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## A method to evaluate the capillary stress tensor at failure in unsaturated soils

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### ABSTRACT

This paper aims to provide a better understanding of the mechanisms that make it difficult to interpret the behavior of unsaturated soils using an effective stress principle. The main difficulty is that some deformed states of the unsaturated medium may not be statically admissible in saturated conditions. Therefore, we introduce the notions of total strain, which is measured at the boundary of the sample; effective strain, which is conjugated to the effective stress of the solid skeleton; and capillary strain, which results from the capillary forces. Recent results using the discrete element method prove that the effective strain can still be identified with the total strain in "small" strains and at failure. Consequently, we propose to interpret the behavior of unsaturated soils on the basis of the principle of effective stress revisited with the theory of multiphase media and to identify the total and effective strain tensors in a first simplified approach. A conceptual experimental procedure is proposed to assess what we call the capillary tensor, using existing experimental devices. We propose certain hypotheses that allow engineers to numerically bound this capillary tensor at failure by knowing the total stress state measured at failure and the effective failure envelope in saturated conditions.

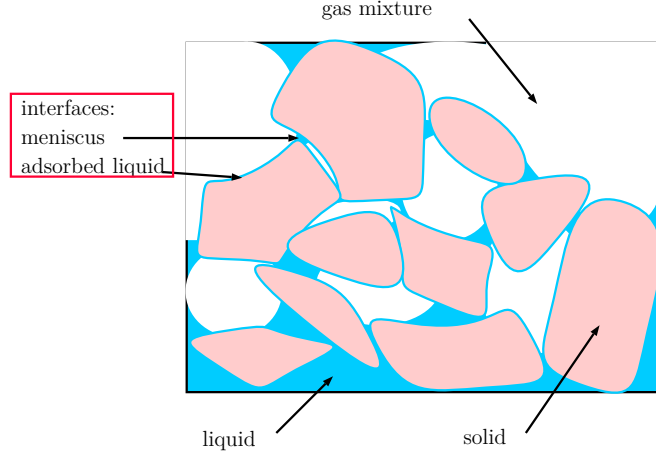
### KEYWORDS

Unsaturated soils, effective stress concept, capillary tensor, triaxial test

## 1. Introduction

To our knowledge, two main approaches have been developed to describe the behavior of unsaturated media in recent decades:

- the extended effective stress principle of Bishop and Blight (1963). It is well established that to consider only the scalar suction and the air pressure does not allow a good description of the actions of the fluid phases on the solid skeleton even by introducing an ad-hoc  $\chi$  function as in (e.g. Arai, Prunier, Djéran-Maigre, & Darve, 2012; Khalili & Khabbaz, 1998; Nikooee, Habibagahi, Hassanizadeh, & Ghahramani, 2013) or by introducing the notion of effective saturation degree (e.g. E. Alonso, Pereira, Vaunat, & Olivella, 2010; Schwan, 2023). The term



**Figure 1.** Conceptual diagram of an unsaturated granular medium  
 Alt Text: Diagram of a porous media highlighting grains and pores filled by air and liquid bridges.

30 "solid skeleton" is used here to refer to the solid material containing its voids and  
 31 excluding any fluid-solid interface as introduced by [Coussy \(2004\)](#).  
 32 • the Barcelona Basic Model (E.E. Alonso, Gens, & Josa, 1990) and its most recent  
 33 extensions, such as (e.g. Bruno, Gallipoli, & Mendes, 2021; Rojas & Chávez, 2013;  
 34 Weber, Romero, & Lloret, 2022), which describe the unsaturated soil behavior  
 35 using generalized stress variables in the form of a net stress  $\boldsymbol{\sigma}^{net} = \boldsymbol{\sigma} - u_a \mathbf{I}$  and  
 36 an effective stress  $\boldsymbol{\sigma}''$ .  $\boldsymbol{\sigma}$  is the total stress tensor,  $u_a$  the air pressure, and  $\mathbf{I}$   
 37 the identity tensor. We note here the effective stress  $\boldsymbol{\sigma}''$ , since the latter is supposed to  
 38 act on the skeleton, which includes the solid skeleton plus the fluid-solid interfaces  
 39 in the form of adsorbed water and menisci (e.g. Coussy, 2004; Coussy & Dangla,  
 40 2002). See figure 1 for a qualitative description of an unsaturated granular medium  
 41 and the fluid-solid interfaces.

42 In fact, it was natural to want to describe the behavior of unsaturated soils with  
 43 measurable quantities, in the same way as Terzaghi (1936) advocated in his principle  
 44 of effective stress for saturated soils:

45  
 46 *All the measurable effects of a change of stress, such as compression, distortion, and a*  
 47 *change of shearing resistance are exclusively due to changes in the effective stresses  $\sigma'_1$ ,*  
 48  *$\sigma'_2$ , and  $\sigma'_3$ .*

49 However, we find that these two approaches are not really used in practice. There are  
 50 several reasons for this:

- 51 • the ratio between the cost of characterizing unsaturated soil behavior and the  
 52 gain from designing a geotechnical structure that takes into account the beneficial  
 53 effects of unsaturated states is generally unsatisfactory.
- 54 • experimental devices for measuring or controlling suction are difficult to imple-  
 55 ment and insufficient. Furthermore, in terms of the Barcelona approach, the consti-  
 56 tutive law for the skeleton (namely the solid skeleton plus its fluid-solid interfaces)  
 57 turns out to be extremely complex.

58 Therefore, we propose to analyze the mechanical behavior of unsaturated soils in the  
 59 framework of the multiphase media theory as developed by [Madeo, dell'Isola, and Darve](#)

60 (2013). Nevertheless, we restrict our work to granular soils and silts, excluding swelling  
 61 clays such as smectites. It is likely that this approach can be extended to clays that are  
 62 not water sensitive, such as kaolinites. As demonstrated in section 2, Terzaghi’s effective  
 63 stress principle for saturated soils fits perfectly into the framework of this theory. For  
 64 unsaturated soils, we observe that this theory allows an extension of an effective stress  
 65 principle, but by introducing a nonlinear decomposition of the strain tensor in the  
 66 following form:

$$\boldsymbol{\varepsilon} = F(\boldsymbol{\varepsilon}', \boldsymbol{\varepsilon}^{cap}) \quad (1)$$

67 In our opinion, this decomposition is one of the original features of this work, and is  
 68 presented in detail in section 2.3. In the rest of the paper, bold letters designate tensors  
 69 whose order will be specified in the text. Here,  $\boldsymbol{\varepsilon}$  is the 2<sup>nd</sup> order strain tensor measured  
 70 at the boundary of a homogeneous soils sample,  $\boldsymbol{\varepsilon}'$  is the strain tensor conjugated to  
 71 the effective stress tensor by the strain energy of the solid skeleton, and  $\boldsymbol{\varepsilon}^{cap}$  is the  
 72 strain tensor resulting from the deformation action of all the various liquid-gas and  
 73 solid-fluid finite interfaces. Nevertheless, this paper demonstrates that the exploitation  
 74 of this theory of multiphase media in an experimental context is currently complicated.  
 75 To our knowledge, there are no experimental devices that allow direct measures of  
 76 the physical quantities involving the constitutive law of each phase: that of the solid  
 77 skeleton, that of the liquid phase, and that of the gas phase. This paper does not  
 78 present any innovation on experimental devices that would allow a better understanding  
 79 of the behavior of unsaturated soils. Nevertheless, in section 2 we make a qualitative  
 80 presentation of the latest theoretical developments that should, in our opinion, enable a  
 81 simpler understanding and description of unsaturated soil behavior. We hope that the  
 82 theoretical developments proposed in section 2 will assist in better understanding this  
 83 complex behavior and could help in future research to:

- 84 • provide new ideas in interpreting data from standard experimental set-ups
- 85 • take these new conceptual developments into account when designing future ad-  
 86 vanced experimental set-ups, such as those that will soon enable us to finely char-  
 87 acterize the evolution of the microstructure of the solid and fluid phases during  
 88 laboratory tests.

89 Therefore, notwithstanding the experimental difficulties specified in the theoretical part,  
 90 we propose in section 3 a conceptual experimental procedure that should make it pos-  
 91 sible to quantitatively assess the capillary effects on the solid skeleton with existing  
 92 experimental devices. This procedure relies on the strong assumption of a pure identifi-  
 93 cation of the total strain tensor with the effective strain tensor. After having presented a  
 94 work mainly in conceptual form, in section 4 we propose certain assumptions that allow  
 95 engineers to quantitatively evaluate the components of the capillary tensor at failure.  
 96 These assumptions allow the strength of the unsaturated soil to be quantified even at  
 97 low saturation, where an assumption of a spherical capillary tensor may no longer be  
 98 relevant. Finally, section 5 presents an experimental campaign carried out on an unsat-  
 99 urated silt using a simple device. Results at failure are analyzed using the assumptions  
 100 previously formulated in section 4.

101 **2. Some remarks about the effective stress concept**

102 The mechanical behavior of "monophasic" (i.e., saturated drained) granular materials  
 103 is currently well understood and simulated by phenomenological elasto-visco-plastic  
 104 constitutive relations (e.g. Christoffersen & Hutchinson, 1979; Colman & Gurtin, 1967;  
 105 Desai & Siriwardane, 1984; DiMaggio & Sandler, 1971), homogenization techniques  
 106 (e.g. Eshelby, 1957; Mori & Tanaka, 1973; Oller, 2014), and discrete element method  
 107 (DEM) modelling (e.g. Froiio & Roux, 2011; Roux & Combe, 2002; Sibille, Hadda,  
 108 Nicot, Tordesillas, & Darve, 2015).

109  
 110 For biphasic granular media, the relevance of Terzaghi's principle has been clearly  
 111 established in geomechanics through experimental and theoretical approaches. The total  
 112 stress  $\boldsymbol{\sigma}$  can be additively decomposed into a so-called effective stress ( i.e. the solid  
 113 skeleton stress)  $\boldsymbol{\sigma}'$  and an interstitial spherical fluid pressure  $u\mathbf{I}$ , with  $\mathbf{I}$  the identity  
 114 tensor:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}' + u\mathbf{I} \quad (2)$$

115 The effective stress appears to be the proper stress variable to express the constitutive  
 116 behavior of the solid skeleton with its energy-conjugated total strain. The energy  
 117 conjugate effective strain is equal here to the total strain, since it is clear that there  
 118 is neither any capillary stress nor capillary strain in a saturated granular material  
 119 without any liquid-gas interfaces. This has been very well verified by experiments  
 120 (e.g. Skempton, 1954; Skempton & Bjerrum, 1957), and analytical homogenization  
 121 techniques (e.g. Biot, 1955; De Buhan & Dormieux, 1996; Lade & De Boer, 1997).  
 122 However, it is demonstrated later that in a general framework, neither the total stress  
 123 nor the total strain (the one measured at the boundary of the soil) are constitutive  
 124 variables; they are independent of each other and are not linked by any constitutive  
 125 relationship. Moreover, undrained triaxial tests are usually simulated in the finite  
 126 element method with a single-phase formulation by imposing the strain path. The  
 127 response of the dry solid skeleton in terms of effective stresses is identical to the  
 128 response of the solid skeleton subjected to the undrained hydro-mechanical path.

129  
 130 Let us consider a body constituted by three phases: a granular porous solid (sub-  
 131 script  $s$ ), an interstitial isotropic liquid (subscript  $w$ ), and an interstitial isotropic gas  
 132 (subscript  $a$ ). The figure 1 provides a conceptual diagram of an unsaturated granular  
 133 medium.

134  
 135 In line with the surface tension phenomena, certain finite separation areas exist be-  
 136 tween these three immiscible phases, as follows:  $S_{sa}$  between solid and gas,  $S_{sw}$  between  
 137 solid and liquid, and  $S_{aw}$  between gas and liquid. Therefore, it is not possible to apply  
 138 the mixture theory, which basically assumes an intimate mixture between both fluids.  
 139 Bishop's equation, in which the parameter  $\chi$  is taken as the saturation degree  $S_r$ , is  
 140 therefore not valid here:

$$\boldsymbol{\sigma}' = \boldsymbol{\sigma} - u_a\mathbf{I} + S_r(u_a - u_w)\mathbf{I} \quad (3)$$

141 where  $u_a$  and  $u_w$  are the gas and liquid pressures, respectively. Furthermore, the more  
 142 general Bishop relationship, where  $\chi$  is an ad hoc coefficient to be measured in experi-

143 ments, seems impossible to define objectively in practice.

$$\boldsymbol{\sigma}' = \boldsymbol{\sigma} - u_a \mathbf{I} + \chi(u_a - u_w) \mathbf{I} \quad (4)$$

144 Indeed, the existence of finite separation surfaces between liquid and gas implies two  
145 consequences:

- 146 (1) The fluids stress tensor is not spherical, being impossible to describe with equa-  
147 tion (4). The fluids stress must be considered as a true capillary stress tensor,  
148 which possesses deviatoric components (i.e., it is a non-spherical matrix). This is  
149 quite obvious from a micro-mechanical point of view if we consider the distribution  
150 of menisci inside the body. In fact, it is well known that the distribution of granu-  
151 lar contacts is not isotropic as soon as a deviatoric loading is applied to a granular  
152 assembly due to the induced anisotropy, which is particularly strong for granular  
153 materials. In the same way, the distribution of menisci will not be isotropic in a  
154 three-phasic medium. Therefore, the capillary stress induced by these menisci will  
155 not be isotropic, thus contradicting Bishop's relationship 4.
- 156 (2) For specific configurations, where the fluid stress tensor is spherical, Bishop's  
157 relationship reduces the interface effects to a function  $\chi$  times the suction  $s =$   
158  $u_a - u_w$ , while micro-mechanical approaches (e.g. Chateau & Dormieux, 1995;  
159 Duriez, Eghbalian, Wan, & Darve, 2017; Farahnak et al., 2021; Zhou et al., 2019)  
160 prove that an additional term that depends on the air-water surface tension  $\gamma_{aw}$   
161 cannot be neglected. This last point is developed in the next section.

### 162 *2.1. Insight into the stress concept for unsaturated soils with a* 163 *multiphasic theory*

164 If the mixture theory cannot be applied for partially saturated granular media, it is  
165 necessary to come back to the general theory of multiphasic media, as done by Madeo  
166 et al. (2013).

167 The first step is to define the deformations of each phase and their energy conjugate  
168 stresses:

- 169 •  $\boldsymbol{\varepsilon}_s$  and  $\boldsymbol{\sigma}_s$  for the anisotropic solid skeleton,
- 170 •  $\varepsilon_w$  and  $u_w$  for the isotropic liquid,
- 171 •  $\varepsilon_a$  and  $u_a$  for the isotropic gas.

172 The constitutive behavior and the associated potential energy of each phase can then  
173 be defined in this second step. For the purposes of simplicity, we can assume here  
174 that the porous solid is hyper-elastic. Cowin and Nunziato (1983) elastic relation is a  
175 possible solution with five constitutive parameters (thus three additional parameters  
176 with respect to Hooke's linear isotropic relation to take into account the variable  
177 porosity). Let us note that the elastic law of the porous solid skeleton must definitely  
178 be distinguished from the elastic law of the homogeneous non-porous solid matter. The  
179 state equations for the liquid and the gas are classical and well established with, for  
180 example, function of their respective quadratic potentials.

181  
182 A third step takes into account the conservation laws for the phase masses and  
183 the momentum balance equations for the phases. The total energy of the system is  
184 assumed to be equal to the sum of the phase potential energies, the interface potential  
185 energies, and the phase kinetic energies minus the externally imposed work. The final

186 equation comes from the application of the least action principle in the framework of  
 187 variational methods.

188

189 This set of equations is, however, not closed, and it is necessary to describe the  
 190 inter-phase interactions, such as, for example, Laplace's equation:

$$s = u_a - u_w = \gamma_{aw} \operatorname{div}(\vec{n}) \quad (5)$$

191 where  $\vec{n}$  is the outward normal of the air-water interface and  $s$  the suction. In the case  
 192 of a spherical surface (air bubble) of radius  $R$ , we would obtain:

$$s = u_a - u_w = 2\gamma_{aw}/R \quad (6)$$

193 Equation (5) or equation (6) is one of the discontinuity equations at the various  
 194 phase separation surfaces that have to be taken into account to obtain a fully closed  
 195 description of the system.

196

197 In order to gain a less abstract understanding of the conjugate quantities that enable  
 198 a pertinent description of the behavior of the solid skeleton in the unsaturated domain,  
 199 it is necessary to return to the granular nature of the medium under consideration.

200 **2.2. Insight into the stress concept for unsaturated soils with the**  
 201 **micro-mechanical "μ-unsat" approach**

202 Denoting  $V_\alpha$  the volume of the three bulk phases  $\alpha = a, s, w$  and  $\alpha\beta$  any interface  
 203 extending over the surface  $S_{\alpha\beta}$  with  $\alpha, \beta \in \{a, w, s\}$ , internal forces exist in all those  
 204 volumes  $V_\alpha$  and interfaces  $S_{\alpha\beta}$  (through surface tension phenomena) and combine to  
 205 form the total stress  $\boldsymbol{\sigma}$  that corresponds to the loads exerted onto the boundaries of an  
 206 unsaturated representative elementary volume (REV):

$$\boldsymbol{\sigma} = \frac{1}{V} \left[ \sum_{\alpha=a,s,w} \int_{V_\alpha} \boldsymbol{\sigma}_\alpha dV + \sum_{\alpha,\beta=a,s,w} \int_{S_{\alpha\beta}} \boldsymbol{\pi}_{\alpha\beta} dS \right] \quad (7)$$

207 As demonstrated by Chateau and Dormieux (1995); Gray and Schrefler (2007); Gurtin  
 208 and Murdoch (1975), appropriate interface stress tensors  $\boldsymbol{\pi}_{\alpha\beta}$  should express according  
 209 to the surface tension coefficient  $\gamma_{\alpha\beta}$  and a projection tensor  $(\mathbf{I} - \vec{n} \otimes \vec{n})$  with  $\vec{n}$  the  
 210 normal to  $S_{\alpha\beta}$ :  $\boldsymbol{\pi}_{\alpha\beta} = \gamma_{\alpha\beta}(\vec{n} \otimes \vec{n} - \mathbf{I})$ . The stresses within the fluid phases  $\alpha = a, w$  are  
 211  $\boldsymbol{\sigma}_\alpha = u_\alpha \mathbf{I}$ , while the stress  $\boldsymbol{\sigma}_s$  inside the solid phase can be classically obtained from  
 212 the tractions acting onto the individual soil grains of total surface  $S_s$  since (assuming  
 213 mechanical equilibrium under no gravity):

$$\int_{V_s} \boldsymbol{\sigma}_s dV = \int_{S_s} (\boldsymbol{\sigma}_s \vec{n}) \otimes \vec{x} dS = \int_{S_s} -\vec{t} \otimes \vec{x} dS \quad (8)$$

After considering all relevant tractions  $\vec{t}$  in Eq. (8), that is, solid-solid contact forces,  
 fluid pressure loads, as well as surface tension forces onto the solid grains, and algebraic  
 derivations assuming rigid solid grains and zero contact surfaces between them, the  
 contribution of solid-fluid interfaces vanishes and the total stress can finally be expressed

as (Duriez et al., 2017; Duriez & Wan, 2016):

$$\begin{aligned} \boldsymbol{\sigma} - u_a \mathbf{I} = \boldsymbol{\sigma}^{cont} + \boldsymbol{\sigma}^{cap} = & \\ & \frac{1}{V} \sum_c \vec{f}^c \otimes \vec{l} \\ & - \frac{1}{V} \left[ s \left( V_w \mathbf{I} + \int_{S_{sw}} \vec{n} \otimes \vec{x} dS \right) + \gamma_{aw} \left( \int_{S_{aw}} (\mathbf{I} - \vec{n} \otimes \vec{n}) dS + \int_{\Gamma} \vec{\nu}_{aw} \otimes \vec{x} dl \right) \right] \end{aligned} \quad (9)$$

214 with:

$$\boldsymbol{\sigma}^{cont} = \frac{1}{V} \sum_c \vec{f}^c \otimes \vec{l} \quad (10)$$

$$\begin{aligned} \boldsymbol{\sigma}^{cap} = & \\ & - \frac{1}{V} \left[ s \left( V_w \mathbf{I} + \int_{S_{sw}} \vec{n} \otimes \vec{x} dS \right) + \gamma_{aw} \left( \int_{S_{aw}} (\mathbf{I} - \vec{n} \otimes \vec{n}) dS + \int_{\Gamma} \vec{\nu} \otimes \vec{x} dl \right) \right] \end{aligned} \quad (11)$$

215 In the rest of this paper, we assume a constant uniform air pressure  $u_a$ , serving as  
216 reference. This assumption covers most geotechnical applications, and we obtain the  
217 following simple expression of  $\boldsymbol{\sigma}$ .

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{cont} + \boldsymbol{\sigma}^{cap} \quad (12)$$

218 This additive relation has been very carefully verified by DEM computations of Duriez  
219 and Wan (2017b) in the case of a pendular regime.

220

221 Eqs (9) to (11) first involves the contact stress  $\boldsymbol{\sigma}^{cont}$ , function in Eq. (10) of the  
222 contact force  $\vec{f}^c$  as sustained by 2 from 1 in a 1-2 contact  $c$  and of the branch vector  $\vec{l}$   
223 pointing from the center of mass of 1 to the center of mass of 2. This stress tensor equals  
224 the total stress in dry conditions, as well as the classical effective stress in saturated  
225 conditions. Its constitutive role in unsaturated conditions is discussed later on. The  
226 remainder of the total stresses are encompassed in a capillary stress  $\boldsymbol{\sigma}^{cap}$  with various  
227 tensorial terms. These tensorial terms describe the internal forces within the fluids and  
228 interface phases, as well as the mechanical interactions between the solid phase and the  
229 other phases. In Eq. (11), for example, one should note the oriented action of suction  
230  $s$  along wetted surfaces  $S_{sw}$  with  $\vec{n}$  the outward solid normal and  $\vec{x}$  being possibly  
231 expressed from each particle centroid; together with an oriented action of surface tension  
232 within the contractile skin  $S_{aw}$  and along contact lines  $\Gamma$  where the three phases meet,  
233 with  $\vec{\nu}$  being the inwards conormal to  $S_{aw}$ . We emphasize that this last term (factor of  
234  $\gamma_{aw}$ ) is not taken into account in Bishop's expression.

### 235 **2.3. Concluding remarks about the constitutive stress and strain** 236 **variables in unsaturated soils**

237  $\boldsymbol{\sigma}^{cont}$  corresponds to the solid skeleton stress. It could be called effective stress and  
238 denoted as  $\boldsymbol{\sigma}'$  in the sense that this stress has to be used to express the solid skeleton



239 constitutive relation. We simplify the notations of equation (12) as follows :

$$\boxed{\boldsymbol{\sigma} = \boldsymbol{\sigma}' + \boldsymbol{\sigma}^{cap}} \quad (13)$$

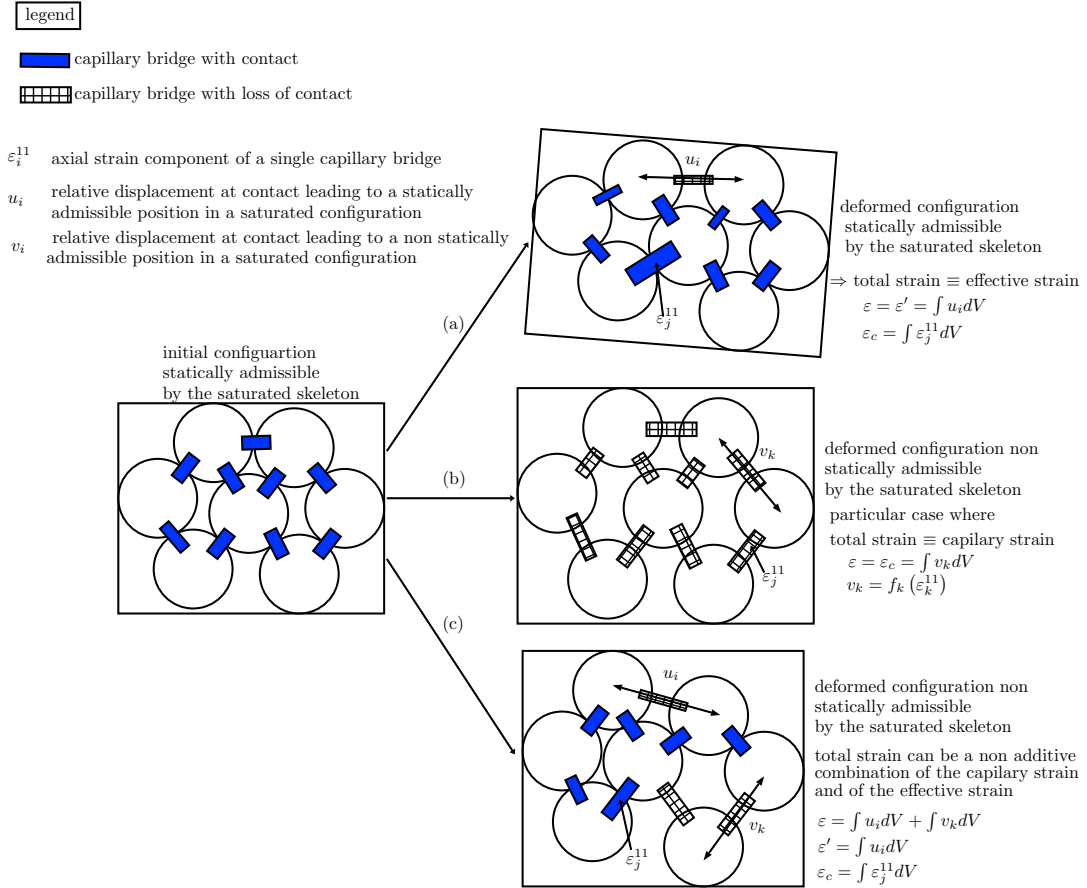
240 Moreover, it should be noted that this stress also corresponds to what is commonly  
241 called "effective stress" in saturated conditions.

242

243  $\boldsymbol{\sigma}^{cap}$  has until now been called the capillary stress. It results from the isotropic  
244 state equations of the liquid and the gas, and the structural strength of the liquid/gas  
245 interfaces. Consequently, there is no energy conjugate capillary strain for this structural  
246 system, which is multiphasic. Therefore, no constitutive equation linking capillary  
247 stress and capillary strain can be considered. As regards the strains of the system,  
248 we can only expect to have a relationship between total strain, effective strain, air  
249 volume variation, water volume variation and liquid-gas interface variations. A second  
250 observation concerns the fact that the capillary stress is decomposed additively into two  
251 stresses according to equation (11): the first is proportional to the suction  $s = u_a - u_w$   
252 and the second is proportional to the surface tension coefficient. This is precisely what  
253 we observe from the point of view of physics. Indeed, the attractive capillary force has  
254 two origins: (i) the pressure difference between liquid and gas, giving rise to the first  
255 term; and (ii) the surface tension stress on the liquid-gas interface, inducing the second  
256 term.

257

258 A final observation relates to the non-additive decomposition of the total strain in  
259 an effective part and a capillary part. A sketch of such a non-additive decomposition  
260 of the total strain is presented in figure 2. The non-additive decomposition of the total  
261 strain tensor is probably one of the main difficulties to overcome from a practical point  
262 of view. In fact, the "effective strain" is almost inaccessible from a practical point of  
263 view (i.e., in the absence of an advanced 3D scanning imagery device). Only the "to-  
264 tal strain" is easily accessible experimentally. A second important difficulty is that the  
265 set of statically admissible geometrical configurations in the saturated case is strictly  
266 included inside the set of statically admissible geometrical configurations in an unsatu-  
267 rated case. Figure 2 also highlights this problem. In other words, a constitutive model  
268 identified and verified in saturated configurations does not encompass all the admissible  
269 geometrical configurations of certain unsaturated cases. Consequently, the existence of  
270 a constitutive law of the solid skeleton under unsaturated conditions becomes a tangible  
271 question that we do not open in this work. This is what makes it so difficult to build  
272 a general constitutive model that is also suitable for unsaturated conditions. Never-  
273 theless, assuming a full identification of the "total strain" with the "effective strain",  
274 verifications and discussions of the use of the contact stress as an effective stress have  
275 been proposed along different loading paths in Duriez and Wan (2018); Duriez, Wan,  
276 Pouragha, and Darve (2018) in the pendular regime with spherical particles. In fact,  
277 this assumption seems quite reasonable, at least for dense soils where few of the con-  
278 tact losses that occur in the unsaturated case can be expected to be inaccessible in the  
279 saturated case.



**Figure 2.** Diagram of strain decomposition in an unsaturated soil showing three possible strain regimes with different proportions of effective ( $\varepsilon'$ ) and capillary ( $\varepsilon_c$ ) strains in the total strain ( $\varepsilon$ ). Grains and capillary bridges are represented schematically by circles and rectangles. The sign  $\int$  schematically denotes a summation operation, but is not properly mathematically defined in this sketch.

Alt Text: 4 pictures of a sample of grains with liquid bridges.

Long description: On the left an initial configuration with liquid bridges at each contact. On the top right a deformed configuration statically admissible in a saturated state. On the middle right a deformed configuration non statically admissible in a saturated configuration: every contact is lost. The equilibrium is done only with liquid bridges in tension. On bottom right a deformed configuration non statically admissible in saturated condition with some contact losses.

280 **2.4. Constitutive role of the effective stress when associated with the**  
281 **total strain**

282 Discrete element method approaches enable a thorough investigation of these stress  
283 tensors in the case of an ideal unsaturated granular soil with spherical particles and  
284 in the pendular regime. They first evidenced a clear tensorial (i.e., spherical and  
285 deviatoric) nature of  $\sigma'$  as soon as the microstructure was not completely isotropic  
286 - during triaxial compressions, for example. See Duriez and Wan (2017a); Scholtes,  
287 Hicher, Nicot, Chareyre, and Darve (2009).

288  
289 The effective stress  $\sigma'$  was also investigated for its constitutive stress-strain-strength  
290 role in unsaturated conditions. Duriez et al. (2018) demonstrated that the limit states  
291 of  $\sigma'$  did coincide in time with the limit states of the total stress  $\sigma$  along conventional  
292 drained triaxial loading paths  $\sigma_2 = \sigma_3 = cst$  and that the corresponding failure line in  
293 terms of effective stress was unique in dry and unsaturated conditions, when examining  
294 two different packing densities. This unique failure description had also been previously  
295 proposed on the basis of DEM by Duriez et al. (2017); Duriez and Wan (2017a);  
296 Scholtes et al. (2009), as well as on the basis of experimental data by E. Alonso et al.  
297 (2010); Khalili and Khabbaz (1998); Lu and Likos (2006), thus making it possible to  
298 consider the effective stress as an appropriate stress-strength effective stress.

299  
300 In terms of stress-strain behavior when assuming that  $\varepsilon' \equiv \varepsilon$  (i.e., when defining  
301 strains from the boundaries of an unsaturated REV), Duriez and Wan (2018);  
302 Duriez et al. (2018) compared the resulting effective stress paths for given strain  
303 paths in dry or unsaturated conditions. The comparison indicated that the effective  
304 stress-total strain behavior was unique in limited cases, involving dense packings  
305 and contractant loading paths only. While this demonstrates that no complete  
306 constitutive stress-strain nature can be claimed for the effective stress associated  
307 with the total strain, a limited constitutive nature can still be considered for these  
308 specific cases. This makes it interesting in practice to estimate initial stiffnesses or possi-  
309 bly oedometric (over-consolidated) loading paths, as argued by E. Alonso et al. (2010).

310  
311 In the rest of this paper, we assume that an effective stress can be associated with  
312 the total strain and propose a procedure for experimentally evaluating this capillary  
313 tensor with the use of standard devices.

314 **3. A conceptual procedure to assess the capillary tensor**

315 It must be stressed that the procedure described in this section is only conceptual, as  
316 we have not yet developed an oedometer or triaxial device to make such experiments  
317 possible. Assuming a complete identification of  $\varepsilon'$  and  $\varepsilon$ , we can state that the effective  
318 stress governs the deformation of the solid skeleton by definition. Hence, if no technology  
319 exists to measure the capillary stress tensor  $\sigma^{cap}$ , then the only way to assess  $\sigma^{cap}$  is to  
320 use its definition and to evaluate the difference  $\sigma - \sigma'$  along a comparable effective strain  
321 path. An experimental procedure to assess the capillary tensor is proposed as follows:  
322 starting from a saturated state, a first desaturation path is followed until reaching  
323 a given initial degree of saturation; a loading path is then performed in unsaturated  
324 conditions. On a second identical soil sample, a second test under saturated conditions  
325 is then controlled by following exactly the same strain path from the initially saturated

326 conditions. The strain path is here understood to be controlled in every direction. In  
 327 theory, the six components of the tensor should be imposed equal to the response of  
 328 the first sample subjected to the unsaturated test. In practice, this will result in the  
 329 three principal components for any homogeneous test in the orthotropy axes. In such  
 330 a conceptual framework, the difference between  $\boldsymbol{\sigma} - \boldsymbol{\sigma}'$  along the same strain path is  
 331 relevant from a constitutive point of view. At the scale of the representative elementary  
 332 volume (REV), expression (13) can be rewritten as follows:

$$\boxed{\boldsymbol{\sigma}' = \boldsymbol{\sigma} - \boldsymbol{\sigma}^{cap}(s, \gamma_{aw}, \boldsymbol{\lambda})} \quad (14)$$

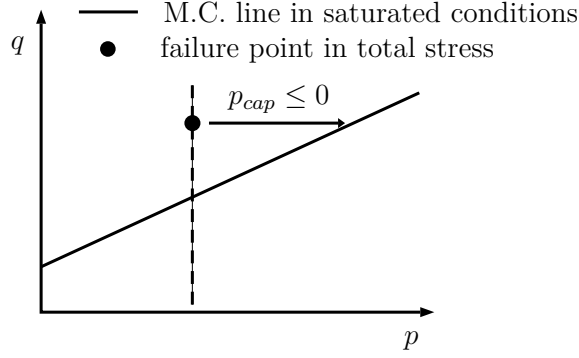
333 where  $\boldsymbol{\lambda}$  is a vector of internal variables that would account for the granular mi-  
 334 crostructure, as well as for the fluid-solid interfaces microstructure and the liquid-gas  
 335 interfaces microstructure. In a first simplified approach,  $\boldsymbol{\lambda}$  can contain components or  
 336 invariants of the plastic strain tensor  $\boldsymbol{\varepsilon}^p$  only. From laboratory tests allowing a control  
 337 or a measure of the suction, the difference between the total stress under unsaturated  
 338 conditions and the effective stress under saturated condition provides the evolution of  
 339 the capillary tensor  $\boldsymbol{\sigma}^{cap}$  as a function of the state variables  $s$  and  $\gamma_{aw}$  and of internal  
 340 variables  $\boldsymbol{\lambda}$ . Nevertheless, the conditions allowing us to identify the total strain with  
 341 the effective strain as a constitutive tensor conjugated to the effective stress tensor have  
 342 to be more precise in future studies. In particular, the question that emerges is whether  
 343 this assumption can be relevant over the whole range of a loading path to failure for  
 344 certain dense or overconsolidated soils. For loose or normally consolidated soils, the  
 345 results of the literature review presented in the first sections up to 2.4 reveal that this  
 346 procedure is valid only for a small range of deformations in a quasi-elastic regime. We  
 347 emphasize that with the procedure described above, the ability of the experimental  
 348 set-up to be fully strain controlled from the initial state to failure is a good indicator  
 349 of the validity of the hypothesis of identifying the total strain with the effective strain.

350  
 351 In concluding this experimental procedure, we observe that it works perfectly for  
 352 saturated soils. Indeed, in this last case, the fluid-solid interface forces are self-balancing.  
 353 Consequently, it is only the fluid pressure that plays a role at the scale of the REV,  
 354 and the effective strain is moreover identified with the total strain. Therefore, drained  
 355 triaxial tests at constant volume make it possible to highlight the phenomenon of dry  
 356 liquefaction. In a saturated consolidated undrained triaxial test (C.U+u), the response  
 357 of the material in effective stresses is exactly the same as in a dry, consolidated and  
 358 drained triaxial test with constant volume (Darve, Arulanandan, & Scott, 1993; Lanier &  
 359 Bloch, 1989). This has been verified in the experiments, and the procedure is commonly  
 360 used in finite elements or discrete elements to reproduce the undrained response of  
 361 saturated soils without implementing a hydro-mechanical coupling.

#### 362 4. Practical assessment of the capillary tensor at failure

363 Measurement or control of suction is not necessary to evaluate the capillary tensor  
 364  $\boldsymbol{\sigma}^{cap}$  at failure, unless identification with the state variables  $(s, \gamma_{aw})$  and the internal  
 365 variables  $\boldsymbol{\lambda}$  is desired. Using classical experimental devices, the main difficulty is to  
 366 know which total and effective stress states are comparable from a constitutive point  
 367 of view.

368



**Figure 3.** Sketch of the image point of effective failure stress states under the basic assumption:  $\sigma_{ij}^{cap} = \sigma^{cap} \delta_{ij}$ . The dashed line serves only to emphasize that  $p_{cap}$  is given by the offset along the  $p$ -axis between  $p_{tot}$  and  $p'$  when  $\sigma_{cap}$  is spheric.

Alt Text:  $q - p$  plane with the effective Mohr-Coulomb line. One point in total stress outside of the effective admissible domain. From this point an horizontal arrow showing the projection permitting the computation of the capillary pressure.

369 If we assume a spherical capillary tensor for saturation degree in the vicinity of 1, we  
 370 have:

$$\sigma_{ij}^{cap} = \sigma^{cap} \delta_{ij} \quad (15)$$

371 Hence, noting  $q = \sigma_1 - \sigma_3$  with  $\sigma_1$  the axial principal stress and  $\sigma_3$  the radial principal  
 372 stress, the classical result holds:

$$q = q' \quad (16)$$

373 When presenting the results in a  $(q - p)$  plane,  $\sigma^{cap}$  is simply deduced by making the  
 374 horizontal projection of the total stress state at failure on the Mohr-Coulomb line ob-  
 375 tained in saturated conditions (see figure 3). In this last case, we have:

$$\boxed{\sigma^{cap} = p - p'} \quad (17)$$

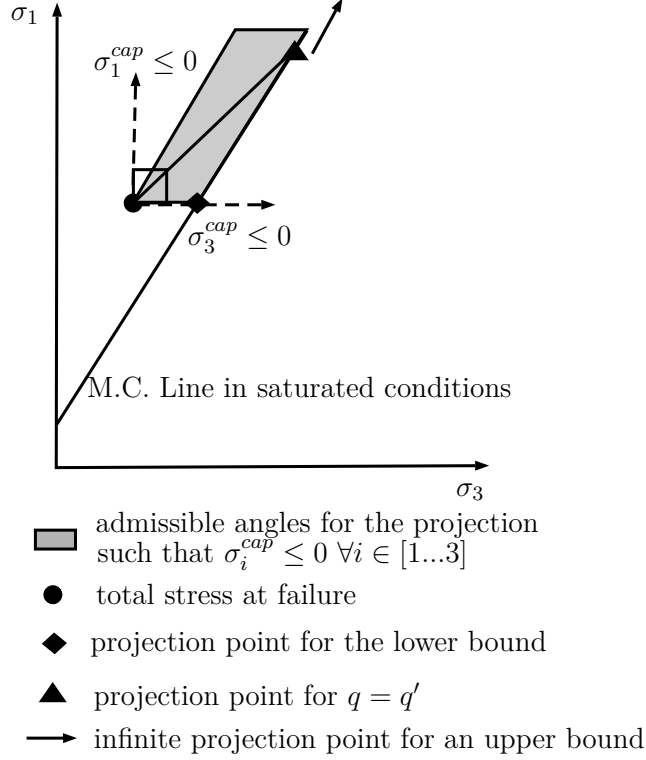
376 The problem becomes more complicated when we want to consider a possible non-  
 377 spherical capillary stress tensor. In that case,  $q \neq q'$ , we therefore propose a way to find  
 378 a range for the values of the capillary tensor. A first basic assumption is to consider  
 379 that

$$p^{cap} = \frac{1}{3} (\sigma_1^{cap} + 2\sigma_3^{cap}) \leq 0 \quad (18)$$

380 In fact, experimental evidence, as well as physical models developed for moistened gran-  
 381 ular media prove that  $p^{cap} < 0$  in the capillary regime (near the saturation regime). For  
 382 example, [Rumpf \(1962\)](#) proposed the following expression :

$$p^{cap} = -\frac{6}{\phi} \frac{S_r}{e} \frac{\gamma_{aw}}{d_p} \cos(\theta) \quad (19)$$

383 with  $\phi$  a particle shape factor,  $d_p$  the apparent particle diameter and  $\theta$  the the liquid



**Figure 4.** Sketch of possible image point of effective failure stress states under a basic assumption:  $\sigma_i^{cap} \leq 0, \forall i \in [1, 3]$ .

Alt Text:  $(\sigma_1 - \sigma_3)$  plane showing the possible projection on the effective More-Coulomb line to compute the components of the capillary tensor.

384 wetting angle. Consistent with DEM observations, we then assume that the total stress  
 385 tensor and the capillary stress tensor are coaxial in the context of laboratory tests per-  
 386 formed in the orthotropic axes, and we extend the first assumption to every eigenvalue  
 387 of the capillary stress tensor:

$$\sigma_i^{cap} \leq 0, \forall i \in [1, 3] \quad (20)$$

388 Indeed, in the case of the pendular regime (low saturation, no continuous liquid phase),  
 389 each capillary bridge gives rise to an attractive resultant force between two grains. It  
 390 therefore seems natural to consider that the sum of all these actions in a given direction  
 391 provides a negative stress component. For the intermediate funicular regime, things seem  
 392 less natural. Here, we assume a kind of continuity between each regime. We assume that  
 393 a positive capillary stress component is possible only in saturated conditions. This set  
 394 of hypotheses is summarized in figure 4. The value of  $\sigma^{cap}$  is denoted by  $\sigma_{LB}^{cap}$ , which  
 395 ensures a minimal value of  $q'$  on the effective Mohr-Coulomb failure line. In the same  
 396 way the value of  $\sigma^{cap}$  is denoted by  $\sigma_{UB}^{cap}$ , which ensures a maximal value of  $q'$  on the  
 397 effective Mohr-Coulomb failure line.

398 **4.1. A lower bound for  $\sigma^{cap}$**

399 In the absence of any micro-mechanical results covering the full saturation range, a  
400 reasonable lower bound for the capillary stress tensor is obtained as follows:

$$q' \leq q \Leftrightarrow q^{cap} \geq 0 \quad (21)$$

401

$$\Leftrightarrow \sigma_1 - \sigma_1^{cap} - (\sigma_3 - \sigma_3^{cap}) \leq \sigma_1 - \sigma_3 \quad (22)$$

402

$$\Leftrightarrow \sigma_3^{cap} - \sigma_1^{cap} \leq 0 \quad (23)$$

403 as  $\sigma_i^{cap} \leq 0 \forall i$  a maximum value of  $q^{cap}$  is reached for

$$\boxed{\sigma_{1LB}^{cap} = 0} \quad (24)$$

404 By using the Mohr-Coulomb relationship that holds for effective stress, we obtain the  
405 following:

$$\sigma'_1 = \frac{1 + \sin \varphi'}{1 - \sin \varphi'} \sigma'_3 + \frac{2C' \cos \varphi'}{1 - \sin \varphi'} \quad (25)$$

$$\Leftrightarrow \sigma_1 = \frac{1 + \sin \varphi'}{1 - \sin \varphi'} (\sigma_3 - \sigma_{3LB}^{cap}) + \frac{2C' \cos \varphi'}{1 - \sin \varphi'} \quad (26)$$

406 which provides:

$$\boxed{\sigma_{3LB}^{cap} = \sigma_3 + \frac{2C' \cos \varphi'}{1 + \sin \varphi'} - \sigma_1 \frac{1 - \sin \varphi'}{1 + \sin \varphi'}} \quad (27)$$

407 **4.2. An upper bound for  $\sigma^{cap}$**

408 An upper bound cannot be derived in the same way. With the help of figure 4, we can  
409 only conclude that when:

$$q' \geq q \Leftrightarrow q^{cap} \leq 0 \quad (28)$$

410

$$\Leftrightarrow \sigma_1 - \sigma_1^{cap} - (\sigma_3 - \sigma_3^{cap}) \geq \sigma_1 - \sigma_3 \quad (29)$$

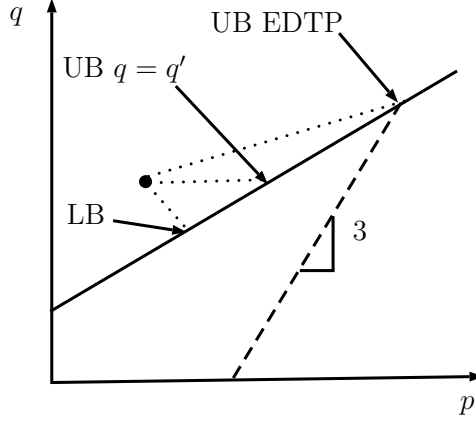
411

$$\Leftrightarrow \sigma_3^{cap} - \sigma_1^{cap} \geq 0 \quad (30)$$

412 and we obtain an infinite minimum value of  $q^{cap}$  for:

$$\sigma_1^{cap} = -\infty \quad (31)$$

413 Therefore, micromechanical considerations about the particle size distribution, the  
414 maximum particle distance acceptable for a capillary bridge, and acceptable statistics



**Figure 5.** Range of effective stresses at failure to assess the capillary stress at failure. [LB]=lower bound, [UB  $q=q'$ ]=upper bound assuming a spherical capillary stress, [UB EDTP]=upper bound assuming an effective drained triaxial path.

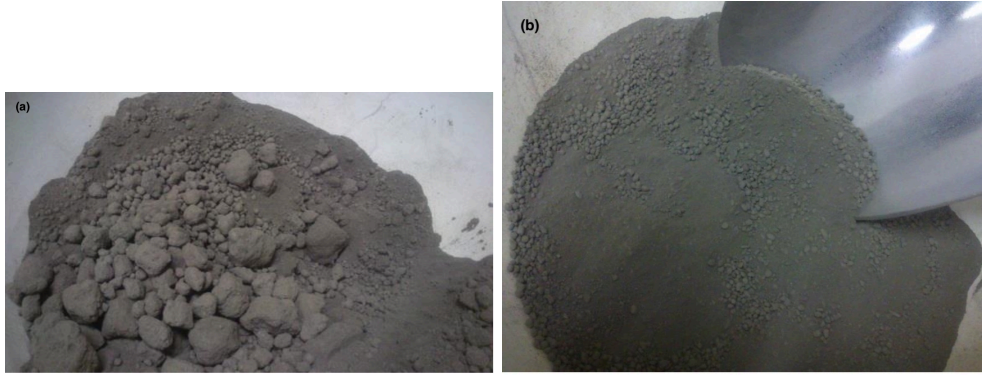
Alt Text:  $(q - p)$  plane showing the three assumed projections on the effective Mohr-Coulomb line to compute the components of the capillary tensor.

415 of the capillary bridge distribution seem to be necessary to derive such an upper  
 416 bound. However, this is beyond the scope of the present work. Facing such difficulty, a  
 417 conservative argument can be proposed in engineering practice. The spherical capillary  
 418 tensor can be used as an upper bound of the capillary effects, since beyond this limit,  
 419  $q' > q$  provides an increasing strength for the soil sample.

420

421 Finally, we present a last assumption to derive an upper bound for  $\sigma^{cap}$ . In the study  
 422 by Duriez et al. (2018), in which several DEM simulations of unsaturated drained tri-  
 423 axial tests under constant suction were performed, it was observed that the response in  
 424 terms of the effective stress path is also a drained triaxial path. This observation needs  
 425 to be confirmed by future research, but if we consider this to be true in all saturation  
 426 regimes, then the next procedure can be used. First, an upper bound for the capillary  
 427 pressure has to be assessed at the end of the isotropic phase. In engineering practice, we  
 428 can assume that the initial value of the suction  $s_0$  before any isotropic compression is  
 429 an upper bound of the capillary pressure  $p_0^{cap}$  at the end of the isotropic compression.  
 430 In fact,  $\text{Tr}(\sigma^{cap})/3 = p^{cap} \leq s$  (e.g. Arai et al., 2012), and we can assume in a first  
 431 approach that  $s$  is not excessively affected by a drained isotropic compression. From  
 432 this initial state, the effective failure state is obtained in a classical manner with the  
 433 intersection of the Mohr-Coulomb envelope and the straight line of slope 3 in a  $(q, p)$   
 434 plane. In figure 5, we present the range of the possible image points that can be used  
 435 on the effective Mohr-Coulomb envelope to assess the effective stress at failure, and  
 436 therefore the capillary stress at failure. In concluding this section, we make the following  
 437 observation. When performing three drained triaxial paths under a separate confining  
 438 pressure with a controlled suction on an unsaturated soil, the failure points can  
 439 sometimes be aligned in a  $(q - p)$  plane or in the Mohr's plane. However, the apparent  
 440 cohesion and the apparent friction angle expressed in total stress have no meaning from  
 441 a constitutive point of view. Indeed, the components of the capillary tensor cannot be  
 442 measured and are distinct in the three tests. This is exactly the same as measuring  
 443 an undrained cohesion  $C_u$  when performing an unconsolidated and undrained test  
 444 (U-U) on a saturated soil sample.  $C_u$  has no meanings from a constitutive point of view.





**Figure 6.** (a) Natural material of Gavet. (b) Passing fraction at 2 mm studied in this work. According to Hoang (2017).

Alt Text: 2 photographs of the Gavet silt. On the left the natural material. On the right the material with only particles smaller than 2 mm.

445

446 Finally, if we could impose the same capillary tensor for all three tests, the expressions  
 447 for its components could be given in terms of apparent cohesion, effective cohesion, and  
 448 effective friction angle, as presented in the Appendix 7. However, these formulas are  
 449 useless in practice.

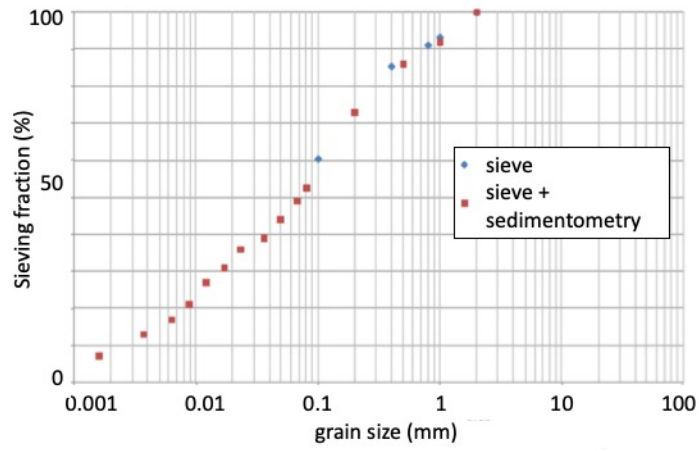
## 450 5. Experimental results analysis

451 In this section, we present the results of the drained triaxial tests that were performed  
 452 on silt from Gavet, a village in the French Alps. A first campaign of characterization  
 453 tests was carried out in the work of Hoang (2017). We then completed these tests with  
 454 a series of unsaturated triaxial tests with measured initial suction.

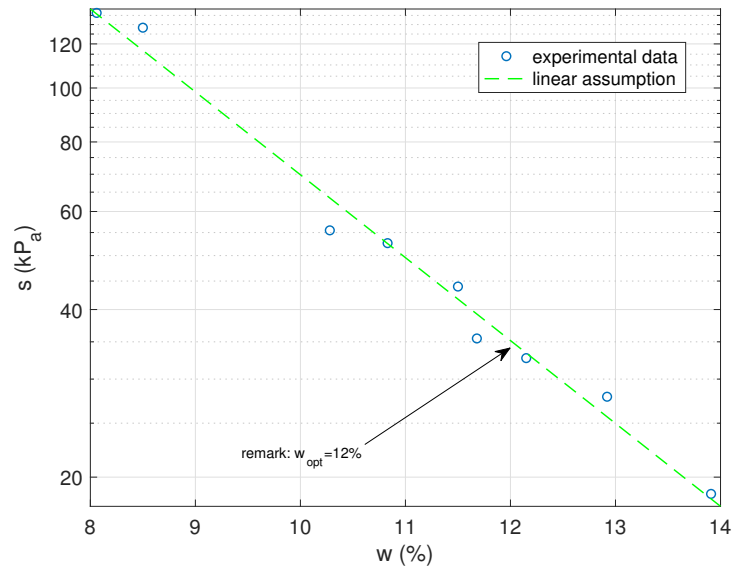
### 455 5.1. Summary of the results from Hoang (2017)

456 A qualitative aspect of the studied material is presented in figure 6. The soil fraction  
 457 finer than 20 mm is given in figure 7. In the present study, only particles smaller than  
 458 2 mm were taken into account in the experimental campaign (see figure 6 (b)). From  
 459 the dry state in powder form, the samples were moistened and subjected to Proctor  
 460 compaction tests under 25 shots. During these tests, suction was measured precisely with  
 461 the filter paper method to determine the suction levels of the samples after compaction.  
 462 This made it possible to plot the graph of the suction versus the [mass water content](#)  
 463 ( $w = M_w/M_s$ ), as presented in figure 8. A series of saturated drained triaxial tests was  
 464 then performed. The samples were first compacted with the Proctor procedure under an  
 465 average [mass water content](#) of  $w = 10.5\%$ , while the optimum Proctor was measured  
 466 at  $w = 12\%$ . This permits to warrant an initial suction level about  $55\text{ kPa}$  and samples  
 467 easy to shape. This led to an initial void ratio  $e \approx 0.5$ . The samples were then saturated  
 468 inside the triaxial device.

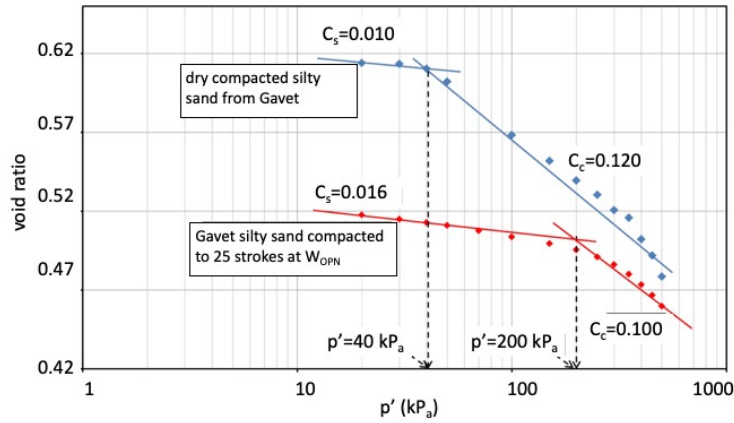
469 A first series of isotropic compression was performed. The results are presented in the  
 470  $(e, p')$  curve (figure 9). This enables us to estimate the pre-consolidation pressure at  
 471 approximately  $200\text{ kPa}$  for the compaction protocol proposed above. Next, a series



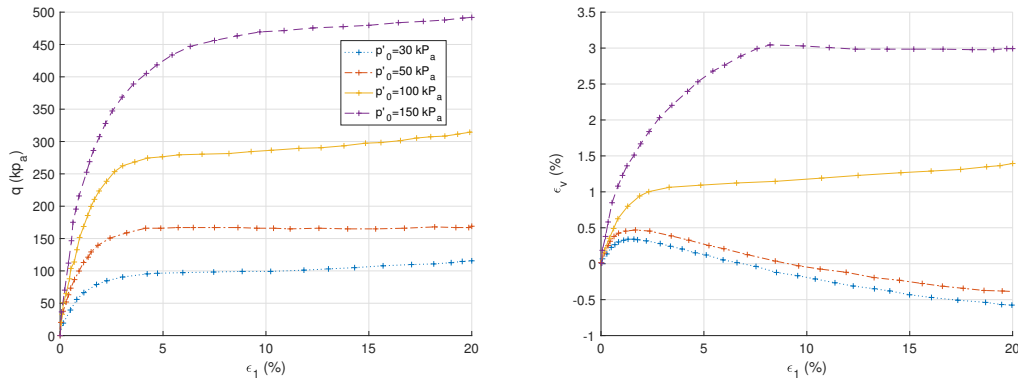
**Figure 7.** Particle size curve of the Gavet silt: fraction under 20 mm. According to Hoang (2017).  
 Alt Text: Particle size curve of the Gavet silt



**Figure 8.** Evolution of the suction with the mass water content after Proctor tests. According to Hoang (2017).  
 Alt Text: Retention curve of the Gavet Silt showing the evolution of the suction as a function of the mass water content.



**Figure 9.** Evolution of the void ratio with the effective isotropic pressure - according to Hoang (2017). In the present paper, the compaction at the optimum Proctor is retained (red curve).  
 Alt Text:  $(e - p')$  curve showing the isotropic response of the Gavet silt.



**Figure 10.** Response of the material under saturated drained triaxial tests - according to Hoang (2017).  
 Alt Text:  $(q - \varepsilon_1)$  curves and  $(\varepsilon_v - \varepsilon_1)$  curves showing the responses of the Gavet silt on drained triaxial test at  $30 \text{ kPa}$ ,  $50 \text{ kPa}$ ,  $100 \text{ kPa}$  and  $150 \text{ kPa}$

472 of four drained triaxial tests was carried out. The initial confining pressures were as  
 473 follows:  $30 \text{ kPa}$ ,  $50 \text{ kPa}$ ,  $100 \text{ kPa}$  and  $150 \text{ kPa}$ . Responses in the planes  $(q, \varepsilon_1)$  and  
 474  $(\varepsilon_v, \varepsilon_1)$  are presented in figure 10. Although all four samples were expected to be in  
 475 the overconsolidation range, the response of the samples under initial consolidation  
 476 pressures of  $100 \text{ kPa}$  and  $150 \text{ kPa}$  indicated a contractive response.

## 477 **5.2. Description of the unsaturated triaxial tests**

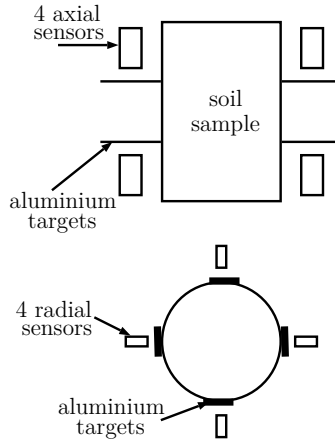
478 After the compaction procedure, the samples at the CBR mold dimensions were packed  
 479 in a plastic film and stored in a standard room in our laboratory for at least 10 days  
 480 in order to be sure that the suction became stable. This delay was empirically observed  
 481 in the study by Hoang (2017). The samples were then shaped with a cutting device,  
 482 as indicated in figure 11. The initial dimensions and characteristics of the samples are  
 483 summarized in table 1. The samples were then placed in a standard triaxial cell. We



**Figure 11.** Cutting device for the soil samples.  
 Alt Text: Photograph of the cutting device of the soil samples.

$\phi$ (mm)	$h$ (mm)	$w$ (%)	$\rho_s$ ( $g/cm^3$ )
54.0	112	10.5	2.72
$\rho_d$ ( $g/cm^3$ )	$e$	$S_r$ (%)	$s$ (kPa)
1.82	0.494	57.7	55.0

**Table 1.** Main dimensions and characteristics of the soil samples.



**Figure 12.** Sketch of the measurement device  
 Alt Text: Sketch showing the position of each contactless sensors.

484 used local inductive sensors to measure the deformations. In fact, the volume change  
 485 cannot be measured with the pore water content changes. A sketch of the measurement  
 486 device is presented in figure 12, and a photograph is presented in figure 13.

### 487 **5.3. Results of the unsaturated triaxial tests**

488 A series of three drained triaxial compressions were performed for three distinct confin-  
 489 ing pressures (in total stress): a simple compression at  $p_0 = 0$ , followed by two confined  
 490 compressions at  $p_0 = 100 \text{ kPa}$  and  $p_0 = 150 \text{ kPa}$ . Pictures of the state of the samples  
 491 after being dismantled are presented shown in figure 14. We can clearly see the effect of  
 492 the confining pressure on the failure mode. For the simple compression, a sub-vertical  
 493 band is observed with an inclination of approximately  $70^\circ$ . The inclination of the shear  
 494 band is approximately  $60^\circ$  for the test at  $100 \text{ kPa}$ , while no macroscopic shear band is  
 495 visible for the test at  $150 \text{ kPa}$ . The responses of this soil in terms of deviatoric stress and  
 496 volume strain are presented in figure 15. During these tests, the initial suction before  
 497 the isotropic compression was approximately  $55 \text{ kPa}$ . The isotropic compression and the  
 498 deviatoric compression were then performed in drained conditions without control or  
 499 measurement of the suction. It was observed that the mass water content remained con-  
 500 stant during the tests. Indeed, measurements of the mass water content on the samples  
 501 dismantled at the end of the tests revealed no water loss.

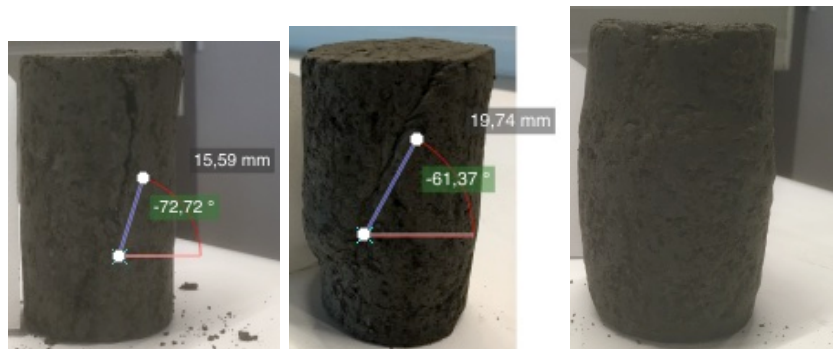
### 502 **5.4. Interpretation of the results**

503 First, we should indicate that we were unable to assess the value of the capillary pressure  
 504 ( $p^{cap} = tr(\boldsymbol{\sigma}^{cap})/3$ ) at the end of the isotropic compression in an objective manner with  
 505 our standard experimental device. Consequently, we focus our work on the assessment  
 506 of the capillary tensor at the failure state. To do so, it is convenient to indicate the  
 507 failure points obtained in saturated and unsaturated conditions in the  $(q - p)$  plane  
 508 (see figure 16). It should be noted that in the present work, we adopt the standard  
 509 definition of failure given by the elastoplasticity theory, i.e. the maximum stress that  
 510 the elastoplastic solid can sustain and not the residual stress after the peak, for the  
 511 following reasons. After the peak, the mathematical solution to the problem posed in



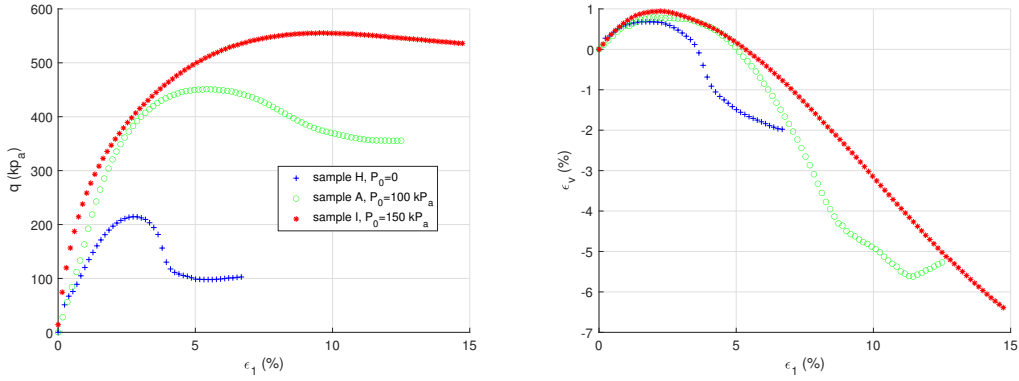
**Figure 13.** Picture of the experimental device.

Alt Text: 2 photographs of the triaxial device. On the left with the cell. On the right the cell is removed to show every sensors on the sample.



**Figure 14.** Pictures of the silt samples after the unsaturated drained triaxial tests: (a) simple compression, shear band angle  $\approx 70^\circ$ , (b) test at  $100 \text{ kPa}$ , shear band angle  $\approx 60^\circ$ , (c) test at  $150 \text{ kPa}$ , no visible macroscopic shear band.

Alt Text: 3 photographs of the 3 samples dismantled at the end of the 3 triaxial trials. The first two samples show a shear band while the third one no.



**Figure 15.** Response of the material under unsaturated drained triaxial tests. The initial suction before the isotropic compression was approximately  $55 \text{ kPa}$ . The isotropic compression and the deviatoric compression were then performed in drained conditions without control or measurement of the suction.  
 Alt Text:  $(q - \varepsilon_1)$  and  $(\varepsilon_v - \varepsilon_1)$  curves showing the response of the Gavet silt to the unsaturated drained triaxial tests at 0, 100  $\text{kPa}$  and 150  $\text{kPa}$

$p_0 = 0 \text{ kPa}$	$p_0 = 100 \text{ kPa}$	$p_0 = 150 \text{ kPa}$
$p_f^{cap} = -64 \text{ kPa}$	$p_f^{cap} = -40 \text{ kPa}$	$p_f^{cap} = -26 \text{ kPa}$

**Table 2.** Results in terms of capillary pressure at failure, using the assumption of a spherical capillary tensor. The initial suction is  $s_0 \approx 55 \text{ kPa}$ , for the three tests.

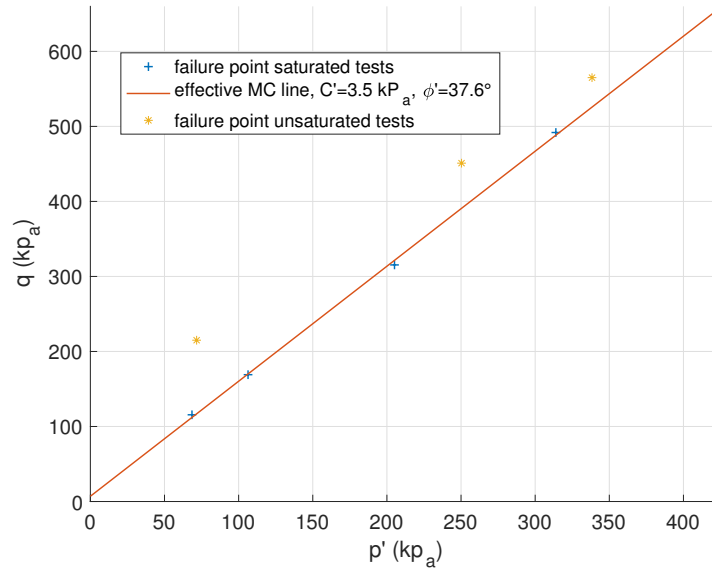
512 the elastoplasticity framework is no longer unique. From a physical point of view, this  
 513 translates into a post-peak response that is highly sensitive to the slightest defects in the  
 514 sample or boundary conditions. Hence, we consider an analysis by working on samples  
 515 deemed to be homogeneous. This approach aims to ensure the consistency of our study  
 516 and minimize variables that could influence our conclusions.

517 In this figure, we did not link the failure points relating to unsaturated tests, since  
 518 these failure points are not comparable from a constitutive point of view. In fact, their  
 519 respective capillary tensors cannot be measured or controlled. Consequently, it is not  
 520 possible to derive a cohesion and a friction angle expressed in total stress from these  
 521 points. To estimate the capillary tensor that holds at failure for each of the three tests,  
 522 we used the hypotheses presented in section 4.

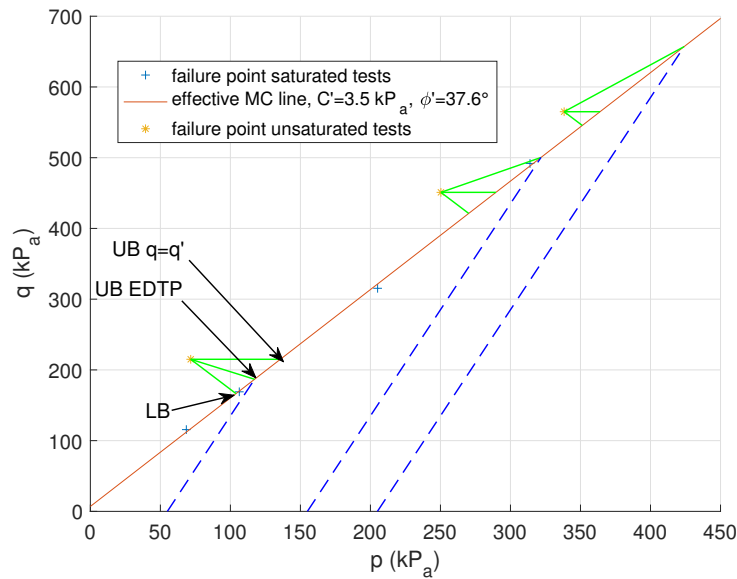
523

524 The most simple approach is to assume a spherical capillary tensor that can be  
 525 used as a conservative upper bound for the capillary stress that holds at failure. In  
 526 such a case,  $q = q'$ , and the effective image point is the horizontal projected point on  
 527 the Mohr-Coulomb envelope. In the present case, the capillary pressures obtained are  
 528 presented in table 2. Next, the lower bound of the capillary stress that holds at failure  
 529 was assessed with the equations (24) and (27), while the upper bound is presented with  
 530 the questionable assumption of a drained triaxial path for the effective stress path, as  
 531 presented in section 4. figure 17 presents the range of the possible image points that  
 532 can be used on the effective Mohr-Coulomb envelope to assess the effective stress at  
 533 failure, and therefore the capillary stress at failure. figure 18 presents the range of the  
 534 acceptable capillary stress at failure. The following observations arise from these results:

- 535 (1) The assumption of a drained triaxial path for the effective stress as a response of  
 536 the unsaturated sample has to be verified in further DEM simulations for a wider

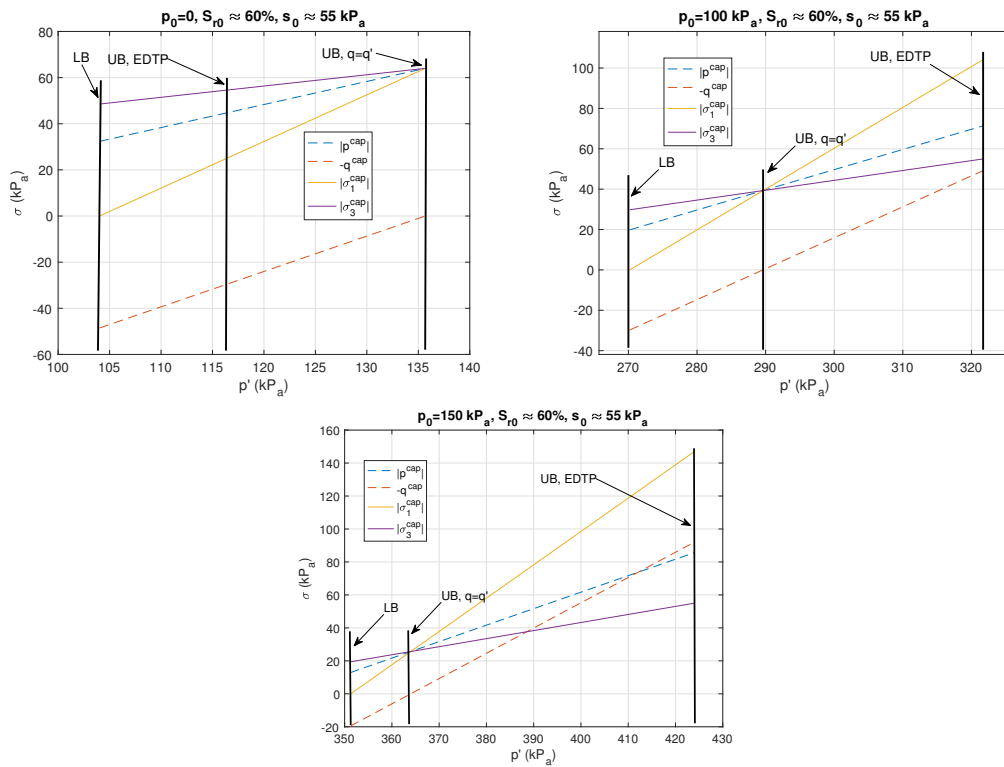


**Figure 16.** Failure points obtained for the saturated drained triaxial tests ( $p'_0 = 30, 50, 100,$  and  $150 \text{ kPa}$ ) and for the unsaturated drained triaxial tests ( $p_0 = 0, 100,$  and  $150 \text{ kPa}$ ).  
 Alt text:  $(q - p)$  plane showing the effective Mohr-Coulomb line and the failure points measured in total stress.



**Figure 17.** Range of effective stresses at failure to assess the capillary stress at failure. [LB]=lower bound, [UB  $q=q'$ ]=upper bound assuming a spherical capillary stress, [UB EDTP]=upper bound assuming an effective drained triaxial path. The dashed lines depict the effective stress path assumed using the upper limit for the initial capillary pressure  $p_0^{capmax} = -s_0 \approx -55 \text{ kPa}$ .  
 Alt Text:  $(q - p)$  plane showing the effective Mohr-Coulomb line and the the failure points measured in total stress as well as green lines that shows the assumed projection procedure to compute the capillary stress components.





**Figure 18.** Range of capillary stresses at failure. [LB]=lower bound, [UB  $q=q'$ ]=upper bound assuming a spherical capillary stress, [UB EDTP]=upper bound assuming an effective drained triaxial path.  
 Alt Text: 3 graphs showing the possible range of the capillary stress components bounded by the assumptions developed in the present work. The 3 graphs show the results for the 3 trials done at 0, 100  $kPa_a$  and 150  $kPa_a$ .

$q'_{0LB}/q'^{sat}_0$	$q'_{100LB}/q'^{sat}_{100}$	$q'_{150LB}/q'^{sat}_{150}$
10.64	1.28	1.13
$q'_{0UB}/q'^{sat}_0$	$q'_{100UB}/q'^{sat}_{100}$	$q'_{150UB}/q'^{sat}_{150}$
15.04	1.38	1.17

**Table 3.** Effective shear strength gain obtained comparing the unsaturated case with the saturated case. [LB]=lower bound. [UB]=upper bound assuming a spherical capillary stress.

537 range of saturation regime and, if possible, in future experimental tests where  
538 proper image analysis is developed for the fluid phases.

539 (2) At present, in view of engineering practice, the proposed lower bound and the  
540 upper bound given by  $q = q'$  seem reasonable enough and are very easy to as-  
541 sess, even with the use of standard triaxial devices. It should be stressed that  
542 the non-spherical lower bound must be considered to ensure that conservative  
543 computations are performed.

544 (3) In the context of our experimental results, we can quantify the gain in shear  
545 strength by considering the effective strength using the non-spherical lower bound  
546 or by avoiding unsaturated effects, as is frequently done in practice. To do so, we  
547 denote  $[q'^{sat}_0, q'^{sat}_{100}, q'^{sat}_{150}]$  as the shear strength when performing saturated drained  
548 triaxial tests for an initial isotropic pressure  $p'_0 = 0, 100 \text{ kPa}$ , and  $150 \text{ kPa}$   
549 and  $[q'_{0LB}, q'_{100LB}, q'_{150LB}]$  the effective shear strength, assessed with the lower  
550 bound assumption, when performing unsaturated drained triaxial tests for an  
551 initial isotropic pressure  $p_0 = 0, 100 \text{ kPa}$ , and  $150 \text{ kPa}$ . In the same way, we  
552 denote  $[q'_{0UB}, q'_{100UB}, q'_{150UB}]$  when the effective shear strength is assessed with  
553 the spherical capillary stress assumption ( $q = q'$ ). The table 3, yields the ratio  
554  $q'_{LB}/q'^{sat}$  and  $q'_{UB}/q'^{sat}$ . It is clear that the gain can be very significant at low  
555 confining pressures and less significant at upper confining pressures. Nevertheless,  
556 it could help to justify some practical analyses in the short term. We emphasize  
557 that considering a computation with the effective soil parameters and introducing  
558 a non-spherical capillary stress can lead to different conclusions than adding the  
559 apparent cohesion in the soil parameters.

## 560 6. Concluding remarks

561 In conclusion, the main results of this work are summarized below.

562 (1) The main difficulty in establishing a constitutive model with an effective stress for-  
563 malism in the unsaturated domain relates to the fact that an unsaturated medium  
564 can be in static equilibrium for geometric configurations that are not acceptable  
565 in the saturated domain. This led us to propose a diagram illustrating the notions  
566 of effective strain (conjugated with its effective stress via the existence of a  
567 potential strain energy for the solid skeleton) and capillary strain resulting from  
568 the liquid-gas interfaces and fluid-solid interfaces (Madeo et al. (2013)). However,  
569 it is easy to observe that, in general, the total strain measured at the boundary  
570 of the sample does not decompose in an additive way into effective and capillary  
571 strains. Moreover, access to an experimental measurement of these two strain ten-  
572 sors remains a challenge at the present time. Nevertheless, the study by Duriez et  
573 al. (2018) tends to confirm that the identification of the total deformation with

574 the effective deformation is possible for compressive small deformations and for  
575 failure states.

576 (2) Based on these conceptual results, we propose an experimental procedure to be  
577 tested in future studies to identify the capillary tensor. It consists of performing  
578 a first desaturation on an initially saturated soil sample to obtain a desired suc-  
579 tion or degree of saturation and then applying a loading path under unsaturated  
580 conditions. On a second identical sample, exactly the same strain path measured  
581 on the first sample is imposed at the boundary of the second sample, but under  
582 saturated conditions. If this strain path can be imposed, the difference between  
583 the total stress path of the first sample and the effective stress path of the second  
584 sample provides the capillary stress response. Otherwise, this means that the total  
585 strain cannot be identified with the effective strain for the unsaturated loading  
586 path considered. To our knowledge, such a procedure has not been tested. In a  
587 second step, we propose certain basic assumptions to evaluate the capillary tensor  
588 at failure. The objective is to identify an effective image point on the failure en-  
589 velope obtained in saturated conditions that is admissible from a constitutive point  
590 of view. The difference between the total stress measured at failure and the effec-  
591 tive stress given by the image point then provides an assessment of the capillary  
592 effects. We first assume that all components of the capillary tensor are negative in  
593 the unsaturated regime. This provides a lower bound of the effective stress image  
594 point. This lower bound is reached for  $q' < q$ . An upper bound such that  $q' > q$   
595 is found to be infinite with the latter assumption. Therefore, we propose using  
596 the assumption of an isotropic capillary tensor to evaluate a conservative upper  
597 bound as a first approach. This is equivalent to considering  $q = q'$ .

598 (3) We carried out a campaign of unsaturated triaxial tests with a standard experi-  
599 mental device and analysed the results at failure, on the basis of the assumptions  
600 given above. For the considered silt under an initial suction level of  $55 kP_a$ , it was  
601 observed, inter alia, that the apparent total strength was 10 times the effective  
602 strength for the compression at  $\sigma_3 = 0$ , while this ratio decreased to 1.3 and 1.1  
603 for compressions at  $\sigma_3 = 100 kP_a$  and  $\sigma_3 = 150 kP_a$ .

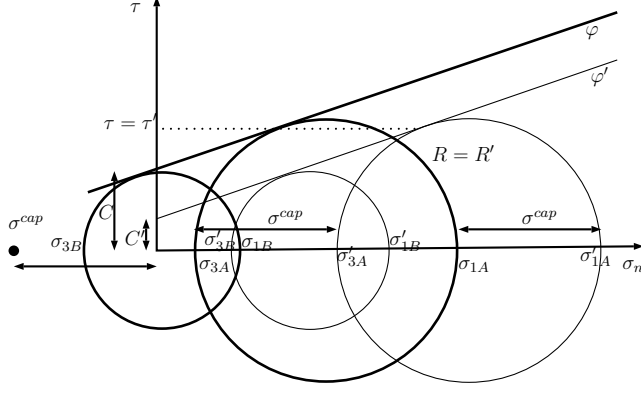
## 604 7. Appendix 1

605 In the next subsections, we propose to develop the mathematical expressions of the  
606 capillary tensor  $\sigma^{cap}$  that holds at failure by adopting a Mohr-Coulomb failure criterion.  
607  $\sigma^{cap}$  is derived from the difference of the total stress  $\sigma$  and of the effective stress  $\sigma'$ ,  
608 assuming that it is possible to experimentally determine two constitutively comparable  
609 states from the unsaturated case and the saturated case.

### 610 7.1. Spherical capillary stress case

611 Let us consider the Mohr diagram in figure 19.  $C'$  and  $\varphi'$  denote the cohesion and the  
612 friction angle measured considering the effective stress with saturated samples, while  $C$   
613 and  $\varphi$  denote the cohesion and the friction angle virtually measured with unsaturated  
614 samples subjected to the same  $\sigma^{cap}$  for distinct lateral stresses  $\sigma_3$ .

615 If we assume a spherical capillary tensor, then  $\sigma^{cap}$  reduces to a single point in the  
616 Mohr diagram and the following relationships hold:



**Figure 19.** Mohr circles considering a spherical capillary tensor. Total stress circles are only translated of the quantity  $\sigma^{cap}$ . Strong lines are used for total stress and thin lines for effective stress.  
 Alt Text: Sketch of Mohr circles considering a spherical capillary tensor for computing the capillary pressure.

$$\tau = \tau' \quad (32)$$

617 and

$$\frac{\sigma_1 - \sigma_3}{2} = \frac{\sigma'_1 - \sigma'_3}{2} \quad (33)$$

618 consequently we have the following result :

$$\boxed{\sigma_{ij}^{cap} = \sigma^{cap} \delta_{ij} \Leftrightarrow R = R'} \quad (34)$$

619 with  $R$  is the radius of the Mohr circle in the unsaturated case and  $R'$  is the radius  
 620 in the saturated case. In fact, it is trivial to prove that  $R = R' \Rightarrow \sigma_1^{cap} = \sigma_3^{cap}$ . In other  
 621 words, the change of the radius of two comparable stress states from a constitutive point  
 622 of view provides a measure of the deviator part of the capillary tensor.

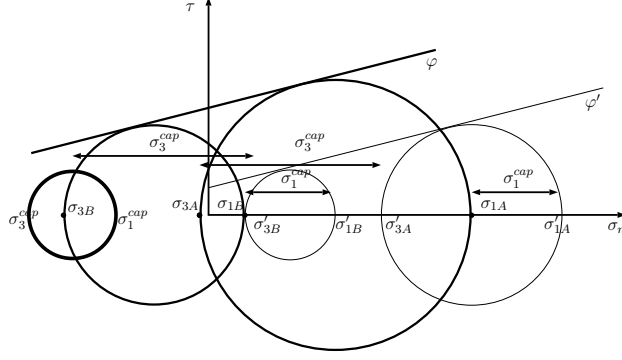
623 Furthermore,

$$\frac{\sigma_1 - \sigma_3}{2} = \frac{\tau}{\cos \varphi} = \frac{\tau'}{\cos \varphi'} \Rightarrow \varphi = \varphi' \quad (35)$$

624 Thus, in the spherical case, we get only a pure horizontal translation of the failure  
 625 envelope. For soils, the translation of the Mohr-Coulomb line is given by the relationship

$$\tan \varphi' = -\frac{C - C'}{\sigma^{cap}} \quad (36)$$

626 and we get the classical final result:



**Figure 20.** Mohr circles considering a non-spherical capillary tensor under axisymmetric conditions.  
 Alt Text: Sketch of Mohr circles considering a non spherical capillary tensor under axisymmetric conditions for computing the components of the capillary stress tensor.

$$\boxed{\sigma^{cap} = -\frac{C - C'}{\tan \varphi'}} \quad (37)$$

627 which provides the apparent cohesion of unsaturated soils.

## 628 **7.2. Non-spherical capillary tensor under axisymmetric conditions**

629 In this case  $R \neq R'$ . The Mohr diagram to be considered is given in figure 20. Because  
 630 the capillary tensor is no longer spherical, we have  $\tau \neq \tau'$ .

631 In the first step, we prove that  $\varphi = \varphi'$  remains true. From a qualitative point of view,  
 632  $R \neq R'$  implies a shift plus a dilation of the failure envelop. This geometrical transfor-  
 633 mation keeps  $\varphi = \varphi'$  for the Mohr-Coulomb envelop. The Mohr-Coulomb relationship  
 634 can also be written in terms of  $t = 1/2(\sigma_1 - \sigma_3)$  and  $s = 1/2(\sigma_1 + \sigma_3)$ , as follows:

$$t = s \sin \varphi + C \cos \varphi \quad (38)$$

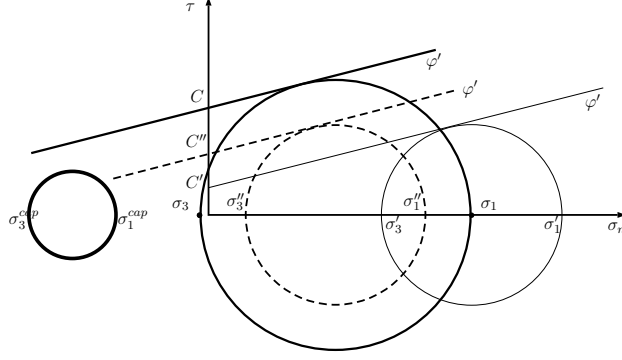
635 Consequently, we get :

$$t'_A + \frac{\sigma_1^{cap} - \sigma_3^{cap}}{2} = \left( s'_A + \frac{\sigma_1^{cap} + \sigma_3^{cap}}{2} \right) \sin \varphi + C \cos \varphi \quad (39)$$

636 and

$$t'_B + \frac{\sigma_1^{cap} - \sigma_3^{cap}}{2} = \left( s'_B + \frac{\sigma_1^{cap} + \sigma_3^{cap}}{2} \right) \sin \varphi + C \cos \varphi \quad (40)$$

637 Making the difference between equations (39) and (40), we get:



**Figure 21.** Mohr circles considering a non-spherical capillary tensor under axisymmetric conditions and the intermediate stress state  $\sigma''$  considered.

Alt Text: Sketch of Mohr circles considering a non spherical capillary tensor under axisymmetric conditions for computing the components of the capillary stress tensor highlighting the intermediate state.

$$t'_A - t'_B = (s'_A - s'_B) \sin \varphi \quad (41)$$

$$\Leftrightarrow (s'_A - s'_B) \sin \varphi' = (s'_A - s'_B) \sin \varphi \quad (42)$$

$$\Leftrightarrow \varphi = \varphi' \quad (43)$$

638 To assess the components of the capillary tensor, we introduce the fictive stress state  
 639  $\sigma''$ , whose Mohr circle is concentric to the one in the unsaturated configuration  
 640 but keeps the radius of the Mohr circle in the saturated configuration. A sketch of this  
 641 fictive stress is displayed in figure 21.

642 Thus, we get

$$\frac{\sigma_1^{cap} + \sigma_3^{cap}}{2} = -\frac{C'' - C'}{\tan \varphi'} \quad (44)$$

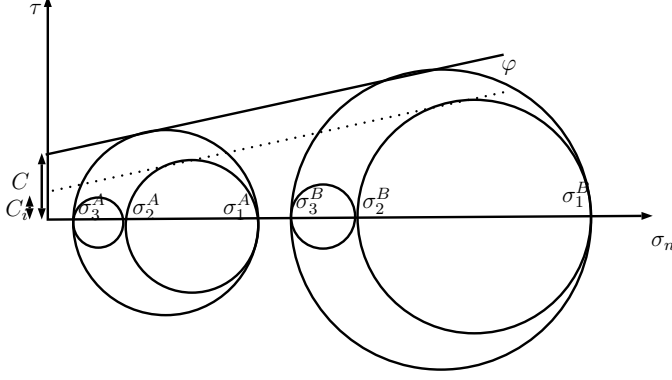
643 and, in virtue of the Mohr-Coulomb relationship, we also have :

$$t' + \frac{\sigma_1^{cap} - \sigma_3^{cap}}{2} = \left( s' + \frac{\sigma_1^{cap} + \sigma_3^{cap}}{2} \right) \sin \varphi' + C \cos \varphi' \quad (45)$$

$$\Leftrightarrow t' + \frac{\sigma_1^{cap} - \sigma_3^{cap}}{2} = s' \sin \varphi' - (C'' - C') \cos \varphi' + C \cos \varphi' \quad (46)$$

$$\Leftrightarrow \frac{\sigma_1^{cap} - \sigma_3^{cap}}{2} = (C - C'') \cos \varphi' \quad (47)$$

644 thus



**Figure 22.** Envelop of the Mohr circle using the major and the intermediate stress under a constant Lode's angle triaxial path.

Alt Text: Sketch of Mohr circles considering a non spherical capillary tensor under 3d conditions for computing the components of the capillary stress tensor.

$$\begin{cases} \frac{\sigma_1^{cap} + \sigma_3^{cap}}{2} = -\frac{C'' - C'}{\tan \varphi'} \\ \frac{\sigma_1^{cap} - \sigma_3^{cap}}{2} = (C - C'') \cos \varphi' \end{cases} \quad (48)$$

645 To conclude this section, equation (48) provides a good estimator of the components  
 646 of the capillary stress tensor that hold at failure, if two failure points under unsaturated  
 647 conditions are observed to be aligned parallel to the effective Mohr-Coulomb line.

### 648 **7.3. Non-spherical capillary stress tensor under general 3D conditions**

649 For the general 3D case, we have to consider a particular true triaxial loading path to be  
 650 able to give an interpretation to the intermediate stress. Let us consider a true drained  
 651 triaxial loading path under constant Lode's angle. Under such a stress path, we get the  
 652 following relationship (see figure 22):

$$\frac{\sigma_1^A - \sigma_2^A}{\sigma_1^A - \sigma_3^A} = \frac{\sigma_1^B - \sigma_2^B}{\sigma_1^B - \sigma_3^B} \quad (49)$$

653 The proof that equation (49)  $\Leftrightarrow \theta^A = \theta^B$  is provided in the appendix 8, where  $\theta^A$   
 654 and  $\theta^B$  are the respective Lode's angles of the stresses  $A$  and  $B$ .

655 Consequently, the Mohr circles made of  $\sigma_1$  and  $\sigma_2$  follows a similar envelop as the  
 656 failure envelop with an expansion factor. For the case of the Mohr-Coulomb envelop,  
 657 it means that the friction angle is maintained. We can introduced an intermediate  
 658 cohesion denoted  $C_i$ , which has no physical meanings but can mathematically describe  
 659 the envelop of the intermediate stress (see figure 22).

660 Using such loading paths, the results obtained in axisymmetrical conditions can be  
 661 extended in a similar manner and lead to the expressions given in equation (50).

$$\left\{ \begin{array}{l} \frac{\sigma_1^{cap} + \sigma_3^{cap}}{2} = -\frac{C'' - C'}{\tan \varphi'} \\ \frac{\sigma_1^{cap} - \sigma_3^{cap}}{2} = (C - C'') \cos \varphi' \\ \frac{\sigma_1^{cap} - \sigma_2^{cap}}{2} = (C_i - C_i'') \cos \varphi' \end{array} \right. \quad (50)$$

662 In practice, the formulae in equation (37), equation (48) and equation (50) may be of  
 663 interest in cases where the experimental conditions allow for the observation of  $\varphi = \varphi'$   
 664 and it has been ensured that  $\sigma_0^{cap}$  at the beginning of the triaxial compression is the  
 665 same for the three tests. Therefore, it seems that  $\sigma_f^{cap}$  at failure is likely to be the same  
 666 for all three tests. Thus, these formulas can be used in a standard way to perform a  
 667 linear regression and to minimize experimental errors.

## 668 8. Appendix 2

669 Proof of :

$$\frac{\sigma_1^A - \sigma_2^A}{\sigma_1^A - \sigma_3^A} = \frac{\sigma_1^B - \sigma_2^B}{\sigma_1^B - \sigma_3^B} \Leftrightarrow \theta^A = \theta^B \quad (51)$$

670 with  $\theta^A$  and  $\theta^B$  the respective Lode's angles of the stresses  $A$  and  $B$ .  
 671 Let us denote :

$$I_1 = \text{tr}(\boldsymbol{\sigma}); \quad \mathbf{s} = \boldsymbol{\sigma} - \frac{I_1}{3}\mathbf{I}; \quad J_2 = \text{tr}(\mathbf{s}^2); \quad J_3 = \text{tr}(\mathbf{s}^3) \quad (52)$$

672 with such notations we have:

$$\cos(3\theta) = \frac{\sqrt{6}J_3}{\sqrt{J_2^3}} = \frac{\sqrt{6}(s_1^3 + s_2^3 + s_3^3)}{\sqrt{(s_1^2 + s_2^2 + s_3^2)^3}} \quad (53)$$

673 for the sake of clarity, we will also denote

$$Q = \frac{\sigma_1^B - \sigma_2^B}{\sigma_1^B - \sigma_3^B} \quad (54)$$

674 consequently we have:

$$\sigma_2^A = \sigma_1^A - Q(\sigma_1^A - \sigma_3^A) \quad (55)$$

675 and after some few algebra we get :



$$\begin{cases} s_1^A = \frac{1+Q}{3} (\sigma_1^A - \sigma_3^A) \\ s_2^A = \frac{1-2Q}{3} (\sigma_1^A - \sigma_3^A) \\ s_3^A = \frac{Q-2}{3} (\sigma_1^A - \sigma_3^A) \end{cases} \quad (56)$$

676 denoting  $C = (\sigma_1^A - \sigma_3^A) / 3$  we get:

$$\cos(3\theta^A) = \frac{\sqrt{6}C^3 [(1+Q)^3 + (1-2Q)^3 + (Q-2)^3]}{C^3 \sqrt{[(1+Q)^2 + (1-2Q)^2 + (Q-2)^2]^3}} \quad (57)$$

677 then it is also easy to show that:

$$\begin{cases} 1+Q = \frac{s_1^B}{(\sigma_1^B - \sigma_3^B)} \\ 1-2Q = \frac{s_2^B}{(\sigma_1^B - \sigma_3^B)} \\ Q-2 = \frac{s_3^B}{(\sigma_1^B - \sigma_3^B)} \end{cases} \quad (58)$$

678 and replacing these expressions into equation (57), we finally get

$$\cos(3\theta^A) = \frac{\sqrt{6} ((s_1^B)^3 + (s_2^B)^3 + (s_3^B)^3)}{\sqrt{((s_1^B)^2 + (s_2^B)^2 + (s_3^B)^2)^3}} = \cos(3\theta^B) \quad (59)$$

## 679 9. Data Availability

680 The authors confirm that the data supporting the findings of this study are available  
681 within the article.

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