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Finite Relational Semantics for Language Kleene Algebra with Complement

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Abstract

We study the equational theory of Kleene algebra (KA) w.r.t. languages by extending the language complement. This extension significantly enhances the expressive power of KA. In this paper, we present a (finite) *relational semantics* completely characterizing the equational theory w.r.t. languages, which extends the relational characterizations known for KA and for KA with top. Based on this relational semantics, we show that the equational theory w.r.t. languages is Π_1^0 -complete for KA with complement (with or without Kleene-star) and is PSPACE-complete if the complement only applies to variables or constants.

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1 Introduction

Kleene algebra (KA) [23, 11, 24] is an algebraic system for regular expressions consisting of identity (1), empty (0), composition (\cdot), union (+), and iteration ($_*$). As iteration frequently appears in computer science, KA has many applications, e.g., the semantics of programs [42], relation algebra [37], graph query language [12, 20], program verification [27, 22, 44], and program logics [25, 38, 48]. In practice, we often consider extensions of KA. One direction of extensions is to extend equations to formulas, e.g., Horn formulas ($t_1 = s_1 \rightarrow \dots \rightarrow t_n = s_n \rightarrow t = s$) for considering hypotheses [9, 26, 14]. Another direction is to extend terms by adding some operators. For example, Kleene algebra with tests (KAT) applies to model Hoare logic [25] and KAT with top (\top) applies to model incorrectness logic [38, 48, 41]. It is also natural to extend KA with language operators, e.g., reverse [3], residual [8], intersection (\cap) [2], top (universality) [48, 41], variable complements (\bar{x}) [35, 36], and combinations of some of them [4, 5]. Note that, whereas the class of regular languages is closed under these operators, such extensions strictly enhance the expressive power of KA w.r.t. languages (see [35, 36] and Sect. 2.2, for complement).

In this paper, we study KA w.r.t. languages by extending the *language complement* ($_^-$). Extending with complement and considering its fragments is a natural comprehensive approach, e.g., in logic, formal language [10, 39], and relation algebra [46, 37] (see also [1, 6, 30, 40, 32]). The language complement in KA w.r.t. languages significantly enhances the expressive power (cf. KAT is also an extension of KA with complement, but this complement does not mean the language complement [27, Sect. 3]). For instance, we can define \top and \cap using complement: $\top = 0^-$ and $t \cap s = (t^- + s^-)^-$. Additionally, we can encode positive quantifier-free formulas by equations of KA terms with complement.

Our main contribution is to present a *finite relational semantics* for KA with complement w.r.t. languages: relational subword models RSUB (Sect. 3). As KA with complement has a high expressive power, our relational semantics can apply to a more broad class of extensions of KA (including KA with \top and \cap) than known relational semantics, e.g., REL (for KA) [42, third page] and GREL (for KA with \top) [48, 41] (see Remark 3.5). A good point of RSUB is its form; each model is finite and totally ordered (with the least and greatest vertices). For instance, the Π_1^0 upper bound result of the equational theory of KA with complement w.r.t. languages is immediate from the finiteness of RSUB. Also, we can give a (polynomial-time)

reduction from the quantifier-free theory w.r.t. RSUB into the equational theory w.r.t. RSUB (Thm. 4.3) by using that each model of RSUB has the least and greatest vertices. Another good point is that we can naturally consider lifting techniques known in REL to LANG. For instance, by lifting the techniques in our previous paper w.r.t. REL [32], we can show the following complexity results: the equational theory w.r.t. languages is Π_1^0 -complete for KA with intersection and variable complements (Thm. 4.9) and for KA with complement and without Kleene-star (i.e., star-free regular expressions w.r.t. LANG) (Thm. 4.12); and PSPACE-complete for KA with variable and constant complements (Thm. 6.10). The PSPACE decidability result above positively settles the open problem posed in [35].

This paper is structured as follows. In Sect. 2, we give basic definitions, including language models (LANG) and generalized relational models (GREL). In Sect. 3, we give RSUB and show that the equational theory w.r.t. LANG coincides with that w.r.t. RSUB (a subclass of GREL). In Sect. 4, by using RSUB, we give a reduction from the quantifier-free theory w.r.t. LANG into the equational theory w.r.t. LANG. Using this reduction, we show that the equational theory w.r.t. LANG is Π_1^0 -complete for KA with intersection and variable complements and for KA with complement and without Kleene-star. In Sect. 5, by using RSUB, we give a graph characterization for KA terms with variable and constant complements. In Sect. 6, by using this characterization, we show that the equational theory for KA terms with variable and constant complements is PSPACE-complete. In Sect. 7, we conclude this paper.

2 Preliminaries

We write \mathbb{N} for the set of non-negative integers. For $l, r \in \mathbb{N}$, we write $[l, r]$ for the set $\{i \in \mathbb{N} \mid l \leq i \leq r\}$. For a set X , we write $\wp(X)$ for the power set of X .

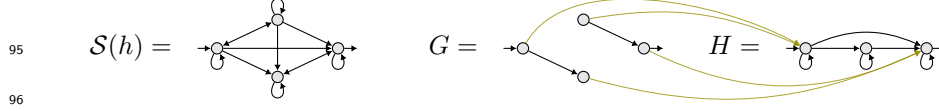
For a set X (of letters), we write X^* for the set of words over X . A *language* over X is a subset of X^* . We use w, v to denote words and use L, K to denote languages, respectively. We write $\|w\|$ for the *length* of a word w . We write 1 for the empty word. We write wv for the concatenation of words w and v . For languages $L, K \subseteq X^*$, the concatenation $L \cdot K$ and the Kleene-star L^* is defined by:

$$L \cdot K \triangleq \{wv \mid w \in L \wedge v \in K\}, \quad L^* \triangleq \{w_0 \dots w_{n-1} \mid \exists n \in \mathbb{N}, \forall i < n, w_i \in L\}.$$

A (*2-pointed*) *graph* G over a set A is a tuple $\langle |G|, \{a^G\}_{a \in A}, 1^G, 2^G \rangle$, where $|G|$ is a non-empty set (of vertices), each $a^G \subseteq |G|^2$ is a binary relation, and $1^G, 2^G \in |G|$ are vertices. Let G, H be graphs over a set A . For a map $f: |G| \rightarrow |H|$, we say that f is a *graph homomorphism* from G to H , written $f: G \rightarrow H$, if for all x, y , and a , $\langle x, y \rangle \in a^G$ implies $\langle f(x), f(y) \rangle \in a^H$, $f(1^G) = 1^H$, and $f(2^G) = 2^H$. We say that f is a *graph isomorphism* from G to H if f is a bijective graph homomorphism and for all x, y , and a , $\langle x, y \rangle \in a^G$ iff $\langle f(x), f(y) \rangle \in a^H$. We say that H is a (canonical) *edge-extension* of G if $|H| = |G|$ and the identity map is a graph homomorphism from G to H . For a set $\{1^G, 2^G\} \subseteq X \subseteq |G|$, the *induced subgraph* of G on X is the graph $\langle X, \{a^G \cap X^2\}_{a \in A}, 1^G, 2^G \rangle$. For an equivalence relation E on $|G|$, the *quotient graph* of G w.r.t. E is the graph $G/E \triangleq \langle |G|/E, \{\langle X, Y \rangle \mid \exists x \in X, y \in Y, \langle x, y \rangle \in a^G\}_{a \in A}, [1^G]_E, [2^G]_E \rangle$ where X/E denotes the set of equivalence classes of X by E and $[x]_E$ denotes the equivalence class of x . Additionally, we use the following operation:

► **Definition 2.1.** For a graph homomorphism $h: G \rightarrow H$ where G, H are graphs over a set A , the *edge-saturation* of G w.r.t. h is the graph $\mathcal{S}(h) \triangleq \langle |G|, \{\langle x, y \rangle \mid \langle h(x), h(y) \rangle \in a^H\}_{a \in A}, 1^G, 2^G \rangle$.

91 ► **Example 2.2.** Let $h: G \rightarrow H$ be the graph homomorphism indicated by green colored
 92 arrows (graphs are depicted as unlabeled graphs for simplicity). Then $\mathcal{S}(h)$ is the following
 93 graph in the left-hand side, which is an edge-extension of G where the extended edges are
 94 derived from edges of H :



97 2.1 Syntax: terms of KA with complement

98 We consider *terms* over the signature $S \triangleq \{1_{(0)}, 0_{(0)}, \cdot_{(2)}, +_{(2)}, -^*_{(1)}, -_{(1)}\}$. Let \mathbf{V} be a
 99 countably infinite set of variables. For a term t over S , let \bar{t} be s if $t = s^-$ for some s and be
 100 t^- otherwise. We use the abbreviations: $\top \triangleq 0^-$ and $t \cap s \triangleq (t^- + s^-)^-$.

101 For $X \subseteq \{\bar{x}, \bar{1}, \top, \cap, -\}$, let KA_X be the minimal set A of terms over S satisfying:

$$102 \quad \frac{y \in \mathbf{V}}{y \in A} \quad \frac{}{1 \in A} \quad \frac{}{0 \in A} \quad \frac{t \in A \quad s \in A}{t \cdot s \in A} \quad \frac{t \in A \quad s \in A}{t + s \in A} \quad \frac{t \in A}{t^* \in A}$$

$$103 \quad \frac{\bar{x} \in X \quad y \in \mathbf{V}}{\bar{y} \in A} \quad \frac{\bar{1} \in X}{\bar{1} \in A} \quad \frac{\top \in X}{\top \in A} \quad \frac{\cap \in X \quad t \in A \quad s \in A}{t \cap s \in A} \quad \frac{- \in X \quad t \in A}{t^- \in A}.$$

104

105 We often abbreviate $t \cdot s$ to ts . We use parentheses in ambiguous situations (where $+$ and \cdot
 106 are left-associative). We write $\sum_{i=1}^n t_i$ for the term $0 + t_1 + \cdots + t_n$.

107 An *equation* $t = s$ is a pair of terms. An *inequation* $t \leq s$ abbreviates the equation
 108 $t + s = s$. The set of *quantifier-free formulas* of KA_X is defined by the following grammar:

$$109 \quad \varphi, \psi ::= t = s \mid \varphi \wedge \psi \mid \neg \varphi. \quad (t, s \in \text{KA}_X)$$

110

111 We use the following abbreviations, as usual: $\varphi \vee \psi \triangleq \neg(\neg\varphi \wedge \neg\psi)$, $\varphi \rightarrow \psi \triangleq \neg\varphi \vee \psi$,
 112 $\varphi \leftrightarrow \psi \triangleq (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$, $\mathbf{f} \triangleq \neg\varphi \wedge \varphi$, and $\mathbf{t} \triangleq \neg\mathbf{f}$. We use parentheses in ambiguous
 113 situations (where \vee and \wedge are left-associative). We write $\bigwedge_{i=1}^n \varphi_i$ for $\mathbf{t} \wedge \varphi_1 \wedge \cdots \wedge \varphi_n$ and
 114 $\bigvee_{i=1}^n \varphi_i$ for $\mathbf{f} \vee \varphi_1 \vee \cdots \vee \varphi_n$.

115 We say that a quantifier-free formula is *positive* if the formula in the following set A :

$$116 \quad \varphi, \psi \in A ::= t = s \mid \varphi \wedge \psi \mid \varphi \vee \psi \quad (t, s \in \text{KA}_X)$$

117

118 where $\varphi \vee \psi$ expresses $\neg(\neg\varphi \wedge \neg\psi)$ in the above. We say that a quantifier-free formula is a
 119 *Horn formula* if the formula is of the form $(\bigwedge_{i=1}^n \varphi_i) \rightarrow \psi$.

120 2.2 Semantics: language models

121 An *S-algebra* \mathcal{A} is a tuple $\langle |\mathcal{A}|, \{f^{\mathcal{A}}\}_{f_{(k)} \in S} \rangle$, where $|\mathcal{A}|$ is a non-empty set and $f^{\mathcal{A}}: |\mathcal{A}|^k \rightarrow |\mathcal{A}|$
 122 is a k -ary map for each $f_{(k)} \in S$. A *valuation* \mathbf{v} of an S -algebra \mathcal{A} is a map $\mathbf{v}: \mathbf{V} \rightarrow |\mathcal{A}|$. For a
 123 valuation \mathbf{v} , we write $\hat{\mathbf{v}}: \text{KA}_{\{-\}} \rightarrow |\mathcal{A}|$ for the unique homomorphism extending \mathbf{v} . Moreover,
 124 for a quantifier-free formula φ , we define $\hat{\mathbf{v}}(\varphi) \in \{\text{true}, \text{false}\}$ by:

$$125 \quad \hat{\mathbf{v}}(t = s) \triangleq (\hat{\mathbf{v}}(t) = \hat{\mathbf{v}}(s)), \quad \hat{\mathbf{v}}(\varphi \wedge \psi) \triangleq (\hat{\mathbf{v}}(\varphi) \text{ and } \hat{\mathbf{v}}(\psi)), \quad \hat{\mathbf{v}}(\neg\varphi) \triangleq (\text{not } \hat{\mathbf{v}}(\varphi)).$$

126

127 For a quantifier-free formula φ and a class of valuations (of S -algebra) \mathcal{C} ,¹ we write

$$128 \quad \mathcal{C} \models \varphi \triangleq \hat{\mathbf{v}}(\varphi) \text{ holds for all valuations } \mathbf{v} \in \mathcal{C}.$$

129

¹ This paper considers classes of valuations rather than classes of S -algebras (cf. Thm. 3.6).

130 We abbreviate $\{\mathbf{v}\} \models \varphi$ to $\mathbf{v} \models \varphi$. The *equational theory w.r.t. \mathcal{C}* is the set of all equations
 131 $t = s$ such that $\mathcal{C} \models t = s$. The *quantifier-free theory w.r.t. \mathcal{C}* is the set of all quantifier-free
 132 formulas φ such that $\mathcal{C} \models \varphi$.

133 The *language model \mathcal{A}* over a set X , written lang_X , is the S-algebra defined by $|\mathcal{A}| = \wp(X^*)$,
 134 $1^{\mathcal{A}} = \{1\}$, $0^{\mathcal{A}} = \emptyset$, and for all $L, K \subseteq X^*$,

$$135 \quad L \cdot^{\mathcal{A}} K = L \cdot K, \quad L +^{\mathcal{A}} K = L + K, \quad L^{*\mathcal{A}} = L^*, \quad L^{-\mathcal{A}} = X^* \setminus L.$$

137 We write LANG_X for the class of all valuations of lang_X and write LANG for $\bigcup_X \text{LANG}_X$.
 138 The *equational theory (resp. quantifier-free theory) w.r.t. languages* expresses that w.r.t.
 139 LANG .

140 The language $[t] \subseteq \mathbf{V}^*$ of a term t is $\hat{\mathbf{v}}_{\text{st}}(t)$ where \mathbf{v}_{st} is the standard valuation on the
 141 language model over the set \mathbf{V} , which is defined by $\mathbf{v}_{\text{st}}(x) = \{x\}$ for $x \in \mathbf{V}$. Since $\mathbf{v}_{\text{st}} \in \text{LANG}$,
 142 we have

$$143 \quad \text{LANG} \models t = s \quad \Rightarrow \quad [t] = [s] \quad (\dagger)$$

145 The converse direction fails, e.g., $[y] \subseteq [\bar{x}]$ and $\text{LANG} \not\models y \leq \bar{x}$ where $x \neq y$, because
 146 $[y] = \{y\} \subseteq \mathbf{V}^* \setminus \{x\} = [\bar{x}]$ and $\hat{\mathbf{v}}(y) = \{1\} \not\subseteq \emptyset = \hat{\mathbf{v}}(\bar{x})$ where \mathbf{v} is a valuation of lang_{\emptyset} s.t.
 147 $\mathbf{v}(x) = \mathbf{v}(y) = \{1\}$. See [35] for more counter-examples.

148 ► **Remark 2.3.** For KA, the equational theory w.r.t. languages coincides with the language
 149 equivalence [2][41]: for (non-extended) KA terms t and s , $[t] = [s] \Leftrightarrow \text{LANG} \models t = s$. This is
 150 an easy consequence of the completeness theorem of KA [24] (see also [35, Appendix A] for
 151 a direct proof). From this, KA with complement (even with variable complements) has a
 152 strictly more expressive power than KA.

153 In the sequel, we consider the equational theory w.r.t. languages.

154 2.3 (Generalized) relational models

155 We write Δ_A for the identity relation on a set A : $\Delta_A \triangleq \{\langle x, x \rangle \mid x \in A\}$. For binary relations
 156 R, S on a set B , the composition $R; S$, the n -th iteration R^n (where $n \in \mathbb{N}$), and the reflexive
 157 transitive closure R^* are defined by:

$$158 \quad R; S \triangleq \{\langle x, z \rangle \mid \exists y, \langle x, y \rangle \in R \wedge \langle y, z \rangle \in S\}, \quad R^n \triangleq \begin{cases} R; R^{n-1} & (n \geq 1) \\ \Delta_B & (n = 0) \end{cases}, \quad R^* \triangleq \bigcup_{n \in \mathbb{N}} R^n.$$

160 Let U be a binary relation on a non-empty set B . A *generalized relational model*² \mathcal{A} on
 161 U is an S-algebra such that $|\mathcal{A}| \subseteq \wp(U)$, $1^{\mathcal{A}} = \Delta_B$, $0^{\mathcal{A}} = \emptyset$, and for all $R, S \subseteq U$,

$$162 \quad R \cdot^{\mathcal{A}} S = R; S, \quad R +^{\mathcal{A}} S = R \cup S, \quad R^{*\mathcal{A}} = R^*, \quad R^{-\mathcal{A}} = U \setminus R.$$

164 We say that \mathcal{A} is a *relational model* if $U = B^2$ and $|\mathcal{A}| = \wp(B^2)$. We write GREL (resp. REL)
 165 for the class of all valuations of generalized relational models (resp. relational models).³

² By definition, for each generalized relational model, U is a preorder: (Reflexivity): By $\Delta_B = 1^{\mathcal{A}} \in |\mathcal{A}| \subseteq \wp(U)$, we have $\Delta_B \subseteq U$; (Transitivity): By $\emptyset = 0^{\mathcal{A}} \in |\mathcal{A}|$, $U = \emptyset^{-\mathcal{A}} \in |\mathcal{A}|$, and $U; U = U \cdot^{\mathcal{A}} U \in |\mathcal{A}| \subseteq \wp(U)$, we have $U; U \subseteq U$.

³ Generalized relational models and relational models are variants of proper relation algebras and full proper relation algebras (see, e.g., [21]), respectively, where B is non-empty set and the converse operator is not introduced (due to this, U is possibly not symmetric, cf. [21, Lem. 3.4]) here.

166 For a generalized relational model \mathcal{A} on a binary relation U on a set A and a non-empty
 167 subset $B \subseteq A$ s.t. $\{z \in A \mid \exists x, y \in B, \langle x, z \rangle, \langle z, y \rangle \in U\} \subseteq B$, the *restriction* $\mathcal{A} \upharpoonright B$ of \mathcal{A}
 168 w.r.t. B is the generalized relational model on the binary relation $U \cap B^2$ on the set B with
 169 the universe $\{R \cap B^2 \mid R \in |\mathcal{A}|\}$. It is easy to see that the map $\kappa_B: R \mapsto R \cap B^2$ forms an
 170 S-homomorphism from \mathcal{A} to $\mathcal{A} \upharpoonright B$ (the condition of B above is for preserving \cdot and $*$). For
 171 a valuation \mathfrak{v} of \mathcal{A} , let $\mathfrak{v} \upharpoonright B$ be the valuation of $\mathcal{A} \upharpoonright B$ given by the map κ_B .

172 3 RSUB: finite relational models for language models

173 In this section, we define the class RSUB of *relational subword models*, for the equational
 174 theory w.r.t. languages of $\text{KA}_{\{-\}}$.

175 ► **Definition 3.1.** Let $n \in \mathbb{N}$. The relational subword language model \mathcal{A} of length n , written
 176 rsub_n , is the generalized relational model on the set $U = \{\langle i, j \rangle \in [0, n]^2 \mid i \leq j\}$ s.t.

$$177 \quad |\mathcal{A}| = \{R \in \wp(U) \mid R \supseteq \Delta_{[0, n]} \vee U \setminus R \supseteq \Delta_{[0, n]}\}.$$

178 We write RSUB_n for the class of all valuations of rsub_n and write RSUB for $\bigcup_{n \in \mathbb{N}} \text{RSUB}_n$. ◻

179 Each rsub_n is defined based on the image of the map $\iota_X: L \mapsto \{\langle w, wv \rangle \mid w \in X^* \wedge v \in L\}$
 180 [42, third page] (called Pratt's embedding or Cayley map), cf. [48, 41], where we restrict the
 181 universe X^* of words into the subwords of a word of length n with *pairwise distinct* letters.

182 Let rlang_X be the generalized relational model on $\iota_X(X^*)$ with the universe $\{\iota_X(L) \mid L \subseteq$
 183 $X^*\}$. It is easy to see that the map ι_X forms an S-isomorphism from lang_X to rlang_X . For a
 184 word w , let $\text{Subw}(w)$ be the set of subwords of w . By Def. 3.1, it is easily shown that

- 185 ■ for a word $w = a_1 \dots a_n \in X$ where a_1, \dots, a_n are pairwise distinct letters, the generalized
- 186 relational model $\text{rlang}_X \upharpoonright \text{Subw}(w)$ is isomorphic to rsub_n ;
- 187 ■ for a word $w \in X^*$ of length n , the generalized relational model $\text{rlang}_X \upharpoonright \text{Subw}(w)$ is
- 188 isomorphic to a subalgebra of rsub_n ,
- 189 by the map $\theta: R \mapsto \{\langle \|w\|, \|v\| \rangle \mid \langle w, v \rangle \in R\}$.

190 We then have that the equational theory w.r.t. languages coincides with that w.r.t. RSUB.
 191

192 ► **Theorem 3.2.** For all $\text{KA}_{\{-\}}$ terms t and s , we have: $\text{LANG} \models t \leq s \Leftrightarrow \text{RSUB} \models t \leq s$.

193 **Proof.** (\Rightarrow): By the *surjective* S-homomorphism given by:

$$194 \quad \text{lang}_{\{a_1, \dots, a_n\}} \xrightarrow{\iota_{\{a_1, \dots, a_n\}}} \text{rlang}_{\{a_1, \dots, a_n\}} \xrightarrow{\kappa_{\text{Subw}(a_1 \dots a_n)}} \text{rlang}_X \upharpoonright \text{Subw}(a_1 \dots a_n) \xrightarrow{\theta} \text{rsub}_n$$

195 where a_1, \dots, a_n are pairwise distinct letters, for each $n \in \mathbb{N}$.

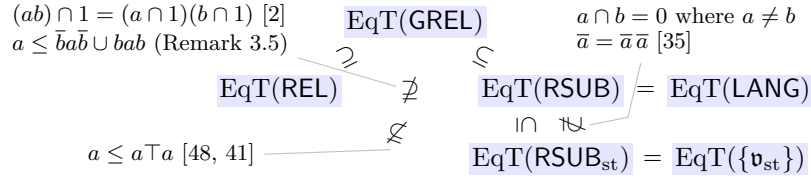
196 (\Leftarrow): We prove the contraposition. By $\text{LANG} \not\models t \leq s$, there are X , $\mathfrak{v} \in \text{LANG}_X$, and
 197 $w_0 \in X^*$ such that $w_0 \in \hat{\mathfrak{v}}(t) \setminus \hat{\mathfrak{v}}(s)$. We then consider the S-homomorphism given by:

$$198 \quad \text{lang}_X \xrightarrow{\iota_X} \text{rlang}_X \xrightarrow{\kappa_{\text{Subw}(w_0)}} \text{rlang}_X \upharpoonright \text{Subw}(w_0) \xrightarrow{\theta} \text{rsub}_{\|w_0\|}$$

199 Let \mathfrak{v}' , \mathfrak{v}'' , and \mathfrak{v}''' be the valuations of rlang_X , $\text{rlang}_X \upharpoonright \text{Subw}(w_0)$, and rsub_n , defined by
 200 $\mathfrak{v}' = \iota_X \circ \mathfrak{v}$, $\mathfrak{v}'' = \kappa_{\text{Subw}(w_0)} \circ \mathfrak{v}'$, and $\mathfrak{v}''' = \theta \circ \mathfrak{v}''$. We then have:

$$\begin{aligned} 201 \quad w_0 \in \hat{\mathfrak{v}}(t) \setminus \hat{\mathfrak{v}}(s) &\Rightarrow \langle 1, w_0 \rangle \in \hat{\mathfrak{v}}'(t) \setminus \hat{\mathfrak{v}}'(s) && \text{(By } w_0 \in L \text{ iff } \langle 1, w_0 \rangle \in \iota_X(L)) \\ 202 \quad &\Rightarrow \langle 1, w_0 \rangle \in \hat{\mathfrak{v}}''(t) \setminus \hat{\mathfrak{v}}''(s) && \text{(By } 1, w_0 \in \text{Subw}(w_0)) \\ 203 \quad &\Rightarrow \langle 0, \|w_0\| \rangle \in \hat{\mathfrak{v}}'''(t) \setminus \hat{\mathfrak{v}}'''(s). && \text{(By } \langle 1, w_0 \rangle \in R \text{ iff } \langle 0, n \rangle \in \theta(R) \text{ for } R \subseteq \text{Subw}(w_0)^2) \end{aligned}$$

204 Hence, $\text{RSUB} \not\models t \leq s$. ◀



■ **Figure 1** Equational theories for $\text{KA}_{\{-\}}$ under GREL.

244 ▶ **Lemma 4.1.** *For all terms t, s , $\text{RSUB} \models t \leq s \Leftrightarrow \forall n \in \mathbb{N}, \forall \mathbf{v} \in \text{RSUB}_n, \langle 0, n \rangle \notin \hat{\mathbf{v}}(t) \setminus \hat{\mathbf{v}}(s)$.*

245 **Proof.** (\Rightarrow) : Trivial. (\Leftarrow) : We revisit the proof of Thm. 3.2. If $\text{RSUB} \not\models t \leq s$, then by
246 $\text{LANG} \not\models t \leq s$ (Thm. 3.2), there is a valuation \mathbf{v} of rsub_n such that $\langle 0, n \rangle \in \hat{\mathbf{v}}(t \cap s^-)$ (by
247 the proof of Thm. 3.2). Hence, this completes the proof. ◀

248 Next, we consider replacing each inequation $u \leq 0$ with $\top u \top \leq 0$. We then have that
249 $\mathbf{v} \models u \leq 0$ iff $\langle 0, n \rangle \notin \hat{\mathbf{v}}(\top u \top)$, where $n \in \mathbb{N}$ and $\mathbf{v} \in \text{RSUB}_n$ (see Lem. 4.2). More generally,
250 for a quantifier-free formula φ , let $\text{Tr}(\varphi)$ be the term defined by:⁴

$$251 \quad \text{Tr}(t = s) \triangleq \top((t \cap s^-) \cup (t^- \cap s))\top, \quad \text{Tr}(\varphi \wedge \psi) \triangleq \text{Tr}(\varphi) \cup \text{Tr}(\psi), \quad \text{Tr}(\neg\varphi) \triangleq \text{Tr}(\varphi)^-.$$

253

254 ▶ **Lemma 4.2.** *Let $n \in \mathbb{N}$ and $\mathbf{v} \in \text{RSUB}_n$. For all quantifier-free formulas φ , we have:*

$$255 \quad \mathbf{v} \models \varphi \Leftrightarrow \langle 0, n \rangle \notin \hat{\mathbf{v}}(\text{Tr}(\varphi)).$$

256 **Proof.** By easy induction on φ . We distinguish the following forms of φ . Case $(t = s)$: Let
257 $u = (t \cap s^-) + (t^- \cap s)$. Then $\mathbf{v} \models t = s$ iff $\hat{\mathbf{v}}(u) = \emptyset$ iff $\langle 0, n \rangle \notin \hat{\mathbf{v}}(\top u \top)$ iff $\langle 0, n \rangle \notin \hat{\mathbf{v}}(\text{Tr}(t = s))$.
258 Case $\psi \wedge \rho$: By $(\langle 0, n \rangle \notin \hat{\mathbf{v}}(\text{Tr}(\psi)) \text{ and } \langle 0, n \rangle \notin \hat{\mathbf{v}}(\text{Tr}(\rho)))$ iff $\langle 0, n \rangle \notin \hat{\mathbf{v}}(\text{Tr}(\psi) + \text{Tr}(\rho))$. Case
259 $\neg\psi$: By $(\text{not } \langle 0, n \rangle \notin \hat{\mathbf{v}}(\text{Tr}(\psi)))$ iff $\langle 0, n \rangle \notin \hat{\mathbf{v}}(\text{Tr}(\psi)^-)$. ◀

260 ▶ **Theorem 4.3.** *For all quantifier-free formulas φ , $\text{RSUB} \models \varphi \Leftrightarrow \text{RSUB} \models \text{Tr}(\varphi) \leq 0$.*

261 **Proof.** By Lems. 4.1 and 4.2. ◀

262 ▶ **Remark 4.4.** “ $\text{RSUB} \models \varphi \Leftrightarrow \text{Tr}(\varphi) \leq 0$ ” fails in general (e.g., when φ is $\neg x \leq 0$), cf. the
263 theorem above and $\text{REL} \models \varphi \Leftrightarrow \text{Tr}(\varphi) \leq 0$ holds by the Schröder-Tarski translation [46,
264 XXXII.][18, p. 390, 391]. This is because $\text{RSUB} \not\models (\neg x \leq 0) \rightarrow \top \leq \top x \top$; when $\mathbf{v} \in \text{RSUB}_1$
265 satisfies $\mathbf{v}(x) = \{\langle 0, 1 \rangle\}$, we have $\hat{\mathbf{v}}(\top) = \{\langle 0, 0 \rangle, \langle 0, 1 \rangle, \langle 1, 1 \rangle\}$ but $\hat{\mathbf{v}}(\top x \top) = \{\langle 0, 1 \rangle\}$. This is
266 why we go via “ $\langle 0, n \rangle \notin \hat{\mathbf{v}}(_)$ ”. ◻

267 ▶ **Corollary 4.5** (Hoare hypothesis elimination). *For all terms t, s, u , we have:*

$$268 \quad \text{RSUB} \models u \leq 0 \rightarrow t \leq s \Leftrightarrow \text{RSUB} \models t \leq s \cup \top u \top.$$

269 **Proof.** By set-theoretic equivalences with Thm. 4.3, we have:

$$\begin{array}{lcl}
270 & \text{RSUB} \models u \leq 0 \rightarrow t \leq s & \Leftrightarrow \text{RSUB} \models \top(t \cap s^-)\top \leq \top u \top & \text{(By Thm. 4.3)} \\
271 & & \Leftrightarrow \text{RSUB} \models t \cap s^- \leq \top u \top & (\Rightarrow: \text{By } 1 \leq \top \Leftarrow: \text{By } \top \top \leq \top) \\
272 & & \Leftrightarrow \text{RSUB} \models t \leq s \cup \top u \top & \blacktriangleleft \\
273 & & &
\end{array}$$

⁴ $\text{Tr}(t = s)$ can be simplified for specific cases, e.g., $\text{Tr}(t \leq s) = \top(t \cap s^-)\top$ and $\text{Tr}(t \leq 0) = \top t \top$.

274 ▶ Remark 4.6. Thm. 4.3 and Cor. 4.5 fail w.r.t. LANG. For instance,

$$275 \quad \text{LANG} \models xx \leq 0 \rightarrow x \leq 0, \quad \text{LANG} \not\models x \leq \top xx \top. \quad \lrcorner$$

277 ▶ Remark 4.7. By Thm. 4.3, there is a polynomial-time reduction from the quantifier-free
278 theory to the equational theory w.r.t. RSUB. Thus by Cor. 3.4, the quantifier-free theory
279 w.r.t. RSUB for $\text{KA}_{\{-\}}$ terms is also in Π_1^0 . However, we leave open the complexity of the
280 quantifier-free theory (resp. Horn theory) w.r.t. LANG for $\text{KA}_{\{-\}}$ terms; particularly, whether
281 there exists a recursive function from the quantifier-free theory (resp. Horn theory) to the
282 equational theory w.r.t. LANG. \lrcorner

283 4.1 Undecidability via Hoare hypothesis elimination

284 Using Hoare hypothesis elimination w.r.t. RSUB (Cor. 4.5), we show the undecidability of
285 the equational theory w.r.t. LANG. The proof can be obtained by the same argument as [32,
286 Lem. 47] where we replace REL with RSUB.

287 A *context-free grammar* (CFG) \mathfrak{C} over a finite set A is a tuple $\langle X, \mathcal{R}, s \rangle$, where

- 288 ■ X is a finite set of non-terminal labels s.t. $A \cap X = \emptyset$;
- 289 ■ \mathcal{R} is a finite set of rewriting rules $x \leftarrow w$ of $x \in X$ and $w \in (A \cup X)^*$;
- 290 ■ $s \in X$ is the start label.

291 The relation $x \vdash_{\mathfrak{C}} w$, where $x \in X$ and $w \in A^*$, is defined as the minimal relation closed
292 under the following rule: for all $n \in \mathbb{N}$, $x, x_1, \dots, x_n \in X$ and $w_0, \dots, w_n, v_1, \dots, v_n \in A^*$, if
293 $x \leftarrow w_0 x_1 w_1 \dots x_n w_n \in \mathcal{R}$, then $\frac{x_1 \vdash_{\mathfrak{C}} v_1 \quad \dots \quad x_n \vdash_{\mathfrak{C}} v_n}{x \vdash_{\mathfrak{C}} w_0 v_1 w_1 \dots v_n w_n}$. The *language* $[\mathfrak{C}]$ is defined by
294 $[\mathfrak{C}] \triangleq \{w \in A^* \mid s \vdash_{\mathfrak{C}} w\}$. It is well-known that the *universality problem* for CFGs—given a
295 CFG \mathfrak{C} , does $[\mathfrak{C}] = A^*$ hold?—is Π_1^0 -complete. We can naturally encode this problem by the
296 quantifier-free theory w.r.t. RSUB as follows.

297 ▶ **Lemma 4.8.** *Let $\mathfrak{C} = \langle X, \mathcal{R}, s \rangle$ be a CFG over a finite set $A = \{a_1, \dots, a_n\}$. Then,*

$$298 \quad [\mathfrak{C}] = A^* \quad \Leftrightarrow \quad \text{RSUB} \models \left(\bigwedge_{(x \leftarrow w) \in \mathcal{R}} w \leq x \right) \rightarrow \left(\left(\sum_{i=1}^n a_i \right)^* \leq s \right).$$

299 **Proof Sketch.** By the same argument as [32, Lem. 47] where we replace REL with RSUB
300 (see Appendix G for more details). \blacktriangleleft

301 ▶ **Theorem 4.9.** *The equational theory w.r.t. languages is Π_1^0 -complete for $\text{KA}_{\{\bar{x}, \cap\}}$.*

302 **Proof.** (in Π_1^0): By Cor. 3.4. (Π_1^0 -hard): Let $\mathfrak{C} = \langle X, \{x_i \leftarrow w_i \mid i \in [1, n]\}, s \rangle$ be a
303 CFG over a finite set A . For Lem. 4.8, by Hoare hypothesis elimination (Cor. 4.5) with
304 $\text{RSUB} \models (\bigwedge_{i=1}^n w_i \leq x_i) \leftrightarrow (\sum_{i=1}^n w_i \cap \bar{x}_i) \leq 0$, we have: $[\mathfrak{C}] = A^*$ iff $\text{RSUB} \models (\sum_{i=1}^n a_i)^* \leq$
305 $s \cup \top (\sum_{i=1}^n w_i \cap \bar{x}_i) \top$. Thus, we can give a reduction from the universality problem of
306 CFGs. \blacktriangleleft

307 Moreover, by using the following equivalence Prop. 4.10, we can eliminate Kleene-star from
308 Lem. 4.8 (Lem. 4.11).

309 ▶ **Proposition 4.10.** $\text{RSUB} \models \bar{1} = (\sum_{i=1}^n a_i) \top \rightarrow (\sum_{i=1}^n a_i)^* = \top$.

310 **Proof.** Let $n \in \mathbb{N}$ and $\mathfrak{v} \in \text{RSUB}_n$. Let $i \in [1, n]$. By $\langle i-1, i-1 \rangle \notin \hat{\mathfrak{v}}(\bar{1}) = \hat{\mathfrak{v}}((\sum_{i=1}^n a_i) \top)$,
311 we have $\langle i-1, i-1 \rangle \notin \hat{\mathfrak{v}}((\sum_{i=1}^n a_i))$. By $\langle i-1, i \rangle \in \hat{\mathfrak{v}}(\bar{1}) = \hat{\mathfrak{v}}((\sum_{i=1}^n a_i) \top)$, we have
312 $\langle i-1, i \rangle \in \hat{\mathfrak{v}}((\sum_{i=1}^n a_i))$ (by $\langle i-1, i-1 \rangle \notin \hat{\mathfrak{v}}((\sum_{i=1}^n a_i))$). Thus $\hat{\mathfrak{v}}((\sum_{i=1}^n a_i)^*) = \{\langle i, j \rangle \mid 0 \leq$
313 $i \leq j \leq n\} = \hat{\mathfrak{v}}(\top)$. \blacktriangleleft

314 ▶ **Lemma 4.11.** *Let $\mathcal{C} = \langle X, \mathcal{R}, s \rangle$ be a CFG over a finite set A . Then,*

$$315 \quad [\mathcal{C}] = A^* \quad \Leftrightarrow \quad \text{RSUB} \models (\bar{1} = (\sum_{i=1}^n a_i)^\top \wedge \bigwedge_{(x \leftarrow w) \in \mathcal{R}} w \leq x) \rightarrow (\top \leq s).$$

316 **Proof Sketch.** By the same argument as Lem. 4.8 using Prop. 4.10 (see Appendix H for
317 more details). ◀

318 Hence, the undecidability still holds even without Kleene-star.

319 ▶ **Theorem 4.12.** *The equational theory w.r.t. languages is Π_1^0 -complete for $\text{KA}_{\{-\}}$ without
320 Kleene-star.*

321 **Proof.** By Lem. 4.11 with Thms. 3.2 and 4.3 (as with Thm. 4.9). ◀

322 5 Graph characterization for $\text{KA}_{\{\bar{x}, \bar{1}, \top, \cap\}}$ terms

323 In Sects. 5 and 6, we show that the equational theory w.r.t. languages for $\text{KA}_{\{\bar{x}, \bar{1}, \top\}}$ is
324 decidable and PSPACE-complete. We recall Sect. 2 for graphs. In this section, we give a
325 graph characterization of the equational theory of RSUB for $\text{KA}_{\{\bar{x}, \bar{1}, \top, \cap\}}$, by generalizing the
326 graph characterization of REL [32, Thm. 18].

327 5.1 Graph languages for $\text{KA}_{\{\bar{x}, \bar{1}, \top, \cap\}}$

328 Let $\tilde{\mathbf{V}} \triangleq \{x, \bar{x} \mid x \in \mathbf{V}\} \cup \{\bar{1}, \top\}$ and $\tilde{\mathbf{V}}_1 \triangleq \tilde{\mathbf{V}} \cup \{1\}$. For a $\text{KA}_{\{\bar{x}, \bar{1}, \top, \cap\}}$ term t , the graph
329 language $\mathcal{G}(t)$ [1, 7, 32] is a set of graphs over $\tilde{\mathbf{V}}_1$ defined by:⁵

$$330 \quad \mathcal{G}(x) \triangleq \{ \rightarrow \circ - x \rightarrow \circ \} \text{ where } x \in \tilde{\mathbf{V}} \quad \mathcal{G}(0) \triangleq \emptyset \quad \mathcal{G}(1) \triangleq \{ \rightarrow \circ \} \\ 331 \quad \mathcal{G}(t \cap s) \triangleq \{ \rightarrow \circ \xrightarrow[G]{H} \circ \} \mid G \in \mathcal{G}(t) \wedge H \in \mathcal{G}(s) \quad \mathcal{G}(t \cup s) \triangleq \mathcal{G}(t) \cup \mathcal{G}(s) \\ 332 \quad \mathcal{G}(t \cdot s) \triangleq \{ \rightarrow \circ - G \rightarrow \circ - H \rightarrow \circ \} \mid G \in \mathcal{G}(t) \wedge H \in \mathcal{G}(s) \quad \mathcal{G}(t^*) \triangleq \bigcup_{n \in \mathbb{N}} \mathcal{G}(t^n). \\ 333$$

334 For a valuation $\mathbf{v} \in \text{GREL}$ on a binary relation on a set B and $\langle x, y \rangle \in \hat{\mathbf{v}}(\top)$, let $\mathbf{G}(\mathbf{v}, x, y)$
335 be the graph defined by: $\mathbf{G}(\mathbf{v}, x, y) \triangleq \langle B, \{\hat{\mathbf{v}}(a)\}_{a \in \tilde{\mathbf{V}}_1}, x, y \rangle$. For a class $\mathcal{C} \subseteq \text{GREL}$, let $\text{GR}_{\mathcal{C}}$
336 be the graph language $\{\mathbf{G}(\mathbf{v}, x, y) \mid \mathbf{v} \in \mathcal{C} \text{ and } \langle x, y \rangle \in \hat{\mathbf{v}}(\top)\}$. We say that a graph language
337 \mathcal{G} is *induced subgraph-closed* if every induced subgraph of every $G \in \mathcal{G}$ is isomorphic to a
338 member of \mathcal{G} . We say that a class $\mathcal{C} \subseteq \text{GREL}$ is *induced subgraph-closed* if $\text{GR}_{\mathcal{C}}$ is induced
339 subgraph-closed. By the form of rsub_n , RSUB is induced subgraph-closed.

340 We recall edge-saturations $\mathcal{S}(h)$ of Def. 2.1. For a graph G and graph language \mathcal{G} , let

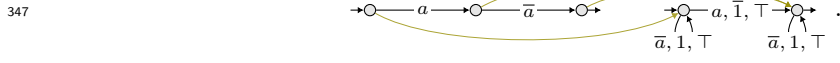
$$341 \quad \mathcal{S}_{\mathcal{C}}(G) \triangleq \{ \mathcal{S}(h) \mid \exists H \in \text{GR}_{\mathcal{C}}, h: G \rightarrow H \} \quad \mathcal{S}_{\mathcal{C}}(\mathcal{G}) \triangleq \bigcup_{H \in \mathcal{G}} \mathcal{S}_{\mathcal{C}}(H). \\ 342$$

343 ▶ **Example 5.1.** The following is an instance of $\mathcal{S}_{\text{RSUB}}(G)$ where $\mathbf{V} = \{a\}$:

$$344 \quad \mathcal{S}_{\text{RSUB}}(\rightarrow \circ - a \rightarrow \circ - \bar{a} \rightarrow \circ) = \left\{ \begin{array}{l} \begin{array}{c} \xrightarrow{\bar{1}, \top} \\ \begin{array}{ccc} \begin{array}{c} \rightarrow \circ - a, \bar{1}, \top \rightarrow \circ - a, \bar{1}, \top \rightarrow \circ \\ \downarrow \quad \downarrow \quad \downarrow \\ a, 1, \top \quad a, 1, \top \quad a, 1, \top \end{array} & , & \begin{array}{c} \rightarrow \circ - a, \bar{1}, \top \rightarrow \circ - \bar{a}, \bar{1}, \top \rightarrow \circ \\ \downarrow \quad \downarrow \quad \downarrow \\ \bar{a}, 1, \top \quad \bar{a}, 1, \top \quad \bar{a}, 1, \top \end{array} \end{array} \\ \begin{array}{c} \xrightarrow{\bar{1}, \top} \\ \begin{array}{ccc} \begin{array}{c} \rightarrow \circ - a, 1, \top \rightarrow \circ - a, \bar{1}, \top \rightarrow \circ \\ \downarrow \quad \downarrow \quad \downarrow \\ a, 1, \top \quad a, 1, \top \quad a, 1, \top \end{array} & , & \begin{array}{c} \rightarrow \circ - a, \bar{1}, \top \rightarrow \circ - \bar{a}, 1, \top \rightarrow \circ \\ \downarrow \quad \downarrow \quad \downarrow \\ \bar{a}, 1, \top \quad \bar{a}, 1, \top \quad \bar{a}, 1, \top \end{array} \end{array} \end{array} \right\}. \text{ (Gray-} \\ 345 \text{colored edges are the edges extended by edge-saturations } \mathcal{S}_{\text{RSUB}}. \text{ We omit unimportant edges.)}$$

⁵ We introduce \top -labeled edges, cf. [32, Def. 6], because \top is not fixed to the full relation.

346 For instance, the below right graph is obtained by the map:



348 ┘

349 Note that by the form of GR_{RSUB} , each $H \in \mathcal{S}_{\text{RSUB}}(\text{GR}_{\text{RSUB}})$ satisfies the following:

- 350 ■ $|H|$ is finite and \top^H is a total preorder (possibly not a total order);
- 351 ■ $a^H \supseteq 1^H$ or $\bar{a}^H \supseteq 1^H$ holds for each $a \in \mathbf{V}$.

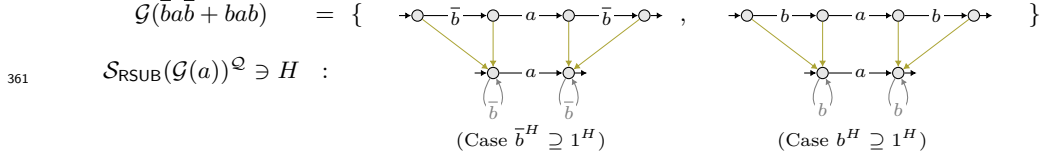
352 Let $H^\mathcal{Q} \triangleq H/(1^H)^\equiv$ and $\mathcal{G}^\mathcal{Q} \triangleq \{H^\mathcal{Q} \mid H \in \mathcal{G}\}$ where R^\equiv denotes the equivalence closure
 353 of R . We then have the following graph language characterization, which is an analog of [32,
 354 Thm. 18], but is generalized for including RSUB (see Appendix I, for a proof).

355 ► **Theorem 5.2.** *Let $\mathcal{C} \subseteq \text{GREL}$ be induced subgraph-closed. For all $\text{KA}_{\{\bar{x}, \bar{1}, \top, \cap\}}$ terms t, s ,*

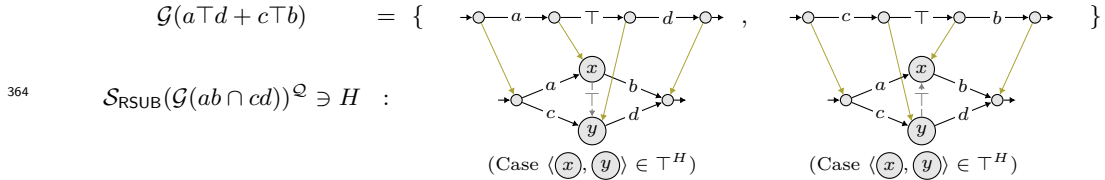
356
$$\mathcal{C} \models t \leq s \iff \forall H \in \mathcal{S}_{\mathcal{C}}(\mathcal{G}(t))^\mathcal{Q}, \exists G \in \mathcal{G}(s), G \longrightarrow H.$$

357 ► **Example 5.3.** (We recall the inequations in Remark 3.5.) Here are examples to show
 358 $\text{KA}_{\{\bar{x}, \bar{1}, \top, \cap\}}$ equations on RSUB using Thm. 5.2. (Gray-colored edges are the edges extended
 359 by edge-saturations $\mathcal{S}_{\text{RSUB}}$. We omit unimportant edges.)

360 ■ $\text{LANG} \models a \leq \bar{b}\bar{a} + bab$: This equation is shown by the following graph homomorphisms:



362 ■ $\text{LANG} \models ab \cap cd \leq a\top d + c\top b$: For each graph $H \in \mathcal{S}_{\text{RSUB}}(\mathcal{G}(ab \cap cd))^\mathcal{Q}$, we can give a
 363 graph homomorphism from some graph in $\mathcal{G}(a\top d + c\top b)$ as follows:

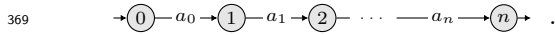


365 Additionally, note that $_^\mathcal{Q}$ is necessary in general, e.g., for $\top \leq 1 + \bar{1}$ [32, Remark 19]. ┘

366 5.2 Word languages for $\text{KA}_{\{\bar{x}, \bar{1}, \top\}}$

367 Particularly for $\text{KA}_{\{\bar{x}, \bar{1}, \top\}}$, Thm. 5.2 can be rephrased by word languages.

368 For a word $w = a_1 \dots a_n$ over $\tilde{\mathbf{V}}$, let $\mathbf{G}(w)$ be the following graph where $|\mathbf{G}(w)| = [0, n]$:



370 $\mathbf{G}(w)$ is the unique graph in $\mathcal{G}(w)$ up to graph isomorphisms.

371 For a $\text{KA}_{\{\bar{x}, \bar{1}, \top\}}$ term t , we write $[t]_{\tilde{\mathbf{V}}}$ for the word language $[t]$ over $\tilde{\mathbf{V}}$ where \bar{x} , $\bar{1}$, and
 372 \top are also viewed as letters; e.g., $[\bar{x}]_{\tilde{\mathbf{V}}} = \{\bar{x}\}$ and $[\bar{x}] = \mathbf{V}^* \setminus \{x\}$ for $x \in \mathbf{V}$. Note that
 373 $\mathcal{G}(t) = \{\mathbf{G}(w) \mid w \in [t]_{\tilde{\mathbf{V}}}\}$. Hence, for $\text{KA}_{\{\bar{x}, \bar{1}, \top\}}$ terms, graph languages are expressible by
 374 using word languages.

375 Additionally, we introduce nondeterministic finite word automata with epsilon transitions
 376 (*NFAs*). NFAs are (2-pointed) graphs over $\tilde{\mathbf{V}}_1$ where the source and target vertices denote
 377 the initial and (single) accepting states, respectively, and 1-labeled edges denote epsilon

378 transitions. For a graph H and a word $w = a_1 \dots a_n$, we write δ_w^H for the binary relation
 379 $(1^H)^*; a_1^H; (1^H)^*; \dots; a_n^H; (1^H)^*$. For $q \in |H|$, we let $\delta_w^H(q) \triangleq \{q' \mid \langle q, q' \rangle \in \delta_w^H\}$. For $Q \subseteq |H|$,
 380 we let $\delta_w^H(Q) \triangleq \bigcup_{q \in Q} \delta_w^H(q)$. The word language $[H]_{\tilde{\mathbf{V}}}$ is defined as $\{w \in \tilde{\mathbf{V}}^* \mid \langle 1^H, 2^H \rangle \in \delta_w^H\}$.
 381 Note that $[H]_{\tilde{\mathbf{V}}} = \{w \in \tilde{\mathbf{V}}^* \mid G(w) \longrightarrow H^{\mathcal{Q}}\}$ if 1^H is an equivalence relation. We then have
 382 the following: a rephrasing of Thm. 5.2 (see Appendix I for a proof).

383 ► **Corollary 5.4.** *Let $\mathcal{C} \subseteq \text{GREL}$ be induced subgraph-closed. For all $\text{KA}_{\{\bar{x}, \bar{1}, \top\}}$ terms t, s ,*

$$384 \quad \mathcal{C} \models t \leq s \iff [t]_{\tilde{\mathbf{V}}} \subseteq \{w \in \tilde{\mathbf{V}}^* \mid \forall H \in \mathcal{S}_{\mathcal{C}}(G(w)), [s]_{\tilde{\mathbf{V}}} \cap [H]_{\tilde{\mathbf{V}}} \neq \emptyset\}.$$

385 ► **Example 5.5.** Here are examples to show $\text{KA}_{\{\bar{x}, \bar{1}, \top\}}$ equations on RSUB using Cor. 5.4.

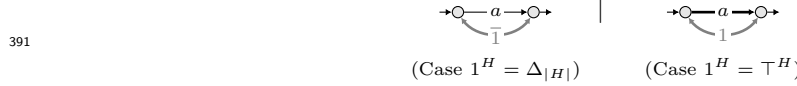
386 ■ **LANG** $\models a \leq \bar{b}\bar{a}\bar{b} + bab$ (the first example in Example 5.3): For all NFAs $H \in \mathcal{S}_{\text{RSUB}}(G(a))$,

387 we have $[\bar{b}\bar{a}\bar{b} + bab]_{\tilde{\mathbf{V}}} \cap [H]_{\tilde{\mathbf{V}}} \ni \begin{cases} \bar{b}\bar{a}\bar{b} & (\bar{b}^H \supseteq 1^H) \\ bab & (b^H \supseteq 1^H) \end{cases}$ by the following paths:

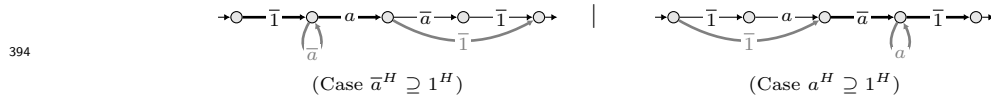


389 ■ **LANG** $\models a \leq \bar{1} + aa$ [32, (3)]: For all NFAs $H \in \mathcal{S}_{\text{RSUB}}(G(a))$, we have $[\bar{1} + aa]_{\tilde{\mathbf{V}}} \cap [H]_{\tilde{\mathbf{V}}} \ni$

390 $\begin{cases} \bar{1} & (1^H = \Delta_{|H|}) \\ aa & (1^H = \top^H) \end{cases}$ by the following paths:



392 ■ **LANG** $\models \bar{1}a\bar{a}\bar{1} \leq \bar{1}\bar{a}\bar{a}\bar{1}$ [36]: For all NFAs $H \in \mathcal{S}_{\text{RSUB}}(G(\bar{1}\bar{a}\bar{a}\bar{1}))$, we have $[\bar{1}\bar{a}\bar{a}\bar{1}]_{\tilde{\mathbf{V}}} \cap [H]_{\tilde{\mathbf{V}}} \ni$
 393 $\bar{1}\bar{a}\bar{a}\bar{1}$ in either $\bar{a}^H \supseteq 1^H$ or $a^H \supseteq 1^H$ by the following paths:



395 In the next section, we use the NFA characterization of Cor. 5.4 for an automata
 396 construction.

397 6 PSPACE decidability for $\text{KA}_{\{\bar{x}, \bar{1}, \top\}}$ terms

398 We consider graphs over $\tilde{\mathbf{V}}_1$ / NFAs, instead of $\text{KA}_{\{\bar{x}, \bar{1}, \top\}}$ terms / regular expressions over
 399 the alphabet $\tilde{\mathbf{V}}$. In this section, relying on the graph characterization (Cor. 5.4), we consider
 400 that, given an NFA J (having the same language as the term s in Cor. 5.4), we give an NFA
 401 recognizing the following word language:

$$402 \quad \mathbf{L}_J \triangleq \{w \in \tilde{\mathbf{V}}^* \mid \exists H \in \mathcal{S}_{\text{RSUB}}(G(w)), [J]_{\tilde{\mathbf{V}}} \cap [H]_{\tilde{\mathbf{V}}} = \emptyset\}.$$

403 Note that $\text{RSUB} \models t \leq s \iff [t]_{\tilde{\mathbf{V}}} \cap \mathbf{L}_J = \emptyset$ when $[s]_{\tilde{\mathbf{V}}} = [J]_{\tilde{\mathbf{V}}}$. We first present an equivalent
 404 notion of “ $w \in \mathbf{L}_J$ ” in Sect. 6.1, and then we give an NFA construction in Sect. 6.2. Our
 405 approach in this section is based on [32] where we replace REL with RSUB .

406 6.1 Saturable paths for RSUB

407 We first give an equivalent notion of $[J]_{\tilde{\mathbf{V}}} \cap [H]_{\tilde{\mathbf{V}}} = \emptyset$ in the definition of \mathbf{L}_J .

408 ► **Definition 6.1.** *Let J and H be NFAs. A map $U: |H| \rightarrow \wp(|J|)$ is an emptiness-witness
 409 for $[J]_{\tilde{\mathbf{V}}} \cap [H]_{\tilde{\mathbf{V}}} = \emptyset$ if the following hold where $U_x \triangleq U(x)$:*

12 Finite Relational Semantics for Language Kleene Algebra with Complement

410 ■ $1^J \in U_{1^H}$ and $\forall a \in \tilde{\mathbf{V}}_1, \forall \langle x, y \rangle \in a^H, \delta_a^J(U_x) \subseteq U_y$;

411 ■ $2^J \notin U_{2^H}$. ┘

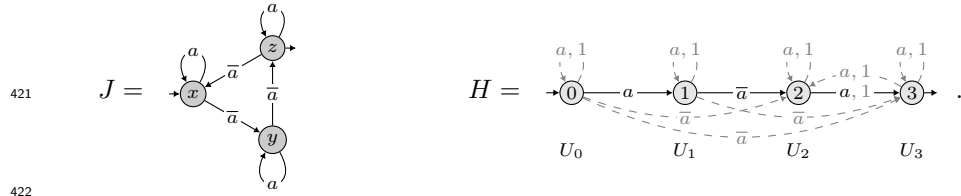
412 Intuitively, the first condition denotes that U is a cover of the reachable states from the pair
413 “ $1^J \in U_{1^H}$ ”. If the second condition holds, we can see that the pair “ $2^J \in U_{2^H}$ ” is unreachable.

414 As expected, we have the following (see Appendix B, for a proof).

415 ► **Proposition 6.2.** *Let J and H be NFAs where 1^H is reflexive. Then*

416 $[J]_{\tilde{\mathbf{V}}} \cap [H]_{\tilde{\mathbf{V}}} = \emptyset \iff \exists U: |H| \rightarrow \wp(|J|), U$ is an emptiness-witness for $[J]_{\tilde{\mathbf{V}}} \cap [H]_{\tilde{\mathbf{V}}} = \emptyset$.

417 ► **Example 6.3.** We consider the following NFAs J and H . The NFA J satisfies $[J]_{\tilde{\mathbf{V}}} = \{w \in$
418 $\{a, \bar{a}\}^* \mid \exists n \in \mathbb{N}, \bar{a}$ occurs $3n + 2$ times in $w\}$ and the NFA H is a graph in $\mathcal{S}_{\text{RSUB}}(\mathbf{G}(a\bar{a}a))$,
419 where \top - or $\bar{1}$ -labeled edges are omitted, and gray-colored edges are the edges edge-saturated
420 from the graph $\mathbf{G}(a\bar{a}a)$. From the form of H , one can see that $[J]_{\tilde{\mathbf{V}}} \cap [H]_{\tilde{\mathbf{V}}} = \emptyset$.



423 If $U_0 = U_1 = \{x\}$ and $U_2 = U_3 = \{y\}$, then this U is an emptiness-witness; e.g., for
424 $\langle 1, 2 \rangle \in \bar{a}^H$, $\delta_a^J(U_1) = \{y\} \subseteq U_2$. By the witnesses, we have $[J]_{\tilde{\mathbf{V}}} \cap [H]_{\tilde{\mathbf{V}}} = \emptyset$. Besides this,
425 if $U_0 = U_1 = \{x\}$ and $U_2 = U_3 = \{x, y\}$, this U is also an emptiness-witness; so, U may
426 not coincide with the reachable states from the pair “ $1^J \in U_{1^H}$ ”. ┘

427 Next, we give an equivalent notion of “ $w \in \mathbf{L}_J$ ”, by forgetting saturated edges (gray-colored
428 edges in Example 6.3) using U (in Prop. 6.2).

429 ► **Definition 6.4.** *Let J be a NFA and w be a word. A pair $P = \langle H, U \rangle$ is a saturable path
430 for $w \in \mathbf{L}_J$ if the following hold:*

431 (P-Ext) H is an edge-extension of $\mathbf{G}(w)$ such that⁶

432 ■ \top^H is a total preorder and $\top^H \supseteq \{\langle i-1, i \rangle \mid i \in [1, n]\}$ where $w = a_1 \dots a_n$;

433 ■ $1^H = \top^H \cap \{\langle j, i \rangle \mid \langle i, j \rangle \in \top^H\}$ and $\bar{1}^H = \top^H \setminus 1^H$;

434 ■ $\forall a \in \mathbf{V}, \langle a^H, \bar{a}^H \rangle$ is either $\langle a^{\mathbf{G}(w)} \cup 1^H, \bar{a}^{\mathbf{G}(w)} \rangle$ or $\langle a^{\mathbf{G}(w)}, \bar{a}^{\mathbf{G}(w)} \cup 1^H \rangle$.

435 (P-Con) H is consistent: $\forall a \in \mathbf{V}, a^{H^\ominus} \cap \bar{a}^{H^\ominus} = \emptyset$.

436 (P-Wit) $U: |H| \rightarrow \wp(|J|)$ is an emptiness-witness for $[J]_{\tilde{\mathbf{V}}} \cap [H]_{\tilde{\mathbf{V}}} = \emptyset$.

437 (P-Sat) H is saturable: $\forall a \in \mathbf{V}, \forall \langle i, j \rangle \in \bar{1}^H, \delta_a^J(U_i) \subseteq U_j$ or $\delta_a^J(U_i) \subseteq U_j$. ┘

438 Then the existence of saturable path can characterize “ $w \in \mathbf{L}_J$ ” as follows.

439 ► **Lemma 6.5** (Appendix C). *Let J be a NFA and w be a word. Then,*

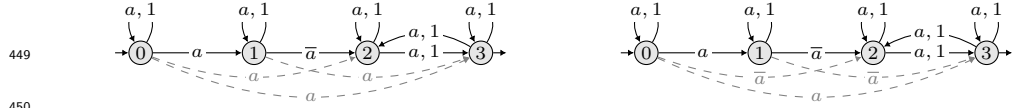
440 $w \in \mathbf{L}_J \iff$ there is a saturable path for $w \in \mathbf{L}_J$.

441 ► **Example 6.6.** We recall the NFAs J and $H \in \mathcal{S}_{\text{RSUB}}(\mathbf{G}(a\bar{a}a))$ in Example 6.3. The
442 following P is a saturable path for $a\bar{a}a \in \mathbf{L}_J$ where \top - or $\bar{1}$ -labeled edges are omitted:

⁶ In this definition, \top^H -, 1^H -, and $\bar{1}^H$ -edges are edge-saturated and a - and \bar{a} -edges in 1^H (for $a \in \mathbf{V}$) are also edge-saturated. This is for preserving (P-Con) easily.

$$P = \left(\begin{array}{cccc} \begin{array}{c} a, 1 \\ \circlearrowleft \\ 0 \end{array} & \begin{array}{c} a, 1 \\ \circlearrowleft \\ 1 \end{array} & \begin{array}{c} a, 1 \\ \circlearrowleft \\ 2 \end{array} & \begin{array}{c} a, 1 \\ \circlearrowleft \\ 3 \end{array} \\ \rightarrow 0 \xrightarrow{a} 1 \xrightarrow{\bar{a}} 2 \xrightarrow{a, 1} 3 \rightarrow & & & \\ \{\circledast x\} & \{\circledast x\} & \{\circledast x, \circledast y\} & \{\circledast x, \circledast y\} \end{array} \right).$$

(P is of the form of a *path graph* by taking the quotient graph w.r.t. 1-labeled edges.) P is an abstraction of edge-saturated graphs. From P , we can construct a graph $H \in \mathcal{S}_{\text{RSUB}}(\mathcal{G}(a\bar{a}a))$ s.t. $[J]_{\bar{\mathbf{V}}} \cap [H]_{\bar{\mathbf{V}}} = \emptyset$. Because both $\delta_a^J(\{\circledast x\}) \subseteq \{\circledast x, \circledast y\}$ and $\delta_a^J(\{\circledast x\}) \subseteq \{\circledast x, \circledast y\}$ hold, in addition to the graph H in Example 6.3, for instance, the following are also possible edge-saturated graphs:



Thanks to saturable paths, we can replace the existence of such gray-colored edges connecting distant vertices with a locally defined witness U . This rephrasing will be useful for our automata construction. \lrcorner

To give an NFA construction, moreover, we replace (P-Sat) with a local condition. Let $\varphi^J(\mathcal{U}, U) \triangleq \forall a \in \mathbf{V}, \forall \langle u, u' \rangle \in \mathcal{U}, \delta_a^J(u) \subseteq U \vee \delta_a^J(u') \subseteq U$.

► **Proposition 6.7.** *Let J and H be graphs. Let $i \in |H|$. Then we have:*

$$(\forall a \in \mathbf{V}, \forall j \text{ s.t. } \langle j, i \rangle \in \bar{1}^H, \delta_a^J(U_j) \subseteq U_i \vee \delta_a^J(U_j) \subseteq U_i) \Leftrightarrow \varphi^J\left(\bigcup_{j; \langle j, i \rangle \in \bar{1}^H} U_j \times U_j, U_i\right).$$

Proof. For each i and j , we have: $(\forall a \in \mathbf{V}, \delta_a^J(U_j) \subseteq U_i \vee \delta_a^J(U_j) \subseteq U_i)$ iff $(\forall a \in \mathbf{V}, (\forall u \in U_j, \delta_a^J(u) \subseteq U_i) \vee (\forall u' \in U_j, \delta_a^J(u') \subseteq U_i))$ iff $\varphi^J(U_j \times U_j, U_i)$ (by taking the prenex normal form). Thus by $(\forall j \text{ s.t. } \langle j, i \rangle \in \bar{1}^H, \varphi^J(U_j \times U_j, U_i))$ iff $\varphi^J(\bigcup_{j; \langle j, i \rangle \in \bar{1}^H} U_j \times U_j, U_i)$, this completes the proof. \blacktriangleleft

6.2 Automata from saturable paths

Let $\mathcal{X} \triangleq \{X \in \wp(\tilde{\mathbf{V}}_1) \mid 1, \top \in X, \bar{1} \notin X, \text{ and } \forall x \in \mathbf{V}, x \in X \leftrightarrow \bar{x} \notin X\}$. (This set is equivalent to the set $\{\{x \in \tilde{\mathbf{V}}_1 \mid 1^H \subseteq x^H\} \mid H \in \text{GR}_{\text{RSUB}}\}$.)

► **Definition 6.8** (NFA construction). *Let ► and ◀ be two fresh symbols. For a graph J and a set $X \in \mathcal{X}$, let J^{S^x} be the graph G defined as follows:*

- $|G| = \{\blacktriangleright, \blacktriangleleft\} \cup \{\langle \mathcal{U}, U \rangle \in \wp(|J| \times |J|) \times \wp(|J|) \mid \varphi^J(\mathcal{U}, U) \wedge \forall x \in X, \delta_x^J(U) \subseteq U\}$;
- $1^G = \{\blacktriangleright\} \times \{\langle \mathcal{U}, U \rangle \in |G| \mid 1^J \in U \wedge \mathcal{U} = \emptyset\} \cup \{\langle \mathcal{U}, U \rangle \in |G| \mid 2^J \notin U\} \times \{\blacktriangleleft\}$;
- $x^G = \{\langle \langle \mathcal{U}, U \rangle, \langle \mathcal{U}', U' \rangle \rangle \in |G| \times |G| \mid \psi_{x, \bar{1}}^X(\mathcal{U}, U, \mathcal{U}', U') \vee \psi_{x, 1}^X(\mathcal{U}, U, \mathcal{U}', U')\}$ for $x \in \tilde{\mathbf{V}}$;
- $1^G = \blacktriangleright$;
- $2^G = \blacktriangleleft$.

Here, $\psi_{x, \bar{1}}^X(\mathcal{U}, U, \mathcal{U}', U')$ and $\psi_{x, 1}^X(\mathcal{U}, U, \mathcal{U}', U')$ are defined as follows:

$$\psi_{x, \bar{1}}^X(\mathcal{U}, U, \mathcal{U}', U') \Leftrightarrow \left(\mathcal{U}' = \mathcal{U} \cup (U \times U) \wedge \bigwedge \left\{ \begin{array}{l} \delta_x^J(U) \subseteq U' \\ \delta_{\top}^J(\{u \mid \langle u, u \rangle \in \mathcal{U}'\}) \subseteq U' \\ \delta_{\bar{1}}^J(\{u \mid \langle u, u \rangle \in \mathcal{U}'\}) \subseteq U' \end{array} \right\} \right);$$

$$\psi_{x, 1}^X(\mathcal{U}, U, \mathcal{U}', U') \Leftrightarrow (\mathcal{U}' = \mathcal{U} \wedge U' = U \wedge x \in X).$$

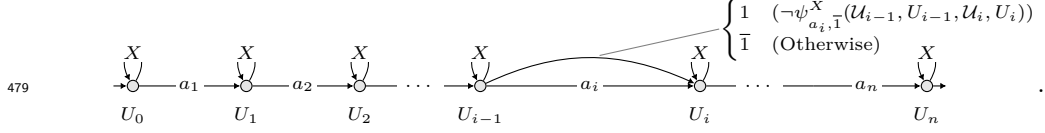
\lrcorner

475 By the form of $J^{\mathcal{S}^x}$, if $a_1 \dots a_n \in [J^{\mathcal{S}^x}]_{\bar{\mathbf{V}}}$, then its run is of the following form:

$$476 \quad \rightarrow \blacktriangleright \xrightarrow{1} \langle \mathcal{U}_0, U_0 \rangle \xrightarrow{a_1} \langle \mathcal{U}_1, U_1 \rangle \xrightarrow{a_2} \langle \mathcal{U}_2, U_2 \rangle \xrightarrow{\dots} \dots \xrightarrow{a_n} \langle \mathcal{U}_n, U_n \rangle \xrightarrow{1} \blacktriangleleft \rightarrow .$$

477 Intuitively, this run corresponds to the following saturable path where some \top -, $\bar{1}$ -, or

478 1-labeled edges are omitted and \bigcup^X denotes that an x -labeled edge exists for each $x \in X$:



480 Here, \mathcal{U}_i is used to denote the set $\bigcup_{j; \langle j, i \rangle \in \bar{1}^H} U_j \times U_j$ (cf. Prop. 6.7) where H is the graph of
 481 the saturable path above. According to this correspondence, from a word $w \in \bigcup_{X \in \mathcal{X}} [J^{\mathcal{S}^x}]_{\bar{\mathbf{V}}}$,
 482 we can construct a saturable path for $w \in L_J$. Conversely, from a saturable path for $w \in L_J$,
 483 we can show $w \in \bigcup_{X \in \mathcal{X}} [J^{\mathcal{S}^x}]_{\bar{\mathbf{V}}}$. Thus, we have the following (see Appendix D, for details).

484 **► Lemma 6.9** (Appendix D). *Let J be a graph. Then we have $L_J = \bigcup_{X \in \mathcal{X}} [J^{\mathcal{S}^x}]_{\bar{\mathbf{V}}}$.*

485 **► Theorem 6.10.** *The equational theory w.r.t. languages for $\text{KA}_{\{\bar{x}, \bar{1}, \top\}}$ is PSPACE-complete.*

486 **Proof.** (in PSPACE): Let t and s be $\text{KA}_{\{\bar{x}, \bar{1}, \top\}}$ terms. Let G and J be NFAs s.t. $[G]_{\bar{\mathbf{V}}} = [t]_{\bar{\mathbf{V}}}$
 487 and $[J]_{\bar{\mathbf{V}}} = [s]_{\bar{\mathbf{V}}}$. By Cor. 5.4 and Lem. 6.9, we have: $\text{RSUB} \models t \leq s \Leftrightarrow [G]_{\bar{\mathbf{V}}} \cap L_J = \emptyset \Leftrightarrow$
 488 $[G]_{\bar{\mathbf{V}}} \cap (\bigcup_{X \in \mathcal{X}} [J^{\mathcal{S}^x}]_{\bar{\mathbf{V}}}) = \emptyset$. Thus we can reduce the equational theory into the emptiness
 489 problem of NFAs of size exponential to the size of the input equation, where we use the
 490 union construction for \cup and the product construction for \cap in NFAs. In this reduction,
 491 using a standard on-the-fly algorithm for the non-emptiness problem of NFAs (essentially the
 492 graph reachability problem), we can give a non-deterministic polynomial space algorithm.
 493 (Note that the membership of “ $a \in |J^{\mathcal{S}^x}|$ ” and “ $\langle a, b \rangle \in x^{J^{\mathcal{S}^x}}$ ” for each $x \in \bar{\mathbf{V}}_1$ can be easily
 494 determined in polynomial space; so, we can construct such an on-the-fly algorithm indeed.)
 495 (Hardness): The equational theory of KA w.r.t. languages coincides with the language
 496 equivalence problem of regular expressions (Remark 2.3), which is PSPACE-complete [29].
 497 Hence, the equational theory of $\text{KA}_{\{\bar{x}, \bar{1}, \top\}}$ is PSPACE-hard. \blacktriangleleft

498 **► Remark 6.11.** It is still open whether the equational theory w.r.t. REL for $\text{KA}_{\{\bar{x}, \bar{1}, \top\}}$ is in
 499 PSPACE [32, Remark 45]. W.r.t. RSUB, each equivalence class induced from 1-labeled edges
 500 is always an interval; so, the problematic case presented in [32, Remark 45] (w.r.t. REL) does
 501 not appear in Thm. 6.10 (w.r.t. RSUB). \perp

502 **7 Conclusion and Future directions**

503 We have introduced RSUB for the equational theory w.r.t. languages for $\text{KA}_{\{-\}}$ terms. Using
 504 RSUB, we have shown some complexity results for the equational theory w.r.t. languages for
 505 fragments of $\text{KA}_{\{-\}}$ terms (Table 1). We leave open the decidability and complexity of the
 506 equational theory w.r.t. languages for $\text{KA}_{\{\bar{1}, \cap\}}$.

507 A natural interest is to consider variants or fragments of $\text{KA}_{\{-\}}$, e.g., with reverse [3],
 508 with tests [27] (by considering guarded strings) or with (anti-)domain [13]. It would also be
 509 interesting to consider the combination of variables and letters (cf. Thms. 3.2 and 3.6) in the
 510 context of language/string constraints.

511 Additionally, to separate the expressive power w.r.t. languages, it would also be interesting
 512 to consider games like Ehrenfeucht-Fraïssé games [15, 16] on RSUB, cf., e.g., on REL for the

	KA	KA _{x}	KA _{I}	KA _{x,I}	KA _{∩}	KA _{I,∩}	KA _{x,∩}	KA _{x,I,∩}	KA _{-}
LANG	PSPACE-c (Thm. 6.10)				EXPSPACE-c [4]	(open)	Π ₁ ⁰ -c (Thm. 4.9 and Cor. 3.4)		
RSUB	PSPACE-c [29]				EXPSPACE-c [17]			TOWER-c [45, 43]	
{v _{st} }	PSPACE-c [32]				in coNEXP [32]	EXPSPACE-c [30, 6, 7]	Π ₀ ¹ -c [33]	Π ₀ ¹ -c [32]	Π ₁ ¹ -c [19]
REL	PSPACE-c [32]				in coNEXP [32]	EXPSPACE-c [30, 6, 7]	Π ₀ ¹ -c [33]	Π ₀ ¹ -c [32]	Π ₁ ¹ -c [19]

■ **Table 1** Summary of our complexity results for equational theories w.r.t. languages, with comparison to other semantics.

513 calculus of relations [31] and on languages for star-free expressions [47]. (Note that while
514 the expressive powers w.r.t. languages can be separated if their language classes are not
515 equivalent, this argument cannot apply even to show, e.g., the expressive power difference
516 between $KA_{\{x\}}$ and KA .)

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624 **A** Proof of Cor. 5.4

625 **Proof.** We have:

$$\begin{aligned}
626 \quad \mathcal{C} \models t \leq s &\Leftrightarrow \forall w \in [t]_{\tilde{V}}, \forall H \in \mathcal{S}_{\mathcal{C}}(\mathbf{G}(w)), \exists v \in [s]_{\tilde{V}}, \mathbf{G}(v) \longrightarrow H^{\mathcal{Q}} \\
&\quad \text{(Thm. 5.2 and } \mathcal{G}(s) = \{\mathbf{G}(v) \mid v \in [s]_{\tilde{V}}\}) \\
627 \quad &\Leftrightarrow \forall w \in [t]_{\tilde{V}}, \forall H \in \mathcal{S}_{\mathcal{C}}(\mathbf{G}(w)), [s]_{\tilde{V}} \cap [H]_{\tilde{V}} \neq \emptyset \quad ([H]_{\tilde{V}} = \{v \in \tilde{V}^* \mid \mathbf{G}(v) \longrightarrow H^{\mathcal{Q}}\}) \\
628 \quad &\Leftrightarrow [t]_{\tilde{V}} \subseteq \{w \in \tilde{V}^* \mid \forall H \in \mathcal{S}_{\mathcal{C}}(\mathbf{G}(w)), [s]_{\tilde{V}} \cap [H]_{\tilde{V}} \neq \emptyset\}. \quad \blacktriangleleft \\
629
\end{aligned}$$

630 **B** Proof of Prop. 6.2

631 Let $R' \subseteq |H| \times |J|$ be the minimal set such that

- 632 ■ $\langle 1^H, 1^J \rangle \in R'$;
- 633 ■ $\forall a \in \tilde{V}_1, \forall x, x' \in |H|, \forall y, y' \in |J|, \langle x, y \rangle \in R' \wedge \langle x, x' \rangle \in \delta_a^H \wedge \langle y, y' \rangle \in \delta_a^J \Rightarrow \langle x', y' \rangle \in R'$.

634 ▷ **Claim B.1.** $[J]_{\tilde{V}} \cap [H]_{\tilde{V}} \neq \emptyset \Leftrightarrow \langle 2^H, 2^J \rangle \in R'$.

635 **Proof.** By definition, R' coincides with the set of all reachable states of the product NFA of
636 H and J . ◁

637 Let $R \subseteq |H| \times |J|$ be the minimal set such that

- 638 ■ $\langle 1^H, 1^J \rangle \in R$;
- 639 ■ $\forall a \in \tilde{V}_1, \forall x, x' \in |H|, \forall y, y' \in |J|, \langle x, y \rangle \in R \wedge \langle x, x' \rangle \in \delta_a^H \wedge \langle y, y' \rangle \in \delta_a^J \Rightarrow \langle x', y' \rangle \in R$.

640 ▷ **Claim B.2.** $R = R'$.

641 **Proof.** (\subseteq): Clear, by $a^H \subseteq \delta_a^H$. (\supseteq): By induction on derivations of R' .

- 642 ■ Case $\langle 1^H, 1^J \rangle \in R'$: Trivial by $\langle 1^H, 1^J \rangle \in R$.
- 643 ■ Case $\langle x, y \rangle \in R' \wedge \langle x, x' \rangle \in \delta_a^H \wedge \langle y, y' \rangle \in \delta_a^J \Rightarrow \langle x', y' \rangle \in R'$: By IH, $\langle x, y \rangle \in R$.
644 ■ Sub-Case $a \neq 1$: Let $x_0, \dots, x_{n-1}, x_n, \dots, x_m$ be s.t. $\langle x, x' \rangle = \langle x_0, x_m \rangle$ and

645 * for all $i \in [1, n-1]$, $\langle x_{i-1}, x_i \rangle \in 1^H$;
 646 * $\langle x_{n-1}, x_n \rangle \in a^H$;
 647 * for all $i \in [n+1, m]$, $\langle x_{i-1}, x_i \rangle \in 1^H$.
 648 Let $y_0 = \dots = y_{n-1} = y$ and $y_n = \dots = y_m = y'$. Then by applying the second rule
 649 multiply, we have $\langle x', y' \rangle \in R$.
 650 ■ Sub-Case $a = 1$: By reflexivity of 1^H , $\langle x, x' \rangle \in (1^H)^+$. Let x_0, \dots, x_m ($m > 0$) be s.t.
 651 $\langle x, x' \rangle = \langle x_0, x_m \rangle$ and
 652 * for all $i \in [1, m]$, $\langle x_{i-1}, x_i \rangle \in 1^H$.
 653 Let $y_0 = y$ and $y_1 = \dots = y_m = y'$. Then by applying the second rule multiply, we
 654 have $\langle x', y' \rangle \in R$. \triangleleft

655 **Proof of Prop. 6.2.** (\Rightarrow): By letting U as the map defined by $U(x) \triangleq \{y \mid \langle x, y \rangle \in R\}$. Here,
 656 $2^J \notin U_{2^H}$ is shown by $[J]_{\check{\mathbf{V}}} \cap [H]_{\check{\mathbf{V}}} = \emptyset$ with Claim B.1 and B.2. (\Leftarrow): Let $R'' \triangleq \{\langle x, y \rangle \mid y \in$
 657 $U(x)\}$. By the minimality of R , we have $R \subseteq R''$. By $\langle 2^H, 2^J \rangle \notin R''$, we have $\langle 2^H, 2^J \rangle \notin R$.
 658 Hence by Claim B.1 and B.2, we have $[J]_{\check{\mathbf{V}}} \cap [H]_{\check{\mathbf{V}}} = \emptyset$. \blacktriangleleft

659 **C Proof of Lem. 6.5**

660 **Proof.** (\Rightarrow): By Prop. 6.2, let $H' \in \mathcal{S}_{\text{RSUB}}(\mathbf{G}(w))$ and let U be an emptiness-witness for
 661 $[J]_{\check{\mathbf{V}}} \cap [H']_{\check{\mathbf{V}}} = \emptyset$. We define the graph H as follows:

- 662 ■ $|H| = |H'|$;
- 663 ■ $a^H = a^{H'}$ for $a \in \{\top, 1, \bar{1}\}$;
- 664 ■ $a^H = a^{\mathbf{G}(w)} \cup (a^{H'} \cap 1^{H'})$ for $a \in \check{\mathbf{V}}_1 \setminus \{\top, 1, \bar{1}\}$.

665 We then have that the pair $P \triangleq \langle H, U \rangle$ is a saturable path for $w \in \mathbf{L}_J$, as follows:

- 666 ■ (P-Ext): By that H' is an edge-saturation w.r.t. RSUB .
- 667 ■ (P-Con): Because H' is consistent by $H' \in \mathcal{S}_{\text{RSUB}}(\mathbf{G}(w))$.
- 668 ■ (P-Wit): Because U is an emptiness-witness for $[J]_{\check{\mathbf{V}}} \cap [H']_{\check{\mathbf{V}}} = \emptyset$.
- 669 ■ (P-Sat): Because $a^{H'} \cup \bar{a}^{H'} = \top^{H'}$ and U is an emptiness-witness for $[J]_{\check{\mathbf{V}}} \cap [H']_{\check{\mathbf{V}}} = \emptyset$.

670 (\Leftarrow): Let $P = \langle H, U \rangle$ be a saturable path for $w \in \mathbf{L}_J$. By (P-Ext), 1^H is an equivalence
 671 relation. We define the graph H' as follows:

- 672 ■ $|H'| = |H|$;
- 673 ■ $a^{H'} = a^H$ for $a \in \{\top, 1, \bar{1}\}$;
- 674 ■ for $a \in \mathbf{V}$ and $\langle x, y \rangle \in \top^H$,
 - 675 ■ if $\langle [x]_{1^H}, [y]_{1^H} \rangle \in a^{H^\circ}$, then $\langle x, y \rangle \in a^{H'} \setminus \bar{a}^{H'}$;
 - 676 ■ else if $\langle [x]_{1^H}, [y]_{1^H} \rangle \in \bar{a}^{H^\circ}$, then $\langle x, y \rangle \in \bar{a}^{H'} \setminus a^{H'}$;
 - 677 ■ else if $U_y \subseteq \delta_a^J(U_x)$, then $\langle x, y \rangle \in a^{H'} \setminus \bar{a}^{H'}$;
 - 678 ■ else $\langle x, y \rangle \in \bar{a}^{H'} \setminus a^{H'}$.

679 By the construction of H' , we have the following:

- 680 ■ H' is an edge-extension of H : By (P-Con), if $\langle [x]_{1^H}, [y]_{1^H} \rangle \in \bar{a}^{H^\circ}$, then $\langle [x]_{1^H}, [y]_{1^H} \rangle \notin$
 681 a^{H° .
- 682 ■ H' is consistent: If $[x]_{1^H} = [y]_{1^H}$ then $U_x = U_y$, because $U_x \subseteq \delta_1^J(U_x) \subseteq U_y \subseteq$
 683 $\delta_1^J(U_y) \subseteq U_x$ by (P-Wit); thus, if $[x]_{1^H} = [x']_{1^H}$ and $[y]_{1^H} = [y']_{1^H}$, then $\langle x, y \rangle \in a^{H'}$ iff
 684 $\langle x', y' \rangle \in a^{H'}$.
- 685 ■ for $a \in \mathbf{V}$, $\bar{a}^{H'} = \top^{H'} \setminus a^{H'}$: Because $a^{H'} \cup \bar{a}^{H'} = \top^{H'}$ and H' is consistent.

686 From them and (P-Ext), we have $H' \in \mathcal{S}_{\text{RSUB}}(\mathbf{G}(w))$. Also, U is an emptiness-witness for
 687 $[J]_{\check{\mathbf{V}}} \cap [H']_{\check{\mathbf{V}}} = \emptyset$ as follows. For edges already in H , it is shown by (P-Wit). For extended
 688 edges from H , it is shown by the construction of H' (for the last case of the four cases above,
 689 by $U_y \not\subseteq \delta_a^J(U_x)$ and (P-Sat), we have $U_y \subseteq \delta_a^J(U_x)$). Hence, this completes the proof. \blacktriangleleft

D Proof of Lem. 6.9

690

691 **Proof.** (\subseteq): Let $w = a_1 \dots a_n \in \mathbb{L}_J$. Let $P = \langle H, U \rangle$ be a saturable path for $w \in \mathbb{L}_J$. Let
 692 $X \triangleq \{a \in \tilde{\mathbb{V}}_1 \mid a^H \supseteq 1^H\}$ (note that $X \in \mathcal{X}$). For each i , let $\mathcal{U}_i \triangleq \bigcup_{j; \langle j, i \rangle \in \bar{1}^H} U_j \times U_j$. Then
 693 we have:

- 694 ■ $\varphi^J(\mathcal{U}_i, U_i)$: By (P-Sat) and Prop. 6.7.
- 695 ■ $\forall a \in X, \delta_a^J(U_i) \subseteq U_i$: By $a^H \supseteq 1^H \supseteq \Delta_{|H|}$ and (P-Wit).

696 Thus $\langle \mathcal{U}_i, U_i \rangle \in |J^{\mathcal{S}^X}|$. We consider the following run of the NFA $J^{\mathcal{S}^X}$ on w :

697 $\rightarrow \blacktriangleright \xrightarrow{1} \langle \mathcal{U}_0, U_0 \rangle \xrightarrow{-a_1} \langle \mathcal{U}_1, U_1 \rangle \xrightarrow{-a_2} \langle \mathcal{U}_2, U_2 \rangle \xrightarrow{\dots} \xrightarrow{-a_n} \langle \mathcal{U}_n, U_n \rangle \xrightarrow{1} \blacktriangleleft \rightarrow \dots$

698 This is indeed a run of the NFA $J^{\mathcal{S}^X}$ as follows:

- 699 ■ $\langle \blacktriangleright, \langle \mathcal{U}_0, U_0 \rangle \rangle \in 1^{J^{\mathcal{S}^X}}$: By $1^J \in U_0$ (P-Wit) and $\mathcal{U}_0 = \emptyset$.
- 700 ■ $\langle \langle \mathcal{U}_n, U_n \rangle, \blacktriangleleft \rangle \in 1^{J^{\mathcal{S}^X}}$: By $2^J \notin U_n$ (P-Wit).
- 701 ■ $\forall i \in [1, n], \langle \langle \mathcal{U}_{i-1}, U_{i-1} \rangle, \langle \mathcal{U}_i, U_i \rangle \rangle \in a_i^{J^{\mathcal{S}^X}}$: We distinguish the following cases:
 702 ■ Case $\langle i-1, i \rangle \in 1^H$:
 703 * $\mathcal{U}_i = \mathcal{U}_{i-1}$: By $\langle j, i \rangle \in \bar{1}^H$ iff $\langle j, i-1 \rangle \in \bar{1}^H$, for all j .
 704 * $U_i = U_{i-1}$: By (P-Wit), we have $U_{i-1} \subseteq \delta_1^J(U_{i-1}) \subseteq U_i \subseteq \delta_1^J(U_i) \subseteq U_{i-1}$.
 705 * $a_i \in X$ ($a_i^H \supseteq 1^H$): By $a_i^H \cap 1^H \neq \emptyset$ and (P-Ext), we have $a_i^H = a_i^{\mathbb{G}(w)} \cup 1^H$ (if not,
 706 this contradicts to (P-Con)).

707 Thus by $\psi_{a_i, 1}^X(\mathcal{U}_{i-1}, U_{i-1}, \mathcal{U}_i, U_i)$, we have $\langle \langle \mathcal{U}_{i-1}, U_{i-1} \rangle, \langle \mathcal{U}_i, U_i \rangle \rangle \in a_i^{J^{\mathcal{S}^X}}$.

- 708 ■ Case $\langle i-1, i \rangle \in \bar{1}^H$:
 709 * $\mathcal{U}_i = \mathcal{U}_{i-1} \cup (U_{i-1} \times U_{i-1})$: By $\langle j, i \rangle \in \bar{1}^H$ iff $j < i$ iff $\langle j, i-1 \rangle \in \bar{1}^H \vee \langle j, i-1 \rangle \in 1^H$,
 710 for all j .
 711 * $\delta_{a_i}^J(U_{i-1}) \subseteq U_i$: By (P-Wit).
 712 * $\delta_1^J(\{u \mid \langle u, u \rangle \in \mathcal{U}_i\}) \subseteq U_i$: We have $\delta_1^J(\{u \mid \langle u, u \rangle \in \mathcal{U}_i\}) = \delta_1^J(\bigcup_{j; \langle j, i \rangle \in \bar{1}^H} U_j) =$
 713 $\bigcup_{j < i} \delta_1^J(U_j) \subseteq U_i$ by (P-Wit).
 714 * $\delta_{\bar{1}}^J(\{u \mid \langle u, u \rangle \in \mathcal{U}_i\}) \subseteq U_i$: We have $\delta_{\bar{1}}^J(\{u \mid \langle u, u \rangle \in \mathcal{U}_i\}) = \delta_{\bar{1}}^J(\bigcup_{j; \langle j, i \rangle \in \bar{1}^H} U_j) =$
 715 $\bigcup_{j < i} \delta_{\bar{1}}^J(U_j) \subseteq U_i$ by (P-Wit).

716 Thus by $\psi_{a_i, \bar{1}}^X(\mathcal{U}_{i-1}, U_{i-1}, \mathcal{U}_i, U_i)$, we have $\langle \langle \mathcal{U}_{i-1}, U_{i-1} \rangle, \langle \mathcal{U}_i, U_i \rangle \rangle \in a_i^{J^{\mathcal{S}^X}}$.

717 Hence, $w \in [J^{\mathcal{S}^X}]$.

718 (\supseteq): Let $X \subseteq \mathcal{X}$ and $w = a_1 \dots a_n \in [J^{\mathcal{S}^X}]_{\tilde{\mathbb{V}}}$. Let the run of $J^{\mathcal{S}^X}$ on w be as follows:

719 $\rightarrow \blacktriangleright \xrightarrow{1} \langle \mathcal{U}_0, U_0 \rangle \xrightarrow{-a_1} \langle \mathcal{U}_1, U_1 \rangle \xrightarrow{-a_2} \langle \mathcal{U}_2, U_2 \rangle \xrightarrow{\dots} \xrightarrow{-a_n} \langle \mathcal{U}_n, U_n \rangle \xrightarrow{1} \blacktriangleleft \rightarrow \dots$

720 Let H be the edge-extension of $\mathbb{G}(w)$ defined as follows:

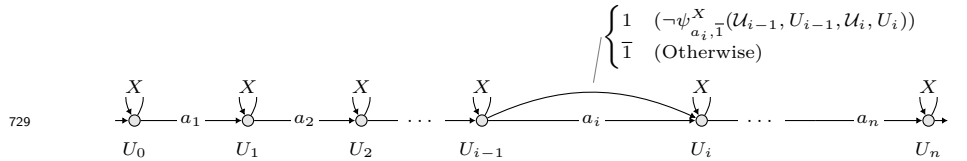
- 721 ■ $\top^H = \{\langle x, y \rangle \in [0, n] \times [0, n] \mid \forall i \in [y+1, x], \neg \psi_{a_i, \bar{1}}^X(\mathcal{U}_{i-1}, U_{i-1}, \mathcal{U}_i, U_i)\}$;
- 722 ■ $1^H = \top^H \cap \{\langle x, y \rangle \mid \langle y, x \rangle \in \top^H\}$ and $\bar{1}^H = \top^H \setminus 1^H$;
- 723 ■ $\forall a \in \mathbb{V} \cap X, \langle a^H, \bar{a}^H \rangle = \langle a^{\mathbb{G}(w)} \cup 1^H, \bar{a}^{\mathbb{G}(w)} \rangle$.

724 Note that by definition of \top^H , we have

- 725 ■ $\top^H \supseteq \{\langle x, y \rangle \mid x \leq y\}$;
- 726 ■ \top^H is transitive by case analysis.

727 Hence, \top^H is a total preorder and each equivalence class w.r.t. 1^H is an interval $[l, r]$.

728 Let $P \triangleq \langle H, U \rangle$ where U is defined as $i \mapsto U_i$ for $i \in [0, n]$. The following depicts P .



730 Then P is a saturable path for $w \in \mathbb{L}_J$ as follows:

731 ■ (P-Ext): By the definition of H .

732 ■ (P-Con): Assume that $a^{H^\circ} \cap \bar{a}^{H^\circ} \neq \emptyset$. Let x, x', y, y' be s.t. $[x]_{1^H} = [x']_{1^H}$, $[y]_{1^H} = [y']_{1^H}$,
733 $\langle x, y \rangle \in a^H$, and $\langle x', y' \rangle \in \bar{a}^H$. WLOG, we can assume that $a \in X$ and $\bar{a} \notin X$. Then, we
734 have the following:

- 735 ■ $\langle x', y' \rangle \in \bar{a}^{G(w)}$ (so, $x' = y' - 1$ and $a_{y'} = \bar{a}$): By $\bar{a}^H = \bar{a}^{G(w)}$ (since $\bar{a} \notin X$).
- 736 ■ $\langle x, y \rangle \in a^{G(w)}$ (so, $x = y - 1$ and $a_y = a$): If not, then by $a^H = a^{G(w)} \cup 1^H$,
737 we have $[x]_{1^H} = [y]_{1^H}$. Thus, $\langle y', y' - 1 \rangle \in 1^H (\subseteq \top^H)$. By the definition of
738 \top^H , we have $\neg \psi_{a_{y'}, \bar{1}}^X(\mathcal{U}_{y'-1}, U_{y'-1}, \mathcal{U}_{y'}, U_{y'})$. By the definition of $a^{J^{S^X}}$, we have
739 $\psi_{a_{y'}, 1}^X(\mathcal{U}_{y'-1}, U_{y'-1}, \mathcal{U}_{y'}, U_{y'})$, so $\bar{a} \in X$. This contradicts $\bar{a} \notin X$.
- 740 ■ $([x, x'] \cup [x', x]) \cap ([y, y'] \cup [y', y]) = \emptyset$ (so, $x = x'$ and $y = y'$): If not, then because the
741 interval between x and x' and that between y and y' have an intersection, we have
742 $[x]_{1^H} = [y]_{1^H}$. Then, in the same manner as above, we have $\bar{a} \in X$. This contradicts
743 $\bar{a} \notin X$.

744 Thus, we reach a contradiction, because $a = a_y = a_{y'} = \bar{a}$ (by $y = y'$). Hence,
745 $a^{H^\circ} \cap \bar{a}^{H^\circ} = \emptyset$.

746 ■ (P-Sat): By the form of J^{S^X} , we have $\mathcal{U}_x = \begin{cases} \mathcal{U}_{x-1} & (\langle x-1, x \rangle \in 1^H) \\ \mathcal{U}_{x-1} \cup (U_{x-1} \times U_{x-1}) & (\langle x-1, x \rangle \in \bar{1}^H) \end{cases}$.

747 Thus, $\mathcal{U}_y = \bigcup_{x; \langle x, y \rangle \in \bar{1}^H} U_x \times U_x$ (★). By Prop. 6.7, this completes the proof.

748 ■ (P-Wit): For $1^J \in U_0$ and $2^J \notin U_n$, they are shown by the form of J^{S^X} . For $\forall a \in$
749 $\tilde{\mathbf{V}}_1, \forall \langle x, y \rangle \in a^H, \delta_a^J(U_x) \subseteq U_y$, we distinguish the following cases:

750 ■ Case $a = 1$: Then we have

751 * $U_x = U_y$: By $\langle x, y \rangle \in 1^H$ and the form of J^{S^X} , we have the following: $\forall z \in$
752 $[y+1, x], \psi_{a_z, 1}^X(\mathcal{U}_{z-1}, U_{z-1}, \mathcal{U}_z, U_z)$. Thus, $U_y = U_{y+1} = \dots = U_x$.

753 * $\delta_1^J(U_x) \subseteq U_x$: By $\langle \mathcal{U}_x, U_x \rangle \in |J^{S^X}|$.

754 Hence, $\delta_1^J(U_x) \subseteq U_y$.

755 ■ Case $a = \bar{1}$: Let $z \in [x+1, y]$ be such that $\psi_{a_z, \bar{1}}^X(\mathcal{U}_{z-1}, U_{z-1}, \mathcal{U}_z, U_z)$ and $\forall z' \in$
756 $[z+1, y], \neg \psi_{a_{z'}, \bar{1}}^X(\mathcal{U}_{z'-1}, U_{z'-1}, \mathcal{U}_{z'}, U_{z'})$. Then we have

$$\begin{aligned} 757 \quad \delta_{\bar{1}}^J(U_x) &\subseteq \delta_{\bar{1}}^J(\{u \mid \langle u, u \rangle \in U_z\}) && \text{(by (★) and } \langle x, z \rangle \in \bar{1}^H \text{ (by } \langle z-1, z \rangle \in \bar{1}^H)) \\ 758 \quad &\subseteq U_z && \text{(by } \psi_{a_z, \bar{1}}^X(\mathcal{U}_{z-1}, U_{z-1}, \mathcal{U}_z, U_z)) \\ 759 \quad &\subseteq U_{z+1} = \dots = U_y. && \text{(by the form of } J^{S^X}, \psi_{a_{z'}, 1}^X(\mathcal{U}_{z'-1}, U_{z'-1}, \mathcal{U}_{z'}, U_{z'})) \end{aligned}$$

760 ■ Case $a = \top$: We distinguish the following two sub-cases:

761 * Case $\langle x, y \rangle \in \bar{1}^H$: By the similar argument as Case $a = \bar{1}$.

762 * Case $\langle x, y \rangle \in 1^H$: By the similar argument as Case $a = 1$, we have $U_x = U_y$ and
763 $\delta_{\bar{1}}^J(U_x) \subseteq U_x$, and thus $\delta_{\bar{1}}^J(U_x) \subseteq U_y$.

764 ■ Case $a \in \{a, \bar{a} \mid a \in \mathbf{V}\}$: We distinguish the following sub-cases:

765 * Case $\langle x, y \rangle \in \bar{1}^H$: By $\langle x, y \rangle \in a^H \cap \bar{1}^H = a^{G(w)}$, we have $x = y - 1$ and $a_y = a$.
766 Thus by $\psi_{a_y, \bar{1}}^X(\mathcal{U}_{y-1}, U_{y-1}, \mathcal{U}_y, U_y)$, we have $\delta_a^J(U_x) \subseteq U_y$.

767 * Case $a \notin X$: By $a^H = a^{G(w)}$, we have $x = y - 1$ and $a_y = a$. By the form of J^{S^X}
768 with $\neg \psi_{a_y, 1}^X(\mathcal{U}_{y-1}, U_{y-1}, \mathcal{U}_y, U_y)$ (since $a_y \notin X$), we have $\psi_{a_y, \bar{1}}^X(\mathcal{U}_{y-1}, U_{y-1}, \mathcal{U}_y, U_y)$.
769 Hence, $\delta_a^J(U_x) \subseteq U_y$.

770 * Case $\langle x, y \rangle \in 1^H$ and $a \in X$: By the similar argument as Case $a = 1$, we have
771 $U_x = U_y$ (by $\langle x, y \rangle \in 1^H$) and $\delta_a^J(U_x) \subseteq U_x$ (by $a \in X$). Thus, $\delta_a^J(U_x) \subseteq U_y$. ◀



774 (Extra Appendices)

775 **E** Extensions of Thm. 3.2

776 In this section, we note that we can extend Thm. 3.2 in the following two:

777 ► **Theorem E.1.** *For all positive quantifier-free formulas φ of $\text{KA}_{\{-\}}$ terms, we have:*
778 $\text{LANG} \models \varphi \Rightarrow \text{RSUB} \models \varphi$.

779 **Proof Sketch.** By the same surjective S-homomorphism in the proof of Thm. 3.2(\Rightarrow). ◀

780 ► **Theorem E.2.** *For all quantifier-free formulas φ of $\text{KA}_{\{-\}}$ terms, we have: $\text{LANG} \models \varphi \Leftarrow$*
781 $\text{RSUB} \models \varphi$.

782 **Proof.** Because the formulas $t = s \leftrightarrow (t \leq s \wedge s \leq t)$ and $t \leq s \leftrightarrow t \cap s^- \leq 0$ are valid
783 on $\text{LANG} \cup \text{SUB}$, without loss of generality, we can assume that each equation in φ is of
784 the form $u \leq 0$. By taking the conjunctive normal form, it suffices to prove when φ is of
785 the form $(\bigvee_{i=1}^n t_i \leq 0) \vee (\bigvee_{j=1}^m \neg s_j \leq 0)$. We prove the contraposition. By $\text{LANG} \not\models \varphi$,
786 there are $X, \mathbf{v} \in \text{LANG}_X, w_1, \dots, w_n \in X^*$ such that $w_i \in \hat{\mathbf{v}}(t_i)$ for $i \in [1, n]$ and $\hat{\mathbf{v}}(s_j) = \emptyset$
787 for $j \in [1, m]$. By letting $w_0 \triangleq w_1 \dots w_n$ and considering the same S-homomorphism as
788 Thm. 3.2(\Leftarrow), we have $\text{RSUB} \not\models \varphi$. ◀

789 **F** SUB: subword models

790 In this section, we also define the class SUB of *subword models*, which is isomorphic to RSUB.
791 Apart from RSUB, intuitively, this model is based on the following property in the standard
792 language valuation (\mathbf{v}_{st}): the membership $w \in \hat{\mathbf{v}}_{\text{st}}(t)$ can be determined from $w' \in \hat{\mathbf{v}}_{\text{st}}(t')$
793 where w' ranges over *subwords* of w (and t' ranges over subterms of t). This situation is the
794 same also for any valuations $\mathbf{v} \in \text{LANG}$. The class SUB is helpful to use this property. These
795 models are language models where the universe is restricted to the set of subwords of a word.

796 ► **Definition F.1.** *Let X be a set and let $w = a_0 \dots a_{n-1} \in X^*$. The subword language*
797 *model sub_w is the S-algebra defined by: $|\mathcal{A}| = \wp(\text{Subw}(w))$, $1^{\mathcal{A}} = \{1\}$, $0^{\mathcal{A}} = \emptyset$, and for all*
798 $L, K \subseteq \text{Subw}(w)$,

$$\begin{aligned} 799 \quad L \cdot^{\mathcal{A}} K &= (L \cdot K) \cap \text{Subw}(w), & L +^{\mathcal{A}} K &= L \cup K, \\ 800 \quad L^{*\mathcal{A}} &= L^* \cap \text{Subw}(w), & L^{-\mathcal{A}} &= \text{Subw}(w) \setminus L. \end{aligned}$$

802 We write SUB_w for the class of all valuations of sub_w and SUB. ◻

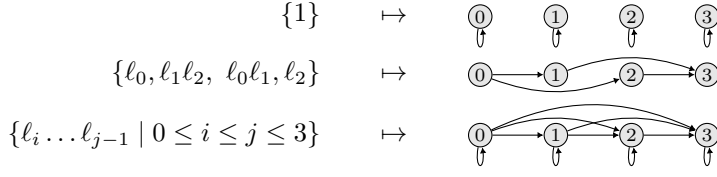
803 Let κ'_w be the map given by $L \mapsto L \cap \text{Subw}(w)$. The map κ'_w forms an S-homomorphism
804 from lang_X to sub_w .

805 We use ℓ_1, ℓ_2, \dots as fixed pairwise distinct letters. We write SUB for $\bigcup_{n \in \mathbb{N}} \text{SUB}_{\ell_1 \ell_2 \dots \ell_n}$.
806 Let χ' be the map given by $L \mapsto \{\ell_1 \dots \ell_n \mid a_1 \dots a_n \in L\}$. For each word $a_1 \dots a_n$, the map
807 χ' forms an S-homomorphism from $\text{sub}_{a_1 \dots a_n}$ to $\text{sub}_{\ell_1 \dots \ell_n}$.

808 ► **Remark F.2.** The map χ' is defined, inspired by words-to-letters valuations [35] where we
809 consider “*letters-to-letters valuations*” by restricting the mapping from words to letters.

810 ► **Proposition F.3.** *For each $n \in \mathbb{N}$, $\text{sub}_{\ell_0 \dots \ell_{n-1}}$ is isomorphic to rsub_n .*

811 **Proof.** By the map $L \mapsto \{\langle i, j \rangle \mid 0 \leq i \leq j \leq n \wedge \ell_i \dots \ell_{j-1} \in L\}$. This map can be also
812 given by $\theta \circ \iota_{\{\ell_0, \dots, \ell_{n-1}\}}$. Fig. 2 presents illustrative instances of this map. ◀



■ **Figure 2** Illustrative instances of the bijective map of Prop. F.3, when $n = 3$.

813 By Prop. F.3, the isomorphism closure of RSUB coincides with that of SUB . Particularly,
814 the quantifier-free theory w.r.t. RSUB coincides with that of SUB .

815 Summarizing the above and Sect. 3, for each $n \in \mathbb{N}$ and word $w \in X^n$, we have considered
816 the following S-homomorphisms:

$$\begin{array}{ccccc}
\text{lang}_X & \xrightarrow{\kappa'_w} & \text{sub}_w & \xrightarrow{\chi': L \mapsto \{w \mid f(w) \in L\}} & \text{sub}_{\ell_0 \dots \ell_{n-1}} \\
\downarrow \iota_X & & \downarrow \iota_X & \searrow R \mapsto \{\langle w, wv \rangle \mid \langle f(w), f(wv) \rangle \in R\} & \downarrow \iota_{\{\ell_0, \dots, \ell_{n-1}\}} \\
\text{rlang}_X & \xrightarrow{\kappa_w} & \text{rlang}_X \upharpoonright \text{Sub}_w(w) & \xrightarrow{\quad} & \text{rlang}_{\{\ell_0, \dots, \ell_{n-1}\}} \upharpoonright \text{Sub}_w(\ell_0 \dots \ell_{n-1}) \\
& & & & \downarrow \theta \\
& & & & \text{rsub}_n
\end{array}$$

818 where f is the monoid homomorphism given by $\ell_i \mapsto a_i$ for $i \in [0, n-1]$.

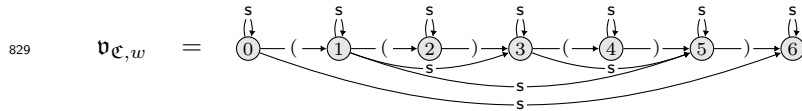
819 **G Detailed proof of Lem. 4.8**

820 (The following proof is almost the same as [32, Section VI] except that we consider RSUB
821 instead of REL .)

822 For a CFG $\mathfrak{C} = \langle X, \mathcal{R}, s \rangle$ and a word $w = a_1 \dots a_n$, let $\mathbf{v}_{\mathfrak{C}, w} \in \text{RSUB}_n$ be the valuation
823 defined by:

- 824 ■ $\mathbf{v}_{\mathfrak{C}, w}(x) = \{\langle i-1, i \rangle \mid i \in [1, n] \wedge x = a_i\}$ for $x \in A$;
- 825 ■ $\{\mathbf{v}_{\mathfrak{C}, w}(x)\}_{x \in X}$ are the minimal sets such that for all $x \in X$, $v \in (X \cup A)^*$, and $i, j \in [0, n]$,
- 826 if $x \leftarrow v \in \mathcal{R}$, then $\frac{\langle i, j \rangle \in \mathbf{v}_{\mathfrak{C}, w}(v)}{\langle i, j \rangle \in \mathbf{v}_{\mathfrak{C}, w}(x)}$.

827 For instance, if $A = \{(\cdot, \cdot)\}$, $\mathfrak{C} = \langle \{s\}, \{s \leftarrow (s)s, s \leftarrow ()\}, s \rangle$ (i.e., $[\mathfrak{C}]$ is the Dyck-1 language),
828 and $w = ((\cdot))$, then $\mathbf{v}_{\mathfrak{C}, w}$ is of the following form:



830 By construction, we have the following:

831 ► **Lemma G.1** (cf. [32, Lem. 47]). *Let $\mathfrak{C} = \langle X, \mathcal{R}, s \rangle$ be a CFG. Let \mathcal{C} be s.t. $\{\mathbf{v}_{\mathfrak{C}, w}\} \subseteq \mathcal{C} \subseteq$
832 GREL . For all $x \in X$ and $w = a_0 \dots a_{n-1} \in A^*$, the following are equivalent:*

- 833 1. $x \vdash_{\mathfrak{C}} w$;
- 834 2. $\mathcal{C} \models (\bigwedge_{(x' \leftarrow w') \in \mathcal{R}} w' \leq x') \rightarrow w \leq x$;
- 835 3. $\langle 0, n \rangle \in \mathbf{v}_{\mathfrak{C}, w}(x)$.

836 **Proof.** $1 \Rightarrow 2$: By induction on the derivation tree of $\vdash_{\mathfrak{C}}$. Let

$$\begin{array}{c}
837 \frac{x_1 \vdash_{\mathfrak{C}} v_1 \quad \dots \quad x_n \vdash_{\mathfrak{C}} v_m}{x \vdash_{\mathfrak{C}} w_0 v_1 w_1 \dots v_m w_m}
\end{array}$$

838 where $x \leftarrow w_0x_1w_1 \dots x_mw_m \in \mathcal{R}$. Let $\mathbf{v} \in \mathcal{C}$ be any s.t. $\mathbf{v} \models \bigwedge_{(x' \leftarrow w') \in \mathcal{R}} w' \leq x'$. Particu-
 839 larly, we have $\mathbf{v} \models w_0x_1w_1 \dots x_mw_m \leq x$. By IH, we have $\mathbf{v} \models v_k \leq x_k$. Combining them
 840 yields $\mathbf{v} \models w_0v_1w_1 \dots v_mw_m \leq x$. (We only need axioms of monoids, so this holds for any
 841 Kleene algebras beyond GREL.)

842 $2 \Rightarrow 3$: By $\mathbf{v}_{\mathcal{C},w} \in \mathcal{C}$, $\mathbf{v}_{\mathcal{C},w} \models \bigwedge_{(x' \leftarrow w') \in \mathcal{R}} w' \leq x'$, and $\langle 0, n \rangle \in \hat{\mathbf{v}}_{\mathcal{C},w}(w)$, we have $\langle 0, n \rangle \in$
 843 $\mathbf{v}_{\mathcal{C},w}(x)$ by 2.

844 $3 \Rightarrow 1$: By induction on the derivation tree induced from the definition of $\mathbf{v}_{\mathcal{C},w}$. Let

$$845 \frac{\langle i, j \rangle \in \hat{\mathbf{v}}_{\mathcal{C},w}(w_0x_1w_1 \dots x_mw_m)}{\langle i, j \rangle \in \mathbf{v}_{\mathcal{C},w}(x)}$$

846 where $x \leftarrow w_0x_1w_1 \dots x_mw_m \in \mathcal{R}$. Let $i = i_0 \leq j_0 \leq \dots \leq i_m \leq j_m = j - 1$ be s.t. $\langle i_k, j_k \rangle \in$
 847 $\hat{\mathbf{v}}_{\mathcal{C},w}(w_k)$ and $\langle j_{k-1}, i_k \rangle \in \mathbf{v}_{\mathcal{C},w}(x_k)$. By construction of $\mathbf{v}_{\mathcal{C},w}$, we have $w_k = a_{i_k} \dots a_{j_k}$. By
 848 IH, $x_k \vdash_{\mathcal{C}} a_{j_{k-1}} \dots a_{i_k}$. Hence by definition of $\vdash_{\mathcal{C}}$, we have $x \vdash_{\mathcal{C}} a_i \dots a_{j-1}$. \blacktriangleleft

849 \blacktriangleright **Lemma G.2** (cf. [32, Lem. 48]). *Let $\mathcal{C} = \langle X, \mathcal{R}, \mathbf{s} \rangle$ be a CFG over a finite set $A =$
 850 $\{a_1, \dots, a_n\}$. Let \mathcal{C} be s.t. $\{\mathbf{v}_{\mathcal{C},w} \mid w \in A^*\} \subseteq \mathcal{C} \subseteq \text{GREL}$. Then we have*

$$851 [\mathcal{C}] = A^* \Leftrightarrow \mathcal{C} \models \left(\bigwedge_{(x \leftarrow w) \in \mathcal{R}} w \leq x \right) \rightarrow \left(\sum_{i=1}^n a_i \right)^* \leq \mathbf{s}.$$

852 **Proof.** We have:

$$853 [\mathcal{C}] = A^* \Leftrightarrow \forall v \in A^*, \mathbf{s} \vdash_{\mathcal{C}} v \Leftrightarrow \forall v \in A^*, \mathcal{C} \models \left(\bigwedge_{(x \leftarrow w) \in \mathcal{R}} w \leq x \right) \rightarrow v \leq \mathbf{s} \quad (\text{By Lem. G.1})$$

$$854 \Leftrightarrow \forall m \in \mathbb{N}, \mathcal{C} \models \left(\bigwedge_{(x \leftarrow w) \in \mathcal{R}} w \leq x \right) \rightarrow \left(\sum_{i=1}^n a_i \right)^m \leq \mathbf{s}$$

$$855 \Leftrightarrow \mathcal{C} \models \left(\bigwedge_{(x \leftarrow w) \in \mathcal{R}} w \leq x \right) \rightarrow \left(\sum_{i=1}^n a_i \right)^* \leq \mathbf{s}. \quad (\text{GREL is } * \text{-continuous: } xy^*z = \sup_{n \geq 0} xy^n z) \quad \blacktriangleleft$$

856 **Proof of Lem. 4.8.** By Lem. G.2. \blacktriangleleft

858 **H** Proof of Lem. 4.11

859 **Proof.** We recall Lem. G.1. Because $\mathbf{v}_{\mathcal{C},w} \models \bar{\mathbf{1}} = \left(\sum_{i=1}^n a_i \right) \top$, Lem. G.1 holds even if we can
 860 replace 2 of Lem. G.1 with

$$861 \mathcal{C} \models \left(\bar{\mathbf{1}} = \left(\sum_{i=1}^n a_i \right) \top \wedge \bigwedge_{(x \leftarrow w) \in \mathcal{R}} w \leq x \right) \rightarrow w' \leq x'.$$

862 Thus for $\mathcal{C} \subseteq \text{RSUB}$, by the same argument as Lem. G.2 with Prop. 4.10, we have

$$863 [\mathcal{C}] = A^* \Leftrightarrow \mathcal{C} \models \left(\bar{\mathbf{1}} = \left(\sum_{i=1}^n a_i \right) \top \wedge \bigwedge_{(x \leftarrow w) \in \mathcal{R}} w \leq x \right) \rightarrow \top \leq \mathbf{s}. \quad \blacktriangleleft$$

864 **I** Detailed Proof of Thm. 5.2

865 Let $\mathbf{v} \in \text{GREL}$. For a graph H and a graph language \mathcal{G} , we write

$$866 \hat{\mathbf{v}}(H) \triangleq \{ \langle x, y \rangle \mid H \longrightarrow \mathbf{G}(\mathbf{v}, x, y) \}, \quad \hat{\mathbf{v}}(\mathcal{G}) \triangleq \bigcup_{H \in \mathcal{G}} \hat{\mathbf{v}}(H).$$

867
 868

869 ▶ **Lemma I.1.** *Let $\mathfrak{v} \in \text{GREL}$. For all $\text{KA}_{\{\bar{x}, \bar{1}, \top, \cap\}}$ terms t , we have $\hat{\mathfrak{v}}(t) = \hat{\mathfrak{v}}(\mathcal{G}(t))$.*

870 **Proof (cf. [32, Prop. 11]).** By easy induction on t . See Appendix I.1 for details. ◀

871 ▶ **Lemma I.2.** *Let $\mathcal{C} \subseteq \text{GREL}$, $\mathfrak{v} \in \mathcal{C}$, and H be a graph. Then $\hat{\mathfrak{v}}(\mathcal{S}_{\mathcal{C}}(H)^{\mathcal{Q}}) = \hat{\mathfrak{v}}(H)$.*

872 **Proof.** (\subseteq): Because, for any $J \in \mathcal{S}_{\mathcal{C}}(H)$, we have $J \rightarrow J^{\mathcal{Q}}$ by $H \rightarrow J$ and $J \rightarrow J^{\mathcal{Q}}$. (\supseteq):
873 Let $h: H \rightarrow \mathbb{G}(\mathfrak{v}, x, y)$. Then $\mathcal{S}(h) \rightarrow \mathbb{G}(\mathfrak{v}, x, y)$. Because $1^{\mathbb{G}(\mathfrak{v}, x, y)}$ is the identity relation,
874 $\mathcal{S}(h)^{\mathcal{Q}} \rightarrow \mathbb{G}(\mathfrak{v}, x, y)$. Hence $\hat{\mathfrak{v}}(\mathcal{S}_{\mathcal{C}}(H)^{\mathcal{Q}}) \supseteq \hat{\mathfrak{v}}(H)$. ◀

875 ▶ **Proposition I.3.** *Let $h: H \rightarrow \mathbb{G}(\mathfrak{v}, x, y)$. Then the graph $\mathcal{S}(h)^{\mathcal{Q}}$ is isomorphic to the
876 induced subgraph of $\mathbb{G}(\mathfrak{v}, x, y)$ on the range of h .*

877 **Proof.** Easy, by construction. Note that since $1^{\mathbb{G}(\mathfrak{v}, x, y)}$ is the identity relation, $1^{\mathcal{S}(h)}$ is an
878 equivalence relation (see, e.g., Example 2.2). ◀

879 **Proof of Thm. 5.2 (cf. [32, Thm. 18]).** We have:

$$\begin{aligned}
880 \quad & \mathcal{C} \models t \leq s \\
881 \quad & \Leftrightarrow \forall \mathfrak{v} \in \mathcal{C}, \hat{\mathfrak{v}}(t) \subseteq \hat{\mathfrak{v}}(s) \Leftrightarrow \forall \mathfrak{v} \in \mathcal{C}, \hat{\mathfrak{v}}(\mathcal{S}_{\mathcal{C}}(\mathcal{G}(t))^{\mathcal{Q}}) \subseteq \hat{\mathfrak{v}}(\mathcal{G}(s)) \quad (\text{Lems. I.1 and I.2}) \\
882 \quad & \Leftrightarrow \forall G \in \mathcal{S}_{\mathcal{C}}(\mathcal{G}(t))^{\mathcal{Q}}, \forall J \in \text{GR}_{\mathcal{C}}, (G \rightarrow J) \text{ implies } (\exists H \in \mathcal{G}(s), H \rightarrow J) \quad (\text{Def. of } \hat{\mathfrak{v}}) \\
883 \quad & \Leftrightarrow \forall G \in \mathcal{S}_{\mathcal{C}}(\mathcal{G}(t))^{\mathcal{Q}}, \exists H \in \mathcal{G}(s), H \rightarrow G. \quad (\heartsuit)
\end{aligned}$$

885 Here, for (\heartsuit), (\Leftarrow): Let $H \in \mathcal{G}(s)$ be s.t. $H \rightarrow G$. Then for all J s.t. $G \rightarrow J$, we have
886 $H \rightarrow J$ by transitivity of \rightarrow . (\Rightarrow): By Prop. I.3 and that $\text{GR}_{\mathcal{C}}$ is induced subgraph-closed,
887 we have $\mathcal{S}_{\mathcal{C}}(\mathcal{G}(t))^{\mathcal{Q}} \subseteq \text{GR}_{\mathcal{C}}$. Thus by letting $J = G$, this completes the proof. ◀

888 I.1 Proof of Lem. I.1

889 (The following proof is almost the same as [32, Prop. 8, 11] except that we consider GREL
890 instead of REL. Similar arguments can also be found, e.g., in [1, 7, 40].)

891 We use the two notations for graphs, series-composition (\cdot) and parallel-composition (\cap):

$$\begin{aligned}
892 \quad & G \cdot H \triangleq \circ \text{---} G \text{---} \circ \text{---} H \text{---} \circ \text{---} \circ, \quad G \cap H \triangleq \circ \text{---} \begin{array}{c} G \\ \text{---} \\ H \end{array} \text{---} \circ \text{---} \circ.
\end{aligned}$$

894 ▶ **Proposition I.4.** *Let $\mathfrak{v} \in \text{GREL}$ and G, H be graphs.*

$$895 \quad \hat{\mathfrak{v}}(G \cap H) = \hat{\mathfrak{v}}(G) \cap \hat{\mathfrak{v}}(H) \quad (\text{Prop. I.4}\cap)$$

$$896 \quad \hat{\mathfrak{v}}(G \cdot H) = \hat{\mathfrak{v}}(G) ; \hat{\mathfrak{v}}(H) \quad (\text{Prop. I.4}\cdot)$$

898 **Proof.** (Prop. I.4 \cap): It suffices to prove that for every x, y ,

$$899 \quad \exists f, f: (G \cap H) \rightarrow \mathbb{G}(\mathfrak{v}, x, y) \Leftrightarrow \exists f_G, f_H, f_G: G \rightarrow \mathbb{G}(\mathfrak{v}, x, y) \wedge f_H: H \rightarrow \mathbb{G}(\mathfrak{v}, x, y).$$

900 (\Rightarrow): By letting $f_G \triangleq \{\langle x', f(x') \rangle \mid x' \in |G|\}$ and $f_H \triangleq \{\langle x', f(x') \rangle \mid x' \in |H|\}$. (\Leftarrow): By
901 letting $f \triangleq f_G \cup f_H$. Note that $f_G(1^G) = x = f_H(1^H)$ and $f_G(2^G) = y = f_H(2^H)$; so f is
902 indeed a map.

903 (Prop. I.4 \cdot): It suffices to prove that for every x, y ,

$$904 \quad \exists f, f: (G \cdot H) \rightarrow \mathbb{G}(\mathfrak{v}, x, y) \Leftrightarrow \exists z, \exists f_G, f_H, f_G: G \rightarrow \mathbb{G}(\mathfrak{v}, x, z) \wedge f_H: H \rightarrow \mathbb{G}(\mathfrak{v}, z, y).$$

905 (\Rightarrow): By letting $z \triangleq f(2^G)$, $f_G \triangleq \{\langle x', f(x') \rangle \mid x' \in |G|\}$, and $f_H \triangleq \{\langle x', f(x') \rangle \mid x' \in |H|\}$.

906 (\Leftarrow): By letting $f \triangleq f_G \cup f_H$. Note that $f_G(2^G) = z = f_H(1^H)$; so f is indeed a map. ◀

907 **Proof of Lem. I.1.** By easy induction on t .

908 Case $t = x$ where $x \in \hat{\mathbf{V}} = \{a, \bar{a} \mid a \in \mathbf{V}\} \cup \{\bar{1}, \top\}$: For every $\langle x, y \rangle \in \hat{\mathbf{v}}(\top)$, we have

$$\begin{aligned}
 909 \quad \langle x, y \rangle \in \hat{\mathbf{v}}(a) &\Leftrightarrow (\rightarrow \circ \text{---} a \text{---} \circ \rightarrow \rightarrow \rightarrow \mathbf{G}(\mathbf{v}, x, y)) \\
 910 &\Leftrightarrow \langle x, y \rangle \in \hat{\mathbf{v}}(\rightarrow \circ \text{---} a \text{---} \circ \rightarrow \rightarrow) && \text{(Def. of } \hat{\mathbf{v}} \text{)} \\
 911 &\Leftrightarrow \langle x, y \rangle \in \hat{\mathbf{v}}(\mathcal{G}(a)). && \text{(Def. of } \mathcal{G} \text{)}
 \end{aligned}$$

913 Case $t = 1$: For every $\langle x, y \rangle \in \hat{\mathbf{v}}(\top)$, we have

$$\begin{aligned}
 914 \quad \langle x, y \rangle \in \hat{\mathbf{v}}(1) &\Leftrightarrow (\rightarrow \circ \text{---} 1 \text{---} \circ \rightarrow \rightarrow \rightarrow \mathbf{G}(\mathbf{v}, x, y)) \\
 915 &\Leftrightarrow \langle x, y \rangle \in \hat{\mathbf{v}}(\rightarrow \circ \text{---} 1 \text{---} \circ \rightarrow \rightarrow) && \text{(Def. of } \hat{\mathbf{v}} \text{)} \\
 916 &\Leftrightarrow \langle x, y \rangle \in \hat{\mathbf{v}}(\rightarrow \circ \rightarrow) && (1^{\mathbf{G}(\mathbf{v}, x, y)} \text{ is the identity relation)} \\
 917 &\Leftrightarrow \langle x, y \rangle \in \hat{\mathbf{v}}(\mathcal{G}(1)). && \text{(Def. of } \mathcal{G} \text{)}
 \end{aligned}$$

919 Case $t = 0$: For every $\langle x, y \rangle \in \hat{\mathbf{v}}(\top)$, we have

$$920 \quad \langle x, y \rangle \in \hat{\mathbf{v}}(0) \Leftrightarrow \text{false} \Leftrightarrow \langle x, y \rangle \in \hat{\mathbf{v}}(\emptyset) \Leftrightarrow \langle x, y \rangle \in \hat{\mathbf{v}}(\mathcal{G}(0)). \quad \text{(Def. of } \hat{\mathbf{v}} \text{ and } \mathcal{G} \text{)}$$

922 Case $t = s \cdot u$:

$$\begin{aligned}
 923 \quad \hat{\mathbf{v}}(s \cdot u) &= \hat{\mathbf{v}}(s) ; \hat{\mathbf{v}}(u) && \text{(Def. of } \hat{\mathbf{v}} \text{)} \\
 924 &= \hat{\mathbf{v}}(\mathcal{G}(s)) ; \hat{\mathbf{v}}(\mathcal{G}(u)) && \text{(IH)} \\
 925 &= \bigcup_{G \in \mathcal{G}(s)} \bigcup_{H \in \mathcal{G}(u)} \hat{\mathbf{v}}(G) ; \hat{\mathbf{v}}(H) && (; \text{ is distributive w.r.t. } \cup) \\
 926 &= \bigcup_{G \in \mathcal{G}(s)} \bigcup_{H \in \mathcal{G}(u)} \hat{\mathbf{v}}(G \cdot H) && \text{(Equation (Prop. I.4.))} \\
 927 &= \hat{\mathbf{v}}(\mathcal{G}(s \cdot u)). && \text{(Def. of } \mathcal{G} \text{)} \\
 928
 \end{aligned}$$

929 Case $t = s \cap u$:

$$\begin{aligned}
 930 \quad \hat{\mathbf{v}}(s \cap u) &= \hat{\mathbf{v}}(s) \cap \hat{\mathbf{v}}(u) && \text{(Def. of } \hat{\mathbf{v}} \text{)} \\
 931 &= \hat{\mathbf{v}}(\mathcal{G}(s)) \cap \hat{\mathbf{v}}(\mathcal{G}(u)) && \text{(IH)} \\
 932 &= \bigcup_{G \in \mathcal{G}(s)} \bigcup_{H \in \mathcal{G}(u)} (\hat{\mathbf{v}}(G) \cap \hat{\mathbf{v}}(H)) && (\cap \text{ is distributive w.r.t. } \cup) \\
 933 &= \bigcup_{G \in \mathcal{G}(s)} \bigcup_{H \in \mathcal{G}(u)} \hat{\mathbf{v}}(G \cap H) && \text{(Equation (Prop. I.4}\cap \text{))} \\
 934 &= \hat{\mathbf{v}}(\mathcal{G}(s \cap u)). && \text{(Def. of } \mathcal{G} \text{)} \\
 935
 \end{aligned}$$

936 Case $t = s + u$:

$$\begin{aligned}
 937 \quad \hat{\mathbf{v}}(s + u) &= \hat{\mathbf{v}}(s) \cup \hat{\mathbf{v}}(u) && \text{(Def. of } \hat{\mathbf{v}} \text{)} \\
 938 &= \hat{\mathbf{v}}(\mathcal{G}(s)) \cup \hat{\mathbf{v}}(\mathcal{G}(u)) && \text{(IH)} \\
 939 &= \hat{\mathbf{v}}(\mathcal{G}(s) \cup \mathcal{G}(u)) = \hat{\mathbf{v}}(\mathcal{G}(s + u)). && \text{(Def. of } \mathcal{G} \text{)} \\
 940
 \end{aligned}$$

941 Case $t = s^*$:

$$942 \quad \hat{\mathbf{v}}(s^*) = \bigcup_{n \in \mathbb{N}} \hat{\mathbf{v}}(s^n) = \bigcup_{n \in \mathbb{N}} \hat{\mathbf{v}}(s)^n \quad (\text{Def. of } \hat{\mathbf{v}})$$

$$943 \quad = \bigcup_{n \in \mathbb{N}} \hat{\mathbf{v}}(\mathcal{G}(s))^n \quad (\text{IH})$$

$$944 \quad = \bigcup_{n \in \mathbb{N}} \bigcup_{G_1, \dots, G_n \in \mathcal{G}(s)} \hat{\mathbf{v}}(G_1); \dots; \hat{\mathbf{v}}(G_n) \quad (; \text{ is distributive w.r.t. } \cup)$$

$$945 \quad = \bigcup_{n \in \mathbb{N}} \bigcup_{G_1, \dots, G_n \in \mathcal{G}(s)} \hat{\mathbf{v}}(G_1 \cdot \dots \cdot G_n) \quad (\text{Equation (Prop. I.4.)})$$

$$946 \quad = \bigcup_{n \in \mathbb{N}} \hat{\mathbf{v}}(\mathcal{G}(s^n)) = \hat{\mathbf{v}}(\mathcal{G}(s^*)). \quad (\text{Def. of } \mathcal{G})$$

948

◀