On the classification of Meromorphic Affine Connections on Complex Compact Surfaces

Alexis Garcia

ON THE CLASSIFICATION OF MEROMORPHIC AFFINE CONNECTIONS ON COMPLEX COMPACT SURFACES

ALEXIS GARCIA

Contents

Date: April 28, 2024.
2020 *Mathematics Subject Classification.* 51, 53.

1. Introduction

1.1. **Uniformization of complex compact manifolds by holomorphic geometric structures.** The classification of compact complex manifolds, by means of geometric and topological invariants, is still of interest. In dimension one, an historical answer is the Riemann's uniformization theorem for complex compact smooth curves (i.e. compact Riemann surfaces). It implies that any such a complex manifolds is the quotient of an open subset U of: either the projective curve $\mathbb{P}^1$, the complex line $\mathbb{C}$, or the Poincaré disk $\mathbb{H}$, by a discrete subgroup Γ of projective transformations preserving U with free left action.

This is a case of *uniformizability*, in the sense that any complex compact manifold of dimension one is the quotient $\Gamma\backslash U$ of an open set $U \subset X$ of a fixed space X by a subgroup Γ of automorphisms of X preserving a fixed holomorphic *geometric structure* and with a free action. The notion of *geometric structures* will not be defined in this paper but may be thought as one of these examples : a holomorphic affine connection (see Definition 2.4), a holomorphic projective connection (see for example [9]) or a holomorphic reduction of a k-th order frame bundle of X. Uniformized complex manifolds are endowed with a holomorphic geometric structure induced by the one on the model X.

1.2. **Meromorphic affine connections on surfaces.** In the opposite way, there are sufficient conditions on a holomorphic geometric structure on a complex compact manifold to obtain a uniformized manifold. In [6], Inoue, Kobayashi and Ochiai classified holomorphic affine connections on compact complex surfaces. It follows from their work that any complex compact surfaces admitting such a geometric structure is also equipped with a flat one, that is a holomorphic affine structure. Many of them, namely Kähler ones, are quotients of an open subset of $\mathbb{C}^2$ by affine transformations. This result brings two remarks: first, there are restrictions on the topology and geometry of a compact complex manifold to be endowed with a holomorphic geometric structure of a fixed type, and second, many known examples of complex compact manifolds endowed with holomorphic geometric structures give rise to uniformized manifolds. This result was completed by Kobayashi and Ochiai in [9], where it appears that any complex compact surface endowed with a holomorphic projective structure is uniformizable by the unit ball in $\mathbb{C}^2$.

It seems thus natural to ask whether any complex compact manifold admits a holomorphic geometric structure. In [5], Biswas, Dumitrescu and McKay gave rather general classification result, asserting that many holomorphic geometric structures (in particular holomorphic affine connections) can't be beared by simply connected compact complex manifolds with constant meromorphic functions.

In view of such a negative answer, we propose to investigate whether allowing the geometric structure to admits some reasonable singularities (namely poles) could enable to get more examples of compact complex manifolds, and investigate for a more general definition of the uniformization in this context. Here is an example of a possible generalization. Though there are few projective manifolds M endowed with holomorphic affine connections, since this implies that all Chern classes are zero, any such manifold is endowed with a finite map $f : M \longrightarrow \mathbb{P}^N$ for some integer $N \geq 1$. The canonical projective structure on $\mathbb{P}^N$ then induces through f a meromorphic (flat) projective connection on M (see [3]).

In this paper, we study the existence of *meromorphic affine connections* (Definition 2.4) on complex compact surfaces of *algebraic dimension* $a(M) = 1$ (the algebraic dimension is defined above (6)). Such pairs, that we call *meromorphic affine complex compact surface of algebraic dimension one*. We classify compact complex surfaces of algebraic dimension one which can be endowed with a such structure.

The steps of this classification are as follow. By the well-known work of Kodaira ([10],[11],[12]), complex compact surfaces of algebraic dimension one are known to be elliptic surfaces. Moreover, we can restrict ourselves to minimal surfaces since a meromorphic affine connection on a minimal surface with $a(M) = 1$ is the *pullback* (Definition 2.3) of a meromorphic affine connection on its minimal model (Lemma 2.3). We first prove the following (Theorem 3.1):

Theorem 1.5. *Any meromorphic affine complex compact surface of algebraic dimension one is an isotrivial surface.*

Up to finite cover, a minimal meromorphic affine complex compact surface is thus a principal elliptic fiber bundle, and we have explicit descriptions of such surfaces in terms of their universal covers (see Proposition 2.2.2). The problem is then split in two : first we have to classify meromorphic affine principal elliptic fiber bundles, and then study the possible finite quotients of such pairs. This is completely done when the base curve is the projective line $\mathbb{P}^1$ (Theorem 4.2 and Theorem 4.3) or an elliptic curve (Theorem 5.4,Theorem 5.5,Theorem 5.2 and Theorem 5.3). In the remaining case (hyperelliptic curve), we describe a subset of codimension 3 in the space of meromorphic affine connections (Theorem 6.1) extending the work by Klingler ([8]). However, we prove that there is no non-trivial quotient of such meromorphic affine surface (Theorem 6.3). So there a no new examples arising from these principal elliptic bundles. These results can be compared to the result of [6] to obtain:

Theorem 1.1. *Any meromorphic affine surface with $a(M) = 1$ endowed with a meromorphic affine connection also admits a flat affine holomorphic connection.*

As an example, no $K3$-surface with $a(M) = 1$ admits a meromorphic affine connection.

1.3. **Organization of the paper.** The paper is organized as follows. In section 2, we recall the notion of meromorphic affine connections. In section 3, we collect classical facts from the work of Kodaira on elliptic surfaces that will be used in the rest of the paper, and prove Theorem 3.1 and Theorem 3.2, reducing the problem of classification to the one of meromorphic affine principal elliptic bundles and their quotients, as explained above. Then, in section 4,section 5 we classify meromorphic affine complex compact surfaces of algebraic dimension one arising as quotients of principal elliptic fiber bundles over $\mathbb{P}^1$ or an elliptic curve. In section 6, we treat the case of an hyperelliptic base curve. We give a description of a codimension three subset in the space of meromorphic affine connections on the corresponding principal elliptic bundle, in terms of meromorphic differential operators, and prove that no other examples arise from a quotient of such meromorphic affine surfaces.

2. Meromorphic affine connections and minimality

2.1. **Meromorphic connections and linearizations.** We begin by notations and definitions. Let M be a complex manifold and $D = \sum_{\alpha\in I} n_\alpha D_\alpha$ an effective divisor of M, where n_α is an integer and D_α is an analytic hypersurface of M. In the rest of the paper, we will denote by TM the sheaf of holomorphic vector fields, Ω^1_M the sheaf of holomorphic one forms. For any $\alpha \in I$ and $k_\alpha \in \mathbb{Z}$, $\mathcal{O}_M(k_\alpha D_\alpha)$ will stand for the sheaf of local meromorphic functions on open subsets $U \subset M$ such that, if z_1 is an equation for $U \cap D_\alpha$, then $z_1^{k_\alpha} f \in \mathcal{O}_M(U)$.

We then define:

$$\mathcal{O}_M(*D) = \bigcup_{(k_\alpha)\in\mathbb{Z}^I} \mathcal{O}_M(\sum_{\alpha\in I} k_\alpha D_\alpha) \tag{1}$$

as the sheaf of local meromorphic functions which are holomorphic when restricted to $M \setminus D$. Given any morphism $\Psi : M \longrightarrow M'$ of complex manifolds, Ψ^* (resp. Ψ_*) stands for the pullback functor for sheaves (resp. the pushforward).

Definition 2.1. Let M be a complex manifold and $\mathcal{E}$ be a $\mathcal{O}_M$-module. A meromorphic connection on M with poles at D is a morphism of $\underline{\mathbb{C}}_M$-sheaves:

$$\nabla \quad : \quad \mathcal{E} \quad \longrightarrow \quad \Omega^1_M \otimes \mathcal{E}(*D)$$

satisfying the Leibniz identity:

$$\forall f \in \mathcal{O}_M(U), \forall s \in \mathcal{E}(U),\ \nabla(fs) = df \otimes s + f\nabla(s)$$

Definition 2.2. Let G be a group and M be a complex G-manifold with right (resp. left) action. Let $\mathcal{E}$ be a $\mathcal{O}_M$-module. A right (resp. left) G-linearization of $\mathcal{E}$ is a family $(\phi_g)_{g\in G}$ of isomorphisms:

$$\phi_g \quad : \quad \mathcal{E} \quad \xrightarrow{\sim} \quad g^*\mathcal{E}$$

with the property:

$$\forall g, g' \in G,\ \phi_{gg'} \quad = \quad g'^*(\phi_g) \circ \phi_{g'}$$

A G-linearized $\mathcal{O}_M$-module is a pair $(\mathcal{E}, (\phi_g)_{g\in G})$.

In the case of a discrete group G, a $\mathcal{O}_M$-module with a G-linearization is a G-equivariant $\mathcal{O}_M$-module as defined in [7]. In this case, if $\overline{M} = G\backslash M$ is a complex manifold, then, denoting by $q : M \longrightarrow \overline{M}$ the quotient map, G acts naturally on $q_*\mathcal{E}(U)$ for any $U \subset \overline{M}$. Hence, there is a functor from the category of G-linearized $\mathcal{O}_M$-modules to the one of $\mathcal{O}_{\overline{M}}$-modules mapping $(\mathcal{E}, (\phi_g)_{g\in G})$ to the sheaf:

$$\overline{\mathcal{E}} = (q_*\mathcal{E})^G \tag{2}$$

of G-invariant sections. It is an equivalence of categories in the case where the action of G is free ([7], Proposition 2.2.5).

Definition 2.3. (1) Let M and M' be two complex manifold, D' an effective divisor of M' and ∇' be a meromorphic connection on a $\mathcal{O}_{M'}$-module $\mathcal{E}'$ with poles at D'. Let $\mathcal{E}$ be a $\mathcal{O}_M$-module, $f : M \longrightarrow M'$ a isomorphism of complex manifold and $\varphi : \mathcal{E} \longrightarrow \mathcal{O}_M \otimes f^*\mathcal{E}'$ an isomorphism of $\mathcal{O}_M$-modules. The corresponding pullback $(f, \varphi)^\star\nabla'$ of ∇' is the meromorphic connection ∇ on $\mathcal{E}$ with poles at $D = f^*D'$ defined by the commutative diagram:

(3)
$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\quad\nabla\quad} & \Omega^1_M \otimes \mathcal{E}(*D) \\ {\scriptstyle\varphi}\downarrow & & \uparrow{\scriptstyle df^*\otimes\varphi^{-1}} \\ \mathcal{O}_M \otimes f^*\mathcal{E}' & \xrightarrow[\overline{f^*\nabla'}]{} & f^*\Omega^1_{M'} \otimes f^*\mathcal{E}'(*D') \end{array}$$

where:

- $df \quad : \quad TM(*D) \quad \longrightarrow \quad \mathcal{O}_M \otimes f^*TM'(*D')$ is the sheaf-theoretic differential of f
- $\overline{f^*\nabla'}$ is the extension of the sheaf-theoretic pullback $f^*\nabla' : f^*\mathcal{E}' \longrightarrow f^*\Omega^1_{M'} \otimes \mathcal{E}'(*D')$ by the Leibniz rule to $\mathcal{O}_M \otimes f^*\mathcal{E}'$.

(2) Let $(\mathcal{E}, (\phi_g)_{g\in G})$ be a G-linearized $\mathcal{O}_M$-module. A meromorphic connection ∇ on $\mathcal{E}$ is invariant by $(\phi_g)_{g\in G}$ if $(g, \phi_g)^\star\nabla = \nabla$ for any $g \in G$.

Since a holomorphic connection on a $\mathcal{O}_M$-module can be alternatively described as a global section of $\mathcal{H}om(\mathcal{E}, J^1\mathcal{E})$ splitting the jet-sequence of $\mathcal{E}$ (see [1]), the correspondance between linearized sheaves and sheaves on the quotient immediately implies the following:

Lemma 2.1. *Let $q : M \longrightarrow M'$ be a ramified cover between two complex manifolds, with Galois group $\Gamma \subset Aut(M)$. Let $\Phi_\Gamma = (\phi_\gamma)_{\gamma\in\Gamma}$ be a Γ-linearization of a locally free $\mathcal{O}_M$-module $\mathcal{E}$. Consider the locally free $\mathcal{O}_{M'}$-module $\overline{\mathcal{E}} = (q_*\mathcal{E})^{\Phi_\Gamma}$ and the natural inclusion:*

$$Id : \mathcal{O}_M \otimes q^*\overline{\mathcal{E}} \hookrightarrow \mathcal{E}$$

Suppose the existence of a holomorphic connection ∇ on $\mathcal{E}$ which is invariant through any ϕ_γ ($\gamma \in \Gamma$). Then:

1. *If q is unramified, then there exists induces a (unique) holomorphic connection ∇' on $\overline{\mathcal{E}}$ such that the pullback $(q, Id)^\star\nabla'$ (see Definition 2.3) coïncides with ∇ when restricted to $q^*\overline{\mathcal{E}}$.*
2. *If q is finite, then there exists a unique meromorphic connection ∇' on $\overline{\mathcal{E}}$, with poles supported on the ramification locus, such that the pullback $(q, Id)^\star\nabla'$ coïncides with the restriction of ∇ to $q^*\overline{\mathcal{E}}$.*

Proof. (1) Consider the restriction ∇' of the morphism of $\underline{\mathbb{C}}_{M'}$-sheaves $q_*\nabla$ to the subsheaf $\overline{\mathcal{E}}$. It satisfies the Leibniz identity, and by definition of $(\gamma, \phi_\gamma)^\star\nabla = \nabla$ (Definition 2.3), the invariance of ∇ by the linearization Φ_Γ implies that ∇' restricts as a morphism:

$$\nabla' \ : \ \overline{\mathcal{E}} \ \longrightarrow \ (q_*\Omega^1_M \otimes \mathcal{E})^\Gamma$$

Now, we have a natural inclusion of sheaves:

$$\Omega^1_{M'} \otimes \overline{\mathcal{E}} = (q_*\Omega^1_M)^\Gamma \otimes (q_*\mathcal{E})^\Gamma \subset (q_*\Omega^1_M \otimes \mathcal{E})^\Gamma \tag{4}$$

where the right handside is the subsheaf of invariant sections through the linearisation obtained by tensorizing the isomorphisms $(d\gamma^*)_{\gamma\in\Gamma}$ and $(\phi_\gamma)_{\gamma\in\Gamma}$. Since q is unramified, we have

$$\mathcal{O}_M \underset{\mathcal{O}_{M'}}{\otimes} q^*\mathcal{O}_{M'} = \mathcal{O}_M$$

and since Γ acts freely, we also have (see [7], Proposition 2.2.5):

$$\mathcal{O}_M \otimes q^*\overline{\mathcal{E}} \simeq \mathcal{E}$$

Hence the pullbacks of the vector bundles corresponding to the left and right handside of (4) coïncide. Hence $\Omega^1_{M'} \otimes \overline{\mathcal{E}} = (q_*\Omega^1_M \otimes \mathcal{E})^\Gamma$, so that ∇' is in fact a holomorphic connection on $\overline{\mathcal{E}}$. Its pullback through (q, Id) coïncides with the restriction of ∇ to $q^*\overline{\mathcal{E}}$ by construction.

(2) The restriction of q to the complement of its ramification locus $S \subset M$ is an unramified cover $q|_{M\setminus S} : M \setminus S \longrightarrow M' \setminus S'$. By (1), we get a holomorphic connection ∇' on $\overline{\mathcal{E}}|_{M'\setminus S'}$ whose pullback through $(q|_{M\setminus S}, Id)$ coïncides with the restriction of $\nabla|_{M\setminus S}$ to $q^*\overline{\mathcal{E}}|_{M\setminus S}$. Denote by j the inclusion of complex manifold of $M' \setminus S'$ in M'. It remains to prove that $j_*\nabla'$ restricts to the subsheaf $\overline{\mathcal{E}} \subset j_*\overline{\mathcal{E}}|_{M'\setminus S'} = (j_*\mathcal{O}_{M'\setminus S'}) \otimes \overline{\mathcal{E}}$ as a morphism with values in the subsheaf $\Omega^1_{M'} \otimes \overline{\mathcal{E}}(*S')$. It is sufficient to prove this property for the restriction of $(\overline{\mathcal{E}}, j_*\nabla')$ to any complex submanifold $\Sigma \subset M$ of dimension one. Therefore, we assume from now on that q is a finite cover between two complex curves M, M'. Pick $x \in S$ and $y = q(x)$. Since Γ is finite, there exists a generator $\gamma \in \Gamma$ leaving a neigborhood U of x invariant, and local coordinates z, z' at neighborhoods of x and y such that:

$$q(z) = z^m \text{ and } \gamma \cdot z = e^{\frac{2i\pi}{m}} z$$

for some integer $m \geq 1$. Moreover, ∇ is a flat connection on $\mathcal{E}$, so there is a local trivialization of $\mathcal{E}$ at a neighborhood U of x such that ∇ identifies with the de Rham differential d on $\mathcal{O}_U^{\oplus r}$ (r stands for the rank of $\mathcal{E}$). By the invariance of ∇ through Φ_Γ, the automorphism ϕ_γ maps the kernel $\underline{\mathbb{C}}_U^{\oplus r}$ of d to its pullback $\gamma^*\underline{\mathbb{C}}_U^{\oplus r}$. It is therefore

given by a constant linear transformation. Since γ is of finite order, up to applying a constant linear change of trivialization, this linear transformation is diagonal with roots of the unity as eigenvalues, that is ϕ_γ corresponds to the isomorphism:

$$\begin{array}{rccc} \phi_\gamma & : & \mathcal{O}_U^{\oplus r} & \longrightarrow & \gamma^*\mathcal{O}_M^{\oplus r} \\ & & (f_1(z),\dots,f_r(z)) & \mapsto & (e^{\frac{2k_1 i\pi}{m}} f_1(e^{-\frac{2i\pi}{m}}z),\dots,e^{\frac{2k_r i\pi}{m}} f_r(e^{-\frac{2i\pi}{m}}z)) \end{array}$$

with $k_j \in \{1,\dots,m\}$. Hence, the restriction of $q^*\overline{\mathcal{E}}$ to U is the subsheaf of r-uples of the form:

$$s = (z^{m-k_1}g_1(z^m),\dots,z^{m-k_r}g_r(z^m))$$

for some holomorphic functions $g_1,\dots,g_r$ on $z^m(U)$. Then, through the trivialization of $\mathcal{E}|_U$ just defined:

$$\nabla(s) = dz \otimes (\frac{m-k_1}{z}z^{m-k_1-1}g_1(z_1^m),\dots,\frac{m-k_r}{z}z^{m-k_r-1}g_r(z^m)) + q^\star(dz')\otimes s'$$

for some $s' \in q^*\overline{\mathcal{E}}(U)$. We can rewrite this section as:

$$\nabla(s) = q^\star(dz')\otimes(\frac{z^{m-k_1}}{z^m}h_1(z^m),\dots,\frac{z^{m-k_r}}{z^m}h_r(z^m))$$

for some meromorphic functions $h_1,\dots,h_r$ on $z^m(U)$ with poles along $z^m(x)$. Hence $\nabla(s)$ is a section of $(\Omega^1_U)^\Gamma \otimes q^*\overline{\mathcal{E}}|_U(*x)$. This being true for any $x \in S$, and since $\Omega^1_{M'} = (q_*\Omega^1_M)^\Gamma$, we get:

$$j_*\nabla'(\overline{\mathcal{E}}) \subset \Omega^1_{M'}\otimes\overline{\mathcal{E}}(*S')$$

as required. This exactly means that ∇' defines a meromorphic connection on $\overline{\mathcal{E}}$, and this connection satisfies the pullback property by construction.

□

2.2. **Meromorphic affine connections and pullback.** We introduce the meromorphic geometric structure considered in this paper:

Definition 2.4. Let M be a complex manifold and D an effective divisor of M. A meromorphic affine connection on (M,D) is a meromorphic connection on TM with poles at D.

The pullback defined in Definition 2.3 defines the category of meromorphic affine connections, with arrows given by the pullbacks through (f,df) for f an isomorphism of complex manfiolds.

Lemma 2.2. *Let $q : \hat{M} \longrightarrow M$ be a morphism of complex manifolds of the same dimension. Let ∇ be a meromorphic affine connection on M and $\hat{\nabla} = q^\star\nabla$. Let Ψ be an automorphism of M and $\hat{\Psi}$ an automorphism of $\hat{M}$ lifting Ψ through q.*

Then $\Psi^\star\nabla = \nabla$ if and only if $\hat{\Psi}^\star\hat{\nabla} = \hat{\nabla}$.

Proof. Since $\hat{\Psi}$ is the lift of Ψ through q, we have the following commutative diagram:

$$\begin{array}{ccc} T\hat{M} & \xrightarrow{dq} & \mathcal{O}_{\hat{M}}\otimes q^*T\hat{M} \\ {\scriptstyle d\hat{\Psi}}\downarrow & & \downarrow{\scriptstyle \hat{\Psi}^*\otimes d\Psi} \\ \hat{\Psi}^*T\hat{M} & \xrightarrow[\hat{\Psi}^*dq]{} & \mathcal{O}_{\hat{M}}\otimes q^*T\hat{M} \end{array} \tag{5}$$

The equivalence asserted is then a direct consequence of the diagram defining a pullback (Definition 2.3). □

2.3. **Algebraic dimension and general property of elliptic surfaces.** Let M be a compact complex manifold of complex dimension $n \geq 1$. Moishezon proved that the field of meromorphic functions $\mathbb{C}(M)$ is a field of finite transcendancy degree over the field of constant functions $\mathbb{C}$. This degree is called the *algebraic dimension* of M and denoted by $a(M)$. In particular $a(M) \leq n$, and there exists a bimeromorphic map $\Psi : M \longrightarrow M'$ and a holomorphic map

$$\pi : M' \longrightarrow N \tag{6}$$

onto a complex compact manifold of dimension $a(M)$, with the property $\mathbb{C}(M) = \pi^*\mathbb{C}(N)$. In this paper, we will focus on complex compact surfaces with $a(M) = 1$.

A *elliptic surface* is a holomorphic fibration $M \xrightarrow{\pi} N$ of a complex compact surface over a (compact) complex smooth curve, such that for a generic $y \in N$, the fiber $M_y := \pi^{-1}(y)$ is a (smooth) complex torus.

We recall the following result from Kodaira ([10]):

Theorem 2.1. *Any compact complex surface with $a(M) = 1$ is the total space of an elliptic surface given by the algebraic reduction map* (6)*.*

Moreover:

Theorem 2.2. *Let $M \xrightarrow{\pi} N$ be an elliptic surface with $a(M) = 1$. Any divisor D of M is of the form $D = \pi^*C$ for some divisor C of N.*

Proof. See [10], Theorem 4.3. □

Now, let $M \xrightarrow{\pi} N$ be a general elliptic surface. Let $(y_\beta)_{\beta\in J}$ denote the set of points in N such that the fiber M_{y_β} of π is not a smooth elliptic curve. For any local coordinate z_β on N centered at y_β, there exists an integer $m_\beta > 0$ and an equation f_β for M_{y_β} with

$$f_\beta = (z_\beta \circ \pi)^{m_\beta} \tag{7}$$

The corresponding y_β are the *singular points* and the above integer will be called the *multiplicty* of y_β (resp. of the *singular fiber* M_{y_β}), according to the work of Kodaira ([10]).

Proposition 2.2.1. *Let $M \xrightarrow{\pi} N$ be an elliptic surface, $(y_\beta)_{\beta\in J}$ the singular points and N' their complement in N. Then:*

(1) For any $y \in N'$, there is a neighborhood $\overline{U}$ of y in N', and a holomorphic function $\tau : \overline{U} \longrightarrow \mathbb{H}$ such that :

$$\pi^{-1}(\overline{U}) \simeq \overline{U} \times \mathbb{C}/\langle\psi_1, \psi_2\rangle$$

where z_1, z_2 are global coordinates adapted to the natural fibration and:

$$\psi_1(z_1, z_2) = (z_1, z_2 + 1) \quad \text{and} \quad \psi_2(z_1, z_2) = (z_1, z_2 + \tau) \tag{8}$$

(2) There exists a global holomorphic function $\tilde{\tau} : \tilde{N'} \longrightarrow \mathbb{H}$ on the universal cover of $\tilde{p} : \tilde{N'} \longrightarrow N'$, such that for any $y \in N'$, and any τ as in 1., $\tau = \tilde{\tau} \circ s$ for some section of $\tilde{p}$ near y.

Proof. (1) See [11], (10.9)
(2) See [11], Theorem 7.2

□

Recall tat for any $\tau, \tau' \in \mathbb{H}$, the tori $\mathbb{C}/\mathbb{Z} \oplus \tau\mathbb{Z}$ and $\mathbb{C}/\mathbb{Z} \oplus \tau'\mathbb{Z}$ are isomorphic exactly when τ, τ' lie in the same $SL_2(\mathbb{Z})$-orbit in $\mathbb{H}$, through the action:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau := \frac{a\tau + b}{c\tau + d}$$

Since the action is free, there exists an associated representation $\rho : \pi_1(N', y) \longrightarrow SL_2(\mathbb{Z})$ such that:

$$\tilde{\tau}(\gamma \cdot \tilde{y}) = \rho(\gamma) \cdot \tilde{\tau}(\tilde{y}) \tag{9}$$

for any $\tilde{y} \in \tilde{p}^{-1}(y)$ and $\gamma \in \pi_1(N', y)$.

The associated local system over N' is called the *homological invariant* of π by Kodaira. The following facts can be recovered from [11]:

Proposition 2.2.2. *Let $M \xrightarrow{\pi} N$ be an elliptic surface, and N' (resp. M') the complement of its singular points $(y_\beta)_{\beta \in J}$ (resp. its singular fibers in M). Then:*

1. *If $J \neq \emptyset$ or the genus $g(N) \geq 1$ then the universal cover $\tilde{N}'$ is an open subset of $\mathbb{C}$ and $H^1(\tilde{N}', \mathcal{O}_{\tilde{N}'}) = \{0\}$. By Proposition 2.2.1, this implies $\tilde{M}' = \tilde{N}' \times \mathbb{C} \subset \mathbb{C}^2$ and we let (z_1, z_2) be the canonical coordinates on $\mathbb{C}^2$.*
2. *Let $y \in N'$. For any $\gamma \in \pi_1(N', y)$, the corresponding automorphism of $\tilde{N}'$ lifts canonically to an automorphism φ_γ of $\tilde{M}'$, normalizing the subgroup spanned by ψ_1 and ψ_2 (see (8))*
3. *In the case $J \neq \emptyset$ or $g(N) \geq 1$, let $\gamma \in \pi_1(N', y)$ and φ_γ the automorphism of the universal cover of M' as in 2. There exists a constant $\mu_\gamma \in \mathbb{C}^*$ and a holomorphic function f_γ on $\tilde{N}'$ such that:*

$$\varphi_\gamma(z_1, z_2) = (\gamma \cdot z_1, \frac{\mu_\gamma}{c_\gamma \tilde{\tau}(z_1) + d_\gamma} z_2 + f_\gamma(z_1)) \tag{10}$$

   where $\rho(\gamma) = \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix}$ (see (9)).
4. *The pullback (see Definition 2.3) through the universal covering $\tilde{p} : \tilde{M}' \longrightarrow M'$ induces a bijection between the set of meromorphic affine connections on M' and the set of meromorphic affine connections $\tilde{\nabla}$ on $\tilde{M}'$ satisfying:*

$$\psi_1^\star \tilde{\nabla} = \psi_2^\star \tilde{\nabla} = \varphi_\gamma^\star \tilde{\nabla} = \tilde{\nabla} \tag{11}$$

   for any $\gamma \in \pi_1(N', y)$, where ψ_1, ψ_2 are defined as in Proposition 2.2.1.

Proof. (1) In either case, $\tilde{N}'$ is isomorphic to $\mathbb{P}^1, \mathbb{C}$ or $\mathbb{H}$. If $\tilde{N}' = \mathbb{P}^1$, then $N' = \tilde{N}' = \mathbb{P}^1$ since N is compact, so that $J = \emptyset$. The converse is clearly true. Now if $J \neq \emptyset$, necessarly $\tilde{N}'$ is not $\mathbb{P}^1$, whence the assertion.

(2) Consider the complex manifold $\overline{M'} = \tilde{N}' \underset{N'}{\times} M'$ fitting in the following diagram:

$$\begin{array}{ccc} \overline{M'} & \xrightarrow{\overline{p'}} & M' \\ {\scriptstyle \overline{\pi}}\downarrow & & \downarrow{\scriptstyle \pi} \\ \tilde{N}' & \xrightarrow{\overline{p'}} & N' \end{array} \tag{12}$$

where $\overline{p'}$ is the projection on the second factor, $\overline{\pi}$ the projection on the first one. Then $\tilde{M}'$ is clearly isomorphic to the universal cover of $\overline{M'}$. We denote by $[\tilde{y}, x]$ the class of $(\tilde{y}, x) \in \tilde{N}' \times M'$ in $\overline{M'}$.

For any $\gamma \in \pi_1(N', y)$ denote by $\overline{\varphi}_\gamma$ the corresponding automorphism of $\overline{p'}$. Then:

$$\overline{\varphi}_\gamma([\tilde{y}, x]) = [\overline{\varphi}_\gamma(\tilde{y}), x] \tag{13}$$

define an automorphism $\overline{\varphi}_\gamma$ of $\overline{M'}$ compaptible with $\overline{p}$ and $\overline{\pi}$. Then $\overline{\varphi}_\gamma$ lifts to an unique automorphism φ_γ of the total space of the universal cover $\tilde{M}'$.

(3) Let (z_1, z_2) be coordinates on $\tilde{M'} = \tilde{N'} \times \mathbb{C}$, and $\tilde{\tau}$ as in Proposition 2.2.1, 2. Consider again the complex manifold $\overline{M'}$ as in the proof of 2. By the proof of Proposition 2.2.1, 1., we have a canonical isomorphism of elliptic fibrations

$$\overline{M'} \xrightarrow{\sim} \tilde{N'} \times \mathbb{C}/\langle \psi_1, \psi_2 \rangle$$

where ψ_1, ψ_2 are the automorphism defined by:

$$\psi_1(z_1, z_2) = (z_1, z_2 + 1) \quad \text{and} \quad \psi_2(z_1, z_2) = (z_1, z_2 + \tilde{\tau}(z_1)) \tag{14}$$

Let $\overline{\varphi}_\gamma$ be the lifting to $\overline{M'}$ of the automorphism of $\tilde{N'}$ corresponding to $\gamma \in \pi_1(N', y)$ (recall that φ_γ is the lifting of $\overline{\varphi}_\gamma$ to the universal cover $\tilde{M'}$ of $\overline{M'}$). Let $\tilde{y} \in \tilde{N'}$ with $\overline{p'}(\tilde{y}) = y$, and $z_1 = z_1(\tilde{y})$. Then, by (9) and the above biholomorphism, we get that the multiplication by $\frac{1}{c_\gamma \tilde{\tau}(z_1) + d_\gamma} \in \mathbb{C}^\times$ induces a biholomorphism between the fibers $\overline{M'}_{\tilde{y}}$ and $\overline{M'}_{\gamma \cdot \tilde{y}}$.

Now, recall that the automorphisms $\mathcal{A}(\overline{M'}_{\tilde{y}})$ are described by the exact sequence:

$$0 \longrightarrow \overline{M'}_{\tilde{y}} \longrightarrow \mathcal{A}(\overline{M'}_{\tilde{y}}) \longrightarrow \mathbb{Z}/n_{\tilde{y}}\mathbb{Z} \longrightarrow 0$$

where $\mathbb{Z}/n_{\tilde{y}}\mathbb{Z}$ corresponds to complex multiplications by a $n_{\tilde{y}}$-th root of the unity inducing an involution on the elliptic curve ($n_{\tilde{y}} \leq 6$) and $\overline{M'}_{\tilde{y}}$ is identified as the subgroup of translations on itself.

Since $\overline{\varphi}_\gamma$ is an automorphism of the elliptic fibration $\overline{\pi} : \overline{M'} \longrightarrow \tilde{N'}$, the above remarks imply the existence of a holomorphic function μ_γ on $\tilde{N'}$, and a holomorphic section $\overline{f}_\gamma$ of $\overline{\pi}$ such that:

$$\forall z_1 \in \tilde{N'},\ \forall z_2 \in \mathbb{C},\ \overline{\varphi}_\gamma([z_1, z_2]) = [\gamma \cdot z_1, \frac{\mu_\gamma(z_1)}{c_\gamma \tilde{\tau}(z_1) + d_\gamma} z_2] + \overline{f}_\gamma(z_1)$$

In particular, μ_γ is a constant, and since $\check{H}^1(\tilde{N'}, \mathcal{O}_{\tilde{N'}}) = \{0\}$, $\overline{f}_\gamma$ lifts to a section of $\tilde{\pi} : \tilde{M'} \longrightarrow \tilde{N'}$, that is a holomorphic function f_γ on $\tilde{N'}$. Then φ_γ is exactly the automorphism described in the statement.

(4) If $\tilde{\nabla}$ is a meromorphic affine connection on $\tilde{M'}$ satisfying (11), then in particular it is invariant through the Galois group of the universal covering $\tilde{\overline{p}} : \tilde{M'} \longrightarrow \overline{M'}$ as in (12). Thus, using Lemma 2.1, we have $\tilde{\nabla} = \tilde{\overline{p}}^\star \overline{\nabla}$ for some meromorphic affine connection on $\overline{M'}$. Since the automorphisms φ_γ are the lifts of the elements $\overline{\varphi}_\gamma$ of the Galois group of the covering $\overline{p'}$, we also have $\overline{\nabla} = \overline{p'}^\star \nabla$ for some meromorphic affine connection on M', that is $\tilde{\nabla} = \tilde{p}^\star \nabla$.

Reciprocally, suppose that $\tilde{\nabla} = \tilde{p}^\star \nabla$ for some meromorphic affine connection ∇ on M'. Then applying Lemma 2.2 to the lifts ψ_1, ψ_2 and φ_γ of the identity of M' gives (11). □

We will therefore use these well-known following facts about elliptic surfaces, due to Kodaira [11]. First, recall that given divisors D_1, D_2 on a complex compact surface M, there is a well defined *intersection number*:

$$D_1 \cdot D_2 := c_1(D_1) c_1(D_2)$$

where $c_1(D) \in H^1(M, \mathbb{Z})$ stands for the first Chern class of the line bundle $\mathcal{O}_M(D)$. An *exceptional curve* is then a rational smooth curve C in M such that $C \cdot C = -1$.

Theorem 2.3. *Let N be a smooth complex curve, $\mathcal{J} : \tilde{N} \longrightarrow SL_2(\mathbb{Z})\backslash\mathbb{H}$ and $\mathcal{G}$ a sheaf of subgroups of $SL_2(\mathbb{Z})$ as above. Then:*

(1) There exists a unique (up to biholomorphisms of elliptic surfaces) elliptic surface $B \xrightarrow{\pi_0} N$ with invariants $\mathcal{J}, \mathcal{G}$ and a global holomorphic section, called the basic member.

(2) *Any minimal elliptic surface* $M' \xrightarrow{\pi'} N$ *with invariants* $\mathcal{J}, \mathcal{G}$ *and no multiple singular fiber is locally isomorphic to* B.

Proof. This immediately follows from Theorem 10.1 in [11].

□

2.4. **Minimal model for meromorphic affine complex surface with algebraic dimension one.** By a well-known result of Grauert (see [2], Theorem 7.1), if C is an exceptional curve, then there exists a complex compact surface M_1, and $x_1 \in M_1$ such that M is isomorphic to the *blow-up* at x_0 of M:

$$\sigma : M \longrightarrow M_1$$

and $\sigma(C) = \{x_1\}$. In this case $a(M_1) = a(M)$, the restriction of σ to $M \setminus C$ is an isomorphism onto $M_1 \setminus \{x_1\}$ and σ maps any fiber of the algebraic reduction of M to a fiber of the algebraic reduction of M_1. Given a complex compact surface M, there is a finite number of exceptional curve, and thus composing the maps σ obtained as above we get a map:

$$\sigma_0 : M \longrightarrow M_0$$

which restricts as an isomorphism between $M \setminus C_0$, where C_0 is the union of the exceptional curves, and $M_0 \setminus \{x_0^1, \ldots, x_0^n\}$ where the x_0^i are points. Again $a(M_0) = a(M)$ and M_0 will be called the *minimal model* of M.

In particular:

Lemma 2.3. *Let* M *be a complex compact surface endowed with a meromorphic affine connection* ∇*, with poles at* D*. Suppose that* M *contains an exceptional curve and let* $\sigma : M \longrightarrow M_0$ *the minimal model of* M*. If* $a(M) = 1$*, then there exists a meromorphic affine connection* ∇_0 *on* M_0 *such that* $\nabla = \sigma_0^\star \nabla_0$ *(see Definition 2.3).*

Proof. First, using the inverse of the restriction of σ_0 to $M \setminus C_0$, we obtain a meromorphic affine connection ∇_0 on $M_0 \setminus \{x_0^1, \ldots, x_0^n\}$ such that $\sigma_0^\star \nabla_0$ is the restriction of ∇ to $M \setminus C_0$.

It remains to prove that ∇_0 extends as a meromorphic connection in a neighborhood of any x_0^i. For, let U_0 be a neighborhood of x_0^i such that $TM_0|_{U_0}$ is trivializable. Then sections of $TM(*D_0)|_{U_0}$ identify with vectors of meromorphic functions on open subsets of U_0. Let $j : U_0 \setminus \{x_0^i\} \longrightarrow U_0$ be the inclusion. The Hartogs' extension theorem thus implies that the restriction morphism:

$$\rho : TM(*D_0)|_{U_0} \longrightarrow j_*(TM(*D_0)|_{U_0 \setminus \{x_0^i\}})$$

is an isomorphism of $\mathcal{O}_{U_0}$-modules. The same property holds for the restriction morphism:

$$\rho' : \Omega^1_{U_0} \otimes TM|_{U_0} \longrightarrow j_*(\Omega^1_{U_0 \setminus \{x_0^i\}} \otimes TM(*D_0)|_{U_0 \setminus \{x_0^i\}})$$

Hence, there is a morphism of $\mathcal{O}_{U_0}$-modules $\overline{\nabla}_0$ such that the diagram:

$$\begin{array}{ccc} TM(*D_0)|_{U_0} & \xrightarrow{\overline{\nabla}_0} & \Omega^1_{U_0} \otimes TM(*D_0)|_{U_0} \\ \downarrow \rho & & \downarrow \rho' \\ j_*(TM(*D_0)|_{U_0 \setminus \{x_0^i\}}) & \xrightarrow[j_*\nabla_0]{} & j_*(\Omega^1_{U_0 \setminus \{x_0^i\}} \otimes TM(*D_0)|_{U_0 \setminus \{x_0^i\}}) \end{array}$$

commutes, where $j_*\nabla_0$ is the sheaf-theoretic pushforward of the morphism ∇_0. Since ∇_0 satisfies the Leibniz identity, the same holds for $\overline{\nabla}_0$. Such an extension is clearly unique, and defines a meromorphic affine connection on M_0.

□

3. Reduction to the classification of principal elliptic surfaces

By Lemma 2.3, we can restrict ourselves to the classification of minimal complex compact surface of algebraic dimension one endowed with meromorphic affine connections.

We now prove that the functional invariant of such a minimal elliptic surface is constant, hence this is a principal elliptic bundle up to a finite ramified covering. This reduces the problem of the classification to the classification of meromorphic affine principal elliptic bundles and their quotients.

3.1. **Reduction to isotrivial elliptic surfaces.** Let (M, D, ∇) be a meromorphic affine complex compact surface with algebraic dimension one, and $M \xrightarrow{\pi} N$ the corresponding elliptic surface. Our aim is to prove that this is an isotrivial surface, meaning the functional invariant τ is constant.

Let S (resp. $\overline{S}$) be the union of the singular fibers (resp. the singular points) in M (resp. N). In view of Proposition 2.2.2,(1), we can and do assume that $J \neq \emptyset$ or $N \neq \mathbb{P}^1$, otherwise M is a Hopf surface, hence a principal elliptic fiber bundle over N. Let $M' = M \setminus S$ (resp. $N' = N \setminus \overline{S}$), $\tilde{M}' = \tilde{N}' \times \mathbb{C} \overset{\pi'=proj_1}{\longrightarrow} \tilde{N}'$ with adapted global coordinates (z_1, z_2) as in Proposition 2.2.2, (1). This is the total space of the universal covering $p' : \tilde{M}' \longrightarrow M'$.

We let $\tilde{\nabla}' = p'^{\star}\nabla$, a meromorphic affine connection on $\tilde{M}'$ with poles D' supported on fibers of π' by Theorem 2.2. Moreover, reemploying the notations from Proposition 2.2.2, it is both invariant through the automorphisms:

$$\psi_1(z_1, z_2) = (z_1, z_2 + 1) \quad \text{and} \quad \psi_2(z_1, z_2) = (z_1, z_2 + \tau(z_1)) \tag{15}$$

and the automorphims $(\varphi_\gamma)_{\gamma \in \pi_1(N', y_0)}$ defined as in (10).

Now we observe:

Lemma 3.1. *Let g be a global holomorphic function on $\mathbb{C}$ with $g(z+1) = g(z)$ for any $z \in \mathbb{C}$. Suppose that there exists $\nu \in \mathbb{H}$ and $\mu \in \mathbb{C}$ such that*

$$\forall z \in \mathbb{C},\ g(z + \nu) = g(z) + \mu \tag{16}$$

Then g is constant and $\mu = 0$.

Proof. The 1-periodicity is equivalent to the existence of a holomorphic function on $\mathbb{C} \setminus \{0\}$ such that $g(z) = \overline{g}(e^{2i\pi z})$ for any $z \in \mathbb{C}$.

Then the property (16) implies

$$\overline{g}(\lambda u) = \overline{g}(u) + \mu \tag{17}$$

where $\lambda = e^{2i\nu}$ satisfies $0 < |\lambda| < 1$. Derivating this relation in u implies $\overline{g}'(\lambda u) = \frac{\overline{g}'(u)}{\lambda}$. But $\overline{g}'$ is a Laurent series at 0, with residue 0. Since $|\lambda| < 1$, $\lambda^{n-1} \neq \frac{1}{\lambda}$ except for $n = 0$, and the identity (17) implies $\overline{g}' = 0$. Hence $\overline{g}$ is a constant, and the same holds for g. In particular $\mu = 0$. □

Corollary 3.1. *Let (M, D, ∇) be a meromorphic affine complex compact surface with algebraic dimension one, and let $(\tilde{M}', \tilde{D}', \tilde{\nabla}')$ be as above, with homological invariant τ. Let (z_1, z_2) be adapted global coordinates as above and $(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2})$ (resp. (dz_1, dz_2)) the corresponding trivialisation of TM' (resp. $\Omega^1_{M'}$). Then either $\tau' = 0$, or $\tilde{\nabla}'$ has matrix:*

$$dz_1 \otimes \begin{pmatrix} b(z_1) & 0 \\ d(z_1) & c(z_1) \end{pmatrix} \quad + \quad dz_2 \otimes \begin{pmatrix} 0 & 0 \\ a(z_1) & 0 \end{pmatrix} \tag{18}$$

for some holomorphic functions a, b, c, d on $z_1(\tilde{N}') \subset \mathbb{C}$.

Proof. We assume $\tau' \neq 0$. Let $(f_{ij}, g_{ij})_{i,j=1,2}$ be the meromorphic functions on $\tilde{M'}$ such that the matrix of $\tilde{\nabla}'$ in $(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2})$ is:

$$dz_1 \otimes \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \quad + \quad dz_2 \otimes \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \tag{19}$$

Recall that, given any automorphism ψ of $\tilde{M'}$, the pullback $\psi^\star\tilde{\nabla}'$ is described by a commutative diagram (Definition 2.3). The differential of the automorphism ψ_1 from (15) corresponds, through the trivialisation $(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2})$, to the post-composition by ψ_1 of functions. In particular, by Theorem 2.2, the invariance of $\tilde{\nabla}'$ through ψ_1 implies that the restrictions of $(f_{ij}, g_{ij})_{i,j=1,2}$ on a generic fiber of π' are 1-periodic holomorphic functions.

Now, by definition of the pullback, the identity $\frac{\partial}{\partial z_2} \lrcorner \psi_2^\star\tilde{\nabla}' = \frac{\partial}{\partial z_2} \lrcorner \tilde{\nabla}'$ rewrites as:

$$\begin{cases} g_{12}(z_1, z_2 + \tau(z_1)) & = & g_{12}(z_1, z_2) \\ g_{ii}(z_1, z_2 + \tau(z_1)) + (-1)^i \tau'(z_1) g_{12}(z_1, z_2 + \tau(z_1)) & = & g_{ii}(z_1, z_2) \\ g_{21}(z_1, z_2 + \tau(z_1))(z_1, z_2 + \tau(z_1)) & = & g_{21}(z_1, z_2) \\ +\tau'(z_1)(g_{11} - g_{22})(z_1, z_2 + \tau(z_1))) & & \\ -(\tau'(z_1))^2 g_{12}(z_1, z_2 + \tau(z_1)) & & \end{cases} \tag{20}$$

Since $\tau' \neq 0$,the first line implies that the restriction of g_{12} to a generic fiber is a holomorphic elliptic function, that is a constant, i.e $g_{12}(z_1, z_2) = g_{12}(z_1)$. Now, the second line and the previous fact show that the restriction of g_{ii} to a generic fiber satisfies the conditions of Lemma 3.1, hence $g_{12} = 0$ and $g_{ii}(z_1, z_2) = g_{ii}(z_1)$. This in turn implies, together with the third line, that the restriction of g_{21} to a generic fiber satisfies the conditions in Lemma 3.1, so that $g_{11}(z_1) = g_{22}(z_1)$ and $g_{21}(z_1, z_2) = a(z_1)$.

Now, we can rewrite similarly the system of functional equations corresponding to $\frac{\partial}{\partial z_1} \lrcorner \psi_2^\star\nabla' = \frac{\partial}{\partial z_1} \lrcorner \nabla'$, taking in account that

$$d\psi_2^*(\psi_2^* \frac{\partial}{\partial z_2}) = \frac{\partial}{\partial z_2} + \tau'(z_1) \frac{\partial}{\partial z_1}$$

Since $g_{12} = 0$, the first line will be indentical to the one of (20), that is $f_{12}(z_1, z_2) = f_{12}(z_1)$. Then the second line show that f_{ii} satisfy conditions of Lemma 3.1, so that $f_{ii}(z_1, z_2) = f_{ii}(z_1)$ and $f_{12} - g_{11} = f_{12} + g_{22} = 0$, while $g_{11} = g_{22}$ by the previous facts. Hence $g_{11} = g_{22} = 0$. Finally, as before, the last line show that f_{21} satisfy conditions of Lemma 3.1, i.e. $f_{21}(z_1, z_2) = f_{21}(z_1)$. □

Theorem 3.1. *Any meromorphic affine complex compact surface of algebraic dimension one is an isotrivial elliptic surface.*

Proof. We reemploy the above notations, and will describe explicitely the identity

$$\frac{\partial}{\partial z_1} \lrcorner \varphi_\gamma^\star\tilde{\nabla}' = \frac{\partial}{\partial z_1} \lrcorner \tilde{\nabla}'$$

for any generator γ of $\pi_1(N', y_0)$.

First introduce the following notations. We let g_γ be the function of z_1 corresponding to the matrix of the differential of the automorphism of $\tilde{N'}$ corresponding to γ. We also let:

$$\delta_\gamma(z_1) = \frac{\mu_\gamma}{c_\gamma \tau(z_1) + d_\gamma} \tag{21}$$

where $\mu_\gamma, c_\gamma, d_\gamma$ are defined as in Proposition 2.2.2.

We first prove that the above identity implies $\delta'_\gamma(z_1) = 0$. Indeed, the matrix of $d\varphi_\gamma$ in the basis $(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2})$ and $(\varphi_\gamma^* \frac{\partial}{\partial z_1}, \varphi_\gamma^* \frac{\partial}{\partial z_2})$ is:

$$C = \begin{pmatrix} g_\gamma(z_1) & 0 \\ z_2\delta'_\gamma(z_1) + f'_\gamma(z_1) & \delta_\gamma(z_1) \end{pmatrix} \tag{22}$$

In particular:

$$\frac{\partial}{\partial z_1} \lrcorner C^{-1} dC = \begin{pmatrix} \frac{g'_\gamma(z_1)}{g_\gamma(z_1)} & 0 \\ z_2(\frac{\delta''_\gamma(z_1)}{\delta(z_1)} - \frac{\delta'_\gamma(z_1)}{\delta(z_1)}\frac{g'_\gamma(z_1)}{g_\gamma(z_1)}) + h(z_1) & \frac{\delta'_\gamma(z_1)}{\delta(z_1)} \end{pmatrix} \tag{23}$$

for some meromorphic function h on $z_1(\tilde{N'}) \subset \mathbb{C}$. Recalling the definition of the pullback (Definition 2.3), and focusing on the coefficient $c(z_1)$ of the matrix of $\tilde{\nabla}'$ (see Corollary 3.1), the invariance by φ_γ leads to:

$$z_2(\tfrac{\delta'_\gamma(z_1)}{\delta_\gamma(z_1)} - \tfrac{\delta'_\gamma(z_1)}{\delta_\gamma(z_1)} b(\gamma \cdot z_1) + \tfrac{\delta'_\gamma(z_1)}{\delta_\gamma(z_1)} c(\gamma \cdot z_1)) - z_2^2(\tfrac{g'_\gamma(z_1)}{g_\gamma(z_1)}\delta''_\gamma(z_1)\tfrac{\delta'_\gamma(z_1)}{\delta_\gamma(z_1)}) + h(z_1) = c(z_1) \tag{24}$$

for some meromorphic function h on $z_1(\tilde{N'})$. In particular:

$$\forall \gamma \in \pi_1(N', y_0), \quad \tfrac{g'_\gamma(z_1)}{g_\gamma(z_1)}\delta''_\gamma(z_1)\tfrac{\delta'_\gamma(z_1)}{\delta_\gamma(z_1)} \;=\; 0 \tag{25}$$

Now we fix a set $((\gamma_\alpha)_{\alpha\in I}, (\gamma_{\epsilon,1}, \ldots, \gamma_{\epsilon,2g})_{\epsilon=1,2})$ of generators for $\pi_1(N', y_0)$, where γ_α is obtained from a loop containing the singular point y_α in its bounded component, and $\gamma_1, \ldots, \gamma_{2g}$ span $\pi_1(N)$ ($g = g(N)$ is the genus). We prove that there are two possible cases:

a) τ is constant.
b) For any $\alpha \in I$, $c_{\gamma_\alpha} = 0$.

Indeed, assume τ is not constant and pick $\alpha \in I$. Clearly, $c_{\gamma_\alpha} \neq 0$ implies that $\delta''_{\gamma_\alpha}\delta'_{\gamma_\alpha} \neq 0$. Hence, in view of (25), either $c_{\gamma_\alpha} = 0$ or $g'_{\gamma_\alpha} = 0$. Now, considering a suitable Möbius transformation ϕ in the connected component of $\mathbb{P}SL_2(\mathbb{R})$, and replacing the coordinate z_1 by $z'_1 = \phi \circ z_1$, the equation (25) remains true when replacing τ by $\overline{\tau} = \tau \circ \phi^{-1}$, g'_{γ_α} by the matrix of the differential of $\gamma_\alpha \cdot$ in the basis $\frac{\partial}{\partial z'_1}$ which can be picked out non-zero, and keeping the same c_{γ_α} (since $\overline{\tau}$ have the same monodromy as τ). Hence necessarly $c_{\gamma_\alpha} = 0$.

Now we prove that case b) can't happen in our situation. Indeed, it is known that there are generators $(\gamma_\alpha)_{\alpha\in I}$ and $(\gamma_{\epsilon,j})_{\substack{j=1,\ldots,g \\ \epsilon=1,2}}$ as above such that:

$$\prod_{j=1}^{g} [\gamma_{1,j}, \gamma_{2,j}] = \prod_{\alpha\in I} \gamma_\alpha \tag{26}$$

But property b) means that the monodromy of τ is a representation ρ with values in the abelian subgroup of translations in $SL_2(\mathbb{R})$. In particular, (26) implies that the composition of the images $\rho(\gamma_\alpha)$ is trivial. Moreover, following the proof of Theorem 7.3 in [11], these are translations by $b_\alpha \geq 0$. From the previous remark, the sum of these positive integers is zero, so that $b_\alpha = 0$, that is A_{γ_α} is the identity for any $\alpha \in I$.

By Theorem 7.3 in [11], this implies that the functional invariant J have no pole on N. It is therefore a constant meromorphic function. As a consequence, τ is constant.

□

3.2. **Reduction to elliptic fiber bundles.** We now prove that the classification reduces to the one of meromorphic affine elliptic fiber bundles over complex curves, and their finite quotients.

From now, assume that $M \xrightarrow{\pi} N$ is a <u>minimal</u> elliptic surface, of algebraic dimension one, with singular fibers $S = \pi^{-1}(\overline{S})$, endowed with a meromorphic affine connection ∇ with poles D. By Theorem 3.1, its invariants are constants. Since N is a smooth compact complex curve, its is clear that there exists a finite cover $\hat{N} \xrightarrow{\overline{q}} N$, ramified at $\overline{S}$, such that the elliptic surface $\hat{M} \xrightarrow{\hat{\pi}} \hat{N}$ obtained from the diagram:

$$\begin{array}{ccc} \hat{M} = M \underset{N}{\times} \hat{N} & \xrightarrow{q} & M \\ \downarrow{\scriptstyle \hat{\pi}} & & \downarrow{\scriptstyle \pi} \\ \hat{N} & \xrightarrow[\overline{q}]{} & N \end{array} \tag{27}$$

where q is the restriction of the first projection, and $\hat{\pi}$ is the restriction of the second projection, is an elliptic surface without multiple singular fiber. Its invariants are respectively $\hat{J} = J \circ q$ and $q^*\mathcal{G}$, that is respectively a constant and a constant sheaf.

We get:

Theorem 3.2. *There is a surjective functor from the category of objects of the form $(\hat{M} \xrightarrow{\hat{\pi}} \hat{N}, \hat{\nabla}, \Gamma)$ where:*

- *$\hat{M} \xrightarrow{\hat{\pi}} \hat{N}$ is a principal elliptic bundle of algebraic dimension one through its algebraic reduction.*
- *$\hat{\nabla}$ is a meromorphic affine connection on $\hat{M}$, with pole $\hat{D}$,*
- *Γ is a finite group of automorphisms of the elliptic surface $\hat{M} \xrightarrow{\hat{\pi}} \hat{N}$ and of the meromorphic affine connection $\hat{\nabla}$, with smooth quotient.*

to the category of <u>minimal</u> meromorphic affine surfaces $(M \xrightarrow{\pi} N, \nabla)$ with $a(M) = 1$. This functor maps $(\hat{M} \xrightarrow{\hat{\pi}} \hat{N}, \hat{\nabla}, \Gamma)$ to (M, ∇) where:

- *$M = \Gamma\backslash\hat{M}$, $N = \overline{\Gamma}\backslash\hat{N}$ and $\overline{\Gamma}$ is the subgroup of finite automorphisms of $\hat{N}$ covered by an element of Γ,*
- *∇ is the meromorphic connection on $TM = (q_*T\hat{M})^{\Gamma}$ with poles at $D = q(\hat{D})$, obtained by applying Lemma 2.1 to the quotient map $q : \hat{M} \longrightarrow M$ corresponding to the action of Γ.*

Moreover, given any $(\hat{M}, \hat{\nabla}, \Gamma)$ in the first category, any $\gamma \in \Gamma$ lifts to the universal cover $\tilde{p} : \tilde{M} \longrightarrow \hat{M}$ as an automorphism:

$$\tilde{\Psi}(z_1, z_2) = (\delta \cdot z_1, \mu z_2 + f_\delta(z_1)) \tag{28}$$

where δ is the lift of an automorphism $\overline{\delta} \in Aut(\hat{N})$ to the universal covering $\tilde{N}$, $\mu \in \mathbb{C}^*$ and f_δ is a holomorphic function on $\tilde{N}$, and (z_1, z_2) are coordinates on $\tilde{M} \subset \mathbb{C}^2$ as in Proposition 2.2.2.

Proof. We begin by proving the equivalence of categories. Suppose that $(\hat{M} \xrightarrow{\hat{\pi}} \hat{N}, \hat{\nabla}, \Gamma)$ is an object and (M, ∇) its image as in the statement. First, we prove that M is of algebraic dimension one. Indeed, suppose that f is a meromorphic function on M. Then $\hat{f} = f \circ q$ is an element of $\hat{\pi}^{\#}\mathbb{C}(\hat{N})$. By definition of $M \xrightarrow{\pi} N$, f is thus an element of $\pi^{\#}\mathbb{C}(N)$. Also, M is a minimal surface. Indeed, if C is an exceptional curve, then $q^*C = \hat{C}$ is a smooth rational curve in $\hat{M}$, contained in a fiber of $\hat{\pi}$, which can't be a principal elliptic fiber bundle. Hence, the functor is well-defined on objects and extends as a functor for the obvious choice of arrows

(namely Γ-equivariant isomorphisms of meromorphic affine connections and isomorphisms of meromorphic affine connections).

Now, if (M, ∇) is an object of the target category, then we define $\hat{M} \xrightarrow{\hat{\pi}} \hat{N}$ as in (27) and $\hat{\nabla} = q^\star \nabla$. We prove that $\hat{M} \xrightarrow{\hat{\pi}} \hat{N}$ is a principal elliptic bundle. First, recall that the invariants are respectively a constant for the functional invariant and a constant sheaf for the homological invariant. The basic member $\hat{B}$ associated to these invariants (see Theorem 2.3) is $\hat{B} = \hat{N} \times \mathbb{C}/\Lambda$ for some lattice Λ. Moreover, $\hat{M}$ is a minimal surface. Indeed, suppose the existence of an exceptional curve $\hat{C}$ in $\hat{M}$. Since the algebraic dimension of $\hat{M}$ is one, the proof of Theorem 4.2 in [10] implies that $\hat{C}$ is a singular fiber of $\hat{\pi}$. But then its image C through q would also be an exceptional curve. Indeed, the restriction of q to the support of $\hat{C}$ is a biholomorphism, so C is a smooth rational curve. It is also a singular fiber of the minimal elliptic surface M. By the proof of Theorem 6.2 in [11], we must have $C \cdot C \leq -1$. In one other hand, since q is a finite cover, we also have

$$-1 = \hat{C} \cdot \hat{C} = deg(\mathcal{N}_{\hat{C}}) = kC \cdot C$$

for some positive integer k, hence $C \cdot C = -1$ contradicting the minimality of M. From the point 2. of the Theorem 2.3, $\hat{M}$ is locally isomorphic, as an elliptic fibration, to $\hat{B}$. Hence, $\hat{M}$ has no singular fiber. Up to considering a finite cover, $\hat{M} \xrightarrow{\hat{\pi}} \hat{N}$ is therefore a principal elliptic fiber bundle. Finally, setting Γ as the (finite) group of automorphisms of q, we get an object mapping to (M, ∇).

Now let's prove the formula (28). Let Ψ be an automorphism of $\hat{\pi}$. It covers an automorphism $\overline{\delta} \in Aut(\hat{N})$ and we define $\delta \in Aut(\tilde{N})$ as its lift to the universal covering $\tilde{\overline{p}} : \tilde{N} \longrightarrow \hat{N}$. By construction, any lift $\tilde{\Psi}$ of Ψ to the universal covering $\tilde{M}$ is an automorphism of $\mathbb{C}$-principal bundle covering δ.

Consider the covering $\overline{q} : \overline{M} \longrightarrow M$, where $\overline{M} = \hat{M} \underset{\hat{N}}{\times} \tilde{N}$, obtained as in the proof of Proposition 2.2.2. Any element of $\overline{M}$ is of the form $[\hat{y}, x]$ for some $x \in \hat{M}$ and $\hat{y}$ in $\hat{N}$. Recall that $\tilde{M}$ is also the universal covering of $\overline{M}$. Moreover, Ψ lifts canonically to an automorphism $\overline{\Psi}$ of $\overline{M}$ defined by:

$$\overline{\Psi}([\hat{y}, x]) = [\overline{\delta}(\hat{y}), \Psi(x)] \tag{29}$$

Then the formula (28) follows as in the proof of point 3. of Proposition 2.2.2.

□

4. Principal elliptic surfaces over projective line

We begin to apply the strategy proposed in the last section, with the case $\hat{N} = \mathbb{P}^1$. We first describe the compact complex surfaces which are principal elliptic bundles of algebraic dimension one on $\hat{N}$, namely the Hopf surfaces with $a(\hat{M}) = 1$. This description is in terms of the universal cover $\tilde{M} = \mathbb{C}^2 \setminus \{0\}$. Then we classify meromorphic affine connections on $\hat{M}$ (Theorem 4.2). Finally we prove that a non trivial finite group of $Aut(\hat{\pi})$ has no fixed curve (Theorem 4.3), any minimal meromorphic affine surface M with $a(M) = 1$ arising from a Hopf surface through the construction (27) is again a Hopf surface.

4.1. **Hopf surfaces of algebraic dimension one.** Recall the following characterization of Hopf surfaces among elliptic surfaces (see for example [2]):

Theorem 4.1. *Let $\hat{M}$ be a complex compact surface. Then the following assertions are equivalent:*

(1) M is a Hopf surface with algebraic dimension one

(2) *The universal cover of* $\hat{M}$ *is* $\mathbb{C}^2 \setminus \{0\}$ *and* M *is of algebraic dimension one*
(3) *There exists* $\lambda \in \mathbb{C}^*$ *with* $|\lambda|^k \to 0$*, an integer* $d \geq 1$ *such that the following diagram commutes:*

$$\begin{array}{ccc} \mathbb{C}^2 \setminus \{0\} & \xrightarrow{\rho_d} & \hat{M} \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathbb{P}^1 & = & \mathbb{P}^1 \end{array} \tag{30}$$

where ρ_d *is the quotient map corresponding to the action of* $\Gamma_d = \langle (z_1, z_2) \mapsto (\lambda z_1, \lambda^{\frac{1}{d}} z_2) \rangle$ *on* $\mathbb{C}^2 \setminus \{0\}$ *and* p_1 *is the bundle map for the tautological bundle of* $\mathbb{P}^1$*.*

The Hopf surface corresponding to some fixed λ *and* $d \geq 1$ *will be denoted by* M_d*.*

As an example, the original Hopf surface is M_1 for $\lambda = \frac{1}{2}$.

4.2. **Meromorphic affine elliptic bundles over the projective line.** Let $\hat{M}$ be a principal elliptic bundle over $\hat{N} = \mathbb{P}^1$, with algebraic dimension one. By Theorem 4.1, there exists $\lambda \in \mathbb{C}^\times$ $d \geq 1$ such that $\hat{M} = M_d$ using the previous notation. In particular, the universal cover $p : \tilde{M} \longrightarrow \hat{M}$ identifies the field of meromorphic functions on $\hat{M}$ with $\mathbb{C}(\frac{z_1}{z_2^d})$, where (z_1, z_2) are the canonical coordinates of $\mathbb{C}^2$ restricted to $\tilde{M} = \mathbb{C}^2 \setminus \{0\}$. From now on we identify $\mathbb{C}(\hat{M})$ with $\mathbb{C}(X)$ through this isomorphism.

Theorem 4.2. *Let* $\hat{M}$*,* $\lambda \in \mathbb{C}^*$ *and* $d \geq 1$ *be as above. Denote by* $\mathcal{A}$ *the affine space* $\mathcal{A}$ *of meromorphic affine connections on* $\hat{M}$*. Then:*

(1) *There is an isomorphism of affine spaces:*

$$\begin{array}{ccc} \mathbb{C}(X)^4 \times \mathbb{C}(X)^4 & \xrightarrow{\sim} & \mathcal{A} \\ (P_{ij}(X), Q_{ij}(X))_{i,j=1,2} & \mapsto & \rho(\tilde{\nabla}_{P_{ij},Q_{ij}}) \end{array} \tag{31}$$

where ρ *is the map sending any meromorphic affine connection on* $\tilde{M}$ *invariant by the Galois group of* p *to the meromorphic affine connection on* $\hat{M}$ *constructed in Lemma 2.1, and:* $\tilde{\nabla}_{P_{ij},Q_{ij}}$ *is the meromorphic affine connection on* $\tilde{M}$ *whose matrix in* $(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2})$ *is:*

$$dz_1 \otimes \begin{pmatrix} P_{11}(\frac{z_1}{z_2^d})\frac{z_2^d}{z_1^2} & P_{12}(\frac{z_1}{z_2^d})\frac{z_2^{d-1}}{z_1} \\ P_{21}(\frac{z_1}{z_2^d})\frac{z_2}{z_1^2} & P_{22}(\frac{z_1}{z_2^d})\frac{z_2^d}{z_1^2} \end{pmatrix} + dz_2 \otimes \begin{pmatrix} Q_{11}(\frac{z_1}{z_2^d})\frac{z_2^{d-1}}{z_1} & Q_{12}(\frac{z_1}{z_2^d})\frac{z_2^d}{z_1} \\ Q_{21}(\frac{z_1}{z_2^d})\frac{z_2^d}{z_1^2} & Q_{22}(\frac{z_1}{z_2^d})\frac{z_2^{d-1}}{z_1} \end{pmatrix} \tag{32}$$

(2) *In particular, there exists non-flat meromorphic affine connections on any Hopf surface of algebraic dimension one, and exactly one holomorphic affine connection on any such manifold, that is the standard affine structure.*

Proof. Pick a meromorphic affine connection ∇ on $\hat{M}$ let $\tilde{\nabla}$ be its pullback on $\mathbb{C}^2 \setminus \{0\}$ through $\tilde{p}$ (see Definition 2.3). Then $\tilde{\nabla}$ is by construction a meromorphic affine connection on $\mathbb{C}^2 \setminus \{0\}$, which is Γ_d-invariant, and there are meromorphic functions $(f_{ij}, g_{ij})_{i,j=1,2}$ on $\mathbb{C}^2 \setminus \{0\}$ such that the matrix of $\tilde{\nabla}$ in $(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2})is$:

$$dz_1 \otimes \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \quad + \quad dz_2 \otimes \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \tag{33}$$

The invariance of $\tilde{\nabla}$ by the Galois group of $\tilde{p}$ is equivalent to the system of functional equations:

$$\left\{\begin{array}{ccccc} \lambda & f_{ii}(\lambda z_1, \lambda^{\frac{1}{d}} z_2) & = & f_{ii}(z_1,z_2) \\ \lambda^{\frac{1}{d}} & f_{12}(\lambda z_1, \lambda^{\frac{1}{d}} z_2) & = & f_{12}(z_1,z_2) \\ \lambda^{2-\frac{1}{d}} & f_{21}(\lambda z_1, \lambda^{\frac{1}{d}} z_2) & = & f_{21}(z_1,z_2) \\ \lambda^{\frac{1}{d}} & g_{ii}(\lambda z_1, \lambda^{\frac{1}{d}} z_2) & = & g_{ii}(z_1,z_2) \\ \lambda^{\frac{2}{d}-1} & g_{12}(\lambda z_1, \lambda^{\frac{1}{d}} z_2) & = & g_{12}(z_1,z_2) \\ \lambda & g_{21}(\lambda z_1, \lambda^{\frac{1}{d}} z_2) & = & g_{21}(z_1,z_2) \end{array}\right. \tag{34}$$

Now, it is straightforward that the functions $(f^0_{ij}, g^0_{ij})_{i,j=1,2}$ given by setting $P_{ij} = Q_{ij} = 1$ in (32) define a solution of the above system. Given any other solution $(f_{ij}, g_{ij})_{i,j=1,2})$, the quotients $h_{ij} = \frac{f_{ij}}{f^0_{ij}}$ and $k_{ij} = \frac{g_{ij}}{g^0_{ij}}$ are clearly elements of $q^{\#}\mathbb{C}(\hat{M})$. Hence $\nabla = \rho(\tilde{\nabla}_{P_{ij},Q_{ij}})$ for some $P_{ij}, Q_{ij} \in \mathbb{C}(X)$. Reciprocally, given any elements $P_{ij}, Q_{ij} \in \mathbb{C}(X)$, the meromorphic connection $\tilde{\nabla}_{P_{ij},Q_{ij}}$ on $\mathbb{C}^2 \setminus \{0\}$ is invariant through the action of the Galois group of $\tilde{p}$, so by Lemma 2.1 there exists a unique ∇ on $\hat{M}$ with $\tilde{p}^\star \nabla = \tilde{\nabla}_{P_{ij},Q_{ij}}$.

The curvature of the connection $\tilde{\nabla} = \tilde{\nabla}_{P_{ij},Q_{ij}}$ can be computed explicitely. As an example, for $Q_{ij} = 0$, $P_{ij} = \delta^{21}_{ij}$ the matrix of the curvature $R_{\tilde{\nabla}}$ in the basis $\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}$ is

$$dA + A \wedge A = dz_1 \wedge dz_2 \otimes \begin{pmatrix} 0 & 0 \\ \frac{1}{z_1^2} & 0 \end{pmatrix} \neq 0$$

whence the assertion.

However, when $\tilde{\nabla}_1$ is holomorphic, then $P_{ij} = Q_{ij} = 0$ that is $\tilde{\nabla}$ is the standard affine structure on $\mathbb{C}^2$. □

4.3. **Quotients of meromorphic affine Hopf surfaces.**

Theorem 4.3. *Let (M, ∇) be a minimal meromorphic affine surface with $a(M) = 1$ and suppose that the finite covering $\hat{M}$ from (27) is a principal elliptic surface over $\mathbb{P}^1$. Then $\hat{M} = M$ and (M, ∇) is classified in Theorem 4.2.*

Proof. Let Γ be the Galois group of $q : \hat{M} \longrightarrow M$ and $\overline{\Gamma}$ the Galois group of $\overline{q} : \hat{N} \longrightarrow N$. Suppose that $\overline{q}$ (and so q) admits a ramification point $y_\beta \in N$. By definition of q, this means that there exists on M a multiple fiber $S_\beta = \pi^{-1}(y_\beta)$ and a non-trivial element $\epsilon \in \Gamma$ fixing $\hat{S}_\beta = q^{-1}(S_\beta)$. Then $\hat{S}_\beta$ is the curve obtained as the quotient of $\{z_1 = 0\}$ or $\{z_2 = 0\}$ in the univeral cover $\mathbb{C}^2 \setminus \{0\}$ of the Hopf surface $\hat{M} = M_d$ (see Theorem 4.1). These are precisely the inverse images of 0 and ∞ through $\hat{\pi} : \hat{M} \longrightarrow \hat{N} = \mathbb{P}^1$. Without loss of generality (up to exchanging z_1 and z_2), we suppose that $\hat{S}_\beta$ is the quotient of $\{z_1 = 0\}$. This implies that the automorphism $\bar{\epsilon}$ of $\hat{N}$ covered by ϵ is an element of $Aut(\mathbb{C}) \subset Aut(\mathbb{P}^1)$. In particular, ϵ lifts to the universal cover $\mathbb{C}^2 \setminus \{0\}$ as an automorphism $\tilde{\epsilon}$ of the form:

$$\tilde{\epsilon}(z_1, z_2) = (\mu(az_1 + bz_2), \mu z_2)$$

for some $a, \mu \in \mathbb{C}^*$ and $b \in \mathbb{C}$. But then $\tilde{\epsilon}^m$ is an element of the Galois group of the universal cover $\rho_d : \mathbb{C}^2 \setminus \{0\} \longrightarrow \hat{M}$ for some $m \geq 1$. Since this Galois group is spanned by $(z_1, z_2) \mapsto (\lambda z_1, \lambda^d z_2)$, this implies:

$$\mu^m = \lambda^{rd} \quad \text{and} \quad b \sum_{k=1}^{m} \mu^k = 0 \quad \text{and} \quad a\mu^m = \lambda^r$$

Since $|\lambda| > 1$ we get $b = 0$, so that $\tilde{\epsilon}(z_1, z_2) = (\lambda^{\frac{1}{r}} z_1, \lambda^{\frac{d}{r}} z_2)$ for some integer r. Now, by definition of q and the remarks above, ϵ fixes the quotient of $\{z_1 = 0\}$. Hence $\lambda^{\frac{d}{r}} = \lambda^{ld}$ for some integer l, so that $\tilde{\epsilon}$ is in fact an element of the Galois group of ρ_d, i.e ϵ is the identity on $\hat{M}$.

We have thus proved that $\overline{q}$ is an unramified finite cover. Hence M has no multiple fiber, so it is a Hopf surface, in particular a principal elliptic bundle. This implies $\hat{M} = M$. □

5. Principal elliptic surfaces over an elliptic curve and quotients

In this section, following Theorem 3.2, we classify meromorphic affine connections on (holomorphic) principal elliptic surfaces over a one torus $\hat{N} = \mathbb{C}/\Lambda'$, as well as their quotients.

Let $\hat{M}$ be a complex compact surface which is a holomorphic principal elliptic bundle over a torus. We first recall a result of Kodaira asserting that $\hat{M}$ corresponds to one of the two following examples:

Definition 5.1. A primary Kodaira surface over a torus $\hat{N} = \mathbb{C}/\Lambda'$ is an elliptic surface $\hat{M} \xrightarrow{\hat{\pi}} \hat{N}$, where $\hat{M} = G\backslash\mathbb{C}^2$, with $\hat{\pi}(z_1, z_2) = [z_1]$ ($[z_1]$ stands for the class of z_1 in $\mathbb{C}/\Lambda'$) and the group $G \subset Aut(\mathbb{C}^2)$ spanned by ψ_1, ψ_2 as in Proposition 2.2.1 (for some $\tau \in \mathbb{H}$) and the automorphisms $(\varphi_{\lambda'})_{\lambda' \in \Lambda'}$ defined by:

$$\varphi_{\lambda'}(z_1, z_2) = (z_1 + \lambda', z_2 + \overline{\lambda'} z_1 + \beta_{\lambda'}) \tag{35}$$

for some $\beta_{\lambda'} \in \mathbb{C}$.

Definition 5.2. A two torus is an elliptic surface $\hat{M} \xrightarrow{\hat{\pi}} \hat{N}$ where $\hat{M}$ is a quotient $G\backslash\mathbb{C}^2$, with $\hat{\pi}(z_1, z_2) = [z_1]$ and G a subgroup of translations in $\mathbb{C}^2$.

Theorem 5.1. *Let $\hat{M}$ be a complex compact surface which is a holomorphic principal elliptic bundle over a torus. Then $\mathcal{K}_{\hat{M}} \simeq \mathcal{O}_{\hat{M}}$ and either:*

a) The first Betti number of $\hat{M}$ is odd if and only if $\hat{M}$ is a primary Kodaira surface.
b) The first Betti number of $\hat{M}$ is even if and only if $\hat{M}$ is a two torus.

Proof. Let (z_1, z_2) be coordinates on the universal cover $\tilde{M}$ of $\hat{M}$ as in Proposition 2.2.2. Then the holomorphic volume form $dz_1 \wedge dz_2$ is clearly invariant through the automorphisms ψ_1, ψ_2 and $(\varphi_\gamma)_{\gamma \in \pi_1(\hat{N}, y)}$. Thus, it is the pullback of a global holomorphic volume form $\hat{\eta}$ on the covering $\overline{M}$ defined as in (12), which is invariant through the Galois group of this covering. Hence, $\hat{\eta}$ is the pullback of a global holomorphic volume form on M, proving $\mathcal{K}_M \simeq \mathcal{O}_M$.

The second assertion is a part of the Kodaira's classification ([12], 6.), where we eliminated the K3 surfaces since these are elliptic surfaces over the projective line.

□

We are thus led to classify meromorphic affine primary Kodaira surfaces, meromorphic affine two tori, and their quotients.

5.1. Meromorphic affine primary Kodaira surfaces.

Let $\hat{M} \xrightarrow{\pi} \hat{N} = \mathbb{C}/\Lambda'$ be a primary Kodaira surface (Definition 5.1) and G the group such that $\hat{M} = G\backslash\mathbb{C}^2$. Suppose the existence of a meromorphic affine connection ∇ on (M, D) for some divisor D. Since π is a principal elliptic bundle, Theorem 2.2 implies $D = \pi^* C$ for some divisor on the one torus N.

Define $\mathcal{E}_0$ as the subspace of Λ'-elliptic functions, that is:

$$\mathcal{E}_0 = \{h \in \mathcal{M}(\mathbb{C}) \,|\, \forall \lambda' \in \Lambda',\, \delta_{\lambda'}(h)(z_1) = 0\} \quad \text{where} \quad \delta_{\lambda'}(h)(z_1) = h(z_1 + \lambda') - h(z_1) \tag{36}$$

Recall that $\mathcal{E}_0$ is the subfield of meromorphic functions obtained as the extension of $\mathbb{C}$ by two elements $\wp(z_1), \wp'(z_1)$, where:

$$\wp(z_1) = \frac{1}{z_1^2} + \sum_{\lambda' \in \Lambda' \setminus \{0\}} \frac{1}{(z_1 - \lambda')^2} - \frac{1}{\lambda'^2} \tag{37}$$

is Weirestrass elliptic function. Then define:

$$\mathcal{E}_1 \;=\; \{h \in \mathcal{M}(\mathbb{C}) \,|\, \exists \chi_h \in Hom_{\mathbb{Z}}(\Lambda', \mathbb{C}),\, \forall \lambda' \in \Lambda',\, \delta_{\lambda'}(h)(z_1) = \chi_h(\lambda')\} \tag{38}$$

equipped with the natural linear map:

$$\begin{array}{rccc} \chi & : & \mathcal{E}_1 & \longrightarrow & Hom_{\mathbb{Z}}(\Lambda', \mathbb{C}) \\ & & h & \mapsto & \delta(h) \end{array} \tag{39}$$

Clearly $h \in \mathcal{E}_1$ if and only if $h' \in \mathcal{E}_0$. In particular, $z_1 \in \mathcal{E}_1$, and the Weirestrass zeta function (a primitive of $\wp$) :

$$\zeta \in \mathcal{E}_1 \tag{40}$$

This implies:

Lemma 5.1. *There is an exact sequence:*

$$0 \longrightarrow \mathcal{E}_0 \longrightarrow \mathcal{E}_1 \xrightarrow{\chi} Hom_{\mathbb{Z}}(\Lambda', \mathbb{C}) \longrightarrow 0 \tag{41}$$

which splits through the linear map:

$$\chi^{-1}(\alpha\chi_\zeta + \beta\chi_{z_1}) = \alpha\zeta(z_1) + \beta z_1 \tag{42}$$

Proof. The fact $ker(\chi) = \mathcal{E}_0$ is immediate by definition of χ. By (37), $Res_0(\zeta) = -1$, while $Res_0(z_1) = Res_0(f) = 0$ for any $f \in \mathcal{E}_0$. As a consequence, $\chi_\zeta = \chi(\zeta)$ and $\chi_{z_1} = \chi(z_1)$ are independant. Since $Hom_{\mathbb{Z}}(\Lambda', \mathbb{C})$ has dimension two, the above sequence is right-exact. □

Consider the pullback $\tilde{\nabla} = \tilde{\rho}^\star \nabla$, which is a meromorphic G-invariant affine connection on $\mathbb{C}^2$. By Theorem 2.2, the pole $\tilde{D}$ of $\tilde{\nabla}$ is supported on a Λ'-invariant union of subvarieties $\{z_1 = y_\alpha + \lambda'\}$. There are meromorphic function f_{ij}, g_{ij} on $\mathbb{C}^2$, with poles supported at $\tilde{D}$, such that :

$$Mat_{\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}}(\tilde{\nabla}) = dz_1 \otimes \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} + dz_2 \otimes \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \tag{43}$$

By the Λ-invariance of $\tilde{\nabla}$, the restriction of f_{ij} and g_{ij} to any fiber of z_1 is constant, that is f_{ij}, g_{ij} are functions of z_1, and we will omit the second variable z_2 in the sequel. Now, given any $\lambda' \in \Lambda'$, we have :

$$\varphi_{\lambda'}^\star(dz_1) = dz_1 \quad \text{and} \quad \varphi_{\lambda'}^\star(dz_2) = dz_2 + b_{\lambda'} dz_1 \tag{44}$$

where $\varphi_{\lambda'}$ are the elements of G as in Definition 5.1. Hence, the invariance of $\tilde{\nabla}$ by G rewrites as (see [16] p.238-239):

$$
\forall \lambda' \in \Lambda', \begin{cases}
\delta_{\lambda'}(g_{12})(z_1) &= 0 \\
\delta_{\lambda'}(g_{ii})(z_1) &= (-1)^{i+1}\overline{\lambda'} g_{12}(z_1) \\
\delta_{\lambda'}(g_{21})(z_1) &= \overline{\lambda'}(g_{22}-g_{11})(z_1) - \overline{\lambda'}^2 g_{12}(z_1) \\
\delta_{\lambda'}(f_{12})(z_1) &= -\overline{\lambda'} g_{12}(z_1) \\
\delta_{\lambda'}(f_{ii})(z_1) &= (-1)^{i+1}\overline{\lambda'} f_{12}(z_1) - \overline{\lambda'} g_{ii}(z_1) \\
&\quad +(-1)^{i}\overline{\lambda'}^2 g_{12}(z_1) \\
\delta_{\lambda'}(f_{21})(z_1) &= \overline{\lambda'}(f_{22}-f_{11}-g_{21})(z_1) \\
&\quad +\overline{\lambda'}^2(g_{22}(z_1)-g_{11}(z_1)-f_{12}(z_1)) + \overline{\lambda'}^3 g_{12}(z_1)
\end{cases} \tag{45}
$$

Reciprocally, any family $(f_{ij}, g_{ij})_{i,j=1,2}$ of meromorphic functions on $\mathbb{C}$ satisfying (45) is the family of the Christoffel symbols for a G-invariant meromorphic affine connection on $\mathbb{C}^2$, and thus defines a meromorphic affine connection on $\hat{M}$.

Now, we compute the solutions (f_{ij}, g_{ij}) of the system of functional equations (45).

Proposition 5.1.1. *Let $\hat{\nabla}$ be a meromorphic connection on a primary Kodaira surface $\hat{M}$ as above. Let $\mathcal{Z} = \alpha\zeta + \beta z_1$ the meromorphic function from Lemma 5.1. Then the matrix of the meromorphic affine connection $\tilde{\nabla} = p^\star\hat{\nabla}$ in the basis $(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2})$ is either:*

a)

$$
\begin{aligned}
& dz_1 \otimes \left(\begin{array}{c|c} -(\mathcal{Z}^2+\gamma_{11})g_{12} & \mathcal{Z} g_{12} \\ \hline -(\mathcal{Z}^3 + c(\mathcal{Z}+k)^2 & ((\mathcal{Z}+\delta_{22}+\gamma_{12})^2 \\ \quad +d\mathcal{Z}+\gamma_{12})g_{12} & \quad +\gamma_{22})g_{12} \end{array}\right) \\
+ \quad & dz_2 \otimes \left(\begin{array}{c|c} -(\mathcal{Z}+\delta_{11})g_{12} & g_{12} \\ \hline -((\mathcal{Z}+(\delta_{22}-\delta_{11}))^2 & \\ \quad +\delta_{21})g_{12} & (\mathcal{Z}+\delta_{22})g_{12} \end{array}\right)
\end{aligned} \tag{46}
$$

with g_{12} a non trivial Λ'-elliptic function, $\gamma_{ij}, \delta_{ij} \in \mathcal{E}_0$, and $h = -\frac{2}{3}\delta_{11}$, $c = \delta_{11}+\delta_{22}+\gamma_{12}$ and d, k satisfying (49)*, or:*

b)

$$
\begin{aligned}
& dz_1 \otimes \left(\begin{array}{c|c} -(g_{11}-f_{12})\mathcal{Z} & f_{12} \\ \quad + \ \gamma_{11} & \\ \hline (g_{11}-g_{22}+f_{12})\mathcal{Z}^2 & -(g_{22}+f_{12})\mathcal{Z} \\ + \ (\gamma_{11}-\gamma_{22}+\delta_{21})\mathcal{Z} & + \ \gamma_{22} \\ + \ \gamma_{21} & \end{array}\right) \\
+ \quad & dz_2 \otimes \left(\begin{array}{c|c} g_{11} & 0 \\ \hline -(g_{22}-g_{11})\mathcal{Z} & \\ \quad +\delta_{21} & g_{22} \end{array}\right)
\end{aligned} \tag{47}
$$

where $g_{11}, g_{22}, f_{12}, \gamma_{ij}, \delta_{21}$ are arbitrary Λ'-elliptic functions.

Reciprocally, any matrix as above is the matrix of the pullback $\tilde{\nabla}$ of some meromorphic affine connection $\hat{\nabla}$ on $\hat{M}$ through its universal covering $\tilde{p} : \tilde{M} \longrightarrow \hat{M}$.

Proof. Recall that the pullback of a meromorphic affine connection on $\hat{M}$ to its universal covering $\tilde{M}$ defines a bijection between meromorphic affine connections on $\hat{M}$ and meromorphic functions $(f_{ij}, g_{ij})_{i,j=1,2}$ on $\mathbb{C}$ solutions of (45). We distinguish the cases $g_{12} = 0$ and $g_{12} \neq 0$:

a) First suppose $g_{12} \neq 0$. In this case, the first line of (45) is equivalent to $g_{12} \in \mathcal{E}_0 \setminus \{0\}$, and applying Lemma 5.1 to $\frac{g_{ii}}{g_{12}}$ shows that the second and fourth one is equivalent to $g_{ii} = (-1)^i(\mathcal{Z} + \delta_{ii})g_{12}$ and $f_{12} = (\mathcal{Z} + \gamma_{12})g_{12}$ for some elliptic functions $\delta_{11}, \delta_{22}, \gamma_{12}$. Rewriting the system (45) in this case, we see that the third line is equivalent to:

$$\forall \lambda' \in \Lambda',\ \delta_{\lambda'}(g_{21})(z_1) = \delta_{\lambda'}(-(\mathcal{Z} + (\delta_{22} - \delta_{11}))^2 g_{12})(z_1) \tag{48}$$

so that this line becomes equivalent to

$$g_{21} = -((\mathcal{Z} + (\delta_{22} - \delta_{11}))^2 + \delta_{21})g_{12}$$

for an arbitrary elliptic function δ_{21}. By the same principle, the fifth and sixth lines are now equivalent to $f_{ii} = (-1)^i((\mathcal{Z} + \delta_{ii} + \gamma_{12})^2 + \gamma_{ii})g_{12}$ and $f_{21} = -(\frac{1}{3}(\mathcal{Z} + h)^3 + c(\mathcal{Z} + k)^2 + \gamma_{21})g_{12}$ for an arbitrary elliptic functions γ_{21} and with $h, c, k \in \mathcal{E}_0$ solutions of the system:

$$\begin{cases} 3h^2 + 2ck &= (\delta_{22} - \delta_{11})^2 + (\delta_{22} + \gamma_{12})^2 + (\delta_{11} + \gamma_{12})^2 + \delta_{21} + \gamma_{11} - \gamma_{22} \\ 2c + 3h &= 4(\delta_{22} + \gamma_{12}) \\ 3h + c &= \delta_{22} - \delta_{11} + \gamma_{12} \end{cases} \tag{49}$$

We get the matrix form (46).

b) Now suppose $g_{12} = 0$. Then using the Lemma 5.1 as before we get that the five first lines of (45) are equivalent to $g_{12} = 0, g_{11}, g_{22}, f_{12} \in \mathcal{E}_0$ and $g_{21} = (g_{11} - g_{22})\mathcal{Z} + \delta_{21}$ and $f_{ii} = -(g_{ii} + (-1)^i f_{12})\mathcal{Z} + \gamma_{ii}$ for some arbitrary elliptic function γ_{ii}, δ_{21}.

Now, the last line is equivalent to:

$$\delta_{\lambda'}(f_{21})(z_1) = \delta_{\lambda'}((g_{11} - g_{22} + f_{12})(\mathcal{Z} + \gamma_{21})^2 + (\gamma_{11} - \gamma_{22} + \delta_{21})\mathcal{Z})$$

We get the matrix form (47).

□

We obtained:

Theorem 5.2. *Let $\pi : M \longrightarrow \mathbb{C}/\Lambda'$ be a primary Kodaira surface and $p : \mathbb{C}^2 \longrightarrow M$ its universal cover. Then:*

(1) The pullback of meromorphic connections through p gives a bijection between the set of meromorphic affine connections on M and the set of meromorphic affine connections on $\mathbb{C}^2$ with matrix as in Proposition 5.1.1.

(2) The only holomorphic affine connections on M are the ∇ corresponding to $\tilde{\nabla}$ with matrix (47) with $g_{22} - g_{11} = f_{12}$ and $\gamma_{11} = \gamma_{22}$ constants (we recover the result by A. Vitter, see [16], 5.b). In particular their curvature identically vanishes, and there are flat holomorphic affine structures on M.

(3) There exists non flat meromorphic affine connections on M.

Proof. (1) By the remark below (45), the set of meromorphic affine connections on M is in bijection with the set of $\tilde{\nabla}$ with matrix form as in Proposition 5.1.1.

(2) Among the matrix forms in (5.1.1), the only possible form with holomorphic one forms as entries is (47) with holomorphic functions as coefficients for each matrix. This is equivalent to the identities from the statement. In particular the curvature is identically zero, and picking $f_{11} = f_{22} = f_{21} = 0$, we get that the standard holomorphic affine structure of $\mathbb{C}^2$ induces a holomorphic affine structure on M, thus recovering the result of Inoue,Kobayashi and Ochiai.

(3) In (47), pick $f_{11} = \wp$, $f_{22} = f_{21} = 0$. Then the curvature of $\tilde{\nabla}$ is

$$R_{\tilde{\nabla}} = -\wp(z_1)dz_1 \wedge dz_2 \otimes dz_1 \otimes \frac{\partial}{\partial z_2} \neq 0$$

We thus get a non flat meromorphic affine connection on M.

□

5.2. **Quotients: Meromorphic affine secondary Kodaira surfaces.** We now classify the quotients of the meromorphic affine primary Kodaira surface $(\hat{M}, \hat{\nabla})$ from the previous section. Using the description of the quotients of primary Kodaira surfaces ([13], Theorem 39), we get the following:

Lemma 5.2. *Let $M \xrightarrow{\pi} N$ be a minimal elliptic surface with $a(M) = 1$, endowed with a meromorphic affine connection, and $\hat{M} \xrightarrow{\hat{\pi}} \hat{N}$ its finite ramified covering as in Theorem 3.2. Suppose $\hat{M}$ has canonical trivial bundle. Then either $\hat{M} = M$ or $\mathcal{K}_M^{\otimes k} = \mathcal{O}_M$ for some $k \geq 2$. Moreover:*

(1) $M = \Gamma \backslash \hat{M}$ where Γ is a cyclic group acting freely and spanned by an automorphism $\tilde{\Psi}$ of the form

$$\tilde{\Psi}(z_1, z_2) = (\nu z_1 + \theta, \mu z_2 + a z_1 + b) \tag{50}$$

where ν is a k-th root of the unity ($k \leq 6$), μ is a power of ν and $a, b \in \mathbb{C}$.

(2) Moreover, if $\hat{M}$ is a two torus and Γ is not trivial, $\mu \neq \nu$ in (50).

Proof. (1) If M is a principal elliptic bundle then $\hat{M} = M$ by construction of $\hat{M}$. We then suppose that M is not a principal elliptic bundle and Γ is not trivial. Moreover $kod(M) \leq kod(\hat{M}) = 0$. Since M is minimal, by the Enriques-Kodaira classification (see [2], Table 10 p.189), if $kod(M) = 0$, then $\mathcal{K}_M^{\otimes k} = \mathcal{O}_M$ for some integer $k \geq 2$. Since $\mathcal{K}_{\hat{M}}$ is trivial, $q : \hat{M} \longrightarrow M$ is isomorphic to the unramified covering associated with $\mathcal{K}_M$, and the formula (50) follows from the proof of Theorem 38 in [13]. If $kod(M) = -\infty$, then M is a Hopf surface with $a(M) = 1$, and therefore a principal elliptic bundle through its algebraic reduction. In particular, it has no singular fiber so that $\hat{M} = M$. This contradicts $\mathcal{K}_{\hat{M}} = 0$ so necessarly $kod(M) = 0$. In particular $\mathcal{K}_M^{\otimes k}$ is trivial for some $k \geq 1$ and the formula (50) can be recovered from Theorem 39 and Theorem 40 in [13].

(2) Suppose $\mu = \nu$. Since Ψ is of order k, $\tilde{\Psi}^k$ belongs to the subgroup spanned by the automorphisms Ψ_1, Ψ_2 and $(\varphi_{\lambda'})_{\lambda' \in \Lambda'}$, where k is the order of Γ. Suppose $k > 1$. The matrix of $d\tilde{\Psi}$ in the basis $(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2})$ and $(\tilde{\Psi}^* \frac{\partial}{\partial z_1}, \tilde{\Psi}^* \frac{\partial}{\partial z_2})$ is:

$$\begin{pmatrix} \nu & 0 \\ a & \nu \end{pmatrix}$$

, while the matrix of any element φ in the subgroup spanned by Ψ_1, Ψ_2 and the $\varphi_{\lambda'}$ in $(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2})$ and $(\varphi^* \frac{\partial}{\partial z_1}, \varphi^* \frac{\partial}{\partial z_2})$ is the identity.

We get immediately $a = 0$. Hence, $\tilde{\Psi}$ is the product of the automorphisms $z_1 \mapsto \nu z_1 + \theta$ and $z_2 \mapsto \nu z_2 + b$, with $\nu \neq 1$. Hence, there exists on $\tilde{M} = \mathbb{C}^2$ an isolated fixed point $\tilde{x}_0 = (z_1^0, z_2^0)$. The coordinates $u_1 = z_1 - z_1^0$ and $u_2 = z_2 - z_2^0$ identify a neighborhood

$\tilde{U}$ of $\tilde{x}_0$, invariant by $\tilde{\Psi}$, with $D(0,1)\times D(0,1)$. It conjugates the action of $\tilde{\Psi}$ and the action of the automorphism Ψ_1 of $D(0,1)\times D(0,1)$ defined by:

$$\Psi_1(u_1,u_2)=(\nu u_1,\nu u_2)$$

For a suitably small $\tilde{U}$, $\tilde{p}$ restricts as a biholomorphism between $\tilde{U}$ and an open neighborhood $\hat{U}$ of $\hat{x}_0=\tilde{p}(\tilde{x}_0)$. Then $q(\hat{U})$ is an open neighborhood of some point $x_0\in M$, which is isomorphic to the analytic space obtained as the quotient of $D(0,1)\times D(0,1)$ by the subgroup spanned by Ψ_1 as above. It is clear since $\nu\neq 1$, that this space is not a smooth complex manifold. Hence $\mu=\nu$ implies $k=1$ that is Γ is the trivial group.

□

Definition 5.3. A secondary Kodaira surface is an elliptic surface $M\xrightarrow{\pi}N$ which admits a primary Kodaira surface $\hat{M}\xrightarrow{\hat{\pi}}\hat{N}$ as a finite unramified cover.

Hence, the classification of (non-trivial) quotients of meromorphic affine primary Kodaira surfaces is reduced to the classification of meromorphic affine secondary Kodaira surfaces. The two following lemmas will be useful to simplify the invariance equations corrsponding to $\tilde{\Psi}^\star\tilde{\nabla}=\tilde{\nabla}$:

Lemma 5.3. *Let $\nu\in\mathbb{C}^\times\setminus\{1\}$ and $\theta\in\mathbb{C}$ such that $z_1\mapsto\nu z_1+\theta$ is an automorphism of the elliptic curve $\mathbb{C}/\Lambda'$, $r:\mathbb{C}/\Lambda'\longrightarrow\mathbb{P}^1$ the quotient by the subgroup spanned by this automorphism, and $\wp_0=Z_1\circ r$ where Z_1 is any primitive element in the field of meromorphic functions on $\mathbb{P}^1$. Then, for any integer $k\geq 0$ the set of Λ'-elliptic functions satisfying:*

$$f(\nu z_1+\theta)=\frac{1}{\nu^k}f(z_1) \tag{51}$$

is $\mathbb{C}(\wp_0)\,\wp_0^{(k)}$.

Proof. Since $\mathbb{C}(\mathbb{P}^1)=\mathbb{C}(Z_1)$, the subfield $\mathbb{C}(\wp_0)$ coïncides with the set of functions of the form $f(Z_1)\circ r$ where $P\in\mathbb{C}(X)$. In one other hand, by definition of r, the last set is the subset of Λ'-elliptic functions invariant through the automorphism from the statement. In particular, taking the derivatives of the invariance equation for $\wp_0$ gives :

$$\wp_0^{(k)}(\nu z_1+\theta)=\frac{1}{\nu^k}\wp_0^{(k)}(z_1) \tag{52}$$

that is $\wp_0^{(k)}$ is a Λ'-elliptic function satisfying (51). Now let f be any function as in the statement. Then $g=\frac{f}{\wp_0^{(k)}}$ is an Λ-elliptic function which is invariant through the automorphism from the statement. Hence, $g\in\mathbb{C}(\wp_0)$ and finally $f\in\mathbb{C}(\wp_0)\,\wp_0^{(k)}$.

□

Lemma 5.4. *Let ν be a non trivial root of the unity and $\theta\in\mathbb{C}$ such that $\delta:z_1\mapsto\nu z_1+\theta$ is a finite automorphism of $\mathbb{C}/\Lambda'$. Let $\mathcal{Z}$ as in Lemma 5.1, and $k\geq 1$. Suppose that there are elements $(h_{i,1},h_{i,0})_{i=0,\dots,k}$ of $\mathcal{E}_0$ such that:*

$$\sum_{i=0}^{k}h_{i,1}(z_1)\mathcal{Z}^i(\delta\cdot z_1)+h_{i,0}(z_1)\mathcal{Z}^i(z_1)=0 \tag{53}$$

Then $h_{0,0}=0$ and $h_{i,0}=-\nu^i h_{i,1}$ for $i=1,\dots,k$. Moreover, if there exists $i\in\{1,\dots,k\}$ with $h_{i,0}\neq 0$ then $\theta=0$.

Proof. We proceed by induction on $k\in\mathbb{N}_{\geq 1}$.

Suppose the relation (53) holds for $k=1$. If $h_{1,1}=0$, then clearly $h_{1,1}=h_{1,0}=h_{0,0}=0$ since $\mathcal{Z}$ is not an element of $\mathcal{E}_0$. Thus, we can assume, without loss of generality, that $h_{1,1}=1$. Then recalling $\mathcal{Z}(z_1)=\alpha\zeta(z_1)+\beta z_1$ with $\zeta'(z_1)=\wp(z_1)$ and taking the derivative of (53) gives :

$$\nu(\alpha\wp+\beta)(\nu z_1+\theta)+\frac{\partial h_{1,0}}{\partial z_1}\mathcal{Z}(z_1)+(\alpha\wp+\beta)h_{1,0}+\frac{\partial h_{0,0}}{\partial z_1}=0$$

As before this implies $h_{1,0}=c\in\mathbb{C}$. But then, (53) becomes:

$$\mathcal{Z}(\nu z_1+\theta)+c\mathcal{Z}(z_1)=h_{0,0}(z_1)$$

If $\theta\notin\Lambda'$, then the left handside has residues summing to a nonzero value. This is impossible by the second Liouville's theorem. Hence $\theta\notin\Lambda'$ implies $h_{1,1}=h_{1,0}=h_{0,0}=0$. If $\theta\in\Lambda'$, then $\mathcal{Z}(\delta z_1)=\nu\mathcal{Z}(z_1)$. Again, since $\mathcal{Z}$ is not an element of $\mathcal{E}_0$, we get that (53) implies $h_{0,0}=0$ and $h_{1,0}=-\nu h_{1,1}$.

Suppose the lemma is true for $k\in\mathbb{N}_{\geq 1}$, and suppose (53) holds for $k+1$ in place of k. Applying the operator $\delta_{\lambda'}$ (see (36)) to this relation gives:

$$\forall\overline{\lambda'}\in\Lambda',\ \overline{\lambda'}^{k+1}(\nu^{k+1}h_{k+1,1}+h_{k+1,0})+\overline{\lambda'}^{k}f_k+\ldots+\overline{\lambda'}f_1+f_0=0$$

where $f_0,\ldots,f_k$ are $\mathbb{C}$-linear combinations of $h_{j,0}$, $h_{j,1}$ and $\mathcal{Z}^i(z_1)$ with $i,j\leq k$. In particular we get $\nu^{k+1}h_{k+1,1}+h_{k+1,0}=0$ and:

$$\sum_{i=0}^{k}h_{i,1}(z_1)\mathcal{Z}^i(\delta\cdot z_1)+h_{i,0}(z_1)\mathcal{Z}^i(z_1)=0 \tag{54}$$

By induction hypothesis, we get that $h_{i,0}=\nu^i h_{i,1}$ for $i=1,\ldots,k$.

Finally, if there exists $i\in\{1,\ldots,k\}$ such that $h_{i,0}\neq 0$, then the induction hypothesis implies $\theta\in\Lambda'$. Also, by (54), if $h_{k+1,0}\neq 0$ and $\theta\notin\Lambda'$, then by the second Livouille's theorem $h_{k+1,1}\mathcal{Z}^{k+1}(\nu z_1+\theta)+h_{k+1,0}\mathcal{Z}^{k+1}(z_1)$ have non trivial poles at the classes of 0 and θ in $\mathbb{C}/\Lambda'$. □

Theorem 5.3. *Let $M\xrightarrow{\pi}N$ be a minimal meromorphic affine elliptic surface with $a(M)=1$, and suppose that the elliptic surface $\hat{M}\xrightarrow{\hat{\pi}}\hat{N}$ from (27) is a primary Kodaira surface. Denote by $p:\tilde{M}\longrightarrow\hat{M}$ the universal covering of $\hat{M}$ and (z_1,z_2) coordinates as in Proposition 2.2.2. Then:*

1. *M is the quotient of $\hat{M}$ by a cyclic group Γ, spanned by an element Ψ, which lifts to an automorphism $\tilde{\Psi}$ of $\tilde{M}$ of the form:*

$$\tilde{\Psi}(z_1,z_2)=(\nu z_1+\theta,\mu z_2+bz_1+c) \tag{55}$$

   where $z_1\mapsto\nu z_1+\theta$ is an automorphism of the elliptic curve $\hat{N}=\mathbb{C}/\Lambda'$, μ is a power of ν, and $b,c\in\mathbb{C}$. Moreover $\nu=1$ if and only if $\hat{M}=M$.
2. *Suppose that $\nu\neq 1$ (i.e. M is a secondary Kodaira surface). Let $\wp_0$ defined as in Lemma 5.3. Then the map $\nabla\mapsto\tilde{\nabla}=\tilde{p}^\star q^\star\nabla$ is a bijection between the set of meromorphic affine connections on M and the set of meromorphic affine connections on $\tilde{M}$ with one of the following matrix forms in $(\frac{\partial}{\partial z_1},\frac{\partial}{\partial z_2})$:*
   a) <u>if $\mu=\nu^2=1$ and $\theta=0$:</u>

$$dz_1\otimes\left(\begin{array}{c|c}\gamma_{11} & 0\\ \hline (\gamma_{11}-\gamma_{22}+\delta_{21})\mathcal{Z} & \\ +\ \frac{a}{1-\nu}\delta_{21}+\gamma_{21} & \gamma_{22}\end{array}\right)+dz_2\otimes\left(\begin{array}{c|c}0 & 0\\ \hline \delta_{21} & 0\end{array}\right)$$

   with $\mathcal{Z}$ as in Lemma 5.1, $\gamma_{ii},\delta_{21}\in\mathbb{C}(\wp_0)\,\wp_0'$ and $\gamma_{21}\in\mathbb{C}(\wp_0)\,\wp_0''$.

b) if $\mu \neq 1$ or $\nu^2 \neq 1$ or $\theta \neq 0$:

$$dz_1 \otimes \left(\begin{array}{c|c} \gamma_{11} & 0 \\ \hline \gamma_{21} & \gamma_{11} \end{array}\right)$$

with $\gamma_{11} \in \mathbb{C}\{\wp_0\}\,\wp_0'$ and $\gamma_{21} \in \mathbb{C}\{\wp_0\}\,\wp_0''$,

In particular, there always exists meromorphic affine flat connections on any secondary Kodaira surfaces.

Proof. (1) It is a consequence of Lemma 5.2 and the definition of $\hat{M}$ in Theorem 3.2. Indeed, $\nu = 1$ implies that $\overline{q} : \hat{N} \longrightarrow N$ is an unramified covering, that is M have no multiple singular fiber. By definition this implies $\hat{M} = M$.

(2) Let ∇ be a meromorphic affine connection on a secondary Kodaira surface $M \xrightarrow{\pi} N$, and $\tilde{\nabla}$ the corresponding pullback to the universal covering $\tilde{M}$ of the primary Kodaira surface $\hat{M}$. Recall that the matrix of $\tilde{\nabla}$ in $(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2})$ was described in Proposition 5.1.1.

Suppose that $\tilde{\nabla}$ has matrix form (46) in Proposition 5.1.1 and let $\tilde{\Psi}$ be as in 1, so that $\nu \neq 1$. Then by Lemma 5.4, the equations corresponding to $\tilde{\Psi}^\star\tilde{\nabla} = \tilde{\nabla}$ imply:

$$\left\{\begin{array}{rcl} \nu\mathcal{Z}^2(\nu z_1+\theta)g_{12}(\nu z_1+\theta) & = & \mathcal{Z}^2(z_1)g_{12}(z_1) \\ \mathcal{Z}(\nu z_1+\theta)g_{12}(\nu z_1+\theta) & = & \mathcal{Z}(z_1)g_{12}(z_1) \\ \frac{\mu}{\nu}g_{12}(\nu z_1+\theta) & = & g_{12}(z_1) \end{array}\right.$$

Since $g_{12} \neq 0$, using again Lemma 5.4 we get:

$$\frac{\nu^3}{\mu} = 1 \quad \text{and} \quad \frac{\nu^2}{\mu} = 1$$

so that $\nu = 1$, that is M is a primary Kodaira surface by 1. This contradicts our assumption on M.

Hence $\tilde{\nabla}$ has matrix form (47) in Proposition 5.1.1. In this case, the equations corresponding to $\tilde{\Psi}^\star\tilde{\nabla} = \tilde{\nabla}$ are:

$$\left\{\begin{array}{lcr} \mu g_{ii}(\nu z_1+\theta) & = & g_{ii}(z_1) \\ f_{12}(\nu z_1+\theta) & = & f_{12}(z_1) \\ -\nu(g_{22}-g_{11})(z_1)(\mathcal{Z})(\nu z_1+\theta) & = & (g_{22}-g_{11})(z_1)\mathcal{Z}(z_1) \\ +\mu\nu\delta_{21}(\nu z_1+\theta) & & +\delta_{21}(z_1) \\ -(g_{ii}+(-1)^i\nu f_{12})(z_1)\mathcal{Z}(\nu z_1+\theta) & = & -(g_{ii}+(-1)^i f_{12})(z_1)\mathcal{Z}(z_1) \\ +\nu\gamma_{ii}(\nu z_1+\theta)+\frac{a}{\mu}g_{ii}(z_1) & & +\gamma_{ii}(z_1) \\ \nu(g_{11}-g_{22}+\nu f_{12})(z_1)\mathcal{Z}^2(\nu z_1+\theta) & = & (g_{11}-g_{22}+f_{12})(z_1)\mathcal{Z}^2(z_1) \\ +-a\mathcal{Z}(g_{11}-g_{22})(\nu z_1+\theta) & & \\ +\nu^2(\gamma_{11}-\gamma_{22}+\delta_{21})(\nu z_1+\theta)\mathcal{Z}(\nu z_1+\theta) & & +(\gamma_{11}-\gamma_{22}+\delta_{21})(z_1)\mathcal{Z}(z_1) \\ +\nu^2\gamma_{21}(\nu z_1+\theta))+a\delta_{21}(\nu\zeta+\theta) & & +\gamma_{21}(z_1) \end{array}\right. \tag{56}$$

Using Lemma 5.4 we get the following restrictions. The third line of (56) implies:

$$\delta_{21}(\nu z_1+\theta) = \frac{1}{\mu\nu}\delta_{21}(z_1) \tag{57}$$

The fourth line implies:

$$(\nu-1)g_{ii}=(-1)^i(1-\nu^2)f_{12}$$

In the same way, the fifth line implies:

$$(\nu^2-1)(g_{11}-g_{22})=(1-\nu^3)f_{12}$$

Since $\nu\neq 1$, we get:

$$g_{ii}=f_{12}=0 \tag{58}$$

The fourth line also implies:

$$\gamma_{ii}(\nu z_1+\theta)=\frac{1}{\nu}\gamma_{ii}(z_1)-\frac{a}{\mu\nu}g_{ii}(z_1)=\frac{1}{\nu}\gamma_{ii}(z_1) \tag{59}$$

Finally the fifth line of (56) implies:

$$\gamma_{21}(\nu z_1+\theta)=\frac{1}{\nu^2}\gamma_{21}(z_1)-\frac{a}{\mu\nu^2}\delta_{21}(z_1) \tag{60}$$

We distinguish between two cases:

- If $\mu\neq\nu$, then $\gamma=-\frac{a}{\nu^2-\mu\nu}\delta_{21}(z_1)$ is a solution of (60). Hence, in view of Lemma 5.3, (60) is equivalent to:

$$\gamma_{21}\in-\frac{a}{\nu^2-\mu\nu}\delta_{21}(z_1)+\mathbb{C}(\wp_0)\wp_0'' \tag{61}$$

- If $\mu=\nu$, then either $\delta_{21}=0$ and Lemma 5.3 shows that (60) is equivalent to $\gamma_{21}\in\mathbb{C}(\wp_0)\wp_0'$, or:

$$\frac{\gamma_{21}}{\delta_{21}}(\nu z_1+\theta)=(\nu-\frac{1}{\nu^2})\frac{\gamma_{21}}{\delta_{21}}(z_1)-\frac{a}{\nu}$$

  In the second subcase, since $z_1\mapsto\nu z_1+\theta$ has finite order k with $\nu^k=1$, we get $(\nu-\frac{1}{v^2})^{k-1}+\ldots+(\nu-\frac{1}{\nu^2})+1=0$. Since moreover $k=2$ or 3, we get a contradiction. Hence if $\mu=\nu$ then:

$$\delta_{21}=0\quad\text{and}\quad\gamma_{21}\in\mathbb{C}(\wp_0)\wp_0' \tag{62}$$

Moreover, the fifth line and Lemma 5.4 imply:

$$\nu(\gamma_{11}-\gamma_{22}+\frac{1}{\mu}\delta_{21})(z_1)=\frac{1}{\nu}(\gamma_{11}-\gamma_{22}+\frac{1}{\mu}\delta_{21})(z_1)$$

and $\theta\in\Lambda'$ whenever $\delta_{21}\neq 0$ or $\gamma_{11}\neq\gamma_{22}$. Comparing with (57) and (59), the above equality implies:

$$(\nu^2=1\text{ and }\mu=1\text{ and }\theta=0)\quad\text{or}\quad\gamma_{11}-\gamma_{22}=\delta_{21}=0 \tag{63}$$

We have proved that $\tilde{\nabla}$ has matrix form as described in the statement.

Reciprocally, suppose that $\tilde{\Psi}$ is the lift of the generator of Γ as in 1. Suppose also tat $\tilde{\nabla}$ is a meromorphic affine connection with matrix form as in the statement. Then (56) is clearly satisfied, i.e. $\tilde{\nabla}$ is $\tilde{\Psi}$-invariant, and $M=\Gamma\backslash\hat{M}$ is a secondary Kodaira surface. This achieves the proof since the matrix form in case b) also appears in case a), and the torsion and curvature both vanish when $\gamma_{11}=\gamma_{22}=0$ in this case.

□

5.3. **Two tori.** In view of Theorem 5.1, to achieve the case $\hat{N} = \mathbb{C}/\Lambda'$ (see Theorem 3.2), it remains to classify meromorphic affine two tori with $a(\hat{M}) = 1$ and their quotients.

Let $\hat{M} \xrightarrow{\hat{\pi}} \mathbb{C}/\Lambda'$ be a two torus with $a(\hat{M}) = 1$, and $\mathcal{E}_0$ the subfield of Λ'-elliptic meromorphic functions. Let $\tilde{p} : \tilde{M} \longrightarrow \hat{M}$ be the universal covering, with coordinates (z_1, z_2) as in Proposition 2.2.2. Then the automorphisms $(\varphi_{\lambda'})_{\lambda' \in \Lambda'}$ are translations in these coordinates.

Hence, for any meromorphic affine connection $\tilde{\nabla}$ on $\tilde{M}$, with matrix:

$$dz_1 \otimes (f_{ij})_{i,j=1,2} + dz_2 \otimes (g_{ij})_{i,j=1,2}$$

the condition $\varphi_{\lambda'}^{\star} \tilde{\nabla} = \tilde{\nabla}$ is equivalent to $f_{ij}, g_{ij} \in \mathcal{E}_0$.

We immediately get:

Theorem 5.4. *Let $\hat{M}$ be a two torus with $a(\hat{M}) = 1$ and $\tilde{p} : \tilde{M} \longrightarrow \hat{M}$ its universal covering with global coordinates (z_1, z_2) as in Proposition 2.2.2. Then the map from the set of meromorphic affine connections on $\hat{M}$ to $\mathcal{E}_0^8$ obtained by mapping $\hat{\nabla}$ to the coefficients (f_{ij}, g_{ij}) of the matrix of $\tilde{\nabla} = \tilde{p}^{\star} \hat{\nabla}$ in $(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2})$ is a bijection.*

5.4. **Quotients of meromorphic affine two tori.**

Theorem 5.5. *Let (M, ∇) be a minimal meromorphic affine complex compact surface of algebraic dimension one, and suppose that the finite ramified covering $(\hat{M}, \hat{\nabla})$ from Equation 27 is a meromorphic affine two torus (of algebraic dimension one). Then either M is a two torus, or (M, ∇) is a meromorphic affine secondary Kodaira surface, and such pairs were classified in Theorem 5.3.*

Proof. If M is not a two torus, then the finite covering $q : \hat{M} \longrightarrow M$ and therefore $\mathcal{K}_M$ is not trivial. The proof of point 1. in Lemma 5.2 only relies on the fact that $\mathcal{K}_{\hat{M}}$ is trivial, which is still satisfied in our situation. As a consequence, we get that the canonical global section $(dz_1 \wedge dz_2)^{\otimes k}$ of $\mathcal{K}_{\tilde{M}}^{\otimes k}$, where k is the order of ν and $\tilde{M}$ is the universal cover of $\hat{M}$, is invariant by the lift $\tilde{\Psi}$ of any automorphism of $q : \hat{M} \longrightarrow M$. Hence $\mathcal{K}_M^{\otimes k}$ is trivial. By Theorem 38 of [13], this implies that a finite unramified cover of M is either a two torus or a primary Kodaira surface. In the second case, since $\mathcal{K}_M$ is not trivial, we immediately get that M is a secondary Kodaira surface. In the first case, we have $kod(M) = 0$ and $a(M) = 1$. By the Enriques-Kodaira classification (see [2], Table 10 p.189), M is either a secondary Kodaira surface, a two torus, or a K3 surface. Again, since $\mathcal{K}_M$ is not trivial, the only possiblity is a secondary Kodaira surface. □

6. Principal elliptic surface over an hyperelliptic compact Riemann surface and quotients

We finally classify meromorphic affine principal ellitpic surfaces over an hyperelliptic compact Riemann surface and their quotients.

6.1. **Non-existence on principal elliptic surfaces with $b_1(\hat{M})$ even.** Let $\hat{M} \longrightarrow \hat{N}$ be a principal elliptic surface over a Riemann surface $\hat{N}$ of genus $g \geq 2$, with $a(\hat{M}) = 1$. Denote by $p : \mathbb{H} \times \mathbb{C} \longrightarrow \hat{M}$ its universal cover. From Proposition 2.2.2, p is the quotient by the automorphisms Ψ_1, Ψ_2 corresponding to a lattice associated with the fibers of $\hat{M}$, and by the automorphisms φ_γ ($\gamma \in \pi_1(\hat{N}, y)$) lifting the desk transformations of the universal cover $\overline{p} : \mathbb{H} \longrightarrow \hat{N}$. The later are of the form:

$$\varphi_\gamma(z_1, z_2) = (\frac{a_\gamma z_1 + b_\gamma}{c_\gamma z_1 + d_\gamma}, z_2 + f_\gamma(z_1))$$

in suitable global coordinates z_1 on $\mathbb{H}$ and z_2 on $\mathbb{C}$, with f_γ a holomorphic function on $\mathbb{H}$.

Proposition 6.0.1. *Let $\hat{M} \longrightarrow \hat{N}$ be a principal elliptic surface as above. If $\hat{M}$ admits a meromorphic affine connection $\hat{\nabla}$, then $b_1(\hat{M})$ is odd.*

Proof. By [11], Theorem 4.17 applied to the basic member

$$B := \mathbb{H} \times \mathbb{C}/\Lambda$$

in the same family as $\hat{M}$, we get that either $b_1(\hat{M})$ is odd or $\hat{M}$ is a deformation of B. Suppose that $b_1(\hat{M})$ is even. In particular, there exists a diffeomorphism between the underlying smooth manifolds:

$$\begin{array}{ccc} \hat{M}^{\mathbb{R}} & \xrightarrow{\varphi} & B^{\mathbb{R}} \\ \downarrow & & \downarrow \\ \hat{N}^{\mathbb{R}} & = & \hat{N}^{\mathbb{R}} \end{array} \tag{64}$$

In particular φ induces an isomorphism of $\pi_1(\hat{N}, y)$-manifolds between the universal covers of $\hat{M}$ and B. Pick $U \subset \hat{N}$ such that there exists a section of p over $\hat{\pi}^{-1}(U)$, and thus a section of the universal cover of B over the corresponding open subset $U \times V$ in B. Then, in the induced coordinates (z_1, z_2) on $\hat{\pi}^{-1}(U)$ and (z_1, u_2) on $U \times V$, φ has the expression:

$$\varphi(z_1, z_2) = (z_1, z_2 + f_U(z_2)) \tag{65}$$

for some $\mathcal{C}^\infty$ function f_U on U. We fix U and omit the subscript U in the sequel.

Now, on B, we can consider a meromorphic affine connection ∇_0 induced by the canonical holomorphic affine connection on $\mathbb{C}/\Lambda$ and any meromorphic affine connection on $\hat{N}$, and denote by S_0 the pole. Then ∇_0 extends canonically as a connection on the sheaf $T^{\mathbb{C}}(B \setminus S_0) = TB|_{B\setminus S_0} \otimes \mathcal{C}^\infty_{M\setminus S,\mathbb{C}}$, where $\mathcal{C}^\infty_{B\setminus S_0,\mathbb{C}}$ stands for the sheaf of complex-valued differentiable functions. We can thus consider the smooth connection

$$\overline{\nabla}_0 := \varphi^\star \nabla_0$$

on the sheaf $T^{\mathbb{C}}(\hat{M} \setminus S) := T\hat{M}|_{\hat{M}\setminus S} \otimes \mathcal{C}^\infty_{\hat{M}\setminus S,\mathbb{C}}$ where $S = \varphi^{-1}(S_0)$. In the basis $\frac{\partial}{\partial z_1}, \frac{\partial}{\partial \overline{z}_1}, \frac{\partial}{\partial z_2}, \frac{\partial}{\partial \overline{z}_2}$ induced by coordinates as above, the matrix of the pullback $\tilde{\nabla}_0 := p^\star \overline{\nabla}_0$ is:

$$dz_1 \otimes \begin{pmatrix} a(z_1) & 0 \\ df \cdot a(z_1) + \frac{\partial}{\partial z_1} df(z_1) & 0 \end{pmatrix} + d\overline{z}_1 \otimes \begin{pmatrix} 0 & 0 \\ \frac{\partial}{\partial \overline{z}_1} df(z_1) & 0 \end{pmatrix} \tag{66}$$

where a is a meromorphic function on $\mathbb{H}$, identified with a $\mathcal{C}^\infty$ function $a : \mathbb{H} \longrightarrow \mathfrak{gl}_2(\mathbb{R})$ and:

$$df(z_1) = \frac{1}{2} \begin{pmatrix} \frac{\partial f}{\partial z_1} & \frac{\partial \overline{f}}{\partial \overline{z}_1} \\ \frac{\partial f}{\partial z_1} & \frac{\partial f}{\partial \overline{z}_1} \end{pmatrix}$$

Suppose that $\hat{\nabla}$ is a meromorphic affine connection on $\hat{M}$, with poles at D, and let $S' = S + D$. Denote by $\hat{\nabla}^{\mathbb{C}}$ (resp. $\tilde{\nabla}^{\mathbb{C}}$) the unique extension of $\hat{\nabla}$ to a smooth connection on the sheaf $T^{\mathbb{C}}(M \setminus S)$ (resp. its pullback to the universal cover $\mathbb{H} \times \mathbb{C}$). Then the matrix A of $\tilde{\nabla}^{\mathbb{C}}$ in $\frac{\partial}{\partial z_1}, \frac{\partial}{\partial \overline{z}_1}, \frac{\partial}{\partial z_2}, \frac{\partial}{\partial \overline{z}_2}$ satisfies:

$$\frac{\partial}{\partial \overline{z}_1} \lrcorner A = \frac{\partial}{\partial \overline{z}_2} \lrcorner A = 0 \tag{67}$$

In one other hand, the difference $\tilde{\nabla}^{\mathbb{C}} - \tilde{\nabla}_0 = p^\star(\hat{\nabla}^{\mathbb{C}} - \overline{\nabla}_0)$ is a Γ-equivariant section of the $\mathcal{C}^\infty_{\hat{M}\setminus S',\mathbb{C}}$-sheaf $T^{\mathbb{C}}(\hat{M} \setminus S')^* \otimes End(T^{\mathbb{C}}\hat{M} \setminus S)$. The matrix of this endormorphism in the above

basis as:

$$\begin{array}{ccc} dz_1 \otimes (f_{i,j}^{(1,0)})_{i,j=1,2} & + & dz_2 \otimes (g_{i,j}^{(1,0)})_{i,j=1,2} \\ d\overline{z}_1 \otimes (f_{i,j}^{(0,1)})_{i,j=1,2} & + & d\overline{z}_2 \otimes (g_{i,j}^{(0,1)})_{i,j=1,2}) \end{array} \tag{68}$$

where $f_{i,j}^{(p,q)}$ and $g_{i,j}^{(p,q)}$ are matrices with global sections of $\mathcal{C}^{\infty}_{\hat{M}\setminus S',\mathbb{C}}$ as entries. Using (66) and (67), we get:

$$f_{21}^{(0,1)} = -\frac{\partial}{\partial \overline{z}_1} df(z_1) \tag{69}$$

But f is a Γ-invariant function on $\mathbb{H}$, so that, for any $\gamma \in \Gamma$:

$$df(\gamma z_1) = \begin{pmatrix} \frac{1}{(c_\gamma z_1 + d_\gamma)^2} & 0 \\ 0 & \frac{1}{\overline{(c_\gamma z_1 + d_\gamma)}^2} \end{pmatrix} \cdot df(z_1)$$

In particular:

$$\frac{\partial}{\partial \overline{z}_1} df(\gamma z_1) = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{2c_\gamma}{(c_\gamma z_1 + d_\gamma)^3} \end{pmatrix} \cdot df(z_1) + \begin{pmatrix} \frac{1}{(c_\gamma z_1 + d_\gamma)^2} & 0 \\ 0 & \frac{1}{\overline{(c_\gamma z_1 + d_\gamma)}^2} \end{pmatrix} \cdot \frac{\partial}{\partial \overline{z}_1} df(z_1)$$

Comparing with (69) and recalling the Γ-invariance of (68), we get that for any $\gamma \in \Gamma$:

$$-\frac{2}{(c_\gamma z_1 + d_\gamma)^3} \frac{\partial f}{\partial \overline{z}_1} = 0$$

Hence f is a holomorphic function on $\mathbb{H}$, which precisely means that φ is a biholomorphism. This contradicts the hypothesis $a(\hat{M}) = 1$.

Hence $\hat{M}$ does not admit any meromorphic affine connection. □

6.2. **Meromorphic differential operators on a root of the canonical bundle and projective structures.** We will see in the next subsection that a family of meromorphic affine connections on a principal elliptic bundle $\hat{M} \xrightarrow{\hat{\pi}} \hat{N}$, with $g(\hat{N}) \geq 2$ and $b_1(\hat{N})$ odd, can be parametrized by meromorphic projective structures on $\hat{N}$. Therefore, we introduce in this subsection notations and basic facts that will be used.

Let $\hat{N}$ be a complex manifold, and $\mathcal{L}$ be a line bundle on $\hat{N}$, identified with its sheaf of sections. We can consider, for any $r \geq 0$, its sheaf of r-jets $J^r\mathcal{L}$ (we refer to [1] and [14] Definition 2.21 for a definition). These sheaves are locally free $\mathcal{O}_{\hat{N}}$-modules and fit in exact sequences for $r \geq 1$:

$$0 \longrightarrow (\Omega^1_{\hat{N}})^{\otimes r} \otimes \mathcal{L} \longrightarrow J^r\mathcal{L} \xrightarrow{\pi^{r-1}} J^{r-1}\mathcal{L} \longrightarrow 0 \tag{70}$$

where π^{r-1} stands for the truncation map. These maps generalize by compositions to maps:

$$\pi_r^s \ : \ J^r\mathcal{L} \longrightarrow J^s\mathcal{L} \tag{71}$$

for $s \leq r$. Each $J^r\mathcal{L}$ contains the subsheaf spanned by the equivalence classes of local sections of $\mathcal{L}$ with the same r-jets, and so there are morphisms of $\underline{\mathbb{C}}_{\hat{N}}$-sheaves:

$$j^r \ : \ \mathcal{L} \longrightarrow J^r\mathcal{L} \tag{72}$$

Given a divisor C of $\hat{N}$, A *meromorphic differential operator of order two and pole at C* on $\mathcal{L}$ is a morphism of $\mathcal{O}_{\hat{N}}$-modules $\delta : J^1\mathcal{L}(*C) \longrightarrow J^2\mathcal{L}(*C)$ such that $\pi_0^2 \circ \delta = \pi_1^0$.

Suppose now that $\hat{N}$ is a compact Riemann surface, and Γ' is a subgroup of $\mathbb{P}SL_2(\mathbb{R})$ such that $\hat{N} = \Gamma'\backslash\mathbb{H}$ and $\overline{p} : \mathbb{H} \longrightarrow \Gamma'\backslash\mathbb{H}$ is the universal cover of $\hat{N}$. Suppose that there exists a Γ'-linearization $\alpha = (\alpha_\gamma)_{\gamma\in\Gamma'}$ of $\mathcal{O}_{\mathbb{H}}$ such that $\mathcal{L} = (\overline{p}_*\mathcal{O}_{\mathbb{H}})^\alpha$ (see Definition 2.2). Then the

linearization α induces isomorphisms $j^2(\alpha_\gamma) : j^2(\mathcal{O}_{\mathbb{H}}) \longrightarrow \gamma^* j^2(\mathcal{O}_{\mathbb{H}})$, and the action of γ gives a natural linearization by differentials on $\Omega^k_{\mathbb{H}}$. We obtain isomorphisms:

$$J^2\alpha_\gamma : J^2\mathcal{O}_{\mathbb{H}} \longrightarrow \gamma^* J^2\mathcal{O}_{\mathbb{H}} \tag{73}$$

defininig a natural linearization $J^2(\alpha)$ on $J^2\mathcal{O}_{\mathbb{H}}$. Since $\overline{p}$ is a unramified covering, $J^2\mathcal{L}$ coïncides with the sheaf $(\overline{p}_* J^2\mathcal{O}_{\mathbb{H}})^{J^2\alpha}$.

Now, fix Γ' as before, identified with a subgroup of $SL_2(\mathbb{R})$, and let $\alpha = (\alpha_\gamma)_{\gamma\in\Gamma'}$ be defined by:

$$\begin{array}{rccc} \alpha_\gamma & : & \mathcal{O}_{\mathbb{H}} & \longrightarrow & \gamma^*\mathcal{O}_{\mathbb{H}} \\ & & f & \mapsto & (c_\gamma z_1 + d_\gamma) f \circ \gamma^{-1} \end{array} \tag{74}$$

and define $\mathcal{L}$ to be the corresponding line bundle on $\hat{N}$ through the equivalence between linearized modules and modules on the quotient:

$$\mathcal{L} = (\overline{p}_* \mathcal{O}_{\mathbb{H}})^\alpha \tag{75}$$

Consider the natural trivialization

$$\Psi : J^2\mathcal{O}_{\mathbb{H}} \xrightarrow{\sim} \mathcal{O}_{\mathbb{H}}^{\oplus 3}$$

given by the global basis $(1 \otimes dz_1^{\otimes 2}, j^1(1 \otimes dz_1), j^2(1))$. Then for any $\gamma \in \Gamma$, $J^2\alpha_\gamma$ is the isomorphism given by the commutative diagram:

$$\begin{array}{ccc} J^2\mathcal{O}_{\mathbb{H}} & \xrightarrow[\Psi]{\sim} & \mathcal{O}_{\mathbb{H}}^{\oplus 3} \\ {\scriptstyle J^2\alpha_\gamma}\downarrow & & \downarrow{\scriptstyle \zeta \mapsto (\zeta\circ\gamma^{-1})\, {}^t A_2(z_1)} \\ \gamma^* J^2\mathcal{O}_{\mathbb{H}} & \xrightarrow[\sim]{\gamma^*\Psi} & \gamma^*\mathcal{O}_{\mathbb{H}}^{\oplus 3} \end{array} \tag{76}$$

where:

$$A_2(z_1) := \begin{pmatrix} \frac{1}{(c_\gamma z_1 + d_\gamma)^5} & -\frac{3c_\gamma}{(c_\gamma z_1 + d_\gamma)^4} & \frac{2c_\gamma^2}{(c_\gamma z_1 + d_\gamma)^3} \\ 0 & \frac{1}{(c_\gamma z_1 + d_\gamma)^3} & -\frac{c_\gamma}{(c_\gamma z_1 + d_\gamma)^2} \\ 0 & 0 & \frac{1}{c_\gamma z_1 + d_\gamma} \end{pmatrix}$$

Similarly, $J^1\mathcal{L}$ can be described by the linearization corresponding to the lower right minor $A_1(z_1)$ of $A_2(z_1)$ as above.

Then the equivalence between linearized sheaves and sheaves on the base gives a bijection between order-two meromorphic differential operators $\delta : J^1\mathcal{L}(*\overline{C}) \longrightarrow J^2\mathcal{L}(*\overline{C})$ and meromorphic differential operators $\tilde{\delta} : J^1\mathcal{O}_{\mathbb{H}}(*\tilde{C}) \longrightarrow J^2\mathcal{O}_{\mathbb{H}}(*\tilde{C})$ which are Γ'-equivariant as morphisms of linearized sheaves. We will denote by $\mathcal{P}_{\mathcal{L}}$ the set of meromorphic differential operators of order two on $\mathcal{L}$, endowed with the natural structure of an $\mathcal{O}_{\hat{N}}(*C)$-vector space (here $\mathcal{O}_{\hat{N}}(*C)$ stands for the field of global meromorphic functions which are holomorphic outside from C). Thus, any $\delta \in \mathcal{P}_{\mathcal{L}}$ can be identified with the matrix of the corresponding meromorphic differential operator $\tilde{\delta} = \overline{p}^*\delta$ on $\mathcal{O}_{\mathbb{H}}$, in the basis $(1 \otimes dz_1, j^1(1))$ of $J^1\mathcal{O}_{\mathbb{H}}$ and $(1 \otimes dz_1^{\otimes 2}, j^1(1 \otimes dz_1), j^2(1)$ of $J^2\mathcal{O}_{\mathbb{H}}$:

$$\Delta(z_1) = \begin{pmatrix} b(z_1) & c(z_1) \\ \nu(z_1) & a(z_1) \\ \lambda(z_1) & \mu(z_1) \end{pmatrix} \tag{77}$$

satisfying:

$$A_2(z_1)\Delta(\gamma z_1)A_1^{-1}(z_1) = \Delta(z_1) \tag{78}$$

The meromorphic affine connections we will study will be first shown to be in bijection with the following operators:

Definition 6.1. Let $\mathcal{L}$ as in (75) and $\mathcal{P}_{\mathcal{L}}$ as above. Then $\mathcal{P}_{\mathcal{L},++}$ is the subspace of the elements $\delta \in \mathcal{P}_{\mathcal{L}}$ satisfying:

(i) $\delta(ker\,\pi_1^0) \subset ker\,\pi_2^0(*C)$ (equivalently the matrix (77) satisfies $\lambda = 0$)
(ii) The matrix (77) satisfies $\mu = \nu$

The interest lies in the following geometric description of $\mathcal{P}_{\mathcal{L},++}$:

Proposition 6.0.2. *The subspace $\mathcal{P}_{\mathcal{L},++}$ is isomorphic, as a $\mathcal{O}_{\hat{N}}(*C)$-affine space, to the product:*

$$\mathcal{O}_{\hat{N}}(*C) \times \mathcal{S}_{\hat{N},C}$$

where $\mathcal{S}_{\hat{N},C}$ is the $\mathcal{O}_{\hat{N}}(*C)$-affine space of meromorphic projective structures on $\mathcal{L}$ with poles at C.

Proof. Consider the hyperplane $\mathcal{P}_{\mathcal{L},1} \subset \mathcal{P}_{\mathcal{L},++}$ of elements such that the matrix (77) satifies $\mu = \nu = 1$. It is the subset of elements $\delta \in \mathcal{P}_{\mathcal{L},++}$ with the property:

$$\pi_1^0 \circ \delta \circ j^1 = Id_{\mathcal{L}}$$

These are exactly the splittings of the meromorphic one jet sequence associated with the vector bundle $J^1\mathcal{L}$, i.e. meromorphic connections on $J^1\mathcal{L}$ with pole at C. This in particular includes the meromorphic connections inducing the canonical connection on $det(J^1\mathcal{L}) = \mathcal{O}_{\hat{N}}$, and inducing an isomorphism between $ker\,\pi_1^0(*C)$ and $J^1\mathcal{L}/ker\,\pi_1^0(*C)$. By [4], Theorem 4.7, the these meromorphic connections are in bijection with the nonempty set of $\mathcal{S}_{\hat{N},C}$ meromorphic projective structures on $\hat{N}$ with poles at C. We then pick $\delta_1 \in \mathcal{P}_{\mathcal{L},1}$ and use the projection of $\mathcal{P}_{\mathcal{L},++}$ on $\mathcal{P}_{\mathcal{L},0} = ker(\nu)$ with respect to δ_1, to obtain an isomorphism:

$$\mathcal{P}_{\mathcal{L},++} \simeq \mathcal{O}_{\hat{N}}(*C)\delta_1 \oplus \mathcal{P}_{\mathcal{L},0}$$

Finally, $\mathcal{P}_{\mathcal{L},0}$ is the space of translations of $\mathcal{P}_{\mathcal{L},1}$, so it is isomorphic, as an affine space, to $\mathcal{S}_{\hat{N},C}$. □

In order to study the quotients of meromorphic affine principal elliptic bundles, we will also need to interpret the line bundle (75) geometrically. This can be made using the theory of meromorphic opers (see [4], Theorem 4.7) that we recall now.

Let $\mathbb{P}^1$ be the complex projective line seen as the homogeneous complex manifold G/P, where $G = SL_2(\mathbb{C})$ and P the subgroup stabilizing the line $\mathbb{C}\mathfrak{e}_1 \subset \mathbb{C}^2$ through the standard representation (($\mathfrak{e}_1, \mathfrak{e}_2$ is the canonical basis of $\mathbb{C}^2$). Let us introduce a notation. If $p : E \longrightarrow M$ is a holomorphic P-principal bundle and $\rho : P \longrightarrow GL(\mathbb{V})$ a P-representation, we let:

$$E(\mathbb{V}) = (p_*\mathcal{O}_E \otimes \mathbb{V})^P \tag{79}$$

where the action of P on $p_*\mathcal{O}_E \otimes \mathbb{V}$ is given by

$$b \cdot p_*(f \otimes A) = p_*(f \circ b^{-1} \otimes \rho(b)(A))$$

Then we have a natural isomorphism (see for example [15]):

$$\mathcal{O}_{\mathbb{P}^1}(1) \simeq G(\mathbb{C}\mathfrak{e}_1) \tag{80}$$

where G is seen as the total space of the holomorphic P-principal bundle $p_{G/P} : G \longrightarrow G/P$. In the rest of the paper we will identify these two modules. In particular there is a natural left G-linearization (see Definition 2.2) of this module defined for any $g \in G$, by:

$$\begin{array}{rccc} \phi_g & : & G(\mathbb{C}\mathfrak{e}_1) & \longrightarrow & g^*G(\mathbb{C}\mathfrak{e}_1) \\ & & p_{G/P*}\,\tilde{s} & \mapsto & p_{G/P*}\,\tilde{s} \circ g^{-1} \end{array} \tag{81}$$

Now we can restrict this line bundle to $\mathbb{H} \subset G/P$, and we get a Γ'-linearisation by considering the isomorphisms $(\phi_\gamma)_{\gamma \in \Gamma'}$ as above. Then:

Lemma 6.1. *The line bundle $\mathcal{L}$ defined as in* (75) *is naturally isomorphic to* $\mathcal{L}_1 = (\overline{p}_* G(\mathbb{C}\mathfrak{e}_1)|_{\mathbb{H}})^{(\phi_\gamma)_{\gamma\in\Gamma'}}$.

Proof. It is sufficient to find a trivialization of $\mathcal{L}_1$ such that the isomorphisms ϕ_γ identifies with the isomorphisms α_γ as in (74). For, recall that there exists a global holomorphic section $\sigma_0 : \mathbb{H} \longrightarrow G$ given by:

$$\sigma_0(z) = \begin{pmatrix} 1 & 0 \\ z & 0 \end{pmatrix}$$

Such a section defines a trivialization of any module obtained as a representation of G, in particular:

$$\begin{array}{rccc} \psi_{\sigma_0} & : & G(\mathbb{C}\mathfrak{e}_1)|_{\mathbb{H}} & \longrightarrow & \mathcal{O}_{\mathbb{H}} \\ & & [(\sigma_0, f\mathfrak{e}_1)] & \mapsto & f \end{array} \tag{82}$$

Now we have:

$$\sigma_0(\gamma z) = \gamma \cdot \sigma_0(z) \cdot \begin{pmatrix} c_\gamma z + d_\gamma & 0 \\ 0 & \frac{1}{c_\gamma z + d_\gamma} \end{pmatrix}$$

This implies the following commutative diagram for any $\gamma \in \Gamma'$:

$$\begin{array}{ccc} G(\mathbb{C}\mathfrak{e}_1)|_{\mathbb{H}} & \xrightarrow{\phi_\gamma} & \gamma^* G(\mathbb{C}\mathfrak{e}_1)|_{\mathbb{H}} \\ \downarrow{\scriptstyle \psi_{\sigma_0}} & & \downarrow{\scriptstyle \gamma^*\psi_{\sigma_0}} \\ \mathcal{O}_{\mathbb{H}} & \xrightarrow[\alpha_\gamma]{} & \gamma^* \mathcal{O}_{\mathbb{H}} \end{array}$$

□

On the model $\mathbb{P}^1 = G/P$, the sheaf of one-jets $J^1(\mathcal{L}^*_{G/P})$, where $\mathcal{L}_{G/P} = G(\mathbb{C}\mathfrak{e}_1)$, is naturally isomorphic to $G(\mathbb{C}^2)$ (it can be seen by considering trivialisations of $G \longrightarrow G/P$ as in the proof of Lemma 6.1). In particular, it contains $\mathcal{L}_{G/P}$ as a locally free submodule. Since $G(\mathbb{C}^2)$ is by definition the sheaf of sections of a homogeneous bundle on G/P,there is a natural linearization for the left action of G (see Definition 2.2) on this sheaf of $\mathcal{O}_{G/P}$-modules, denoted by $(\phi^J_g)_{g\in G}$. Moreover, it admits a global trivialization induced by the two P-equivariant maps $\tilde{s}_i : G \longrightarrow \mathbb{C}^2$ defined by:

$$\tilde{s}_i(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \mathfrak{e}_i$$

for $i = 1, 2$. The corresponding flat connection of trivial module $\nabla^J_{G/P}$ on $G(\mathbb{C}^2)$ is invariant through the isomorphisms $(\phi^J_g)_{g\in G}$ since the above functions are invariant through these isomorphisms. Moreover, since G is a $SL_2(\mathbb{C})$-reduction of the bundle of basis of $G(\mathbb{C}^2)$, there is a natural isomorphism $\bigwedge^2 G(\mathbb{C}^2) \simeq \mathcal{O}_{\mathbb{P}^1}$ and the canonical connection coïncides with the connection induce by $\nabla^J_{G/P}$. The key property of $\nabla^J_{G/P}$ is that the induced morphism of line bundles:

$$\begin{array}{rccc} [\nabla^J_{G/P}] & : & \mathcal{L}_{G/P} & \longrightarrow & \mathcal{K}_{\mathbb{P}^1} \otimes G(\mathbb{C}^2)/\mathcal{L}_{G/P} \end{array} \tag{83}$$

is an isomorphism. This indeed enables to recover that

$$\mathcal{L}^{\otimes 2}_{G/P} = \mathcal{L}_{G/P} \otimes (G(\mathbb{C}^2)/\mathcal{L}_{G/P}) \otimes \mathcal{K}_{\mathbb{P}^1} = \mathcal{K}_{\mathbb{P}^1} \tag{84}$$

that is $\mathcal{L}_{G/P} = \mathcal{O}_{\mathbb{P}^1}(1)$. The restriction of $\nabla^J_{G/P}$ over $\mathbb{H}$ is Γ'-invariant, so it induces a connection $\hat{\nabla}^J_0$ on:

$$\mathcal{E} := (\overline{q}_* G(\mathbb{C}^2)|_{\mathbb{H}})^{\Gamma'} \tag{85}$$

Since the $SL_2(\mathbb{C})$-reduction of $G(\mathbb{C}^2)$ is also Γ'-invariant, we get $\bigwedge^2 \mathcal{E} = \mathcal{O}_{\hat{N}}$. By construction, we have the same isomorphism as in (83), so we get:

$$\mathcal{E} = J^1(\mathcal{L}^*) \tag{86}$$

and recover that

$$\mathcal{L}^{\otimes 2} = \mathcal{K}_{\hat{N}} \tag{87}$$

Definition 6.2. The pair $(\mathcal{E}, \hat{\nabla}_0^J)$ is called the *holomorphic $SL_2(\mathbb{C})$-oper corresponding to the projective structure* $\hat{N} = \Gamma' \backslash \mathbb{H}$.

6.3. **Principal elliptic surfaces with $b_1(\hat{M})$ odd.** Let $\hat{M} \longrightarrow \hat{N}$ be a principal elliptic surface over a Riemann surface $\hat{N}$ of genus $g \geq 2$, with $a(\hat{M}) = 1$ and odd first Betti number. Denote by $p : \tilde{M} \longrightarrow \hat{M}$ and $\overline{p} : \mathbb{H} \longrightarrow \hat{N}$ the respective universal covers, and (z_1, z_2) global coordinates on $\tilde{M}$ as in Proposition 2.2.2. We will also use the notation $\Gamma = \pi_1(\hat{N}, y_0)$ for the fundamental group of $\hat{N}$ at a fixed point.

By [8], Proposition 5.2, up to finite unramified cover of the elliptic surface $\hat{M}$, for any $\gamma \in \pi_1(\hat{N}, y_0)$, the automorphism φ_γ from Equation 10 is of the form:

$$\varphi_\gamma(z_1, z_2) = (\frac{a\gamma z_1 + b_\gamma}{c_\gamma z_1 + d_\gamma}, z_2 + log_\gamma(c_\gamma z_1 + d_\gamma)) \tag{88}$$

for some $\begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix} \in SL_2(\mathbb{R})$ and some determination log_γ of the logarithm on $c_\gamma \mathbb{H} + d_\gamma$. Moreover, there exists a holomorphic affine connection $\hat{\nabla}_0$ on $\hat{M}$. As in the previous sections, taking the matrix of the pullback $p^\star \hat{\nabla} - p^\star \hat{\nabla}_0$ in $(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2})$ gives a bijection between the set of meromorphic affine $\hat{\nabla}$ connections on $\hat{M}$ and the set of solutions (f_{ij}, g_{ij}) of the following system of functional equations, for any $\gamma \in \pi_1(\hat{N}, y_0)$:

$$\begin{cases} g_{12}(z_1) &= (c_\gamma z_1 + d_\gamma)^2 g_{12}(\gamma z_1) \\ g_{ii}(z_1) &= g_{ii}(\gamma z_1) + (-1)^i c_\gamma (c_\gamma z_1 + d_\gamma) g_{12}(\gamma \cdot z_1) \\ g_{21}(z_1) &= \frac{1}{(c_\gamma z_1 + d_\gamma)^2} g_{21}(\gamma z_1) + \frac{c_\gamma}{c_\gamma z_1 + d_\gamma}(g_{22} - g_{11})(\gamma z_1) - c_\gamma^2 g_{12}(\gamma z_1) \end{cases} \tag{89}$$

and:

$$\begin{cases} f_{12}(z_1) &= f_{12}(\gamma \cdot z_1) + \frac{c_\gamma}{c_\gamma z_1 + d\gamma} g_{12}(\gamma z_1) \\ f_{ii}(z_1) &= \frac{1}{(c_\gamma z_1 + d_\gamma)^2} f_{ii}(\gamma z_1) + (-1)^i \frac{c_\gamma}{c_\gamma z_1 + d_\gamma} f_{12}(\gamma z_1) + \frac{c_\gamma}{c_\gamma z_1 + d_\gamma} g_{ii}(\gamma z_1) \\ f_{21}(z_1) &= \frac{1}{(c_\gamma z_1 + d_\gamma)^4} f_{21}(\gamma z_1) + \frac{c_\gamma}{(c_\gamma z_1 + d_\gamma)^3}(f_{22} - f_{11})(\gamma z_1) \\ & \quad - \frac{c_\gamma^2}{(c_\gamma z_1 + d_\gamma)^2} f_{12}(\gamma z_1) + \frac{c_\gamma}{c_\gamma z_1 + d_\gamma} g_{21}(\gamma z_1) \end{cases} \tag{90}$$

We now describe, in terms of meromorphic projective structures on $\hat{N}$, the affine subspace $\mathcal{A}_{\hat{M},+}$ of meromorphic affine connections on $\hat{M}$ whose associated solutions (f_{ij}, g_{ij}) of the above system satisfy:

$$g_{12} = g_{21} = f_{12} + g_{11} - g_{22} = 0 \tag{91}$$

Then:

Theorem 6.1. *Let $\hat{M}$ be a principal elliptic bundle with odd first Betti number over a complex compact curve $\hat{N}$ with genus $g(\hat{N}) \geq 2$. Let C be an effective divisor of $\hat{N}$ and $\mathcal{O}_{\hat{N}}(*C)$ the field of global meromorphic functions with poles in C. Then there exists a non-empty $\mathcal{O}_{\hat{N}}(*C)$-affine subspace $\mathcal{A}_{\hat{M},+}$ in the space of meromorphic affine connection on $\hat{M}$ with poles at C, which is isomorphic to the $\mathcal{O}_{\hat{N}}(*C)$-affine space:*

$$\mathcal{O}_{\hat{N}}(*C)^2 \times \mathcal{S}_{\hat{N},C}$$

where $\mathcal{S}_{\hat{N},C}$ is the affine space of meromorphic projective structures on $\hat{N}$ with poles at C.

Proof. Let $\mathcal{P}_{\mathcal{L},++}$ be the subspace of meromorphic differential operators from Definition 6.1. Then the property (78) implies that the following map is well-defined:

$$\begin{array}{rccccc} \Psi & : & \mathcal{A}^+_{\hat{N},C} & \longrightarrow & \mathcal{O}_{\hat{N}}(*C) & \times & \mathcal{P}_{\mathcal{L},++} \\ & & \hat{\nabla} & \mapsto & (g_{11} & , & \begin{pmatrix} f_{22} & f_{21} \\ g_{22}-g_{11} & -\frac{1}{3}f_{11} \\ 0 & g_{22}-g_{11} \end{pmatrix}) \end{array}$$

It is clearly $\mathcal{O}_{\hat{N}}(*C)$-linear and bijective. The statement then follows from Proposition 6.0.2. □

6.4. **Quotients of principal elliptic bundles over higher genus curves.** We now classify the minimal meromorphical affine surfaces with $a(M) = 1$ such that the associated finite cover $\hat{M}$ is a principal elliptic bundle with odd first Betti number over a compact curve with genus $g(\hat{N}) \geq 2$.

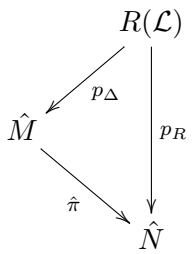
Let $\mathcal{L}$ the line bundle as in (75). We reemploy notations from subsection 6.2 and denote by $R(\mathcal{L})$ the $\mathbb{C}^*$-principal bundle whose fiber over $\hat{y} \in \hat{N}$ is the set of non-zero vectors of the fiber $\mathcal{L}(\hat{y})$. Then it is immediate that $R(\mathcal{L})$ is the quotient of $R(G(\mathbb{C}\mathfrak{e}_1)|_{\mathbb{H}})$ by the action of Γ' corresponding to the isomorphisms ϕ_γ from (81). Moreover, we have a natural isomorphism:

$$R(G(\mathbb{C}\mathfrak{e}_1)|_{\mathbb{H}}) = G|_{\mathbb{H}}/P^+ \subset G/P^+$$

where P^+ is the kernel of the representation of P on $\mathbb{C}\mathfrak{e}_1$ (i.e. the unipotent radical of P). Through this identification, the action of Γ' is the natural left action of $\Gamma' \subset G$ on G/P^+ (note that G/P^+ is biholomorphic to an open subset of $\mathbb{C}^2 \setminus \{0\}$ invariant through Γ' for the standard action).

Let $\mathbb{Z} \simeq \Delta \subset \mathbb{C}^*$ be a lattice, identified with a subgroup of the standard torus of G (namely the diagonal elements). Since the right action of Δ on G and the left action of Γ' on G commute, there is an induced right action of Δ on $R(\mathcal{L})$ covering the identity on $\hat{N}$, and the quotient map is a unramified cover of the complex manifold $\hat{M}$:

(92)



where p_Δ is the quotient map for the action of Δ.

Theorem 6.2 ([8], Proposition 5.2)**.** *Let $\hat{M} \xrightarrow{\hat{\pi}} \hat{N}$ be a principal elliptic fiber bundle over a compact Riemann surface with $g(\hat{N}) \geq 2$. Then, up to a finite Galois cover, there exists a lattice $\mathbb{Z} \simeq \Delta \subset \mathbb{C}^*$ such that $\hat{M}$ is the complex surface obtained as in* (92)*.*

In particular, we get the following:

Lemma 6.2. *Let $\hat{M} \longrightarrow \hat{N}$ be a principal elliptic surface as in* (92)*. Let $\overline{q}_R : (G/P^+)|_{\mathbb{H}} \longrightarrow R(\mathcal{L})$ be the map of the quotient by the left action of $\Gamma' \subset G$. Then any automorphism φ of $\hat{M}$ lifts through $p_\Delta \circ \overline{q}_R$ as the automorphism of G/P^+ corresponding to the left action of an element $A \in SL_2(\mathbb{C})$.*

Proof. The composition $p_\Delta \circ \overline{q}_R$ is an unramified cover of $\hat{M}$, and any automorphism φ of $\hat{\pi} : \hat{M} \longrightarrow \hat{N}$ admits a lift to the total space of this cover $\tilde{\varphi}$. In particular, $\tilde{\varphi}$ normalizes the Galois group of $\overline{q}_R$, that is Γ', that is :

$$\forall \gamma \in \Gamma', \exists \gamma' \in \Gamma',\ \tilde{\varphi} \circ \gamma = \gamma' \circ \tilde{\varphi} \tag{93}$$

Such an automorphism covers an automorphism of $\mathbb{H}$, that is the action of some $A_1 \in SL_2(\mathbb{R})$. Hence, through the trivialization $G/P^+ \simeq \mathbb{H} \times \mathbb{C}^*$ induced by the section σ_0 from the proof of Lemma 6.1, we have:

$$\tilde{\varphi}(z, b) = (A_1 \cdot z, \lambda(z)b)$$

for some holomorphic function $\lambda : \mathbb{H} \longrightarrow \mathbb{C}^*$. Then (93) rewrites as:

$$\begin{aligned} (A_1 \cdot \gamma \cdot z \quad &, \quad \lambda(\gamma \cdot z)(c_{A_1}\gamma \cdot z + d_{A_1})^{-1}(c_\gamma z + d_\gamma)^{-1} b) \\ = \quad (\gamma' \cdot A_1 \cdot z \quad &, \quad \lambda(z)(c_{\gamma'}A_1 \cdot z + d_{\gamma'})^{-1}(c_{A_1} z + d_{A_1})^{-1} b) \end{aligned}$$

In particular $\gamma' = A_1 \gamma A_1^{-1}$ so that, using that $\gamma \mapsto \alpha_\gamma$ is an automorphy factor, λ is a Γ'-invariant holomorphic function, that is a constant. This implies the first assertion. The second one is obtained by applying Lemma 2.1. □

Using these facts, we can prove:

Proposition 6.2.1. *Let $\hat{M} \xrightarrow{\hat{\pi}} \hat{N}$ be a principal elliptic bundle as in* (92)*. Let $\overline{q} : \hat{N} \longrightarrow N$ be a Galoisian finite cover such that $\hat{M}$ is the pullback of a an elliptic surface $\pi : M \longrightarrow N$. Suppose that the sum of the multiplicities $(m_\alpha)_{\alpha \in I}$ of $\overline{q}$ at the ramification points is a multiple of the degree $k = deg(\overline{q})$, and that the Galois group Γ of $\overline{q}$ fixes the ramification points. Then $\overline{q}$ is an unramified cover.*

Proof. Let Γ be the Galois group of $\overline{q}$. We identify Γ with a subgroup of $SL_2(\mathbb{R})$ normalizing the holonomy Γ' of the uniform projective structure on $\hat{N}$ as described before. By Lemma 6.2, the action of the Galois group Γ of $\overline{q}$ on $\hat{N}$ lifts to a left action of Γ on the cover $R(\mathcal{L}) \xrightarrow{q_\Delta} \hat{M}$ (see (92)), obtained from the natural left action of Γ on G. In particular, there is an induced Γ-linearization (see Definition 2.2) $(\phi^J_\epsilon)_{\epsilon \in \Gamma}$ on $J^1(\mathcal{L}^*)$. By construction, the holomorphic $SL_2(\mathbb{C})$-oper $(\mathcal{E}, \hat{\nabla}^J_0) = (J^1(\mathcal{L}^*), \hat{\nabla}^J_0)$ associated with the projective structure $\hat{N} = \Gamma' \backslash \mathbb{H}$ (see Definition 6.2) is invariant by this Γ-linearization.

Consider the line bundle

$$\mathcal{L}' = (\overline{q}_* \mathcal{L})^\Gamma \tag{94}$$

on N. It is a submodule of the locally free $\mathcal{O}_N$-module

$$\mathcal{E}' = (\overline{q}_* \mathcal{E})^\Gamma \tag{95}$$

We prove the existence of a holomorphic connection ∇^J_0 on $\mathcal{E}'$ with the same properties as in (83). For, we first prove that the universal cover of N factorizes by $\overline{q} : \hat{N} \longrightarrow N$. Pick a

system of generators $\gamma_1, \ldots, \gamma_r$ for Γ. By Lemma 6.2, these generators lifts through $p_\Delta \circ \overline{q}_R$ to the left actions of elements $A_1, \ldots, A_r \in SL_2(\mathbb{R})$ on $G/P^+|_{\mathbb{H}}$. By definition of Γ, N is the quotient of $\mathbb{H}$ by the subgroup $\Gamma'' \subset SL_2(\mathbb{R})$ spanned by the A_j and Γ'. Suppose that there exists $j \in \{1, \ldots, r\}$ such that the Möbius transformation corresponding to A_j fixes some $z \in \mathbb{H}$. Recall that γ_j fixes the fibers of $\overline{p}(z) \in \hat{N}$ in $\hat{M} = R(\mathcal{L})/\Delta$. Since γ_j have finite order and $\Delta \simeq \mathbb{Z}$, the induced action on $R(\mathcal{L})$ fixes the fiber of $\overline{p}(z)$. Hence A_j fixes the fiber of z in $G/P^+|_{\mathbb{H}}$. Thus $A_j = Id$, that is Γ'' acts freely and properly discontinusly, in other words $\mathbb{H} \longrightarrow \Gamma''\backslash\mathbb{H} = N$ is the universal cover. Since the A_j normalizes Γ', we have a factorization:

$$\begin{array}{ccc} \mathbb{H} & \xrightarrow{\overline{q}''} & N \\ & \searrow^{\overline{p}} \quad \nearrow_{\overline{q}} & \\ & \hat{N} & \end{array} \tag{96}$$

where $\overline{q}''$ is the quotient by Γ''. In particular, $\mathcal{E}' = (\overline{q}''_* G(\mathbb{C}^2)|_{\mathbb{H}})^{\Gamma''}$ and the quotient of $G|_{\mathbb{H}}$ by Γ'' is a holomorphic P-reduction of its bundle of basis.

Moreover, the holomorphic connection $\nabla^J_{G/P}$ is invariant by the action of Γ'', and since $\overline{q}''$ is an unramified cover, Lemma 2.1 implies the existence of an induced holomorphic connection ∇'^J on $\mathcal{E}'$. By construction, the induced morphism:

$$[\nabla'^J] : \mathcal{L}' \longrightarrow \Omega^1_N \otimes \mathcal{E}'/\mathcal{L}'$$

is a non-trivial morphism. We denote by S the effective divisor corresponding to the line bundle $End(\mathcal{L}', \mathcal{K}_N \otimes \mathcal{E}/\mathcal{L}')$.

In the sequel, we employ the notation $deg(\mathcal{L}) = \int_N c_1(\mathcal{L})$ for any line bundle $\mathcal{L}$ on a compact complex curve N. Using $deg(\mathcal{E}') = 0$ (by the existence of a $SL_2(\mathbb{C})$-holomorphic reduction of its bundle of basis), we get:

$$deg(S) = deg(\mathcal{K}_N) - 2deg(\mathcal{L}') \tag{97}$$

In one other hand, consider the sheaf of sections of the pullback line bundle, that is $\mathcal{O}_{\hat{N}} \otimes \overline{q}^*\mathcal{L}'$. Since $\overline{q}^*\mathcal{L}'$ is by definition the subsheaf of $\mathcal{L}$ spanned by the sections invariant by the action of Γ, we have a well-defined non-trivial morphism of modules:

$$\begin{array}{rccc} \iota : & \mathcal{O}_{\hat{N}} \otimes \overline{q}^*\mathcal{L}' & \longrightarrow & \mathcal{L} \\ & f \otimes s & \mapsto & fs \end{array} \tag{98}$$

Recall that the action of Γ on $\hat{M} = \Delta\backslash R(\mathcal{L})$ fixes the fibers of the ramification locus of $\overline{q}$. Hence, in a neighborhood U_α of any component D_α of the ramification locus of $\overline{q}$ in $\hat{N}$, we can find a coordinate z and a trivialization of $\mathcal{L}$, such the action of the automorphism $\epsilon \in \Gamma$ corresponding to a generator of $\pi_1(U_\alpha \setminus D_\alpha, \hat{y})$, on a section $s \in \mathcal{L}(U_\alpha)$, is given by:

$$\epsilon \cdot s(z) = \nu_\epsilon s \circ \epsilon^{-1}(z)$$

for some $\nu_\epsilon \in \Delta$. But ϵ has finite order m_α and Δ contains no non-trivial cyclic element, so $\nu = 1$. As a consequence, the section corresponding to 1 in the choosen trivialisation of $\mathcal{L}$ is a local invariant section of $\mathcal{L}$ on U_α. This in turn implies that Equation 98 is an isomorphism. Hence, since $\overline{q}$ is a finite cover of degree k, we get:

$$deg(\mathcal{L}) = deg(\mathcal{O}_{\hat{N}} \otimes q^*\mathcal{L}') = kdeg(\mathcal{L}') \tag{99}$$

Now, by the Riemann-Hurwitz formula, we also have:

$$-deg(\mathcal{K}_{\hat{N}}) = -kdeg(\mathcal{K}_N) + |I| - \sum_{\alpha \in I} m_\alpha \tag{100}$$

By (99), and (97), we also have:

$$\begin{aligned} -deg(\mathcal{K}_{\hat{N}}) &= -2deg(\mathcal{L}) \\ &= -2k\, deg(\mathcal{L}') \\ &= -k\, deg(\mathcal{K}_N) + k\, deg(S) \end{aligned}$$

Combining the above equality with (100) we get:

$$k\, deg(S) + \sum_{\alpha \in I} m_\alpha = |I|$$

By the assumption on the degree k of $\overline{q}$ the above equality rewrites as:

$$|I| = k' \sum_{\alpha \in I} m_\alpha$$

with $k' \geq 0$ and $m_\alpha \geq 2$ for any $\alpha \in I$. This is possible only if $k' = 0$, i.e $I = \emptyset$ and $\overline{q}$ is unramified.

□

Theorem 6.3. *Let (M, ∇) be a minimal meromorphic affine surface with $a(M) = 1$ and $(\hat{M} \xrightarrow{\hat{\pi}} \hat{N}, \hat{\nabla})$ the meromorphic affine principal elliptic bundle obtained as in* (27)*. If the genus $g(\hat{N}) \geq 2$, then $\hat{M} = M$.*

Proof. By Proposition 6.2.1, $\hat{M}$ has an odd first Betti number. Hence, by Theorem 6.2, up to finite Galois cover, it is a quotient of $R(\mathcal{L})$ by a lattice $\mathbb{Z} \simeq \Delta \subset \mathbb{C}^*$ as in (92). We can assume from now on that $\hat{M}$ coïncides with this Galois cover.

By construction, the Galois cover $\overline{q} : \hat{N} \longrightarrow N$ (resp. $q : \hat{M} \longrightarrow M$) in (27) is a composition:

$$\overline{q} = \overline{q}' \circ \overline{q}_1 \, (\text{resp.} q = q' \circ \overline{q}_1)$$

where $\overline{q}' : \hat{N}_1 \longrightarrow N$ is a composition of cyclic covers and $\overline{q}_1$ is an unramified finite cover (the maps q' and q_1 are the corresponding pullbacks of elliptic bundles).

Denote by $\Gamma_1 \subset \Gamma$ the Galois group of $\overline{q}_1$, identified with the Galois group of q_1. Then by construction $\hat{M}_1 = \Gamma_1 \backslash \hat{M}$ is the quotient of $R(\mathcal{L}_1)$ by Δ where

$$\mathcal{L}_1 = (\overline{q}_{1*}\mathcal{L})^{\Gamma_1}$$

and by Lemma 6.2, the action of Γ_1 lifts to the natural action of a subgroup $\tilde{\Gamma}_1 \subset SL_2(\mathbb{C})$ on $G(\mathbb{C}\mathfrak{e}_1)|_{\mathbb{H}}$. Hence $\Gamma'' = \langle \Gamma', \Gamma_1 \rangle$ is the holonomy group of a uniform $(G, G/P)$-structure

$$\overline{p}_1 : \mathbb{H} \longrightarrow \hat{N}_1 = \Gamma'' \backslash \mathbb{H}$$

and

$$\mathcal{L}_1 = (\overline{p}_{1*} G(\mathbb{C}\mathfrak{e}_1)|_{\mathbb{H}})^{\Gamma''}$$

so $\hat{M}_1$ is obtained as in (92). In particular, it is a principal elliptic bundle over $g(\hat{N}_1) \geq 2$ with odd first Betti number. Without loss of generality we can and will further assume that $\hat{N}_1 = \hat{N}$ and $\hat{M}_1 = \hat{M}$.

In this situation, by construction, the Galois group of $\overline{q}$ fixes its ramification locus, and $k = deg(\overline{q})$ is a multiple of $\sum_{\alpha \in I} m_\alpha$. Then Theorem 6.3 proves that $\overline{q} : \hat{N} \longrightarrow N$ is unramified. Since moreover $\overline{q} = \overline{q}'$, we get $\hat{M} = M$ by definition of $\overline{q}'$. □

7. Conclusion

Comparing with [6], the Theorem 3.2 together with Theorem 4.3, Theorem 5.3, Theorem 5.5, and Proposition 6.2.1 prove Theorem 1.1. From the point of view of the uniformization, this suggests that we have to consider other meromorphic geometric structures to encompass more complex compact surfaces. It could be interesting to extend the technics of this paper to meromorphic projective connections (see [9]) by adopting the Cartan point of view.

References

[1] M. F. Atiyah. Complex analytic connections in fibre bundles. *Trans. Amer. Math. Soc.*, 85(1):181–181, 1957.
[2] W. Barth, K. Hulek, C. Peters, and A. van de Ven. *Compact complex surfaces*. Springer, Berlin, Heidelberg, 2004.
[3] I. Biswas and S. Dumitrescu. Branched holomorphic Cartan geometries and Calabi-Yau manifolds. *Int. Math. Res. Not.*, 23(9):7428–7458, 2019.
[4] I. Biswas, S. Dumitrescu, and S. Gupta. Branched projective structures on a Riemann surface and logarithmic connections. *Doc. Math.*, 24:2159–2197, 2019.
[5] I. Biswas, S. Dumitrescu, and B. McKay. Cartan geometries on complex manifolds of algebraic dimension zero. *Épijournal Geom. Algèbrique*, 3(19), 2019.
[6] M. Inoue, S. Kobayashi, and T. Ochiai. Holomorphic affine connections on compact complex surfaces. *J. Fac. Sci. Univ. Tokyo*, 1980.
[7] V. Lunts J. Bernstein. *Equivariant sheaves and functors*. Springer, Berlin, Heidelberg, 1994.
[8] B. Klingler. Structures affines et projectives sur les surfaces complexes. *Ann. Inst. Fourier (Grenoble)*, 48(2):441–477, 1998.
[9] S. Kobayashi and T. Ochiai. Holomorphic projective structures on compact complex surfaces. *Math. Ann.*, 249(1):75–94, 1980.
[10] K. Kodaira. On compact complex analytic surfaces, I. *Ann. Math.*, 71(1):111–152, 1960.
[11] K. Kodaira. On compact analytic surfaces, II. *Ann. Math.*, 77(3):563–626, 1963.
[12] K. Kodaira. On the structure of compact complex analytic surfaces, I. *Amer. J. Math.*, 86(4):751–798, 1964.
[13] K. Kodaira. On the structure of compact complex analytic surfaces, II. *Amer. J. Math.*, 88(3):682–721, 1966.
[14] B. Pym M. Gualtieri. The Stokes Groupoids. *J. reine angew. Math.*, 739:81–119, 2018.
[15] D. M. Snow. On the ampleness of homogeneous vector bundles. *Trans. Amer. Math. Soc.*, 294(2):585–585, 1986.
[16] A. Vitter. Affine structures on compact complex manifoids. *Invent. Math.*, 17:231–244, 1972.

Université d'Avignon
Email address: alexis.garcia@univ-avignon.fr