

Electronic Supplementary Material of "Inertia-gravity waves in geophysical vortices"

J eremie Vidal*  & Yves Colin de Verdi ere†

February 14, 2024

Abstract

We present the derivation of the boundary-value pressure problem governing the normal modes of an incompressible fluid, which is rotating and stratified in density under the Boussinesq approximation. Next, we discuss for which conditions this is an elliptic boundary-value problem (using microlocal analysis). Applications to ellipsoidal vortices with a constant Brunt-V ais al a frequency are then considered. Finally, we introduce a boundary-layer analysis to determine the (leading-order) diffusive decay rates of the normal modes in an ellipsoid.

1 Boundary-value pressure problem

We consider a diffusionless and incompressible fluid, which is enclosed in a smooth bounded domain V (whose boundary is denoted by ∂V). The fluid is rotating at the angular velocity $\boldsymbol{\Omega} = (\Omega_x, \Omega_y, \Omega_z)^\top$ with respect to an inertial frame, and is stratified in density under the imposed gravity field $\mathbf{g} = -g(z)\mathbf{1}_z$ within the Boussinesq approximation. At rest, the pressure P_0 and the density satisfy the hydrostatic balance $\nabla P_0 = [\rho_* + \rho_0(z)]\mathbf{g}$, where ρ_* is the typical (mean) density of the fluid within the Boussinesq approximation. Then, we seek small-amplitude oscillatory perturbations (where $\omega \in \mathbb{R}$ is the angular frequency) for the velocity $\mathbf{u} \exp(i\omega t)$, the pressure $\Phi \exp(i\omega t)$, and the density $\zeta \exp(i\omega t)$ upon the hydrostatic basic state. These perturbations are governed in the rotating frame by the incompressible Euler equations

$$i\omega \mathbf{u} + 2(\boldsymbol{\Omega} \times \mathbf{u}) = -\nabla \Phi - (\zeta/\rho^*)g(z)\mathbf{1}_z, \quad \nabla \cdot \mathbf{u} = 0, \quad (1a,b)$$

and the buoyancy equation

$$i\omega \zeta = \frac{\rho_* N^2(z)}{g(z)} (\mathbf{u} \cdot \mathbf{1}_z), \quad N^2(z) := -\frac{g(z)}{\rho_*} \frac{d\rho_0}{dz}, \quad (2a,b)$$

where $N(z)$ is the Brunt-V ais al a (BV) frequency. The latter equations can be reduced, as long as $\omega \neq 0$, to a quadratic equation in ω for the velocity given by

$$-\omega^2 \mathbf{u} + i\omega (2\boldsymbol{\Omega} \times \mathbf{u}) + N^2(z) (\mathbf{u} \cdot \mathbf{1}_z) \mathbf{1}_z = -i\omega \nabla \Phi. \quad (3)$$

*Universit e Grenoble Alpes, CNRS, ISTerre, 38000 Grenoble, France

†Universit e Grenoble Alpes, CNRS, Institut Fourier, 38000 Grenoble, France

BOX 1.1: Poincaré problem

When $N = 0$, the velocity is given by (see formula 2.7.2 in Greenspan [1])

$$(4|\boldsymbol{\Omega}|^2 - \omega^2)\mathbf{u} = -i\omega\nabla\Phi - \frac{4}{i\omega}(\boldsymbol{\Omega} \cdot \nabla\Phi)\boldsymbol{\Omega} + 2\boldsymbol{\Omega} \times \nabla\Phi. \quad (\text{B1.1.1})$$

The divergence of the latter equation yields the so-called Poincaré equation [1]

$$\omega^2\nabla^2\Phi = (2\boldsymbol{\Omega} \cdot \nabla)^2\Phi. \quad (\text{B1.1.2})$$

The associated BC, obtained from equation (15) below, is given by

$$\omega^2\nabla\Phi \cdot \mathbf{n} = i\omega(2\boldsymbol{\Omega} \times \mathbf{n}) \cdot \nabla\Phi + 4(\boldsymbol{\Omega} \cdot \nabla\Phi)(\boldsymbol{\Omega} \cdot \mathbf{n}) \quad \text{on } \partial V, \quad (\text{B1.1.3})$$

which is identical to equation (2.7.4) in Greenspan [1].

Finally, equation (3) is supplemented with the no-penetration boundary condition (BC) given by $\mathbf{u} \cdot \mathbf{n}|_{\partial V} = 0$ (i.e. \mathbf{u} is tangent to ∂V), where \mathbf{n} is the unit vector normal to the boundary.

Our goal is to find an alternative formulation of the problem in terms of the pressure Φ . An equation for the pressure can be obtained by taking the divergence of equation (3), which gives

$$-i\omega\nabla^2\Phi = i\omega\nabla \cdot (2\boldsymbol{\Omega} \times \mathbf{u}) + (\mathbf{u} \cdot \mathbf{1}_z)\partial_z N^2 + N^2(z)\partial_z(\mathbf{u} \cdot \mathbf{1}_z). \quad (4)$$

The associated boundary condition (BC) is then obtained by taking the scalar product of equation (3) with \mathbf{n} , which gives

$$-i\omega(\nabla\Phi \cdot \mathbf{n}) = i\omega(2\boldsymbol{\Omega} \times \mathbf{u}) \cdot \mathbf{n} + N^2(z)(\mathbf{u} \cdot \mathbf{1}_z)(\mathbf{1}_z \cdot \mathbf{n}) \quad \text{on } \partial V. \quad (5)$$

However, equations (4)-(5) still depend on \mathbf{u} . Therefore, an expression of \mathbf{u} in terms of Φ only is required before obtaining the equation for the pressure (e.g. the Poincaré problem, see Box 1.1). We will give below the explicit formula(s) accounting for misaligned rotation and gravity with a varying BV frequency $N(z)$.

1.1 Second-order equation in the volume

To find the equation satisfied by the pressure Φ in V , we first find the equation relating \mathbf{u} and Φ (Proposition 1.1). Then, we enforce the incompressibility condition to obtain the pressure equation (Proposition 1.2). Finally, we give in Proposition 1.3 the pressure equation when N is constant. It will be of prime importance for the microlocal analysis presented in Section 2.

Proposition 1.1. The velocity \mathbf{u} is related to the pressure Φ by the equation $\mathcal{D}_\omega(z)\mathbf{u} = \mathbf{f}$ when $\mathcal{D}_\omega(z) = (\omega^2 - \omega_+^2)(\omega^2 - \omega_-^2) \neq 0$ with

$$\omega_\pm^2 = \frac{1}{2}(N^2(z) + 4|\boldsymbol{\Omega}|^2) \pm \sqrt{\frac{1}{4}(N^2(z) + 4|\boldsymbol{\Omega}|^2)^2 - 4N^2(z)(\boldsymbol{\Omega} \cdot \mathbf{1}_z)^2}, \quad (6)$$

and with

$$\begin{aligned} \mathbf{f} = & i\omega^3\nabla\Phi - 2\omega^2(\boldsymbol{\Omega} \times \nabla\Phi) - 4i\omega(\boldsymbol{\Omega} \cdot \nabla\Phi)\boldsymbol{\Omega} \\ & + N^2(z)[(2\boldsymbol{\Omega} \cdot \mathbf{1}_z)(\mathbf{1}_z \times \nabla\Phi) + i\omega\mathbf{1}_z \times (\mathbf{1}_z \times \nabla\Phi)]. \end{aligned} \quad (7)$$

Proof. We recast Euler equation (1a) and buoyancy equation (2) as a linear system given by

$$\mathbf{M} \begin{pmatrix} u_x \\ u_y \\ u_z \\ \zeta/\rho_* \end{pmatrix} = - \begin{pmatrix} \partial_x \Phi \\ \partial_y \Phi \\ \partial_z \Phi \\ 0 \end{pmatrix}, \quad \mathbf{M} := \begin{pmatrix} i\omega & -2\Omega_z & 2\Omega_y & 0 \\ 2\Omega_z & i\omega & -2\Omega_x & 0 \\ -2\Omega_y & 2\Omega_x & i\omega & g(z) \\ 0 & 0 & -N^2(z)/g(z) & i\omega \end{pmatrix}, \quad (8a,b)$$

and with $\mathbf{u} = (u_x, u_y, u_z)^\top$. The system admits non-zero solutions when $\det(\mathbf{M}) := \mathcal{D}_\omega(z) \neq 0$. In this case, we can invert the linear system and the result follows. Note that it reduces to equation (2.7.2) in Greenspan [1] when $N = 0$. \blacksquare

Proposition 1.2. When $\mathcal{D}_\omega(z) \neq 0$, the pressure obeys the second-order equation given by

$$\nabla \cdot (\mathbf{f}/\mathcal{D}_\omega) = 0 \iff \nabla \cdot \mathbf{f} - (1/\mathcal{D}_\omega) f_z \partial_z \mathcal{D}_\omega = 0, \quad (9)$$

with $\partial_z \mathcal{D}_\omega = -(\omega^2 - \omega_-^2) \partial_z \omega_+^2 - (\omega^2 - \omega_+^2) \partial_z \omega_-^2$.

When $N(z)$ is variable and $\boldsymbol{\Omega} = \Omega_z \mathbf{1}_z$, equation (9) reduces to

$$\nabla^2 \Phi + \frac{\partial}{\partial z} \left(\frac{N^2 - 4\Omega_z^2}{\omega^2 - N^2} \frac{\partial \Phi}{\partial z} \right) = 0, \quad (10)$$

which is identical to equation (5.1) in Friedlander & Siegmann [2]. For oscillatory motions with $\omega \neq 0$, equation (11) is thus equivalent to equation (3) when $|\omega| \neq \omega_\pm(z)$. However, the pressure must obey another equation $\mathcal{D}_\omega(z) = 0$ (see below).

Proposition 1.3. When N is constant, equation (9) reduces to $\mathcal{P}_\omega(\Phi) = 0$ with

$$\mathcal{P}_\omega := -\nabla^2 - \frac{N^2}{\omega^2 - N^2} (\mathbf{1}_z \cdot \nabla)^2 + \frac{4}{\omega^2 - N^2} (\boldsymbol{\Omega} \cdot \nabla)^2. \quad (11)$$

The operator \mathcal{P}_ω is elliptic when $0 < |\omega| < |\omega_-|$, and its principal symbol \mathbf{p} is given by

$$\mathbf{p} := Ak_x^2 + Bk_y^2 + Ck_z^2 + 2Dk_y k_z + 2Ek_x k_z + 2Fk_x k_y \quad (12)$$

with

$$A = \frac{\omega^2 - N^2 - 4\Omega_x^2}{\omega^2 - N^2}, \quad B = \frac{\omega^2 - N^2 - 4\Omega_y^2}{\omega^2 - N^2}, \quad C = \frac{\omega^2 - 4\Omega_z^2}{\omega^2 - N^2}, \\ D = \frac{-4\Omega_y \Omega_z}{\omega^2 - N^2}, \quad E = \frac{-4\Omega_x \Omega_z}{\omega^2 - N^2}, \quad F = \frac{-4\Omega_x \Omega_y}{\omega^2 - N^2}.$$

Finally, it is worth explicitly writing the Cartesian components of $\mathbf{f} = (f_x, f_y, f_z)^\top$ when N is constant (as we will reuse them below). They are given by

$$\frac{f_x}{\omega^2 - N^2} = i\omega[A\partial_x \Phi + F\partial_y \Phi + E\partial_z \Phi] - \frac{2\omega^2 \Omega_y}{\omega^2 - N^2} \partial_z \Phi + 2\Omega_z \partial_y \Phi, \quad (13a)$$

$$\frac{f_y}{\omega^2 - N^2} = i\omega[F\partial_x \Phi + B\partial_y \Phi + D\partial_z \Phi] + \frac{2\omega^2 \Omega_x}{\omega^2 - N^2} \partial_z \Phi - 2\Omega_z \partial_x \Phi, \quad (13b)$$

$$\frac{f_z}{\omega^2 - N^2} = i\omega[E\partial_x \Phi + D\partial_y \Phi + C\partial_z \Phi] + \frac{2\omega^2}{\omega^2 - N^2} (\Omega_y \partial_x \Phi - \Omega_x \partial_y \Phi). \quad (13c)$$

where the coefficients (A, B, C, D, E, F) appear in the principal symbol of \mathcal{P}_ω (Proposition 1.3).

BOX 1.2: Aligned case

When $\boldsymbol{\Omega} = \Omega_z \mathbf{1}_z$ and N is constant, pressure equation (10) simplifies into

$$-\nabla^2 \Phi - \frac{N^2 - 4\Omega_z^2}{\omega^2 - N^2} (\mathbf{1}_z \cdot \nabla)^2 \Phi = 0, \quad (\text{B1.2.1})$$

which is identical to equation (2.21) in Friedlander & Siegmann [3]. If required, the velocity is related to the pressure as

$$[4\Omega_z^2 - \omega^2] \mathbf{u} = -i\omega \nabla \Phi + \frac{i\omega}{\omega^2 - N^2} [4\Omega_z^2 - N^2] (\mathbf{1}_z \cdot \nabla \Phi) \mathbf{1}_z + 2\Omega_z (\mathbf{1}_z \times \nabla \Phi). \quad (\text{B1.2.2})$$

To find the pressure BC, we rearrange equation (15) as

$$i\omega [\omega^2 - N^2] \nabla \Phi \cdot \mathbf{n} = 2\Omega_z [\omega^2 - N^2] (\mathbf{1}_z \times \nabla \Phi) \cdot \mathbf{n} + i\omega [4\Omega_z^2 - N^2] (\mathbf{1}_z \cdot \mathbf{n}) \frac{\partial \Phi}{\partial z},$$

which simplifies into

$$\nabla \Phi \cdot \mathbf{n} = \frac{2\Omega_z}{i\omega} (\mathbf{n} \times \mathbf{1}_z) \cdot \nabla \Phi - \left[\frac{N^2 - 4\Omega_z^2}{\omega^2 - N^2} \right] (\mathbf{1}_z \cdot \mathbf{n}) \frac{\partial \Phi}{\partial z} \quad \text{on } \partial V. \quad (\text{B1.2.3})$$

BC (B1.2.3) is identical to formula (2.22) in Friedlander & Siegmann [3].

1.2 Boundary condition

Proposition 1.4. The pressure BC is given by $\mathbf{f} \cdot \mathbf{n}|_{\partial V} = 0$, where the components of \mathbf{f} are given in equations (13a-c).

For the sake of comparison with prior studies, the pressure BC can be rewritten as

$$i\omega^3 \nabla \Phi \cdot \mathbf{n} = 2\omega^2 (\boldsymbol{\Omega} \times \nabla \Phi) \cdot \mathbf{n} + 4i\omega (\boldsymbol{\Omega} \cdot \nabla \Phi) (\boldsymbol{\Omega} \cdot \mathbf{n}) - N^2(z) [(2\boldsymbol{\Omega} \cdot \mathbf{1}_z) (\mathbf{1}_z \times \nabla \Phi) + i\omega \mathbf{1}_z \times (\mathbf{1}_z \times \nabla \Phi)] \cdot \mathbf{n}. \quad (14)$$

Since we have $\mathbf{1}_z \times (\mathbf{1}_z \times \nabla \Phi) = (\mathbf{1}_z \cdot \nabla \Phi) \mathbf{1}_z - \nabla \Phi$, we can rewrite the pressure BC as

$$i\omega [\omega^2 - N^2(z)] \nabla \Phi \cdot \mathbf{n} = 2 [\omega^2 (\boldsymbol{\Omega} \times \nabla \Phi) - N^2(z) (\boldsymbol{\Omega} \cdot \mathbf{1}_z) (\mathbf{1}_z \times \nabla \Phi)] \cdot \mathbf{n} + i\omega [4(\boldsymbol{\Omega} \cdot \nabla \Phi) \boldsymbol{\Omega} - N^2(z) (\mathbf{1}_z \cdot \nabla \Phi) \mathbf{1}_z] \cdot \mathbf{n}. \quad (15)$$

BC (15) can be strongly simplified in some cases (see Box 1.2).

1.3 First-order equation when $\mathcal{D}_\omega(z) = 0$

The pressure is no longer given by equation (9) when $\mathcal{D}_\omega(z) = 0$. The latter defines the equation of turning surfaces, on which the mathematical problem change of type. Onto such surfaces, the pressure obeys another partial differential equation. Assuming that the field unknowns remain non-singular across such surfaces, this equation is found by using vector manipulations of the primitive equations (Proposition 1.5).

Proposition 1.5. When $\mathcal{D}_\omega(z) = 0$ (i.e. on turning surfaces), the pressure is given by the first-order equation

$$(2\boldsymbol{\Omega} \cdot \mathbf{1}_z)(2\boldsymbol{\Omega} \cdot \nabla\Phi) + 2i\omega \boldsymbol{\Omega} \cdot (\mathbf{1}_z \times \nabla\Phi) - \omega^2(\mathbf{1}_z \cdot \nabla\Phi) = 0, \quad (16)$$

which is a particular case of equation (3.5) in Friedlander & Siegmann [2].

Proof. We apply $2\boldsymbol{\Omega} \times$ to equation (1a), which gives

$$\begin{aligned} i\omega(2\boldsymbol{\Omega} \times \mathbf{u}) &= -2\boldsymbol{\Omega} \times (2\boldsymbol{\Omega} \times \mathbf{u}) - 2\boldsymbol{\Omega} \times \nabla\Phi - g(z)(\zeta/\rho^*)(2\boldsymbol{\Omega} \times \mathbf{1}_z), \\ &= -4[(\boldsymbol{\Omega} \cdot \mathbf{u})\boldsymbol{\Omega} - |\boldsymbol{\Omega}|^2\mathbf{u}] - 2\boldsymbol{\Omega} \times \nabla\Phi - g(z)(\zeta/\rho^*)(2\boldsymbol{\Omega} \times \mathbf{1}_z). \end{aligned}$$

The latter equation can be rearranged to give

$$(4|\boldsymbol{\Omega}|^2 - \omega^2) \mathbf{u} = -i\omega \nabla\Phi + 4(\boldsymbol{\Omega} \cdot \mathbf{u})\boldsymbol{\Omega} + 2\boldsymbol{\Omega} \times \nabla\Phi + \frac{N^2(z)}{i\omega}(\mathbf{1}_z \cdot \mathbf{u}) [2\boldsymbol{\Omega} \times \mathbf{1}_z - i\omega\mathbf{1}_z], \quad (17)$$

where we have used equation (1a) to simplify $2\boldsymbol{\Omega} \times \mathbf{u}$ and used equation (2a) to simplify the buoyancy term. Thus, it only remains to express $\boldsymbol{\Omega} \cdot \mathbf{u}$ and $\mathbf{1}_z \cdot \mathbf{u}$ as a function of Φ only. We take the scalar product of equation (1a) with $i\omega\boldsymbol{\Omega}$, which gives

$$\omega^2(\boldsymbol{\Omega} \cdot \mathbf{u}) = i\omega(\boldsymbol{\Omega} \cdot \nabla\Phi) + N^2(z)(\boldsymbol{\Omega} \cdot \mathbf{1}_z)(\mathbf{1}_z \cdot \mathbf{u}). \quad (18)$$

Then, we take the scalar product of equation (1a) with $i\omega\mathbf{1}_z$ to obtain

$$[N^2(z) - \omega^2] (\mathbf{1}_z \cdot \mathbf{u}) = i\omega 2\boldsymbol{\Omega} \cdot (\mathbf{1}_z \times \mathbf{u}) - i\omega(\mathbf{1}_z \cdot \nabla\Phi). \quad (19)$$

Since we have also from equation (1a)

$$i\omega(\mathbf{1}_z \times \mathbf{u}) = -\mathbf{1}_z \times (2\boldsymbol{\Omega} \times \mathbf{u}) - \mathbf{1}_z \times \nabla\Phi = -2[(\mathbf{1}_z \cdot \mathbf{u})\boldsymbol{\Omega} - (\mathbf{1}_z \cdot \boldsymbol{\Omega})\mathbf{u}] - \mathbf{1}_z \times \nabla\Phi, \quad (20)$$

we combine equations (18) to (20) to obtain

$$-\mathcal{D}_\omega(z)(\mathbf{1}_z \cdot \mathbf{u}) = i\omega(2\boldsymbol{\Omega} \cdot \mathbf{1}_z)(2\boldsymbol{\Omega} \cdot \nabla\Phi) - 2\omega^2 \boldsymbol{\Omega} \cdot (\mathbf{1}_z \times \nabla\Phi) - i\omega^3(\mathbf{1}_z \cdot \nabla\Phi). \quad (21)$$

From equation (21), we see that the pressure must satisfy equation (16) when $\mathcal{D}_\omega(z) = 0$. ■

2 Microlocal analysis of boundary-value problems

We investigate, using microlocal analysis, under which conditions a quite general boundary-value scalar problem is elliptic in §2.1. Then, we revisit the ellipsoidal model with a constant N in §2.2 to prove that the low-frequency spectrum σ_2 is essential.

2.1 Elliptic boundary-value problem

Remark. In this subsection, the Cartesian coordinates are denoted by (x_1, x_2, x_3) to make use of index notations.

We consider a smooth bounded domain V in \mathbb{R}^d , and a symmetric elliptic differential operator \mathcal{P}_ω of order 2 with smooth coefficients and of principal symbol \mathfrak{p} . We also assume that the Dirichlet extension of \mathcal{P}_ω is > 0 . The scalar Φ is governed by the boundary-value problem given by

$$\mathcal{P}_\omega(\Phi) = 0 \text{ in } V, \quad \mathcal{B}(\Phi) := \mathbf{n}^* \cdot \nabla \Phi + \mathcal{V}(\Phi) = 0 \text{ on } \partial V, \quad (22a,b)$$

where \mathbf{n}^* is the unit vector normal computed with respect to the metric \mathfrak{g} induced on ∂V by the principal symbol of \mathcal{P}_ω (see Box 2.1), and \mathcal{V} is a differential operator of degree 1 tangent¹ to ∂V . To solve system (22), we introduce the Dirichlet-to-Neumann operator \mathcal{N} in Proposition 2.1.

Proposition 2.1. If $\Psi : \partial V \rightarrow \mathbb{C}$ is a field defined on the boundary, then the Dirichlet-to-Neumann operator is defined by $\mathcal{N}(\Psi) := \partial_{\mathbf{n}^*} \Phi$ where $\partial_{\mathbf{n}^*} = \mathbf{n}^* \cdot \nabla$ is the normal derivative with respect to the vector \mathbf{n}^* , and where Φ is the solution of the Dirichlet problem in V given by $\mathcal{P}_\omega(\Phi) = 0$ in V and $\Phi|_{\partial V} = \Psi$ on ∂V .

Note that \mathcal{N} is a self-adjoint operator acting on functions defined in the space $L^2(\partial V, d\sigma)$ for the area $d\sigma$ defined by the metric \mathfrak{g} restricted on ∂V [5]. To solve problem (22), we first solve the boundary problem $\mathcal{N}(\Psi) + \mathcal{V}(\Psi) = 0$ on ∂V , and then consider the Dirichlet problem in V given the boundary field Ψ . In practice, this amounts to solving the problem on the boundary ∂V and, then, propagating the boundary solution in the interior V by using the Dirichlet problem. Actually, the nature of the boundary-value problem crucially depends on the nature of the boundary equation $\mathcal{N}(\Psi) + \mathcal{V}(\Psi) = 0$ on ∂V . This can be analysed using microlocal analysis. Indeed, \mathcal{N} is a nice pseudo-differential operator of principal symbol is given in Proposition 2.2. The ellipticity of boundary-value problem (22) is then given by Proposition 2.3. Finally, the nature is given by Proposition 2.4 below.

Proposition 2.2. The principal symbol of the Dirichlet-to-Neumann operator \mathcal{N} , defined for an elliptic operator with a positive principal symbol $\mathfrak{p} > 0$, is given by $\sqrt{\mathfrak{g}^*|_{\partial V}}$, where $\mathfrak{g}^*|_{\partial V}$ is the dual of the restriction on ∂V of the metric \mathfrak{g} induced by \mathfrak{p} in V .

Proof. It is given in the Appendices of Taylor's book [6]. Note that the principal symbol is defined as $-\sqrt{-\mathfrak{g}}|_{\partial V}$ in [6]. The first minus sign comes from the fact that an inward vector \mathbf{n}^* was considered, and the second from the fact that the other convention $\mathfrak{p} \leq 0$ was used. ■

Proposition 2.3. Let us denote by \mathfrak{v} the principal symbol of \mathcal{V} . Then, boundary-value problem (22) is elliptic if and only if the principal symbol $\sqrt{\mathfrak{g}^*|_{\partial V}} + \mathfrak{v}$ is elliptic on ∂V , that is if it does not vanish for non-zero covectors $\mathbf{k} \in \mathbb{R}^2$ on the boundary ∂V .

Proof. See Chapter 9 in [7]. This is also known as the Shapiro-Lopatinskii condition [7]. ■

Proposition 2.4. If boundary-value problem (22) is elliptic in the interval $[\omega_1, \omega_2]$, then its spectrum is discrete (or empty) in this interval. Otherwise, the spectrum is essential in $[\omega_1, \omega_2]$.

Proof. If the boundary-value pressure problem is elliptic, the operator \mathcal{P}_ω admits a parametrix \mathcal{L} (i.e. an approximate inverse) such that $\mathcal{P}_\omega \circ \mathcal{L} = \text{Id} + \mathcal{R}$, where \mathcal{R} is a pseudo-differential operator of degree -1 . Hence, \mathcal{R} is a compact operator. Moreover, \mathcal{P}_ω and \mathcal{R} are also analytic functions. Then, the result follows from the Fredholm analytic theorem (see Theorem VI.14 in [9]). Otherwise, the values $\omega \in [\omega_1, \omega_2]$ belong to the essential spectrum [10]. ■

¹*tangent* is here understood in the language of differential geometry [4]. For instance, if ∂V is described by $q_1 = \text{constant}$ for a set of curvilinear coordinates (q_1, q_2, q_3) , then \mathcal{V} only involves derivatives $(\partial_{q_2}, \partial_{q_3})$ on the tangent plane at a given point.

BOX 2.1: Normal vector induced by the metric on ∂V

The normal vector $\hat{\mathbf{n}}$ defined from the metric on ∂V is computed as follows. The principal symbol \mathfrak{p} defines a metric given by $\mathbf{g}^* := (\mathbf{g}^*)^{ij} k_i k_j$ for the covectors $\mathbf{k} \in \mathbb{R}^3$ in the cotangent space, whose inverse $\mathbf{g} := (\mathbf{g}^*)^{-1}$ induces a metric in V defined by $\mathbf{g} := \mathbf{g}_{ij} dx_i dx_j$. Then, the normal vector \mathbf{n}^* computed with respect to \mathbf{g}^* is given by

$$\mathbf{n}^* \propto \mathbf{g}^* \mathbf{n}, \quad \mathbf{g}_{ij} n_i^* n_j^* = 1,$$

where $\mathbf{n} = (n_1, n_2, n_3)^\top$ is the unit vector normal to ∂V in Cartesian coordinates with $n_1^2 + n_2^2 + n_3^2 = 1$. The constant of proportionality is such that \mathbf{n}^* has a unit norm with respect to the metric \mathbf{g} . The vector \mathbf{n}^* is also called the **conormal** vector (with respect to \mathbf{g}^*) in the study of partial differential equations [8]. Note that \mathbf{g}^* is a contravariant metric tensor associated with the operator $(\mathbf{g}^*)^{ij} \partial_{x_i} \partial_{x_j}$. Thus, the conormal \mathbf{n}^* is the contravariant description (in the space with the metric defined by \mathbf{g}) of the covariant normal vector \mathbf{n} to ∂V (in the space with the Euclidean metric).

In practice, the difficult step is to evaluate the principal symbol of the Dirichlet-to-Neumann operator (Proposition 2.2). In practice, this can be done as follows. The principal symbol \mathfrak{p} induces a metric in V given by $\mathbf{g} := \mathbf{g}_{ij} dx_i dx_j$. Let us assume that $x_i = x_i(q_j)$ with the curvilinear coordinates (q_1, q_2, q_3) such that the boundary ∂V is given by $q_1 = \text{constant}$ (for instance). Then, we can rewrite the two-dimensional metric restricted on ∂V as $\mathbf{g}|_{\partial V} = \tilde{\mathbf{g}}_{ij} dq_i dq_j$. Then, the dual of the metric induced by \mathbf{g} on ∂V is given by $\mathbf{g}^*|_{\partial V} = (\tilde{\mathbf{g}}^*)^{ij} k_i k_j$ with $\tilde{\mathbf{g}}^* = \tilde{\mathbf{g}}^{-1}$.

2.2 Back to the pressure equation

We aim to apply the above microlocal analysis to the boundary-value pressure problem when N is constant, which is elliptic in V when $0 < |\omega| < \omega_-$. The principal symbol \mathfrak{p} , defined in equation (12), induces the metric \mathbf{g} in V and its dual \mathbf{g}^* characterised by the metric tensors

$$\mathbf{g}^* = \begin{pmatrix} A & F & E \\ F & B & D \\ E & D & C \end{pmatrix}, \quad \mathbf{g} = \frac{1}{\det(\mathbf{g}^*)} \begin{pmatrix} BC - D^2 & -CF + DE & -BE + DF \\ -CF + DE & AC - E^2 & -AD + EF \\ -BE + DF & -AD + EF & AB - F^2 \end{pmatrix}, \quad (23a,b)$$

with $\det(\mathbf{g}^*) = ABC - AD^2 - BE^2 - CF^2 + 2DEF$. For a given normal vector $\mathbf{n} = (n_1, n_2, n_3)^\top$ in Cartesian coordinates, the conormal vector \mathbf{n}^* is thus given by

$$\mathbf{n}^* = \frac{1}{\alpha} \left[n_1 \begin{pmatrix} A \\ F \\ E \end{pmatrix} + n_2 \begin{pmatrix} F \\ B \\ D \end{pmatrix} + n_3 \begin{pmatrix} E \\ D \\ C \end{pmatrix} \right], \quad (24)$$

with $\alpha = [An_1^2 + Bn_2^2 + Cn_3^2 + 2(Dn_2n_3 + En_1n_3 + Fn_1n_2)]^{1/2}$ such that $\mathbf{g}_{ij} n_i^* n_j^* = 1$. Thanks to expression (24), we clearly see that the conormal naturally appears in equations (13a-c). Thus, we can apply Proposition 2.3 to determine the nature of the boundary-value pressure problem.

Let us consider here the aligned case with $\boldsymbol{\Omega} = (0, 0, \Omega_z)^\top$, for which the equations are then greatly simplified (see Box 1.2). In this case, the two metrics reduce to

$$\mathbf{g}^* = k_x^2 + k_y^2 + Ck_z^2, \quad \mathbf{g} = dx^2 + dy^2 + C^{-1}dz^2, \quad (25a,b)$$

and the conormal vector is given by $\mathbf{n}^\star = (1/\alpha)(n_1, n_2, n_3 C)^\top$ with $\alpha = \sqrt{1 + n_3^2(C-1)}$, where the Euclidian normal vector is $\mathbf{n} = (n_1, n_2, n_3)^\top$ with $n_1^2 + n_2^2 + n_3^2 = 1$. To demonstrate that the pressure problem is not elliptic for a given frequency $0 < |\omega| < \omega_-$, it is sufficient to find one location on ∂V where Proposition 2.3 is not valid. We look at the behaviour at the equator, where the tangent plane is vertical with $\mathbf{n} = \mathbf{1}_x$ (for instance) such that the pressure BC is simply $f_x = 0$. The restriction of the metrics on ∂V gives

$$\mathbf{g}|_{\partial V} = dy^2 + C^{-1}dz^2, \quad \mathbf{g}^\star|_{\partial V} = k_y^2 + Ck_z^2, \quad (26a,b)$$

and the conormal vector is simply given by $\mathbf{n}^\star = \mathbf{1}_x$. The principal symbol of the pressure BC is then given by

$$\sqrt{k_y^2 + Ck_z^2} = -\frac{2\Omega_z}{\omega}k_y, \quad (27)$$

which reduces to

$$\underbrace{\left(\frac{4\Omega_z^2 - \omega^2}{\omega^2}\right)}_{\geq 0} k_y^2 = \underbrace{C}_{\geq 0} k_z^2 \quad (28)$$

where the two prefactors are positive when $0 < |\omega| < \omega_- = \min(N, 2\Omega_z)$ in the aligned case. Therefore, we can find a real-valued covector $\mathbf{k} = (0, k_y, k_z)^\top$ such that the principal symbol of the BC vanishes. The boundary-value pressure problem is thus not elliptic in this case (Proposition 2.2), and the low-frequency spectrum is essential (Proposition 2.4). Similarly, we conjecture that the spectrum is always essential when $0 < |\omega| < \omega_-$ when $\boldsymbol{\Omega}$ and $\mathbf{1}_z$ are misaligned. The corresponding calculations (which are more lengthy) will be included in a forthcoming paper [5], in which we will also further explore the properties of the low-frequency (surface) modes.

3 Boundary-layer analysis

We introduce diffusion in the problem to investigate the diffusive decay rates of IGMs for a constant BV frequency in an ellipsoid. We use the same model as in the main text except that, here, the fluid has a non-zero kinematic viscosity ν and diffusivity κ . We employ boundary-layer theory (BLT) to approach the low-diffusive regime (relevant for geophysical vortices), and introduce dimensionless variables below. We take Ω_s^{-1} as the time scale, L as the length scale (e.g. either a or c), and $\rho_* L N^2 / g$ as the density scale. For the sake of concision, the dimensionless variables are written below using the same symbols as the dimensional variables in the main text. We seek modal solutions as $[\mathbf{v}, \pi, \rho](\mathbf{r}, t) = [\mathbf{u}, \Phi, \zeta](\mathbf{r}) \exp(\lambda t)$, where $\lambda \in \mathbb{C}$ is the diffusive eigenvalue. The dimensionless linearised Navier-Stokes equations are then given in the rotating frame by

$$\lambda \mathbf{u} + 2(\mathbf{1}_\Omega \times \mathbf{u}) = -\nabla \Phi - \tilde{N}^2 \zeta \mathbf{1}_z + E \nabla^2 \mathbf{u}, \quad \lambda \zeta = (\mathbf{u} \cdot \mathbf{1}_z) + E_\kappa \nabla^2 \zeta, \quad (29a,b)$$

with the (dimensionless) Ekman number $E = \nu / (\Omega_s L^2)$ and $E_\kappa = E / Pr$ where $Pr = \nu / \kappa$ is the Prandtl number, and with the normalised BV frequency $\tilde{N} = N / \Omega_s$. Multiplying equation (29a) by \mathbf{u}^\dagger and the complex-conjugate of equation (29b) by $\tilde{N}^2 \zeta$, we obtain after volume integration

$$\lambda \|\mathbf{u}\|^2 + \lambda^\dagger \tilde{N}^2 \|\zeta\|^2 = E \langle \mathbf{u}, \nabla^2 \mathbf{u} \rangle + \tilde{N}^2 E_\kappa \langle \nabla^2 \zeta, \zeta \rangle, \quad (30)$$

where the scalar product between two scalar quantities is defined as $\langle \zeta_1, \zeta_2 \rangle = \int_V \zeta_1^\dagger \cdot \zeta_2 dV$. Note that formula (30) is also valid in a sphere with a radial gravity [11]. Finally, the problem is supplemented with BC. We assume that the velocity field satisfies the stress-free BCs

$$\mathbf{u} \cdot \mathbf{n}|_{\partial V} = 0, \quad [\boldsymbol{\epsilon}(\mathbf{u}) \cdot \mathbf{n}] \times \mathbf{n}|_{\partial V} = \mathbf{0} \quad (31a,b)$$

where $\boldsymbol{\epsilon}(\mathbf{u}) = (1/2) [\nabla\mathbf{u} + (\nabla\mathbf{u})^\top]$ is the strain-rate tensor. SF-BCs (31a,b) allow the tangential velocity to slip on ∂V . For the density perturbation, we enforce a Neumann BC $\nabla\zeta \cdot \mathbf{n}|_{\partial V} = 0$. A density jump is thus possible between the vortex and the ambient fluid (as considered in idealised models [12, 13, 14]), but without exchange of mass between the two fluids.

We can simplify equation (31) using BLT when $E \ll 1$. Formally, classical BLT for rotating fluids is not expected to be strongly modified by stratification when \tilde{N} is not too large (e.g. for rigid boundaries [15]). Moreover, it is known that the boundary-layer flow is $E^{1/2}$ smaller than the bulk flow for SF-BCs [16, 17]. This considerably simplifies the BLT, because explicit expressions for the boundary-layer solutions are no longer required to estimate the decay rate of the modes. We seek the variables at the leading order in E for the SF-BCs as

$$\lambda \simeq i\omega_0 + E\lambda_1, \quad [\mathbf{u}, \zeta] \simeq [\mathbf{u}_0, \zeta_0] + E^{1/2} [\mathbf{u}_1, \zeta_1], \quad (32a,b)$$

where $[\omega_0, \mathbf{u}_0]$ is the eigenvalue-eigenvector pair of a diffusionless IGM, $\lambda_1 \in \mathbb{C}$ is the first-order correction of the eigenvalue, and $[\mathbf{u}_1, \zeta_1]$ are the first-order corrections within the boundary layer such that $\mathbf{u}_0 + E^{1/2}\mathbf{u}_1$ satisfies SF-BCs (31a,b) and $\zeta_0 + E^{1/2}\zeta_1$ satisfies the Neumann BC for the density. We substitute the above asymptotic expansions into equation (30) and we obtain

$$\lambda_1 \|\mathbf{u}_0\|^2 + \lambda_1^\dagger \tilde{N}^2 \|\zeta_0\|^2 \simeq - \left(2 \int_V \boldsymbol{\epsilon}(\mathbf{u}_0) : \boldsymbol{\epsilon}(\mathbf{u}_0^\dagger) dV + \frac{\tilde{N}^2}{Pr} \int_V (\nabla\zeta_0)^2 dV \right) \quad (33)$$

at the order E , where we have used the SF-BCs and the Neumann BC to simplify the volume integrals on the right-hand side. Since the latter is real-valued and negative, there is no frequency correction at the leading order in E due to SF-BCs such that $\Im_m(\lambda_1) = 0$ and $\Re_e(\lambda_1) = \tau_1 \leq 0$. We recover from equation (33) the viscous decay rate of pure inertial modes with SF-BCs [17].

Finally, we can crudely estimate whether forced IGMs would quickly decay or not in large-scale stratified vortices by plugging geophysical estimates into equation (33). Typical values for the kinematic viscosity are $\nu \sim 10^{-6} \text{ m}^2.\text{s}^{-1}$ for water and $\nu \sim 4 \times 10^{-7} \text{ m}^2.\text{s}^{-1}$ for gas giants (according to ab-initio simulations [18]). Using Table 1 in the main text, typical values for the Ekman number are thus $E \sim 10^{-8} - 10^{-12}$ for Mediterranean eddies and $E \sim 10^{-12} - 10^{-17}$ for Jovian vortices. Estimating the Prandtl number depends on whether the stratification is due to thermal effects or compositional ones (for which $Pr \gg 1$ in both water and gas giants). Heat diffusion in Mediterranean eddies is characterised by $Pr \simeq 0.7$, and typical values $Pr \sim 10^{-2} - 1$ are expected for thermal diffusion in gas giants [18]. Hence, we have $\tilde{N}E_\kappa \sim 10^{-4} - 10^{-8}$ for Mediterranean eddies and $\tilde{N}E_\kappa \sim 10^{-6} - 10^{-11}$ for Jovian vortices. Consequently, the largest-scale IGMs are not expected to be strongly damped by diffusion in stratified vortices.

References

- [1] Greenspan HP. 1968 *The theory of rotating fluids*. Cambridge (UK): Cambridge University Press.
- [2] Friedlander S, Siegmann WL. 1982a Internal waves in a rotating stratified fluid in an arbitrary gravitational field. *Geophys. Astrophys. Fluid Dyn.* **19**, 267–291. (doi:10.1080/03091928208208959).
- [3] Friedlander S, Siegmann WL. 1982b Internal waves in a contained rotating stratified fluid. *J. Fluid Mech.* **114**, 123–156. (doi:10.1017/S002211208200007X).

- [4] Schutz BF. 1980 *Geometrical methods of Mathematical Physics*. Cambridge (UK): Cambridge University Press.
- [5] Colin de Verdière Y, Vidal J. 2024 On gravito-inertial surface waves. Preprint.
- [6] Taylor M. 2013 *Partial differential equations II: Qualitative studies of linear equations*. New York (USA): Springer.
- [7] Grubb G. 2008 *Distributions and operators*. Graduate Texts in Mathematics. Springer, New York (USA): Springer.
- [8] Miranda C. 1970 *Partial differential equations of elliptic type*. Springer-Verlag.
- [9] Reed M, Simon B. 1972 *Methods of modern Mathematical Physics. I. Functional analysis*. San Diego (USA): Academic Press.
- [10] Colin de Verdière Y. 2020 Spectral theory of pseudodifferential operators of degree 0 and an application to forced linear waves. *Anal. PDE* **13**, 1521–1537.
- [11] Dintrans B, Rieutord M, Valdettaro L. 1999 Gravito-inertial waves in a rotating stratified sphere or spherical shell. *J. Fluid Mech.* **398**, 271–297. (doi:10.1017/S0022112099006308).
- [12] Hassanzadeh P, Marcus PS, Le Gal P. 2012 The universal aspect ratio of vortices in rotating stratified flows: theory and simulation. *J. Fluid Mech.* **706**, 46–57. (doi:10.1017/jfm.2012.180).
- [13] Aubert O, Le Bars M, Le Gal P, Marcus PS. 2012 The universal aspect ratio of vortices in rotating stratified flows: experiments and observations. *J. Fluid Mech.* **706**, 34–45. (doi:10.1017/jfm.2012.176).
- [14] Lemasquerier D, Facchini G, Favier B, Le Bars M. 2020 Remote determination of the shape of Jupiter’s vortices from laboratory experiments. *Nat. Phys.* **16**, 695–700. (doi:10.1038/s41567-020-0833-9).
- [15] Friedlander S. 1989 Asymptotic behaviour of decay rates of internal waves in a rotating stratified spherical shell. *Geophys. J. Int.* **96**, 245–252. (doi:10.1111/j.1365-246X.1989.tb04448.x).
- [16] Rieutord M, Zahn JP. 1997 Ekman pumping and tidal dissipation in close binaries: a refutation of Tassoul’s mechanism. *Astrophys. J.* **474**, 760. (doi:10.1086/303494).
- [17] Vidal J, Cébron D. 2023 Precession-driven flows in stress-free ellipsoids. *J. Fluid Mech.* **954**, A5. (doi:10.1017/jfm.2022.976).
- [18] French M, Becker A, Lorenzen W, Nettelmann N, Bethkenhagen M, Wicht J, Redmer R. 2012 Ab initio simulations for material properties along the Jupiter adiabat. *Astrophys. J. Suppl. Ser.* **202**, 5. (doi:10.1088/0067-0049/202/1/5).