FINITENESS OF HYPERBOLIC ENTROPY FOR HOLOMORPHIC FOLIATIONS WITH NON-DEGENERATE SINGULARITIES

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ABSTRACT. Consider $\mathscr{F} = (M, \mathscr{L}, E)$ a Brody-hyperbolic foliation on a compact complex surface M. Suppose that the singularities of $\mathscr{F}$ are all non-degenerate. We show that the hyperbolic entropy of $\mathscr{F}$ is finite.

1. INTRODUCTION

There has been a lot of progress in the dynamical theory of laminations by Riemann surfaces during the last two decades. More precisely, much of attention has been focused on building an ergodic theory when the leaves are hyperbolic. To have such a setup, the case of the projective spaces is very typical. Indeed, every polynomial vector field on $\mathbb{C}^n$ can be compactified naturally into a holomorphic foliation on $\mathbb{P}^n$. This foliation is always singular. Let $d, n \in \mathbb{N}$ with $n \geqslant 2$, denote by $\mathscr{F}_d(\mathbb{P}^n)$ the space of singular holomorphic foliations of degree d on $\mathbb{P}^n$. Lins Neto and Soares [14], using a work of Jouanolou [11], show that a generic foliation $\mathscr{F} \in \mathscr{F}_d(\mathbb{P}^n)$ has only non-degenerate singularities. Moreover, by a result of Lins Neto [12] and Glutsyuk [9], such a foliation is hyperbolic if $d \geqslant 2$. It is even Brody-hyperbolic in the sense of [6]. Loray and Rebelo [15] also build a non-empty open subset of these foliations, the leaves of which are all dense in $\mathbb{P}^n$. When $n = 2$, Nguyên [19] uses the integrability of the holonomy cocycle in [16] to compute the Lyapunov exponent of a generic foliation $\mathscr{F} \in \mathscr{F}_d(\mathbb{P}^2)$. We recall briefly some recent studies and refer the reader to the survey articles [7, 8, 17, 18] for a more detailed exposition.

By solving heat equations with respect to harmonic currents, Dinh, Nguyên and Sibony are able in [4] to prove abstract ergodic theorems for laminations and foliations. This new approach enables them to develop an effective ergodic theory for laminations and foliations, and in particular, geometric versions of Birkhoff's theorem in this context. In two articles [5, 6], the three authors study a modulus of continuity for the leafwise Poincaré metric. More precisely, they show that it is Hölder in the case of a compact regular hyperbolic foliation, and Hölder with a logarithmic slope towards the origin in the case of linearizable singularities. Somehow, their work on the heat equation implicitly studies the dynamics of foliations in a canonical time, which is measured by the Poincaré distance in the universal covering. From this viewpoint, they introduce a canonical notion of hyperbolic entropy and prove the following finiteness results.

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Theorem 1.1. *(1)* (Dinh–Nguyên–Sibony [5, Theorem 3.10]) *Let $\mathscr{F} = (X, \mathscr{L})$ be a smooth compact lamination by hyperbolic Riemann surfaces. Then, the hyperbolic entropy of $\mathscr{F}$ is finite.*

(2) (Dinh–Nguyên–Sibony [6, Theorem 1.1]) *Let $\mathscr{F} = (M, \mathscr{L}, E)$ be a Brody-hyperbolic singular holomorphic foliation on a compact complex surface. Suppose that all the singularities of $\mathscr{F}$ are linearizable. Then, the hyperbolic entropy of $\mathscr{F}$ is finite.*

The finiteness of the hyperbolic entropy in this theorem is strongly dependent on their previous result on the modulus of continuity of the leafwise Poincaré metric (see [5, Theorem 2.1] and [6, Theorem 3.2]). In our previous work [1, 2], we generalize this regularity result to foliations with non-degenerate singularities. In this article, we obtain the following generalization of Theorem 1.1 (2).

Theorem 1.2. *Let $\mathscr{F} = (M, \mathscr{L}, E)$ be a Brody-hyperbolic singular holomorphic foliation on a compact complex surface. Suppose that all the singularities of $\mathscr{F}$ are non-degenerate. Then, the hyperbolic entropy of $\mathscr{F}$ is finite.*

Let us explain briefly the method of our proof. We follow the general strategy of the three authors in [6] for linearizable singularities. They are able to ensure that two points are at small Bowen distance by solving a Beltrami equation for a map that is obtained by gluing local orthogonal projections from a leaf to another. To control the Beltrami coefficient that could explode near the singularities, they need to correct this function and make it holomorphic when approaching the singular set. They use a small hyperbolic step and holonomy mappings to carry information from a transversal to another. By a crucial refinement lemma, they construct a covering such that they can define such an orthogonal projection for two points in the same cell, up to a hyperbolic time R. Of course, they need to estimate the cardinality of the covering all along the refinement process to show that the entropy is finite.

To adapt their proof, we use a classification of non-degenerate singularities in dimension 2 in three types.

- The linearizable singularities.
- The singularities with two separatrices and real negative characteristic number.
- The resonant singularities.

By Poincaré linearization theorem and Briot–Bouquet theorem, every non-degenerate singularity is of one of these types. Moreover, Poincaré–Dulac theorem enables us to have an explicit form for resonant singularities. We show the same kind of estimates as [6] for both cases with separatrices, and stronger ones for the resonant case. This gives us an initial covering on which we control the behaviour of close leaves in small hyperbolic time. To obtain such estimates, we use a generalization of Grönwall Lemma for non-linear differential equations, due to Lins Neto and Canille Martins [13]. We also use their estimation of the Poincaré metric near the singularities to compare the flow time and the hyperbolic time. To correct the orthogonal projection in the neighbourhood of singularities, we fix a point in the initial leaf, and move in the second leaf with the same time of flow. This is what the three authors do, without naming it as such, since everything is explicit for linearizable singularities. It remains to check and adapt each technical element of [6]. This is easily said, but everything is more difficult than it is in the linearizable case. Sometimes the resonant case and sometimes the two separatrices

case need new arguments. Some of our techniques are slightly different from the three authors', but the main structure of our proof is very similar. Most of the time, our work on the linearizable case is just reproving what is already there in [6] but in a slightly different setting. This enables us to clarify our work and statements in both other cases.

The article is organised as follows. In Section 2, we introduce the hyperbolic entropy following [5]. Moreover, we recall our previous work on local orthogonal projections from a leaf to another and the generalization of the Grönwall Lemma. In Section 3, we study the flow in a small step of hyperbolic time for the three types of singularities. We obtain a first cell decomposition. In Section 4, we show a sufficient condition for the entropy to be finite. This criterion involves the orthogonal projections and their corrections near the singularities. In Section 5, we build a hyperbolically dense mesh of transversals and the initial covering that is refined later to obtain the Bowen cells. In Section 6, we study the holonomy mappings in small hyperbolic time to carry information during the refinement process. In Section 7, we consider trees that encode the dynamics on the universal cover $\mathbb{D}$, and are compatible with our mesh of transversals. Section 8 ends the proof by exposing more precisely our refinement algorithm and building the orthogonal projection that is needed for our criterion.

Notations. Throughout this paper, we denote by $\mathbb{D}$ the unit disk of $\mathbb{C}$, and $r\mathbb{D}$ the open disk of radius $r \in \mathbb{R}_+^*$ for the standard Euclidean metric of $\mathbb{C}$. For $R \in \mathbb{R}_+^*$, we denote by $\mathbb{D}_R$ the open disk of hyperbolic radius R in $\mathbb{D}$, so that $\mathbb{D}_R = r\mathbb{D}$ with $r = \frac{e^R-1}{e^R+1}$, or if $r \in [0,1)$, with $R = \ln \frac{1+r}{1-r}$. More generally, for $\rho \in \mathbb{R}_+^*$ and U a subset of a vector space with a marked point z_0, ρU denotes the image of U by the homothety $z \mapsto z_0 + \rho(z - z_0)$. In particular, if $D \subset \mathbb{C}$ is a disk of radius r, ρD is the disk of same center and radius $r\rho$.

We consider several distances on a complex manifold M. For g_M a Hermitian metric on M, the distance induced by g_M is denoted by d. Consider a singular holomorphic foliation $\mathscr{F} = (M, \mathscr{L}, E)$. If L is a leaf of $\mathscr{F}$, g_M induces a distance on L that we denote by d_L. If L is hyperbolic, then L is endowed with the Poincaré metric denoted g_P and the induced distance denoted d_P. We use the same notation for the Poincaré metric and distance on $\mathbb{D}$. If $x \in M \backslash E$ is such that the leaf through x, denoted L_x, is hyperbolic, we note $\phi_x \colon \mathbb{D} \to L_x$ a uniformization of L_x such that $\phi_x(0) = x$. For u, v two functions from K to M, to a leaf or to $\mathbb{D}$, we denote by

$$d_K(u,v) = \sup_{x \in K} d(u(x), v(x)), \quad d_{P,K}(u,v) = \sup_{x \in K} d_P(u(x), v(x)).$$

We try to make our notations different for different contexts. If $z \in \mathbb{C}$ and $r \in \mathbb{R}_+^*$, we note $D(z,r)$ the disk of center z and radius r. Inside a metric space, we denote by $B(x,r)$ the ball of center x and radius r. We try to keep this notation for ambiant Hermitian distances. Inside a leaf, we denote by $L_x[r] = \{y \in L_x;\ d_{L_x}(x,y) < r\}$. Inside the Poincaré disk, if $\xi \in \mathbb{D}$ and $R \in \mathbb{R}_+$, we note $\mathbb{D}_R(\xi) = \{\zeta \in \mathbb{D};\ d_P(\xi,\zeta) < R\}$.

The term "constant" means a real positive number that does not depend on a point $x \in M \backslash E$, nor on the hyperbolic radius R that will go to $+\infty$. Most of our proof relies on the fact that some constants $h, h_1, \hbar$ are sufficiently small, independently on R, given that it is sufficiently large. We may have forgotten to say it somewhere and the reader can suppose it is in the hypotheses of every statement. When we do not care about constants, we simply denote them by C, C', C'', ... When we want to keep track of them to clarify our arguments, we denote them by C_0, C_1, C_2, ...

Finally, we denote by $\lceil a \rceil$ the smallest integer k such that $k \geqslant a$, for $a \in \mathbb{R}$. We also denote $\Re(a)$ (resp. $\Im(a)$) the real (resp. imaginary) part of a complex number a.

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2. Preliminaries

2.1. **Leafwise Poincaré metric.** In all this section, we let $\mathscr{F} = (M, \mathscr{L}, E)$ be a singular holomorphic foliation on a complex manifold M. Suppose that M is endowed with a Hermitian metric g_M and for $x \in M \backslash E$, consider

$$\eta(x) = \sup \left\{ \|\alpha'(0)\|_{g_M} ; \, \alpha \colon \mathbb{D} \to L_x \text{ holomorphic such that } \alpha(0) = x \right\}.$$

Above, $\|v\|_{g_M}$ is the norm of a vector $v \in T_x L_x$ with respect to the Hermitian metric g_M. That is, $\|v\|_{g_M} = (g_{M,x}(v,v))^{1/2}$. The function η was introduced by Verjovsky in [20]. It is designed to satisfy the following facts.

Proposition 2.1. *(1) For $x \in M \backslash E$, $\eta(x) < +\infty$ if and only if the leaf L_x is hyperbolic, that is, it is uniformized by the Poincaré disk $\mathbb{D}$.*
(2) If L_x is hyperbolic, we have $\eta(x) = \|\phi'(0)\|_{g_M}$, where $\phi \colon \mathbb{D} \to L_x$ is any uniformization of L_x such that $\phi(0) = x$.
(3) If L_x is hyperbolic, then $\frac{4 g_M}{\eta^2}$ induces the Poincaré metric on L_x.

In this article, we are interested in the case of hyperbolic leaves and we need to specify our global setting. We follow [6] in our vocabulary.

Definition 2.2. If all the leaves of $\mathscr{F}$ are hyperbolic, we say that $\mathscr{F}$ is *hyperbolic*. If moreover there exists a constant $c_0 > 0$ such that $\eta(x) < c_0$ for all $x \in M \backslash E$, we say that $\mathscr{F}$ is *Brody-hyperbolic*.

From now on, we suppose that $\mathscr{F}$ is hyperbolic. We also need to define the type of singularities we deal with.

Definition 2.3. Near a singularity $a \in E$, there exists a vector field X defining $\mathscr{F}$. In coordinates $(z_1, \ldots, z_n)$ centered at a, we can write

$$X(z) = \sum_{j=1}^{n} F_j(z) \frac{\partial}{\partial z_j}.$$

The functions F_j can be developed as a power series $F_j = \sum_{\alpha \in \mathbb{N}^n} c_{\alpha,j} z^\alpha$. The *1-jet* of X at a is defined in the chart (U, z) as $X_1 = \sum_{j=1}^{n} \sum_{|\alpha| \leqslant 1} c_{\alpha,j} z^\alpha \frac{\partial}{\partial z_j}$. See [10, Chapter I] for more details. If the 1-jet of X has an isolated singularity at a, we say that a is a *non-degenerate singularity of $\mathscr{F}$*.

We use the following estimate of η for non-degenerate singularities. It can be found in [1, Proposition 4.2] and its proof is basically the same as Dinh, Nguyên and Sibony's one [6, Proposition 3.3], together with a local estimate that is due to Lins Neto and Canille Martins [13, Theorem 2].

Proposition 2.4. *Note d the distance induced by g_M. Suppose that M is compact and that $\mathscr{F}$ is Brody-hyperbolic with only non-degenerate singularities. Then, there exists a constant $C \geqslant 1$ such that*

$$C^{-1} d(x,E) \log^\star d(x,E) \leqslant \eta(x) \leqslant C d(x,E) \log^\star d(x,E), \qquad x \in M \backslash E,$$

where $\log^\star = 1 + |\log|$ is a $\log$-type function.

2.2. **Hyperbolic entropy.** For $x \in M\backslash E$, denote by $\phi_x \colon \mathbb{D} \to L_x$ a uniformization of L_x such that $\phi_x(0) = x$. To unify notations, set also $\phi_a(\zeta) = a$, for $a \in E$ and $\zeta \in \mathbb{D}$. The idea of Dinh, Nguyên and Sibony [5] is to consider the Poincaré distance in $\mathbb{D}$ to be a canonical time. More precisely, for $R \geqslant 0$, consider the Bowen distance

$$d_R(x,y) = \inf_{\theta \in \mathbb{R}} \sup_{\xi \in \overline{\mathbb{D}}_R} d(\phi_x(\xi), \phi_y(e^{i\theta}\xi)), \quad x,y \in M.$$

It measures the distance between the orbits of x and y up to time R. It is clear that it is independent on the choice of ϕ_x. This enables us to define the entropy of $\mathscr{F}$. For $x \in M$, $R, \varepsilon > 0$, denote by $B_R(x,\varepsilon) = \{y \in M \,;\, d_R(x,y) < \varepsilon\}$ the *Bowen ball* of radius ε and center x up to time R. For $Y \subset M$, $R, \varepsilon > 0$ and $F \subset Y$, we say that F is (R,ε)-*dense* in Y if $Y \subset \cup_{x \in F} B_R(x,\varepsilon)$. Denote by $N(Y,R,\varepsilon)$ the minimal cardinality of an (R,ε)-dense subset in Y. The *hyperbolic entropy* of Y is defined as

$$h(Y) = \sup_{\varepsilon > 0} \limsup_{R \to +\infty} \frac{1}{R} \log N(Y,R,\varepsilon).$$

For $Y = M$, we denote it by $h(\mathscr{F})$. If M is compact, then it does not depend on the choice of g_M. A similar and equivalent definition can be made with maximal (R,ε)-separated sets, but we do not need it. The interested reader can see [5] for more details.

2.3. **Local orthogonal projection.** In order to show Theorem 1.2, we need to build cells in sufficiently small cardinality, such that two points $x, y \in M$ in the same cell are close up to time R. To ensure such a proximity, we build a smooth map $\psi \colon \mathbb{D}_R \to L_y$ close to ϕ_x, take its lifting $\Psi \colon \mathbb{D}_R \to \mathbb{D}$ *via* ϕ_y, slightly correct it into a close holomorphic map $v \colon \mathbb{D}_R \to \mathbb{D}$, and finally correct v into a close rotation $r_\theta \colon \xi \mapsto e^{i\theta}\xi$. That way, we are able to show that ϕ_x and $\phi_y \circ r_\theta$ are close up to large time. Actually, the last two steps of this proof are hidden behind a result of Dinh, Nguyên and Sibony [6, Proposition 3.6]. For the first step of this construction, we need to recall our previous work [1] on local orthogonal projections from a leaf to another.

For $x \in M\backslash E$, the metric g_M can be restricted to L_x and induces a distance d_{L_x} on it. For $x \in M\backslash E$ and $r > 0$, denote by $L_x[r] = \{x' \in L_x \,;\, d_{L_x}(x,x') < r\}$. Suppose that M is compact and that all the singularities of $\mathscr{F}$ are non-degenerate.

Lemma 2.5 ([1, Lemma 4.3])**.** *There exist constants ε_0, ε_1, k and K such that for two points $x, y \in M\backslash E$, if $d(x,y) \leqslant \varepsilon_1 d(x,E)$, then there exists a local orthogonal projection*

$$\Phi_{xy} \colon L_x[\varepsilon_0 d(x,E)] \to L_y[k\varepsilon_0 d(y,E)],$$

satisfying

(1) $d_{L_y}(y, \Phi_{xy}(x)) \leqslant k d(x,y)$,
(2) for $x_1, x_2 \in L_x[\varepsilon_0 d(x,E)]$, $d_{L_y}(\Phi_{xy}(x_1), \Phi_{xy}(x_2)) \leqslant k d_{L_x}(x_1,x_2)$.

(3) Φ_{xy} *is smooth and in a finite set of charts,*

$$\|\Phi_{xy} - \mathrm{id}\|_{\infty} \leqslant e^K d(x, \Phi_{xy}(x)), \quad \|\Phi_{xy} - \mathrm{id}\|_{\mathscr{C}^1} \leqslant e^K \frac{d(x, \Phi_{xy}(x))}{d(x, E)},$$

$$\|\Phi_{xy} - \mathrm{id}\|_{\mathscr{C}^2} \leqslant e^K \frac{d(x, \Phi_{xy}(x))}{d(x, E)^2}.$$

(4) If $x' \in L_x[\varepsilon_0 d(x, E)]$, $y' \in L_y[k\varepsilon_0 d(x, E)]$ *and* $d(x', y') \leqslant \varepsilon_1 d(x', E)$, *then* $\Phi_{x'y'} = \Phi_{xy}$ *on the intersection of their domains of definition.*

More precisely, we build the local orthogonal projection by solving an implicit equation on the flow. That way, in singular charts, we are able to estimate the flow time that is needed to join y and $\Phi_{xy}(x)$, with the notations of the previous lemma.

Lemma 2.6 ([1, Lemma 3.1])**.** *Consider* X *a holomorphic vector field on a neighbourhood of* $\overline{\mathbb{D}}^2$ *with a non-degenerate singularity at the origin. Suppose that* $\mathbb{D}^2$ *is endowed with the standard Hermitian metric on* $\mathbb{C}^2$. *Let* $x, y \in \frac{3}{4}\mathbb{D}^2 \backslash \{0\}$ *be such that* Φ_{xy} *exists in the sense of Lemma 2.5. Denote by* φ_y *the flow of* X *starting at* y *from an open neighbourhood of* 0 *in* $\mathbb{C}$ *to* L_y. *Then, there exists* $t \in \mathbb{C}$, *with* $t = O\left(\|x - y\| \, \|x\|^{-1}\right)$ *such that* $\Phi_{xy}(x) = \varphi_y(t)$.

2.4. **Variations on the Grönwall Lemma.** In order to obtain thorough estimates, we use several generalizations of the Grönwall Lemma, including some non-linear cases. First, let us state it in a form that contains these various versions and then do some remarks.

Proposition 2.7 (Lins Neto–Canille Martins [13, Proposition 6])**.** *Let* $F \colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ *be a continuous function such that* $F(t, x) \leqslant F(t, y)$ *if* $x \leqslant y$. *Suppose that for any* $x_0 \in \mathbb{R}_+$, *the Cauchy problem*

$$x'(t) = F(t, x(t)), \qquad x(0) = x_0, \tag{2.1}$$

has a unique maximal solution in a neighbourhood of $t = 0$. *Let* $x \colon [0, r_x) \to \mathbb{R}_+$ *be a continuous function that satisfies*

$$x(t) \leqslant x_0 + \int_0^t F(s, x(s)) ds, \quad t \in [0, r_x)\,,$$

and $y \colon [0, r_y) \to \mathbb{R}_+$ *be the unique maximal solution of* (2.1) *starting at* x_0. *Then, for* $t \in [0, \min(r_x, r_y))$, $x(t) \leqslant y(t)$.

Remark 2.8. Actually, Lins Neto and Canille Martins prove a stronger estimate for partial orders on $\mathbb{R}^n_+$, but we only need it for $n = 1$. We use this result in two contexts. The first one is of an autonomous system, sometimes in the non-linear case and sometimes in the linear case (i.e. $F(t, x) = Cx$, which gives the classical Grönwall Lemma). The second one is a linear but non-autonomous system. More precisely, for $F(t, x) = Cx + f(t)$. In that case, note that we have

$$y(t) = \left(x_0 + \int_0^t f(s) e^{-Cs} ds\right) e^{Ct}.$$

3. Local cell decomposition

3.1. **First estimates.** We begin by some local work near a non-degenerate singularity. We want to establish estimates of the divergence of orbits in a (small) step of hyperbolic time. More precisely, we want to decompose the singular open sets into small cells in which we have a good control of the flow in this hyperbolic time. Of course, we need a bound on the cardinality of this covering by cells. Our main result in this section (see Proposition 3.11) is close to [6, Proposition 2.7].

We consider $\mathscr{F} = (M, \mathscr{L}, E)$ a Brody-hyperbolic singular holomorphic foliation on a compact complex surface M (that is $\dim_{\mathbb{C}} M = 2$) with only non-degenerate singularities. Take $a \in E$ and $U_a \simeq \mathbb{D}^2$ a neighbourhood of a on which $\mathscr{F}$ is generated by a vector field X. We suppose that the coordinates of U_a and X extend to a neighbourhood of $\overline{\mathbb{D}}^2$. We need to clarify our vocabulary.

Definition 3.1. For $z \in \frac{1}{2}\mathbb{D}^2\backslash\{0\}$, note φ_z the flow of the vector field X starting at z, defined on a maximal open subset of $\mathbb{C}$ such that $\varphi_z(t)$ stays in $\mathbb{D}^2$.

A *flow path* for z and X is a $\mathscr{C}^1$ map $\gamma\colon [0,T] \to \mathbb{C}$, where $T \in \mathbb{R}_+$, $\gamma(0) = 0$ and $\varphi_z(\gamma(t))$ is well defined for all $t \in [0,T]$ and belongs to $\frac{3}{4}\mathbb{D}^2$. Most of the time, T is implicit. The length of γ as a path in $\mathbb{C}$ will be denoted by $\ell(\gamma)$. The *Poincaré length* $\ell_P(\gamma)$ of γ is by definition the Poincaré length of $\varphi_z \circ \gamma$ in L_z. The notation $\ell_P(\gamma)$ is used only if there is no confusion possible for the point z and the vector field X.

Let $\delta\colon [0,T] \to L_z \cap \frac{3}{4}\mathbb{D}^2$ be a $\mathscr{C}^1$ map such that $\delta(0) = z$. A flow path $\gamma\colon [0,T] \to \mathbb{C}$ for z and X is said to *correspond to* δ if $\delta(t) = \varphi_z(\gamma(t))$. Fix a uniformization ϕ_z of L_z such that $\phi_z(0) = z$. Let $\xi \in \mathbb{D}$ be such that $\phi_z([0,\xi]) \subset \frac{3}{4}\mathbb{D}^2$. We say that a flow path $\gamma\colon [0,1] \to L_z$ *represents* ξ if $\varphi_z(\gamma(t)) = \phi_z(t\xi)$. Since the flow is a local biholomorphism, it is clear that for such a ξ, there is a unique flow path representing it. Similarly, for such a δ, there is a unique flow path corresponding to it.

Let $z, w \in \frac{1}{2}\mathbb{D}^2\backslash\{0\}$ and $R, \delta > 0$. We say that z and w are (R,δ)*-relatively close following the flow of* X if for all $\xi \in \mathbb{D}_R$, $\phi_z(\xi) \in \frac{3}{4}\mathbb{D}^2$ and for γ the flow path for z and X representing ξ, $\varphi_w(\gamma(t))$ belongs to $\mathbb{D}^2$ and for all $t \in [0,1]$, $\|\varphi_z(\gamma(t)) - \varphi_w(\gamma(t))\|_1 \leqslant \delta \|\varphi_z(\gamma(t))\|_1$; and if we also have the same properties when switching the roles of z and w. Here, we have denoted by $\|z\|_1 = \max(|z_1|, |z_2|)$ for $z = (z_1, z_2) \in \mathbb{D}^2$.

We need a classification of non-degenerate singularities in dimension 2.

Theorem 3.2 (Briot–Bouquet, Poincaré–Dulac)**.** *Let a be a non-degenerate singularity of a foliation $\mathscr{F}$ on a compact complex surface. There exist local coordinates $(z_1, z_2) \in \mathbb{D}^2$ centered at a, such that $\mathscr{F}$ is generated on $\mathbb{D}^2$ by one of the vector fields*

$$
\begin{aligned}
X_1 &= z_1\frac{\partial}{\partial z_1} + \lambda z_2\frac{\partial}{\partial z_2}, && \lambda \in \mathbb{C}^*;\\
X_2 &= z_1\frac{\partial}{\partial z_1} + (mz_2 + \mu z_1^m)\frac{\partial}{\partial z_2}, && m \in \mathbb{N}^*,\ \mu \in \mathbb{C}^*,\ |\mu| < \frac{1}{2};\\
X_3 &= z_1\frac{\partial}{\partial z_1} - \alpha z_2\left(1 + z_1 z_2^{q+1} f(z_1, z_2)\right)\frac{\partial}{\partial z_2}, && q \in \mathbb{N},\ \|f\|_\infty < 1,\\
& && \alpha \in (0,1] \cap \left[(q+1)^{-1}, q^{-1}\right).
\end{aligned}
$$

In X_3, if $q = 0$, then we just have $\alpha = 1$.

Proof. Let β be a characteristic number of the singularity. If $\beta \in \mathbb{C}^* \setminus \left(\mathbb{R}_- \cup \mathbb{N}^* \cup \frac{1}{\mathbb{N}^*}\right)$, then Poincaré linearization theorem implies that the singularity is linearizable and $\mathscr{F}$ is generated by some X_1. If $\beta \in \mathbb{N}^* \cup \frac{1}{\mathbb{N}^*}$, then by Poincaré–Dulac theorem (see for both [10, Chapter I, Section 5]) gives either X_1 or X_2, whether the singularity is linearizable or not. Finally, if $\alpha \in \mathbb{R}_-$, taking α^{-1} if necessary, we can suppose that $\alpha \in (0,1] \cap [(q+1)^{-1}, q^{-1})$ for some $q \in \mathbb{N}$. By Briot–Bouquet theorem and a refinement by Camacho–Kuiper–Palis [3, Lemma 7], we have the form X_3. The estimates on μ and f can be obtained by homothety or transformations of the type $(z_1, z_2) \mapsto (Az_1, z_2)$ for some $A \in \mathbb{C}^*$. □

In what follows, we suppose that $\mathscr{F}$ is generated on a neighbourhood of $\overline{\mathbb{D}}^2$ by one of the vector fields $X = X_j$, for $j \in \{1,2,3\}$. If $j = 1$, we talk about the *linearizable case*, if $j = 2$ about the *Poincaré–Dulac case* and if $j = 3$ about the *Briot–Bouquet case*. For X_3, we also consider

$$\widehat{X}_3 = -\frac{1}{\alpha} z_1 \left(1 + z_1 z_2^{q+1} f(z_1, z_2)\right)^{-1} \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2} = -\frac{1}{\alpha} z_1 \left(1 + z_1 z_2^{q+1} g(z_1, z_2)\right) \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2}.$$

Making an other homothety, we can still suppose that $\|g\|_\infty < 1$. Exchanging z_1 and z_2 if necessary, note that both X_3 and $\widehat{X}_3$ are of the form

$$\widetilde{X}_3 = z_1 \frac{\partial}{\partial z_1} - \alpha z_2 \left(1 + z_1^k z_2 f(z_1, z_2)\right) \frac{\partial}{\partial z_2}, \tag{3.1}$$

with $\alpha \in \mathbb{R}_+$, $k \in \mathbb{N}^*$ and $k \geqslant \alpha$. We often use only this hypothesis. In that case, we are able to make a unified proof of our estimates for X_3 and $\widehat{X}_3$. The Briot–Bouquet case is basically the only one for which we specify that flow and flow paths are for X_3, $\widehat{X}_3$ or $\widetilde{X}_3$ (that is, for any of X_3 or $\widehat{X}_3$).

Now, we want to establish some first useful results that we use uniformly in all three cases and throughout our proof. Since 0 is a non-degenerate singularity of the vector field X, we know that there exist constants $C_0, C_1, C_2 > 0$ such that

$$C_0^{-1} \|z\|_1 \leqslant \|X(z)\|_1 \leqslant C_0 \|z\|_1, \qquad z \in \mathbb{D}^2; \tag{3.2}$$

$$C_1^{-1} \|z - w\|_1 \leqslant \|X(z) - X(w)\|_1 \leqslant C_1 \|z - w\|_1, \qquad z, w \in \mathbb{D}^2; \tag{3.3}$$

$$C_2^{-1} \|z\|_1 |\ln \|z\|_1| \leqslant \eta(z) \leqslant C_2 \|z\|_1 |\ln \|z\|_1|, \qquad z \in \frac{3}{4}\mathbb{D}^2. \tag{3.4}$$

The last inequality is a consequence of Proposition 2.4.

Lemma 3.3. *Let z be a point in $\frac{1}{2}\mathbb{D}^2 \setminus \{0\}$ and γ be a flow path for z. We suppose that $\ell_P(\gamma) \leqslant R$. There exists a constant $C_3 > 0$ such that for all t,*

$$|\gamma(t)| \leqslant C_0^{-1} |\ln \|z\|_1| \left(e^{C_3 R} - 1\right).$$

Proof. Consider a reparametrization $\widetilde{\gamma}\colon [0,T] \to \mathbb{C}$ of γ such that $|\widetilde{\gamma}'(t)| = 1$ for all $t \in [0,T]$. It is clear that $T \geqslant \sup_{u \in [0,T]} |\widetilde{\gamma}(u)| \geqslant |\gamma(t)|$ for all t. Let us translate the bound $\ell_P(\widetilde{\gamma}) \leqslant R$ in terms of an integral.

$$R \geqslant 2 \int_0^T \frac{\|X(\varphi_z(\widetilde{\gamma}(t)))\|}{\eta(\varphi_z(\widetilde{\gamma}(t)))} dt \geqslant C \int_0^T \frac{dt}{|\ln \|\varphi_z(\widetilde{\gamma}(t))\|_1|},$$

where we used first the relation $g_P = \frac{4 g_M}{\eta^2}$ and the fact that $|\widetilde{\gamma}'(t)| = 1$, and second the equivalence of Hermitian metrics, (3.2) and (3.4). Now, by (3.2), we have

$\left\|\left(\varphi_z \circ \widetilde{\gamma}\right)'(t)\right\|_1 \leqslant C_0 \left\|\left(\varphi_z \circ \widetilde{\gamma}\right)(t)\right\|$. Then, Grönwall Lemma gives $\left\|\varphi_z(\widetilde{\gamma}(t))\right\|_1 \leqslant \|z\|_1 e^{C_0 t}$ and the same argument on the reverse path ensures that $\left\|\varphi_z(\widetilde{\gamma}(t))\right\|_1 \geqslant \|z\|_1 e^{-C_0 t}$. Hence,

$$R \geqslant C \int_0^T \frac{dt}{|\ln \|z\|_1| + C_0 t} = \frac{C}{C_0} \ln\left(1 + \frac{C_0 T}{|\ln \|z\|_1|}\right).$$

Thus, $|\gamma(t)| \leqslant T \leqslant C_0^{-1} |\ln \|z\|_1| \left(e^{C_0 C^{-1} R} - 1\right)$. □

The next result describes the Bowen ball of a singularity. It is close to [6, Lemma 2.5].

Lemma 3.4. *Let $R > 0$ and $\varepsilon \in \left(0, \frac{1}{2}\right)$. If $0 < \|z\|_1 < \exp\left(\ln(\varepsilon) e^{C_3 R}\right)$, then $\phi_z(\mathbb{D}_R) \subset \varepsilon \mathbb{D}^2$.*

Proof. The proof is by contradiction. Take $z' \in \phi_z(\mathbb{D}_R)$ such that $\frac{1}{2} > \|z'\|_1 > \varepsilon$. Let $\gamma\colon [0,1] \to \mathbb{C}$ be a flow path with respect to z', of Poincaré length less than R and such that $\varphi_{z'}(\gamma(1)) = z$. By Lemma 3.3, $|\gamma(1)| \leqslant C_0^{-1} |\ln \|z'\|_1| \left(e^{C_3 R} - 1\right)$. On the other hand, using the same arguments as the previous lemma, we have the contradiction

$$\|z\|_1 \geqslant \|z'\|_1 e^{-C_0 |\gamma(1)|} \geqslant \|z'\|_1 \exp\left(\ln \|z'\|_1 \left(e^{C_3 R} - 1\right)\right) \geqslant \exp\left(\ln(\varepsilon) e^{C_3 R}\right). \qquad \square$$

Fix $\varepsilon \in \left(0, \frac{1}{2}\right)$ and denote by $r_{\mathrm{sing}}(R) = \exp\left(\ln\left(\frac{\varepsilon}{4}\right) e^{C_3 R}\right)$ and $U_{\mathrm{sing}}(R) = r_{\mathrm{sing}}(R)\mathbb{D}^2$, for R sufficiently large. The next paragraphs are devoted to prove our cell decomposition. We need to distinguish three cases for the three vector fields X_j, for $j \in \{1,2,3\}$.

3.2. **Linearizable case.** What we show in this subsection is quite easy, but it clarifies our wishes, methods and notations for the two following subsections. Indeed, our idea for the Briot–Bouquet and Poincaré–Dulac cases is to compare them to the corresponding linearizable cases with $\lambda = -\alpha$ or $\lambda = m$.

Take the vector field $X = X_1$, with the notations of Theorem 3.2. For $z, w \in \frac{1}{2}\mathbb{D}^2 \backslash \{0\}$, denote by $z(t) = (z_1(t), z_2(t)) = \varphi_z(t)$ and $w(t) = (w_1(t), w_2(t)) = \varphi_w(t)$ the coordinates of the flow trajectories. Set $\lambda_1 = 1$ and $\lambda_2 = \lambda$. We have

$$z_j(t) = z_j e^{\lambda_j t}, \qquad w_j(t) = w_j e^{\lambda_j t}.$$

The cell decomposition is a consequence of the following estimate, together with analogous ones in both non-linearizable cases. In that case, it is somehow a weaker version of [6, Proposition 2.7]. However, for the other two cases, we would not have something as strong as what Dinh, Nguyên and Sibony obtain in the linearizable case.

Lemma 3.5. *Let $h, \delta \in (0,1)$ be sufficiently small. For $z, w \in \frac{1}{2}\mathbb{D}^2 \backslash \left(\frac{1}{2} U_{\mathrm{sing}}(R)\right)$, if for each $j \in \{1,2\}$ we are in one of the following configurations,*

(C1) $|z_j|, |w_j| \leqslant \delta r_{\mathrm{sing}}(R)^2$,

(C2) $w_j, z_j \neq 0$, $\left|1 - \frac{z_j}{w_j}\right| \leqslant \delta$ *and* $\left|1 - \frac{w_j}{z_j}\right| \leqslant \delta$,

then z and w are (h,δ)-relatively close following the flow.

Proof. Since our hypotheses are symmetric in z and w, it is sufficient to prove only the assertions about flow paths with respect to z. Fix $\xi \in \mathbb{D}_h$, γ a flow path representing ξ and $j \in \{1,2\}$. If z_j and w_j are in configuration (C2), then

$$|z_j(\gamma(t)) - w_j(\gamma(t))| = \left|1 - \frac{w_j}{z_j}\right| |z_j(\gamma(t))| \leqslant \delta \left\|z(\gamma(t))\right\|_1.$$

Next, suppose that z_j and w_j are in configuration (C1). By Lemma 3.3 and since h is small, we have $|\gamma(t)| \leqslant Ch\,|\ln\|z\|_1|$. It follows by Grönwall Lemma and (3.2) that $\|z(\gamma(t))\|_1 \geqslant r_{\mathrm{sing}}(R)^{1+Ch}$. On the other hand, if h is sufficiently small,

$$|z_j(\gamma(t)) - w_j(\gamma(t))| \leqslant (|z_j| + |w_j|)\, e^{|\lambda_j||\gamma(t)|} \leqslant 2\delta r_{\mathrm{sing}}(R)^{2-Ch} \leqslant \delta\,\|z(\gamma(t))\|_1\,. \qquad \square$$

3.3. **Poincaré–Dulac case.** Now, suppose that $X = X_2$, with the notations of Theorem 3.2. With the same notations as before, we have the explicit flow

$$z_1(t) = z_1e^t,\ z_2(t) = (z_2 + \mu t z_1^m)\, e^{mt}, \qquad w_1(t) = w_1e^t,\ w_2(t) = (w_2 + \mu t w_1^m)\, e^{mt}.$$

The analogous of Lemma 3.5 is the following.

Lemma 3.6. *Let $h, \delta \in (0,1)$ be sufficiently small. For $z, w \in \frac{1}{2}\mathbb{D}^2 \setminus \left(\frac{1}{2}U_{\mathrm{sing}}(R)\right)$, if*
(C1.1) $|z_1|\,, |w_1| \leqslant \delta r_{\mathrm{sing}}(R)^2$ *or*
(C1.2) $w_1, z_1 \neq 0$, $\left|1 - \frac{z_1}{w_1}\right| \leqslant |\ln\|w\|_1|^{-1}\,\frac{\delta}{4}$ *and* $\left|1 - \frac{w_1}{z_1}\right| \leqslant |\ln\|z\|_1|^{-1}\,\frac{\delta}{4}$,
and
(C2.1) $|z_2 - w_2| \leqslant \frac{\delta}{2}\max\left(|z_1|^m\,, |w_1|^m\right)$ *or*
(C2.2) $z_2, w_2 \neq 0$, $\left|1 - \frac{z_2}{w_2}\right| \leqslant |\ln\|w\|_1|^{-1}\,\frac{\delta}{4}$ *and* $\left|1 - \frac{w_2}{z_2}\right| \leqslant |\ln\|z\|_1|^{-1}\,\frac{\delta}{4}$,
then z and w are (h,δ)-relatively close following the flow.

Proof. Similarly to Lemma 3.5, it is sufficient to prove the assertions concerning flow paths for z. Take also the same notations for ξ, γ. By the same arguments as in the linearizable case, we have $|z_1(\gamma(t)) - w_1(\gamma(t))| \leqslant \frac{\delta}{4}\,\|z(\gamma(t))\|_1$. Next, consider

$$|z_2(\gamma(t)) - w_2(\gamma(t))| \leqslant |z_2 - w_2|\, e^{m\Re(\gamma(t))} + |\mu\gamma(t)\,(z_1^m - w_1^m)|\, e^{m\Re(\gamma(t))}. \tag{3.5}$$

First, focus on the second term in the right hand side. If z_1 and w_1 are in configuration (C1.1), the same arguments as in the linearizable case show that it is bounded above by $\frac{\delta}{2}\,\|z(\gamma(t))\|_1$. On the other hand, if z_1 and w_1 are in configuration (C1.2), we get

$$|\mu\gamma(t)|\,|z_1^m - w_1^m|\, e^{m\Re(\gamma(t))} \leqslant Ch\,|\ln\|z\|_1|\left|1 - \frac{w_1^m}{z_1^m}\right||z_1(\gamma(t))|^m \leqslant |\ln\|z\|_1|\left|1 - \frac{w_1}{z_1}\right||z_1(\gamma(t))|\,,$$

if h is sufficiently small. In any case, the second term of (3.5) is bounded above by $\frac{\delta}{2}\,\|z(\gamma(t))\|_1$. Now, consider the first term. If z_2 and w_2 are in configuration (C2.1), then

$$|z_2 - w_2|\, e^{m\Re(\gamma(t))} \leqslant \frac{|z_2 - w_2|}{|z_1|^m}\,|z_1(\gamma(t))|^m \leqslant \frac{\delta}{2}\,\|z(\gamma(t))\|_1\,.$$

Next, suppose that z_2 and w_2 are in configuration (C2.2). We distinguish two cases.

(i) $|z_2 + \mu\gamma(t)z_1^m| \leqslant \frac{|z_2|}{2}$. In particular, $z_1 \neq 0$ and $|\mu\gamma(t)z_1^m| \geqslant \frac{|z_2|}{2}$. Hence,

$$|z_2 - w_2|\, e^{m\Re(\gamma(t))} = |z_2 - w_2|\,\frac{|z_1(\gamma(t))|^m}{|z_1|^m} \leqslant 2\left|1 - \frac{w_2}{z_2}\right||\mu\gamma(t)|\,|z_1(\gamma(t))|^m \leqslant \frac{\delta}{2}\,|z_1(\gamma(t))|\,,$$

if h is sufficiently small, using Lemma 3.3 and $|z_1(\gamma(t))| \leqslant \frac{3}{4}$.

(ii) $|z_2 + \mu\gamma(t)z_1^m| > \frac{|z_2|}{2}$. In particular, $|z_2(\gamma(t))| \geqslant \frac{|z_2|}{2}e^{m\Re(\gamma(t))}$. Thus,

$$|z_2 - w_2|\, e^{m\Re(\gamma(t))} \leqslant 2\left|1 - \frac{w_2}{z_2}\right||z_2(\gamma(t))|\,.$$

In any case, we get $|z_2(\gamma(t)) - w_2(\gamma(t))| \leqslant \delta\,\|z(\gamma(t))\|_1$. $\square$

3.4. **Briot–Bouquet case.** This is the most delicate one. With the notations of (3.1), consider $X = \widetilde{X}_3$, i.e. both X_3 and $\widehat{X}_3$. Note $z(t) = (z_1(t), z_2(t)) = \varphi_z(t)$, and also $\widetilde{z}_2(t) = z_2(t)e^{\alpha t}$ to compare it with the corresponding linearizable case, where we would have $\widetilde{z}_2(t)$ being constant equal to z_2. Take also the same notations for w. We begin by some study of z alone.

Lemma 3.7. *With the notations above, if* $|z_1(t)| \leqslant \frac{3}{4}$*,*

$$|\widetilde{z}_2'(t)| \leqslant \alpha \left(\frac{3}{4}\right)^{k-\alpha} |z_1|^\alpha |\widetilde{z}_2(t)|^2 .$$

Proof. By a simple computation, we get

$$\widetilde{z}_2'(t) = -\alpha \widetilde{z}_2(t) z_1(t)^k z_2(t) f(z_1(t), z_2(t)) = -\alpha \widetilde{z}_2(t)^2 z_1^k e^{(k-\alpha)t} f(z_1(t), z_2(t)). \tag{3.6}$$

Now, since $k \geqslant \alpha$ and $\frac{3}{4} \geqslant |z_1(t)| = |z_1| e^{\Re(t)}$, we obtain our result. □

Lemma 3.8. *If* $h > 0$ *is sufficiently small, then for any* $z \in \frac{1}{2}\mathbb{D}^2\backslash\{0\}$ *and any flow path* γ *such that* $\ell_P(\gamma) \leqslant h$*,* $\frac{1}{2}|z_2| \leqslant |\widetilde{z}_2(\gamma(t))| \leqslant 2|z_2|$*.*

Proof. Since h is small and by Lemma 3.3, $|\gamma(t)| \leqslant Ch|\ln \|z\|_1|$. Integrating the inequality of Lemma 3.7 along a radius gives for $t \neq 0$,

$$|\widetilde{z}_2(t)| \leqslant |z_2| + C \int_0^{|t|} |z_1|^\alpha \left|\widetilde{z}_2\left(s\frac{t}{|t|}\right)\right|^2 ds.$$

We want to apply Proposition 2.7. With its notations, $F(s,x) = C|z_1|^\alpha x^2$ and the unique solution $y(s)$ is given by $\frac{x_0}{1-C|z_1|^\alpha x_0 s}$. Hence,

$$|\widetilde{z}_2(\gamma(t))| \leqslant \frac{|z_2|}{1 - C|z_1|^\alpha |z_2|\,|\gamma(t)|} \leqslant \frac{|z_2|}{1 - Ch|\ln \|z\|_1|\,|z_1|^\alpha |z_2|}.$$

Since $x \mapsto x\ln x$ is bounded on $\left[0, \frac{3}{4}\right]$, we get $|\widetilde{z}_2(\gamma(t))| \leqslant 2|z_2|$ for h small enough. We argue the same on the reverse path from $z_2(\gamma(t))$ to z_2 to obtain the other inequality. □

We can control the distance between $z_2(\gamma(t))$ and $w_2(\gamma(t))$ in small hyperbolic time.

Lemma 3.9. *Let* $h > 0$ *be sufficiently small and* $z, w \in \frac{1}{2}\mathbb{D}^2\backslash\{0\}$ *be such that* $\|z\|_1 \leqslant 2\|w\|_1$*. Let* $\gamma\colon [0,T] \to \mathbb{C}$ *be a flow path for* z *with* $\ell_P(\gamma) \leqslant h$*. Then,*

$$|\widetilde{z}_2(\gamma(t)) - \widetilde{w}_2(\gamma(t))| \leqslant 2|z_2 - w_2| + |z_2| \sup_{u\in[0,T]} |z_1(\gamma(u)) - w_1(\gamma(u))| .$$

Proof. Let us bound $|\widetilde{z}_2'(t) - \widetilde{w}_2'(t)|$ by (3.6).

$$|\widetilde{z}_2'(t) - \widetilde{w}_2'(t)| = \alpha e^{\alpha\Re(t)} \left|z_1(t)^k z_2(t)^2 f(z_1(t), z_2(t)) - w_1(t)^k w_2(t)^2 f(w_1(t), w_2(t))\right| .$$

Name $g(a,b) = a^k b^2 f(a,b)$. It is clear that there exists a constant $C > 0$ such that $\left|\frac{\partial g}{\partial a}\right| \leqslant C|a|^{k-1}|b|^2$ and $\left|\frac{\partial g}{\partial b}\right| \leqslant C|a|^k|b|$. Integrating along direction a and then b, we get

$$\begin{aligned}
|\widetilde{z}_2'(t) - \widetilde{w}_2'(t)| \leqslant & Ce^{\alpha\Re(t)}(|z_2(t)|^2 \max(|z_1(t)|, |w_1(t)|)^{k-1} |z_1(t) - w_1(t)| \\
& + |w_1(t)|^k \max(|z_2(t)|, |w_2(t)|)\,|z_2(t) - w_2(t)|), \\
\leqslant & C\left(|z_2(t)|\,|\widetilde{z}_2(t)|\,|z_1(t) - w_1(t)| + |w_1|^\alpha \max(|z_2|, |w_2|)\,|\widetilde{z}_2(t) - \widetilde{w}_2(t)|\right),
\end{aligned}$$

if $|t| \leqslant Ch|\ln \|z\|_1|$ and h is sufficiently small. Here, we used first that $\varphi_z(t), \varphi_w(t)$ stay in $\frac{3}{4}\mathbb{D}^2$ if h is sufficiently small, since $\ln \|z\|_1$ and $\ln \|w\|_1$ have bounded quotient; and the

same trick as in Lemma 3.7. Furthermore, by Grönwall Lemma, $|z_2(t)| \leqslant |z_2|^{\frac{1}{2}}$ and by Lemma 3.8, $|\widetilde{z}_2(t)| \leqslant 2|z_2|$ if h is sufficiently small. Thus,

$$|\widetilde{z}_2'(t) - \widetilde{w}_2'(t)| \leqslant C\left(|z_2|^{\frac{3}{2}}|z_1(t) - w_1(t)| + |w_1|^{\alpha}\max(|z_2|, |w_2|)|\widetilde{z}_2(t) - \widetilde{w}_2(t)|\right). \tag{3.7}$$

Applying the last inequality to a reparametrization $\widetilde{\gamma}$ of γ such that $|\widetilde{\gamma}'(u)| = 1$, integrating along $\widetilde{\gamma}$ and applying Proposition 2.7, we obtain for $\beta = C|w_1|^{\alpha}\max(|z_2|, |w_2|)$,

$$|\widetilde{z}_2(\gamma(t)) - \widetilde{w}_2(\gamma(t))| \leqslant \left(|z_2 - w_2| + C|z_2|^{\frac{3}{2}}\sup_{\gamma}|z_1(\cdot) - w_1(\cdot)|\int_0^{|\gamma(t)|} e^{-\beta s}ds\right)e^{\beta|\gamma(t)|}.$$

Now, since $\|w\|_1 \leqslant 2\|z\|_1$, we have $\beta|\gamma(t)| \leqslant Ch|w_1|^{\alpha}$. Hence, if h is sufficiently small, $e^{\beta|t|} \leqslant 2$. For $I = e^{\beta|\gamma(t)|}\int_0^{|\gamma(t)|} e^{-\beta s}ds$, we also deduce that $I \leqslant C|\gamma(t)|$. Finally, it is quite clear that $|z_2|^{\frac{1}{2}}|\gamma(t)| \leqslant Ch$ so we conclude by putting together all these observations. □

This enables us to prove a similar result to the other cases.

Lemma 3.10. *Let $h, \delta \in (0,1)$ be sufficiently small. For $z, w \in \frac{1}{2}\mathbb{D}^2 \setminus \left(\frac{1}{2}U_{\text{sing}}(R)\right)$, if for each $j \in \{1,2\}$ we are in one of the following configurations,*

(C1) $|z_j|, |w_j| \leqslant \delta r_{\text{sing}}(R)^2$,

(C2) $w_j, z_j \neq 0$, $\left|1 - \frac{z_j}{w_j}\right| \leqslant \frac{\delta}{8}$ *and* $\left|1 - \frac{w_j}{z_j}\right| \leqslant \frac{\delta}{8}$,

then z and w are (h,δ)-relatively close following the flow of $\widetilde{X}_3$.

Proof. As usual, we only deal with flow paths for z. We also keep the same notations as Lemmas 3.5 and 3.6. By the same arguments as in the linearizable case, we still have $|z_1(\gamma(t)) - w_1(\gamma(t))| \leqslant \frac{\delta}{4}\|z(\gamma(t))\|_1$, and $|z_2(\gamma(t)) - w_2(\gamma(t))| \leqslant \delta\|z(\gamma(t))\|_1$ if z_2 and w_2 are in configuration (C1). Now, suppose that z_2 and w_2 are in configuration (C2). By Lemmas 3.8 and 3.9,

$$\left|\frac{z_2(\gamma(t)) - w_2(\gamma(t))}{z_2(\gamma(t))}\right| = \left|\frac{\widetilde{z}_2(\gamma(t)) - \widetilde{w}_2(\gamma(t))}{\widetilde{z}_2(\gamma(t))}\right| \leqslant 4\left|1 - \frac{w_2}{z_2}\right| + \frac{\delta}{2} \leqslant \delta. \qquad \square$$

3.5. **Cell decomposition.** Now, we wish to cover the bidisk by smaller bidisks such that we have a good control of the divergence of the flow in small hyperbolic time of two points in a cell. Let us take any $j \in \{1,2,3\}$ and $X = X_j$ if $j \neq 3$, $X = \widetilde{X}_3$ if $j = 3$. Fix $\varepsilon > 0$ and let $R > 0$ be sufficiently large (depending on ε). Let C_4 be a strictly positive constant that is specified by our further computations. Name $r_0 = e^{-\exp(C_4 R)}$, $r_n = r_0 e^{n\exp(-C_4 R)}$, for $n \in [\![1, N]\!]$, $N = \left\lceil e^{2C_4 R}\right\rceil$, and $\theta_k = \frac{2k\pi}{N}$, for $k \in [\![1, N']\!]$, and $N' = \left\lceil 4\pi e^{C_4 R}\right\rceil$. Let $D_0 = r_0\mathbb{D}$ and $D_{nk} = D\left(r_{n-1}e^{i\theta_k}, r_n - r_{n-1}\right)$, for $n \in [\![1, N]\!]$ and $k \in [\![1, N']\!]$. Define also the collection $\mathcal{D} = \{D_0\} \cup \{D_{nk}; n \in [\![1, N]\!], k \in [\![1, N']\!]\}$. It is easy to see that $\mathcal{D}$ is a covering of the disk if R is sufficiently large. This is actually about the same covering as [6, p. 602].

Proposition 3.11. *Let h be sufficiently small and R sufficiently large. For $D^{(1)}, D^{(2)} \in \mathcal{D}$, let $U = D^{(1)} \times D^{(2)}$ and $z, w \in 2U \cap \left(\frac{1}{2}\mathbb{D}^2 \setminus \{0\}\right)$. If C_4 is well chosen,*

(1) If z and w belong to $U_{\text{sing}}(R)$, then z and w are (R,ε)-close;

(2) If z or w does not belong to $U_{\text{sing}}(R)$, then z and w are (h, e^{-2R})-relatively close following the flow.

Proof. Point (1) is a direct consequence of Lemma 3.4 and the definition of $U_{\text{sing}}(R)$. Let us suppose by symmetry that $z \notin U_{\text{sing}}(R)$. It is clear that if $C_4 > C_3 + 2$ and R is sufficiently large, then $w \notin \frac{1}{2}U_{\text{sing}}(R)$. Similarly, if $C_4 > C_3$ and $D^{(j)} = D_0$, z_j and w_j are in configuration (C1) of Lemmas 3.5, 3.6 and 3.10. On the other hand, if $D^{(j)}$ is D_{nk}, then $\left|1 - \frac{z_j}{w_j}\right|, \left|1 - \frac{w_j}{z_j}\right| \leqslant Ce^{-C_4 R}$. Since $z, w \notin \frac{1}{2}U_{\text{sing}}(R)$, z_j and w_j are in configuration (C2) of the three lemmas. We conclude by applying them. □

Remark 3.12. Actually, configuration (C2.1) in the Poincaré–Dulac case is most of the time far weaker than the configurations corresponding to separatrices in any other cases. On some sublevel $\{|z_2| \leqslant C\,|z_1|^m\}$, we can replace the disks in the second coordinate by disks of radius $e^{-2R}\,|z_1|^m$. We do so later.

4. General strategy and reductions

4.1. **Geometric setup.** First, let us describe the general geometric assumptions we make to simplify our arguments. Since the entropy $h(\mathscr{F})$ does not depend on the choice of the Hermitian metric g_M, we build one that satisfies some suitable conditions. Let $(U_r, U_a)_{r\in\mathcal{R}, a\in E}$ be a finite open covering of M by

- Singular flow boxes $U_a \simeq \mathbb{D}^2$ such that $\mathscr{F}$ is generated on a neighbourhood of $\overline{U}_a$ by one of the vector fields X_j, for $j \in \{1, 2, 3\}$. We also suppose that the Hermitian metric g_M is given on U_a by $\|dz\|^2$.
- Regular flow boxes $U_r \simeq \mathbb{D} \times \mathbb{T}_r$ such that $2U_r$ is still a flow box and $2\overline{U}_r \cap E = \varnothing$. We often identify $\mathbb{T}_r$ with $\{0\} \times \mathbb{T}_r$. We also suppose that the regular flow boxes cover $M \setminus (\cup_{a\in E}\rho U_a)$. Here, $\rho > 0$ is fixed below. For this section, we need $\rho < \frac{1}{4}$.

4.2. **Reduction to studying orthogonal projections.** We want to do some reductions to a criterion involving an orthogonal projection. Let us begin by the following.

Proposition 4.1 (Dinh–Nguyên–Sibony [6, Proposition 4.1])**.** *Denote by* $\mathbb{T} = \cup_{r\in\mathcal{R}}\mathbb{T}_r$. *If* $h(\mathbb{T}) < \infty$, *then* $h(\mathscr{F}) < \infty$.

Whereas the three authors prove it in the setup of linearizable singularities, it is implicit in [6] that it is enough to prove the following lemma.

Lemma 4.2. *Let* $R, \varepsilon > 0$ *and* $x \in M\setminus E$ *be such that* $\phi_x(\mathbb{D}_{2R}) \subset \frac{1}{2}U_a$. *If* R *is sufficiently large, then* $\phi_x(\mathbb{D}_R) \subset \frac{\varepsilon}{2}U_a$.

Proof. We prove that for $\varepsilon > 0$, there exists $K_\varepsilon > 0$ such that for any $z \in U_a$ with $\|z\|_1 \geqslant \varepsilon$, there is $z' \in \left(U_a \setminus \frac{1}{2}U_a\right) \cap L_z$ with $d_P(z, z') \leqslant K_\varepsilon$. It is easy to see that it implies the lemma. We have to distinguish the vector field we are dealing with.

Briot–Bouquet or linearizable case. Note $z = (z_1, z_2)$. Let $j \in \{1, 2\}$ be the coordinate such that $|z_j| = \|z\|_1$. By symmetry, considering $\widehat{X}_3$ if necessary, we can suppose that $j = 1$. Then, $\varphi_z(t)$, for $t \in \mathbb{R}_+$, stays in L_z. On the other hand, it reaches $U_a \setminus \frac{1}{2}U_a$ in flow time less than $\ln \frac{1}{4\varepsilon}$ and in hyperbolic time less than $\frac{C}{\varepsilon}$, for some $C > 0$.

Poincaré–Dulac case. With the same notations, if $|z_1| \geqslant |z_2|^2$, then $|z_1| \geqslant \frac{\varepsilon^2}{2}$ and we argue similarly. If $|z_2|^2 > |z_1|$, then $\frac{1}{m}\,|\ln|z_2||\,|\mu|\,|z_1|^m \leqslant \frac{1}{2}\,|z_2|$. In particular, $\varphi_z(t)$ escapes $\frac{1}{2}U_a$ in positive real time $\frac{1}{m}\,|\ln|z_2|| \leqslant \frac{1}{m}\,|\ln\varepsilon|$. We conclude by the same observations. □

Let us recall a notion of [5] that clarifies our work on the orthogonal projection.

Definition 4.3. Let $R, \delta > 0$ be such that $\delta \leqslant e^{-2R}$. Two points $x, y \in M \backslash E$ are said to be (R, δ)-*conformally close* if the following properties, and the same when exchanging the roles of x and y, are satisfied.

(a) There exists a smooth function $\psi \colon \overline{\mathbb{D}}_R \to L_y$ without critical points such that $d_{L_y}(\psi(0), y) \leqslant \delta$ and $d_{\overline{\mathbb{D}}_R}(\psi, \phi_x) \leqslant \delta$.
(b) $\|d\psi\|_\infty \leqslant 2A$ for a constant A such that $\eta \leqslant A$ on M, and the norm is considered for the Poincaré metric at the source $\overline{\mathbb{D}}_R$ and g_M on the goal L_y.
(c) Denote by $y' = \psi(0)$. There exists a map $\Psi \colon \overline{\mathbb{D}}_R \to \mathbb{D}$ such that $\Psi(0) = 0$, $\phi_{y'} \circ \Psi = \psi$ and the Beltrami coefficient μ_Ψ satisfies $\|\mu_\Psi\|_{\mathscr{C}^1} \leqslant \delta$.

Recall that the *Beltrami coefficient* is defined to have $\frac{\partial \Psi}{\partial \bar{t}} = \mu_\Psi \frac{\partial \Psi}{\partial t}$.

What interests us with this notion is the following lemma. What is hidden behind is that we correct Ψ into a close holomorphic map by solving a Beltrami equation, and then correct this holomorphic map into a rotation.

Lemma 4.4 (Dinh–Nguyên–Sibony [6, Proposition 3.6])**.** *There exists a constant $C > 0$ such that if R is large enough and $x, y \in \mathbb{T}$ are (R, e^{-2R})-conformally close, then they are $\big(R/3, Ce^{-R/3}\big)$-close,* i.e. $d_{R/3}(x, y) \leqslant Ce^{-R/3}$.

Now, let us state the criterion we apply to show the finiteness of the entropy. Its proof occupies most of the end of the section.

Proposition 4.5. *Let $h_1 > 0$ be sufficiently small, $R > 0$ be sufficiently large and $(V_i)_{i \in I}$ be a covering of $\mathbb{T}$ such that $\operatorname{card}(I) \leqslant e^{gR}$ and satisfying the following. Let $i \in I$ and $x, y \in V_i$.*

(1) There exists a subset $F \subset \mathbb{D}$ with $\mathbb{D}_R \subset \mathbb{D}_{h_1}(F) = \cup_{\xi \in F} \mathbb{D}_{h_1}(\xi)$,
(2) There exists a map $\psi \colon \mathbb{D}_{h_1}(F) \to L_y$ without critical points, that is locally near $\xi \in \mathbb{D}_{h_1}(F)$ given as the orthogonal projection from L_x near $\phi_x(\xi)$ to L_y near $\psi(y)$,
(3) If $\xi \in F$ is such that $\phi_x(\xi) \in 2\rho U_a$, then $\phi_x(\xi)$ and $\psi(\xi)$ are $\big(3h_1, e^{-2R}\big)$-relatively close following the flow, for $\widetilde{X}_3$ in the Briot–Bouquet case.

Then, $h(\mathbb{T}) \leqslant 3g$.

Take $\alpha > 0$ and an open covering $(V_i)_{i \in I}$ that satisfies the hypotheses of Proposition 4.5 for $R(1+\alpha)$. It is enough to show that for $\varepsilon > 0$, if $x, y \in V_i$ for some $i \in I$, and R is sufficiently large, then $d_{R/3}(x, y) \leqslant \varepsilon$. By Lemma 4.4, it is even enough to show that x, y are (R, e^{-2R})-conformally close. We fix $x, y \in V_i$ and $R > 0$. Let us also take F and ψ given by the hypotheses of Proposition 4.5. We need to control paths in singular flow boxes and therefore need to meet F often enough. We use the two following lemmas.

Lemma 4.6. *Let $a \in E$ and V_a be a connected component of $\mathbb{D}_{R(1+\alpha)} \cap \phi_x^{-1}(\rho U_a)$. For $\zeta_1, \zeta_2 \in V_a$, there exist paths $\lambda_1, \dots, \lambda_N \colon [0,1] \to \mathbb{D}_{h_1}(F) \cap \phi_x^{-1}(2\rho U_a)$ with $\lambda_1(0) = \zeta_1$, $\lambda_N(1) = \zeta_2$, $\lambda_j(1) = \lambda_{j+1}(0) \in F$, for $j \in [\![1, N-1]\!]$ and $\ell_P(\lambda_j) \leqslant 3h_1$, $j \in [\![1, N]\!]$.*

Proof. Since $\mathscr{F}$ is Brody-hyperbolic and h_1 is sufficiently small, if $\phi_x(\xi) \in \rho U_a$, then $\phi_x\left(\mathbb{D}_{h_1}(\xi)\right) \subset 2\rho U_a$. Since V_a is connected, there exists a path connecting ζ_1 and ζ_2. Since F is h_1-dense, one can correct this path into $\lambda_1, \dots, \lambda_N$ asked by the lemma. □

Lemma 4.7. *Keep the notations of Lemma 4.6. Let λ be the concatenation of $\lambda_1, \dots, \lambda_N$, $z = \phi_x(\zeta_1)$ and $w = \psi(\zeta_1)$. There is a flow path γ (resp. δ) for z (resp. w) such that $\phi_x(\lambda(t)) = \varphi_z(\gamma(t))$ (resp. $\psi(\lambda(t)) = \varphi_w(\delta(t))$) and $|\gamma(t) - \delta(t)| \leqslant e^{-(2+\alpha)R}$.*

Proof. The existence of the flow paths is a consequence of the fact that the flow is a local biholomorphism. We need to show the estimate $|\gamma(t) - \delta(t)| \leqslant e^{-(2+\alpha)R}$. Note that λ meets F at the ends of each λ_j and that they satisfy $\ell_P(\lambda_j) \leqslant 3h_1$. Hence, we can show by induction and using Lemma 2.6 that

$$|\gamma(t) - \delta(t)| \leqslant C \left\|\varphi_w(\gamma(t)) - \varphi_z(\gamma(t))\right\|_1 \left\|\varphi_z(\gamma(t))\right\|_1^{-1} \leqslant Ce^{-2R(1+\alpha)} \leqslant e^{-(2+\alpha)R},$$

since $\phi_x(\xi)$ and $\psi(\xi)$ are $(3h_1, e^{-2(R+\alpha)})$-relatively close following the flow. Here, we prove by induction and use that the orthogonal projection from L_x near $\varphi_z(\gamma(t))$ onto L_y near $\varphi_w(\delta(t))$ coincide with the one from L_x near $\varphi_z(\gamma(t))$ onto L_y near $\varphi_w(\gamma(t))$. □

We need to control some monodromy phenomena for the flow by the next result.

Lemma 4.8. *There exists $\varepsilon_2 > 0$ such that if $z, w \in \mathbb{D}^2\backslash\{0\}$, for $U_a \simeq \mathbb{D}^2$ a linearizable or Poincaré–Dulac singular flow box, are such that $\|z - w\|_1 \leqslant \frac{1}{2}\|z\|_1$, then the following holds. If t_1, t_2, u_1, u_2 satisfy $\varphi_z(t_1) = \varphi_z(t_2)$, $\varphi_w(u_1) = \varphi_w(u_2)$ and $|(t_1 - t_2) - (u_1 - u_2)| \leqslant \varepsilon_2$, then $t_1 - t_2 = u_1 - u_2$.*

For a Briot–Bouquet singularity, we have the same result for at least one of the vector fields X_3 or $\widehat{X}_3$, depending on z, w (but not on t_1, t_2, u_1, u_2).

Proof. Name $z = (z_1, z_2)$ and $w = (w_1, w_2)$. Note that the hypothesis $\|z - w\|_1 \leqslant \frac{1}{2}\|z\|_1$ implies that there is a coordinate $j \in \{1, 2\}$ with $z_j, w_j \neq 0$. Without loss of generality, we suppose in the linearizable case that $j = 1$. Since we can choose X_3 or $\widehat{X}_3$, we do the same in the Briot–Bouquet case. The explicit form of the flow shows that $t_1 - t_2, u_1 - u_2 \in 2i\pi\mathbb{Z}$. We conclude for $\varepsilon_2 < 2\pi$. We argue similarly in the Poincaré–Dulac case. □

End of proof of Proposition 4.5. We correct the orthogonal projection ψ into a function $\widetilde{\psi}$, which coincides with ψ far from the singular set, is holomorphic near the singularities, and still satisfies (a) and (b) of being (R, e^{-2R})-conformally close. Then, condition (c) is clear and it is sufficient to conclude. It is enough to construct this $\widetilde{\psi}$ on V_a such that it coincides with ψ on $V_a \cap \phi_x^{-1}\left(\rho U_a \backslash \frac{\rho}{2} U_a\right)$ and make this correction for each a and each connected component V_a. We can also do so only if $V_a \cap \phi_x^{-1}\left(\frac{\rho}{2} U_a\right) \neq \varnothing$.

Fix any $\xi \in F \cap \phi_x^{-1}(\rho U_a)$. Such a ξ exists because $\mathscr{F}$ is supposed to be Brody-hyperbolic, $V_a \cap \phi_x^{-1}\left(\frac{\rho}{2} U_a\right) \neq \varnothing$ and h_1 is small. Set $z = \phi_x(\xi)$ and $w = \psi(\xi)$. Let $\zeta \in V_a$ and take paths $\lambda\colon [0, 1] \to \mathbb{D}$ given by Lemma 4.6 and γ, δ given by Lemma 4.7. For the Briot–Bouquet case, we choose flow paths for the vector field that satisfies Lemma 4.8 for z and w. Let $\chi\colon [0, 1] \to [0, 1]$ be a smooth function such that $\chi = 0$ on $\left[\frac{\rho}{2}, 1\right]$ and $\chi = 1$ on $\left[0, \frac{\rho}{4}\right]$. Define

$$\widetilde{\psi}(\zeta) = \varphi_w\left(\chi\left(\|\varphi_z(\gamma(1))\|\right)(\gamma(1) - \delta(1)) + \delta(1)\right).$$

Lemmas 4.8 and 4.7 imply that this definition does not depend on the choice of λ. The correction $\widetilde{\psi}$ is holomorphic on $V_a \cap \phi_x^{-1}\left(\frac{\rho}{4} U_a\right)$ and we conclude with the first remarks. □

Above, the correction process for ψ is close to [6, Lemma 2.12], but its proof is more difficult in our context, because it is less explicit. To ensure that a h_1-dense subset F has its image in an appropriate cell, we use the following refinement result, due to Dinh, Nguyên and Sibony, on holonomy mappings.

Lemma 4.9 (Dinh–Nguyên–Sibony [6, Lemma 4.4])**.** *Let Ω be a subset of a finite union of copies of $\mathbb{C}$. Let $\mathcal{V}_1, \ldots, \mathcal{V}_n$ be coverings of Ω by disks, all satisfying $\operatorname{card} \mathcal{V}_i \leqslant K$, for some*

uniform $K > 0$. Then, there exists a covering $\mathcal{V}$ of Ω by disks, with $\operatorname{card}\mathcal{V} \leqslant 200^n K$ such that for any $D \in \mathcal{V}$, there are $D_i \in \mathcal{V}_i$, $i \in [\![1, n]\!]$, satisfying $2D \subset 2D_1 \cap \cdots \cap 2D_n$.

5. Mesh of transversals and initial covering

5.1. **Mesh of transversals.** At the end of our argument, we wish to apply Proposition 4.5. Hence, we need to build a covering of $\mathbb{T}$, the union of some regular transversals. However, to build such a covering, we also need to control how leaves from these transversals behave when approaching the singular set. Therefore, we also build a mesh of transversals, covering almost all the manifold, which is in some sense hyperbolically dense in the foliation. That way, we are able to control at the beginning what happens to leaves in small hyperbolic time near all of these transversals, and then refine to control what happens all along the orbit in large hyperbolic time.

Let us fix some constant $h_1 > 0$ which is sufficiently small to have all the results of Section 3 for $h = 3h_1$. We also take a constant $K \in (0, 1)$ and name $\hbar = Kh_1$. The constant K is fixed in the following sections. Note that since we are only concerned in what happens to points in $\mathbb{T}$ in time R, we let $\varepsilon = d(\mathbb{T}, E)$ and $r_{\text{sing}}(R) = \exp\left(\ln\left(\frac{\varepsilon}{4}\right) e^{C_3 R}\right)$. Hence, $\phi_x\left(\mathbb{D}_R\right) \cap B\left(a, r_{\text{sing}}(R)\right) = \varnothing$, for $a \in E$ and $x \in \mathbb{T}$, by Lemma 3.4. First, let us state what we ask of our mesh of transversals.

Proposition 5.1. *There exists a constant $K' > 0$ such that for all $R > 0$ sufficiently large, there exists a mesh of transversals $\widetilde{\mathbb{T}} = (\mathbb{T}_i)_{i \in I_{\mathbb{T}}}$ satisfying*

- *(HT1) For $x \in M \backslash E$, if $d(x, E) \geqslant r_{\text{sing}}(R)$, there is $y \in (\cup_{i \in I_{\mathbb{T}}} \mathbb{T}_i) \cap L_x$ with $d_P(x, y) \leqslant \hbar$;*
- *(HT2) For $i \in I_{\mathbb{T}}$, there are at most K' elements $j \in I_{\mathbb{T}}$ such that there are $x \in \mathbb{T}_i$, $y \in L_x \cap \mathbb{T}_j$ with $d_P(x, y) \leqslant 2h_1$.*

This is a difference in the structure of our argument and the one of the three authors of [6]. They build transversals for each of the singular cells and have to consider some multiplicity of transversals. The property (HT2) enables us to avoid this because the refinement Lemma 4.9 applies to a bounded number of transversals.

In regular flow boxes $U_r \simeq \mathbb{D} \times \mathbb{T}_r$, the Poincaré metric is equivalent to the standard Hermitian metric. Then, we consider a lattice $\Lambda = B\hbar\left(\mathbb{Z} + i\mathbb{Z}\right) \cap \mathbb{D}$ and $\mathbb{T}_{r,\lambda} = \{\lambda\} \times \mathbb{T}_r$. The union of these transversals clearly satisfy (HT1) and (HT2) if B is sufficiently small.

So, the difficulty of Proposition 5.1 is for the singular flow boxes. We have to distinguish different cases for the different types of singularities. First, let $a \in E$ be a singularity of linearizable or Briot–Bouquet type. Define $r_0^{\mathbb{T}} = r_{\text{sing}}(R)$ and for $j \in \mathbb{N}$, $r_j^{\mathbb{T}} = \left(r_0^{\mathbb{T}}\right)^{(1-C_5\hbar)^j}$, for some $C_5 \in \left(0, \frac{1}{\hbar}\right)$ that our further computation specifies. Let also $\theta_1^{\mathbb{T}}, \ldots, \theta_P^{\mathbb{T}}$ be some angles that are $C_5\hbar$-dense in $[0, 2\pi]$, for $P = \left\lceil \frac{\pi}{C_5\hbar} \right\rceil$. Define

$$\mathbb{T}_{a,j,k,u} = \left\{ z = (z_1, z_2) \in \mathbb{D}^2 \simeq U_a;\ z_u = r_j^{\mathbb{T}} e^{i\theta_k^{\mathbb{T}}},\ \|z\|_1 \leqslant \frac{3}{2} r_j^{\mathbb{T}} \right\},$$

for $j \in [\![0, N_a(R)]\!]$, $k \in [\![1, P]\!]$, $u \in \{1, 2\}$ and $N_a(R) = \max\left\{j \in \mathbb{N};\ r_j^{\mathbb{T}} \leqslant 2\rho\right\}$. Here, we need $\rho < \frac{1}{4}$ so that the biggest transversal is still contained in $\frac{3}{4}\mathbb{D}^2$. Let us prove (HT1) and (HT2) for these transversals. This is what is done in the following statements.

Lemma 5.2. *Let $a \in E$ be a singularity as above. If C_5 is sufficiently small, then for any $z \in \mathbb{D}^2$ with $\|z\|_1 \in \left[r_{j-1}^{\mathbb{T}}, r_j^{\mathbb{T}}\right]$, for some $j \in [1, N_a(R)]$, there exists $z' \in L_z$ satisfying $\|z'\|_1 = r_j^{\mathbb{T}}$ and $d_P(z, z') \leqslant \frac{\hbar}{2}$.*

Proof. Considering $\widehat{X}_3$ if necessary, we may assume that $\|z\|_1 = |z_1|$ without loss of generality. For the flow $\varphi_z(t)$ starting from z in time $t \in \mathbb{R}_+$, let

$$T = \inf\left\{t \in \mathbb{R}_+;\ \|\varphi_z(t)\|_1 = r_j^{\mathbb{T}}\right\}.$$

It is clear that $T \leqslant \ln \frac{r_j^{\mathbb{T}}}{r_{j-1}^{\mathbb{T}}}$. Moreover, the path $\gamma\colon [0,T] \to L_z$ defined by $\gamma(t) = \varphi_z(t)$ satisfies $\|\gamma(t)\|_1 \leqslant r_j^{\mathbb{T}}$ and $\|\gamma(T)\|_1 = r_j^{\mathbb{T}}$. Then, its Poincaré length is bounded by

$$\ell_P(\gamma) = 2\int_0^T \frac{\|\gamma'(t)\|}{\eta(\gamma(t))}dt \leqslant \frac{CT}{\left|\ln r_j^{\mathbb{T}}\right|} \leqslant C\left(\frac{\ln r_{j-1}^{\mathbb{T}}}{\ln r_j^{\mathbb{T}}} - 1\right) = \frac{CC_5\hbar}{1 - C_5\hbar} \leqslant \frac{\hbar}{2},$$

by (3.2) and (3.4) and if C_5 is sufficiently small. □

Once we have reached the right norm, we need to turn around to a transversal angle.

Lemma 5.3. *Let $a \in E$ be a singularity as above. If C_5 is sufficiently small, $z \in \mathbb{D}^2$ and $j \in [\![1, N_a(R)]\!]$ are such that $\|z\|_1 = r_j^{\mathbb{T}}$, then there exist $k \in [\![1,P]\!]$, $u \in \{1,2\}$ and a $z' \in L_z \cap \mathbb{T}_{a,j,k,u}$ with $d_P(z,z') \leqslant \frac{\hbar}{2}$.*

Proof. Similarly to the previous lemma, we can assume that $\|z\|_1 = |z_1|$. Note $z_1 = r_j^{\mathbb{T}} e^{i\theta}$ and let $k \in [\![1,P]\!]$ be such that $\left|\theta - \theta_k^{\mathbb{T}}\right| \leqslant C_5\hbar$. Consider the path $\gamma(t) = \varphi_z\left(it\left(\theta_k^{\mathbb{T}} - \theta\right)\right)$. We have $\gamma(0) = z$ and $\gamma(1) = \left(r_j^{\mathbb{T}} e^{i\theta_k^{\mathbb{T}}}, z_2'\right)$. By Grönwall Lemma, it is easy to see that $|z_2| \leqslant \frac{3}{2} r_j^{\mathbb{T}}$. Thus, $z' \in \mathbb{T}_{a,j,k,1}$. On the other hand, if C_5 is sufficiently small,

$$\ell_P(\gamma) \leqslant \frac{C'\left|\theta - \theta_k^{\mathbb{T}}\right|}{\left|\ln r_j^{\mathbb{T}}\right| - C\left|\theta - \theta_k^{\mathbb{T}}\right|} \leqslant \frac{\hbar}{2}.$$ □

For these singularities, the condition (HT1) is clear, and we need to prove (HT2).

Lemma 5.4. *Let $a \in E$ be a singularity as above. There is a constant $K' > 0$, independent on R such that for any $j \in [\![0, N_a(R)]\!]$, $k \in [\![1,P]\!]$, $u \in \{1,2\}$, there are at most K' elements $(j',k',u') \in [\![0, N_a(R)]\!] \times [\![1,P]\!] \times \{1,2\}$ such that there exist $x \in \mathbb{T}_{a,j,k,u}$, $y \in \mathbb{T}_{a,j',k',u'} \cap L_x$ with $d_P(x,y) \leqslant 2h_1$.*

Proof. Note that $\|x\|_1 \in \left[r_j^{\mathbb{T}}, \frac{3}{2} r_j^{\mathbb{T}}\right]$. On the other hand, $\|y\|_1 \in \left[\left(r_j^{\mathbb{T}}\right)^{1+Ch_1}, \left(\frac{3}{2} r_j^{\mathbb{T}}\right)^{1-Ch_1}\right]$ by Grönwall Lemma. A direct computation implies that there is a bounded number of possible j' if j is given. The integers k', u' are also in bounded quantity. □

We finish building the singular transversals, with a singularity $a \in E$ of Poincaré–Dulac type. We take the same radii $r_j^{\mathbb{T}} = (r_{\text{sing}}(R))^{(1-C_5\hbar)^j}$, but this time for $j \in \mathbb{Z}$. We also keep the same angles $\theta_k^{\mathbb{T}}$, for $k \in [\![1,P]\!]$. Consider the transversals

$$\mathbb{T}_{a,j,k,1} = \left\{(z_1,z_2) \in \mathbb{D}^2 \simeq U_a;\ z_1 = r_j^{\mathbb{T}} e^{i\theta_k^{\mathbb{T}}},\ |z_2| \leqslant \frac{3}{2}\left(r_j^{\mathbb{T}}\right)^m \left|\ln r_j^{\mathbb{T}}\right|\right\};$$

$$\mathbb{T}_{a,j,k,2} = \left\{(z_1,z_2) \in \mathbb{D}^2 \simeq U_a;\ z_2 = \left(r_j^{\mathbb{T}}\right)^m \left|\ln r_j^{\mathbb{T}}\right| e^{i\theta_k^{\mathbb{T}}},\ |z_1| \leqslant \left(\frac{3}{2}\right)^{1/m} r_j^{\mathbb{T}}\right\};$$

for $j \in \mathbb{Z}$ such that $r_{\text{sing}}(R) \leqslant \max\left(r_j^{\mathbb{T}}, \left(r_j^{\mathbb{T}}\right)^m \left|\ln r_j^{\mathbb{T}}\right|\right)$ and $\min\left(r_j^{\mathbb{T}}, \left(r_j^{\mathbb{T}}\right)^m \left|\ln r_j^{\mathbb{T}}\right|\right) \leqslant 2\rho$. We denote the maximal j by $N_a(R)$ and the minimal j by $N_a'(R)$. For a unification purpose, denote by $N_a'(R) = 0$ for the other singularities. Now, we need ρ so that

$$2\max\left(r_{N_a(R)}^{\mathbb{T}}, \left(r_{N_a(R)}^{\mathbb{T}}\right)^m \left|\ln r_{N_a(R)}^{\mathbb{T}}\right|\right) \leqslant 1.$$

That way, the biggest transversal is contained in $\frac{3}{4}\mathbb{D}^2$. The next results are analogous to Lemmas 5.2, 5.3 and 5.4.

Lemma 5.5. *Let $a \in E$ be as above and C_5 be small enough. If $j \in [\![N'_a(R)+1, N_a(R)]\!]$ and $z=(z_1,z_2) \in \mathbb{D}^2$ are such that we are in one of the following configurations,*

(1) Either $|z_1| \in \left[r_{j-1}^{\mathbb{T}}, r_j^{\mathbb{T}}\right]$ and $|z_2| \leqslant \left(r_j^{\mathbb{T}}\right)^m \left|\ln r_j^{\mathbb{T}}\right|$,

(2) Or $|z_1| \leqslant r_{j-1}^{\mathbb{T}}$ and $|z_2| \in \left[\left(r_{j-1}^{\mathbb{T}}\right)^m \left|\ln r_{j-1}^{\mathbb{T}}\right|, \left(r_j^{\mathbb{T}}\right)^m \left|\ln r_j^{\mathbb{T}}\right|\right]$.

then, there is $z'=(z'_1,z'_2) \in L_z$, with $d_P(z,z') \leqslant \frac{\hbar}{2}$ and one of the following configurations.

(i) Either $|z_1| = r_j^{\mathbb{T}}$ and $|z_2| \leqslant \left(r_j^{\mathbb{T}}\right)^m \left|\ln r_j^{\mathbb{T}}\right|$,

(ii) Or $|z_1| \leqslant r_j^{\mathbb{T}}$ and $|z_2| = \left(r_j^{\mathbb{T}}\right)^m \left|\ln r_j^{\mathbb{T}}\right|$.

Proof. The situation is similar to Lemma 5.2, the proof of which we keep some notations. Consider the flow $\varphi_z(t)$ starting at z in positive real time. Note $\varphi_z(t)=(z_1(t),z_2(t))$. Set

$$T = \inf\left\{t \in \mathbb{R}_+;\ |z_1(t)| = r_j^{\mathbb{T}} \text{ or } |z_2(t)| = \left(r_j^{\mathbb{T}}\right)^m \left|\ln r_j^{\mathbb{T}}\right|\right\}.$$

Similarly, it is enough to show that $T \leqslant C \ln \frac{r_j^{\mathbb{T}}}{r_{j-1}^{\mathbb{T}}}$. Indeed, for $t \leqslant T$, $|z_2(t)| \leqslant \left(r_j^{\mathbb{T}}\right)^{\frac{1}{2}}$. In configuration (1), the same arguments still work. Let us suppose the setup of configuration (2). Denote by $t = 2\ln \frac{r_j^{\mathbb{T}}}{r_{j-1}^{\mathbb{T}}}$. We prove that $t \geqslant T$ by ensuring that $|z_2(t)| \geqslant \left(r_j^{\mathbb{T}}\right)^m$.

$$|z_2(t)| \geqslant |z_2| \left(1 - \frac{|\mu t z_1^m|}{|z_2|}\right) \left(\frac{r_j^{\mathbb{T}}}{r_{j-1}^{\mathbb{T}}}\right)^{2m} \geqslant \left(r_j^{\mathbb{T}}\right)^m \left|\ln r_{j-1}^{\mathbb{T}}\right| \left(1 - \frac{\ln\left(r_j^{\mathbb{T}}/r_{j-1}^{\mathbb{T}}\right)}{\left|\ln r_{j-1}^{\mathbb{T}}\right|}\right) \left(\frac{r_j^{\mathbb{T}}}{r_{j-1}^{\mathbb{T}}}\right)^m.$$

Here, we have used both hypotheses of configuration (2). Now, the right hand side is equal to $\left(r_j^{\mathbb{T}}\right)^m \left|\ln r_j^{\mathbb{T}}\right| \left(\frac{r_j^{\mathbb{T}}}{r_{j-1}^{\mathbb{T}}}\right)^m$ and we conclude. □

Again, once the right radius found, we have to turn to the right angle.

Lemma 5.6. *Let $a \in E$ be as above. If C_5 is small enough and $z \in \mathbb{D}^2$ is in configuration* (i) *or* (ii)*, then there are $k \in [\![1,P]\!]$, $j \in \{1,2\}$ and $z' \in L_z \cap \mathbb{T}_{a,j,k,u}$ with $d_P(z,z') \leqslant \frac{\hbar}{2}$.*

Proof. In configuration (i), the proof is the same as for other singularities. Let us suppose we are in configuration (ii). Note $z_2 = \left(r_j^{\mathbb{T}}\right)^m \left|\ln r_j^{\mathbb{T}}\right| e^{i\theta}$ and $k \in [\![1,P]\!]$ with $\left|\theta - \theta_k^{\mathbb{T}}\right| \leqslant C_5\hbar$. We search a small t such that

$$f(t) = \left(1 + e^{-i\theta}\mu z_1^m \left(r_j^{\mathbb{T}}\right)^{-m} \left|\ln r_j^{\mathbb{T}}\right|^{-1} t\right) e^{mt} = e^{i\left(\theta_k^{\mathbb{T}} - \theta\right)}. \tag{5.1}$$

It is easy to see that $|f'(0)| \geqslant m - |\mu| \left|\ln r_j^{\mathbb{T}}\right|^{-1} \geqslant \frac{1}{2}$, shrinking again ρ if necessary. It is also clear that $|f(t) - f(0) - tf'(0)| \leqslant C|t|^2$ for $|t| \leqslant 1$ and some uniform constant $C>0$. Hence, f is injective on a small uniform disk and by Koebe $\frac{1}{4}$-Theorem, (5.1) admits a solution $t = O(C_5\hbar)$, if C_5 is small enough. We conclude the same as in Lemma 5.3. □

So, (HT1) is also true for this type of singularity. The following statement finishes the proof of Proposition 5.1, by showing (HT2) for Poincaré–Dulac singularities.

Lemma 5.7. *Let $a \in E$ be a singularity as above. There is a constant $K'>0$, independent on R such that for any $j \in [\![0, N_a(R)]\!]$, $k \in [\![1,P]\!]$, $u \in \{1,2\}$, there are at most K' elements $(j',k',u') \in [\![0,N_a(R)]\!] \times [\![1,P]\!] \times \{1,2\}$ such that there exist $x \in \mathbb{T}_{a,j,k,u}$, $y \in \mathbb{T}_{a,j',k',u'} \cap L_x$ with $d_P(x,y) \leqslant 2h_1$.*

Proof. The arguments are similar to those of Lemma 5.4, using that

$$\left(r_j^{\mathbb{T}}\right)^m \left|\ln r_j^{\mathbb{T}}\right| \in \left[C^{-1}\left(r_j^{\mathbb{T}}\right)^m, C\left(r_j^{\mathbb{T}}\right)^{\frac{1}{2}}\right]. \qquad \square$$

Actually, we correct a bit the movement that makes us end on a transversal to control monodromy processes below. For $u \in \{1,2\}$, we consider a vector field X_j^u, for $j \in \{1,2,3\}$. For the linearizable case, we put $X_1^1 = \frac{\partial}{\partial z_1} + \lambda z_2 \frac{\partial}{\partial z_2}$, $X_1^2 = \lambda^{-1} z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2}$. For the Poincaré–Dulac case, we just put $X_2^1 = X_2^2 = X_2$. For the Briot–Bouquet case, we put $X_3^1 = X_3$ and $X_3^2 = \widehat{X}_3$. When the $j \in \{1,2,3\}$ is implicitly determined by a singularity a, we simply denote the vector field by X^u. This enables to have the following. We use it below to obtain Lemma 7.8. Somehow, this is a way to deal with the movement to reach a transversal. Since it is not so standard (compared with what we do in Section 7), we need to control the behaviour in both coordinates. Hence, we need it not to be too neutral, and avoid turning around inside a transversal level $\{|z_u| = r_j^{\mathbb{T}}\}$.

Lemma 5.8. *Let $z \in 2\rho U_a$ be such that $d(x,E) \geqslant r_{\mathrm{sing}}(R)$. There are a transversal $\mathbb{T}_{a,j,k,u}$ and flow paths $\gamma^v\colon [0,1] \to \mathbb{C}$ for X^v, $v \in \{1,2\}$, such that $\varphi_z^1(\gamma^1(t)) = \varphi_z^2(\gamma^2(t))$, where φ_z^v denotes the flow of X^v, $\varphi_z^v(\gamma^v(1)) \in \mathbb{T}_{a,j,k,u}$, $\ell_P(\gamma) \leqslant \hbar$ and*

- *In the linearizable case, there is a constant $c > 0$ such that*

$$|\Im(\gamma^v(1))| \leqslant c\,|\Re(\gamma^v(1))| + O\left(C_5\hbar\right), \qquad v \in \{1,2\}.$$

- *In the Briot–Bouquet and Poincaré–Dulac cases, $|\Im(\gamma^v(1))| = O(C_5\hbar)$.*

Proof. We change slightly the processes of Lemmas 5.2, 5.3, 5.5 and 5.6. Note that they already give us a flow path γ, with $\gamma(1) = t_1 + t_2$, where t_1 is given by Lemma 5.2 or 5.5, and t_2 by Lemma 5.3 or 5.6. In any case, $|t_2| = O\left(C_5\hbar\right)$ so it is enough to show that

- We can choose $|\Re(t_1)| \geqslant c\,|\Im(t_1)|$ and $|\Re(\lambda t_1)| \geqslant c\,|\Im(\lambda t_1)|$ in the linearizable case.
- $\Im(t_1^v) = O(C_5\hbar)$, where t_1^v is the flow time corresponding to X^v and Lemma 5.3 for the Briot–Bouquet or Poincaré–Dulac case.

For the Poincaré–Dulac case, t_1 is real. For the linearizable case, we consider a complex number ω of modulus 1 such that $\Re(\omega) > 0$ and $\Re(\lambda\omega) \neq 0$. For $\varphi_z(\omega t)$ and with notations of Lemma 5.2, $T \leqslant \frac{1}{\Re(\omega)} \ln \frac{r_j^{\mathbb{T}}}{r_{j-1}^{\mathbb{T}}}$. We obtain our result if C_5 is small enough. For the Briot–Bouquet case, our process is in real flow time for X^1. We have

$$\varphi_z^1(\gamma^1(t)) = \left(z_1 e^{\gamma^1(t)}, z_2(\gamma^1(t))\right) = \varphi_z^2(\gamma^2(t)) = \left(z_1(\gamma^2(t)), z_2 e^{\gamma^2(t)}\right).$$

Note $\widetilde{z}_2(t) = z_2(t)e^{\alpha t}$ and $\widehat{z}_2(t) = \widetilde{z}_2(t) - z_2$. Locally, it is clear that

$$\gamma^2(t) = -\alpha\gamma^1(t) + \ln\left(1 + \frac{\widehat{z}_2(\gamma^1(t))}{z_2}\right).$$

Since $\widehat{z}_2'(t) = \widetilde{z}_2'(t)$, by Lemmas 3.7 and 3.8, we get $|\widehat{z}_2'(t)| \leqslant C\,|z_1|^\alpha\,|z_2|^2$. It follows that $|\widehat{z}_2(t)| = O\left(C_5\hbar\,|z_2|^2\right)$ and since $C_5\hbar$ is small, that $|\Im\left(\gamma^1(t)\right)| = O\left(C_5\hbar\right)$. $\square$

5.2. **Initial covering.** Now, we want to build an initial covering of $\widetilde{\mathbb{T}}$ on which we control the orthogonal projection and the flow in hyperbolic time $3h_1$. This is the covering that we progessively refine by Lemma 4.9 in order to apply Proposition 4.5. As we did for the transversals, let us begin by stating what we wish for our covering.

Proposition 5.9. *There exists a constant $C_6 > 0$ such that for R sufficiently large we have the following. For each $i \in I_{\mathbb{T}}$, there exists a covering by disks $\mathcal{V}_i$ of $\mathbb{T}_i$ such that*

(HD1) If $\mathbb{T}_i$ is a regular transversal, $D \in \mathcal{V}_i$ and $x, y \in 2D$, the orthogonal projection Φ_{xy} exists on $\phi_x(\mathbb{D}_{3h_1})$ and satisfies $d(x', \Phi_{xy}(x')) \leqslant e^{-2R}$.

(HD2) If $\mathbb{T}_i$ is a singular transversal, $D \in \mathcal{V}_i$ and $x, y \in 2D \backslash U_{\mathrm{sing}}(R)$, then the orthogonal projection Φ_{xy} exists in a neighbourhood of x and satisfies $d(x, \Phi_{xy}(x)) \leqslant e^{-2R}$. Moreover, x and $\Phi_{xy}(x)$ are $(3h_1, e^{-2R})$-relatively close following the flow.

(HD3) $\max_{i \in I_{\mathbb{T}}} \operatorname{card} \mathcal{V}_i \leqslant e^{C_6 R}$.

In the regular case, we can simply cover $\mathbb{T}_{r,\Lambda}$ by Ce^{4R} disks of radius Be^{-2R}, for B sufficiently small. The difficulty is again the singular case and we address particularly different types of singularities. For a linearizable or Briot–Bouquet singularity, we consider the covering $\mathcal{D}$ of Subsection 3.5 of a transversal $\mathbb{T}_{a,j,k,u}$. More precisely, we consider only the disks of $\mathcal{D}$ that intersect $\mathbb{T}_{a,j,k,u}$.

For a Poincaré–Dulac singularity, we consider the same covering if $u = 2$. On the other hand, if $u = 1$, we consider a covering of $\mathbb{T}_{a,j,k,1}$ by $O\left(e^{2C_4 R}\right)$ disks of radius $\left(r_j^{\mathbb{T}}\right)^m \left|\ln r_j^{\mathbb{T}}\right| e^{-C_4 R}$. Note that $\left|\ln r_j^{\mathbb{T}}\right| e^{-C_4 R} \leqslant e^{-2R}$ because $C_4 > C_3 + 2$. Actually, we need something stronger and need to enlarge again C_4.

Proof of Proposition 5.9. Points (HD1) and (HD3) are clear. We have to prove (HD2). For a transversal $\mathbb{T}_i = \mathbb{T}_{a,j,k,u}$, $D \in \mathcal{V}_i$ and $x = (z_1, z_2), y = (w_1, w_2) \in 2D \backslash U_{\mathrm{sing}}(R)$, we have $z_u = w_u$. In any case, it is then clear that $\frac{\|z-w\|_1}{\|z\|_1} = O\left(\left|\ln \|z\|_1\right| e^{-C_4 R}\right) = O\left(e^{-2R}\right)$. Thus, the orthogonal projection exists and $y' = (w_1', w_2') = \Phi_{xy}(x) = \varphi_y(t)$, for some $t = O(|\ln \|z\|_1| e^{-C_4 R})$. To show that x and y' are $(3h_1, e^{-2R})$-relatively close following the flow, we have to consider different cases. Note $v \in \{1, 2\}$ such that $v \neq u$, so that the coordinate on the transversal is z_v.

$D = D_0$ for a linearizable, Briot–Bouquet or Poincaré–Dulac singularity with $u = 2$. Note that $d(x, y') \leqslant C r_0$. Since $C_4 > C_3$ and $|z_u| \geqslant (r_{\mathrm{sing}}(R))^m$, it is easy to check that z_u and w_u' are in configuration (C2) and z_v, w_v' in configuration (C1) of Lemmas 3.5, 3.6 and 3.10.

$D = D_{nk}$ with the same singularities. Here, we have $\frac{\|z-w\|_1}{\|z\|_1} = O\left(e^{-C_4 R}\right)$ and t is a $O\left(e^{-C_4 R}\right)$. Since $\{z_v = 0\}$ is a separatrix, Grönwall Lemma implies that z_v and w_v' are in configuration (C2). For a linearizable or Briot–Bouquet singularity, this is the same for z_u and w_u'. For the Poincaré–Dulac case,

$$z_2 - w_2' = z_2 - (z_2 + \mu t w_1^m)\, e^{mt}.$$

The condition $|w_1|^m \leqslant C\,|z_2|\,|\ln \|z\|_1|^{-1}$ and $t = O\left(e^{-C_4 R}\right)$ give us that z_2 and w_2' are in configuration (C2.2) if $C_4 > C_3 + 2$.

A Poincaré–Dulac singularity with $u = 1$. Since $t = O\left(\left|\ln \|z\|_1\right| e^{-C_4 R}\right)$, z_1 and w_1' are in configuration (C1.2) if $C_4 > 2C_3 + 2$. On the other hand,

$$z_2 - w_2' = z_2 - (w_2 + \mu t z_1^m)\, e^{mt}.$$

This time, $|z_2 - w_2| = O\left(|z_1|^m \left|\ln \|z\|_1\right| e^{-C_4 R}\right)$ and again $t = O\left(e^{-2R}\right)$. Hence, $|z_2 - w_2'|$ is a $O\left(|z_1|^m e^{-2R}\right)$ and z_2, w_2' are in configuration (C2.1). □

6. Holonomy and flow in hyperbolic time

6.1. **Comparison of three motion processes.** We need to control the behaviour of an orthogonal projection. Since we work on a universal cover, it will be very convenient

to have processes that are invariant under homotopy and that is why we study holonomy. Moreover, this motion enables us to carry information step by step from far away transversals to the regular transversals where we want to build orthogonal projections. Still, we need it to be well defined on an initial disk and a way to keep disks in the process. Here, estimating the flow is crucial to guarantee these technical elements. The end of the proof relies deeply on comparing these three motions. We need some preparation.

Lemma 6.1. *Let $a \in E$ be some singularity of $\mathscr{F}$. There exists a constant $\varepsilon_3 > 0$ such that for any $z_0 \in \frac{3}{4}U_a$, there exists a regular flow box $U \simeq \mathbb{D} \times \mathbb{T}$ with $B\left(z_0, \varepsilon_3 \left\|z_0\right\|_1\right) \subset U$.*

Proof. If $z \in U_a$ satisfies $\left\|z - z_0\right\|_1 \leqslant \frac{1}{2C_0C_1}\left\|z_0\right\|_1$, then $\left\|X(z) - X(z_0)\right\|_1 \leqslant \frac{1}{4}\left\|X(z_0)\right\|_1$ by (3.2) and (3.3). Write $X = X_1\frac{\partial}{\partial z_1} + X_2\frac{\partial}{\partial z_2}$. Without loss of generality, we can suppose that $\left\|X(z_0)\right\|_1 = |X_1(z_0)|$. Considering the exponential map of $\frac{X}{X_1}$, we conclude. □

This preparation enables us to ensure that the holonomy maps we consider are not only germs, but are defined on the whole disks of the initial covering.

Proposition 6.2. *Let $\mathbb{T}_i, \mathbb{T}_j \in \widetilde{\mathbb{T}}$, $z \in \mathbb{T}_i$ and $\lambda\colon [0,1] \to L_z$ be such that $\lambda(0) = z$, $\lambda(1) \in \mathbb{T}_j$ and $\ell_P(\lambda) < 2h_1$. Let $D \in \mathcal{V}_i$ be a disk of the initial covering containing z. If R is sufficiently large, the holonomy map Hol_λ along λ from $\mathbb{T}_i$ to $\mathbb{T}_j$ is well defined on $2D$.*

If $w \in 2D$, there exists a unique map $\widetilde{\Phi}_{zw}\colon \mathbb{D}_{2h_1} \to L_w$ such that near $\zeta \in \mathbb{D}_{2h_1}$, if $z' = \phi_z(\zeta)$ and $w' = \widetilde{\Phi}_{zw}(\zeta)$, then $\widetilde{\Phi}_{zw} = \Phi_{z'w'} \circ \phi_z$, where $\Phi_{z'w'}$ is the orthogonal projection from $L_{z'}$ near z' onto $L_{w'}$ near w'; and moreover $\widetilde{\Phi}_{zw} = \Phi_{zw} \circ \phi_z$ in a neighbourhood of 0.

Finally, if $w'' = \mathrm{Hol}_\lambda(w)$ and $\zeta \in \mathbb{D}_{2h_1}$ is such that $\phi_z(\zeta) = z' = \lambda(1)$, then $\widetilde{\Phi}_{zw} = \Phi_{z'w''} \circ \phi_z$ in a neighbourhood of ζ.

Proof. Since $\mathscr{F}$ is Brody-hyperbolic and h_1 is small, the conclusions are clear for $\mathbb{T}_i$ a regular transversal. If $\mathbb{T}_i$ is a singular transversal, then note that z and w are $(3h_1, e^{-2R})$-relatively close following the flow by Proposition 5.9. Considering a flow path γ for z representing λ and using Lemma 6.1, we can cut γ in parts where $\lambda(t)$ and $\varphi_w(\gamma(t))$ are in the same flow box to ensure that the holonomy is well defined. The existence of $\widetilde{\Phi}_{zw}$ and its link with $\Phi_{z'w''}$ can be addressed similarly using Lemma 2.6. □

6.2. **Image of disks by holonomy.** Now that we know the holonomy maps are well defined on the disks of the covering, we need a method to keep disks to apply Lemma 4.9. We need the following notion, which is close to one that may be found in [6, p. 617].

Definition 6.3. Let U be an open subset of $\mathbb{C}$ and $\sigma > 1$. We call U *σ-quasi-round* if there exists a disk D with $\sigma^{-1}D \subset U \subset \sigma D$.

Given a σ-quasi-round open set, we use the following to make it round.

Lemma 6.4. *There exists $\sigma_0 > 1$ such that for $\sigma \in (1, \sigma_0)$ and U σ-quasi-round, there exist four disks D_1, D_2, D_3, D_4 with $U \subset \cup_{k=1}^4 D_k$ and $2D_k \subset 2U$, for $k \in [\![1,4]\!]$.*

Proof. Let D be a disk with $\sigma^{-1}D \subset U \subset \sigma D$. By an affine transformation of $\mathbb{C}$, one can suppose that $D = \mathbb{D}$. Il is enough to build the D_k satisfying $2D_k \subset 2\sigma^{-1}\mathbb{D}$ and $\sigma\mathbb{D} \subset \cup_{k=1}^4 D_k$. Easy computations show that for $\sigma < \sigma_0 = \sqrt{\sqrt{6} - \sqrt{2}} \approx 1.02$ and $D_k = D\left(\left(4\frac{\sqrt{3}}{3} - 2\right)\sigma^{-1}i^k, \left(2 - 2\frac{\sqrt{3}}{3}\right)\sigma^{-1}\right)$, we have the result. □

Fix $\sigma_1 \in (0, \sigma_0)$. We want to prove the following. Note that this is where being in dimension 2 is crucial for the case of linearizable singularities [6]. Indeed, for transversals of higher dimension, one would have polydisks of the form $\Delta = r_1\mathbb{D} \times \cdots \times r_n\mathbb{D}$, with possibly very degenerate quotients $\frac{r_i}{r_j}$ and no refinement lemma would be available.

Proposition 6.5. *We keep the notations of Proposition 6.2. If h_1 is sufficiently small, R is sufficiently large and $D' \subset 2D$ is a disk, then $\mathrm{Hol}_\lambda(D')$ is σ_1-quasi-round.*

Let us begin with the easiest cases.

Proposition 6.5's beginning of proof. First, suppose that $\mathbb{T}_i$ or $\mathbb{T}_j$ is a *regular* transversal. Note that Hol_λ is holomorphic without critical points. Hence, it is close to homotheties on small disks and we conclude. Here, we study a bounded number of maps, independently of R. Moreover, these arguments still work if we consider singular transversals that do not approach too much (independently of R) the singularities.

Now, suppose that $\mathbb{T}_i$ and $\mathbb{T}_j$ are singular transversals, for a *linearizable* singularity. Let us introduce many notations, that we keep for the other types of singularities. Note $z^0 = (z_1^0, z_2^0)$ the center of the disk D', and $z^1 = (z_1^1, z_2^1) = \mathrm{Hol}_\lambda(z^0)$. Let $\gamma\colon [0,1] \to L_{z^0}$ be a flow path corresponding to λ. Here, we can suppose that λ is a path on L_{z^0}, since we have a sequence of flow boxes that contain the image of λ and corresponding plaques for z^0. Note $w^0 = (w_1^0, w_2^0) \in D'$, $w^1 = (w_1^1, w_2^1) = \mathrm{Hol}_\lambda(w^0)$, which is well defined by Proposition 6.2. Let $w'^1 = (w_1'^1, w_2'^1) = \varphi_{w^0}(\gamma(1))$. Finally, denote $\mathbb{T}_i = \mathbb{T}_{a,j_0,k_0,u_0}$ and $\mathbb{T}_j = \mathbb{T}_{a,j_1,k_1,u_1}$. By definition, note that $z_{u_0}^0 = w_{u_0}^0$ and $z_{u_1}^1 = w_{u_1}^1$. Let us go back to the linearizable case. For $j \in \{1,2\}$, $\lambda_1 = 1$ and $\lambda_2 = \lambda$, we have $w_j'^1 - z_j^1 = (w_j^0 - z_j^0)e^{\lambda_j \gamma(1)}$. Hence, if $u_0 = u_1$, $\mathrm{Hol}_\lambda(D')$ is actually a disk. On the other hand, if $u_0 \neq u_1$,

$$w^1 = \varphi_{w'^1}\left(\frac{1}{\lambda_{u_1}} \ln \frac{z_{u_1}^1}{w_{u_1}'^1}\right) = \varphi_{w'^1}\left(\frac{1}{\lambda_{u_1}} \ln \frac{z_{u_1}^0}{w_{u_1}^0}\right).$$

Denote by $t_w = \frac{1}{\lambda_{u_1}} \ln \frac{z_{u_1}^0}{w_{u_1}^0}$. We keep this notation such that $w^1 = \varphi_{w'^1}(t_w)$ in the other cases. Here, we have $t_w = \frac{1}{\lambda_{u_1}} \frac{z_{u_1}^0 - w_{u_1}^0}{z_{u_1}^0} + O\left(\left|\frac{z_{u_1}^0 - w_{u_1}^0}{z_{u_1}^0}\right|^2\right)$. This implies that

$$w_{u_0}^1 - z_{u_0}^1 = z_{u_0}^0 e^{\lambda_{u_0}\gamma(1)}\left(e^{\lambda_{u_0} t_w} - 1\right) = \frac{\lambda_{u_0}}{\lambda_{u_1}} z_{u_0}^0 e^{\lambda_{u_0}\gamma(1)} \frac{z_{u_1}^0 - w_{u_1}^0}{z_{u_1}^0}\left(1 + O\left(\left|\frac{w_{u_1}^0 - z_{u_1}^0}{z_{u_1}^0}\right|\right)\right).$$

Finally, since $u_0 \neq u_1$, it is easy to see that we can not come from a separatrix disk, *i.e.* $D \neq D_0$. Therefore, $\left|\frac{w_{u_1}^0 - z_{u_1}^0}{z_{u_1}^0}\right| \leqslant Ce^{-C_4 R}$ and we conclude because $w_{u_1}^0$ runs through the disk D' of center $z_{u_1}^0$.

Next, consider a *Poincaré–Dulac* type singularity. We need to distinguish all four cases for starting and arrival transversals.

$u_0 = u_1 = 1$. As in the linearizable case, $w'^1 = w^1$ and $w_2^1 - z_2^1 = (w_2^0 - z_2^0)e^{m\gamma(1)}$. It follows that $\mathrm{Hol}_\lambda(D')$ is a disk.

$u_0 = 2$ *and* $u_1 = 1$. Here, we have $t_w = \ln \frac{z_1^0}{w_1^0}$ and $D \neq D_0$. As before, it follows that $t_w = \frac{z_1^0 - w_1^0}{z_1^0}\left(1 + O\left(\left|\frac{z_1^0 - w_1^0}{z_1^0}\right|\right)\right)$ and $\left|\frac{z_1^0 - w_1^0}{z_1^0}\right| \leqslant Ce^{-C_4 R}$. Let us compute.

$$w_2'^1 = \left(z_2^0 + \mu\gamma(1) w_1^{0m}\right) e^{m\gamma(1)} = z_2^1 - m\mu\gamma(1) z_1^{1m} \frac{z_1^0 - w_1^0}{z_1^0}\left(1 + O\left(e^{-C_4 R}\right)\right).$$

By the estimate on t_w and since $w^1 = \varphi_{w'^1}(t_w)$, we get

$$\begin{aligned} w_2^1 - w_2'^1 &= \left(mw_2'^1 + \mu w_1'^{1m}\right) \frac{z_1^0 - w_1^0}{z_1^0} \left(1 + O\left(e^{-C_4 R}\right)\right) \\ &= \left(mz_2^1 + \mu z_1^{1m}\right) \frac{z_1^0 - w_1^0}{z_1^0} \left(1 + O\left(e^{-2R}\right)\right), \end{aligned}$$

because z^0 and w^0 are $(3h_1, e^{-2R})$-relatively close following the flow. Here, we used that since $u_0 = 2$, we still have $|z_1^1|^m = O\left(|\ln \|z^0\|_1|^{-1} |z_2^1|\right)$. Putting together the last two equations, we conclude if R is sufficiently large by

$$w_2^1 - z_2^1 = \left(m\mu\gamma(1) z_1^{1m} - mz_2^1 + \mu z_1^{1m}\right) \frac{z_1^0 - w_1^0}{z_1^0} \left(1 + O\left(e^{-2R}\right)\right).$$

$u_0 = 1$ and $u_1 = 2$. Since $u_1 = 2$, $\left|m - \mu \frac{z_1^{1m}}{z_2^1}\right| > \frac{1}{2}$. Arguing as in Lemma 5.6, we get

$$t_w = \frac{z_2^1 - w_2'^1}{mw_2'^1 + \mu w_1'^{1m}} \left(1 + O\left(\left|\frac{z_2^1 - w_2'^1}{z_2^1}\right|\right)\right) = \frac{z_2^1 - w_2'^1}{mz_2^1 + \mu z_1^{1m}} \left(1 + O\left(e^{-2R}\right)\right), \tag{6.1}$$

since $w_1'^1 = z_1^1$ and

$$\left|w_2'^1 - z_2^1\right| = \left|z_2^0 - w_2^0\right| e^{m\Re(\gamma(1))} \leqslant \frac{|z_2^0 - w_2^0|}{|z_1^0|^m} \left|z_1^1\right|^m = O\left(e^{-2R} \left|z_2^1\right|\right).$$

Hence, we conclude with the estimate

$$\frac{z_1^1 - w_1^1}{z_1^1} = \frac{e^{m\gamma(1)}}{mz_2^1 + \mu z_1^{1m}} (z_2^0 - w_2^0) \left(1 + O\left(e^{-2R}\right)\right).$$

$u_0 = u_1 = 2$. This time, $z_2^0 = w_2^0$. Thus,

$$z_2^1 - w_2'^1 = \mu\gamma(1)(z_1^{1m} - w_1'^{1m}) \quad \text{and} \quad z_1^1 - w_1'^1 = e^{\gamma(1)}(z_1^0 - w_1^0).$$

Moreover, the following bound and a similar for $z_1^{1m} - w_1'^{1m}$ show that (6.1) still holds.

$$\left|z_2^1 - w_2'^1\right| = |\mu\gamma(1)| \left|z_1^{1m} - w_1^{1m}\right| \leqslant C |\mu\gamma(1)| \left|z_1^1\right|^m e^{-C_4 R} \leqslant C \left|z_2^1\right| e^{-C_4 R},$$

if h_1 is sufficiently small, because $u_1 = 2$. Here, we have considered the case $D \neq D_0$ but $D = D_0$ is even simpler. It follows that

$$z_1^1 - w_1^1 = e^{\gamma(1)}(z_1^0 - w_1^0) \left(1 - \frac{\mu\gamma(1)}{mz_2^1 + \mu z_1^{1m}} \sum_{k=0}^{m-1} \left(z_1^1\right)^k \left(w_1'^1\right)^{m-k} \left(1 + O\left(e^{-2R}\right)\right)\right).$$

Since $u_1 = 2$, note that $|z_2^1| \geqslant \frac{1}{3} |\mu\gamma(1)| |z_1^1|^m$ if h_1 is sufficiently small. Therefore,

$$\frac{z_1^1 - w_1^1}{z_1^1} = \frac{z_1^0 - w_1^0}{z_1^0} \times \frac{mz_2^1 + \mu(1 - \gamma(1)) z_1^{1m}}{mz_2^1 + \mu z_1^{1m}} \left(1 + O\left(e^{-2R}\right)\right).$$

We have our result for R sufficiently large. □

It leaves us with the Briot–Bouquet case, for which we still need some preparation. We need to ensharpen the estimates of Subsection 3.4. Let us reintroduce some of its notations. Consider the flow in $\widetilde{X}_3$, for $z \in \frac{1}{2}\mathbb{D}^2$, denote by $\varphi_z(t) = (z_1(t), z_2(t))$, with $z_1(t) = z_1 e^t$. Denote by $\widetilde{z}_2(t) = z_2(t) e^{\alpha t}$ and $\widehat{z}_2(t) = \widetilde{z}_2(t) - z_2$, as in Lemma 5.8.

Lemma 6.6. *Take the notations above for z^0 and w^0 in the proof of Proposition 6.5. If R is sufficiently large and h_1 is sufficiently small, then*

(i) If $z_1^0 = w_1^0$, then $|\widehat{z}_2^0(\gamma(1)) - \widehat{w}_2(\gamma(1))| = O\left(|z_2^0|\,|z_2^0 - w_2^0|\right)$.
(ii) If $z_2^0 = w_2^0$, then $|\widehat{z}_2^0(\gamma(1)) - \widehat{w}_2^0(\gamma(1))| = O\left(|z_2^0|\,|z_1^0 - w_1^0|\right)$.

Proof. Note that $\widehat{z}_2^{0\prime}(t) = \widetilde{z}_2^{0\prime}(t)$ and by (3.7),

$$\left|\widehat{z}_2^{0\prime}(t) - \widehat{w}_2^{0\prime}(t)\right| \leqslant C\left(\left|z_2^0\right|^{\frac{3}{2}}\left|z_1^0(t) - w_1^0(t)\right| + \left|w_1^0\right|^{\alpha}\max\left(\left|z_2^0\right|,\left|w_2^0\right|\right)\left|\widetilde{z}_2^0(t) - \widetilde{w}_2^0(t)\right|\right). \tag{6.2}$$

Now, consider the two cases of our statement. In the first one, $z_1^0(t) = w_1^0(t)$ and using Lemma 3.9, $|\widetilde{z}_2^0(t) - \widetilde{w}_2^0(t)| \leqslant 2\,|z_2^0 - w_2^0|$. Hence,

$$\left|\widehat{z}_2^0(\gamma(1)) - \widehat{w}_2^0(\gamma(1))\right| \leqslant C\left|\ln\left\|z^0\right\|_1\right|\left|z_1^0\right|^{\alpha}\left|z_2^0\right|\left|z_2^0 - w_2^0\right|.$$

Then, our result is a consequence of the fact that $|z_1^0|^{\alpha}\,|\ln\|z^0\|_1|$ is bounded.

In the second case, we have $\widehat{z}_2^0(t) - \widehat{w}_2^0(t) = \widetilde{z}_2^0(t) - \widetilde{w}_2^0(t)$. Applying the refined Grönwall Lemma 2.7 to (6.2), denoting by $\beta = C\,|w_1^0|^{\alpha}\max\left(|z_2^0|,|w_2^0|\right)$, we obtain

$$\left|\widehat{z}_2^0(\gamma(1)) - \widehat{w}_2^0(\gamma(1))\right| \leqslant C\left|z_2^0\right|^{\frac{3}{2}} e^{|\gamma(1)|}\left|z_1^0 - w_1^0\right|\int_0^{|\gamma(1)|} e^{\beta(|\gamma(1)|-s)}ds.$$

Since $\beta\,|\gamma(1)|$ is uniformly bounded, the last integral is bounded above by $C\,|\gamma(1)|$. On the other hand $e^{|\gamma(1)|} \leqslant |z_2^0|^{-\frac{1}{2}}$ if h_1 is sufficiently small. The result follows. □

Proposition 6.5's end of proof. We just have to consider a Briot–Bouquet case and with our arguments in the regular case, we can suppose that $\|z^0\|_1$ is sufficiently small (independently of R). Considering $\widehat{X}_3$ if necessary, we can suppose that $u_1 = 1$. We have to distinguish whether $u_0 = 1$ or $u_0 = 2$.

Case $u_0 = 1$. In that case, $t_w = 0$. It follows that

$$z_2^1 - w_2^1 = e^{-\alpha\gamma(1)}\left(z_2^0 - w_2^0 + \widehat{z}_2^0(\gamma(1)) - \widehat{w}_2^0(\gamma(1))\right) = e^{-\alpha\gamma(1)}\left(z_2^0 - w_2^0\right)\left(1 + O\left(\left|z_2^0\right|\right)\right).$$

For $O\left(|z_2^0|\right) \leqslant \sigma_1 - 1$, we have our result.

Case $u_0 = 2$. As for other singularities, $t_w = \ln\frac{z_1^0}{w_1^0} = \frac{z_1^0 - w_1^0}{z_1^0}\left(1 + O\left(e^{-C_4R}\right)\right)$, since $D \neq D_0$. On the other hand, $z_2^1 - w_2'^1 = e^{-\alpha\gamma(1)}(\widehat{z}_2^0(\gamma(1)) - \widehat{w}_2^0(\gamma(1)))$. We get

$$z_2^1 - w_2^1 = e^{-\alpha\gamma(1)}\left(\widehat{z}_2^0(\gamma(1)) - \widehat{w}_2^0(\gamma(1)) - \alpha\widetilde{z}_2^1(1 + z_1^1z_2^1f(z_1^1,z_2^1))\frac{z_1^0 - w_1^0}{z_1^0}\left(1 + O\left(e^{-2R}\right)\right)\right),$$

because z_2^0 and w_2^0 are $(3h_1, e^{-2R})$-relatively close following the flow. Here, we have denoted by $\widetilde{z}_2^1 = \widetilde{z}_2^0(\gamma(1))$. By Lemma 6.6 and Lemma 3.8,

$$z_2^1 - w_2^1 = -\alpha e^{-\alpha\gamma(1)}\widetilde{z}_2^1(1 + z_1^1z_2^1f(z_1^1,z_2^1))\frac{z_1^0 - w_1^0}{z_1^0}\left(1 + O\left(e^{-2R}\right) + O\left(\left|z_1^0\right|\right)\right).$$

Choosing $|z_1^0|$ sufficiently small and R sufficiently large, we conclude. □

7. Hyperbolic motion trees

7.1. **Covering the Poincaré disk.** We still face some serious problems. Whereas a transversal can only interact with a finite number of others by (HT2), it can do so in many ways. Indeed, a transversal that is close to the singularity has around e^{C_3R} monodromy paths of Poincaré length lower than $2h_1$. To tackle this, we consider some standard flow motion. We need a refinement of Lemma 3.3, the proof of which is the same.

Lemma 7.1. *Let $z \in \frac{1}{2}U_a\backslash\{a\}$, $\gamma\colon [0,1] \to \mathbb{C}$ be a flow path for z and h_1 be sufficiently small. We denote by $\ell_P(\gamma)$ the Poincaré length of $\varphi_z \circ \gamma$ and $\ell(\gamma)$ the length of γ for the usual Hermitian metric of $\mathbb{C}$. If $\ell_P(\gamma) \leqslant h_1$, then there exists a constant $C_7 > 1$ with*

$$C_7^{-1}\ell(\gamma) \leqslant \ell_P(\gamma)\,|\ln \|z\|_1| \leqslant C_7\ell(\gamma).$$

Fix $\hbar = \frac{1}{3C_8^2}h_1$ and an integer $p > 0$, for $C_8 > C_7$ that our further computation specifies. For $z \in \mathbb{T}_{a,j,k,u}$ in a singular transversal, denote by

$$\varphi_k(z) = \varphi_z\left(C_8^{-1}h_1\left|\ln r_j^{\mathbb{T}}\right| e^{\frac{2ik\pi}{p}}\right), \qquad k \in [\![0, p-1]\!].$$

These maps are standart flows. In what follows, we use criteria close to [6, Lemma 4.6].

Lemma 7.2. *Let $z \in \mathbb{T}_{a,j,k,u}$ be in a singular transversal. If p is sufficiently large, h_1 is sufficiently small and C_8 is well chosen, there exist $\zeta_0, \dots, \zeta_{p-1} \in \mathbb{D}_{h_1}$ with $\phi_z(\zeta_k) = \varphi_k(z)$ for $k \in [\![0, p-1]\!]$. Moreover,*

$$\mathbb{D}_{h_1+\hbar} \subset \mathbb{D}_{h_1} \cup \bigcup_{k=0}^{p-1} \mathbb{D}_{h_1-\hbar}(\zeta_k).$$

In particular, if $\xi_0, \dots, \xi_{p-1} \in \mathbb{D}_{h_1+\hbar}$ and satisfy $d_P(\zeta_k, \xi_k) \leqslant \hbar$, $k \in [\![0, p-1]\!]$, then

$$\mathbb{D}_{h_1+\hbar} \subset \mathbb{D}_{h_1} \cup \bigcup_{k=0}^{p-1} \mathbb{D}_{h_1}(\xi_k).$$

Proof. The second statement is indeed a direct consequence of the first one. Note that by definition of our transversals, we have $c^{-1}\left|\ln r_j^{\mathbb{T}}\right| \leqslant |\ln \|z\|_1| \leqslant c\left|\ln r_j^{\mathbb{T}}\right|$. Define $C_8 = cC_7$. If h_1 is sufficiently small, Grönwall Lemma implies that the flow is defined on the whole disk $C_8h_1\left|\ln r_j^{\mathbb{T}}\right|\mathbb{D}$. Consider the paths $\lambda_k\colon [0,1] \to L_z$ defined by

$$\lambda_k(t) = \varphi_z\left(C_8^{-1}th_1\left|\ln r_j^{\mathbb{T}}\right| e^{\frac{2ik\pi}{p}}\right).$$

Take the lifting $\widetilde{\lambda}_k$ in $\mathbb{D}$ *via* ϕ_z such that $\widetilde{\lambda}_k(0) = 0$ and define $\zeta_k = \widetilde{\lambda}_k(1)$. Thus, we have $\ell(\lambda_k) = C_8^{-1}h_1\left|\ln r_j^{\mathbb{T}}\right|$, so $\ell_P(\lambda_k) \leqslant h_1$ by Lemma 7.1 and by definition of C_8. Consequently, $\zeta_k \in \mathbb{D}_{h_1}$. Let $\xi \in \mathbb{D}_{h_1+\hbar}$. If $\xi \in \mathbb{D}_{h_1}$, there is nothing to prove. If $d_P(0,\xi) \geqslant h_1$, define the radius $r_1 = \frac{e^{h_1}-1}{e^{h_1}+1}$ and $\xi_1 = r_1\frac{\xi}{|\xi|}$. Let $\gamma\colon [0, |\xi|] \to \mathbb{C}$ be the flow path representing ξ and $\lambda(t) = \varphi_z(t\gamma(r_1))$, which is homotopic to the path $\varphi_z \circ \gamma_{|[0,r_1]}$. Hence, its lifting *via* ϕ_z with $\widetilde{\lambda}(0) = 0$ satisfies $\widetilde{\lambda}(1) = \xi_1$. By definition, we have $\ell(\lambda) = |\gamma(r_1)|$ and $\ell_P(\lambda) \geqslant h_1$. Thus, $|\gamma(r_1)| \geqslant C_8^{-1}h_1\left|\ln r_j^{\mathbb{T}}\right|$ and there exists $r_1' \in [0, r_1]$ with $|\gamma(r_1')| = C_8^{-1}h_1\left|\ln r_j^{\mathbb{T}}\right|$. Note $\xi_1' = r_1'\frac{\xi}{|\xi|}$ and the hyperbolic radius $h_1' = \ln\frac{1+r_1'}{1-r_1'}$. Applying Lemma 7.1, we get

$$h_1' = \ell_P\left(\gamma_{|[0,r_1']}\right) \geqslant C_8^{-1}\left|\ln r_j^{\mathbb{T}}\right|^{-1}\ell\left(\gamma_{|[0,r_1']}\right) \geqslant C_8^{-1}\left|\ln r_j^{\mathbb{T}}\right|^{-1}|\gamma(r_1')| = 3\hbar.$$

Hence, $d_P(\xi, \xi_1') < h_1 - 2\hbar$. It follows that it is enough to find $k \in [\![0, p-1]\!]$ with $d_P(\xi_1', \zeta_k) \leqslant \hbar$. It is clear that we can find k such that

$$\left|\gamma(r_1') - |\gamma(r_1')|\, e^{\frac{2ik\pi}{p}}\right| \leqslant \frac{\pi}{p}\,|\gamma(r_1')| = \frac{\pi}{p}C_8^{-1}h_1\left|\ln r_j^{\mathbb{T}}\right|.$$

Denote by $z_k = \varphi_k(z)$, $u_1' = \gamma(r_1')$ and $t_k = C_8^{-1}h_1\left|\ln r_j^{\mathbb{T}}\right| e^{\frac{2ik\pi}{p}}$. Consider the flow path which goes straight from 0 to u_1' and then travels the arc from u_1' to t_k. It is homotopic to

the straight line from 0 to t_k. Thus, $\phi_z(\xi_1') = \varphi_{z_k}(u_1' - t_k)$. Grönwall Lemma implies that $|\ln \|z_k\|_1| \geqslant \frac{1}{2} |\ln \|z\|_1| \geqslant \frac{1}{2c} \left|\ln r_j^{\mathbb{T}}\right|$ if h_1 is sufficiently small. By Lemma 7.1,

$$d_P(\xi_1', \zeta_k) \leqslant \frac{\pi}{p} C_7 C_8^{-1} h_1 \left|\ln \|z_k\|_1\right|^{-1} \left|\ln r_j^{\mathbb{T}}\right| \leqslant \frac{2\pi h_1}{p}.$$

For $p \geqslant \frac{2\pi h_1}{\hbar} = 6\pi C_8^2$, we have our result. □

Far from the singular set, we use a slightly different criterion.

Lemma 7.3. *Note $r_1 = \frac{e^{h_1}-1}{e^{h_1}+1}$ and $\zeta_k = r_1 e^{\frac{2ik\pi}{p}}$, for $k \in [\![0, p-1]\!]$. If h_1 is sufficiently small,*

$$\mathbb{D}_{h_1+\hbar} \subset \mathbb{D}_{h_1} \cup \bigcup_{k=0}^{p-1} \mathbb{D}_{h_1-\hbar}(\zeta_k).$$

In particular, if $\xi_0, \dots, \xi_{p-1} \in \mathbb{D}_{h_1+\hbar}$ and satisfy $d_P(\zeta_k, \xi_k) \leqslant \hbar$, $k \in [\![0, p-1]\!]$, then

$$\mathbb{D}_{h_1+\hbar} \subset \mathbb{D}_{h_1} \cup \bigcup_{k=0}^{p-1} \mathbb{D}_{h_1}(\xi_k).$$

Proof. Let $\xi \in \mathbb{D}_{h_1+\hbar}$. If $d_P(0, \xi) < h_1$, then $\xi \in \mathbb{D}_{h_1}$. Next, suppose that $d_P(0, \xi) \geqslant h_1$. Define $\xi = \frac{r_1}{|\xi|}\xi$ so that $d_P(\xi, \xi_1) < \hbar$. Note that $\ell_P(\partial \mathbb{D}_{h_1}) = \pi(e^{h_1} - e^{-h_1}) \leqslant 4\pi h_1$ if h_1 is sufficiently small. So, there exists $k \in [\![0, p-1]\!]$ with $d_P(\xi_1, \zeta_k) \leqslant \frac{2\pi h_1}{p} < \hbar$. Since $C_8 > 1$, $d_P(\xi, \zeta_k) < 2\hbar \leqslant h_1 - \hbar$. □

7.2. **Encoding the hyperbolic dynamics.** Here, the notion we introduce is something that exists but is not precisely defined in [6, p. 619–623]. It encodes the transversals through which a leaf L_x goes in time R, and ensures that we get a h_1-dense subset.

Definition 7.4. For $H \in \mathbb{N}$, we define

$$\mathcal{A}_H = \bigsqcup_{j=0}^{H} [\![0, p-1]\!]^j,$$

where by convention $[\![0, p-1]\!]^0 = \{\varnothing\}$. We see $\mathcal{A}_H$ as a tree, the directed edges of which are $(i_1, \dots, i_k) \to (i_1, \dots, i_{k+1})$, for $k \in [\![0, H-1]\!]$ and $i_1, \dots, i_{k+1} \in [\![0, p-1]\!]$. A *hyperbolic motion tree of deepness H* is a map $\Theta \colon \mathcal{A}_H \to \mathbb{D}$ with $\Theta(\varnothing) = 0$ and for any vertex $i_1, \dots, i_k \in [\![0, p-1]\!]$,

$$\mathbb{D}_{h_1+\hbar}\left(\Theta\left(i_1, \dots, i_k\right)\right) \subset \mathbb{D}_{h_1}\left(\Theta\left(i_1, \dots, i_k\right)\right) \cup \bigcup_{i_{k+1}=0}^{p-1} \mathbb{D}_{h_1}\left(\Theta\left(i_1, \dots, i_{k+1}\right)\right),$$

and

$$\Theta(i_1, \dots, i_{k+1}) \in \overline{\mathbb{D}}_{h_1+\hbar}(\Theta(i_1, \dots, i_k)), \quad i_{k+1} \in [\![0, p-1]\!].$$

The *cut-off definition tree of* Θ is defined as the subset $\mathcal{A}_H^\Theta \subset \mathcal{A}_H$, made of elements $(i_1, \dots, i_k) \in \mathcal{A}_H$, for $k \in [\![0, H]\!]$ such that for all $j \in [\![0, k]\!]$, $\Theta(i_1, \dots, i_j) \in \mathbb{D}_{2h_1+H\hbar}$.

These trees are designed to get h_1-covering subsets.

Lemma 7.5. *Let $\Theta \colon \mathcal{A}_H \to \mathbb{D}$ be a hyperbolic motion tree of deepness $H \in \mathbb{N}$. Then,*

$$\mathbb{D}_{h_1+H\hbar} \subset \bigcup_{S \in \mathcal{A}_H^\Theta} \mathbb{D}_{h_1}(\Theta(S)).$$

Proof. The proof is by induction on H. If $H = 0$, this is a definition. Suppose that the lemma is true for deepness up to H, let Θ be a hyperbolic motion tree of deepness $H+1$ and $\Xi = \Theta_{|\mathcal{A}_H}$. Note $\mathbb{D}_{h_1}(\Theta) = \cup_{S\in\mathcal{A}^{\Theta}_{H+1}} \mathbb{D}_{h_1}(\Theta(S))$. Let $\zeta \in \mathbb{D}_{h_1+(H+1)\hbar}$. If ζ belongs to $\mathbb{D}_{h_1+H\hbar}$, then the induction hypothesis applies on Ξ and gives $\zeta \in \mathbb{D}_{h_1}(\Theta)$. Next, suppose that $d_P(0,\zeta) \geqslant h_1 + H\hbar$, define $r_H = \frac{e^{h_1+H\hbar}-1}{e^{h_1+H\hbar}+1}$, $\zeta = re^{i\theta}$ and $\zeta_H = r_H e^{i\theta}$. The induction hypothesis on ζ_H and the condition $d_P(\zeta,\zeta_H) < \hbar$ give us $\xi = \Theta(i_1,\dots,i_k)$, with $(i_1,\dots,i_k) \in \mathcal{A}^{\Xi}_H$ such that $d_P(\zeta,\xi) \leqslant h_1 + \hbar$. By definition, there exists $i_{k+1} \in [\![0, p-1]\!]$ such that $\xi' = \Theta(i_1,\dots,i_{k+1})$ satisfies $d_P(\xi,\zeta) < h_1$. Since $(i_1,\dots,i_k) \in \mathcal{A}^{\Xi}_H$ and

$$d_P(\xi',0) \leqslant d_P(\zeta,\xi') + d_P(\zeta,0) < 2h_1 + (H+1)\hbar,$$

$(i_1,\dots,i_{k+1}) \in \mathcal{A}^{\Theta}_{H+1}$ and the induction is finished. □

These trees emerge from our work on the mesh of transversals.

Lemma 7.6. *Let $x \in \mathbb{T}$ and $H = \left\lceil \frac{R-h_1}{\hbar} \right\rceil$. If h_1 is small enough, then there is a hyperbolic motion tree $\Theta_x \colon \mathcal{A}_H \to \mathbb{D}$ such that for $S \in \mathcal{A}^{\Theta_x}_H$, there exists $\mathbb{T}_S \in \widetilde{\mathbb{T}}$ with $\phi_x(\Theta_x(S)) \in \mathbb{T}_S$. Moreover, if $\phi_x(S) \in 2\rho U_a$ for $S \in \mathcal{A}^{\Theta_x}_H$, then $\mathbb{T}_S$ is a singular transversal $\mathbb{T}_{a,j,k,u}$.*

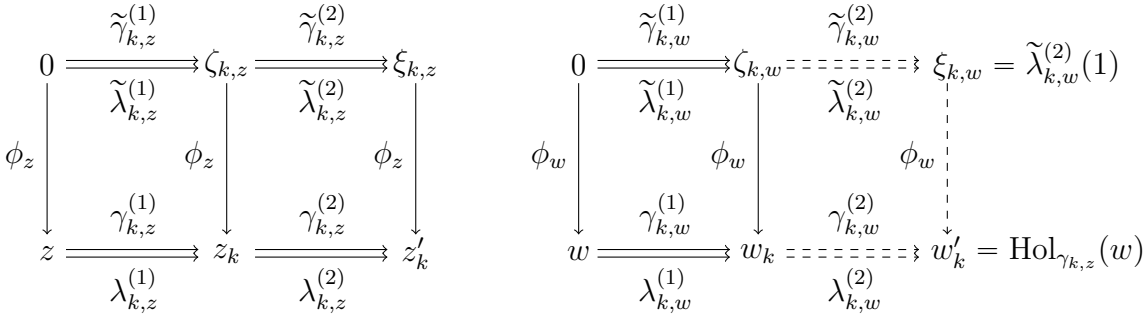
Proof. The proof is by induction on H. If $\mathbb{T}_S$ is regular, then we apply Lemma 7.3 to first find ζ_k for $k \in [\![0, p-1]\!]$. If $\mathbb{T}_S$ is singular, we find similar ζ_k by Lemma 7.2. To find $\xi_k = \Theta_x(S\cdot k)$, we distinguish whether $\|\phi_x(\zeta_k)\|_1 \leqslant 2\rho$ or not. In the first case, we apply Lemma 5.8. In the second case, we choose ξ_k in a transversal such that $\|\phi_x(\xi_k)\|_1 > 2\rho$. Note also that if $d_P(0,\zeta_k) \geqslant 2h_1 + H\hbar$, we can keep the $\xi_k = \zeta_k$. In order to apply the lemmas, we need to check that $\|\phi_x(\zeta_k)\|_1 \geqslant r_{\text{sing}}(R)$. If h_1 is sufficiently small, since we have $d_P(0,\zeta_k) < 2h_1 + H\hbar < R + 2h_1$, we get this condition using Lemma 3.4. □

With the notations of the previous proof, we want to make the choice of ξ_k in the singular case in some sense uniform on a disk of the initial covering. Let us introduce some notations. Let $\mathbb{T}_i$ be a singular transversal, $D \in \mathcal{V}_i$ and $z, w \in 2D$. Note $z_k = \varphi_k(z)$ and $\zeta_{k,z}$ the corresponding point in the disk that we obtain in Lemma 7.2. Suppose that $z_k \in 2\rho U_a$ and follow the procedure of Lemma 5.8 to find a corresponding $z'_k \in L_{z_k}$ and $\xi_{k,z} \in \mathbb{D}_{h_1+\hbar}$, with z'_k on a transversal and $\phi_z(\xi_{k,z}) = z'_k$. Denote by $\widetilde{\gamma}^{(1)}_{k,z}$ the geodesic from 0 to $\zeta_{k,z}$, $\widetilde{\gamma}^{(2)}_{k,z}$ the geodesic from $\zeta_{k,z}$ to $\xi_{k,z}$ and $\widetilde{\gamma}_{k,z} = \widetilde{\gamma}^{(2)}_{k,z} \cdot \widetilde{\gamma}^{(1)}_{k,z}$ the concatenation of the two paths. For each of these, denote without tildas the projection on L_z, that is, for $\star \in \{(1),(2),\varnothing\}$, $\gamma^{\star}_{k,z} = \phi_z \circ \widetilde{\gamma}^{\star}_{k,z}$.

On the other hand, consider $\lambda^{(1)}_{k,z}(t) = \varphi_z\left(tC_8^{-1}h_1 \left|\ln r^{\mathbb{T}}_j\right|\right)$, for $t \in [0,1]$. That way, we have $\lambda^{(1)}_{k,z}(0) = z$, $\lambda^{(1)}_{k,z}(1) = z_k$. Since it is how we have built $\zeta_{k,z}$, $\gamma^{(1)}_{k,z}$ and $\lambda^{(1)}_{k,z}$ are homotopic. Consider then the lifting $\widetilde{\lambda}^{(1)}_{k,z}$ of $\lambda^{(1)}_{k,z}$ *via* ϕ_z such that $\widetilde{\lambda}^{(1)}_{k,z}(0) = 0$. It follows that $\widetilde{\lambda}^{(1)}_{k,z}(1) = \zeta_{k,z}$. Finally, the procedure of Lemma 5.8 gives us a flow path and then a path $\lambda^{(2)}_{k,z}$ on L_z joining z_k and z'_k, by definition of $\xi_{k,z}$. Its lifting $\widetilde{\lambda}^{(2)}_{k,z}$ *via* ϕ_z such that $\widetilde{\lambda}^{(2)}_{k,z}(0) = \zeta_{k,z}$ clearly satisfies $\widetilde{\lambda}^{(2)}_{k,z}(1) = \xi_{k,z}$. Let us also denote $\widetilde{\lambda}_{k,z} = \widetilde{\lambda}^{(2)}_{k,z} \cdot \widetilde{\lambda}^{(1)}_{k,z}$ and $\lambda_{k,z} = \lambda^{(2)}_{k,z} \cdot \lambda^{(1)}_{k,z}$ the concatenation of these paths.

We use analogous notations for w, with slight subtelty. To be more precise, let us denote by $w_k = \varphi_k(w)$, $\zeta_{k,w}$ its preimage by ϕ_w obtained by Lemma 7.2, $\widetilde{\gamma}^{(1)}_{k,w}$ the geodesic joining 0 and $\zeta_{k,w}$, $\gamma_{k,w} = \phi_w \circ \widetilde{\gamma}^{(1)}_{k,w}$, $\lambda^{(1)}_{k,w}(t) = \varphi_w\left(tC_8^{-1}h_1 \left|\ln r^{\mathbb{T}}_j\right|\right)$ and $\widetilde{\lambda}^{(1)}_{k,w}$ its lifting *via* ϕ_w such that $\widetilde{\lambda}^{(1)}_{k,w}(0) = 0$. We stop here with the strict analogy. Indeed, we wish to show that

the construction for z also holds for w to have a uniform choice. Note $w'_k = \mathrm{Hol}_{\gamma_{k,z}}(w)$, which is well defined by Proposition 6.2. We want to find a path $\lambda^{(2)}_{k,w}$ joining w_k and w'_k with peculiar properties (see below). Once we have it, define $\widetilde{\lambda}^{(2)}_{k,w}$ its lifting *via* ϕ_w such that $\widetilde{\lambda}^{(2)}_{k,w}(0) = \zeta_{k,w}$ and denote by $\xi_{k,w}$ its endpoint. Then, take analogous notations with $\widetilde{\gamma}^{(2)}_{k,w}$ the geodesic joining $\zeta_{k,w}$ with $\xi_{k,w}$, $\gamma^{(2)}_{k,w} = \phi_w \circ \widetilde{\gamma}^{(2)}_{k,w}$ and $\lambda_{k,w}$, $\widetilde{\lambda}_{k,w}$, $\gamma_{k,w}$, $\widetilde{\gamma}_{k,w}$ the usual concatenations. See the diagramm below for a summary of all these notations.



Lemma 7.7. *With the notations above, there exists a path $\lambda^{(2)}_{k,w}$ with the following properties.*

(i) If C_5 is sufficiently small, $\ell_P\left(\lambda^{(2)}_{k,w}\right) \leqslant \hbar$.

(ii) $\mathrm{Hol}_{\gamma_{k,z}} = \mathrm{Hol}_{\gamma_{k,w}}$ on $2D$.

Proof. Note $\delta_{k,z}$ the flow path for z corresponding to $\lambda_{k,z}$, that we cut in two parts $\delta^{(j)}_{k,z}\colon [0,1] \to \mathbb{C}$, for $j \in \{1,2\}$ corresponding to $\lambda^{(j)}_{k,z}$. Note that $w_k = \varphi_w\left(\delta^{(1)}_{k,z}(1)\right)$. We build $\lambda^{(2)}_{k,w}$ by concatenating the two following parts.

(a) $\lambda^{(2,1)}_{k,w}(t) = \varphi_{w_k}\left(\delta^{(2)}_{k,z}(t)\right)$, for $t \in [0,1]$. Note w''_k its endpoint.

(b) $\lambda^{(2,2)}_{k,w}(t) = \varphi_{w''_k}(tt_w)$, for $t \in [0,1]$ and the notations of the proof of Proposition 6.5. Its endpoint is w'_k by definition of t_w.

Since $t_w = O(e^{-2R})$ and $\frac{\|z\|_1 - \|w\|_1}{\|z\|_1} = O\left(e^{-C_4 R}\right)$, the second part is of length smaller than $\frac{\hbar}{2}$ if R is sufficiently large. Moreover, using Lemma 7.1, we can compare the Poincaré length of the second part as a flow path for z_k or w_k in L_z or L_w. Choosing C_5 small enough, the four lemmas when we had not fixed $\hbar$ show that (i) holds.

To prove (ii), note that $\lambda_{k,w}$ is the same flow path as the one for z, just completed by adding t_w. Hence, we can cut $\lambda_{k,z}$ and $\lambda_{k,w}$ into parts on which they stay in the same flow box. This implies that $\mathrm{Hol}_{\lambda_{k,w}} = \mathrm{Hol}_{\lambda_{k,z}}$ on $2D$. Note that on the one hand $\gamma_{k,z}$ and $\lambda_{k,z}$ and on the other hand $\gamma_{k,w}$ and $\lambda_{k,w}$ have same starting and ending point in their lifting in $\mathbb{D}$. Therefore, they are homotopic and we also have $\mathrm{Hol}_{\gamma_{k,z}} = \mathrm{Hol}_{\gamma_{k,w}}$ on $2D$. □

In practice, to determine a hyperbolic motion tree for a point x, we do not follow exactly the proof of Lemma 7.6. Instead, we fix a point in the cell (say the center), follow the proof for this point and apply Lemma 7.7 to all points in the same cell. Lemma 7.7 implies that we do not break any symmetry with this choice because we could have done the same with any point by (i), with the same holonomy map by (ii). We need to check the following. This plays a similar role to [6, Lemma 2.9] in our refinement process.

Lemma 7.8. *Let $z, w \in \mathbb{T}_{a,j_0,k_0,u_0}$ and $k \in [\![0, p-1]\!]$ be such that $z'_k, w'_k \in \mathbb{T}_{a,j_1,k_1,u_1}$ arrive on the same transversal, with the notations above. Here, we do not suppose that z, w are in the same disk, so w'_k is indeed obtained by exactly the same process as z'_k. Suppose that $z'_k, w'_k \in 2D \in \mathcal{V}_{a,j_1,k_1,u_1}$ belong to a same disk of the initial covering. Then, $\mathrm{Hol}^{-1}_{\lambda_{k,z}} = \mathrm{Hol}^{-1}_{\lambda_{k,w}}$ on $2D$, with $\lambda_{k,z}, \lambda_{k,w}$ defined before.*

Proof. Note that both holonomies are defined on $2D$ by Proposition 6.2. Let us introduce some notations. Denote by $z' = \mathrm{Hol}^{-1}_{\lambda_{k,w}}(z'_k)$. Note δ_z (resp. δ_w) the flow time for X^{u_0} such that $z'_k = \varphi_z(\delta_z)$ (resp. $w'_k = \varphi_w(\delta_w)$). It is clear that $z' = \varphi_z(\delta_z - \delta_w + t)$, with $|t| = O(e^{-2R})$. Moreover, δ_z and δ_w can be written as a sum of two terms: $\delta_z = t_k + t_z$, $\delta_w = t_k + t_w$, where $t_k = C_8^{-1} h_1 \left|\ln r_j^{\mathbb{T}}\right| e^{\frac{2ik\pi}{p}}$, and t_z, t_w are defined by Lemma 5.8. In the linearizable case, since z and z' belong to the same transversal, we have $\Re(t_z - t_w + t) = 0$. It follows that $|\Im(t_z - t_w + t)| < 2\pi$ and hence is 0. So, $t_z = t_w - t$ and the condition on t imply that we have the same holonomy, by the same argument as Proposition 6.2. For the Briot–Bouquet case or the Poincaré–Dulac case with $u_0 = 1$, we argue the same. Consider a Poincaré–Dulac singularity with $u_0 = 2$. Note $z = (z_1, z_2)$. It is enough to show that

$$z_2 = (z_2 + \mu t z_1^m)\, e^{mt},$$

with $\Im(t) = O(C_5\hbar)$ and $C_5\hbar$ small enough implies that $t = 0$. If $\Re(t) \geqslant |\Im(t)|$, we have

$$1 = \left|1 + \mu t \frac{z_1^m}{z_2}\right| e^{m\Re(t)} \geqslant \left(1 - \frac{3}{2}\Re(t)\left(\left|\ln \left\|r_{j_0}^{\mathbb{T}}\right\|_1\right|\right)^{-1}\right) e^{m\Re(t)}.$$

Here, we have used that $\frac{|z_1|^m}{|z_2|} \leqslant \frac{3}{2}\left|\ln \left\|r_{j_0}^{\mathbb{T}}\right\|_1\right|^{-1}$, since $u_0 = 2$. Studying the function $x \mapsto \left(1 - 3x/\left(2\left|\ln r_{j_0}^{\mathbb{T}}\right|\right)\right) e^{mx}$, it is easy to see that it is not possible if h_1 is sufficiently small. Using similarly that $\left|1 + \mu t \frac{z_1^m}{z_2}\right| \leqslant 1 + \left|\mu t \frac{z_1^m}{z_2}\right|$, one can show that $\Re(t) \leqslant -|\Im(t)|$ is not possible either. Finally, we argue as in Lemma 5.6 to show that $t = 0$. □

8. End of proof of Theorem 1.2

8.1. **Refining the initial covering.** We are ready to finish the proof by exposing the refinement algorithm. Our different setting makes it slightly different to the one of the three authors [6, p. 617–619] but the ideas are close. Consider the initial covering $\mathcal{V}_i$ of each transversal $\mathbb{T}_i \in \widetilde{\mathbb{T}}$, for $i \in I_{\mathbb{T}}$. We denote it by $\mathcal{V}_i^0$, because we refine it by induction. Name also $H = \left\lceil \frac{R - h_1}{\hbar} \right\rceil$. For $H' \in [\![0, H]\!]$, we build a covering $\mathcal{V}_i^{H'}$ by disks that satisfies properties to ensure the existence of the orthogonal projection. What follows is the induction process, and contains some peremptory assertions to define the construction that we prove just after. Suppose that $\mathcal{V}_i^{H'}$ is already built.

I) If $\mathbb{T}_i \cap \left(M \setminus \cup_{a \in E} \frac{\rho}{2} U_a\right) \neq \varnothing$,
   (1) For each $D \in \mathcal{V}_i^0$, we denote by J_D the set of indices $j \in I_{\mathbb{T}}$ such that there exists $x \in D$, $x' \in \mathbb{T}_j \cap L_x$ with $d_P(x, x') \leqslant 2h_1$. If h_1 is sufficiently small, the holonomy map along the geodesic from x to x' only depends on $\mathbb{T}_i$ and $\mathbb{T}_j$ and not on x and x'. We denote it by π_{ij}, which is well defined on D.
   (2) For all disk $D' \in \mathcal{V}_j^{H'}$ such that $\pi_{ij}(D) \cap D' \neq \varnothing$, $\pi_{ij}^{-1}(D')$ is σ_1-quasi-round, for σ_1 fixed below Lemma 6.4. We cover it by four disks $D'_1, \ldots, D'_4$ using Lemma 6.4. Denote by $\mathcal{V}_{i,j,D}^{H'} = \left\{D'_j;\ j \in [\![1, 4]\!],\ \pi_{ij}(D) \cap D' \neq \varnothing\right\}$.

(3) Define J'_D as the set of indices $j \in J_D$ such that $\mathcal{V}^{H'}_{i,j,D}$ is a covering of D. Let $J' = \cup_{D\in\mathcal{V}^0_i} J'_D$ and $\mathcal{V}^{H'}_{i,j} = \cup_{D\in\mathcal{V}^0_i}\mathcal{V}^{H'}_{i,j,D}$. Let $\mathcal{V}^{H'+1}_i$ be the covering of $\mathbb{T}_i$ obtained by applying Lemma 4.9 to $\mathcal{V}^{H'}_i$ and $\left(\mathcal{V}^{H'}_{i,j}\right)_{j\in J'}$. Note that each $\mathcal{V}^{H'}_{i,j}$ does not necessarily cover entirely $\mathbb{T}_i$, but we apply successively Lemma 4.9 on the subset each covers.

II) If $\mathbb{T}_i \subset \frac{\rho}{2}U_a$, for some $a \in E$,

(1) For each $D \in \mathcal{V}^0_i$, fix one $z \in D$. For $k \in [\![0, p-1]\!]$, name by $z_k = \varphi_k(z)$ and $z'_k \in \mathbb{T}_{k,D}$ the point obtained by Lemma 5.8 if we have $\|z_k\|_1 \geqslant r_{\mathrm{sing}}(R)$. Consider the path $\gamma_{k,D}$ defined before Lemma 7.7 and note $\pi_{k,D} = \mathrm{Hol}_{\gamma_{k,D}}$, which is well defined on D. The holonomy map $\pi^{-1}_{k,D}$ only depends on $\mathbb{T}_i$, $\mathbb{T}_{k,D}$ and k, but not on D. Name $J_D = \{k \in [\![0, p-1]\!]\,;\, \|z_k\|_1 \geqslant r_{\mathrm{sing}}(R)\}$.

(2) For all disk $D' \in \mathcal{V}^{H'}_{k,D}$ such that $\pi_{k,D}(D) \cap D' \neq \varnothing$, $\pi^{-1}_{k,D}(D')$ is σ_1-quasi-round and we cover it by four disks $D'_1, \dots, D'_4$ using Lemma 6.4. We denote by $\mathcal{V}^{H'}_{i,k,D} = \left\{D'_j;\, j \in [\![1, 4]\!]\,,\, \pi_{k,D}(D) \cap D' \neq \varnothing\right\}$ and $\mathcal{V}^{H'}_{i,k} = \cup_{D\in\mathcal{V}^0_i}\mathcal{V}^{H'}_{i,k,D}$.

(3) For each $k \in J_D$, $\mathcal{V}^{H'}_{i,k,D}$ is a covering of D. Note $\mathcal{V}^{H'+1}_i$ the covering of $\mathbb{T}_i$ obtained by applying Lemma 4.9 to $\mathcal{V}^{H'}_i$ and $\left(\mathcal{V}^{H'}_{i,k}\right)_{k\in[\![0,p-1]\!]}$. Note that $\mathcal{V}^{H'}_{i,k}$ covers at least the disks D for which $\|z_k\|_1 \geqslant r_{\mathrm{sing}}(R)$.

Lemma 8.1. *This algorithm works well. That is, the peremptory assertions used to define it hold. Moreover, there exists a constant $C_9 > 0$ such that $\max_{i\in I_{\mathbb{T}}} \operatorname{card} \mathcal{V}^H_i \leqslant e^{C_9 R}$.*

Proof. Let us shortly list our assertions: in (I.1) and (II.1), the holonomy does not depend on x and x' or on D; in (I.1) and (II.1), the holonomy is well defined on D; in (I.2) and (II.2), the inverse images of D' are σ_1-quasi-round; in (II.3), $\mathcal{V}^{H'}_{i,k,D}$ is a covering of D. Note that by Lemma 4.9 and by induction, we always have $2D' \subset 2D^0$, for some $D^0 \in \mathcal{V}^0_i$. By Propositions 6.2 and 6.5, it follows that the holonomies are well defined on $2D$ and that the inverse images of D' are σ_1-quasi-round. The fact that the holonomy does not depend on x and x' in (I.1) follows from the following observation. Since in case (I), we have $d(x, E) > c$ for some $c > 0$ and all $x \in \mathbb{T}_i$, we can cover $\mathbb{T}_i$ by a finite number of flow boxes. Reducing h_1 if necessary, we can suppose that $\mathbb{D}_{2h_1}$ is still contained in these flow boxes. The uniqueness of a point of a plaque belonging to a transversal gives us that the holonomy only depends on $\mathbb{T}_i$ and $\mathbb{T}_j$. In (II.1), the similar result follows from Lemma 7.8. Finally, note that Lemmas 5.3 and 5.6 make us end far from the boundary of $\mathbb{T}_{k,D}$. Hence, $\pi_{k,D}$ is fully contained in $\mathbb{T}_{k,D}$ and $\mathcal{V}^{H'}_{i,k,D}$ is a covering of D if R is sufficiently large. Therefore, the algorithm works well.

It remains to prove the control of the cardinality. Note $K_{H'} = \max_{i\in I_{\mathbb{T}}} \operatorname{card} \mathcal{V}^{H'}_i$. By Proposition 5.9, $K_0 \leqslant e^{C_6 R}$. Note that $\operatorname{card} J' \leqslant K'$, by (HT2). Since the holonomy maps only depend on the transversals, we have by construction $\operatorname{card} \mathcal{V}^{H'}_{i,j} \leqslant 4K_{H'}$ and $\operatorname{card} \mathcal{V}^{H'}_{i,k} \leqslant 4K'K_{H'}$. By Lemma 4.9, we obtain for $C = \max\left(4 \times 200^{K'+1}, 4K' \times 200^{p+1}\right)$, $\operatorname{card} \mathcal{V}^{H'+1}_i \leqslant CK_{H'}$. By definition of H, we get $K_H \leqslant C^{R/\hbar} e^{C_6 R}$. □

8.2. **Proof of the existence of an orthogonal projection.** The construction of the covering $\mathcal{V}^H_i$ and of the hyperbolic motion tree clearly imply the following result.

Lemma 8.2. *Let $\mathbb{T}_i \in \mathbb{T}$ be a regular transversal, $D \in \mathcal{V}^H_i$ and $x \in 2D$. Then, there exists a hyperbolic motion tree $\Theta_x \colon \mathcal{A}_H \to \mathbb{D}$ satisfying the conditions of Lemma 7.6 and the*

following. If $S_k = (i_1, \dots, i_k) \in \mathcal{A}_H^{\Theta_x}$*, for* $j \in [\![0, k-1]\!]$*, we denote by* $S_j = (i_1, \dots, i_j)$*. Let* $\xi_j = \Theta_x(S_j)$*,* γ_j *be the geodesic from* ξ_j *to* ξ_{j+1}*,* $\lambda_j = \phi_x \circ \gamma_j$ *and* $\lambda_{S_k} = \lambda_{k-1} \dots \lambda_0$*. Then,* $\mathrm{Hol}_{\lambda_{S_k}}$ *is well defined on* $2D$*, with image in* $2D_k$*, for some* $D_k \in \mathcal{V}_{S_k}$*.*

This enables us to conclude the proof of Theorem 1.2 by checking our criterion.

Proposition 8.3. *The covering* $\left(\mathcal{V}_i^H\right)_{i \in I_{\mathbb{T}}}$ *satisfies the hypotheses of Proposition 4.5.*

Proof. The cardinality condition is an immediate consequence of Lemma 8.1. Let D be in $\mathcal{V}_i^H$ with $\mathbb{T}_i \in \mathbb{T}$ and $x, y \in D$. Let $\Theta_x \colon \mathcal{A}_H \to \mathbb{D}$ be the hyperbolic motion tree of Lemma 8.2 and $F = \{\Theta_x(S);\ S \in \mathcal{A}_H^{\Theta_x}\}$. The set F is h_1-dense by Lemma 7.5. For $S \in \mathcal{A}_H^{\Theta_x}$, note $x_S = \phi_x(\Theta_x(S))$, $y_S = \mathrm{Hol}_{\lambda_S}(y)$, with the notations of Lemma 8.2. We build ψ by gluing all the $\widetilde{\Phi}_{x_S y_S}$ of Proposition 6.2. If we can do so, Lemma 8.2, (HD1) and (HD2) imply points (2) and (3) of Proposition 4.5. By Proposition 6.2 and Lemma 8.2, these maps patch up well on a branch of Θ_x. On the other hand, if given two S_1 and S_2 such that $\mathbb{D}_{h_1}(\Theta_x(S_1)) \cap \mathbb{D}_{h_1}(\Theta_x(S_2)) \neq \varnothing$, we can build homotopic paths from x to the image of a point on the intersection because they have the same starting and ending point in $\mathbb{D}$. The corresponding holonomies coincide as germs and coincide on $2D$ by analytic continuation. Hence, it is clear that $\widetilde{\Phi}_{x_{S_1} y_{S_1}}$ and $\widetilde{\Phi}_{x_{S_2} y_{S_2}}$ coincide. □

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