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Effects of Wariness on Economic Growth in Overlapping Generations Models

Hai-Ha PHAM* Ngoc-Sang PHAM†

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Abstract

We introduce the notion of wariness in overlapping generations models and explore its effects on economic growth. In an exogenous growth model, under standard assumptions, we prove that the capital stock converges to a steady state. We then explore conditions under which this steady state is increasing (or decreasing) in the wariness level. We also provide a necessary and sufficient condition for the dynamic efficiency of the intertemporal equilibrium. In an endogenous growth model à la [Romer \(1986\)](#), we show that the growth rate of capital stock per capita in the economy with wariness is lower (higher, respectively) than that in the economy without wariness if and only if the capital return is high (low, respectively).

Keywords: wariness, overlapping generations, dynamic efficiency, economic growth, endogenous growth.

JEL Classifications: D14, D5, E71, O41.

1 Introduction

Uncertainty plays an important role in our society, including economic activities. An example of uncertainty is the emergence of the COVID-19 pandemic.¹ In a world with uncertainty, some people may worry about the future, prefer to live better now, and enjoy the present. However, others may save more in order to face uncertainties in the future. This behavior concerns the notion of wariness. Motivated by these observations, we aim to investigate the role of wariness in economic growth.

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¹See [Baker et al. \(2016\)](#) and the website <http://www.policyuncertainty.com> for the economic Policy Uncertainty Index, [Jurado et al. \(2015\)](#) for macroeconomic uncertainty index for 12 month horizon, and [Jurado et al. \(2015\)](#), [Ludvigson et al. \(2021\)](#) for financial uncertainty index for 12 month horizon.

To model the wariness, we consider that individuals take care not only of the discounted sum of utilities but also the minimum utility across time. Formally, when a consumer lives for two periods, we assume that her intertemporal utility is given by

$$\text{Intertemporal utility} = \left(u(c) + \beta u(d) \right) + \gamma \min \{ u(c), u(d) \}, \quad (1)$$

where c, d represent the consumer's consumption in the present and the future respectively, β is the rate of time preference. The parameter $\gamma \in [0, \infty)$ represents the wariness of the household. When $\lambda = 0$, we recover the standard case.

In order to understand whether wariness harms or promotes economic development, we embed the modeling (1) of wariness in a standard overlapping generations model (OLG model, henceforth). We then study the role of wariness on the dynamics of capital stocks and the interplay between wariness and economic development.

First, we study how the saving function of the household (with the capital return given) depends on the wariness. Interestingly, we find that the household's saving under the presence of wariness can be higher or lower than that in the case without wariness. It depends on the relationship between the capital return and the wariness level. The intuition is the following. When the capital return is high, the household's income when old would be high while her income when young would be low. So, in the presence of wariness, the household cares more about her consumption when young, which implies that the household consumes more and saves less when young with respect to the case without wariness. By contrast, the household saves more when the capital return is low.

Second, we investigate the dynamics of intertemporal equilibrium to understand how it depends on wariness. Under mild conditions, we prove that the equilibrium capital path converges to the unique steady state and provide comparative statics. We call the stationary capital return R^* the capital return of the economy with stationary consumption (i.e., the consumption when young equals the consumption when old and it does not depend on time). We show that when the wariness is high enough, the steady-state capital stock equals the capital of the economy with stationary consumption. The intuition behind this is that when households really care about the worst consumption over two periods of life, they should choose the same level of consumption when young and old.

The more interesting case is when the wariness level is low. In this case, the steady-state capital stock is increasing (respectively, decreasing) in the wariness level if the stationary capital return is low (respectively, high). Interestingly, the steady-state capital in the economy with wariness may be higher or lower than that in the economy without wariness. It is higher if either (1) the wariness level and the stationary capital return are low or (2) the wariness level is high and the capital return of the economy without wariness is low. We illustrate our theoretical findings with several simulations.

Third, we analyze the effects of wariness in an endogenous growth model à la [Romer \(1986\)](#). We point out that when the wariness level is quite low, the growth rate of capital is increasing (respectively, decreasing) in the wariness level if the productivity is low (respectively, high). However, when the wariness level is high, the growth rate of capital does not depend on the wariness. We also prove that the capital growth rate in the economy with wariness is higher than that in the economy without wariness if and only if the capital return is low.

Our results in both exogenous and endogenous growth models lead to an important implication: Taking into account the worst situation does not necessarily harm economic growth because it may enhance savings and economic growth, particularly in cases of low capital returns.

Related literature

Our paper is related to the literature on endogenous discounting. [Araujo et al. \(2011\)](#), [Araujo et al. \(2016\)](#) consider the utility function of the form $\sum_{t \geq 0} \beta^t u(c_t) + a \inf_{t \geq 0} u(c_t)$ in an infinite-horizon general equilibrium model. However, they focus on the effects of parameter a on asset bubbles while we investigate the growth issue.

[Ha-Huy and Nguyen \(2022\)](#) investigate the optimal capital path of an infinite-horizon model with Ramsey-Rawls criterion where the objective function is $\sum_{t \geq 0} \beta^t u(c_t) + a \inf_{t \geq 0} u(c_t)$. They interpret parameter a as the importance of equity in the inter-temporal generational evaluation in the choice of the economic agent. They provide conditions under which the optimal capital stock is constant over time or it coincides with the solution of the Ramsey problem (i.e., when the parameter $a = 0$). Our paper differs from [Ha-Huy and Nguyen \(2022\)](#) in several points. First, we employ a OLG model instead of a model with an infinitely-lived agent as in [Ha-Huy and Nguyen \(2022\)](#); by consequence, the economic meaning of the wariness in our paper is quite different from that of parameter a in [Ha-Huy and Nguyen \(2022\)](#). Second, and more importantly, the capital path of equilibrium in our paper is never constant over time (excepted the case where the initial capital is at the steady state). Third, we also investigate the issue of endogenous growth while [Ha-Huy and Nguyen \(2022\)](#) do not.

Our paper is also related to the literature on the impact of uncertainty. Several papers document a negative relationship between high uncertainty and firms' investment. [Kumar et al. \(2023\)](#) use a survey of firms in New Zealand and show that firms' perceived macroeconomic uncertainty leads them to reduce their investment. [Bloom et al. \(2022\)](#) use a survey measure of subjective uncertainty collected by the U.S. Census Bureau for approximately 25,000 manufacturing plants and find that there is a strong and robust negative relationship between investment and high subjective uncertainty of plant managers.

[Fukuda \(2008\)](#) uses a OLG model but focuses on the profit maximization of the firm under Knightian uncertainty. He considers that the productivity of firm is random and the producer faces "uncertainty" in its productivity. By analyzing the effects of this uncertainty on the dynamic properties of equilibrium, he finds that there may be multiple steady states and a poverty trap may arise. [Bianchi et al. \(2023\)](#) find sizable negative effects of stochastic changes in the volatility of demand-side (rate of time preference) and supply-side (TFP) shocks on consumption, investment, and output.

Different from these papers focusing on the behavior of firms, we study how the wariness affects the savings of households and then the economic growth. Our novel point is that if uncertainties generate a wariness in the households' preferences, then, in some cases, it may be beneficial to the economy because households may save more and by the way improve the investment and economic growth.

The remainder of the paper is organized as follows. In Section 2, we introduce wariness in an exogenous growth model and provide basic properties of households'

saving. Section 3 presents the effects of wariness in the exogenous growth model while Section 4 discusses the role of wariness in an endogenous growth model à la Romer (1986). Section 5 concludes. Formal proofs are gathered in the appendix section.

2 An OLG Model with wariness

We introduce wariness in a standard two-period OLG model.²

Household and wariness. At period t , N_t individuals are born. We assume that the population growth rate is constant over time and denote $n \equiv N_{t+1}/N_t$. Each consumer-worker lives two periods. When young, she supplies one unit of labor, earns a labor income, consumes c_t and saves s_t . When old, she receives the income from her saving and consumes d_{t+1} .

Assume that the utility of this agent represents by the following function

$$(1 - \lambda)(u(c_t) + \beta u(d_{t+1})) + \lambda \min \{u(c_t), u(d_{t+1})\}$$

where $\lambda \in [0, 1]$ represents the wariness of this individual. When $\lambda = 0$, we recover the standard case. When $\lambda = 1$, the agent only cares about the minimum of her(his) consumption $\min(u(c_t), u(d_{t+1}))$.

Denote $\gamma = \frac{\lambda}{1-\lambda}$. Then, γ varies between 0 and $+\infty$. It also represents the wariness of the household. The maximization problem of household born at date t is given by

$$(P_{c,t}) : \quad \max_{(c_t, d_{t+1}, s_t)} \left[U(c_t, d_t) \equiv u(c_t) + \beta u(d_{t+1}) + \gamma \min(u(c_t), u(d_{t+1})) \right]$$

$$c_t + s_t \leq w_t \tag{2}$$

$$d_{t+1} \leq R_{t+1} s_t \tag{3}$$

$$c_t, d_{t+1}, s_t \geq 0, \tag{4}$$

where R_t represents the capital return between time t and $t + 1$.

We assume that:

Assumption 1. *The function u is twice continuously differentiable, strictly increasing, strictly concave. $u'(0) = \infty$ and $u'(\infty) = 0$.*

Production. Technology is represented by a constant returns to scale production function $F(K, L)$ where K_t and L_t are the aggregate capital and the labor forces.

The firm's profit maximization problem is

$$(P_{f,t}) : \quad \max_{K_t, L_t \geq 0} \left(F(K_t, L_t) - R_t K_t - w_t L_t \right) \tag{5}$$

where R_t and w_t represent the return on capital and the wage rate.

Denote $k_t \equiv K_t/L_t$ denotes the capital intensity, $f(k_t) \equiv F(k_t, 1)$. We require standard assumptions.

Assumption 2. *The function f is twice continuously differentiable, strictly increasing, strictly concave. $f(0) = 0$, $f'(0) = \infty$ and $f'(\infty) = 0$.*

²See de la Croix and Michel (2002) for an introduction of OLG models.

The profit maximization implies that

$$R_t = f'(k_t) \text{ and } w_t = \omega(k_t), \text{ where we denote } \omega(k_t) \equiv f(k_t) - k_t f'(k_t) \quad (6)$$

Under a Cobb-Douglas technology represented by $F(K, L) \equiv AK^\alpha L^{1-\alpha}$ where $A > 0$ represents the total factor productivity. In this case, we have $R_t = \alpha Ak_t^{\alpha-1}$ and $w_t = (1 - \alpha) Ak_t^\alpha$.

2.1 Intertemporal equilibrium

Definition 1. An intertemporal equilibrium is a positive sequence $(R_t, w_t, c_t, d_{t+1}, s_t, K_{t+1}, L_t)_{t \geq 0}$ which satisfies the following conditions:

- (1) given the sequence (R_t, w_t) , the allocation (K_t, L_t) is a solution to the problem $(P_{f,t})$ and the allocation (c_t, s_t, d_{t+1}) is a solution to the problem (P_t) ,
- (2) market clearing conditions:

$$\begin{aligned} \text{physical capital: } & K_{t+1} = N_t s_t \\ \text{labor: } & L_t = N_t \\ \text{consumption good: } & s_t + c_t + d_t/n = f(k_t), \end{aligned}$$

2.2 Properties of the individual saving function

Definition 2. Given $\beta > 0, w > 0, R > 0$, we define $s_\beta(w, R)$ the unique solution of the following equation $u'(w - s) = \beta R u'(Rs)$.

Lemma 1. Let $w > 0, R > 0$ be given. If $\beta > \beta' > 0$, then $s_\beta(w, R) > s_{\beta'}(w, R)$.

Proof. Denote $H(s, \beta) = u'(w - s) - \beta R u'(Rs)$. Then, we have $\frac{\partial H}{\partial s}(s, \beta) = -u''(w - s) - \beta R^2 u''(Rs) > 0$. We also observe that $\frac{\partial H}{\partial \beta}(s, \beta) < 0$. So, $s_\beta(w, R)$ is strictly increasing in β . \square

Denote $\beta_1 \equiv \beta + \gamma, \beta_2 \equiv \frac{\beta}{1+\gamma}$. The following result shows the optimal solution of households.

Proposition 1. The optimal saving of the household problem $(P_{c,t})$ is given by

$$s_t = \begin{cases} s_{\beta_1}(w_t, R_{t+1}) & \text{if } R_{t+1} < \frac{1}{\gamma+\beta} \\ \frac{w_t}{1+R_{t+1}} & \text{if } \frac{1}{\gamma+\beta} \leq R_{t+1} \leq \frac{1+\gamma}{\beta} \\ s_{\beta_2}(w_t, R_{t+1}) & \text{if } R_{t+1} > \frac{1+\gamma}{\beta} \end{cases} \quad (7)$$

Proof. See Appendix A. \square

When there is no wariness, the optimal saving is $s_t = s_\beta(w_t, R_{t+1})$. Since $\beta_1 \geq \beta \geq \beta_2$, Lemma 1 implies that

$$s_{\beta_1}(w_t, R_{t+1}) \geq s_\beta(w_t, R_{t+1}) \geq s_{\beta_2}(w_t, R_{t+1}). \quad (8)$$

It means that the saving of the household under the presence of wariness can be higher or lower than that in the case without wariness. It depends on the relationship between the interest rate R_t and the wariness level γ , as shown in (7).

Let us look at the condition $R_{t+1} < \frac{1}{\gamma+\beta}$ in (7). This condition can be rewritten as $R_{t+1} < \frac{1}{\beta}$ (the capital return is low) and $\gamma < \frac{1}{R_{t+1}} - \beta$ (the wariness is low). So, when the capital return and wariness are low, the saving is $s_{\beta_1}(w_t, R_{t+1})$ which is higher than the saving in the case without wariness $s_{\beta}(w_t, R_{t+1})$. The intuition is that when the capital return is low, the household's expected income when old would be low while her income when young would be high. So, in the presence of low wariness, the household cares more about her consumption when old. This implies that the household consumes less and saves more when young.

A similar interpretation applies for the case $R_{t+1} > \frac{1+\gamma}{\beta}$ (the capital return is high $\beta R_{t+1} - 1 > 0$ and the wariness is low $\gamma < \beta R_{t+1} - 1$).

Condition $\frac{1}{\gamma+\beta} \leq R_{t+1} \leq \frac{1+\gamma}{\beta}$ is equivalent to $\gamma \geq \max(\beta R_{t+1} - 1, 1 - \beta R_{t+1})$ which can be interpreted as the *high wariness*. In this case, the saving equals $\frac{w_t}{1+R_{t+1}}$ at which the consumptions when young and old are the same.

The following result explores the effects of the wariness on the optimal saving.

Proposition 2. *Given $\omega_t > 0$ and $R_{t+1} > 0$. We denote $s_t(\gamma)$ the optimal saving of the household with the wariness level γ . Let $\gamma_1 < \gamma_2$. We have different situations.*

1. If $R_{t+1} < \frac{1}{\gamma_1+\beta}$ then $s_t(\gamma_1) > s_t(\gamma_2)$.
2. If $\frac{1}{\gamma_1+\beta} \leq R_{t+1} \leq \frac{1+\gamma_1}{\beta}$ then $s_t(\gamma_1) = s_t(\gamma_2) = \frac{w_t}{1+R_{t+1}}$
3. If $\frac{1+\gamma_1}{\beta} < R_{t+1}$, then $s_t(\gamma_1) > s_t(\gamma_2)$.

Proof. See Appendix A. □

By comparing βR_{t+1} with 1, we obtain the following result showing the monotonicity of the saving function with respect to the wariness level γ .

Corollary 1. (1) *If $\beta R_{t+1} < 1$ then the optimal saving is increasing in γ for $\gamma > 0$.*
(2) *If $\beta R_{t+1} > 1$ then the optimal saving is decreasing in γ for $\gamma > 0$.*

Proof. See Appendix A. □

In some particular cases, we can explicitly compute the household's saving. Let us firstly consider the logarithmic utility function.

Corollary 2. *If $u(c) = \ln(c)$, then $s_{\beta}(w_t, R_{t+1}) = \frac{\beta w_t}{1+\beta}$, $\forall \beta > 0$. In this case, we obtain the optimal saving*

$$s_t = \begin{cases} \frac{\beta_1 w_t}{1+\beta_1} & \text{if } R_{t+1} < \frac{1}{\gamma+\beta} \\ \frac{w_t}{1+R_{t+1}} & \text{if } \frac{1}{\gamma+\beta} \leq R_{t+1} \leq \frac{1+\gamma}{\beta} \\ \frac{\beta_2 w_t}{1+\beta_2} & \text{if } R_{t+1} > \frac{1+\gamma}{\beta} \end{cases}.$$

When there is no wariness ($\gamma = 0$), we recover the standard function of savings: $s_t = \frac{\beta w_t}{1+\beta}$. Let us denote the optimal allocation when there is no wariness.

$$c_t^o \equiv \frac{w_t}{1+\beta}, \quad d_{t+1}^o \equiv \frac{\beta w_t}{1+\beta} R_{t+1}, \quad s_t^o \equiv \frac{\beta w_t}{1+\beta}. \quad (9)$$

Corollary 2 provides an interesting insight. If the return is high in the sense that $R_{t+1} > \frac{1+\gamma}{\beta}$, then

$$c_t = \frac{1+\gamma}{1+\beta+\gamma} w_t > c_t^o \equiv \frac{w_t}{1+\beta} \quad (10)$$

$$d_{t+1} = \frac{\beta}{1+\beta+\gamma} R_{t+1} w_t < d_{t+1}^o \equiv \frac{\beta}{1+\beta} R_{t+1} w_t \quad (11)$$

$$s_t = \frac{\beta}{1+\beta+\gamma} w_t < s_t^o \equiv \frac{\beta}{1+\beta} R_{t+1} w_t. \quad (12)$$

By contrast, when the return rate is low, in the sense that $R_{t+1} < \frac{1}{\beta+\gamma}$, then $c_t = \frac{1}{1+\beta+\gamma} w_t > d_{t+1} = \frac{\beta+\gamma}{1+\beta+\gamma} R_{t+1} w_t$, $s_t = \frac{\beta+\gamma}{1+\beta+\gamma} w_t$.

When the return rate R_{t+1} takes a middle value, i.e., $\frac{1}{\beta+\gamma} \leq R_{t+1} \leq \frac{1+\gamma}{\beta}$, then households consume the same amount in both date: $c_t = d_{t+1} = w_t \frac{R_{t+1}}{1+R_{t+1}}$, $s_t = \frac{w_t}{1+R_{t+1}}$.

Under the CRRA utility function, we have a similar result.

Corollary 3. *If $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$ for $0 < \sigma$, then, by solving the equations $(w_t - s)^{-\sigma} = \beta R_{t+1} (R_{t+1} s)^{-\sigma}$, we obtain that $s_{\beta}(w_t, R_{t+1}) = \frac{w_t}{1+R_{t+1}^{-\frac{1}{\sigma}+1} \beta^{\frac{-1}{\sigma}}}, \forall \beta$. By consequence, the optimal saving is determined as follows:*

$$s_t = \begin{cases} \frac{w_t}{1+R_{t+1}^{-\frac{1}{\sigma}+1} \beta_1^{\frac{-1}{\sigma}}} & \text{if } R_{t+1} < \frac{1}{\gamma+\beta} \\ \frac{w_t}{1+R_{t+1}} & \text{if } \frac{1}{\gamma+\beta} \leq R_{t+1} \leq \frac{1+\gamma}{\beta} \\ \frac{w_t}{1+R_{t+1}^{-\frac{1}{\sigma}+1} \beta_2^{\frac{-1}{\sigma}}} & \text{if } R_{t+1} > \frac{1+\gamma}{\beta} \end{cases} .$$

3 Wariness and exogenous growth

3.1 Intertemporal equilibrium: convergence and steady state

We now investigate the intertemporal equilibrium. The market clearing condition requires that $nk_{t+1} = s_t$ which becomes

$$nk_{t+1} = s(w_t, R_{t+1}) = \begin{cases} s_{\beta_1}(w_t, R_{t+1}) & \text{if } R_{t+1} < \frac{1}{\gamma+\beta} \\ \frac{w_t}{1+R_{t+1}} & \text{if } \frac{1}{\gamma+\beta} \leq R_{t+1} \leq \frac{1+\gamma}{\beta} \\ s_{\beta_2}(w_t, R_{t+1}) & \text{if } R_{t+1} > \frac{1+\gamma}{\beta} \end{cases} . \quad (13)$$

We are interested in the convergence of (k_t) . To deal with this question, we need intermediate steps.

Lemma 2. *Assume that $kf'(k)$ is increasing on the interval $(0, \infty)$. Given k_t , let k_{t+1} determined by consider the Euler equation $u'(w_t - nk_{t+1}) = \beta_1 R_{t+1} u'(R_{t+1} nk_{t+1})$ and $R_{t+1} = f'(k_{t+1}) < \frac{1}{\gamma+\beta}$. Then k_{t+1} is a strictly increasing function of $w_t = f(k_t) - k_t f'(k_t)$, and so k_t .*

Proof. It suffices to prove that w_t is increasing in k_{t+1} . Taking the derivative of both sides of the equation $u'(w_t - nk_{t+1}) = \beta_1 R_{t+1} u'(R_{t+1} nk_{t+1})$ with respect to k_{t+1} and noting that $R_{t+1} = f'(k_{t+1})$, we have

$$\left(\frac{\partial w_t}{\partial k_{t+1}} - n\right)u''(c_t) = \beta_1 \frac{\partial \left(R_{t+1} u'(R_{t+1} nk_{t+1})\right)}{\partial k_{t+1}}$$

or, equivalently,
$$\frac{\partial w_t}{\partial k_{t+1}} u''(c_t) = nu''(c_t) + \beta_1 \frac{\partial \left(R_{t+1} u'(R_{t+1} nk_{t+1})\right)}{\partial k_{t+1}}$$

Since $u'' < 0$ and $f'(k)u'(nkf'(k))$ is decreasing in k for any $k > 0$, we have $\frac{\partial w_t}{\partial k_{t+1}} > 0$. Note that $w_t = f(k_t) - k_t f'(k_t)$ is increasing in k_t . By consequence, we get that k_{t+1} is an increasing function of k_t . \square

Lemma 3. *Assume that $f'(k)u'(nkf'(k))$ is decreasing in k for any $k > 0$. Then k_{t+1} is a strictly increasing function, denoted by $G(k_t)$, of k_t . By consequence, the capital path (k_t) converges.*

Proof. By using the same argument in Lemma 2, we can prove that k_{t+1} is an increasing function of k_t in each case in the formula (13). \square

TO Add: We must determine the image of the function G to ensure that $a_1 > 0, a_2 > 0$.

Lemma 4. *Assume that $f'(k)u'(nkf'(k))$ is decreasing in k for any $k > 0$. At equilibrium, the dynamics of capital is characterized by the following system*

$$nk_{t+1} = s_\gamma(w_t, R_{t+1}) = \begin{cases} s_{\beta_1}(w_t, R_{t+1}) & \text{if } k_t > a_1 \\ \frac{w_t}{1+R_{t+1}} & \text{if } a_2 \leq k_t \leq a_1 \\ s_{\beta_2}(w_t, R_{t+1}) & \text{if } k_t < a_2 \end{cases} \quad (14)$$

where $a_1 \equiv G^{-1}\left((f')^{-1}\left(\frac{1}{\gamma+\beta}\right)\right)$ and $a_2 \equiv G^{-1}\left((f')^{-1}\left(\frac{1+\gamma}{\beta}\right)\right)$.

Proof. By Lemma 3, we have $k_{t+1} = G(k_t)$ is an increasing function in k_t . The rate of return $R_{t+1} = f'(k_{t+1})$ is an decreasing function in k_{t+1} and then decreasing in k_t . Hence the conditions $R_{t+1} \equiv f'(G(k_t)) < \frac{1}{\gamma+\beta}$ and $R_{t+1} > \frac{1+\gamma}{\beta}$ are respectively equivalent to $k_t > a_1 \equiv G^{-1}\left((f')^{-1}\left(\frac{1}{\gamma+\beta}\right)\right)$ and $k_t < a_2 \equiv G^{-1}\left((f')^{-1}\left(\frac{1+\gamma}{\beta}\right)\right)$. Thus, the system (13) becomes (14). \square

Lemma 5 (finding steady state). *Assume that the function $\frac{w(k)}{k(1+f'(k))}$ is strictly decreasing in k . Let k_i^* , $i = 1, 2, 3$ be satisfied the following equations*

$$\begin{aligned} nk_1^* &= s_{\beta_1}\left(\omega(k_1^*), f'(k_1^*)\right) \\ nk_2^* &= s_{\beta_2}\left(\omega(k_2^*), f'(k_2^*)\right) \\ nk_3^* &= \frac{\omega(k_3^*)}{1+f'(k_3^*)}. \end{aligned}$$

We have that: (1) $f'(k_3^*) < \frac{1}{\gamma+\beta}$ if and only if $f'(k_1^*) < \frac{1}{\gamma+\beta}$, and (2) $f'(k_3^*) > \frac{1+\gamma}{\beta}$ if and only if $f'(k_2^*) > \frac{1+\gamma}{\beta}$.

Proof. See Appendix A. □

The level k_1^* and k_2^* represent the capital steady state of the economy with discount rate β_1 and β_2 respectively. k_3^* is the capital steady state of the economy where the consumptions of young and old generations are the same.

We now state the main result of this section, which is a direct consequence of Lemmas 2-5.

Proposition 3. *Suppose that $f'(k)u'(nkf'(k))$ is decreasing in k for all $k > 0$. Then k_{t+1} is an increasing function, denoted by $G(k_t)$, of k_t . By consequence, the capital path (k_t) monotonically converges.*

Assume, in addition, that the function $\frac{w(k)}{k(1+f'(k))}$ is strictly decreasing in k . Then, the steady state k^ is determined as follows.*

1. *If $f'(k_3^*) < \frac{1}{\gamma+\beta}$, then k^* is a solution to the equation $nk = s_{\beta_1}(\omega(k), f'(k))$.*
2. *If $f'(k_3^*) > \frac{1+\gamma}{\beta}$, then k^* is a solution to the equation $nk = s_{\beta_2}(\omega(k), f'(k))$.*
3. *If $\frac{1+\gamma}{\beta} \geq f'(k_3^*) \geq \frac{1}{\gamma+\beta}$, then $k^* = k_3^*$, where k_3^* is the unique solution to the equation $n = \frac{w(k)}{k(1+f'(k))}$.³*

Proposition 3 provides a general condition for the global convergence of the equilibrium capital stock. It should be notice that assumptions in Proposition 3 are satisfied in standard specifications. Indeed, if the function $cu'(c)$ is increasing in c or $kf'(k)$ is increasing in k , then $f'(k)u'(nkf'(k))$ is decreasing in k .⁴

Moreover, if $f(k) = Ak^\alpha$, then $\frac{w(k)}{k(1+f'(k))} = \frac{(1-\alpha)Ak^{\alpha-1}}{1+\alpha Ak^{\alpha-1}}$, which is strictly increasing in k . Since k_3^* is the solution to the equation $\frac{w(k)}{k(1+f'(k))} = n$, we have that $n = \frac{(1-\alpha)A(k_3^*)^{\alpha-1}}{1+f'(k_3^*)}$, i.e., $f'(k_3^*) = \frac{\alpha n}{1-\alpha-\alpha n}$. Therefore, we see that

$$\frac{1+\gamma}{\beta} \geq f'(k_3^*) \geq \frac{1}{\gamma+\beta} \iff \frac{\beta}{1+\gamma} + 1 \leq \frac{1-\alpha}{\alpha n} \leq 1+\gamma+\beta$$

3.2 Effects of wariness on the equilibrium capital path

In the following, we study the effect of wariness on the equilibrium capital path and the steady state.

Definition 3. *Denote k^s the steady state in the standard economy without wariness, i.e., k^s is determined by*

$$nk^s = s(\omega(k^s), f'(k^s)).$$

³It should be noticed that k_3^* does not depend on γ .

⁴Observe that $\frac{\partial(f'(k)u'(nkf'(k)))}{\partial k} = f''(k)u'(nkf'(k)) + f'(k)u''(nkf'(k))n(f'(k) + kf''(k))$. This is negative if $f'(k) + kf''(k) > 0$. Moreover, if $u'(c) + cu''(c) > 0, \forall c$, then we have

$$\frac{\partial(f'(k)u'(nkf'(k)))}{\partial k} < u''(nkf'(k))nf'(k)(-kf''(k) + f'(k) + kf''(k)) < 0.$$

Following de la Croix and Michel (2002) (Proposition 1.11), we introduce an additional assumption.

Assumption 3. For each $\beta > 0$, the function $\frac{s_\beta(\omega(k), f'(k))}{k}$ is strictly decreasing in k and

$$\lim_{k \rightarrow \infty} \frac{s_\beta(\omega(k), f'(k))}{k} < n < \lim_{k \rightarrow 0} \frac{s_\beta(\omega(k), f'(k))}{k} \quad (15)$$

Since $s \leq \omega(k) \leq f(k)$ and $\lim_{k \rightarrow \infty} \frac{f(k)}{k} < n$, the first inequality in (15) always holds. It should also be noticed that Assumption 3 holds under standard specifications. Indeed, when $f(k) = Ak^\alpha$, $\alpha \in (0, 1)$ and $u(c) = \ln(c)$, we have $\frac{s_\beta(\omega(k), f'(k))}{k} = \frac{\beta(1-\alpha)Ak^{\alpha-1}}{1+\beta}$, which decreases from ∞ to 0 when k increases from 0 to ∞ . When $f(k) = Ak^\alpha$, $\alpha \in (0, 1)$ and $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$ with $\sigma > 0$, we have $\frac{s_\beta(\omega(k), f'(k))}{k} = \frac{(1-\alpha)Ak^{\alpha-1}}{1+(\alpha Ak^{\alpha-1})^{-\frac{1}{\sigma}} + 1\beta^{-\frac{1}{\sigma}}}$, which decreases from ∞ to 0 when k increases from 0 to ∞ .

Under Assumption 3, we have the following property.

Lemma 6. Let $\beta > 0$ be given. Under Assumption 3, then there exists a unique k satisfying $n = \frac{s_\beta(\omega(k), f'(k))}{k}$. Moreover, such a k is increasing in β .

We are now ready to state the main result of this section, which provides a full comparative statics showing the effect of the wariness on the capital steady state.

Proposition 4. Let assumptions in Proposition 3 be satisfied.

1. If $f'(k_3^*) < \frac{1}{\gamma+\beta}$ and Assumption 3 holds, then the capital path converges to $k^* = k_1^*$ which is the unique solution to the equation $nk = s_{\beta_1}(\omega(k), f'(k))$. Moreover, k_1^* is strictly increasing in the wariness level γ and $k_1^* > k^s$.
2. If $f'(k_3^*) > \frac{1+\gamma}{\beta}$ and Assumption 3 holds, then the capital path converges to $k^* = k_2^*$ which is the unique solution to the equation $nk = s_{\beta_2}(\omega(k), f'(k))$. Moreover, k_2^* is strictly decreasing in the wariness level γ and $k_2^* < k^s$.
3. If $\frac{1+\gamma}{\beta} \geq f'(k_3^*) \geq \frac{1}{\gamma+\beta}$, then the capital path converges to $k^* = k_3^*$, where k_3^* is the unique solution to the equation $n = \frac{w(k)}{k(1+f'(k))}$.
Moreover, $k_3^* > k^s$ if and only if the interest rate of the economy without wariness is low (in the sense that $\beta f'(k^s) < 1$)

Proof. 1. If $f'(k_3^*) < \frac{1}{\gamma+\beta}$, then k^* is a solution to the equation $nk = s_{\beta_1}(\omega(k), f'(k))$. By Lemma 6, we get that $k^* = k_1^*$ is increasing in β_1 and hence in γ .

2. If $f'(k_3^*) > \frac{1+\gamma}{\beta}$, we use the same argument as in point 1.

3. Denote $H(k) \equiv \frac{w(k)}{k(1+f'(k))}$. Since $H(k)$ is strictly decreasing and $H(k_3^*) = n$, condition $k_3^* > k^s$ is equivalent to $H(k_s) > H(k_3^*) = n$, i.e., $w(k^s) - nk^s > f'(k^s)nk^s$. This is equivalent to $u'(\omega(k^s) - s) < u'(f'(k^s)s)$, i.e., $\beta f'(k^s) < 1$. \square

First, condition $f'(k_3^*) < \frac{1}{\gamma+\beta}$ can be rewritten as $\beta f'(k_3^*) < 1$ and $\gamma < \frac{1}{f'(k_3^*)} - \beta$, which can be interpreted as *low capital return* and *low wariness* respectively. Second, condition $f'(k_3^*) > \frac{1+\gamma}{\beta}$ is equivalent to $0 < \gamma < \beta f'(k_3^*) - 1$, which can be interpreted as *high capital return* and *low wariness*. Note that k_3^* is the capital return of the economy with stationary consumption (i.e., the consumption when young equals the consumption when old and it does not depend on time).

Therefore, Proposition 4 indicates that under low wariness and low capital return (respectively, high capital return), the steady state capital stock is increasing (respectively, decreasing) in the wariness level.

However, if the wariness level is high enough (in the sense that $\frac{1+\gamma}{\beta} \geq f'(k_3^*) \geq \frac{1}{\gamma+\beta}$, or equivalently, $\gamma \geq \max\left(\beta f'(k_3^*) - 1, \frac{1-\beta f'(k_3^*)}{\beta f'(k_3^*)}\right)$), then the steady state capital stock equals k_3^* which does not depend on the wariness level. The intuition behind this is that when households really care about the worst consumption over two periods of life, they should choose the same level of consumption when young and old.

Notice that k_3^* is lower than the steady state k^s of the economy without wariness if and only if the interest rate of the economy without wariness is high (in the sense that $\beta f'(k^s) > 1$).

Proposition 4 leads to a direct consequence which compares the steady states of the economies with and without wariness.

Corollary 4. 1. $k^* \geq k^s$ if and only if one of the two following conditions hold:

$$(1) f'(k_3^*) < \frac{1}{\gamma+\beta}; (2) \frac{1+\gamma}{\beta} \geq f'(k_3^*) \geq \frac{1}{\gamma+\beta} \text{ and } \beta f'(k^s) \leq 1.$$

2. $k^* < k^s$ if and only if one of the two following conditions hold: (1) $f'(k_3^*) > \frac{1+\gamma}{\beta}$;

$$(2) \frac{1+\gamma}{\beta} \geq f'(k_3^*) \geq \frac{1}{\gamma+\beta} \text{ and } \beta f'(k^s) > 1.$$

Condition $\beta f'(k^s) > 1$ (respectively, $\beta f'(k^s) < 1$) can be named as "high (respectively, low) real interest rate condition". Indeed, we can write $\beta = \frac{1}{1+\rho}$ where ρ is the subjective interest rate, and $f'(k^s) = 1 + r_s$ where r_s represents the real capital return at the steady state of the economy without wariness. By consequence, $\beta f'(k^s) > 1$ becomes $\frac{1+r_s}{1+\rho} > 1$, or equivalently, $r_s > \rho$.

3.2.1 Tractable cases

In this section, we consider some tractable cases where we can explicitly find the equilibrium capital path and compute the steady state. Denote $\sigma_1 \equiv \frac{1}{1+\beta+\gamma}$, $\sigma_2 \equiv \frac{1+\gamma}{1+\beta+\gamma}$. Observe that σ_1 is decreasing but σ_2 is increasing in γ . When there is no wariness $\gamma = 0$, we have $\sigma_1 = \sigma_2 = \frac{1}{1+\beta}$.

Corollary 5. Assume that $f(k) = Ak^\alpha$ and $u(c) = \ln c$. At equilibrium, the dynamics of capital is characterized as follows:

$$nk_{t+1} = \begin{cases} (1 - \sigma_2)(1 - \alpha)Ak_t^\alpha & \text{if } k_t < a_2 \\ (1 - \sigma_1)(1 - \alpha)Ak_t^\alpha & \text{if } k_t > a_1 \end{cases} \quad (16)$$

$$nk_{t+1} + n\alpha Ak_{t+1}^\alpha = (1 - \alpha)Ak_t^\alpha \text{ if } k_t \in [a_2, a_1] \quad (17)$$

where $a_i \equiv \left(\frac{n \left(\frac{\alpha A (1 - \sigma_i)}{\sigma_i} \right)^{\frac{1}{1-\alpha}}}{(1 - \sigma_i)(1 - \alpha)A} \right)^{\frac{1}{\alpha}}$ for $i = 1, 2$.⁵

The unique steady state is determined by

$$k^* = \begin{cases} k_1^* \equiv \left(\frac{(1 - \sigma_1)(1 - \alpha)A}{n} \right)^{\frac{1}{1-\alpha}} & \text{if } \sigma_1 \geq \frac{\alpha n}{1 - \alpha}, \text{ i.e., } \frac{1 - \alpha}{\alpha n} \geq 1 + \beta + \gamma \\ k_3^* \equiv \left(\frac{(1 - \alpha - n\alpha)A}{n} \right)^{\frac{1}{1-\alpha}} & \text{if } \sigma_1 > \frac{\alpha n}{1 - \alpha} > \sigma_2, \text{ i.e., } 1 + \frac{\beta}{1 + \gamma} < \frac{1 - \alpha}{\alpha n} < 1 + \beta + \gamma \\ k_2^* \equiv \left(\frac{(1 - \sigma_2)(1 - \alpha)A}{n} \right)^{\frac{1}{1-\alpha}} & \text{if } \sigma_2 \leq \frac{\alpha n}{1 - \alpha}, \text{ i.e., } \frac{1 - \alpha}{\alpha n} \leq 1 + \frac{\beta}{1 + \gamma} \end{cases} \quad (18)$$

Proof. This is just a direct consequence of Lemma 4. See Appendix A for detailed computations. \square

The following graphic illustrates the relationship $k_{t+1} = G(k_t)$.

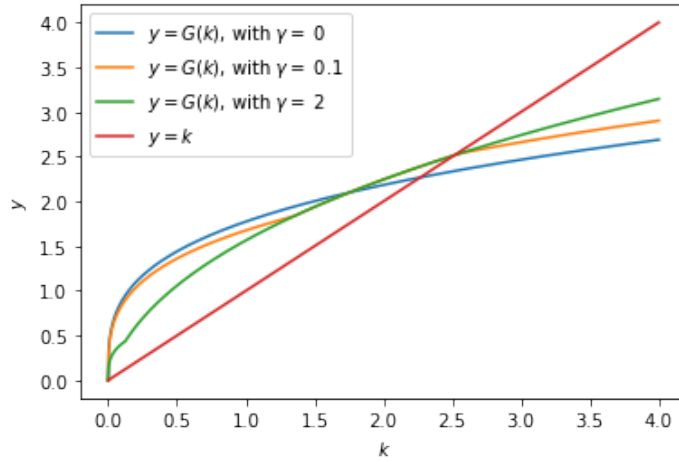


Figure 1: $G(k)$ with different wariness levels for $A = 8, \alpha = 0.3, \beta = 0.7, n = 1.3$.

When there is no wariness (i.e., $\gamma = 0$), we have $\sigma_1 = \sigma_2 = \sigma = \frac{1}{1 + \beta}$ and hence we recover the standard system $nk_{t+1} = \frac{\beta}{1 + \beta}(1 - \alpha)Ak_t^\alpha$. In this case, we have

$$\lim_{t \rightarrow \infty} k_t = k^s \equiv \left(\frac{\beta(1 - \alpha)A}{n(1 + \beta)} \right)^{\frac{1}{1-\alpha}}. \quad (19)$$

We can verify that condition $k_3^* < k^s$, i.e., $\frac{1 - \alpha}{\alpha n} < 1 + \beta$ is equivalent to $\beta f'(k^s) > 1$.⁶

Proposition 5 (comparative statics). *Assume that $f(k) = Ak^\alpha$ and $u(c) = \ln c$.*

1. When γ is low: $\gamma \leq \frac{1 - \alpha}{\alpha n} - (1 + \beta)$, or, equivalently, $\beta f'(k^s) \leq \frac{1 + \beta}{1 + \beta + \gamma}$ then the capital path (k_t) converges to k_1^* which is higher than k^s .

Notice that $\beta f'(k^s) \leq \frac{1 + \beta}{1 + \beta + \gamma}$ requires a low real interest rate, i.e., $\beta f'(k^s) < 1$.

⁵We see that a_i is decreasing in σ_i , for $i = 1, 2$. Therefore, a_1 is increasing but a_2 is decreasing in γ .

⁶Indeed, we observe that $f'(k^s) = \alpha A(k^s)^{\alpha-1} = \alpha A \frac{n(1 + \beta)}{\beta(1 - \alpha)A} = n \frac{\alpha(1 + \beta)}{\beta(1 - \alpha)}$. Thus, $\beta f'(k^s) = \frac{n(1 + \beta)\alpha}{(1 - \alpha)}$.

2. When $\frac{1-\alpha}{\alpha n} \leq 1 + \frac{\beta}{1+\gamma}$, or, equivalently, $\beta f'(k^s) \geq \frac{(1+\beta)(1+\gamma)}{1+\beta+\gamma}$, the capital path (k_t) converges to k_2^* which is lower than k^s .

Here, we need that $\beta f'(k^s) > 1$.

3. When $1 + \frac{\beta}{1+\gamma} < \frac{1-\alpha}{\alpha n} < 1 + \beta + \gamma$, or, equivalently,

$$\frac{1+\beta}{1+\beta+\gamma} < \beta f'(k^s) < \frac{(1+\beta)(1+\gamma)}{1+\beta+\gamma} \quad (20)$$

the capital path (k_t) converges to k_3^* . Moreover,

$$k_3^* \geq k^s \Leftrightarrow \frac{\alpha n}{1-\alpha} \leq \frac{1}{1+\beta} \Leftrightarrow \beta f'(k^s) \leq 1. \quad (21)$$

Note that we need $\frac{\beta}{1+\gamma} - \beta = \beta \frac{\gamma}{1+\gamma} < \frac{1-\alpha}{\alpha n} - (1+\beta) < \gamma$.

Proof. k_2^* is decreasing in γ because it is decreasing in σ_2 and σ_2 is increasing in γ . k_1^* is increasing in γ because it is decreasing in σ_1 and σ_1 is decreasing in γ . \square

Figure 2 shows the capital path for different values of the wariness level.

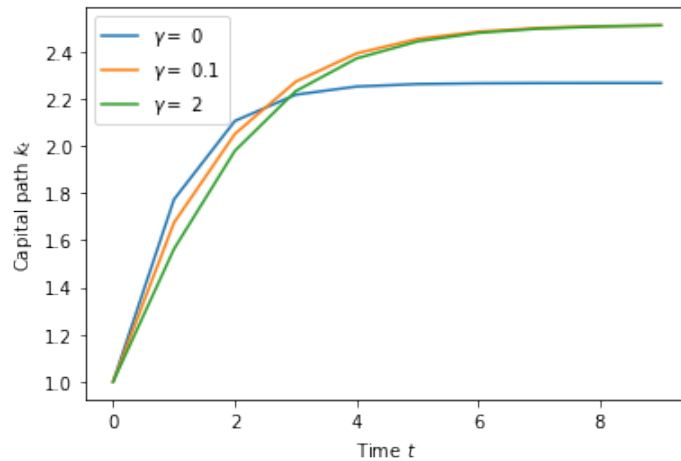


Figure 2: Dynamic capital with different wariness levels γ for $\beta = 0.7, \alpha = 0.3, n = 1.3, A = 8$.

At the steady state, $k^* \geq k^s$ if and only if one of the two following conditions hold: (1) $\gamma \leq \frac{1-\alpha}{\alpha n} - (1+\beta)$; (2) $1 + \frac{\beta}{1+\gamma} < \frac{1-\alpha}{\alpha n} < 1 + \beta + \gamma$ and $\frac{1-\alpha}{\alpha n} \geq 1 + \beta$. Condition (2) is equivalent to $1 + \beta \leq \frac{1-\alpha}{\alpha n} < 1 + \beta + \gamma$. By consequence, we obtain the following result.

Corollary 6. Assume that $f(k) = Ak^\alpha$ and $u(c) = \ln c$. We have that: $k^* \geq k^s$ if and only if $1 + \beta \leq \frac{1-\alpha}{\alpha n}$, or equivalently $\beta f'(k^s) \leq 1$.

Figure 3 illustrates how the physical capital at the steady state depends on the wariness level γ . On the right hand side, we have $1 + \beta > \frac{1-\alpha}{\alpha n}$ and $k^* < k^s, \forall \gamma > 0$. On the left hand side, we have $1 + \beta < \frac{1-\alpha}{\alpha n}$ and $k^* > k^s, \forall \gamma > 0$.

When we work with a CRRA utility function, we have the following result which is a direct consequence of Propositions 3 and 4.

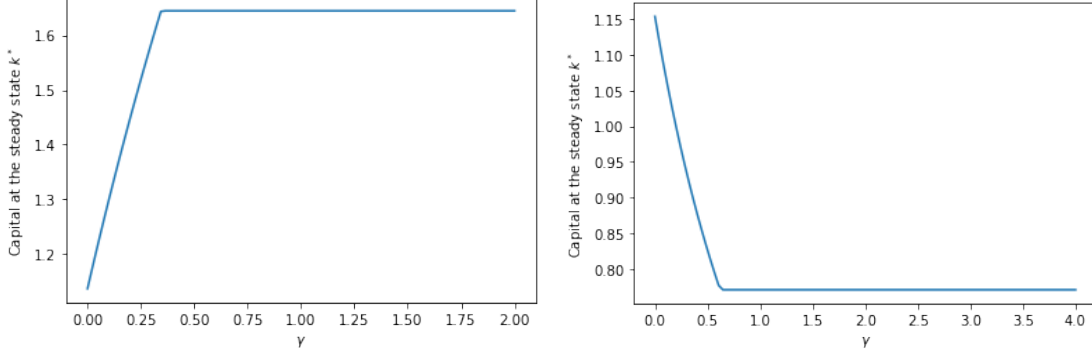


Figure 3: The capital at the steady state versus wariness. On the left: $A = 5, n = 1.2, \beta = 0.6, \alpha = 0.3$. On the right: $A = 5, n = 1.6, \beta = 0.9, \alpha = 0.3$.

Corollary 7. *If $f(k) = Ak^\alpha$ with $0 < \alpha < 1$, and $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$ with $0 < \sigma \neq 1$, then at equilibrium, the dynamics of capital is characterized as follows*

$$nk_{t+1} = s_\gamma(w_t, R_{t+1}) = \begin{cases} \frac{(1-\alpha)Ak_t^\alpha}{1+(\alpha Ak_{t+1}^{\alpha-1})^{-\frac{1}{\sigma}+1}\beta_1^{\frac{-1}{\sigma}}} & \text{if } k_t > a_1 \\ \frac{(1-\alpha)Ak_t^\alpha}{1+\alpha Ak_{t+1}^{\alpha-1}} & \text{if } a_2 \leq k_t \leq a_1 \\ \frac{(1-\alpha)Ak_t^\alpha}{1+(\alpha Ak_{t+1}^{\alpha-1})^{-\frac{1}{\sigma}+1}\beta_2^{\frac{-1}{\sigma}}} & \text{if } k_t < a_2 \end{cases} \quad (22)$$

where $a_i \equiv \left(\frac{n \left(\frac{\alpha A (1-\sigma_i)}{\sigma_i} \right)^{\frac{1}{1-\alpha}}}{(1-\sigma_i)(1-\alpha)A} \right)^{\frac{1}{\alpha}}$ for $i = 1, 2$.

The unique steady state is determined by

$$k^* = \begin{cases} k_1^* & \text{if } \frac{1-\alpha}{\alpha n} > 1 + \beta + \gamma \\ k_3^* \equiv \left(\frac{(1-\alpha-n\alpha)A}{n} \right)^{\frac{1}{1-\alpha}} & \text{if } 1 + \frac{\beta}{1+\gamma} \leq \frac{1-\alpha}{\alpha n} \leq 1 + \beta + \gamma \\ k_2^* & \text{if } \frac{1-\alpha}{\alpha n} \leq 1 + \frac{\beta}{1+\gamma} \end{cases} \quad (23)$$

where k_i^* is the positive solution of the equation $nk_i^* = \frac{(1-\alpha)A(k_i^*)^\alpha}{1+(\alpha A(k_i^*)^{\alpha-1})^{-\frac{1}{\sigma}+1}\beta_i^{\frac{-1}{\sigma}}}$, for $i = 1, 2$.

1. When $\sigma_1 \equiv \frac{1}{1+\beta+\gamma} > \frac{\alpha n}{1-\alpha}$, i.e., γ is low: $\gamma < \frac{1-\alpha}{\alpha n} - (1 + \beta)$, then the capital path (k_t) converges to k_1^* which is higher than k^s .
2. When $\sigma_2 \leq \frac{\alpha n}{1-\alpha}$, i.e., $\frac{1-\alpha}{\alpha n} \leq 1 + \frac{\beta}{1+\gamma}$, the capital path (k_t) converges to k_2^* which is lower than k^s .
3. When $\sigma_1 < \frac{\alpha n}{1-\alpha} < \sigma_2$, i.e., $1 + \frac{\beta}{1+\gamma} < \frac{1-\alpha}{\alpha n} < 1 + \beta + \gamma$, the capital path (k_t) converges to k_3^* . Moreover,

$$k_3^* \geq k^s \Leftrightarrow \beta f'(k^s) \leq 1 \Leftrightarrow \frac{\alpha n}{1-\alpha} \leq \frac{1}{1+\beta}. \quad (24)$$

By combining 3 cases, we observe that: $k^* \geq k^s$ if and only if $1 + \beta \leq \frac{1-\alpha}{\alpha n}$, or equivalently $\beta f'(k^s) \leq 1$.

3.3 Effect of wariness on dynamic efficiency

Following [Malinvaud \(1953\)](#) and [Cass \(1972\)](#), we define the notion of dynamic efficiency.

Definition 4. 1. A sequence of capital stock (k_t) is a feasible path if $f(k_t) - nk_{t+1} \geq 0, \forall t \geq 0$.

2. A feasible path (k_t) is efficient if there does not exist another feasible path (k'_t) satisfying $f(k'_t) - nk'_{t+1} \geq f(k_t) - nk_{t+1}, \forall t$ and there exists at least $t_0 \geq 0$ with $f(k'_{t_0}) - nk'_{t_0+1} > f(k_{t_0}) - nk_{t_0+1}$.

3. An intertemporal equilibrium is dynamically efficient if its capital path is efficient.

According to the consumption good market clearing condition $nk_{t+1} + c_t + \frac{dt}{n} = f(k_t)$, any equilibrium capital path is feasible. An interesting question is whether it is efficient.

By using the same argument in Proposition 2.4 in [de la Croix and Michel \(2002\)](#), we obtain the following result.

Lemma 7. Under conditions in Proposition 3, an intertemporal equilibrium is dynamically efficient if $f'(k^*) > n$, and dynamically inefficient if $f'(k^*) < n$.

When f' is strictly decreasing and $f'(0) > n > f'(\infty)$, there exists a unique so-called golden rule capital stock $k = k_{GR}$ determined by $f'(k_{GR}) = n$ (note that k_{GR} does not depend on γ). In such a case, the conditions $f'(k^*) > n$ and $f'(k^*) < n$ are respectively equivalent to $k^* < k_{GR}$ and $k^* > k_{GR}$. Hence, we can investigate dynamic efficiency by comparing the steady state k^* with the golden rule capital stock.

The following result is a direct consequence of Proposition 4 and Lemma 7.

Proposition 6. Let assumptions in Proposition 3 be satisfied.

1. If $f'(k_3^*) < \frac{1}{\gamma+\beta}$ and Assumption 3 holds, then the intertemporal equilibrium is dynamically efficient if $f'(k_1^*) > n$ and dynamically inefficient if $f'(k_1^*) < n$.

By consequence, if $f'(k_3^*) < \frac{1}{\gamma+\beta} < n$, then intertemporal equilibrium is dynamically inefficient.

2. If $f'(k_3^*) > \frac{1+\gamma}{\beta}$ and Assumption 3 holds, then the intertemporal equilibrium is dynamically efficient if $f'(k_2^*) > n$ and dynamically inefficient if $f'(k_2^*) < n$.

By consequence, if $f'(k_3^*) > \frac{1+\gamma}{\beta} > n$, then intertemporal equilibrium is dynamically efficient.

3. If $\frac{1+\gamma}{\beta} \geq f'(k_3^*) \geq \frac{1}{\gamma+\beta}$, the intertemporal equilibrium is dynamically efficient if $f'(k_3^*) > n$ and dynamically inefficient if $f'(k_3^*) < n$.

Proposition's point 3 indicates that when the capital return $f'(k_3^*)$ and the wariness have a middle value, the equilibrium is dynamically efficient (inefficient) if there is under-accumulation (over-accumulation) at k_3^* .

Points 1 and 2 are more interesting because k_1^* and k_2^* depend on the wariness level γ_2 . Proposition's point 1 suggests that when the capital return $f'(k_3^*)$ is lower than $\frac{1}{\gamma+\beta}$, the equilibrium is dynamically efficient if there is under-accumulation at k_1^* , i.e., $k_1^* < k_{GR}$. According to Proposition 4, k_1^* is strictly increasing in the wariness level γ .

So, in this case, the equilibrium is more likely to be dynamically efficient if the wariness is low enough. By contrast, when the capital return $f'(k_3^*)$ is higher than $(1 + \gamma)/\beta$, then the equilibrium is more likely to be dynamically efficient if the wariness is quite high.

When $f(k) = Ak^\alpha$, we find that $k_{GR} = \left(\frac{\alpha A}{n}\right)^{\frac{1}{1-\alpha}}$. Therefore, we can explicitly find conditions for the dynamic efficiency.

Corollary 8. *Assume that $f(k) = Ak^\alpha$, $u(c) = \frac{u^{1-\sigma}}{1-\sigma}$, $\forall 0 < \sigma \neq 1$, and $u(c) = \ln(c)$, for $\sigma = 1$. We have that:*

1. *If $\frac{1-\alpha}{n\alpha} > 1 + \gamma + \beta$, then the inter-temporal equilibrium is dynamically efficient if $\frac{1-\alpha}{\alpha} < 1 + n^{1-\frac{1}{\sigma}}\left(\frac{\beta}{1+\gamma}\right)^{-\frac{1}{\sigma}}$ and dynamically inefficient if $\frac{1-\alpha}{\alpha} > 1 + n^{1-\frac{1}{\sigma}}\left(\frac{\beta}{1+\gamma}\right)^{-\frac{1}{\sigma}}$.*
2. *If $\frac{1-\alpha}{n} < 1 + \frac{\beta}{1+\gamma}$, then the inter-temporal equilibrium is dynamically efficient if $\frac{1-\alpha}{\alpha} < 1 + n^{1-\frac{1}{\sigma}}(\beta + \gamma)^{-\frac{1}{\sigma}}$ and dynamically inefficient if $\frac{1-\alpha}{\alpha} > 1 + n^{1-\frac{1}{\sigma}}(\beta + \gamma)^{-\frac{1}{\sigma}}$.*
3. *If $1 + \frac{\beta}{1+\gamma} \leq \frac{1-\alpha}{n} \leq 1 + \gamma + \beta$, then the inter-temporal equilibrium is dynamically efficient if $\alpha > \frac{1}{n+2}$ and inefficient if $\alpha < \frac{1}{n+2}$.*

Proof. See Appendix A. □

4 Wariness and endogenous growth

We now introduce endogenous growth à la Romer (1986). Assume that the production function at date t is $f_t(k) = A_t f(k)$, where the productivity A_t is taken as given by the firm. But, in equilibrium, due to a positive externality, A_t is a function of the aggregate capital k_t : $A_t = A(k_t)$. By consequence, we obtain the dynamical system of (k_t)

$$nk_{t+1} = s_t = \begin{cases} s_{\beta_1}(w_t, R_{t+1}) & \text{if } R_{t+1} < \frac{1}{\gamma+\beta} \\ \frac{w_t}{1+R_{t+1}} & \text{if } \frac{1}{\gamma+\beta} \leq R_{t+1} \leq \frac{1+\gamma}{\beta} \\ s_{\beta_2}(w_t, R_{t+1}) & \text{if } R_{t+1} > \frac{1+\gamma}{\beta} \end{cases}$$

$$= \begin{cases} s_{\beta_1}\left(A(k_t)(f(k_t) - k_t f'(k_t)), A(k_{t+1})f'(k_{t+1}))\right) & \text{if } A(k_{t+1})f'(k_{t+1}) < \frac{1}{\gamma+\beta} \\ \frac{A(k_t)(f(k_t) - k_t f'(k_t))}{1+A(k_{t+1})f'(k_{t+1})} & \text{if } \frac{1}{\gamma+\beta} \leq A(k_{t+1})f'(k_{t+1}) \leq \frac{1+\gamma}{\beta} \\ s_{\beta_2}\left(A(k_t)(f(k_t) - k_t f'(k_t)), A(k_{t+1})f'(k_{t+1}))\right) & \text{if } A(k_{t+1})f'(k_{t+1}) > \frac{1+\gamma}{\beta} \end{cases}.$$

For the sake of simplicity, we consider the logarithmic utility function $u(c) = \ln(c)$ and the Cobb-Douglas production function $f(k) = k^\alpha$. In this case, the wage equals $w_t = (1 - \alpha)A_t k^\alpha$ and the capital return equals $R_t = \alpha A_t k_t^{\alpha-1}$.

According to Lemma 2, the saving of household is given by

$$s_t = \begin{cases} \frac{\beta_1 w_t}{1+\beta_1} & \text{if } R_{t+1} < \frac{1}{\gamma+\beta} \\ \frac{w_t}{1+R_{t+1}} & \text{if } \frac{1}{\gamma+\beta} \leq R_{t+1} \leq \frac{1+\gamma}{\beta} \\ \frac{\beta_2 w_t}{1+\beta_2} & \text{if } R_{t+1} > \frac{1+\gamma}{\beta} \end{cases}$$

where recall that $\beta_1 \equiv \beta + \gamma$, $\beta_2 \equiv \frac{\beta}{1+\gamma}$.

Assume that the endogenous productivity A_t is determined by $A_t = Ak_t^{1-\alpha}$, where $A > 0$ represents the exogenous productivity of our model. Hence, the wage is $w_t = (1-\alpha)Ak_t^{1-\alpha}k_t^\alpha = (1-\alpha)Ak_t$ while the capital return equals $R_t = \alpha Ak_t^{1-\alpha}k_t^{\alpha-1} = \alpha A$, $\forall t$. Therefore, we obtain the following result.

Proposition 7. *Consider an endogenous growth model described above. The rate of growth of the capital stock per capita equals*

$$\frac{k_{t+1}}{k_t} = g(\gamma) \equiv \begin{cases} \frac{\beta_1(1-\alpha)A}{n(1+\beta_1)} & \text{if } \alpha A < \frac{1}{\gamma+\beta} \\ \frac{(1-\alpha)A}{n(1+\alpha A)} & \text{if } \frac{1}{\gamma+\beta} \leq \alpha A \leq \frac{1+\gamma}{\beta} \\ \frac{\beta_2(1-\alpha)A}{n(1+\beta_2)} & \text{if } \alpha A > \frac{1+\gamma}{\beta} \end{cases}. \quad (25)$$

Notice that when there is no wariness ($\gamma = 0$), we recover the standard saving $s_t = \frac{\beta w_t}{1+\beta}$ and the rate of capital growth equals $g_0 \equiv \frac{\beta(1-\alpha)A}{n(1+\beta)}$. So, our result leads to interesting implications.

1. If $(\gamma + \beta)\alpha A < 1$ (the wariness and productivity are quite low), then the growth rate of physical per capital $g(\gamma)$ is increasing in the wariness level γ and $g(\gamma) > g_0$.
2. If $\alpha A > \frac{1+\gamma}{\beta}$ (the wariness is quite low but the productivity is quite high), then the growth rate of physical per capital $g(\gamma)$ is decreasing in the wariness level γ and $g(\gamma) < g_0$.
3. If $\frac{1}{\gamma+\beta} \leq \alpha A \leq \frac{1+\gamma}{\beta}$ (the productivity is moderate and the wariness is high), then $g(\gamma) \geq g_0$ if and only if $\alpha\beta A \leq 1$ (i.e., the capital return is low).

Observe that $g(\gamma) \leq g_0$ if and only if one of the following conditions holds: (1) $\alpha A > \frac{1+\gamma}{\beta}$; (2) $\frac{1}{\gamma+\beta} \leq \alpha A \leq \frac{1+\gamma}{\beta}$ and $\alpha\beta A \geq 1$. The second condition means that $\max(\frac{1}{\gamma+\beta}, \frac{1}{\beta}) = \frac{1}{\beta} \leq \alpha A \leq \frac{1+\gamma}{\beta}$.

Inversely, $g(\gamma) > g_0$ if and only if one of the following conditions holds: (1) $(\gamma + \beta)\alpha A < 1$; (2) $\frac{1}{\gamma+\beta} \leq \alpha A \leq \frac{1+\gamma}{\beta}$ and $\alpha\beta A < 1$. The second condition means that $\frac{1}{\gamma+\beta} \leq \alpha A < \frac{1}{\beta}$.

To sum up, we get the following result:

Corollary 9. (1) *The growth rate of capital stock per capita in the economy with wariness is less than that in the economy without wariness, i.e., $g(\gamma) \leq g_0$ if and only if the capital return is high in the sense that $\alpha\beta A \geq 1$.*

(2) *$g(\gamma) > g_0$ if and only if $\alpha\beta A < 1$ (the capital return is low).*

5 Conclusion

We have investigated the effects of wariness on the economic dynamics of the economy in OLG models. Under standard assumptions of utility and production functions, we have proved the capital stock of intertemporal equilibrium, in the exogenous growth economy, converges to a steady state. We have also studied how this steady state

depends on the wariness. We have shown that whether the effects are positive or negative depend on the structure of the economy, specially the interest rate of the economy without wariness. This insight holds not only for the exogenous growth model but also for the endogenous growth framework à la Romer (1986).

A promising avenue for future research involves extending our work to encompass a stochastic economy with incomplete markets.

A Appendix

Proof of Proposition 1. For the sake of simplicity, we remove the subscript and consider the problem

$$P(w, R) : \max_{(c,d,s)} \left\{ U(c, d) \equiv u(c) + \beta u(d) + \gamma \min(u(c), u(d)) \right\} \quad (26)$$

$$c + s \leq w, \quad d \leq Rs, \quad c, d, s \geq 0, \quad (27)$$

Denote U^* the maximum value of the problem $P(w, R)$. We also observe that

$$U^* = \max \left(\max_{\substack{c+s \leq w, \\ d \leq Rs \\ c \geq d}} \left\{ u(c) + (\beta + \gamma)u(d) \right\}, \max_{\substack{c+s \leq w, \\ d \leq Rs \\ c \leq d}} \left\{ (1 + \gamma) \left(u(c) + \frac{\beta}{1 + \gamma} u(d) \right) \right\} \right). \quad (28)$$

Denote $\beta_1 \equiv \beta + \gamma$, $\beta_2 \equiv \frac{\beta}{(1+\gamma)}$. We have $\beta_1 > \beta > \beta_2$.

Consider the first problem

$$(P_1) : \max_{(c,d,s)} u(c) + \beta_1 u(d) \quad (29)$$

$$c + s \leq w, \quad d \leq Rs, \quad c \geq d \geq 0 \quad (30)$$

In optimal, we must have $c + s = w$ and $d = Rs$. So, solving (P_1) is equivalent to solving the following problem

$$(P'_1) : \max_s u(w - s) + \beta_1 u(Rs) \quad (31)$$

$$w - s \geq Rs \geq 0 \Leftrightarrow 0 \leq s \leq \frac{w}{1 + R} \quad (32)$$

Denote $G(s) \equiv u(w - s) + \beta_1 u(Rs)$. We have $G'(s) = -u'(w - s) + \beta_1 R u'(Rs)$ and $G''(s) = u''(w - s) + \beta_1 R^2 u''(Rs) < 0$.

Observe that $G'(0) = +\infty$ and $G'(\frac{w}{1+R}) = u'(\frac{Rw}{1+R})(\beta_1 R - 1)$. By consequence, we obtain that:

1. If $\beta_1 R \geq 1$, then $c = d = w \frac{R}{1+R}$ and $s = \frac{w}{1+R}$.
2. If $\beta_1 R < 1$, then $s = s_{\beta_1}$ which is a unique solution to the equation $G'(s) = 0$.

Consider the second problem

$$(P_2) : \max_{(c,d,s)} u(c) + \beta_2 u(d), \quad (33)$$

$$c + s \leq w, \quad d \leq Rs, \quad 0 \leq c \leq d. \quad (34)$$

By using the similar argument, the solution of (P_2) is given by

1. If $\beta_2 R \leq 1$, then $c = d = w \frac{R}{1+R}$ and $s = \frac{w}{1+R}$.
2. If $\beta_2 R > 1$, then $s = s_{\beta_2}$ which is a unique solution to the equation $-u'(w-s) + \beta_2 R u'(Rs) = 0$.

We now come back to the original problem.

$$U^* = \max \left(\max_{\substack{c+s \leq w, \\ d \leq Rs \\ c \geq d}} \{u(c) + (\beta + \gamma)u(d)\}, \max_{\substack{c+s \leq w, \\ d \leq Rs \\ c \leq d}} \left\{ (1 + \gamma) \left(u(c) + \frac{\beta}{1 + \gamma} u(d) \right) \right\} \right). \quad (35)$$

We consider three cases.

1. If $\beta_1 R < 1$, then the solution of (P_1) satisfies $s = s_{\beta_1}$. Since $\beta_1 > \beta_2$, we have $\beta_2 R < 1$ which implies that the solution of (P_2) satisfies $c = d = w \frac{R}{1+R}$ and $s = \frac{w}{1+R}$. Note that the allocation $c = d = w \frac{R}{1+R}$ and $s = \frac{w}{1+R}$ is feasible for the problem (P_1) . This follows that the maximum value of the problem (P_1) is higher than $(1 + \gamma) \left(u(w \frac{R}{1+R}) + \frac{\beta}{1+\gamma} u(w \frac{R}{1+R}) \right) = (1 + \gamma + \beta) u(\frac{Rw}{1+R})$. So, the solution of the problem $(P(w, R))$ satisfies $s = s_{\beta_1}$.
2. If $\beta_2 R > 1$, then by using the same argument, we obtain that $s = s_{\beta_2}$.
3. If $\beta_1 R \geq 1 \geq \beta_2 R$, then $c = d = w \frac{R}{1+R}$ and $s = \frac{w}{1+R}$.

□

Proof of Proposition 2. Given γ , denote $\beta_1(\gamma) \equiv \beta + \gamma$, $\beta_2(\gamma) \equiv \frac{\beta}{1+\gamma}$.

Since $0 < \gamma_1 < \gamma_2$, we have $\frac{1}{\gamma_2 + \beta} < \frac{1}{\gamma_1 + \beta} < \frac{1 + \gamma_1}{\beta} < \frac{1 + \gamma_2}{\beta}$. So, we consider 5 cases.

1. If $R_{t+1} < \frac{1}{\gamma_2 + \beta}$ then $R_{t+1} < \frac{1}{\gamma_1 + \beta}$. According to Proposition 1, we have $s_t(\gamma_1) = s_{\beta_1(\gamma_1)}(\omega_t, R_{t+1})$ and $s_t(\gamma_2) = s_{\beta_1(\gamma_2)}(\omega_t, R_{t+1})$. Thanks to Lemma 1, we have

$$s_{\beta_1(\gamma_1)}(\omega_t, R_{t+1}) < s_{\beta_1(\gamma_2)}(\omega_t, R_{t+1}).$$

Hence $s_t(\gamma_1) < s_t(\gamma_2)$.

2. If $\frac{1}{\gamma_2 + \beta} \leq R_{t+1} < \frac{1}{\gamma_1 + \beta}$ then we have $s_t(\gamma_1) = s_{\beta_1(\gamma_1)}(\omega_t, R_{t+1}) < \frac{w_t}{1 + R_{t+1}}$. Since $\frac{1}{\gamma_2 + \beta} \leq R_{t+1} < \frac{1 + \gamma_2}{\beta}$, Proposition 1 implies that $s_t(\gamma_2) = \frac{w_t}{1 + R_{t+1}}$. So $s_t(\gamma_1) < s_t(\gamma_2)$.
3. It is a direct consequence of Proposition 1.
4. If $\frac{1 + \gamma_1}{\beta} < R_{t+1} \leq \frac{1 + \gamma_2}{\beta}$, we have $s_t(\gamma_1) = s_{\beta_2(\gamma_1)}(\omega_t, R_{t+1})$ and $s_t(\gamma_2) = \frac{w_t}{1 + R_{t+1}}$. So, $s_t(\gamma_1) > s_t(\gamma_2)$.
5. If $R_{t+1} > \frac{1 + \gamma_2}{\beta}$, by using the similar argument as point 1, we get that $s_t(\gamma_1) > s_t(\gamma_2)$.

□

Proof of Lemma 5. For the sake of simplicity, we write $s^{(i)}(k_i^*)$ instead of $s_{\beta_i}(\omega(k_i^*), f'(k_i^*))$, for $i = 1, 2$.

1. If $f'(k_1^*) < \frac{1}{\gamma+\beta}$.

By definition of $s^{(1)}(k_1^*)$, we have that

$$u'(w(k_1^*) - s^{(1)}(k_1^*)) = \beta_1 f'(k_1^*) u'(f'(k_1^*) s^{(1)}(k_1^*))$$

Then we have $u'(w(k_1^*) - s^{(1)}(k_1^*)) < u'(f'(k_1^*) s^{(1)}(k_1^*))$ which is equivalent to $w(k_1^*) - s^{(1)}(k_1^*) > f'(k_1^*) s^{(1)}(k_1^*)$, i.e., $s^{(1)}(k_1^*) = nk_1^* < \frac{w(k_1^*)}{1+f'(k_1^*)}$.

If the function $\frac{w(k)}{k(1+f'(k))}$ is strictly decreasing for any $k > 0$, then we get that $k_1^* < k_3^*$ because $n = \frac{\omega(k_3^*)}{k_3^*(1+f'(k_3^*))} < \frac{w(k_1^*)}{k_1^*(1+f'(k_1^*))}$. So, $f'(k_3^*) < f'(k_1^*) < \frac{1}{\gamma+\beta}$.

2. By using the same argument, we have that: If $f'(k_1^*) > \frac{1}{\gamma+\beta}$, then $f'(k_3^*) > f'(k_1^*) > \frac{1}{\gamma+\beta}$.
3. If $f'(k_1^*) = \frac{1}{\gamma+\beta}$, then $n = \frac{\omega(k_3^*)}{k_3^*(1+f'(k_3^*))} = \frac{w(k_1^*)}{k_1^*(1+f'(k_1^*))}$, then $k_1^* = k_3^*$.
4. If $f'(k_2^*) > \frac{1+\gamma}{\beta}$, then $s^{(2)}(k_2^*) = nk_2^* > \frac{w(k_2^*)}{1+f'(k_2^*)}$. So, $n = \frac{\omega(k_3^*)}{k_3^*(1+f'(k_3^*))} > \frac{w(k_2^*)}{k_2^*(1+f'(k_2^*))}$. Thus, $k_2^* > k_3^*$, $f'(k_2^*) < f'(k_3^*)$.
5. If $f'(k_2^*) < \frac{1+\gamma}{\beta}$, then $s^{(2)}(k_2^*) = nk_2^* < \frac{w(k_2^*)}{1+f'(k_2^*)}$. So, $n = \frac{\omega(k_3^*)}{k_3^*(1+f'(k_3^*))} < \frac{w(k_2^*)}{k_2^*(1+f'(k_2^*))}$. Thus, $k_3^* > k_2^*$, $f'(k_3^*) < f'(k_2^*) < \frac{1+\gamma}{\beta}$.
6. If $f'(k_2^*) = \frac{1+\gamma}{\beta}$, then $s^{(2)}(k_2^*) = nk_2^* = \frac{w(k_2^*)}{1+f'(k_2^*)}$. So, $k_2^* = k_3^*$.

□

Proof of Corollary 1. (1) Let $\beta R_{t+1} < 1$. We consider two cases:

- For $\gamma \in (0, \frac{1}{R_{t+1}} - \beta)$, Proposition 1 implies that $s_t(\gamma) = s_{\beta_1(\gamma)}$ which is increasing in γ .
- For $\frac{1}{R_{t+1}} - \beta \leq \gamma$, we have $\frac{1}{\beta+\gamma} \leq R_{t+1} < \frac{1}{\beta} < \frac{1+\gamma}{\beta}$. Then Proposition 1 implies that $s_t = \frac{w_t}{1+R_{t+1}}$ which does not depend on γ .

(2) Let $\beta R_{t+1} > 1$. We consider two cases:

- For $\gamma < \beta R_{t+1} - 1$, i.e., $\frac{1+\gamma}{\beta} < R_{t+1}$, Proposition 1 implies that $s_t(\gamma) = s_{\beta_2(\gamma)}$ which is decreasing in γ .
- For $\gamma \geq \beta R_{t+1} - 1$, we have $\frac{1}{\beta} < R_{t+1} < \frac{1+\gamma}{\beta}$. Then Proposition 1 implies that $s_t = \frac{w_t}{1+R_{t+1}}$ which does not depend on γ .

□

Proof of Corollary 5. First we verify that the conditions Lemma 4 hold when $u(c) = \ln c$ and $f(k) = Ak^\alpha$. In fact, we have $\omega(k) = (1 - \alpha)Ak^\alpha$. Thus the function $h(k) = \frac{\omega(k)}{k} = (1 - \alpha)Ak^{\alpha-1}$ is decreasing in k and then $\frac{\omega(k)}{k(1+f'(k))}$ is also decreasing in k . The function $f'(k)u'(nkf'(k)) = \frac{1}{nk}$ is also decreasing in k .

The rate of return

$$R_{t+1} = f'(k_{t+1}) = \alpha Ak_{t+1}^{\alpha-1}.$$

From Lemma 4, the dynamic of capital at equilibrium satisfies the following equation

$$nk_{t+1} = \begin{cases} \frac{\beta_1 \omega(k_t)}{1+\beta_1} = \frac{\beta_1(1-\alpha)Ak_t^\alpha}{1+\beta_1} = (1-\sigma_1)(1-\alpha)Ak_t^\alpha & \text{if } \alpha Ak_{t+1}^{\alpha-1} < \frac{1}{\gamma+\beta} \\ \frac{(1-\alpha)Ak_t^\alpha}{1+\alpha Ak_{t+1}^{\alpha-1}} & \text{if } \frac{1}{\gamma+\beta} \leq \alpha Ak_{t+1}^{\alpha-1} \leq \frac{1+\gamma}{\beta} \\ (1-\sigma_2)(1-\alpha)Ak_t^\alpha & \text{if } \alpha Ak_{t+1}^{\alpha-1} > \frac{1+\gamma}{\beta} \end{cases}.$$

When $\alpha Ak_{t+1}^{\alpha-1} < \frac{1}{\gamma+\beta}$, we have $k_{t+1} = G(k_t)$ with $G(k) = G_1(k) = \frac{1}{n}(1-\sigma_1)(1-\alpha)Ak^\alpha$. By consequence, we can check that condition $\alpha Ak_{t+1}^{\alpha-1} < \frac{1}{\gamma+\beta}$ becomes $k_1 > a_1$.

Similarly, $\alpha Ak_{t+1}^{\alpha-1} > \frac{1+\gamma}{\beta}$ becomes $k_t < a_2$ while $\frac{1}{\gamma+\beta} \leq \alpha Ak_{t+1}^{\alpha-1} \leq \frac{1+\gamma}{\beta}$ becomes $a_2 \leq k_t \leq a_1$. As a consequence, we obtain (16) and (17). \square

Proof of Corollary 7. Remind that k_1^* , k^s and k_2^* are respectively the solution of the following equations

$$\begin{aligned} g_1(k) &\equiv \frac{(1-\alpha)Ak^{\alpha-1}}{1 + (\alpha Ak^{\alpha-1})^{-\frac{1}{\sigma}+1} \beta_1^{-\frac{1}{\sigma}}} = n \\ g_s(k) &\equiv \frac{(1-\alpha)Ak^{\alpha-1}}{1 + (\alpha Ak^{\alpha-1})^{-\frac{1}{\sigma}+1} \beta^{-\frac{1}{\sigma}}} = n \\ g_2(k) &\equiv \frac{(1-\alpha)Ak^{\alpha-1}}{1 + (\alpha Ak^{\alpha-1})^{-\frac{1}{\sigma}+1} \beta_2^{-\frac{1}{\sigma}}} = n \end{aligned}$$

Since the function $\alpha Ak^{\alpha-1}$ is strictly decreasing in k and the function $\frac{x}{1+\beta^{-\frac{1}{\sigma}} x^{1-\frac{1}{\sigma}}}$ is strictly increasing in $x > 0$, then the function $g_1(k)$, $g_s(k)$, $g_2(k)$ are strictly decreasing in k . Since $\beta_1 = \beta + \gamma$ and $\beta_2 = \frac{\beta}{1+\gamma}$, we obtain that k_1^* is strictly decreasing in γ and k_2^* is strictly increasing in γ . Moreover, $k_1^* > k^s > k_2^*$. \square

Proof of Corollary 8. We firstly consider the case $u(c) = \ln c$.

1. If $\frac{1-\alpha}{n\alpha} > 1 + \gamma + \beta$ then from Corollary 5, we have $k^* = k_1^* = \left(\frac{(1-\sigma_1)(1-\alpha)A}{n}\right)^{\frac{1}{1-\alpha}}$. The condition $k_{GR} > k^*$ is equivalent to

$$\frac{(1-\sigma_1)(1-\alpha)A}{n} < \frac{\alpha A}{n}.$$

Using the fact that $\sigma_1 = \frac{1}{1+\gamma+\beta}$, we obtain the equivalent condition $\frac{1-\alpha}{\alpha} < 1 - \sigma_1 = 1 + \frac{1}{\beta+\gamma}$. Hence $k^* < k_{GR}$ if and only if $\frac{1-\alpha}{\alpha} < 1 + \frac{1}{\beta+\gamma}$. Inversely, we see that $k^* > k_{GR}$ if and only if $\frac{1-\alpha}{\alpha} > 1 + \frac{1}{\beta+\gamma}$.

2. If $\frac{1-\alpha}{\alpha n} \leq 1 + \frac{\beta}{1+\gamma}$, then from Corollary 5, we have $k^* = k_2^* = \left(\frac{(1-\sigma_2)(1-\alpha)A}{n}\right)^{\frac{1}{1-\alpha}}$. So, we find

$$f'(k^*) = \alpha A \left(\frac{(1-\sigma_2)(1-\alpha)A}{n}\right)^{\frac{\alpha-1}{1-\alpha}} = \frac{\alpha n}{(1-\sigma_2)(1-\alpha)}.$$

By consequence, condition $f'(k^*) > n$ is equivalent to $\frac{1-\alpha}{\alpha} < \frac{1}{1-\sigma_2}$.

Recall that $\sigma_2 \equiv \frac{1+\gamma}{1+\beta+\gamma}$. So, $\frac{1}{1-\sigma_2} = \frac{1+\beta+\gamma}{\beta} = 1 + \frac{1+\gamma}{\beta}$. Hence, $f'(k^*) > n$ becomes $\frac{1-\alpha}{\alpha} < 1 + \frac{1+\gamma}{\beta}$ while $f'(k^*) < n$ becomes $\frac{1-\alpha}{\alpha} > 1 + \frac{1+\gamma}{\beta}$.

3. If $1 + \frac{\beta}{1+\gamma} \leq \frac{1-\alpha}{n} \leq 1 + \gamma + \beta$ then $k^* = k_3^* = \left(\frac{(1-\alpha-n\alpha)A}{n}\right)^{\frac{1}{1-\alpha}}$. Hence the condition $k^* < k_{GR}$ satisfies if and only if $\frac{(1-\alpha-n\alpha)A}{n} < \frac{\alpha A}{n}$ which is equivalent to $\alpha > \frac{1}{n+2}$. So, we obtain our result.

We now consider the case $u(c) = \frac{u^{1-\sigma}}{1-\sigma}$, $\forall 0 < \sigma \neq 1$.

1. If $\frac{1-\alpha}{n} > 1 + \gamma + \beta$ then $k^* = k_1^*$ is the solution of the equation $g_1(k) \equiv \frac{(1-\alpha)Ak^{\alpha-1}}{1+(\alpha Ak^{\alpha-1})^{-\frac{1}{\sigma}+1}\beta_1^{-\frac{1}{\sigma}}} = n$. Here, recall that $\beta_1 \equiv \beta + \gamma, \beta_2 \equiv \frac{\beta}{1+\gamma}$.

We can verify that $g_1(k)$ is a strictly decreasing function and $g_1(k_{GR}) = \frac{1-\alpha}{\alpha} \frac{n}{1+n^{1-\frac{1}{\sigma}}\beta_1^{-\frac{1}{\sigma}}} < n = g_1(k_1^*)$ if and only if $\frac{1-\alpha}{\alpha(1+n^{1-\frac{1}{\sigma}}\beta_1^{-\frac{1}{\sigma}})} < 1$. Hence $k_1^* < k_{GR}$ and then

the inter-temporal equilibrium is dynamically efficient if $\frac{1-\alpha}{\alpha} < 1 + n^{1-\frac{1}{\sigma}}\beta_1^{-\frac{1}{\sigma}}$. Inversely, the inter-temporal equilibrium is inefficient.

2. If $\frac{1-\alpha}{n} < 1 + \frac{\beta}{1+\gamma}$ then $k^* = k_2^*$ is the solution of the equation

$$g_2(k) \equiv \frac{(1-\alpha)Ak^{\alpha-1}}{1+(\alpha Ak^{\alpha-1})^{-\frac{1}{\sigma}+1}\beta_2^{-\frac{1}{\sigma}}} = n.$$

We can verify that $g_2(k)$ is a strictly decreasing function and then we have $g_2(k_{GR}) = \frac{n}{1+n^{1-\frac{1}{\sigma}}\beta_2^{-\frac{1}{\sigma}}} \frac{1-\alpha}{\alpha} < n = g_2(k_2^*)$ if and only if $\frac{(1-\alpha)}{\alpha(1+n^{1-\frac{1}{\sigma}}\beta_2^{-\frac{1}{\sigma}})} < 1$.

Hence $k_2^* < k_{GR}$ and then the inter-temporal equilibrium is dynamically efficient if $\frac{1-\alpha}{\alpha} < 1 + n^{1-\frac{1}{\sigma}}\beta_2^{-\frac{1}{\sigma}}$ and inefficient if $\frac{1-\alpha}{\alpha} > 1 + n^{1-\frac{1}{\sigma}}\beta_2^{-\frac{1}{\sigma}}$.

3. We use the same argument in the proof of part 3 of the case $u(c) = \ln(c)$.

□

References

- Araujo, A., Novinski, R., Pascoa, M.R., 2011. *General equilibrium, wariness and efficient bubbles*. Journal of Economic Theory 46, pp. 785-811.
- Araujo, A., Gama-Torres, J.P., Novinski, R., Pascoa, M.R., 2016. *On the efficiency of monetary equilibrium when agents are wary*. Discussion Papers in Economics, No. DP 04/16, University of Surrey.
- Baker, S., Bloom, N. , Davis, S. (2016). *Measuring economic policy uncertainty*. The Quarterly Journal of Economics, 131(4), 1593-1636.
- Bewley, T., 1986. *Knightian decision theory: Part I*. Discussion paper, Cowles Foundation.
- Bianchi, F., Howard, K., and Mikhail, T. (2023). *The Origins and Effects of Macroeconomic Uncertainty*. Quantitative Economics Volume 14, Issue 3, 855-896.
- Bloom, N., Davis, S. J., Foster, L. S., Ohlmacher, S. W. & Saporta-Eksten, I. (2022). *Investment and Subjective Uncertainty*. Quantitative Economics (forthcoming). NBER working paper 30654.
- Cass, D., 1972. *On capital overaccumulation in the aggregative, neoclassical model of economic growth: A complete characterization*, Journal of Economic Theory 4, 200-223.
- Cozzi, G., Giordan, P. E., 2011. *Ambiguity attitude, R&D investments and economic growth*. Journal of Evolutionary Economics 21, pp. 303-319.
- de la Croix, D., Michel, P., 2002. *A Theory of Economic Growth : Dynamics and Policy in Overlapping Generations, 1st edition*. Cambridge University Press.
- Jurado, K., Ludvigson, S. C., and Ng, S., 2015. *Measuring uncertainty*. American Economic Review, 105(3), 1177-1216.
- Ludvigson, S. C., Ma, S., and Ng, S., 2021. *Uncertainty and business cycles: Exogenous impulse or endogenous response?*. American Economic Journal: Macroeconomics, 13(4), 369-410.
- Fukuda, S.-I., 2008. *Knightian uncertainty and poverty trap in a model of economic growth*. Review of Economic Dynamics 11 (3), pp. 652-663.
- Ghiglino, C., Tabasso N., 2016. *Risk aversion in a model of endogenous growth*. Journal of Mathematical Economics 64, pp. 30-40.
- Gilboa, I., Schmeidler D., 1989. *Maxmin expected utility with a non-unique prior*, Journal of Mathematical Economics 18(2), pp. 605-639.
- Ha-Huy, T., Nguyen, T.T.M. (2022). *Saving and dissaving under Ramsey-Rawls criterion*, Journal of Mathematical Economics, Volume 103, December 2022, 102783.

- Kumar, S., Gorodnichenko, Y., Coibion, O. (2023). *The Effect of Macroeconomic Uncertainty on Firm Decisions*. *Econometrica* 91, p. 1297-1332.
- Malinvaud, E., 1953. *Capital accumulation and efficient allocation of resources*, *Econometrica* 21, p. 233-268.
- Romer, P. M. (1986). *Increasing returns and long-run growth*, *Journal of Political Economy*, 94(5):1002-37.
- Romer, P. M. (1990). *Endogenous technological change*, *Journal of Political Economy*, 98(5):S71-S102.
- Schmeidler D., 1989. *Subjective probability and expected utility without additivity*. *Econometrica* 57, pp. 571-587.