant i})_{i\geqslant 0}$ is an algebra filtration of $R_{(\Delta,\Gamma)}$ follows from equation (1.2) since the degree ε is decreased by Δ and preserved by Γ thanks to Lemma 2.21. More precisely, if $f, g \in A$ are such that $\varepsilon(f) = i$ and $\varepsilon(g) = j$, then $f * g = fg + u$ for some $u \in A^{\leqslant i+j-1}$. This also implies that the associated graded algebras $\operatorname{gr}(A)$ and $\operatorname{gr}(R)$ have the same multiplication, so that they are indeed equal.

(3). The fact that $\{-,-\}'$ is a Poisson bracket on $\operatorname{gr}(R)$ follows the filtered version of the semi-classical limit construction, see [Goo08, Section 2.4]. The fact that $\{-,-\}''$ is a well-defined biderivation satisfying the Jacobi identity on $\operatorname{gr}(A)$ follows by tedious but straightforward computation from the fact that $\{-,-\}$ is a filtered Poisson bracket on A. Finally $\{-,-\}'''$ is a Poisson bracket since $(\overline{\Delta}, \overline{\Gamma})$ is a solvable pair of derivations of $\operatorname{gr}(A)$.

(4). The Poisson brackets $\{-,-\}'$ and $\{-,-\}''$ are the same since both $f * g - g * f$ and $\{f, g\}$ belong to $A^{\leqslant i+j-1}$ combined with $f * g - g * f - \{f, g\} \in A^{\leqslant i+j-2}$ thanks to assertion 4 of Lemma 2.21. Finally, for homogeneous elements $\overline{f}$ and $\overline{g}$ of ε-degree i and j we have

$$\begin{aligned}
\{\overline{f}, \overline{g}\}''' &= \overline{\Delta}(\overline{f})\overline{\Gamma}(\overline{g}) - \overline{\Gamma}(\overline{f})\overline{\Delta}(\overline{g}) \\
&= (\Delta(f) + A^{\leqslant i-2})(\Gamma(g) + A^{\leqslant i-1}) - (\Delta(g) + A^{\leqslant i-2})(\Gamma(f) + A^{\leqslant i-1}) \\
&= (\Delta(f)\Gamma(g) + A^{\leqslant i+j-2}) - (\Delta(g)\Gamma(f) + A^{\leqslant i+j-2}) \\
&= (\Delta(f)\Gamma(g) - \Delta(g)\Gamma(f)) + A^{\leqslant i+j-2} = \{f, g\} + A^{\leqslant i+j-2} = \{\overline{f}, \overline{g}\}''
\end{aligned}$$

Hence $\{-,-\}''$ and $\{-,-\}'''$ are the same. □

In fact it is enough to assume that $\ker \Delta$ is a domain to obtain the previous result. More precisely we have the following corollary.

Corollary 2.23. *The following assertions are equivalent:*

(1) $\ker\Delta$ *is a domain,*
(2) A *is a domain,*
(3) $\mathrm{gr}(A)=\mathrm{gr}(R)$ *is a domain,*
(4) R *is a domain.*

Proof. (2) $\Longrightarrow$ (3) is part of assertion (1) of Proposition 2.22.
(3) $\Longrightarrow$ (4) is classical.
(4) $\Longrightarrow$ (1) since $\ker\Delta$ is a subring of R.
(1) $\Longrightarrow$ (2). With the notation of the proof of assertion (3) of Lemma 2.21, we have $\Delta^i(f)\in\ker\Delta$ and $\Delta^j(g)\in\ker\Delta$. So $\binom{i+j}{i}\Delta^i(f)\Delta^j(g)\neq 0$ implies that $\Delta^{i+j}(fg)\neq 0$, hence $fg\neq 0$. □

Example 2.24. Consider $\Delta=X_0\partial_{X_1}$ and $\Gamma=aX_0\partial_{X_0}+(a+1)X_1\partial_{X_1}$ on $A=\Bbbk[X_0,X_1]$. We have $A^{\leqslant i}=\ker\Delta^{i+1}=\bigoplus_{\ell\leqslant i}\Bbbk[X_0]X_1^\ell$. Then $\mathrm{gr}(A)=\bigoplus_{i\geqslant 0}A^{\leqslant i}/A^{\leqslant i-1}\cong\bigoplus_{i\geqslant 0}\Bbbk[X_0]X_1^i=A$. Observe that $X_0\in A^{\leqslant 0}$ and $X_1\in A^{\leqslant 1}$ and

$$X_0*X_1-X_1*X_0=-aX_0*X_0\in A^{\leqslant 0}$$

so that the induced Poisson bracket on $\mathrm{gr}(A)$ is given by

$$\{X_0+A^{\leqslant -1},X_1+A^{\leqslant 0}\}=-aX_0*X_0+A^{\leqslant -1}.$$

To lighten notation in the following example, we will denote by $\mathrm{gr}(f)=f+A^{\leqslant i-1}$ the image in $\mathrm{gr}(A)$ of any element $f\in A$ with $\epsilon(f)=i\in\mathbb{N}$.

Example 2.25. Consider the derivation $\Delta=X_0\partial_{X_1}+X_1\partial_{X_2}$ of $A=\Bbbk[X_0,X_1,X_2]$. It is classical that $A^{\leqslant 0}=\ker\Delta=\Bbbk[X_0,F_2]$ where $F_2=2X_0X_2-X_1^2$, see for instance [No94, Example 6.7.1.]. Recall that $\varepsilon(fg)=\varepsilon(f)+\varepsilon(g)$ and that $\varepsilon(X_k)=k$ for $k\in\{0,1,2\}$. Therefore, for all $i\geqslant 1$, we have $\varepsilon(X_1{X_2}^{i-1})=2i-1$ and $\varepsilon({X_2}^i)=2i$. Hence the family $\mathcal{B}=(1,X_1{X_2}^{i-1},{X_2}^{2i})_{i\geqslant 1}$ is free over $A^{\leqslant 0}$. Moreover observe that the Hilbert series of $A^{\leqslant 0}\bigoplus_{i\geqslant 1}(A^{\leqslant 0}X_1{X_2}^{i-1}\oplus A^{\leqslant 0}{X_2}^i)$ is the same that the one of $\Bbbk[X_0,X_1,X_2]$. Hence $\mathcal{B}$ is a basis of $\Bbbk[X_0,X_1,X_2]$ over $A^{\leqslant 0}$. which implies that $\mathrm{gr}(A)=A^{\leqslant 0}\oplus\bigoplus_{i\geqslant 1}A^{\leqslant 0}\ \mathrm{gr}(X_1{X_2}^{i-1})\oplus A^{\leqslant 0}\ \mathrm{gr}({X_2}^i)$. Observe that $\mathrm{gr}(X_2)\in\mathrm{gr}(A)$ generates a polynomial ring over $A^{\leqslant 0}$ and that we have the relation $X_1^2=2X_0X_2-F_2$. Therefore $\mathrm{gr}(A)\cong k[X_0,F_2,X_2][T]/(T^2-(2X_0X_2-F_2))\cong k[X_0,X_2,T]\cong A$ since $T^2-(2X_0X_2-F_2)$ is a degree one polynomial in F_2.

2.5. **Normal elements.** The filtration introduced in the previous section allow us to study Poisson automorphisms and Poisson derivations of $\mathrm{gr}(R)=\mathrm{gr}(A)$ induced by (strongly) normal elements of R. This will be useful in Section 8. Recall that if σ and τ are endomorphisms of a ring T a (σ,τ)-derivation of T is an endomorphism δ of T such that $\delta(rs)=\sigma(r)\delta(s)+\delta(r)\tau(s)$ for all $r,s\in T$.

Lemma 2.26. *Let* (Γ,Δ) *be a solvable pair on* A *and consider an automorphism* Φ *of* $R=(A,*)$ *that is compatible with the filtration* ε*, i.e. such that* $\varepsilon(\Phi(f))=\varepsilon(f)$ *for all* $f\in R$*. Then* Φ *induces a Poisson automorphism* $\overline{\Phi}$ *of* $\mathrm{gr}(R)=\mathrm{gr}(A)$ *given by* $\overline{\Phi}(f+A^{\leqslant i})=\Phi(f)+A^{\leqslant i}$ *for all* $f\in R$ *with* $\varepsilon(f)=i$*. Moreover if* $\overline{\Phi}=\mathrm{id}$*, then* $\Phi-\mathrm{id}$ *of* R *induces a Poisson derivation of* $\mathrm{gr}(R)=\mathrm{gr}(A)$ *of degree* -1*.*

Proof. Since $\varepsilon(\Phi(f))=\varepsilon(f)$ for all $f\in R$ it is clear that $\overline{\Phi}$ is a well-defined automorphism of $\mathrm{gr}(R)$. Consider $f,g\in R$ such that $\varepsilon(f)=i$ and $\varepsilon(g)=j$ and denote by $\overline{f},\overline{g}$ their images in $\mathrm{gr}(R)$. We have

$$\overline{\Phi}(\{\overline{f},\overline{g}\})=\overline{\Phi}(f*g-g*f+A^{\leqslant i+j-2})=\Phi(f)*\Phi(g)-\Phi(g)*\Phi(f)+A^{\leqslant i+j-2}=\{\overline{\Phi}(\overline{f}),\overline{\Phi}(\overline{g})\}$$

hence $\overline{\Phi}$ is a Poisson automorphism of $\mathrm{gr}(A)$.

The equality $\overline{\Phi}=\mathrm{id}$ means that if $\varepsilon(f)=i$ then $\varepsilon(\Phi(f)-f)\leqslant i-1$. Hence $\Phi-\mathrm{id}$ induces a well-defined vector space endomorphism $\overline{\Phi-\mathrm{id}}$ of $\mathrm{gr}(R)$ of degree -1 by setting $\overline{(\Phi-\mathrm{id})}(\overline{f})=\Phi(f)-f+A^{\leqslant i-2}$ for

all $\overline{f} \in \mathrm{gr}(R)$ where $f \in R$ is such that $\varepsilon(f) = i$. By definition we have $\overline{(\Phi - \mathrm{id})}(\{\overline{f}, \overline{g}\}) = (\Phi - \mathrm{id})(f * g - g * f) + A^{\leqslant i+j-3}$. Since $\Phi - \mathrm{id}$ is both a (Φ, id)-derivation and a (id, Φ)-derivation of R we obtain

$$(\Phi - \mathrm{id})(f * g - g * f) = \big(\Phi(f) * (\Phi(g) - g) - (\Phi(g) - g) * \Phi(f)\big) + \big((\Phi(f) - f) * g - g * (\Phi(f) - f)\big)\,.$$

Hence $\overline{(\Phi - \mathrm{id})}(\{\overline{f}, \overline{g}\}) = \{\overline{\Phi}(\overline{f}), \overline{(\Phi - \mathrm{id})}(\overline{g})\} + \{\overline{(\Phi - \mathrm{id})}(\overline{f}), \overline{g}\}$ and we get the result using the fact that $\overline{\Phi} = \mathrm{id}$. □

Definition 2.27. Let T be a $\Bbbk$-algebra that is a domain and let $N \in T$ be a normal element. For every $u \in T$ there exists a unique element $\Phi(u) \in T$ such that $uN = N\Phi(u)$. Then Φ is a $\Bbbk$-algebra automorphism of T and is called the automorphism associated to N and denoted by Φ_N.

Lemma 2.28. *Let (Γ, Δ) be a solvable pair on a domain A and consider $R = (A, *)$. Let N be a normal element in R and consider Φ_N its associated automorphism. We denote by $\overline{N}$ the image of N in $\mathrm{gr}(R)$. Then Φ_N is compatible with ε. Moreover $\overline{\Phi_N} = \mathrm{id}$ and $\overline{(\Phi_N - \mathrm{id})}$ verifies that $\overline{N}\ \overline{(\Phi_N - \mathrm{id})} = \{\,\cdot\,, \overline{N}\}$. In particular, $\overline{N}$ is Poisson normal in $\mathrm{gr}(R)$.*

Proof. Let $f \in R$ and set $i = \varepsilon(f)$ and $j = \varepsilon(N)$. Since $f * N = N * \Phi_N(f)$ and A is a domain, we have $i + j = \varepsilon(\Phi_N(f)) + j$. Therefore Φ_N is compatible with ε.

The definition of the product $*$ implies that $f*N = fN+u$ where $u \in A^{\leqslant i+j-1}$. Similarly $N*\Phi_N(f) = N\Phi_N(f) + v$ where $v \in A^{\leqslant i+j-1}$. Hence $N(\Phi_N(f) - f) \in A^{\leqslant i+j-1}$. Since A is a domain, we obtain that $\Phi_N(f) - f \in A^{i-1}$ i.e. $\overline{\Phi_N} = \mathrm{id}$. Moreover, we have the following equalities

$$0 = N * \Phi_N(f) - f * N = N(\Phi_N(f) - f) + \Delta(N)\Gamma(\Phi_N(f)) - \Delta(f)\Gamma(N) + u$$

for some $u \in A^{\leqslant i+j-2}$ Hence $N(\Phi_N(f) - f) = \Delta(f)\Gamma(N) - \Delta(N)\Gamma(\Phi_N(f)) - u$ with $\Delta(f)\Gamma(\Phi_N(f)) - \Delta(N)\Gamma(f) \in A^{\leqslant i+j-1}$ since Φ_N is compatible with ε. This implies the desired equality since $\overline{\Phi_N} = \mathrm{id}$. □

Definition 2.29 (Strongly normal element)**.** Let (Δ, Γ) be a solvable pair on A. An element $f \in A$ is said to be strongly normal when $\Delta(f) = 0$ and there exists $\alpha \in \Bbbk$ such that $\Gamma(f) = \alpha f$.

Lemma 2.30. *Let (Δ, Γ) be a solvable pair on A. If f is strongly normal in A then f is normal in R and Poisson normal in A. Moreover if A is a domain, the automorphism of R associated to N is ϕ_α and $\overline{(\phi_\alpha - \mathrm{id})} = \alpha\overline{\Delta}$, where α is the scalar given by $\Gamma(f) = \alpha f$.*

Proof. Since $f \in \ker\Delta$, we have $f * g = fg$ for all $g \in R$, hence $f * R = fA$. Also we have

$$g * f = \sum_{k\geq 0} \Delta^k(g)\binom{\Gamma}{k}(f) = \sum_{k\geq 0} \Delta^k(g)\binom{\alpha}{k} f = f\sum_{k\geq 0}\binom{\alpha}{k}\Delta^k(g) = f\phi_\alpha(g)\,.$$

Recall from Theorem 2.17 that ϕ_α is an automorphism of R. Hence the equality $g * f = f\Phi_\alpha(g)$ implies that $R*f = Af$. Therefore we have $f*R = fA = R*f$ and f is normal in R. Moreover, for every $g \in A$, we have $\{f, g\} = \Delta(f)\Gamma(g) - \Gamma(f)\Delta(g) = -\alpha f\Delta(g) \in fA$ and we conclude that f is Poisson normal.

Assume that A is a domain. Since $f \in \ker\Delta$ and $\Gamma(f) = \alpha f$, we have $\overline{\Delta}(f) = 0$ and $\overline{\Gamma}(\overline{f}) = \alpha\overline{f}$. Hence $\{\overline{g}, \overline{f}\} = \alpha\overline{\Delta}(\overline{g})\overline{f}$. Thanks to Lemma 2.28 we have $\{\overline{g}, \overline{f}\} = \overline{(\Phi_\alpha - \mathrm{id})}(\overline{g})\overline{f}$. Since $\mathrm{gr}(A)$ is a domain we obtain the relation $\overline{(\Phi_\alpha - \mathrm{id})} = \alpha\overline{\Delta}$. □

The converse of the preceding lemma will be proved in section 5 : in the case of a linear solvable pair, every normal element in R or Poisson normal element in A is strongly normal (Proposition 5.7).

2.6. **A canonical form up to localization.** It is well know that locally nilpotent derivations with slices feature very interesting properties. For instance, if A is a $\Bbbk$-algebra that is a domain and Δ is a nilpotent derivation of A admitting an element $s \in A$ such that $\Delta(s) = 1$ (s is called a slice for Δ), then A is a polynomial ring in the variable s over the base ring $A^\Delta = \ker\Delta$, see [DF94] and [VdE95]. In this section we exploit this fact to show that, up to localization, the algebras $A_{(\Delta,\Gamma)}$ and $R_{(\Delta,\Gamma)}$ are (Poisson-) Ore extensions of the (localized) kernel of Δ by the (Poisson) derivation Γ.

Let (Δ, Γ) be a solvable pair on A with $\Delta \neq 0$. Since $\Delta \neq 0$ there exists $r \in A$ such that $\Delta(r) \neq 0$ and $\Delta^2(r) = 0$ (any element of $A^{\leqslant 1} \setminus A^{\leqslant 0}$). We consider the localization $A^\circ = S_r^{-1}A$ where $S_r = \{\Delta(r)^i \mid i \geqslant 0\}$. The pair of derivations (Δ, Γ) extends uniquely by localization into a solvable pair of derivations on A° again denoted by (Δ, Γ). Note that Δ is locally nilpotent on A° since $\Delta(r) \in A^\Delta$. Observe that the element $s := r\Delta(r)^{-1} \in A^\circ$ is a slice for Δ since:

$$\Delta(s) = \Delta(r)\Delta(r)^{-1} - r\Delta(r)^{-2}\Delta^2(r) = 1.$$

2.6.1. *The Poisson case.* We consider the Poisson algebra $(A^\circ)_{(\Delta,\Gamma)}$ arising from the solvable pair (Δ, Γ) on A°. Note that the kernel $(A^\circ)^\Delta$ is a Poisson commutative subalgebra of A° that is stable by Γ since for any $a \in (A^\circ)^\Delta$ we have $\Delta\Gamma(a) = \Gamma\Delta(a) + \Delta(a) = 0$. Hence Γ induces a Poisson derivation of $(A^\circ)^\Delta$. This allows us to define the so-called Poisson-Ore extension $(A^\circ)^\Delta[X;\Gamma]_P$ which is equal to the polynomial ring $(A^\circ)^\Delta[X]$ as a commutative algebra and whose Poisson bracket is defined to be the unique extension of the Poisson bracket of $(A^\circ)^\Delta$ such that $\{X, a\} = \Gamma(a)$ for all $a \in (A^\circ)^\Delta$, see [Oh06].

Proposition 2.31. *The map $\phi : (A^\circ)^\Delta[X;\Gamma]_P \to (A^\circ)_{(\Delta,\Gamma)}$ given by*

$$\phi\left(\sum_{i=0}^{n} a_i X^i\right) = \sum_{i=0}^{n} a_i s^i$$

where $a_i \in (A^\circ)^\Delta$ for all i, is a Poisson algebra isomorphism.

Proof. The inclusion $(A^\circ)^\Delta \to A^\circ$ is a Poisson algebras homomorphism. Moreover for $a \in (A^\circ)^\Delta$, we have $\{s, a\} = \Delta(s)\Gamma(a) - \Gamma(s)\Delta(a) = \Gamma(a)$ since $\Delta(s) = 1$. Thanks to the universal property of the Poisson-Ore extension $((A^\circ)^\Delta)[X;\Gamma]_P$ (see [Lec15, Proposition 1.1.15]), there exists a unique Poisson algebra homomorphism from $((A^\circ)^\Delta)[X;\Gamma]_P$ to $(A^\circ)_{(\Delta,\Gamma)}$ sending $a \in (A^\circ)^\Delta$ to a and X to s. It is given by ϕ. Since $A^\circ = (A^\circ)^\Delta[s]$ is a polynomial ring in the slice s over $(A^\circ)^\Delta$ ([VdE95, Corollary 1.2]), it is clear that ϕ is an isomorphism. □

2.6.2. *The noncommutative case.* We consider the algebra $R^\circ = (R^\circ)_{(\Delta,\Gamma)} = (A^\circ, *)$ arising from the solvable pair (Δ, Γ) on A°. Recall that

$$a * b = ab + \sum_{i \geqslant 1} \Delta^i(a)\binom{\Gamma}{i}(b)$$

for all $a, b \in A^\circ$. In particular $(A^\circ)^\Delta$ is a commutative subalgebra of R°. Since Γ leaves $(A^\circ)^\Delta$ invariant, we can consider the Ore extension $(A^\circ)^\Delta[X;\Gamma]$ which is, by definition [GW04, Chapter 2], a free left $(A^\circ)^\Delta$-module with basis $\{X^i \mid i \geqslant 0\}$ and whose multiplication extends the multiplication of $(A^\circ)^\Delta$ by the rule $Xa = aX + \Gamma(a)$ for all $a \in (A^\circ)^\Delta$. For all integer $i \geqslant 0$ let us denote by s^{*i} the i^{th} power of s with respect to the product $*$.

Proposition 2.32. *The map $\phi : (A^\circ)^\Delta[X;\Gamma] \to R^\circ$ given by*

$$\phi\left(\sum_{i=0}^{n} a_i X^i\right) = \sum_{i=0}^{n} a_i s^{*i}$$

where $a_i \in (A^\circ)^\Delta$ for all i, is an algebra isomorphism.

Proof. The inclusion map $(A^\circ)^\Delta \subseteq R^\circ$ is an algebra homomorphism. Moreover for $a \in (A^\circ)^\Delta$, we have $a * s = as$ and $s * a = sa + \Gamma(a)$ since $\Delta(s) = 1$ and $\Delta^2(s) = 0$. Hence from the universal property (see [GW04, Exercise 2.F]) of $(A^\circ)^\Delta[X;\Gamma]$ there exists a unique algebra homomorphism from $(A^\circ)^\Delta[X;\Gamma]$ to R° sending $a \in (A^\circ)^\Delta$ to $a \in R^\circ$ and X to s. It is given by ϕ. So it remains to show that ϕ is bijective. For, we will show by induction on $i \geqslant 0$ that there exist elements $a_{j,i} \in (A^\circ)^\Delta$ with $0 \leqslant j < i$ such that

$s^{*i} = s^i + \sum_{j=0}^{i-1} a_{j,i} s^j$. Cases $i = 0$ and $i = 1$ are trivial. Recall that $\Delta(s) = 1$ and $\Delta^2(s) = 0$ in order to compute $s^{*(i+1)}$:

$$\begin{aligned} s^{*(i+1)} = s * s^{*i} =& s * s^i + s * \sum_{j=0}^{i-1} a_{j,i} s^j = s^{i+1} + \Gamma(s^i) + \sum_{j=0}^{i-1} a_{j,i} s^{j+1} + \sum_{j=0}^{i-1} \Gamma(a_{j,i} s^j) \\ =& s^{i+1} + i(\Gamma(s) - s)s^{i-1} + is^i + \sum_{j=1}^{i} a_{j-1,i} s^j + \sum_{j=0}^{i-1} \Gamma(a_{j,i}) s^j + \sum_{j=0}^{i-1} a_{j,i} \Gamma(s^j) \\ =& s^{i+1} + i(\Gamma(s) - s)s^{i-1} + \sum_{j=0}^{i} b_{j,i} s^j + \sum_{j=0}^{i-1} j a_{j,i} (\Gamma(s) - s)s^{j-1} + \sum_{j=0}^{i-1} j a_{j,i} s^j \end{aligned}$$

where the $b_{j,i}$ for $0 \leqslant j \leqslant i$ are elements of $(A^\circ)^\Delta$ expressed in terms of i, $a_{j-1,i}$ and $\Gamma(a_{j,i})$. Moreover we have $\Delta(\Gamma(s) - s) = \Gamma\Delta(s) = \Gamma(1) = 0$. Hence $\Gamma(s) - s \in (A^\circ)^\Delta$ and $s^{*(i+1)}$ has the desired form. We then deduce that $(s^{*i})_{i \geqslant 0}$ is a basis of R° over $(A^\circ)^\Delta$ since $(s^i)_{i \geqslant 0}$ is ([VdE95, Corollary 1.2]) and we conclude that ϕ is bijective. □

2.7. **Skewfield of fractions.** When A is a polynomial ring and the action of Γ is diagonal, the results of the previous section allow us to relate R° to a particular Lie algebra and to described the skewfield of fractions of $R_{(\Delta,\Gamma)}$. Recall that, given a field F, the Weyl skewfield $D_1(F)$ is the skewfield of fractions of the algebra generated over F by two elements u and v such that $uv - vu = 1$. Linear diagonalizable solvable pairs will be defined in Section 6.

Proposition 2.33. *Assume that A is a polynomial ring and that (Δ, Γ) is a nonzero linear diagonalizable solvable pair on A. Then $R_{(\Delta,\Gamma)}$ is, up to localization, isomorphic to the enveloping algebra of a completely solvable Lie algebra $\mathfrak{g}$. In particular, if $\mathfrak{g}$ is algebraic and non abelian, the skewfield of fractions of $R_{(\Delta,\Gamma)}$ is isomorphic to the Weyl skewfield $D_1(F)$ for a purely transcendental extension F of $\Bbbk$.*

Proof. Since $\Delta(X_1) = X_0$ and $\Delta(X_0) = 0$ we can apply the slice construction with the element $s = X_1 X_0^{-1}$ of $A^\circ = A[X_0^{-1}]$. In particular, the map $\pi : A^\circ \to A^\circ$ given by $\pi(a) = \sum_{p \geqslant 0} \frac{(-s)^p}{p!} \Delta^p(a)$ is an algebra homomorphism such that $(A^\circ)^\Delta = \pi(A^\circ)$, see [VdE95]. Therefore the algebra $(A^\circ)^\Delta$ is generated by elements $Y_0^{\pm 1}, Y_2, \ldots, Y_n$ where Y_i denote the image of X_i by π (note that $\pi(X_0) = X_0$ and $\pi(X_1) = 0$). Observe that $Y_0, Y_2, \ldots, Y_n$ is a set of algebraically independent elements of A° since, for $i \geqslant 2$, we can write $Y_i = \pi(X_i) = X_i + F_i$ where $F_i \in \Bbbk[X_0^{\pm 1}, X_1, \ldots, X_{i-1}]$. Moreover each Y_i is an eigenvector for Γ because one can verify that Γ and π commute thanks to the equality $\Delta\Gamma - \Gamma\Delta = \Delta$.

Denote by $\mathfrak{g}$ the completely solvable Lie algebra with basis $\{y_0, y_2, \ldots, y_n, x\}$ and nonzero brackets $[x, y_i] = \lambda_i y_i$ for all i, where $\lambda_i \in \Bbbk$ is the eigenvalue of Y_i (equivalently X_i) for Γ. The element y_0 is normal in $U(\mathfrak{g})$ and is its clear that $(A^\circ)^\Delta[X; \Gamma]$ is isomorphic to the localization $U(\mathfrak{g})[y_0^{-1}]$. The first part of the proof is now complete since $R^\circ \cong (A^\circ)^\Delta[X; \Gamma]$ thanks to Proposition 2.32.

The second assertion follows firstly from [Mc74]: skewfields of fractions of enveloping algebras of algebraic completely solvable Lie algebras are isomorphic to Weyl skewfields. Moreover, the transcendence degree of the center of the enveloping skewfield of $\mathfrak{g}$ is equal to the index of $\mathfrak{g}$ whose value can be easily computed to be $n - 1$ thanks to [Oo06]. The result follows. □

Remark 2.34. With similar arguments (using [TY10]) the following Poisson version can be obtained: under the assumption of Proposition 2.33, the Poisson field of $A_{(\Delta,\Gamma)}$ is isomorphic to a Poisson Weyl field over a purely transcendental extension F of $\Bbbk$, that is, a rational functions field $F(U, V)$ with $\{U, V\} = 1$.

3. Solvable pairs over polynomial rings

From this section included and until the end of this article we focus on the case of solvable pairs on polynomial algebras.

3.1. **The rank of $A_{(\Delta,\Gamma)}$.** We recall the notion of rank of an affine Poisson variety following [Van01]. Let $V = \Bbbk^{n+1}$ be a Poisson variety, that is, the variety V whose coordinate ring $A = \mathcal{O}(\Bbbk^{n+1}) = \Bbbk[X_0, \dots, X_n]$ is endow with a Poisson structure. The Poisson matrix of A is the skew-symmetric matrix $\Pi(A) = \big(\{X_i, X_j\}\big)_{0 \leqslant i,j \leqslant n}$. For any $p \in V$ the rank $\mathrm{Rk}_p(A)$ of the Poisson bracket at p is the rank of the Poisson matrix $\Pi(A)$ evaluated at p. Finally, the rank of A, denoted by $\mathrm{Rk}(A)$, is the maximum of the $\mathrm{Rk}_p(A)$ for $p \in V$. Note that A is Poisson commutative if and only of $\mathrm{Rk}(A) = 0$.

If (Δ, Γ) is a solvable pair on A then we have

$$\Pi(A_{(\Delta,\Gamma)}) = \begin{pmatrix} \Delta(X_0) \\ \Delta(X_1) \\ \vdots \\ \Delta(X_n) \end{pmatrix} \begin{pmatrix} \Gamma(X_0) \\ \Gamma(X_1) \\ \vdots \\ \Gamma(X_n) \end{pmatrix}^T - \begin{pmatrix} \Gamma(X_0) \\ \Gamma(X_1) \\ \vdots \\ \Gamma(X_n) \end{pmatrix} \begin{pmatrix} \Delta(X_0) \\ \Delta(X_1) \\ \vdots \\ \Delta(X_n) \end{pmatrix}^T .$$

In particular the rank of $A_{(\Delta,\Gamma)}$ is at most 2. Thanks to [Van01, Proposition 2.17] the rank of A is invariant under isomorphism. This shows that not every Poisson bracket on A can be obtained from a solvable pair since Poisson structures with higher ranks do exist.

3.2. **Application: non isomorphism theorem.** The notion of rank recalled in the previous section allow us to prove that a polynomial Poisson algebra arising from a solvable pair cannot be obtained non trivially by tensor products of algebras of the same type. See Appendix B for the definition of the Poisson structure on the tensor product of Poisson algebras.

Proposition 3.1. *Let $A = A_{(\Delta,\Gamma)}$ be non Poisson commutative. Then A is not isomorphic to a non trivial tensor product of the form $\bigotimes_{i=1}^{s} A_{(\Delta_i,\Gamma_i)}$ in the sense that if A is isomorphic to such a tensor product, then only one factor has nonzero rank, thus making A into a Poisson central extension of a rank two Poisson algebra of the form $A_{(\tilde{\Delta},\tilde{\Gamma})}$.*

The above proposition is a corollary of the following lemma.

Lemma 3.2. *Let $s \geqslant 1$ be an integer and for $1 \leqslant i \leqslant s$ let $A^{n_i}_{(\Delta_i,\Gamma_i)}$ be a polynomial Poisson algebra in n_i indeterminates for some solvable pairs (Δ_i, Γ_i) on A^{n_i}. Set $B = \bigotimes_{i=1}^{s} A^{n_i}_{(\Delta_i,\Gamma_i)}$ and assume that B has rank 2. Set $n = \sum_{i=1}^{s} n_i$. Then there exists a solvable pair (Δ', Γ') on a B such that the Poisson structure of B is obtained by the solvable pair (Δ', Γ').*

Proof. Thanks to [Van01, Proposition 2.21] we have $\mathrm{Rk}(B) = \sum_{i=1}^{s} \mathrm{Rk}(A^{n_i}_{(\Delta_i,\Gamma_i)})$. In particular the rank of all $A^{n_i}_{(\Delta_i,\Gamma_i)}$ but one, say $A^{n_1}_{(\Delta_1,\Gamma_1)}$, should be zero, i.e. $A^{n_i}_{(\Delta_i,\Gamma_i)}$ is Poisson commutative for all $i > 1$. Hence $B = A^{n_1}_{(\Delta_1,\Gamma_1)} \otimes C$ where $C = \bigotimes_{i=2}^{s} A^{n_i}_{(\Delta_i,\Gamma_i)}$ is Poisson commutative. One easily verify the conclusion of the lemma holds by setting $\Delta' = \Delta_1 \otimes \mathrm{id}_C$ and $\Gamma' = \Gamma_1 \otimes \mathrm{id}_C$. □

4. Linear solvable pairs

The polynomial algebra $A = \Bbbk[X_0, \dots, X_n]$ is a connected graded algebra once endowed with the grading d in which $d(X_k) = 1$ for every $0 \leqslant k \leqslant n$. We have $A = \bigoplus_{i \geqslant 0} A_i$ where, for all $i \geqslant 0$, we denote by A_i the vector space of homogeneous polynomials of degree i. A Poisson structure on A is called *homogeneous* if $\{A_i, A_j\} \subseteq A_{i+j}$ and in that case A is said to be an *homogeneous polynomial Poisson algebra*. We denote by $\mathrm{P.Der}_{\mathrm{gr}}(A)$ the Lie algebra of linear Poisson derivations of A, that is, Poisson derivations stabilizing A_1. Note that linear derivations are in one to one correspondence with endomorphisms of the $\Bbbk$-vector space A_1 or equivalently with square matrices of order $n+1$ (acting on A_1 by left multiplication). Moreover, locally nilpotent derivations of A correspond to nilpotent endomorphisms/matrices. We now provide a couple of invariants for homogeneous polynomial Poisson algebras.

Proposition 4.1. *Let A and B be two isomorphic homogeneous polynomial Poisson algebras.*

(1) *There exists a Lie algebra isomorphism between* $\mathrm{P.Der_{gr}}(A)$ *and* $\mathrm{P.Der_{gr}}(B)$ *that sends locally nilpotent derivations to locally nilpotent derivations.*
(2) *There exists a vector space isomorphism between* $A_1 \cap Z_P(A)$ *and* $B_1 \cap Z_P(B)$.

Proof. Thanks to [LPV12, Proposition 8.8] a Poisson isomorphism between homogeneous polynomial Poisson structures can always be realized by a linear isomorphism. We denote by $\varphi : A \to B$ such an isomorphism. For (1), the map $\Psi : \mathrm{P.Der_{gr}}(A) \to \mathrm{P.Der_{gr}}(B)$ given by $\Psi = \varphi\delta\varphi^{-1}$ is the desired isomorphism and for (2), the restriction of φ to $A_1 \cap Z_P(A)$ provides us with the desired isomorphism since $\varphi(A_1) = B_1$ and $\varphi(Z_P(A)) = Z_P(B)$. □

Definition 4.2. A solvable pair (Δ, Γ) is said to be linear if Δ and Γ are linear derivation of A.

Note that a linear solvable pair is completely determined by its action on A_1. Hence in the rest of the article, when considering a linear solvable pair we will focus on its action on A_1. The article [LS19] consider the case where Δ is the maximal Jordan block. In the following we will investigate algebras $A_{(\Delta,\Gamma)}$ and $R_{(\Delta,\Gamma)}$ arising from linear derivations Δ with more than one Jordan block.

Remark 4.3. In a solvable pair, the derivation Δ is locally nilpotent. It implies, when the solvable pair is linear, that Δ is a nilpotent endomorphism of A_1. We remark that when Δ and Γ are linear derivations of A verifying $[\Delta, \Gamma] = \Delta$ then Δ is automatically locally nilpotent. The proof goes as follows. We have $[P(\Delta), \Gamma] = (TP')[\Delta]$ for every polynomial $P \in \Bbbk[T]$ by assertion (2) of Lemma 2.21. If P is the minimal polynomial of Δ acting on the finite dimensional space A_1, we obtain that $TP'(T)$ is collinear to P. Hence $P = T^\ell$ for some ℓ and Δ acting on A_1 is nilpotent. Therefore Δ is locally nilpotent on A and (Δ, Γ) a linear solvable pair.

The following example illustrate the facts that not all homogeneous Poisson algebras of rank 2 can be obtain by a linear solvable pair.

Example 4.4. Let $A = \Bbbk[X, Y]$ with $\{X, Y\} = XY$. As seen in Example 2.5, this Poisson bracket cannot be realized by any solvable pair, thus any linear solvable pair. Note that the linear case can be treated by a direct computation (see also 4.6) since XY is not collinear to the square of an element of A_1.

So let (Δ, Γ) be a linear solvable pair on A. Thanks to Lemma 2.1 one can always assume that Δ is in canonical Jordan form, up to a linear automorphism of A. Recall that in canonical Jordan form, a nilpotent matrix is made of diagonal blocks of the form $J_{n_i}(0)$, i.e. blocks of size n_i with 0s everywhere except on the upper diagonal, where the entries are 1. It can be assumed that the size of the diagonal blocks $J_{n_i}(0)$ is decreasing. So, given such a Δ, what are the possible choices for Γ ?

Lemma 4.5. *The set of endomorphism* $\Gamma \in \mathrm{End}_\Bbbk(A_1)$ *such that* $[\Delta, \Gamma] = \Delta$ *is given by*

$$\Gamma_0 + \mathcal{C}(\Delta)$$

where $\mathcal{C}(\Delta) = \{f \in \mathrm{End}_\Bbbk(A_1)\ ;\ [f, \Delta] = 0\}$ *is the commutant of* Δ *in* $\mathrm{End}_\Bbbk(A_1)$ *and where* $\Gamma_0 \in \mathrm{End}_\Bbbk(A_1)$ *is such that* $[\Delta, \Gamma_0] = \Delta$.

Proof. Follows from the fact that for two solvable pairs (Δ, Γ) and $(\Delta, \tilde{\Gamma})$ we have $[\Delta, \Gamma - \tilde{\Gamma}] = 0$. □

4.1. **The case of Δ maximal.** When Δ is a maximal Jordan block one can choose $\Gamma_0 = \mathrm{diag}(0, 1, \dots, n)$. Recall the algebras $A(n, a)$ and $R(n, a)$ from Example 1.3.

Theorem 4.6. *Given a solvable pair* (Δ, Γ) *with* Δ *a maximal Jordan block, there exists* $u \in \mathrm{GL}(A_1) \cap \mathcal{C}(\Delta)$ *such that* $u\Gamma u^{-1} = a\mathrm{Id}_{A_1} + \Gamma_0$, *where* $a \in \Bbbk$ *is such that* $\mathrm{tr}(\Gamma) = (n+1)(a + n/2)$. *In particular* $A_{(\Delta,\Gamma)} \cong A(n, a)$ *and* $R_{(\Delta,\Gamma)} \cong R(n, a)$.

The proof relies on the following classical facts. Recall that $A_1 = \Bbbk X_0 \oplus \cdots \oplus \Bbbk X_n$ is a $\Bbbk$-vector space of dimension $n+1$.

(1) We have $\mathcal{C}(\Delta) = \Bbbk[\Delta] = \mathrm{Span}(\mathrm{id}, \Delta, \Delta^2, \ldots, \Delta^n)$ as a subalgebra of $\mathrm{End}_{\Bbbk}(A_1)$.

(2) Let $u \in \mathcal{C}(\Delta)$ and $P \in \Bbbk[X]$ such that $u = P(\Delta)$. Then $u \in \mathrm{GL}(A_1)$ if and only if $P(0) \neq 0$.

Proof of Theorem 4.6. The relation $[\Delta^i, \Gamma] = i\Delta^i$ shows that $\ker \Delta^i$ is stable by Γ. So let us consider a basis $(e_0, \ldots, e_n)$ of A_1 in which Δ has canonical Jordan form. In such a basis Γ is upper triangular. Let us denote by $(\lambda_0, \ldots, \lambda_n)$ the diagonal part of Γ. We have

$$\Gamma(e_i) = \Gamma(\Delta^{n-i}(e_n)) = \Delta^{n-i}\Gamma(e_n) - (n-i)\Delta^{n-i}(e_n)\,.$$

Thus by computing modulo $\mathrm{vect}(e_0, \ldots, e_{i-1})$, we get that $\lambda_i = \lambda_n - (n-i)$ for all $0 \leqslant i < n$. In particular Γ has $n+1$ distinct eigenvalues and thus a basis of eigenvectors. We then deduce that there exists an eigenvector e'_n of Γ which is not in $\ker \Delta^n$. Using the relation $[\Delta^i, \Gamma] = i\Delta^i$, we obtain that $(\Delta^n(e'_n), \Delta^{n-1}(e'_n), \ldots, e'_n)$ is a basis of A_1 in which Δ has canonical Jordan form and Γ is diagonal $(a, a+1, \ldots, a+n)$ where $a = \lambda_0 = \lambda_n - n$. Finally we get

$$\mathrm{tr}(\Gamma) = a(n+1) + \frac{n(n+1)}{2}$$

showing that a is uniquely determined by Γ. □

Example 4.7. Let $n = 1$. There exists only two isomorphism classes for the algebras $A = A_{(\Delta,\Gamma)}$ and $R = R_{(\Delta,\Gamma)}$ with non necessarily maximal linear solvable pairs. If $\Delta = 0$ then $A = R = \Bbbk[X_0, X_1]$ is a (Poisson) commutative algebra. If $\Delta \neq 0$ then it is necessarily a maximal Jordan block since $n = 1$. Therefore, thanks to Theorem 4.6, we have $A \cong A(1, a)$ and $R \cong R(1, a)$ with $a \in \Bbbk$. If $a = 0$ we retrieve the commutative case (this is the example of Remark 2.13) and when $a \neq 0$, A is isomorphic to the Poisson algebra $\Bbbk[X_0, X_1]$ with Poisson bracket $\{X_0, X_1\} = X_0^2$ and R is isomorphic to the Jordan plane, an algebra given by two generators X_0, X_1 and the relation $X_0X_1 - X_1X_0 = X_0^2$.

4.2. **Center in the linear case.** The following result is a generalization of Lemma 2.9 in the linear case. We describe the Poisson center of A and the center of R which allow us to determine exactly when A is Poisson commutative and R is commutative. The results of this section will be improved by Proposition 5.7 but are stated here since they will be needed in the proof of Proposition 5.7. In the next proposition the rank of Δ seen as an endomorphism of A_1 is denoted by $\mathrm{rank}(\Delta)$.

Proposition 4.8. *Let (Δ, Γ) be a linear solvable pair on A. Assume that $\mathrm{rank}(\Delta) \geq 2$. Then we have $A_1 \cap Z_P(A) = \ker \Delta \cap \ker \Gamma \cap A_1$ and $A_1 \cap Z(R) = \ker \Delta \cap \ker \Gamma \cap A_1$.*

Proof. Since in this proof we only consider restrictions of Δ and Γ to A_1, we simply denote by $\ker \Delta$ the set $\ker \Delta \cap A_1$ and similarly for Γ.

Since we have $\ker \Gamma \cap \ker \Delta \subseteq Z_P(A)$ and $\ker \Gamma \cap \ker \Delta \subseteq Z(R)$ with equality if $\ker \Gamma \neq \ker \Delta$ thanks to Lemma 2.9, it only remains to study the case $\ker \Gamma = \ker \Delta$.

Consider $z \in Z_P(A) \cap A_1$. Let us first show that $z \in \ker \Delta^2$. If not, $z, \Delta(z)$ and $\Delta^2(z)$ are linearly independent over $\Bbbk$. Then $0 = \{z, \Delta(z)\} = \Delta(z)\Gamma\Delta(z) - \Delta^2(z)\Gamma(z)$. Since $\ker \Delta = \ker \Gamma$ and $z \notin \ker \Delta^2$ we get that $\Gamma\Delta(z) \neq 0$. Moreover $\Delta(z), \Gamma\Delta(z), \Delta^2(z)$ and $\Gamma(z)$ are homogeneous element of A of degree 1, hence are irreducible element of the unique factorization domain A. Since $\Delta(z)$ and $\Delta^2(z)$ are not collinear, they are non associated irreducible elements of A. The unique factorization of $\Delta(z)\Gamma\Delta(z) = \Delta^2(z)\Gamma(z)$ ensures us that there exists $\alpha \in \Bbbk$ such that $\Gamma\Delta(z) = \alpha\Delta^2(z)$ and $\Gamma(z) = \alpha\Delta(z)$. Computing $\Delta(z) = [\Delta, \Gamma](z) = \alpha\Delta^2(z) - \alpha\Gamma\Delta(z) = 0$, we get a contradiction. Hence $z \in \ker \Delta^2$.

Before pursuing the computation of the Poisson center of A, we switch for a moment to the study of the centre of R. Consider $z \in Z(R) \cap A_1$. As in the Poisson case, we first show that $z \in \ker \Delta^2$. If not, the integer such that $\Delta^n(z) \neq 0$ et $\Delta^{n+1}(z) = 0$ is greater than 1. We now compute $z * \Delta^{n-1}(z) = \Delta^{n-1}(z) * z$. For $\Delta((\Gamma - \mathrm{id})(\Delta^{n-1}(z)) = \Gamma\Delta^n(z) + \Delta^n(z) - \Delta^n(z) = 0$. Indeed $\Delta^n(z) \in \ker \Delta = \ker \Gamma$.

So $(\Gamma - \mathrm{id})(\Delta^{n-1}(z)) \in \ker\Delta = \ker\Gamma$. Hence $\Delta^{n-1}(z) \in \ker\binom{\Gamma}{2}$. Hence the relation $z * \Delta^{n-1}(z) = \Delta^{n-1}(z) * z$ becomes $\Delta(z)\Gamma(\Delta^{n-1}(z)) = \Gamma(z)\Delta^n(z)$. Since $n > 1$ the element $\Delta(z)$ and $\Delta^n(z)$ are linearly independents. As in the preceding paragraph, we deduce that there exists $\alpha \in \Bbbk$ such that $\alpha\Delta(z) = \Gamma(z)$ and $\alpha\Delta^n(z) = \Gamma(\Delta^{n-1}(z))$. The relation $[\Delta^{n-1}, \Gamma](z) = (n-1)\Delta^{n-1}(z)$ can then be written as $\alpha\Delta^n(z) - \alpha\Delta^n(z) = (n-1)\Delta^{n-1}(z)$. Hence $(n-1)\Delta^{n-1}(z) = 0$, which is absurd since $n > 1$ and $\Delta^{n-1}(z) \neq 0$. We deduce that $z \in \ker\Delta^2$ and therefore that $z = \Delta^{n-1}(z) \in \ker\binom{\Gamma}{2}$.

We now finish the proof by showing that in both cases $z \in \ker\Delta$ when $\mathrm{rank}\Delta \geq 2$. If $z \notin \ker\Delta$ then there exists y such that $\Delta(y)$ is not collinear to $\Delta(z)$. For the Poisson case, we have $0 = \{z, y\} = \Delta(z)\Gamma(y) - \Delta(y)\Gamma(z)$. For the case of R, we have $z * y = zy + \Delta(z)\Gamma(y)$ since $z \in \ker\Delta^2$. But $y * z = yz + \Delta(y)\Gamma(z)$ since $z \in \ker\binom{\Gamma}{2}$. Finally, we also have the equality $\Delta(z)\Gamma(y) = \Delta(y)\Gamma(z)$. Since $\ker\Delta = \ker\Gamma$, we get that $\Gamma(y) \neq 0$ and $\Gamma(z) \neq 0$. Using the fact that $\Delta(y)$ and $\Delta(z)$ are not collinear, we can mimic the argument of the two preceding paragraphs to get the relation $\Gamma(z) = \alpha\Delta(z)$ for some $\alpha \in \Bbbk$. We then obtain $\Delta(z) = [\Delta, \Gamma](z) = \alpha\Delta^2(z) - \Gamma\Delta(z)$. But $z \in \ker\Delta^2$ and hence $\Delta(z) \in \ker\Delta = \ker\Gamma$. We then conclude that $\Delta(z) = 0$ and $z \in \ker\Delta \cap \ker\Gamma$. □

Corollary 4.9. *Let (Δ, Γ) be a linear solvable pair on A. Then $A_1 \cap Z_P(A) \neq \ker\Delta \cap \ker\Gamma \cap A_1$ implies that A is Poisson commutative and $A_1 \cap Z(R) \neq \ker\Delta \cap \ker\Gamma \cap A_1$ implies that R is commutative.*

Moreover A is Poisson commutative if and only if R is commutative if and only if $\Delta = 0$ or (Δ, Γ) is conjugated to (Δ_0, Γ_0) where

$$\Delta_0 = \begin{pmatrix} 0 & 1 & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix} = X_0\partial_{X_1} \qquad \textit{and} \qquad \Gamma_0 = \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & 0 & \\ & & & \ddots \\ & & & & 0 \end{pmatrix} = X_1\partial_{X_1}$$

Proof. Assume that $A_1 \cap Z_P(A) \neq \ker\Delta \cap \ker\Gamma \cap A_1$ or $A_1 \cap Z(R) \neq \ker\Delta \cap \ker\Gamma \cap A_1$, Lemma 2.9 and Proposition 4.8 show that $\Delta = 0$ (and $\Gamma \neq 0$) or Δ is of Jordan type $(2, 1, \ldots, 1)$ and $\ker\Gamma = \ker\Delta$.

In the first case A is clearly Poisson commutative and R is commutative. In the second case, we can choose a basis $(e_0, \ldots, e_n)$ of A_1 such that $\Delta(e_1) = e_0$ and $(e_0, e_2 \ldots, e_n)$ is a basis of $\ker\Delta = \ker\Gamma$. In such a basis, the matrix of Δ and Γ are given by

$$\begin{pmatrix} 0 & 1 & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & \alpha_0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & \alpha_n & 0 & \cdots & 0 \end{pmatrix}$$

The relation $[\Delta, \Gamma] = \Delta$ ensures that $\alpha_1 = 1$. Changing the basis $(e_0, \ldots, e_n)$ to the basis $(e_0, e_1 + \alpha_0 e_0 + \alpha_2 e_2 + \cdots + \alpha_n e_n, e_2, \ldots, e_n)$ we get the matrices Δ_0 and Γ_0. In such a basis, we have that $\{X_i, X_j\} = 0$ and $X_i * X_j = X_i X_j$ as soon as i or j belongs to $\{0, 2, \ldots, n\}$ (since $\ker\Gamma = \ker\Delta$). In particular R is commutative and A is Poisson commutative.

It remains so show if A is Poisson commutative or R is commutative then $\Delta = 0$ or (Δ, Γ) is conjugated to (Δ_0, Γ_0). If $A_1 \cap Z_P(A) \neq \ker\Delta \cap \ker\Gamma \cap A_1$ or $A_1 \cap Z(R) \neq \ker\Delta \cap \ker\Gamma \cap A_1$ then the first part of the proof allows us to conclude. If $A_1 \cap Z_P(A) = \ker\Delta \cap \ker\Gamma \cap A_1$ or $A_1 \cap Z(R) = \ker\Delta \cap \ker\Gamma \cap A_1$ then $A_1 = \ker\Delta \cap \ker\Gamma \cap A_1$ and $\Delta = 0$. □

4.3. **A finer filtration for the linear case.** In section 2.4, we defined a filtration ε on any A with a solvable pair. When A is a polynomial ring and (Δ, Γ) a linear solvable pair on A, we can define a finer filtration $\widetilde{\varepsilon}$ on A. The new filtration is defined as the extension to A of the restriction to A_1 of the filtration ε. Recall that A_1 denote the set of homogeneous polynomial of degree 1 which is stable by Δ and Γ since the pair is linear (Δ, Γ). Appendix C is devoted to the construction of this filtration $\widetilde{\varepsilon}$ and

the study of its properties, see in particular Example C.3. When Δ is a maximal Jordan block, then $\widetilde{\varepsilon}$ is the filtration used in [LS19, Section 3].

For this filtration $\widetilde{\varepsilon}$, analogs of Lemma 2.21 and Proposition 2.22 are valid. Precisely, we have the following proposition.

Proposition 4.10. *Let (Δ, Γ) be a linear solvable pair on $A = \Bbbk[X_0, \dots, X_n]$ and fix integers $i, j \geqslant 0$. Set $A^{\leqslant i,\widetilde{\varepsilon}} = \{x \in A,\ \widetilde{\varepsilon}(x) \leqslant i\}$.*

(1) We have $\Delta(A^{\leqslant i,\widetilde{\varepsilon}}) \subseteq A^{\leqslant i-1,\widetilde{\varepsilon}}$ and so $\widetilde{\varepsilon}(\Delta(f)) \leqslant \widetilde{\varepsilon}(f) - 1$ for every $f \in A$.

(2) We have $\Gamma(A^{\leqslant i,\widetilde{\varepsilon}}) \subseteq A^{\leqslant i,\widetilde{\varepsilon}}$. Hence $\widetilde{\varepsilon}(P(\Gamma)(f)) \leqslant \widetilde{\varepsilon}(f)$ for every $P \in \Bbbk[T]$ and $f \in A$.

(3) If $f, g \in A$ are such that $\widetilde{\varepsilon}(f) = i$ and $\widetilde{\varepsilon}(g) = j$, then $\widetilde{\varepsilon}(fg) = i + j$.

(4) The family $(A^{\leqslant i,\widetilde{\varepsilon}})_{i\in\mathbb{N}}$ is a Poisson algebra filtration of $A_{(\Delta,\Gamma)}$ of degree -1, meaning that it is an algebra filtration of A together with $\{A^{\leqslant i,\widetilde{\varepsilon}}, A^{\leqslant j,\widetilde{\varepsilon}}\} \subseteq A^{\leqslant i+j-1,\widetilde{\varepsilon}}$ for all $i, j \geqslant 0$.

(5) The family $(A^{\leqslant i})_{i\in\mathbb{N}}$ is an algebra filtration of $R_{(\Delta,\Gamma)}$. Moreover, the associated graded algebra $\mathrm{gr}^{\widetilde{\varepsilon}}(R)$ is equal to $\mathrm{gr}^{\widetilde{\varepsilon}}(A)$.

(6) For any $f, g \in A$ we have

$$f * g - g * f = \Delta(f)\Gamma(g) - \Delta(g)\Gamma(f) + \sum_{k\geqslant 2} \left(\Delta^k(f) \binom{\Gamma}{k}(g) - \Delta^k(g) \binom{\Gamma}{k}(f) \right)$$

In particular, if $\widetilde{\varepsilon}(f) = i$ and $\widetilde{\varepsilon}(g) = j$ then $\widetilde{\varepsilon}(f * g - g * f) \leqslant i + j - 1$.

(7) The commutative algebra $\mathrm{gr}^{\widetilde{\varepsilon}}(R) = \mathrm{gr}^{\widetilde{\varepsilon}}(A)$ can be endowed with the following three Poisson brackets

(a) $\{\overline{f}, \overline{g}\}' := (f * g - g * f) + A^{\leqslant i+j-2,\widetilde{\varepsilon}}$

(b) $\{\overline{f}, \overline{g}\}'' := \{f, g\} + A^{\leqslant i+j-2,\widetilde{\varepsilon}}$

(c) $\{\overline{f}, \overline{g}\}''' := \overline{\Delta}(\overline{f})\overline{\Gamma}(\overline{g}) - \overline{\Gamma}(\overline{f})\overline{\Delta}(\overline{g}) \in A^{\leqslant i+j-1,\widetilde{\varepsilon}}/A^{\leqslant i+j-2,\widetilde{\varepsilon}}$

for homogeneous elements $\overline{f}$ and $\overline{g}$ of respective $\widetilde{\varepsilon}$-degree i and j. In (c) the pair of maps $(\overline{\Delta}, \overline{\Gamma})$ is the solvable pair of homogeneous derivations of $\mathrm{gr}^{\widetilde{\varepsilon}}(A)$ of respective degree -1 and 0 which is induced by the solvable pair of filtered derivations Δ and Γ of A (see Proposition C.2).

(8) The Poisson structures defined in (3) are all equal and make $\mathrm{gr}^{\widetilde{\varepsilon}}(A) = \mathrm{gr}^{\widetilde{\varepsilon}}(R)$ into a graded Poisson algebra of degree -1.

Proof. The proof is very close from the one of Lemma 2.21 and of Proposition 2.22 but since the construction of the filtration is more complicated the arguments should be given in a somewhat different order. We start with assertion (3). Proposition C.1 shows that $\mathrm{gr}^{\widetilde{\varepsilon}}(A)$ is a domain. This shows (3). To prove (1) and (2), we start with $x \in A_1$. By definition of $\widetilde{\varepsilon}$ we have $\widetilde{\varepsilon}(\Delta(x)) \leqslant \widetilde{\varepsilon}(x) - 1$. Moreover, Lemma 2.20 shows that $\ker \Delta^i \cap A_1$ is stable by Γ, hence by $P(\Gamma)$ showing that $\widetilde{\varepsilon}(P(\Gamma)(x)) \leqslant \widetilde{\varepsilon}(x)$. We write now $f \in A$ as a sum of products of elements of A_1 and (1) and (2) follow from (3) and the fact that Δ and Γ are derivations. assertion (4) follows from 1.1 and then assertions (1) and (2) and (3). Assertion (5) and (6) follow from (1.2) and then we get assertions (1) and (2) and (3).

(7). The fact that $\{-,-\}'$ is a Poisson bracket on $\mathrm{gr}(R)$ follows the filtered version of the semi-classical limit construction, see [Goo08, Section 2.4]. The fact that $\{-,-\}''$ is a well-defined biderivation satisfying the Jacobi identity on $\mathrm{gr}(A)$ follows by tedious but straightforward computation from the fact that $\{-,-\}$ is a filtered Poisson bracket on A. Finally $\{-,-\}'''$ is a Poisson bracket since $(\overline{\Delta}, \overline{\Gamma})$ is a solvable pair of derivations of $\mathrm{gr}^{\widetilde{\varepsilon}}(A)$.

(8). The Poisson brackets $\{-,-\}'$ and $\{-,-\}''$ are the same since both $f * g - g * f$ and $\{f, g\}$ belong to $A^{\leqslant i+j-1,\widetilde{\varepsilon}}$ combined with $f * g - g * f - \{f, g\} \in A^{\leqslant i+j-2,\widetilde{\varepsilon}}$ thanks to Assertion (1), (2), (3) and (6). Finally $\{-,-\}''$ and $\{-,-\}'''$ are the same since for homogeneous elements $\overline{f}$ and $\overline{g}$ of $\widetilde{\varepsilon}$-degree i and j

we have

$$\begin{aligned}\{\overline{f},\overline{g}\}''' &= \overline{\Delta}(\overline{f})\overline{\Gamma}(\overline{g}) - \overline{\Gamma}(\overline{f})\overline{\Delta}(\overline{g})\\ &= (\Delta(f) + A^{\leqslant i-2,\widetilde{\varepsilon}})(\Gamma(g) + A^{\leqslant j-1,\widetilde{\varepsilon}}) - (\Delta(g) + A^{\leqslant j-2,\widetilde{\varepsilon}})(\Gamma(f) + A^{\leqslant i-1,\widetilde{\varepsilon}})\\ &= (\Delta(f)\Gamma(g) + A^{\leqslant i+j-2,\widetilde{\varepsilon}}) - (\Delta(g)\Gamma(f) + A^{\leqslant i+j-2,\widetilde{\varepsilon}})\\ &= (\Delta(f)\Gamma(g) - \Delta(g)\Gamma(f)) + A^{\leqslant i+j-2,\widetilde{\varepsilon}} = \{f,g\} + A^{\leqslant i+j-2,\widetilde{\varepsilon}} = \{\overline{f},\overline{g}\}''.\end{aligned}$$

□

Remark 4.11 (Matrix version). Assertion (8) says that the Poisson structure on $\mathrm{gr}^{\widetilde{\varepsilon}}(A)$ is associated to the solvable pair $(\overline{\Delta},\overline{\Gamma})$. On A_1, the filtrations $\widetilde{\varepsilon}$ and ε coincide, hence by an appropriate choice of basis of A_1 and by using the Jordan reduction theorem adapted to this filtration, we obtain the following block decompositions

$$\Delta = \begin{pmatrix} 0 & I_{m_0,m_1} & & \\ & \ddots & \ddots & \\ & & \ddots & I_{m_{r-2},m_{r-1}} \\ & & & 0 \end{pmatrix} \quad \text{and} \quad \Gamma = \begin{pmatrix} \Gamma_1 & \Gamma_{1,2} & \cdots & \Gamma_{1,r} \\ & \ddots & \ddots & \vdots \\ & & \ddots & \Gamma_{r-1,r} \\ & & & \Gamma_r \end{pmatrix}$$

where for $p \geq r$, we denote by

$$I_{p,r} = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \in M_{p,r}(\Bbbk)\,.$$

The corresponding $\overline{\Delta}$ and $\overline{\Gamma}$ are then given by

$$\overline{\Delta} = \begin{pmatrix} 0 & I_{m_0,m_1} & & \\ & \ddots & \ddots & \\ & & \ddots & I_{m_{r-2},m_{r-1}} \\ & & & 0 \end{pmatrix} \quad \text{and} \quad \overline{\Gamma} = \begin{pmatrix} \Gamma_1 & & \\ & \ddots & \\ & & \Gamma_r \end{pmatrix}.$$

5. Trigonalizable linear solvable pair

We continue with the notations of Section 4 : $A = k[X_0,\dots,X_n]$ is a polynomial algebra with standard grading and, for $k \geqslant 0$, A_k is the set of degree k polynomials.

Definition 5.1. A linear solvable pair (Δ,Γ) is said to be trigonalizable if the linear map induced by Γ on A_1 is.

Proposition 5.2. *Let (Δ,Γ) be a linear solvable pair. Then it is trigonalizable if and only if the linear map Γ_1 induced by Γ on $A_1 \cap \ker\Delta$ is.*

Proof. Using the notations of Remark 4.11, the linear map induced by Γ on $A_1 \cap \ker\Delta$ is Γ_1. Hence if Γ is trigonalizable then Γ_1 is.

Reciprocally, computing the superdiagonal blocks of the relation $\Delta\Gamma - \Gamma\Delta = \Delta$ shows that for every $1 \leqslant i \leqslant r-1$ we have the following triangular block decomposition

$$\Gamma_i = \begin{pmatrix} \Gamma_{i+1} - I & C_i \\ & B_i \end{pmatrix}$$

where I denote the identity matrix of the appropriate size. Hence $\chi_{\Gamma_{i+1}}(X+1)$ divides $\chi_{\Gamma_i}(X)$, where χ_M denote the characteristic polynomial of the square matrix M. We then deduce that when χ_{Γ_1} is trigonalizable, every χ_{Γ_i} is. Hence Γ acts on A_1 as a trigonalizable linear map. □

5.1. **The Artin-Schelter regular property.** In this section we prove that the algebra $R = R_{(\Delta,\Gamma)}$ is Artin-Schelter regular when (Δ, Γ) is a linear solvable pair. Recall that Artin-Schelter regular algebras are thought to be noncommutative analogue of commutative polynomial rings in the following sense [AS87]:

Definition 5.3. A connected $\mathbb{N}$-graded $\Bbbk$-algebra R is called *Artin-Schelter regular* or *AS-regular* if:

(1) gldim $R < \infty$;
(2) R has finite Gelfand-Kirillov dimension;
(3) $\mathrm{Ext}^i_R(\Bbbk_R, R_R) \cong \begin{cases} 0 & \text{if } i \neq \text{gldim } R \\ {}_R\Bbbk[\ell] & \text{if } i = \text{gldim } R. \end{cases}$

where $\Bbbk[\ell]$ means that the module is degree-shifted by some amount $\ell \in \mathbb{Z}$.

Since both Δ and Γ are linear derivations we have $\Delta(A_k) \subseteq A_k$ and $\Gamma(A_k) \subseteq A_k$ for all integer $k \geqslant 0$. Therefore $A_k * A_\ell \subseteq A_{k+\ell}$ for any $k, \ell \geqslant 0$ thanks to equation (1.2). Thus the algebra $R = (A, *)$ is d-graded, generated in degree 1 and with Hilbert series given by

$$\mathrm{hilb}(R) = \mathrm{hilb}(A) = \frac{1}{(1-t)^{n+1}}.$$

The following result is the key argument for our inductive proof of Theorem 5.5. Since this result relates algebras $R_{(\Delta,\Gamma)}$ (resp. $A_{(\Delta,\Gamma)}$) of different dimensions we will specify the (projective) dimension of the underlying polynomial ring by using the notation $R^n_{(\Delta,\Gamma)}$ (resp. $A^n_{(\Delta,\Gamma)}$).

Proposition 5.4. *Let (Δ, Γ) be a linear solvable pair on $A = \Bbbk[X_0, \ldots, X_n]$ such that $\Delta(X_0) = 0$ and there exists $\alpha \in \Bbbk$ verifying $\Gamma(X_0) = \alpha X_0$. Set $A = A^n_{(\Delta,\Gamma)}$ and $R = R^n_{(\Delta,\Gamma)}$ and denote by Δ and Γ the matrices of Δ and Γ acting on A_1 with respect to a basis starting by X_0.*

(1) The quotient algebra $R/\langle X_0\rangle$ is isomorphic to $R^{n-1}_{(\Delta',\Gamma')}$ for the solvable pair (Δ', Γ') of $A^{n-1} = \Bbbk[Y_0, \ldots, Y_{n-1}]$ obtained from the solvable pair (Δ, Γ) by deleting the first rows and first columns.
(2) X_0 is Poisson normal in A and the Poisson algebra $A/\langle X_0\rangle$ is isomorphic to $A^{n-1}_{(\Delta',\Gamma')}$ for the same solvable pair (Δ', Γ') of A^{n-1}.

In particular, $R/\langle X_0\rangle$ is a deformation of the Poisson algebra $A/\langle X_0\rangle$.

Proof. (1) First note that X_0 is strongly normal in R and Poisson normal in A. The algebras $R/\langle X_0\rangle$, A/X_0A and $R^{n-1}_{(\Delta',\Gamma')}$ can all be identified as graded vector spaces. Let $\pi : R \to R/\langle X_0\rangle$ be the quotient map and set $\overline{X}_i = \pi(X_i)$ for all $0 \leqslant i \leqslant n$. Then the product in $R/\langle X_0\rangle$ is given by

$$\overline{X}_i * \overline{X}_j = \sum_{\ell \geqslant 0} \overline{\Delta}^\ell(\overline{X}_i) \binom{\overline{\Gamma}}{\ell} (\overline{X}_j)$$

where $\overline{\Delta}$ and $\overline{\Gamma}$ denote the derivations induced by$\overline{\Delta}$ and $\overline{\Gamma}$ on the quotient $A/\langle X_0\rangle \cong \Bbbk[Y_0, \ldots, Y_{n-1}]$. It is now a simple verification to check that $(\overline{\Delta}, \overline{\Gamma})$ is a solvable pair on $A/\langle X_0\rangle$ that agrees with the solvable pair (Δ', Γ') of $\Bbbk[Y_0, \ldots, Y_{n-1}]$ under the identification $\overline{X}_{i+1} \mapsto Y_i$ for all $0 \leqslant i < n$.

(2) X_0 is Poisson normal since it is in the kernel of Δ and is an eigenvector of Γ. It is direct that the Poisson bracket on A/X_0A is given by the solvable pair $(\overline{\Delta}, \overline{\Gamma})$ and we conclude as in (1). □

We can now prove now prove that $R_{(\Delta,\Gamma)}$ is Artin-Schelter regular.

Theorem 5.5. *Let (Δ, Γ) be a linear solvable trigonalizable pair. The algebra $R = R^n_{(\Delta,\Gamma)}$ is Artin-Schelter regular of global dimension $n + 1$.*

Proof. The proof follows by induction on n, exactly as in the proof of [LS19, Theorem 3.8] since the only hypotheses used by the authors are that R is connected graded generated in degree 1 with polynomial growth and admits a normal sequence Ω of homogeneous regular elements with $R/\Omega R \cong \Bbbk$. Here we can

also use $\Omega = (X_0, X_1, \dots, X_n)$ where $(X_0, \dots, X_n)$ is a basis of A_1 as in Remark 4.11 and Γ is triangular (Proposition 5.2). □

Corollary 5.6. *Let (Δ, Γ) be a linear solvable pair. The algebra $R = R^n_{(\Delta,\Gamma)}$ is Artin-Schelter regular of global dimension $n+1$.*

Proof. Consider $\widetilde{\Bbbk}$ a finite field extension of $\Bbbk$ such that the characteristic polynomial of Γ acting on A_1 is split. Thanks to Theorem 5.5 the algebra $\widetilde{R} = \widetilde{\Bbbk} \otimes R$ is Artin-Schelter regular since the unique extension of (Δ, Γ) to $\widetilde{R}$ is a trigonalizable solvable pair.

Since $\widetilde{\Bbbk} \otimes -$ is an exact functor which sends projective modules over R to projective modules over $\widetilde{R}$, for any R-modules M, N, we have

$$\mathrm{Ext}^*_{\widetilde{R}}(\widetilde{\Bbbk} \otimes M, \widetilde{\Bbbk} \otimes N) \cong \mathrm{Ext}^*_R(M, \widetilde{\Bbbk} \otimes N) \cong \mathrm{Ext}^*_R(M, N)^{[\widetilde{\Bbbk}:\Bbbk]}$$

Therefore we have $\mathrm{gldim}\, R \leqslant \mathrm{gldim}\, \widetilde{R} < \infty$. Since R and $\widetilde{R}$ are $\mathbb{N}$-graded connected algebras, the case $M = N = k$ and [Ei56, Theorem 11, Theorem 13 and Proposition 15] show that $\mathrm{gldim}\, R = \mathrm{gldim}\, \widetilde{R}$. By considering the case $M = k$ and $N = R$, we obtain that $\mathrm{Ext}^*_{\widetilde{R}}(\widetilde{\Bbbk}, \widetilde{R}) \cong \mathrm{Ext}^*_R(k, R)^{[\widetilde{\Bbbk}:\Bbbk]}$. In particular, if $\mathrm{Ext}^*_{\widetilde{R}}(\widetilde{\Bbbk}, \widetilde{R}) \cong \widetilde{k}$ then $\mathrm{Ext}^*_R(k, R) \cong k$ since the field extension is finite. It follows that the algebra R is Artin-Schelter regular. □

5.2. **Normal elements.** The next proposition completely describes normal elements of R and Poisson normal elements in A in the case of a linear solvable pair: they are the strongly normal element (see Definition 2.29). It generalises the results of Lemma 2.9, Proposition 4.8 and is a converve of Lemma 2.30 in the linear case.

Proposition 5.7. *Let (Δ, Γ) be linear solvable pair. Assume that A is not Poisson commutative (see Corollary 4.9). Then*

(1) N is normal in R iff N is strongly normal iff N is Poisson normal in A.
(2) N is central in R iff $N \in \ker \Delta \cap \ker \Gamma$ iff N is Poisson central in A.

Proof. The proof splits in two cases according to the value of the rank of Δ acting on A_1. When this rank is equal to 1, the proof relies on an explicit computation. When this rank is strictly greater than one, the proof can be mimicked from the one of [LS19, Proposition 3.21] by noticing that this proof only relies on the existence of an $\widetilde{\varepsilon}$-homogeneous, irreducible, strongly normal element G which is not in $\ker \Gamma$. If this is the case, it can verify that the ideal $G * R = R * G$ of R is completely prime since G is an irreducible element of the unique factorization domain A. Notice that the existence of such a G shows that assertion (2) can also be seen as a consequence of Lemma 2.9.

First assume that (Δ, Γ) is a trigonalizable solvable pair. In the following, we only consider actions of (Δ, Γ) on A_1 and its subspaces. Using the notations of Remark 4.11 and Proposition 5.2, we see that Γ_1 is trigonalizable. If Γ_1 has a nonzero eigenvalue λ, then an eigenvector G of Γ_1 with respect to λ satisfies the desired properties. Hence we can assume that Γ_1 is nilpotent. As in the proof of Proposition 5.2, the relation $\Delta\Gamma - \Gamma\Delta = \Delta$ shows that for every $1 \leqslant i \leqslant r-1$ we have the following triangular block decomposition

$$\Gamma_i = \begin{pmatrix} \Gamma_{i+1} - I & C_i \\ & B_i \end{pmatrix}$$

where I denote the identity matrix of the appropriate size. Hence $i-1$ is the only eigenvalue of Γ_i and thus $\Gamma_i - (i-1)\mathrm{id}$ is nilpotent. The triangular shape of the block decomposition of Γ then shows that $\ker \Delta^i = \bigoplus_{j=0}^{i-1} N_j$ where N_j is the generalized eigenspace of Γ with respect to j. The relation $\Delta\Gamma - \Gamma\Delta = \Delta$ can be rewritten $\Delta(\Gamma - j\mathrm{id}) = (\Gamma - (j-1)\mathrm{id})\Delta$ and then $\Delta(\Gamma - j\mathrm{id})^k = (\Gamma - (j-1)\mathrm{id})^k\Delta$ for all integer $k \geqslant 0$. Hence Δ maps N_j into N_{j-1}. Moreover the relation $\ker \Delta^i = \bigoplus_{j=0}^{i-1} N_j$ shows that for all integer $j \geqslant 1$, Δ maps injectively N_j into N_{j-1}.

Assume now that Δ admits a Jordan block of size at least 3. Then 2 is an eigenvalue of Γ. If we consider X_2 such that $\Gamma(X_2) = 2X_2$, then $X_1 = \Delta(X_2) \in \ker(\Gamma - \mathrm{id})$, $X_0 = \Delta(X_1) \in \ker(\Gamma)$, the family (X_0, X_1, X_2) is linearly independent and then $G = 2X_2X_0 - {X_1}^2 \neq 0$ verifies $\Delta(G) = 0$ and $\Gamma(G) = 2G$ as desired.

Hence we can now assume that Δ admits only Jordan blocks of size 2. If Δ admits at least two such blocks then $\dim(\ker \Delta^2 / \ker \Delta) \geq 2$. Hence $\dim N_1 \geq 2$. If $\dim \ker(\Gamma - \mathrm{id}) \geq 2$ then choose $(X_1, Y_1) \in \ker(\Gamma - \mathrm{id})$ linearly independent then $G = X_1\Delta(Y_1) - Y_1\Delta(X_1) \neq 0$ verifies $\Delta(G) = 0$ and $\Gamma(G) = G$ as wanted. For $\Delta(Y_1), \Delta(X_1) \in \ker \Delta \cap \ker \Gamma$. If $\dim \ker(\Gamma - \mathrm{id}) = 1$, since $\dim N_1 \geq 2$, there exists $(X_1, Y_1) \in {N_1}^2$ such that $\Gamma(X_1) = X_1$ and $\Gamma(Y_1) = Y_1 + X_1$ (apply canonical Jordan form to Γ acting on N_1). Set then $X_0 = \Delta(X_1)$ and $Y_0 = \Delta(Y_1)$. We have $X_0, Y_0 \in \ker \Delta$ and $\Gamma(X_0) = 0$ and $\Gamma(Y_0) = X_0$. Hence $G = X_1Y_0 - X_0Y_1 \neq 0$ verifies $\Gamma(G) = G$ and $\Delta(G) = 0$.

It remains to consider the case where Δ admits only one Jordan block which is of size 2 that is to say the rank of Δ is 1. In this case, we have

$$\Delta = \begin{pmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{pmatrix} \qquad \text{and} \qquad \Gamma = \begin{pmatrix} 0 & L_1 & 0 \\ & \widetilde{\Gamma_1} & 0 \\ & & 1 \end{pmatrix}$$

where the second column of the matrices Δ and Γ represents matrices with $(n-1)$ columns. By choosing the corresponding basis $(X_0, \ldots, X_n)$ for A_1, we have $\Delta = X_0\partial_{X_n}$ and $\Gamma(X_0) = 0$, $\Gamma(X_j) \in \Bbbk[X_0, \ldots, X_{n-1}]$ for $j \in \{1, \ldots, n-1\}$ and $\Gamma(X_n) = X_n$. We will show that every Poisson normal element Q lies in $\ker \Delta \cap \ker \Gamma$ and that every normal element Q in R also lies in $\ker \Delta \cap \ker \Gamma$. For, write $Q = \sum_{i=0}^{s} P_i {X_n}^i$ with $P_i \in \Bbbk[X_0, \ldots, X_{n-1}]$ and an integer $s \geqslant 0$. We have

$$\Delta(Q) = \sum_{i=0}^{s} iP_iX_0{X_n}^{i-1} \qquad \text{and} \qquad \Gamma(Q) = \sum_{i=0}^{s} (\Gamma(P_i) + iP_i){X_n}^i .$$

Moreover there exists $j_0 \in \{1, \ldots, n-1\}$ such that $\Gamma(X_{j_0}) \neq 0$, otherwise, from Corollary 4.9, we have that A is Poisson commutative. Assume that Q is Poisson normal and compute $\{Q, X_{j_0}\} = \Delta(Q)\Gamma(X_{j_0})$. Since Q is Poisson normal there exists $F \in A$ such that $\Delta(Q)\Gamma(X_{j_0}) = FQ$. If $F \neq 0$ then by comparing the degree in X_n of the two sides of the equality leads to something impossible. Hence $F = 0$ and $\Delta(Q) = 0$ since $\Gamma(X_{j_0}) \neq 0$. So $Q = P_0 \in k[X_0, \ldots, X_{n-1}]$. By computing $\{Q, X_n\} = -\Gamma(Q)X_0 = -\Gamma(P_0)X_0$, from the Poisson normality of Q, we obtain that there exists $F \in A$ such that $\Gamma(P_0)X_0 = FP_0$. We then write $P_0 = {X_0}^{\ell}R$ with $X_0 \nmid R$. Since $X_0 \in \ker \Gamma$, we obtain the relation $\Gamma(R)X_0 = FR$. Hence $X_0 \mid F$, so $\Gamma(R) = F_1R$ for some $F_1 \in A$. By comparing the total degree we obtain that $F_1 \in \Bbbk$, and by multiplying by ${X_0}^{\ell}$ we obtain that $Q = P_0$ verifies $\Gamma(P_0) = F_1P_0$. But Γ is locally nilpotent on $\Bbbk[X_0, \ldots, X_{n-1}]$ since it acts as a nilpotent linear map on the span of $X_0, \ldots, X_{n-1}$, hence $F_1 = 0$ and $Q = P_0 \in \ker \Gamma \cap \ker \Delta$ is strongly normal.

Let us now consider that $Q \neq 0$ is normal in R and compute $X_{j_0} * Q = X_{j_0}Q$. Since Q is normal, there exists $f \in R$ such that $X_{j_0} * Q = Q * f$. By applying the ε-degree, we deduce that $\varepsilon(f) = \varepsilon(X_{j_0}) = 0$, that is to say $f \in \ker \Delta$. By sending the preceding equality in $\mathrm{gr}(R)$ which is a domain thanks to Corollary 2.23, we obtain that $\overline{X_{j_0}} = \overline{f} \in \ker(A^{\leqslant 0}/\{0\})$ and then $X_{j_0} = f$. The relation $X_{j_0} * Q = Q * f$ can then be rewritten as

$$\sum_{i \geqslant 1} \Delta^i(Q) \binom{\Gamma}{i}(X_{j_0}) = 0 \qquad \text{with} \qquad \binom{\Gamma}{i}(X_{j_0}) \in \ker \Delta$$

By considering the smallest integer $N \geqslant 1$ such that $\Delta^N(Q) = 0$ and by composing the preceding relation with Δ^{N-2}, we obtain that $\Delta^{N-1}(Q)\Gamma(X_{j_0}) = 0$. But $\Gamma(X_{j_0}) \neq 0$, hence $N = 1$ and $Q \in \ker \Delta = k[X_0, \ldots, X_{n-1}]$. We now compute $X_n * Q = X_nQ + X_0\Gamma(Q)$. Since Q is normal, there exists f such that $X_n * Q = Q * f = Qf$. Hence $Qf = X_nQ + X_0\Gamma(Q)$ and we deduce that the total degree of f is not greater than 1 and that $\deg_{X_n}(f) = 1$. By writing $f = aX_n + f_1$ with $f_1 \in \Bbbk[X_0, \ldots, X_{n-1}]$ and

$a \in \Bbbk$, we obtain $a = 1$ and then $Qf_1 = X_0\Gamma(Q)$. By applying the same argument that the one in the Poisson normal case, we obtain that $\Gamma(Q) = 0$. So $Q \in \ker\Gamma = \ker\Delta$ is strongly normal.

To conclude, it remains to consider the case where the pair (Δ, Γ) is not trigonalizable. Consider an element Q that is either Poisson normal in A or normal in R. Let $\overline{\Bbbk}$ be the algebraic closure of $\Bbbk$ and consider the scalar extensions $\overline{A} = \overline{\Bbbk} \otimes A$ and $\overline{R} = \overline{\Bbbk} \otimes R$ of A and R respectively. The pair (Δ, Γ) is a trigonalizable linear pair on $\overline{A}$ and $1 \otimes Q$ is Poisson normal in $\overline{A}$ or normal in $\overline{R}$. Hence $1 \otimes Q$ is strongly normal. Hence $1 \otimes \Delta(Q) = 0$ and $1 \otimes \Gamma(Q) = \lambda \otimes Q$ for some $\lambda \in \overline{\Bbbk}$. But $1 \otimes \Gamma(Q) \in 1 \otimes A$ so $\lambda \in \Bbbk$ and $\Gamma(Q) = \lambda Q$. So Q is strongly normal. □

6. Diagonalizable linear solvable pair : Poisson derivations

Definition 6.1. A linear solvable pair (Δ, Γ) is said to be diagonalizable if there exists a basis of A consisting of eigenvectors for Γ.

Remark 6.2. The existence of a basis of A consisting of eigenvectors for Γ is equivalent to the fact that the restriction of Γ to A_1 is diagonalizable.

Indeed, if we consider a basis of eigenvectors of Γ acting on A_1 then the set of monomials formed with elements of this basis of A_1 is a basis of A consisting of eigenvectors for Γ.

Reciprocally, assume that there is a basis $\mathcal{B}$ of A consisting of eigenvectors for Γ. Let us show that the eigenvectors of Γ contained in A_1 generates A_1. For this, let us consider $v \in A_1$. We express v as a linear combination of elements of $\mathcal{B}$ and gather the eigenvectors associated to the same eigenvalue to write $v = w_1 + \cdots + w_r$ where w_i is an eigenvector for Γ associated to the eigenvalue $\lambda_i \in \Bbbk$ and where $\lambda_i \neq \lambda_j$ when $i \neq j$. By considering the family $(v, \Gamma(v), \ldots, \Gamma^{r-1}(v)) \in {A_1}^r$ and inverting the Vandermonde matrix associated to $(\lambda_1, \ldots, \lambda_r)$, we get that $w_i \in A_1$ for all i. This conclude the proof.

For a diagonalizable linear solvable pair (Δ, Γ) we denote by $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_r$ the size of the Jordan blocks of Δ acting on A_1. Such a Δ is said to be of Jordan type $\overline{n} = (n_1, \ldots, n_r)$ where we set $n + 1 = n_1 + \cdots + n_r$.

Proposition 6.3. *Let (Δ, Γ) be a diagonalizable linear solvable pair. Assume that Δ is of Jordan type $\overline{n} = (n_1, \ldots, n_r)$. Then there exists a basis of A_1 and $\overline{a} = (a_1, \ldots, a_r) \in \Bbbk^r$ such that Δ is in canonical Jordan form and Γ is diagonal of the form $\Gamma = \operatorname{diag}(a_1, \ldots, a_1+n_1-1, a_2, \ldots, a_2+n_2-1, \ldots, a_r, \ldots, a_r+n_r-1)$.*

Proof. This is an adaptation of the proof of the existence of the canonical Jordan form. For any integer $i \geqslant 0$ we get that $F_i := \ker\Delta^i$ is stable by Γ by using the relation $[\Delta^i, \Gamma] = i\Delta^i$. There exists $s \geqslant 0$ such that $F_s = A_1$ since Δ is nilpotent on A_1. We construct by a decreasing induction subspaces $(G_i)_{1 \leqslant i \leqslant s}$ such that

(1) $F_{i-1} \oplus G_i = F_i$;
(2) Δ maps injectively G_{i+1} into G_i;
(3) G_i is stable by Γ.

Since F_{s-1} is stable by Γ and Γ is diagonalizable, there exists G_s which is stable by Γ such that $F_s = A_1 = F_{s-1} \oplus G_s$. Assume that $G_s, \ldots, G_{s+1}$ have been constructed satisfying (1), (2) and (3). Then clearly Δ maps G_{i+1} into F_i. Moreover $\Delta(G_{i+1}) \cap F_{i-1} = \{0\}$ (since $G_{i+1} \cap F_i = \{0\}$) and $G_{i+1} \cap \ker\Delta = \{0\}$. Moreover $\Delta(G_{i+1})$ is stable by Γ. Hence $F_{i-1} \oplus \Delta(G_{i+1})$ too and there exists a subspace W_i of F_i stable by Γ (since Γ is diagonalizable) such that $F_{i-1} \oplus \Delta(G_{i+1}) \oplus W_i = F_i$. It suffices now to consider $G_i = \Delta(G_{i+1}) \oplus W_i$ to get the result.

Let us now remark that if x is an eigenvector for Γ (associated to λ) and $\Delta(x) \neq 0$ then $\Delta(x)$ is an eigenvector for Γ (associated to $\lambda - 1$).

Thus if we consider a basis of G_s composed of eigenvectors of Γ, then we consider the image of this basis by Δ and complete into a basis of G_{s-1} (adding eigenvector of Γ belonging to W_{s-1}). We proceed by induction to produce the basis we are looking for. □

Definition 6.4. Thanks to Lemma 2.1 and Proposition 6.3 one can assume without loss of generalities that for any diagonalizable solvable pair (Δ, Γ) the Poisson algebra $A_{(\Delta,\Gamma)}$ is presented in the basis, denoted by $(X_0, \dots, X_n)$, given by Proposition 6.3. We denoted by $A(\overline{n}, \overline{a})$ the Poisson algebra $A_{(\Delta,\Gamma)}$.

Remark 6.5. It will sometimes be convenient to denote by λ_ℓ the eigenvalue for Γ of X_ℓ in $A(\overline{n}, \overline{a})$. The Poisson bracket of $A(\overline{n}, \overline{a})$ is then given by

$$\{X_i, X_j\} = \lambda_j \Delta(X_i) X_j - \lambda_i \Delta(X_j) X_i$$

for all $0 \leqslant i, j \leqslant n$.

Remark 6.6. Note that when (Δ, Γ) is a linear solvable pair with Δ maximal, then (Δ, Γ) is automatically a diagonalizable linear solvable pair by Theorem 4.6.

6.1. **Poisson derivations for diagonalizable linear solvable pairs.** In this section, we study Poisson derivations for diagonalizable linear solvable pairs. We have seen in Lemma 2.15 that Δ is always a Poisson derivation. In the linear case, the so-called *Eulerian derivation* $\mathcal{E}$ sending X_i to itself for all $0 \leqslant i \leqslant n$ is also a Poisson derivation. Our aim is to recover Δ, or more precisely the conjugacy class of Δ (acting on A_1), from the algebraic structure of A. We start with the case of a maximal Jordan block and then generalizes to every Δ under certain generic conditions on Γ.

6.1.1. *Poisson derivation for maximal Jordan block.* In this paragraph we suppose that $A = A_{(\Delta,\Gamma)}$, where (Δ, Γ) is a linear solvable pair with Δ a maximal Jordan block. We will show that there are no other linear Poisson derivations apart from those in the $\Bbbk$-linear span of Δ and $\mathcal{E}$.

Lemma 6.7. *Assume that $n \geqslant 2$ or that $n = 1$ and $a \neq 0$. For any $\delta \in \mathrm{P.Der}_{\mathrm{gr}}(A)$ there exists a scalar λ such that $\delta(X_0) = \lambda X_0 \in \langle X_0 \rangle$.*

Proof. First if $a = 0$ (so $n \geqslant 2$) then X_0 is, up to scalar multiple, the unique Poisson central element in A_1 thanks to [LS19, Proposition 3.1(2)]. The result follows since $\delta(X_0)$ is also a Poisson central element in A_1, see Remark 2.16.

Now assume $a \neq 0$ and let $\delta = (u_{ij}) \in M_{n+1}(\Bbbk)$ so that for all $0 \leqslant \ell \leqslant n$ we have $\delta(X_\ell) = \sum_{i=0}^n u_{i\ell} X_i$. Set $X_{-1} = 0$. By applying δ to the Poisson bracket $\{X_0, X_n\} = -aX_0X_{n-1}$ we obtain

$$\begin{aligned}\delta(\{X_0, X_n\}) &= -a\delta(X_0)X_{n-1} - aX_0\delta(X_{n-1}) \\ &= -aX_{n-1}\sum_{i=0}^n u_{i0}X_i - aX_0\sum_{i=0}^n u_{i,n-1}X_i,\end{aligned}$$

and

$$\begin{aligned}\{\delta(X_0), X_n\} + \{X_0, \delta(X_n)\} &= \sum_{i=0}^n u_{i0}\{X_i, X_n\} + \sum_{i=0}^n u_{in}\{X_0, X_i\} \\ &= \sum_{i=0}^n u_{i0}\big((a+n)X_{i-1}X_n - (a+i)X_iX_{n-1}\big) - a\sum_{i=0}^n u_{in}X_0X_{i-1}.\end{aligned}$$

For $0 < i < n$ by comparing the coefficients of X_iX_{n-1} we have $iu_{i0} = 0$, i.e. $u_{i0} = 0$. By comparing the coefficients of $X_{n-1}X_n$ we obtain $au_{n0} = 0$ so that $u_{n0} = 0$ since $a \neq 0$. Thus $\delta(X_0) = u_{00}X_0 \in \langle X_0 \rangle$. □

Theorem 6.8. *Assume that $n \geqslant 2$ or that $n = 1$ and $a \neq 0$. Then $\mathrm{P.Der}_{\mathrm{gr}}(A) = \Bbbk\mathcal{E} \oplus \Bbbk\Delta$. In particular the set of locally nilpotent derivations of $\mathrm{P.Der}_{\mathrm{gr}}(A)$ is the one dimensional subspace spanned by Δ.*

Proof. We proceed by induction on $n \geqslant 1$. First assume that $a \neq -(n-1)$. Let $n = 1$ (so that $a \neq 0$) and $\delta \in \mathrm{P.Der}_{\mathrm{gr}}(A)$. By Lemma 6.7 we know that $\delta(X_0) = \lambda X_0$ for a scalar λ. If $\delta(X_1) = \mu X_0 + \nu X_1$ for some scalars $\mu, \nu \in \Bbbk$, we have $\delta(\{X_0, X_1\}) = -2a\lambda X_0^2$ and $\{\delta(X_0), X_1\} + \{X_0, \delta(X_1)\} = -2a(\lambda + \nu)X_0^2$ so that $\lambda = \nu$. The result is proved since we have $\delta = \lambda\mathcal{E} + \mu\Delta$.

Let $n > 1$ and assume true the statement of the theorem for all algebras $A(n-1, b)$ with $b \neq -(n-2)$. Thanks to Lemma 6.7 there exists $\lambda \in \Bbbk$ such that $\delta(X_0) = \lambda X_0 \in \langle X_0 \rangle$. Therefore δ induces a graded Poisson derivation of the Poisson algebra $A/\langle X_0 \rangle \cong A(n-1, a+1)$, see [LS19, Proposition 3.7(1)] for the isomorphism. Since $a + 1 \neq -(n-2)$ we can apply the induction hypothesis to get that there exist $\mu, \mu', \nu_i \in \Bbbk$ such that

$$\delta(X_i) = \begin{cases} \mu X_i + \mu' X_{i-1} + \nu_i X_0 & 2 \leqslant i \leqslant n, \\ \mu X_1 + \nu_1 X_0 & i = 1. \end{cases}$$

To conclude the proof it remains to show that $\mu = \lambda$, that $\nu_1 = \mu'$ and that $\nu_i = 0$ for all $2 \leqslant i \leqslant n$ since then $\delta = \mu\mathcal{E} + \mu'\Delta$. If $a \neq 0$ then by applying δ to the Poisson brackets $\{X_0, X_1\}$ (resp. $\{X_0, X_2\}$) a straightforward computation yields $\mu = \lambda$ (resp. $\nu_1 = \mu'$). If $a = 0$ then by applying δ to the Poisson brackets $\{X_1, X_2\}$ a straightforward computation yields $\mu = \lambda$ and $\nu_1 = \mu'$. Let $2 < i \leqslant n$. On one hand we have

$$\begin{aligned} \delta(\{X_1, X_i\}) &= \delta\big((a+i)X_0X_i - (a+1)X_1X_{i-1}\big) \\ &= 2(a+i)\mu X_0X_i + (i-1)\mu' X_0X_{i-1} - (a+1)\nu_{i-1}X_0X_1 \\ &\quad - 2(a+1)\mu X_1X_{i-1} - (a+1)\mu' X_1X_{i-2} + (a+i)\nu_i X_0^2. \end{aligned}$$

On the other hand

$$\begin{aligned} \{\delta(X_1), X_i\} + \{X_1, \delta(X_i)\} = 2\mu\big((a+i)\mu X_0X_i - (a+1)X_1X_{i-1}\big) \\ + (i-1)\mu' X_0X_{i-1} - (a+1)\mu' X_1X_{i-2} + a\nu_i X_0^2. \end{aligned}$$

By comparing the coefficients of X_0^2 we get $i\nu_i = 0$, i.e. $\nu_i = 0$. If $i = 2$ a similar but simpler computation provides $\nu_2 = 0$.

It remains to deal with the case $a = -(n-1)$. We proceed again by induction on $n \geqslant 2$. It is only necessary to prove the initialization (for $A(2, -1)$) because the induction step proceeds exactly as in the case $a \neq -(n-1)$.

We have $\delta(X_0) = \lambda X_0$ and there exists $u, v, w, t, \nu_1, \nu_2 \in \Bbbk$ such that $\delta(X_1) = uX_2 + vX_1 + \nu_1 X_0$ and $\delta(X_2) = wX_2 + tX_1 + \nu_2 X_0$. Direct computations from the Poisson brackets $\{X_0, X_1\} = X_0^2$, $\{X_0, X_2\} = X_0X_1$ and $\{X_1, X_2\} = X_0X_2$ yield $u = w = \lambda$, $u = \nu_2 = 0$ and $t = \nu_1$ so that $\delta = \lambda\mathcal{E} + \nu_1\Delta$. □

6.1.2. *When Δ is not necessarily maximal.* We now assume that (Δ, Γ) is a diagonalizable linear solvable pair but we do not assume that Δ is a maximal Jordan block. This situation is more complicated as the following example illustrate.

Example 6.9. Let Δ be of Jordan type $(2, 2)$ and $\Gamma = \mathrm{diag}(a, a+1, b, b+1)$. We obtain the Poisson brackets

$$\begin{aligned} &\{X_0, X_1\} = -aX_0^2, \quad \{X_0, X_2\} = 0, \quad \{X_0, X_3\} = -aX_0X_2 \\ &\{X_1, X_2\} = bX_0X_2, \quad \{X_1, X_3\} = (b+1)X_0X_3 - (a+1)X_1X_2, \quad \{X_2, X_3\} = -bX_2^2. \end{aligned}$$

It is easy to verify that $\mathrm{P.Der}_{\mathrm{gr}}(A_{(\Delta,\Gamma)}) = \Bbbk\Delta \oplus \Bbbk\mathcal{E}_{01} \oplus \Bbbk\mathcal{E}_{23}$, where $\mathcal{E}_{01}$ is the derivation sending X_0 to X_0 and X_1 to X_1 and both X_2, X_3 to 0 (a similar definition applies to $\mathcal{E}_{23}$). Therefore $\dim \mathrm{P.Der}_{\mathrm{gr}}(A_{(\Delta,\Gamma)}) = 3$ which is 1 more the number of blocks of Δ.

Definition 6.10. Let (Δ, Γ) be a linear solvable pair. We say that (Δ, Γ) is generic or that Γ is generic if all the eigenvalues of Γ acting on A_1 are distinct and non zero. In particular a generic linear solvable pair is a diagonalisable linear solvable pair.

Recall from Definition 6.4 that without loss of generality we have $A_{(\Delta,\Gamma)} = A(\overline{n},\overline{a})$ where the derivation Δ can be chosen to be in canonical Jordan from (with $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_r$ denoting the size of the blocks of Δ) form and that Γ can be chosen diagonal (with the notation of Remark 6.5 for its eigenvalues).

Theorem 6.11. *Assume that (Δ,Γ) is a generic linear solvable pair with $r \geqslant 2$.*

(1) *If $n_1 = 1$ (i.e $\Delta = 0$), then $\mathrm{P.Der}_{\mathrm{gr}} A(\overline{n},\overline{a}) = \mathfrak{gl}_{n+1}(\Bbbk)$.*

(2) *If $\overline{n} = (2,1,\ldots,1)$, then $\mathrm{P.Der}_{\mathrm{gr}} A(\overline{n},\overline{a})$ is the $2n$-dimensional solvable non nilpotent Lie algebra $\mathscr{L}$ given by*

$$\mathscr{L} = \left\{ \delta_{u,v,c_2,\ldots,c_n,d_2,\ldots,d_n} := \begin{pmatrix} u & v & & & \\ 0 & u & & & \\ 0 & c_2 & d_2 & & \\ \vdots & \vdots & & \ddots & \\ 0 & c_n & & & d_n \end{pmatrix}, \quad u,v,c_2,\ldots,c_n,u_2,\ldots,u_n \in \Bbbk \right\}$$

Moreover, we have $Z\mathscr{L} = \Bbbk\mathrm{id} \oplus \Bbbk\Delta$ and $[\mathscr{L},\mathscr{L}] = \{\delta_{0,0,c_2,\ldots,c_n,0,\ldots,0}, c_2,\ldots,c_n \in \Bbbk\}$. The ascending and descending central series of $\mathscr{L}$ are stationary from rank 1.

(3) *In the other case, $\mathrm{P.Der}_{\mathrm{gr}} A(\overline{n},\overline{a})$ is the $(r+1)$-dimensional commutative Lie algebra given by*

$$\Bbbk\Delta \oplus \bigoplus_{i=1}^{r} \Bbbk\mathcal{E}_i$$

where $\mathcal{E}_i$ is the block diagonal matrix where the diagonal block are 0 except the i^{th} which is id_{n_i}.

In particular, up to a scalar factor, Δ is the only locally nilpotent derivation in the center of $\mathrm{P.Der}_{\mathrm{gr}} A$, except when $\Delta = 0$.

Proof. (1). The case where $\Delta = 0$ is trivial.

(2) and (3). We now assume that $n_1 \geqslant 2$. Let $\delta \in \mathrm{P.Der}_{\mathrm{gr}} A(\overline{n},\overline{a})$. We first prove the following claim: if $n_i \geqslant 2$ for some i, then δ stabilizes all the blocks except possibly the i^{th}.

It is enough to consider the case $i = 1$. To prove the claim we compute $\delta(\{X_1, X_j\})$ for $j \geqslant n_1$ in two different ways and compare the coefficients of X_0X_ℓ. We set $\delta(X_j) = \sum_{i=0}^{n} u_{ij}X_i$ for all j.

It clearly suffices to consider the case of the second block i.e. $n_1 \leqslant j < n_1 + n_2$. Let us prove by induction on $j \in \{n_1,\ldots,n_1+n_2-1\}$ that $\delta(X_j) \in \bigoplus_{j=n_1}^{n_1+n_2-1} \Bbbk X_i$.

With $j = n_1$ we have $\{X_1, X_{n_1}\} = a_2X_0X_{n_1}$. On one hand we obtain $\delta(\{X_1, X_{n_1}\}) = a_2\delta(X_0)X_{n_1} + a_2X_0\delta(X_{n_1})$. On the other hand we have

$$\{\delta(X_1), X_{n_1}\} + \{X_1, \delta(X_{n_1})\} = a_2X_{n_1}\Delta(\delta(X_1)) + X_0\Gamma(\delta(X_{n_1})) - (a_1+1)X_1\Delta(\delta(X_{n_1})).$$

By comparing the coefficients of X_0X_ℓ for $\ell \neq 1, n_1$ we obtain $a_2u_{\ell n_1} = \lambda_\ell u_{\ell n_1}$. Since Γ is generic $\lambda_\ell \neq a_2$ which is the eigenvalue of Γ associated to X_{n_1} (we have $\ell \neq n_1$). The coefficient of X_0X_1 is $a_2u_{1n_1} = (a_1+1)u_{1n_1} - (a_1+1)u_{1n_1}$. Thus $\delta(X_{n_1})$ is collinear to X_{n_1} (which is much more than needed) since $a_2 \neq 0$.

We now assume that the result is true for $\delta(X_{j-1}) \in \bigoplus_{i=n_1}^{n_1+n_2-1} \Bbbk X_i$ (with $n_1 < j < n_1+n_2$). We have $\{X_1, X_j\} = \lambda_j X_0X_j - \lambda_1X_1X_{j-1}$. By applying δ to this Poisson bracket we obtain on one hand

$$\delta(\{X_1, X_j\}) = \lambda_j\delta(X_0)X_j + \lambda_jX_0\delta(X_j) - \lambda_1\delta(X_1)X_{j-1} - \lambda_1X_1\delta(X_{j-1}).$$

On the other hand we have

$$\{\delta(X_1), X_j\} + \{X_1, \delta(X_j)\} = \lambda_jX_j\Delta(\delta(X_1)) - \Gamma(\delta(X_1))X_{j-1} + X_0\Gamma(\delta(X_j)) - \lambda_1X_1\Delta(\delta(X_j)).$$

By computing the coefficient of X_0X_ℓ for $\ell \notin \{1, n_1,\ldots,n_1+n_2-1\}$, the induction hypothesis gives that $\lambda_ju_{\ell j} = \lambda_\ell u_{\ell j}$. Hence $u_{\ell j} = 0$ since $\ell \neq j$ and Γ is generic. The coefficient of X_0X_1 gives the relation $\lambda_ju_{1j} = \lambda_1u_{1j} - \lambda_1u_{1j} = 0$ and the claim is proved.

We now split the argument in two cases:

(i) there are at least two blocks of size larger or equal to 2,
(ii) $n_i = 1$ for all $i > 1$ and $n_1 \geqslant 2$.

In case (i) by applying the claim to two different blocks of size larger or equal than 2 we see that δ must stabilize every blocks. In particular δ induces a Poisson derivation on each Poisson subalgebra B_i generated by the variables X_j corresponding to a block. It is clear that $B_i \cong A(n_i - 1, a_i)$ for each $1 \leqslant i \leqslant r$, so that the induced derivation has a form $\delta|_{B_i} = u\mathcal{E}_{n_i} + v\Delta|_{B_i}$ thanks to Theorem 6.8. Hence $\delta = \sum_{i=1}^{r} u_i\mathcal{E}_{n_i} + v_i\Delta|_{B_i}$ for some $u_i, v_i \in \Bbbk$. We will show that the v_i are all equal, thus proving assertion (3) in case (i). First note that there are nothing to check if the corresponding block is of size 1 since in that case $\Delta_{|B_i}$ is the zero map. Now if $n_i \geqslant 2$ then we compare the coefficients of $X_0X_{n_i}$ after applying δ to the Poisson bracket $\{X_1, X_{n_i+1}\}$. On one hand we have

$$\begin{aligned}\delta(\{X_1, X_{n_i+1}\}) &= \delta\big((a_{n_i}+1)X_0X_{n_i+1} - (a_1+1)X_1X_{n_i}\big)\\ &= \big(v_i(a_{n_i}+1) - v_1(a_1+1)\big)X_0X_{n_i} + \text{other terms without } X_0X_{n_1}\end{aligned}$$

whereas on the the other hand

$$\begin{aligned}\{\delta(X_1), X_{n_i+1}\} + \{X_1, \delta(X_{n_i+1})\} &= (u_1+u_i)\{X_1, X_{n_i+1}\} + v_1\{X_0, X_{n_i+1}\} + v_i\{X_1, X_{n_i}\}\\ &= (v_ia_{n_i} - v_1a_1)X_0X_{n_i} + \text{other terms without } X_0X_{n_1}.\end{aligned}$$

Thus we have $v_i = v_1$, as desired.

In case (ii) we first show that $\delta(X_i) \in \bigoplus_{j=0}^{n_1-1} \Bbbk X_j$ for all $0 \leqslant i \leqslant n_1 - 2$. Let $0 < i < n_1 \leqslant \ell \leqslant n$. We then have $\Delta(X_\ell) = 0$ and $\delta(X_\ell) = u_\ell X_\ell$. Hence

$$\delta(\{X_i, X_\ell\}) = \lambda_\ell \sum_{k=0}^{n} u_{k,i-1}X_kX_\ell + \lambda_\ell u_\ell X_{i-1}X_\ell$$

and

$$\{\delta(X_i), X_\ell\} + \{X_i, \delta(X_\ell)\} = \sum_{k=0}^{n_1-2} \lambda_\ell u_{k+1,i}X_kX_\ell + \lambda_\ell u_\ell X_{i-1}X_\ell.$$

By comparing the coefficients of X_kX_ℓ for all $k \geqslant n_1 - 1$ we obtain that $\lambda_\ell u_{k,i-1} = 0$ so that $u_{k,i-1} = 0$ as desired.

When $n_1 = 2$ the previous computation provides us $u_{10} = 0$ so that $\delta(X_0) = u_{00}X_0$. Moreover by looking at the Poisson bracket $\{X_0, X_1\}$ we obtain that $u_{00} = u_{11}$. No further restriction can be imposed on δ and we indeed obtain the form given in (2). The final assertions of (2) follow from simple computations.

When $n_1 \geqslant 3$ we can moreover prove that $\delta(X_{n_1-1}) \in \bigoplus_{j=0}^{n_1-1} \Bbbk X_j$. We showed that δ sends the Poisson subalgebra generated by $X_0, \ldots, X_{n_1-2}$ into the Poisson subalgebra generated by $X_0, \ldots, X_{n_1-1}$, hence

$$\begin{aligned}\delta(\{X_1, X_{n_1-1}\}) =& \delta(\lambda_{n_1-1}X_0X_{n_1-1} - \lambda_1X_1X_{n_1-2})\\ &\in\ \lambda_{n_1-1}X_0\delta(X_{n_1-1})\ +\ \Bbbk[X_0, \ldots, X_{n_1-1}]\end{aligned}$$

and

$$\{\delta(X_1), X_{n_1-1}\} + \{X_1, \delta(X_{n_1-1})\}\ \in\ \sum_{k=0}^{n} u_{k,n_1-1}\lambda_kX_0X_k - a_1X_1\Delta(\delta(X_{n_1-1}))\ +\ \Bbbk[X_0, \ldots, X_{n_1-1}].$$

By comparing the coefficients of X_0X_k for $k \geqslant n_1$ we found that $(\lambda_k - \lambda_{n_1-1})u_{k,n_1-1} = 0$ so that $u_{k,n_1-1} = 0$ thanks to our genericity hypothesis. In conclusion, δ stabilizes the subalgebra generated by $X_0, \ldots, X_{n_1-1}$ which is isomorphic to $A(n_1 - 1, a_1)$ and the result follows from Theorem 6.8. This concludes the proof of (3). □

Corollary 6.12. *Let (Δ, Γ) is a diagonalizable linear solvable pair with Γ generic and $\Delta \neq 0$. Then Δ may be recovered from the algebraic structure of A since it is, up to a scalar, the only locally nilpotent derivation contained in the center of* $\mathrm{P.Der}_{\mathrm{gr}}A$.

6.1.3. *A remark for non necessarily diagonal* Γ. In general the commutant of Δ is not necessarily made of blocks corresponding to commutant of the Jordan blocks of Δ. We illustrate this fact in the case where Δ has Jordan blocks of size $(2,1)$. Then the general form for Γ is

$$\begin{pmatrix} a & b & c \\ 0 & a+1 & 0 \\ 0 & d & e \end{pmatrix}$$

In $A_{(\Delta,\Gamma)}$ and $R_{(\Delta,\Gamma)}$ we have

$$\begin{aligned} \{X_0, X_1\} &= -aX_0^2, & [X_0, X_1] &= -aX_0 * X_0, \\ \{X_0, X_2\} &= 0, & [X_0, X_2] &= 0, \\ \{X_1, X_2\} &= eX_0X_2 + cX_0^2, & [X_1, X_2] &= eX_0 * X_2 + cX_0 * X_0. \end{aligned}$$

Since both X_0 and X_2 are in the kernel of Δ we remark that $\Gamma(X_1)$ does not appear in the formulae of the Poisson bracket and of the product $*$. Therefore, without loss of generality we can assume that $b = d = 0$. In particular Γ is triangular and its eigenvalues are $a, a+1$ and e.

Case 1. If $e \neq a$ and $e \neq a+1$ then Γ is diagonalizable and one can assume $c = 0$ by Proposition 6.3.

Case 2. Assume that $e = a+1$. The solvable pair (Δ, Γ) is conjugate to the diagonal solvable pair $(\Delta, \mathrm{diag}(a, a+1, a+1))$ via the automorphism of A fixing X_0 and X_1 and sending X_2 to $X_2 + cX_0$. Again we can assume Γ diagonalizable.

Case 3. Assume then that $e = a$. If $c \neq 0$ then Γ is not diagonalizable so that the solvable pair (Δ, Γ) is not diagonalizable (Remark 6.2). Note that the case $c \neq 0$ is isomorphic to the case $c = 1$ via the change of variable $X_1' = \frac{1}{c}X_1$.

As a corollary of the following lemma we obtain that Case 3 above with $c \neq 0$ cannot be realized by a solvable diagonalizable pair (Δ', Γ'). Recall that the Lie algebra of linear Poisson derivations and its subset of locally nilpotent derivations is an invariant of homogeneous Poisson polynomial algebras by Proposition 4.1.

Lemma 6.13. *Consider the Poisson algebra $A = A_{(\Delta,\Gamma)}$ from Case 3. If $a \neq 0$ then the space of linear Poisson derivations is of dimension* 4

$$\mathrm{P.der}_{\mathrm{gr}}(A) = \left\{ \begin{pmatrix} u_{00} & u_{01} & u_{02} \\ 0 & u_{00} & 0 \\ 0 & u_{21} & u_{00} \end{pmatrix} \right\}$$

In particular $\mathrm{P.der}_{\mathrm{gr}}(A)$ *contains a* 3*-dimensional subspace of locally nilpotent derivations. If $a = 0$ then the space of linear Poisson derivations is of dimension* 6

$$\mathrm{P.der}_{\mathrm{gr}}(A) = \left\{ \begin{pmatrix} u_{00} & u_{01} & u_{02} \\ 0 & u_{11} & u_{12} \\ 0 & u_{21} & 2u_{00} - u_{11} \end{pmatrix} \right\}$$

Proof. The computational proof is omitted. □

Now if $A_{(\Delta,\Gamma)}$ is realized by the solvable diagonalizable pair (Δ', Γ'), then by Theorem 6.8, we obtain that Δ' is of Jordan type $(2,1)$, since otherwise $\mathrm{P.der}_{\mathrm{gr}}(A)$ is of dimension 2 or 9. Consider $\Gamma' = \mathrm{diag}\,(\alpha, \alpha+1, \varepsilon)$ (with $\alpha \neq 0$ or $\varepsilon \neq 0$ otherwise A would be Poisson commutative). If $\alpha \neq \varepsilon$ we compute that

$$\mathrm{P.der}_{\mathrm{gr}}(A) = \left\{ \begin{pmatrix} u_{00} & u_{01} & 0 \\ 0 & u_{00} & 0 \\ 0 & u_{21} & u_{22} \end{pmatrix} \right\}$$

is 4 dimensional but contains a 2 dimensional subspace of locally nilpotent Poisson derivation, contrary to $A_{(\Delta,\Gamma)}$. If $\alpha = \varepsilon$ then we compute that

$$\mathrm{P.der}_{\mathrm{gr}}(A) = \left\{ \begin{pmatrix} u_{00} & u_{01} & u_{02} \\ 0 & u_{00} & 0 \\ 0 & u_{21} & u_{22} \end{pmatrix} \right\}$$

is of dimension 5, contrary to $A_{(\Delta,\Gamma)}$ as desired.

Remark 6.14. The fact that the case $\Gamma = \begin{pmatrix} a & 0 & 0 \\ 0 & a+1 & 0 \\ 0 & d & a+1 \end{pmatrix}$ with $d \neq 0$ can be realized by a diagonal solvable pair is a consequence of the fact that $\Gamma(X_1)$ never appears in the formulae. This is because all the other variables are in the kernel of Δ. Therefore we should expect more non diagonalizable cases when the rank of Δ is a least 2.

7. Diagonalizable linear solvable pair: the algebra $R_{(\Delta,\Gamma)}$

The aim of this section is to give a presentation for the algebra R in the case of a diagonalizable linear solvable pair (Δ, Γ). According to Proposition 6.3 we can choose a basis $(X_0, \dots, X_n)$ of A_1 such that Δ is in canonical Jordan form (we denote its Jordan type by $\overline{n} = (n_1, \dots, n_r)$ with $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_r$ and $n+1 = \sum_{i=1}^{r} n_i$) and Γ is diagonal of the form $\Gamma = \mathrm{diag}\,(a_1, \dots, a_1 + n_1 - 1, a_2, \dots, a_2 + n_2 - 1, \dots, a_r, \dots, a_r + n_r - 1)$.

For any $k \in \{1, \dots, r\}$ and $0 \leqslant j < n_k$ we denote by $Y_{j,k} = X_{n_1+\cdots+n_{k-1}+j}$ the j^{th} variable of the k^{th} block. Fix $k \in \{1, \dots, r\}$ and $0 \leqslant j < n_k$. Thanks to equation (1.2) we have for any $f \in R$ that

$$f * Y_{j,k} = \sum_{\ell \geq 0} \Delta^{\ell}(f) \binom{\Gamma}{\ell}(Y_{j,k}) = \sum_{\ell \geqslant 0} \Delta^{\ell}(f) \binom{a_k + j}{\ell} Y_{j,k} = \phi_{a_k+j}(f) Y_{j,k}\,.$$

In particular, for any $k' \in \{1, \dots, r\}$ and $0 \leqslant j' < n_{k'}$ we obtain the relations

$$\phi_{-a_k-j}(Y_{j',k'}) * Y_{j,k} = Y_{j',k'} Y_{j,k} = \phi_{-a_{k'}-j'}(Y_{j,k}) * Y_{j',k'}\,.$$

These relations can be rewritten as

$$\sum_{\ell=0}^{j'} \binom{-a_k - j}{\ell} Y_{j'-\ell,k'} * Y_{j,k} = \sum_{\ell=0}^{j} \binom{-a_{k'} - j'}{\ell} Y_{j-\ell,k} * Y_{j',k'} \tag{7.1}$$

and we have the following proposition.

Proposition 7.2. *The algebra R is given by generators $Y_{j,k}$ for $1 \leqslant k \leqslant r$ and $1 \leqslant j \leqslant n_k$, and the homogeneous relations (7.1).*

Proof. We first show that the set $\mathcal{G} = \{Y_{j,k} \mid 1 \leqslant k \leqslant r, 1 \leqslant j \leqslant n_k\}$ is a generating set for R. From Proposition C.1, $\mathrm{gr}^{\widetilde{\varepsilon}}(A)$ is generated by the image of the $Y_{j,k}$ in $\mathrm{gr}^{\widetilde{\varepsilon}}(A)$. But from Proposition 4.10, $\mathrm{gr}^{\widetilde{\varepsilon}}(A) = \mathrm{gr}^{\widetilde{\varepsilon}}(R)$. Hence $\mathcal{G}$ is a set of generators for R as desired.

Let us denote by T the algebra given by generators $Y_{j,k}$ for $1 \leqslant k \leqslant r$ and $1 \leqslant j \leqslant n_k$, and relations (7.1). From the preceding argument R is a quotient of T. Moreover note that relations (7.1) are homogeneous with respect to the degree $\deg(Y_{j,k}) = 1$ for all j, k. Hence the natural map from T onto R is graded. We endow the set $\mathcal{G}$ with the order given by $Y_{j,k} \leqslant Y_{j',k'}$ if $k < k'$, or if $k = k'$ and $j \leqslant j'$. This induces an order on the monomials in the $Y_{j,k}$'s. The relations (7.1) can be rewritten as

$$Y_{j,k} * Y_{j',k'} = Y_{j',k'} * Y_{j,k} + \text{lower monomials}$$

for all $(j,k), (j',k')$. Hence every element of T is a linear combination of monomials $Y_{j_1,k_1} \cdots Y_{j_s,k_s}$ in the $Y_{j,k}$'s where $Y_{j_\ell,k_\ell} \leqslant Y_{j_{\ell+1},k_{\ell+1}}$ for all $\ell \in \{1, \dots, s-1\}$. In particular, the dimension of the homogeneous component of degree d of T is smaller than the one of R. But since R is a graded quotient of T the canonical map from T onto R must be one-to-one. □

Example 7.3. Let Δ be of Jordan type $(2,2)$ and let $\Gamma = \mathrm{diag}(a, a+1, b, b+1)$ be as in Example 6.9. Thanks to Proposition 7.2 we obtain the following complete set of relations between the generators of $R = R_{(\Delta,\Gamma)}$:

$$\begin{aligned}
&X_0 * X_1 - X_1 * X_0 = -aX_0 * X_0, \qquad X_2 * X_3 - X_3 * X_2 = -bX_2 * X_2,\\
&X_0 * X_2 - X_2 * X_0 = 0,\\
&X_0 * X_3 - X_3 * X_0 = -aX_0 * X_2, \qquad X_1 * X_2 - X_2 * X_1 = bX_0 * X_2\\
&X_1 * X_3 - X_3 * X_1 = (b+1)X_0 * X_3 - (a+1)X_2 * X_1\\
&\qquad\qquad = (b+1)X_0 * X_3 - (a+1)X_1 * X_2 + (a+1)bX_0 * X_2
\end{aligned}$$

8. Unimodarity and Calabi-Yau property

In this section we study the unimodularity of the Poisson algebra $A = A_{(\Delta,\Gamma)}$ in the polynomial case and determine when the algebra $R = R_{(\Delta,\Gamma)}$ is Calabi-Yau. Whereas the unimodularity of A is completely characterized by Proposition 8.1, we only compute the Nakayama automorphism of R in the case of a generic diagonalizable linear solvable pair (Theorem 8.9). This result generalizes [LS19, Theorem 4.16].

8.1. **Unimodularity of $A_{(\Delta,\Gamma)}$ in the polynomial case.** Unimodular Poisson algebras are thought to be Poisson analogous of Calabi-Yau algebras, see [Do09]. Recall that a polynomial Poisson algebra $A = \Bbbk[X_0, \ldots, X_n]$ is called unimodular if its modular derivation is zero, where the modular derivation m of A is given in the polynomial case by

$$m(f) = \sum_{k=0}^{n} \frac{\partial\{X_k, f\}}{\partial X_k}$$

for all $f \in A$, see [LWW15, Lemma 2.4].

Proposition 8.1. *Let $A = \Bbbk[X_0, \ldots, X_n]$ and (Δ, Γ) be a solvable pair on A. The modular derivation m of the Poisson algebra $A_{(\Delta,\Gamma)}$ is equal to $m = (1 - \mathrm{div}(\Gamma))\Delta$. In particular, as long as $\Delta \neq 0$, the Poisson algebra $A_{(\Delta,\Gamma)}$ is unimodular if and only if $\mathrm{div}(\Gamma) = 1$. When Γ is linear this is equivalent to say that $\mathrm{tr}(\Gamma) = 1$.*

Proof. Thanks to [LPV12, Equality 4.22 p.108] the modular derivation m of $A_{(\Delta,\Gamma)}$ is equal to minus the divergence of the Poisson bivector field $\pi = \Delta \wedge \Gamma$ of $A_{(\Delta,\Gamma)}$. Thanks to [LPV12, Proposition 4.16] and the fact that the divergence of a locally nilpotent derivation is zero ([VdE00, Proposition 1.3.51]) we have

$$\begin{aligned}
m &= -\mathrm{div}(\pi)\\
&= -\mathrm{div}(\Delta \wedge \Gamma)\\
&= -\big(\mathrm{div}(\Gamma)\Delta - \mathrm{div}(\Delta)\Gamma - [\Delta, \Gamma]\big)\\
&= -\mathrm{div}(\Gamma)\Delta + \Delta\\
&= (1 - \mathrm{div}(\Gamma))\Delta
\end{aligned}$$

as desired. For the assertion in the linear case, recall that $\mathrm{div}(\Gamma) = \sum_{i=0}^{n} \dfrac{\partial\Gamma(X_i)}{\partial X_i}$. □

We illustrate this result with several examples in the linear case.

Example 8.2. If (Δ, Γ) is a diagonalizable linear solvable pair (Δ, Γ), recall that

(1) its Jordan type is $\overline{n} = (n_1, \ldots, n_r)$ where $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_r$ are the size of the Jordan blocks of Δ acting on A_1 ;

(2) there exists a basis of A_1 and $\overline{a} = (a_1, \dots, a_r) \in \Bbbk^r$ such that Δ is in canonical Jordan form and Γ is diagonal of the form $\Gamma = \operatorname{diag}(a_1, \dots, a_1 + n_1 - 1, a_2, \dots, a_2 + n_2 - 1, \dots, a_r, \dots, a_r + n_r - 1)$

(3) we denote by $A(\overline{n}, \overline{a})$ the corresponding $A_{(\Delta,\Gamma)}$;

(4) we set $n + 1 = n_1 + \cdots + n_r$.

The parameter space of unimodular Poisson algebras $A(\overline{n}, \overline{a})$ is a $(r-1)$-dimensional affine subspace of $\Bbbk^r$ since the Poisson algebra $A(\overline{n}, \overline{a})$ is unimodular if and only if $\operatorname{tr}(\Gamma) = 1$ if and only if

$$\sum_{t=0}^{r-1} n_{t+1} a_{t+1} + \binom{n_{t+1}}{2} = 1.$$

Example 8.3. If $r = 1$ (a single Jordan block) then A is unimodular if and only if

$$(n+1)a + \binom{n+1}{2} - 1 = 0 \iff a = \frac{-(n+2)(n-1)}{2(n+1)}$$

as expected from [LS19], since $n + 1 = n_1 + \cdots + n_r = n_1$.

Example 8.4. Let Δ be of Jordan type $(2, 1)$ and Γ non necessarily diagonalizable. The general form for Γ is $\begin{pmatrix} a & b & c \\ 0 & a+1 & 0 \\ 0 & d & e \end{pmatrix}$, hence the Poisson algebra $A_{(\Delta,\Gamma)}$ is unimodular if and only if $2a + e = 0$.

8.2. **Calabi-Yau property for $R_{(\Delta,\Gamma)}$ in the generic diagonalizable case.** The aim of this section is to compute the Nakayama automorphism of R in order to determine when R is a Calabi-Yau algebra. We denote by $R^e = R \otimes_{\Bbbk} R^{\mathrm{op}}$ the enveloping algebra algebra of R.

Definition 8.5. We say that R is *skew Calabi-Yau* (or *skew CY*) if

(i) R is *homologically smooth*: R has a finite projective resolution as a left R^e-module such that each term is finitely generated;

(ii) There are an algebra automorphism μ of R and an integer d such that

$$\operatorname{Ext}^i_{R^e}(R, R^e) \cong \begin{cases} 0 & \text{if } i \neq 0 \\ {}^1R^\mu & \text{if } i = d \end{cases}$$

where ${}^1R^\mu$ is the R-bimodule which is isomorphic to R as a $\Bbbk$-vector space and such that $r \cdot s \cdot t = rs\mu(t)$.

If R is skew CY, the automorphism μ is called the *Nakayama automorphism* of R. If μ is inner, then R is *Calabi-Yau* or *CY*.

By [RRZ14, Lemma 1.2], any AS-regular connected graded algebra is skew CY. In particular, the algebras $R_{(\Delta,\Gamma)}$ are skew CY in the linear case thanks to Theorem 5.5.

Proposition 8.6. *Let (Δ, Γ) be a linear solvable pair such that A is not Poisson commutative (see Corollary 4.9). Let Φ be an automorphism of R compatible with $\widetilde{\varepsilon}$ such that $\overline{\Phi} = \mathrm{id}$ (see Lemma 2.26). Assume that there exists a strongly normal element N of R whose eigenvalue with respect to Γ is nonzero. Then Φ commute with Δ and ϕ_c for every $c \in \Bbbk$.*

Proof. The proof decomposes into three steps. The first step consists to show that Φ commutes with one Φ_λ by considering the normal element $N \in R$. In the second step, we deduce from the first that Φ commutes with Δ by considering that Φ commutes with $\Phi_{k\lambda}$ for $k \in \mathbb{N}$. The third step is easy : since Φ commute with Δ, it commutes with every formal power series in Δ. Hence with ϕ_c for every $c \in \Bbbk$.

First step. Consider $N \in \ker \Delta \cap \ker(\Gamma - \lambda \mathrm{id})$ with $\lambda \neq 0$. Then N is a normal element, hence $\Phi(N)$ is too. Since every normal element in R is strongly normal thanks to Proposition 5.7, we deduce that there exists $\mu \in \Bbbk$ such that $\Gamma(\Phi(N)) = \mu\Phi(N)$. Since Lemma 2.28 and Lemma 2.30 extends with the same proof to the filtration $\widetilde{\epsilon}$ and the automophism associated to the normal element $\Phi(N)$ is

$\Phi \circ \phi_\lambda \circ \Phi^{-1} = \phi_\mu$, we then deduce that the Poisson derivation associated to N is $\mu\overline{\Delta} = \overline{\Phi} \circ (\lambda\Delta) \circ \overline{\Phi}^{-1}$ (where the $\overline{\cdot}$ notation is related to the filtration $\widetilde{\epsilon}$). However we have $\overline{\Phi} = \mathrm{id}$, hence $\mu\overline{\Delta} = \lambda\overline{\Delta}$. Since $\Delta \neq 0$ and then $\overline{\Delta} \neq 0$, we obtain $\mu = \lambda$. This shows that Φ and ϕ_λ commute.

Second step. For $f \in R$, set n such that $\Delta^{n+1}(f) = 0$ and $\Delta^{n+1}(\Phi(f)) = 0$. The aim is to write $\Delta(f)$ as a linear combination of $(\phi_{k\alpha}(f))_{0\leqslant k\leqslant n}$. Since $\lambda \neq 0$, the matrix of binomial coefficients

$$A = \left(\binom{\lambda k}{\ell}\right)_{0\leqslant k,\ell\leqslant n}$$

is invertible (see Lemma A.8). Hence there exists $(\alpha_0, \ldots, \alpha_n) \in \Bbbk^{n+1}$ such that

$$A \begin{pmatrix} \alpha_0 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

We then deduce that $\Delta(f) = \sum_{k=0}^{n} \alpha_k \phi_{k\lambda}(f)$ and $\Delta(\Phi(f)) = \sum_{k=0}^{n} \alpha_k \phi_{k\lambda}(\Phi(f))$. But Φ commutes with ϕ_λ, therefore it commutes with $\phi_{k\lambda} = {\phi_\lambda}^k$ for any $k \in \{0, \ldots, n\}$. And the preceding equalities show that $\Phi(\Delta(f)) = \Delta(\Phi(f))$. □

Remark 8.7. When the solvable pair (Δ, Γ) is not necessarily linear, the preceding proposition can be adapted with the following statement. Let Φ be an automorphism of R compatible with ε and such that $\overline{\Phi} = \mathrm{id}$. Assume that every normal element of R is strongly normal and that there exists a strongly normal element N of R whose eigenvalue with respect to Γ is nonzero. Then Φ commute with Δ and ϕ_c for every $c \in \Bbbk$.

The preceding proposition applies in particular to the Nakayama automorphism of R as detailed in next example.

Example 8.8. Let (Δ, Γ) be a linear solvable pair. From Proposition C.1, $\mathrm{gr}^{\widetilde{\epsilon}}(A)$ is a polynomial algebra, hence we can apply [WZ01, Theorem 5.7] to get that the Nakayama automorphism μ of R is compatible with $\widetilde{\epsilon}$ and verifies $\overline{\mu} = \mathrm{id}$ and $\overline{(\mu - \mathrm{id})} = (1 - \mathrm{tr}(\overline{\Gamma}))\overline{\Delta} = (1 - \mathrm{tr}(\Gamma))\overline{\Delta}$.

Theorem 8.9. *Assume that (Δ, Γ) is a generic diagonalizable linear solvable pair on $A = k[X_0, \ldots, X_n]$ (see Definition 6.10). Then the Nakayama automorphism of R is $\phi_{1-\mathrm{tr}(\Gamma)}$.*

Proof. Let μ be the Nakayama automorphism of R. It is a graded automorphism for the standard degree. Since $\ker\Delta$ is stable by Γ and Γ is diagonalizable, it induces a diagonalizable endomorphism of $\ker\Delta$. Hence there exists a strongly normal element N whose eigenvalue with respect to Γ is nonzero since 0 is not an eigenvalue of Γ. From Proposition 8.6 and Example 8.8, μ commute with Δ and with ϕ_c for all $c \in \Bbbk$. In particular, Proposition 5.7 shows that every strongly normal element of A_1 is sent by μ to a strongly element of A_1 with the same eigenvalue with respect to Γ.

Let us consider a basis $(X_0, \ldots, X_n)$ of A_1 such that Δ is in canonical Jordan form of type $\overline{n} = (n_1, \ldots, n_r)$ and Γ is diagonal (see Proposition 6.3). For $1 \leqslant k \leqslant r$ and $0 \leqslant i \leqslant n_r - 1$ we denote by $Y_{i,k} = X_{i+n_1+\cdots+n_{k-1}}$ the i^{th} element of the k^{th} block. In particular $\ker\Delta$ is the linear span of the $Y_{0,k}$ for $1 \leqslant k \leqslant r$.

Since the family $\mathcal{G} = \{Y_{j,k} \mid 1 \leqslant k \leqslant r, 1 \leqslant j \leqslant n_k\}$ is a generating set of R, it is enough to prove that μ and $\phi_{1-\mathrm{tr}(\Gamma)}$ coincide on $\mathcal{G}$.

Since $\overline{\mu} = \mathrm{id}$ (Example 8.8) and $\widetilde{\varepsilon}(Y_{0,k}) = 0$, we obtain $\mu(Y_{0,k}) = Y_{0,k} = \phi_{1-\mathrm{tr}(\Gamma)}(Y_{0,k})$. Since $\varepsilon(Y_{1,k}) = 1$, [WZ01, Theorem 5.7] and Proposition 8.1 show that $\mu(Y_{1,k}) = Y_{1,k} + (1 - \mathrm{tr}(\Gamma))\Delta(Y_{1,k}) = \phi_{1-\mathrm{tr}(\Gamma)}(Y_{1,k})$.

Therefore we have the desired equalities for every k such that $n_k \leqslant 2$. Assume now that $n_k \geqslant 3$. We prove by induction on $i \geqslant 2$ that $\mu(Y_{i,k}) = \phi_{1-\mathrm{tr}(\Gamma)}(Y_{i,k})$. Assume that $i \geqslant 1$ and $\mu(Y_{j,k}) = \phi_{1-\mathrm{tr}(\Gamma)}(Y_{j,k})$ for $j \leqslant i$. Since both μ and $\phi_{1-\mathrm{tr}(\Gamma)}$ commutes with Δ, we have

$$\Delta((\mu - \phi_{1-\mathrm{tr}(\Gamma)})(Y_{i+1,k})) = (\mu - \phi_{1-\mathrm{tr}(\Gamma)})(\Delta(Y_{i+1,k})) = (\mu - \phi_{1-\mathrm{tr}(\Gamma)})(Y_{i,k}) = 0\,.$$

Hence there exist scalars $\alpha_0, \dots, \alpha_r$ such that

$$\mu(Y_{i+1,k}) = \phi_{1-\mathrm{tr}(\Gamma)}(Y_{i+1,k}) + \sum_{\ell=0}^{r} \alpha_\ell Y_{0,\ell} \tag{8.10}$$

If $f \in R$ is an eigenvector for Γ associated to the eigenvalue λ, then for every $g \in R$ we have $g*f = \phi_\lambda(g)f$. Set $\Gamma(Y_{0,k}) = \lambda Y_{0,k}$ where $\lambda \in \Bbbk$. Then $\Gamma(Y_{1,k}) = (\lambda+1)Y_{1,k}$ and we obtain $\Gamma(Y_{i+1,k}) = (\lambda+i+1)Y_{i+1,k}$ by an easy induction on $i \in \{0, \dots, n_k - 1\}$. Hence we obtain that

$$\phi_{-\lambda-1}(Y_{i+1,k}) * Y_{1,k} = Y_{i+1,k}Y_{1,k} = \phi_{-\lambda-i-1}(Y_{1,k}) * Y_{i+1,k} \tag{8.11}$$

For the sake of simplicity we set $c = 1 - \mathrm{tr}(\Gamma)$. By applying μ and ϕ_c to the relation (8.11) and by using the commutation of μ with ϕ_α, of ϕ_β with ϕ_α for every $\alpha, \beta \in \Bbbk$, and the relation (8.10), we obtain

$$\begin{aligned}
0 =& \phi_{-\lambda-1}(\mu(Y_{i+1,k})) * \mu(Y_{1,k}) - \phi_{-\lambda-i-1}(\mu(Y_{1,k})) * \mu(Y_{i+1,k}) \\
=& \phi_{-\lambda-1}\left(\phi_c(Y_{i+1,k}) + \sum_{\ell=0}^{r} \alpha_\ell Y_{0,\ell}\right) * \phi_c(Y_{1,k}) - \phi_{-\lambda-i-1}(\phi_c(Y_{1,k})) * \left(\phi_c(Y_{i+1,k}) + \sum_{\ell=0}^{r} \alpha_\ell Y_{0,\ell}\right) \\
=& \phi_{-\lambda-1}\left(\sum_{\ell=0}^{r} \alpha_\ell Y_{0,\ell}\right) * \phi_c(Y_{1,k}) - \phi_{-\lambda-i-1}(\phi_c(Y_{1,k})) * \sum_{\ell=0}^{r} \alpha_\ell Y_{0,\ell}.
\end{aligned}$$

By applying ϕ_{-c} to the last equality, we obtain the relation

$$\left(\sum_{\ell=0}^{r} \alpha_\ell Y_{0,\ell}\right) * Y_{1,k} = \phi_{-\lambda-i-1}(Y_{1,k}) * \sum_{\ell=0}^{r} \alpha_\ell Y_{0,\ell}$$

since $f = \sum_{\ell=0}^{r} \alpha_\ell Y_{0,\ell} \in \ker \Delta$ and so $\phi_\alpha(f) = f$ for all $\alpha \in \Bbbk$. By using the expression of the product $*$, this last relation can be rewritten as $Y_{0,k} \sum_{\ell=0}^{r} \alpha_\ell(\lambda_\ell - (\lambda + i + 1))Y_{0,\ell} = 0$ where λ_ℓ is the eigenvalue for Γ associated to the eigenvector $Y_{0,\ell}$. Since Γ has distinct eigenvalues, we obtain that $\alpha_\ell = 0$ for all ℓ and $\mu(Y_{i+1,k}) = \phi_{1-\mathrm{tr}(\Gamma)}(Y_{i+1,k})$. This concludes the proof. □

Corollary 8.12. *Let (Δ, Γ) be a generic diagonalizable linear solvable pair on $A = k[X_0, \dots, X_n]$. Then A is unimodular if and only if R is Calabi-Yau if and only if* $\mathrm{tr}(\Gamma) = 1$.

Proof. Since R is a connected graded algebra, every inner automorphism is trivial. Hence R is Calabi-Yau if and only if its Nakayama automorphism is trivial. The result follows from Proposition 8.1 and Theorem 8.9 since $\Phi_c = \mathrm{id}$ if and only if $c = 0$. □

Appendix A. Combinatorial relations

Definition A.1. Let B be a $\Bbbk$-algebra (associative) and $b \in B$. For $k \in \mathbb{N}$, we define $\binom{b}{k} = \frac{b(b-1)\cdots(b-k+1)}{k!}$ and for $k \in \mathbb{Z}$ with $k < 0$, we set $\binom{b}{k} = 0$.

Lemma A.2. *Let B be a $\Bbbk$-algebra, $a, b \in B$ with $ab = ba$. For any $n \geqslant 0$ we have*

$$\binom{a}{n} + \binom{a}{n-1} = \binom{a+1}{n} \tag{A.3}$$

$$\sum_{\ell=0}^{n} \binom{a}{\ell}\binom{b}{n-\ell} = \binom{a+b}{n} \qquad \textit{(Chu-Vandermonde identity)} \tag{A.4}$$

$$\binom{a}{n} = (-1)^n \binom{n-a-1}{n} \tag{A.5}$$

We need a better understanding of commutation relations in the enveloping algebra of the two dimensional solvable Lie algebra spanned by Δ and Γ.

Lemma A.6. *Let B be a $\Bbbk$-algebra (associative) and $\Gamma, \Delta \in B$ such that $[\Delta, \Gamma] = \Delta$. For any $i \geqslant 0$ and $k \geqslant 0$ we have*

$$\left[\Delta, \binom{\Gamma}{k}\right] = \binom{\Gamma}{k-1}\Delta, \qquad \left[\Delta^i, \binom{\Gamma}{k}\right] = \sum_{\ell=1}^{i} \binom{i}{\ell}\binom{\Gamma}{k-\ell}\Delta^i$$

Proof. The first assertion is clear for $k = 0$. Let $k \geqslant 1$. Notice that $\Delta(\Gamma - u) = (\Gamma - (u-1))\Delta$ for any $u \in \Bbbk$. Hence one has

$$\begin{aligned} k!\left[\Delta, \binom{\Gamma}{k}\right] &= \big(\Delta\Gamma(\Gamma-1)\cdots(\Gamma-(k-1)) - \Gamma(\Gamma-1)\cdots(\Gamma-(k-1))\Delta\big) \\ &= \big((\Gamma+1)\Gamma\cdots(\Gamma-(k-2))\Delta - \Gamma(\Gamma-1)\cdots(\Gamma-(k-1))\Delta\big) \\ &= \Gamma\cdots(\Gamma-(k-2))(\Gamma+1-(\Gamma-(k-1)))\Delta \\ &= k\Gamma\cdots(\Gamma-(k-2))\Delta \\ &= k(k-1)!\binom{\Gamma}{k-1}\Delta \\ &= k!\binom{\Gamma}{k-1}\Delta \end{aligned}$$

and the result is proved.

We now prove the second assertion by induction on $i \geqslant 0$. The case $i = 0$ is clear. The initialization $i = 1$ has just been proved. Assume that the result is true for some $i \geqslant 1$ and all $k \geqslant 0$. Then

$$\begin{aligned} \left[\Delta^{i+1}, \binom{\Gamma}{k}\right] &= \Delta^i\Delta\binom{\Gamma}{k} - \binom{\Gamma}{k}\Delta^{i+1} \\ &= \Delta^i\left(\binom{\Gamma}{k}\Delta + \binom{\Gamma}{k-1}\Delta\right) - \binom{\Gamma}{k}\Delta^{i+1} \\ &= \left[\Delta^i, \binom{\Gamma}{k}\right]\Delta + \Delta^i\binom{\Gamma}{k-1}\Delta \\ &= \left(\sum_{\ell=1}^{i}\binom{i}{\ell}\binom{\Gamma}{k-\ell} + \sum_{\ell=0}^{i}\binom{i}{\ell}\binom{\Gamma}{k-\ell-1}\right)\Delta^{i+1} \\ &= \sum_{\ell=1}^{i+1}\binom{i+1}{\ell}\binom{\Gamma}{k-\ell}\Delta^{i+1}. \end{aligned}$$

□

Remark A.7. The second relation of Lemma A.6 can be rewritten

$$\Delta^i\binom{\Gamma}{k} = \sum_{\ell=0}^{i}\binom{i}{\ell}\binom{\Gamma}{k-\ell}\Delta^i = \sum_{\ell=0}^{\min(i,k)}\binom{i}{\ell}\binom{\Gamma}{k-\ell}\Delta^i = \sum_{\ell=0}^{k}\binom{i}{\ell}\binom{\Gamma}{k-\ell}\Delta^i = \sum_{\ell=0}^{k}\binom{i}{k-\ell}\binom{\Gamma}{\ell}\Delta^i$$

Indeed, for $\ell > k$, $\binom{\Gamma}{k-\ell} = 0$ and for $\ell > i$, $\binom{i}{\ell} = 0$.

We finish this combinatorial appendix by the computation of the following determinant which will be useful for the determination of the Nakayama automorphism of R.

Lemma A.8. *Let $\Bbbk$ be a field of characteristic 0 and A be a $\Bbbk$-algebra. For $a \in A$ and $n \in \mathbb{N}$ set*

$$M(a) = \left(\binom{ka}{\ell}\right)_{0\leqslant k,\ell\leqslant n} \in M_{n+1}(A)$$

and $m(a) = \det(M(a))$. Then

$$m(a) = a^{n(n+1)/2}.$$

Proof. We can write $\binom{ka}{\ell} = \sum_{i=0}^{\ell} c_{i\ell}(ka)^i$ with $c_{i\ell} \in \mathbb{Q}$. Set $c_{i\ell} = 0$ for $\ell < i \leqslant n$. We have $M(a) = V(a)C$ where $C = (c_{k\ell})_{0\leqslant k,\ell\leqslant n} \in M_{n+1}(\mathbb{Q})$ and $V(a) = ((ka)^\ell)_{0\leqslant k,\ell\leqslant n} \in M_{n+1}(A)$. Since C is an upper triangular matrix with diagonal coefficients $c_{\ell\ell} = 1/\ell!$ and $V(a)$ is a Vandermonde matrix, we obtain the desired equality. □

Appendix B. Tensor product of Poisson algebras

Let A and B be Poisson algebras. The tensor product $A \otimes B$ is a Poisson algebra for the Poisson bracket given by

$$\{a \otimes b, a' \otimes b'\}_{A\otimes B} = \{a, a'\}_A \otimes bb' + aa' \otimes \{b, b'\}_B$$

for $a, a' \in A$ and $b, b' \in B$. A derivation δ of A extends to a derivation $\widehat{\delta}$ of $A \otimes B$ by setting

$$\widehat{\delta}(a \otimes b) = \delta(a) \otimes b$$

for $a \in A$ and $b \in B$. Moreover, if δ is a Poisson derivation of A then $\widehat{\delta}$ is a Poisson derivation of $A \otimes B$. The same is true for a (Poisson) derivation of B with the obvious modification.

Appendix C. Symmetric Algebra of a filtered vector space

Let us consider $V = \cup_{i\in\mathbb{N}} V_i$ a filtration of the vector space V, that is to say, for every $i \in \mathbb{N}$, V_i is a subspace of V and $V_i \subseteq V_{i+1}$.

This filtration on V defines an algebra filtration on $T(V)$ (the tensor algebra of V) and $S(V)$ (the symmetric algebra of V) given by

$$T^n_\alpha = \sum_{\substack{(\alpha_1,\dots,\alpha_n)\in\mathbb{N}^n \\ \alpha_1+\cdots+\alpha_n=\alpha}} V_{\alpha_1} \otimes \cdots \otimes V_{\alpha_n} = \sum_{\substack{(\alpha_1,\dots,\alpha_n)\in\mathbb{N}^n \\ \alpha_1+\cdots+\alpha_n\leqslant\alpha}} V_{\alpha_1} \otimes \cdots \otimes V_{\alpha_n} \subseteq V^{\otimes n} \quad \text{and} \quad S^n_\alpha = \sigma(T^n_\alpha)$$

where $\sigma\colon T(V) \to S(V)$ is the canonical map.

Let us set $M_0 = V_0$ and for $i \geqslant 1$, we consider M_i a supplementary space of V_{i-1} in V_i. We thus get that $V_i = \bigoplus_{j\leqslant i} M_j$ and $V = \bigoplus_{i\in\mathbb{N}} M_i$ can be seen as a graded vector space which we denote by $\mathrm{gr}(V)$. Using this graduation, we get a bigraduation on $T(V) = \bigoplus_{(n,\alpha)\in\mathbb{N}^2} T^{n,\alpha}$ and $S(V) = \bigoplus_{(n,\alpha)\in\mathbb{N}^2} S^{n,\alpha}$ given by

$$T^{n,\alpha} = \bigoplus_{\substack{(\alpha_1,\dots,\alpha_n)\in\mathbb{N}^n \\ \alpha_1+\cdots+\alpha_n=\alpha}} M_{\alpha_1} \otimes \cdots \otimes M_{\alpha_n} \quad \text{and} \quad S^{n,\alpha} = \bigoplus_{\substack{r\in\mathbb{N},\ (n_1,\dots,n_r)\in\mathbb{N}^r \\ n_1+\cdots+n_r=n \\ n_1\alpha_1+\cdots+n_r\alpha_r=\alpha}} S^{n_1}(M_{\alpha_1}) \otimes \cdots \otimes S^{n_r}(M_{\alpha_r})$$

See [Bo07, Algèbre chap III, §5.5 Proposition 7, §6.6 Proposition 10]. We set $T_\alpha = \bigoplus_{n\in\mathbb{N}} T^n_\alpha$ and $S_\alpha = \bigoplus_{n\in\mathbb{N}} S^n_\alpha$.

Proposition C.1. *The graded rings associated to the filtration $T(V) = \cup_{\alpha\in\mathbb{N}} T_\alpha$ and $S(V) = \cup_{\alpha\in\mathbb{N}} S_\alpha$ are isomorphic to $T(\mathrm{gr}(V))$ and $S(\mathrm{gr}(V))$.*

Proof. For $(\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, let us define $\pi^{\beta_1,\dots,\beta_n}_{\alpha_1,\dots,\alpha_n}\colon V_{\alpha_1} \otimes \cdots \otimes V_{\alpha_n} \to M_{\beta_1} \otimes \cdots \otimes M_{\beta_n}$ the map associated to the decomposition of $V_{\alpha_1} \otimes \cdots \otimes V_{\alpha_n}$ as a direct sum

$$V_{\alpha_1} \otimes \cdots \otimes V_{\alpha_n} = \bigoplus_{\substack{(\beta_1,\dots,\beta_n)\in\mathbb{N}^n \\ \forall\, i,\ 1\leqslant\beta_i\leqslant\alpha_i}} M_{\beta_1} \otimes \cdots \otimes M_{\beta_n}$$

Note that if there exists i such that $\beta_i > \alpha_i$ then $\pi^{\beta_1,\dots,\beta_n}_{\alpha_1,\dots,\alpha_n} = 0$. When $(\alpha_1, \dots, \alpha_n) = (\beta_1 \dots, \beta_n)$, we simply denote $\pi_{\alpha_1,\dots,\alpha_n}$ instead of $\pi^{\alpha_1,\dots,\alpha_n}_{\alpha_1,\dots,\alpha_n}$

Let us now define

$$\pi = \oplus \pi_{\alpha_1,\dots,\alpha_n} : \bigoplus_{\substack{(\alpha_1,\dots,\alpha_n)\in\mathbb{N}^n \\ \alpha_1+\cdots+\alpha_n=\alpha}} V_{\alpha_1}\otimes\cdots\otimes V_{\alpha_n} \longrightarrow \bigoplus_{\substack{(\alpha_1,\dots,\alpha_n)\in\mathbb{N}^n \\ \alpha_1+\cdots+\alpha_n=\alpha}} M_{\alpha_1}\otimes\cdots\otimes M_{\alpha_n}$$

We want to show that π induces a map from T^n_α to $T^{n,\alpha}$. It suffices to show that if

$$w = (w_{\alpha_1,\dots,\alpha_n})_{(\alpha_1,\dots,\alpha_n)} \in \bigoplus_{\substack{(\alpha_1,\dots,\alpha_n)\in\mathbb{N}^n \\ \alpha_1+\cdots+\alpha_n=\alpha}} V_{\alpha_1}\otimes\cdots\otimes V_{\alpha_n}$$

verifies $\sum_{(\alpha_1,\dots,\alpha_n)} w_{\alpha_1,\dots,\alpha_n} = 0 \in T^n(V)$ then $\pi(w) = 0$. But using the direct sum decomposition of the V_i we get that

$$T^n_\alpha = \bigoplus_{\substack{(\beta_1,\dots,\beta_n)\in\mathbb{N}^n \\ \beta_1+\cdots+\beta_n\leqslant\alpha}} M_{\beta_1}\otimes\cdots\otimes M_{\beta_n}$$

Hence the hypothesis $\sum_{(\alpha_1,\dots,\alpha_n)} w_{\alpha_1,\dots,\alpha_n} = 0 \in T^n(V)$ may be written for all $(\beta_1,\dots,\beta_n)$ such that $\beta_1+\cdots+\beta_n \leqslant \alpha$, we get $\sum_{(\alpha_1\dots,\alpha_n)} \pi^{\beta_1,\dots,\beta_n}_{\alpha_1,\dots,\alpha_n}(w_{\alpha_1,\dots,\alpha_n}) = 0$.

In particular, when $\beta_1+\cdots+\beta_n = \alpha$ and $(\alpha_1,\dots,\alpha_n) \neq (\beta_1,\dots,\beta_n)$ there exists i such that $\beta_i > \alpha_i$. Hence $0 = \sum_{(\alpha_1\dots,\alpha_n)} \pi^{\beta_1,\dots,\beta_n}_{\alpha_1,\dots,\alpha_n}(w_{\alpha_1,\dots,\alpha_n}) = \pi_{\alpha_1,\dots,\alpha_n}(w_{\alpha_1,\dots,\alpha_n})$ as wanted. We still denote by $\pi\colon T^n_\alpha \to T^{n,\alpha}$ the induced map. We clearly have $\ker\pi = T^n_{\alpha-1}$ showing the first result since the kernel of $\pi_{\alpha_1,\dots,\alpha_n}$ is included into $T^n_{\alpha-1}$ (see [Bo07, Algèbre, chap.2, §3, Proposition 6]).

Let us now consider the case of the symmetric algebra. We denote by $\sigma^n_\alpha\colon T^n_\alpha \to S^n_\alpha$ and by $\sigma^{n,\alpha}\colon T^{n,\alpha} \to S^{n,\alpha}$ the maps induced by $\sigma\colon T(V) \to S(V)$.

The surjective map $\pi : T^n_\alpha \to T^{n,\alpha}$ sends $\ker\sigma^n_\alpha$ onto $\ker\sigma^{n,\alpha}$. Hence π induces a surjective map also denoted by $\pi\colon S^n_\alpha \to S^{n,\alpha}$. To conclude, it remains to show that $\ker\pi = S^n_{\alpha-1}$. This is clear that $S^n_{\alpha-1} \subseteq \ker\pi$. The reverse inclusion follows from diagram chasing in the following diagram (where the column are exact and the second row too)

$$\begin{array}{ccccc}
 & & \ker\sigma^n_\alpha & \xrightarrow{\pi} & \ker\sigma^{n,\alpha} \\
 & & \downarrow & & \downarrow \\
T^n_{\alpha-1} & \hookrightarrow & T^n_\alpha & \xrightarrow{\pi} & T^{n,\alpha} \\
\downarrow{\scriptstyle \sigma^n_{\alpha-1}} & & \downarrow{\scriptstyle \sigma^n_\alpha} & & \downarrow{\scriptstyle \sigma^{n,\alpha}} \\
S^n_{\alpha-1} & \hookrightarrow & S^n_\alpha & \xrightarrow{\pi} & S^{n,\alpha}
\end{array}$$

□

Proposition C.2. *Assume that there exists linear maps $\Delta : V \to V$, $\Gamma : V \to V$ verifying $\Delta(V_i) \subseteq V_{i-1}$ and $\Gamma(V_i) \subseteq V_i$ for every $i \in \mathbb{N}$, and $[\Delta,\Gamma] = \Delta$. The extensions of Δ as derivation of $T(V)$ and $S(V)$ verify $\Delta(T^n_\alpha) \subseteq \Delta(T^n_{\alpha-1})$ and $\Delta(S^n_\alpha) \subseteq \Delta(S^n_{\alpha-1})$. The extensions of Γ as derivation of $T(V)$ and $S(V)$ stabilise the preceding filtrations on $T(V)$ and $S(V)$. Hence they induces graded derivations $\overline{\Delta}$ of degree -1 and $\overline{\Gamma}$ of degree 0 on $T(\mathrm{gr}(V))$ and $S(\mathrm{gr}(V))$. Moreover $[\overline{\Delta},\overline{\Gamma}] = \overline{\Delta}$.*

In addition, Δ induces a degree -1 map on $\mathrm{gr}(V)$ and Γ induces a degree 0 map on $\mathrm{gr}(V)$ and their extensions as derivations on $T(\mathrm{gr}(V))$ and $S(\mathrm{gr}(V))$ coincide with $\overline{\Delta}$ and $\overline{\Gamma}$.

Proof. The extension of a linear map $f : V \to V$ as a derivation of $T(V)$ acts on $V_{\alpha_1}\otimes\cdots\otimes V_{\alpha_n}$ as $f\otimes\mathrm{id}\otimes\cdots\otimes\mathrm{id}+\cdots+\mathrm{id}\otimes\cdots\otimes\mathrm{id}\otimes f$. Hence if $f = \Delta$ then T^n_α is mapped into $T^n_{\alpha-1}$ and if $f = \Gamma$ then T^n_α is mapped into T^n_α showing that they induces graded endomorphisms $\overline{\Delta}$ of degree -1 and $\overline{\Gamma}$ of degree 0 of $T(\mathrm{gr}(V))$ and $S(\mathrm{gr}(V))$. Moreover these endomorphisms are derivations of the graded associated rings and verify $[\overline{\Delta},\overline{\Gamma}] = \overline{\Delta}$.

For the last part of the proof, since the maps considered are derivations of $T(\mathrm{gr}(V))$ and $S(\mathrm{gr}(V))$ it suffices to show the coincidences on $\mathrm{gr}(V)$. But they follow readily from the commutative diagrams

$$\begin{array}{ccc} V_\alpha & \xrightarrow{\Delta} & V_{\alpha-1} \\ \downarrow & & \downarrow \\ V_\alpha/V_{\alpha-1} \cong M_\alpha & \xrightarrow{\overline{\Delta}} & V_{\alpha-1}/V_{\alpha-2} \cong M_{\alpha-1} \end{array} \qquad \text{and} \qquad \begin{array}{ccc} V_\alpha & \xrightarrow{\Gamma} & V_\alpha \\ \downarrow & & \downarrow \\ V_\alpha/V_{\alpha-1} \cong M_\alpha & \xrightarrow{\overline{\Gamma}} & V_\alpha/V_{\alpha-1} \cong M_\alpha \end{array}$$

□

Example C.3. Let us consider V a finite dimensional vector space and Δ and Γ two linear endomorphisms of V verifying $[\Delta, \Gamma] = \Delta$. Since Δ is nilpotent and $\ker \Delta^i$ is stable by Γ (see the proof of Theorem 4.6). The family of subspaces $V_i = \ker \Delta^i$ is a filtration of V satisfying the hypothesis of Proposition C.2.

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