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THE SAITO CRITERION AND SOME AVATARS

DANIELE FAENZI, MARCOS JARDIM, AND JEAN VALLÈS

We dedicate this paper to Enrique Arrondo on his 60th birthday.

Abstract: Let k be a field of characteristic zero and $R = k[x_1, \dots, x_n]$ be a graded commutative ring. To any set of $1 \leq s \leq n - 1$ algebraically independent polynomials

$$f_1, \dots, f_s \in R$$

one associates a rank- $(n + 1 - s)$ R -module $\text{Der}(f_1, \dots, f_s)$ of logarithmic derivations; these logarithmic derivations $\delta \in \text{Der}(f_1, \dots, f_s)$ verify

$$\frac{\delta(f_1)}{f_1} = \dots = \frac{\delta(f_s)}{f_s} \in R.$$

The aim of this note is to extend the Saito criterion, which characterizes freeness of the module $\text{Der}(f)$ of tangent vector fields along a reduced divisor $V(f)$, to the module $\text{Der}(f_1, \dots, f_s)$. More precisely, let δ_i be $(n + 1 - s)$ independent derivations in $\text{Der}(f_1, \dots, f_s)$ and $U \subset \text{Der}(f_1, \dots, f_s)$ the sub-module generated by the δ_i . We prove that $U = \text{Der}(f_1, \dots, f_s)$ if and only if $\text{codim}_k V(\bigwedge^{n+1-s} U) > 1$ when $s > 1$ or $V(\bigwedge^n U) = V(f_1)$ when $s = 1$.

1. INTRODUCTION

Let $R = k[x_1, \dots, x_n]$ be a graded commutative ring over a field k of characteristic zero. The partial derivatives $\frac{\partial}{\partial x_i}$ are denoted ∂_i . The R -module Der_R of polynomial derivations is a free rank n module:

$$\text{Der}_R \simeq \bigoplus_{1 \leq i \leq n} R \partial_i.$$

To a square-free polynomial $f \in R$, one associates its module of tangent derivations (or module of tangent vector fields):

$$\text{Der}(f) = \{\delta \in \text{Der}_R \mid \delta(f) \in (f)\},$$

where (f) is the principal ideal generated by f . The module $\text{Der}(f)$ is a rank n reflexive R -module (see [Sai80, Cor. 1.7]) that could also be defined by the canonical exact sequence

$$0 \longrightarrow \text{Der}(f) \longrightarrow \text{Der}(R) \simeq R^n \longrightarrow R/(f)$$

where the last map is

$$\text{Der}(R) \simeq R^n \longrightarrow R/(f), \quad \delta \mapsto \delta(f) \bmod (f).$$

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Remark 1.1. When $V(f)$ is a normal crossing divisor (NCD for short) then $\text{Der}(f)$ is locally free (see [Del70] for instance). However, the converse is not true.

Remark 1.2. The map $\text{Der}(R) \simeq R^n \rightarrow R/(f)$ is the composition of $R^n \rightarrow R$ sending δ on $\delta(f)$ and the quotient map $R \rightarrow R/(f)$ sending a polynomial on its reduction modulo f . The image of the first map is the Jacobian ideal $J \subset R$ generated by the partial derivatives $\partial_i f$. Its reduction modulo (f) is $\frac{J+(f)}{(f)}$; when f is homogeneous, $J+(f) = J$. It defines a subscheme in $V(f)$ supported by the singular locus of $V(f)$. This locus lives in codimension > 1 since f is square free.

Remark 1.3. If f is homogeneous then

$$\delta_E := \sum_i x_i \partial_i \in \text{Der}(f).$$

The derivation δ_E is known as Euler derivation. This induces a splitting

$$\text{Der}(f) = R\delta_E \oplus \text{Der}_0(f) \text{ where } \text{Der}_0(f) = \{\delta \in \text{Der}(R) \mid \delta(f) = 0\};$$

to be precise, if f is homogeneous and $\delta \in \text{Der}(f)$ then $\delta - \frac{1}{\deg(f)} \frac{\delta(f)}{f} \delta_E \in \text{Der}_0(f)$. The module $\text{Der}_0(f)$ is the Syzygy module of $\nabla f = (\partial_1 f, \dots, \partial_n f)$,

$$0 \rightarrow \text{Der}_0(f) \rightarrow R^n \xrightarrow{\nabla f} R$$

Remark 1.4. $\text{Der}(f)$ being reflexive, it must be locally free on the affine plane \mathbb{A}^2 (resp. \mathbb{P}^2).

Remark 1.5. When $V(f)$ is an hyperplane arrangement in \mathbb{P}^n , that is $V(f) = \cup_i H_i$ is a union of $N+1$ distinct hyperplanes in \mathbb{P}^n then $\mathcal{T}_f = \widetilde{\text{Der}(f)}$ is the dual of a Steiner sheaf, more precisely it arises as the kernel of a matrix of linear forms that could be described by the linear relations between the H_i (see [DK93], [FV14], [Arr10], [AO01], etc.):

$$0 \rightarrow \mathcal{T}_f \rightarrow \mathcal{O}(-1)^{N-1} \rightarrow \mathcal{O}^{N-n-1}.$$

As first pointed out by Saito in [Sai80], there are divisors $V(f)$ not NCD such that $\text{Der}(f)$ is a free R -module (meaning that $\text{Der}(f) \simeq R^n$). In the same paper Saito gave a determinantal characterization of freeness, commonly called Saito criterion. The aim of this note is to prove again this criterion (see Theorem 3.3) with basic algebraic arguments in order to generalize it to non reduced divisor (see Theorem 4.1) and overall to finite families of polynomials (see Theorem 6.1). All these avatars of Saito criterion are direct consequences of a elementary statement about reflexive module which is explained in the next section 2.

Except when it is specified, the polynomials considered in this text are not necessarily homogeneous.

2. ALGEBRAIC PRELUDE: KEY PROPOSITION

We begin by a elementary proposition concerning reflexive modules. This proposition will be the key argument in the next proofs of Saito criterion and its generalization.

Proposition 2.1. *Let M be a rank $s \leq n$ reflexive R -module and Φ a monomorphism $M \hookrightarrow R^n$. Let $N = \sum_{1 \leq i \leq s} Re_i$ be a free sub-module of M and $\iota : N \hookrightarrow M$ be the inclusion map. Then*

$$\bigwedge^s(e_1, \dots, e_s) = \bigwedge^s \Phi \circ \iota = \det(\iota) \bigwedge^s \Phi.$$

In particular, the following statements are equivalent:

- (1) $M = N$
- (2) $\det(\iota) \in k^*$
- (3) $V(\bigwedge^s(\Phi \circ \iota)) = V(\bigwedge^s \Phi)$

Proof. This is a consequence of the commutative square:

$$(1) \quad \begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ N & \xlongequal{\quad} & N \\ \downarrow \iota & & \downarrow (e_1, \dots, e_s) \\ 0 \longrightarrow M & \xrightarrow{\Phi} & R^n \end{array}$$

Indeed, N and M being reflexive modules of rank s , $\bigwedge^s N \simeq \bigwedge^s N^{\vee\vee}$ and $\bigwedge^s M \simeq \bigwedge^s M^{\vee\vee}$ are reflexive module of rank one that is free modules and there is a commutative square

$$(2) \quad \begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ \bigwedge^s N & \xlongequal{\quad} & \bigwedge^s N \\ \downarrow \det(\iota) & & \downarrow \bigwedge^s(e_1, \dots, e_s) \\ 0 \longrightarrow \bigwedge^s M & \xrightarrow{\bigwedge^s \Phi} & \bigwedge^s R^n \end{array}$$

showing that

$$\bigwedge^s(e_1, \dots, e_s) = \bigwedge^s(\Phi \circ \iota) = \det(\iota) \bigwedge^s \Phi.$$

This proves our proposition. Let us point out that if $s = n$ then $\det(\Phi) \mid \det(e_1 \dots, e_s)$. \square

3. FREE DIVISORS: SAITO CRITERION

Definition 3.1. The module $\text{Der}(f)$ is free of rank n if and only if there exist n derivations

$$\delta_i = \sum_{1 \leq j \leq n} P_{ij} \partial_j \in \text{Der}(f)$$

such that

$$\text{Der}(f) = \oplus_i R\delta_i.$$

Let us first point out the following crucial fact (see [Sai80, 1.5]):

Lemma 3.2. *Let $\delta_i \in \text{Der}(f)$ be n independent derivations tangent to $V(f)$. Then,*

$$f \mid \det(\delta_1, \dots, \delta_n).$$

Proof. Since δ_i are n independent derivations tangent to $V(f)$ the R -module $\oplus R\delta_i$ is a free sub-module of rank n of $\text{Der}(f)$. Calling $N = \oplus R\delta_i$ and $M = \text{Der}(f)$ we have, according to Remark 1.2, $\det(\Phi) = f$ and according to Proposition 2.1,

$$\det(\Phi) = f \mid \det(\Psi) = \det(\delta_1, \dots, \delta_n)$$

\square

As direct consequence we get the classical Saito criterion (see [Sai80, Theorem (ii), page 270]):

Theorem 3.3. $\text{Der}(f) = \oplus_i R\delta_i$ if and only if $\det(\delta_1, \dots, \delta_n) = uf$, $u \in k^*$.

3.1. First proof of Theorem 3.3.

Proof. Denoting $N = \oplus R\delta_i$ and $M = \text{Der}(f)$ we have, according to Proposition 2.1, $M = N$ if and only if $\det(\iota) = u \in k^*$. \square

3.2. Second proof of Theorem 3.3.

Proof. Let $U = \oplus_{j=1, \dots, n} R\delta_j$ be a free sub-module of $\text{Der}(f)$. We get a commutative diagram:

$$(3) \quad \begin{array}{ccccccc} & & 0 & & & & \\ & & \downarrow & & & & \\ & 0 & \longrightarrow & U & \longrightarrow & R^n & \longrightarrow & K & \longrightarrow & 0 \\ & & & \downarrow & & \parallel & & \downarrow & & \\ & 0 & \longrightarrow & \text{Der}(f) & \longrightarrow & R^n & \longrightarrow & \frac{J+(f)}{(f)} & \longrightarrow & 0 \\ & & & \downarrow & & & & \downarrow & & \\ & & & \text{Der}(f)/U & & & & 0 & & \\ & & & \downarrow & & & & & & \\ & & & 0 & & & & & & \end{array}$$

where the singular locus of the cokernel K is defined by $V(\wedge^n U) = V(\det(\delta_i))$.

Since $\text{Der}(f)$ is reflexive, U is free and $\text{rk}(\text{Der}(f)) = \text{rk}(U)$, then $\text{Der}(f)/U = 0$ when $\text{Der}(f) = U$ or $\text{Der}(f)/U$ is a torsion sheaf when $\text{Der}(f) \neq U$.

— If $U = \text{Der}(f)$ then it is free and $K = \frac{J+(f)}{(f)}$ by the snake lemma. This proves that the singular locus $V(\wedge^n U)$ of K is exactly $V(f)$. Actually, this proves more. It proves that the Jacobian ideal J modulo f coincide with the first Fitting ideal defined by $V(\wedge^{n-1} U)$.

— If $\text{Der}(f) \neq U$ then $\text{Der}(f)/U$ is a torsion sheaf. By the snake lemma we get

$$0 \longrightarrow \text{Der}(f)/U \longrightarrow K \longrightarrow \frac{J+(f)}{(f)} \longrightarrow 0$$

proving that the singular locus of K defined by $V(\det(\delta_i))$ differs from $V(f)$. \square

4. MULTIDERIVATIONS

The previous material can be extend to non reduced divisors and their associated modules of multiderivations. Indeed a multiple polynomial $f = f_1^{m_1} \cdots f_r^{m_r}$ where f_i are square free polynomials of degree $r_i \geq 1$ being given one can define the following sub-module of $\text{Der}(R)$ (see [Zie89]):

$$\text{Der}_{\mathbf{m}}(f) := \{\delta \in \text{Der}_R \mid \delta(f_i) \in (f_i^{m_i}) \ 1 \leq i \leq r\}.$$

This R module which is also reflexive, can be defined in by the canonical exact sequence

$$0 \longrightarrow \text{Der}_{\mathbf{m}}(f) \longrightarrow \text{Der}(R) \simeq R^n \longrightarrow R/(f)$$

where the last map is

$$\mathrm{Der}(R) \simeq R^n \longrightarrow R/(f), \quad \delta \mapsto \delta\left(\prod_i f_i\right) \bmod \left(\prod_i f_i^{m_i}\right).$$

In particular this shows that its rank is n . Moreover, there is a natural inclusion

$$\mathrm{Der}_{\mathbf{m}}(f) \subset \mathrm{Der}\left(\prod_i f_i\right).$$

Saito's criterion is still pertinent for this module, as this is noted in [Zie89]:

Theorem 4.1. $\mathrm{Der}_{\mathbf{m}}(f) = \oplus_i R\delta_i$ if and only if $\det[\delta_1, \dots, \delta_n] = uf$, $u \in k^*$.

Proof. Same proof than Theorem 3.3. □

4.1. Points in \mathbb{P}^1 . Assume now that the polynomials $f_i \in k[x, y]$ are linear forms (i.e. homogeneous of degree 1). Then $V(f)$ consists in r distinct points p_1, \dots, p_r with multiplicity m_1, \dots, m_r .

Lemma 4.2. $\mathrm{Der}(f_1 \cdots f_r) = R[-1] \oplus R[1 - r]$.

Proof. The Euler derivation and the Koszul derivation $\partial_y f \partial_x - \partial_x f \partial_y$ define a free sub-module in $\mathrm{Der}(f_1 \cdots f_r)$. By Theorem 3.3 this sub-module is the whole $\mathrm{Der}(f_1 \cdots f_r)$. □

Since $\mathrm{Der}_{\mathbf{m}}(f)$ is reflexive it must be free on $R = k[x, y]$. Then there are two integers $0 \leq a \leq b$ such that $\mathrm{Der}_{\mathbf{m}}(f) = R[-a] \oplus R[-b]$ and $a + b = \sum_i m_i$.

Since $\mathrm{Der}_{\mathbf{m}}(f) \subset \mathrm{Der}(f_1 \cdots f_r)$ we have an exact sequence:

$$0 \longrightarrow R[-a] \oplus R[-b] \longrightarrow R[-1] \oplus R[1 - r] \longrightarrow \oplus_i R/(f_i^{m_i - 1})$$

5. EXAMPLES OF APPLICATIONS

We give here two kinds of applications, some in the homogeneous (or projective) case and some in the non homogeneous case (or affine). The first case is in some sense easier since the degree of the derivations are or are not the good ones, separating the free from the non free divisors. In the affine case, the criterion appears to be stronger since the degrees are not enough to distinguish which set of derivations is a basis or not.

Example 5.1. The arrangement of $3n + 3$ lines (the reflection arrangement denoted by $\mathcal{A}_3^3(n)$ in [OT92]) on \mathbb{P}^2 defined by the equation

$$f(x, y, z) := xyz(x^n - y^n)(x^n - z^n)(y^n - z^n) = 0$$

is free with exponents $(n + 1, 2n + 1)$. Indeed the derivations

$$\delta_1 = x^{n+1}\partial_x + y^{n+1}\partial_y + z^{n+1}\partial_z \quad \text{and} \quad \delta_2 = x^{2n+1}\partial_x + y^{2n+1}\partial_y + z^{2n+1}\partial_z$$

belong to $\mathrm{Der}(f)$ and

$$\det \begin{bmatrix} x & x^{n+1} & x^{2n+1} \\ y & y^{n+1} & y^{2n+1} \\ z & z^{n+1} & z^{2n+1} \end{bmatrix} = xyz \det \begin{bmatrix} 1 & x^n & x^{2n} \\ 1 & y^n & y^{2n} \\ 1 & z^n & z^{2n} \end{bmatrix} = xyz(x^n - y^n)(x^n - z^n)(y^n - z^n).$$

Let us give two examples now of smooth conics in the affine plane having a basis consisting in polynomials of degree 2 for the first one and of degree one for the other.

Example 5.2. This example is due to Yohan Genzmer. He shows that even if $\text{Der}(f)$ contains a derivation of minimal degree this derivation is not necessarily a generator of $\text{Der}(f)$. Let $f(x, y) = x^2 + y^2 + x$. Then the derivation $2y\partial_x + (2x + 1)\partial_y \in \text{Der}(f)$ in particular it belongs to $\text{Der}_0(f)$ but a basis of $\text{Der}(f)$ is given by

$$\begin{cases} \delta_1 &= 2y(2x + 1)\partial_x + (4y^2 - 1)\partial_y \\ \delta_2 &= x(x + 1)\partial_x + (\frac{y}{2} + yx)\partial_y \end{cases}$$

Indeed $\delta_1(f) = 2yf$, $\delta_2(f) = (2x + 1)f$ and $\det(\delta_1, \delta_2) = f$. Actually the curve $V(f) \subset \mathbb{A}_2$ is free but the curve $V(x^2 + y^2 + xz) \subset \mathbb{P}^2$ is not free.

Example 5.3. The curve $C \cup L = \{z(x^2 + yz) = 0\}$ union of a smooth conic and one tangent line in \mathbb{P}^2 is free with exponents $(1, 1)$. Indeed two derivations in $\text{Der}_0(C \cup L)$ are $z\partial_x - 2x\partial_y$ and $x\partial_x + 4y\partial_y - 2z\partial_z$. and

$$\det \begin{pmatrix} x & z & x \\ y & -2x & 4y \\ z & 0 & -2z \end{pmatrix} = 6z(x^2 + yz).$$

Removing the line $L = \{z = 0\}$ we get

$$0 \longrightarrow \mathcal{T}_{C \cup L} = \mathcal{O}_{\mathbb{P}^2}(-1)^2 \longrightarrow \mathcal{T}_C \longrightarrow \mathcal{O}_L(\alpha) \longrightarrow 0,$$

for some integer α . Then on the affine space $\mathbb{P}^2 \setminus L$, the curve becomes $\{y + x^2 = 0\}$ which is free with two derivations of degree 1:

$$\begin{cases} \delta_1 &= \partial_x - 2x\partial_y \\ \delta_2 &= \frac{x}{2}\partial_x + y\partial_y \end{cases}$$

verifying $\det(\delta_1, \delta_2) = y + x^2$.

6. SAITO CRITERION FOR SEVERAL POLYNOMIALS

Let $S = V(f_1, \dots, f_s)$ be a locally complete intersection of $1 < s \leq n - 1$ reduced hypersurfaces. We denote by $I_S = (f_1, \dots, f_s)$ the ideal defining S . The module of logarithmic derivations associated to S , say $\text{Der}(S)$, consists in derivations δ such that $\delta(g) \in I_S$ for all $g \in \text{Der}(S)$. This is a rank n module; one can verify easily that it contains $\text{Der}(\prod_i f_i)$.

In this section we are interested in the module:

$$\text{Der}(f_1, \dots, f_s) = \{\delta \in \text{Der}(R) = R^n \mid f_j\delta(f_i) = f_i\delta(f_j), 1 \leq i < j \leq s\}.$$

We remark first that this module depends on the choice of the polynomials generating I_S .

Theorem 6.1. *Let $f_1, \dots, f_s \in R$ be $1 < s \leq n - 1$ algebraically independent polynomials. Then,*

- (1) $\text{Der}(f_1, \dots, f_s)$ is a reflexive module of rank $n - s + 1$.
- (2) Let $\delta_1, \dots, \delta_{n+1-s}$ be independent derivations in $\text{Der}(f_1, \dots, f_s)$. Then,

$$\text{Der}(f_1, \dots, f_s) = \oplus_{j=1, \dots, n+1-s} R\delta_j \Leftrightarrow \text{codim}_{k^n} V \left(\bigwedge^{n+1-s} [\delta_1, \dots, \delta_{n+1-s}] \right) > 1.$$

Proof. Let us consider the R -module map:

$$\text{Der}R = R^n \longrightarrow R^{s-1}, \mu \mapsto (f_i\mu(f_1) - f_1\mu(f_i))_{2 \leq i \leq s}.$$

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DANIELE FAENZI. INSTITUT DE MATHÉMATIQUES DE BOURGOGNE, UMR 5584 CNRS, UNIVERSITÉ DE BOURGOGNE, 9 AVENUE ALAIN SAVARY, BP 47870, 21078 DIJON CEDEX, FRANCE
Email address: `daniele.fauenzi@u-bourgogne.fr`

MARCOS JARDIM. UNIVERSIDADE ESTADUAL DE CAMPINAS (UNICAMP), INSTITUTO DE MATEMÁTICA, ESTATÍSTICA E COMPUTAÇÃO CIENTÍFICA (IMECC), DEPARTAMENTO DE MATEMÁTICA, RUA SÉRGIO BUARQUE DE HOLANDA, 651, 13083-859 CAMPINAS-SP, BRAZIL
Email address: `jardim@unicamp.br`

JEAN VALLÈS. UNIVERSITÉ DE PAU ET DES PAYS DE L'ADOUR, LMAP-UMR CNRS 5142, AVENUE DE L'UNIVERSITÉ - BP 1155 - 64013 PAU CEDEX, FRANCE
Email address: `jean.valles@univ-pau.fr`