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HEAT DIFFUSIONS ON HOLOMORPHIC FOLIATIONS WITH NON-DEGENERATE SINGULARITIES

FRANÇOIS BACHER

ABSTRACT. Consider a Brody hyperbolic foliation with non-degenerate singularities on a compact complex manifold. We show that the leafwise heat diffusions and the abstract heat diffusions coincide. In particular, this will imply that the abstract heat diffusions are unique.

1. INTRODUCTION

The global dynamics of Riemann surface laminations have recently received much attention. Much progress has been focused on developing an adapted ergodic theory. This could be a powerful tool to understand the global behaviour of Riemann surface laminations. The case of the projective space is maybe the most typical, since polynomial vector fields can be compactified naturally into foliations on \mathbb{P}^n . Lins Neto and Soares in [17] and Jouanolou in [14] have shown that a generic foliation \mathcal{F} of a given degree $d \geq 2$ on \mathbb{P}^n has only non-degenerate singularities. By a result of Lins Neto [15] and Glutsyuk [12], a foliation having only non-degenerate singularities is necessarily hyperbolic, and even Brody hyperbolic in the sense of [7]. Therefore, many authors have studied hyperbolic Riemann surface laminations with singularities and in particular hyperbolic singular holomorphic foliations. We present briefly the results of some recent work and refer the reader to the surveys [9, 10, 19, 20] for more details.

To set up an ergodic theory, one needs at least some notions of time, time average, space average (*i.e.* measure), invariant and ergodic measure. It turns out that for foliations, it will also be convenient to use currents instead of measures and forms instead of functions. In the case of hyperbolic foliations, the Poincaré distance in a universal covering can be seen as a canonical time. As an analogue to invariant measures, it is natural to consider directed closed currents. This has led to various results and dynamical methods (see for example Rebelo [22]), but a large class of foliations do not have any. That is why Garnett has considered the weaker notions of harmonic currents and harmonic measures in [11]. This approach has led to strong results, see for example the survey [10] of Fornæss and Sibony or the unique ergodicity result of Dinh, Nguyen, Sibony in [8]. The notion of quasi-harmonic measure was also introduced by Nguyen in [18] to establish an Oseledec multiplicative ergodic theorem for laminations. Then, the natural processes to take the average of a function on the leaves (that is, on the orbits) becomes the heat

UNIVERSITÉ DE LILLE, LABORATOIRE DE MATHÉMATIQUES PAUL PAINLEVÉ, CNRS U.M.R. 8524, 59655 VILLENEUVE D'ASCQ CEDEX, FRANCE.

E-mail address: francois.bacher@univ-lille.fr.

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ORCID: 0000-0001-8212-7109.

diffusions. Two approaches to define such diffusions have been tried and both have given a series of ergodic theorems. It is natural to wonder whether these two approaches lead to the same object, that is, whether both heat diffusions coincide. Though the equations they satisfy look similar, it is not clear and not proven if this is true in general. In his survey [20], Nguyễn has worked on this question. His result is the following.

Theorem 1.1 ([20, Corollary 5.23]). *Let $\mathcal{F} = (M, \mathcal{L}, E)$ be a Brody hyperbolic compact singular holomorphic foliation. Suppose that all the singularities of \mathcal{F} are linearizable and hyperbolic. Then, for every harmonic measure that does not give mass to any leaf, the abstract heat diffusions and the leafwise heat diffusions coincide. In particular, the abstract heat diffusions are unique, i.e., they do not depend on the considered harmonic measures.*

He mentions as a question (see [20, Problem 5.24]) to find better sufficient conditions for the two heat diffusions to coincide. Our main result gives an improvement of Theorem 1.1.

Theorem 1.2. *Let $\mathcal{F} = (M, \mathcal{L}, E)$ be a Brody hyperbolic compact singular holomorphic foliation. Suppose that all the singularities of \mathcal{F} are non-degenerate. Then, for every harmonic measure, the abstract heat diffusions coincide with the leafwise heat diffusions. In particular, the abstract heat diffusions are unique, i.e., they are independent of the harmonic measures.*

Let us explain the method of our proof. When we study non-degenerate singularities, we encounter more singular objects than in the case of linearizable weakly hyperbolic singularities. In our context, the current associated to the harmonic measure may have non-vanishing Lelong numbers on the singularities. Indeed, Chen has shown in [5] that the Lelong number of a directed harmonic current on a non-hyperbolic singularity may be positive. On the other hand, Nguyễn has shown in [21] that it cannot if the singularity is linearizable and weakly hyperbolic. To deal with this new difficulty, we have to control the heat diffusions near the singularities. More precisely, given an initial heat distribution far from the singular set, we show that it essentially does not reach the neighbourhoods of a non-degenerate singularity in finite time (see Lemma 6.3). This result relies on estimates of the Poincaré metric near the singularities.

The article is organized as follows. In Section 2, we introduce the leafwise Poincaré metric g_P . We specify the type of singularities we will consider. In the case where a foliation $\mathcal{F} = (M, \mathcal{L}, E)$ is also endowed with a Riemannian metric g_M , we define the function η . It is a quantitative way to compare g_P and g_M so that $\eta^2 g_P = 4g_M$. This is the main tool we use to estimate the diffusions near the singularities. In Section 3, we describe the heat kernel and the leafwise heat diffusions, as well as their behaviours with respect to a uniformization of a leaf. In Section 4, we explain the link between harmonic measures and harmonic currents. In our context, we show that the Poincaré mass of a directed harmonic current is finite. In Section 5, we recall the construction of the abstract heat diffusions. We state some intermediate lemmas of [6] to show its existence, because we will need them for our proof. Section 6 is devoted to the proof of Theorem 1.2. We state and prove an abstract criterion, analogous to [20, Theorem 5.17] in Theorem 6.1. In most of the article, we follow the method and notations of [20].

Notations. Throughout this paper, we will denote by \mathbb{D} the unit disk of \mathbb{C} and \mathbb{B} the unit ball in \mathbb{C}^k . We denote by $r\mathbb{D}$ the disk of radius r in \mathbb{C} and $r\mathbb{B}$ the ball of radius r in \mathbb{C}^k .

When we consider the hyperbolic distance in \mathbb{D} , we denote by \mathbb{D}_R the disk of hyperbolic radius R in \mathbb{D} , so that $r\mathbb{D} = \mathbb{D}_R$ for $r = \frac{e^R - 1}{e^R + 1}$ or $R = \ln \frac{1+r}{1-r}$.

If $\mathcal{F} = (M, \mathcal{L}, E)$ is a singular holomorphic foliation and $x \in M \setminus E$, we denote by L_x the leaf of \mathcal{F} through x . Moreover, if L_x is hyperbolic, we denote by $\phi_x: \mathbb{D} \rightarrow L_x$ a uniformization of L_x such that $\phi_x(0) = x$.

Given a Hermitian metric g_M on a complex manifold M , we denote by $d_M(\cdot, \cdot)$ the distance induced by g_M . Similarly, we denote by $d_P(\cdot, \cdot)$ the Poincaré distance if one is given a Poincaré metric denoted g_P .

Recall that $d^c = \frac{i}{2\pi} (\bar{\partial} - \partial)$ so that $dd^c = \frac{i}{\pi} \partial\bar{\partial}$.

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2. LEAFWISE POINCARÉ METRIC

Let $\mathcal{F} = (M, \mathcal{L}, E)$ be a singular holomorphic foliation. Fix g_M a Hermitian metric on M . Define

$$(2.1) \quad \eta(x) = \sup \left\{ \|\alpha'(0)\|_{g_M} \mid \alpha: \mathbb{D} \rightarrow L_x \text{ holomorphic such that } \alpha(0) = x \right\},$$

where $\|v\|_{g_M}$ is the norm of $v \in T_x L_x$ with respect to the Hermitian metric g_M . That is, $\|v\|_{g_M} = \sqrt{g_{M,x}(v, v)}$.

Definition 2.1. A leaf L of \mathcal{F} is called *hyperbolic* if it is uniformized by the Poincaré disk \mathbb{D} . The foliation \mathcal{F} is called *hyperbolic* if all the leaves of \mathcal{F} are hyperbolic.

We will be mostly interested in the case of hyperbolic foliations. Using Schwarz' Lemma, it is quite clear that L_x is hyperbolic iff $\eta(x) < \infty$. In that case, $\eta(x) = \|\phi'_x(0)\|_{g_M}$, where $\phi_x: \mathbb{D} \rightarrow L_x$ is a uniformization of L_x such that $\phi_x(0) = x$. By pushing forward the Poincaré metric on \mathbb{D} by ϕ_x , we obtain the so-called *Poincaré metric* g_P on the leaf L_x . Since ϕ_x can be chosen up to pre-composition by a rotation, and since the biholomorphisms of \mathbb{D} are isometries of (\mathbb{D}, g_P) , the Poincaré metric is intrinsically defined. It is quite clear that we have

$$(2.2) \quad \eta^2 g_P = 4g_M.$$

We recall from [7, Definition 3.1] the following.

Definition 2.2. We say that \mathcal{F} is *Brody hyperbolic* if there exists a positive constant c_0 such that $\eta < c_0$ on $M \setminus E$.

Let us define the type of singularities we will study.

Definition 2.3. Let $\mathcal{F} = (M, \mathcal{L}, E)$ be a singular holomorphic foliation. Near a singularity $a \in E$, there exists a vector field X defining \mathcal{F} . In coordinates (z_1, \dots, z_n) centered at a , we can write

$$X(z) = \sum_{j=1}^n F_j(z) \frac{\partial}{\partial z_j}.$$

The functions F_j can be developed as a power series $F_j = \sum_{\alpha \in \mathbb{N}^n} c_{\alpha,j} z^\alpha$. The *1-jet* of X at a is defined in the chart (U, z) as $X_1 = \sum_{j=1}^n \sum_{|\alpha| \leq 1} c_{\alpha,j} z^\alpha \frac{\partial}{\partial z_j}$. See for example [13, Chapter I] for more details. If the 1-jet of X has an isolated singularity at a , we say that a is a *non-degenerate singularity* of \mathcal{F} .

The following result is a direct consequence of a local study of non-degenerate vector fields in [16]. We denote by $\log^* = 1 + |\log|$ a log-type function and by $d_M(x, E)$ the distance from $x \in M$ to the singular set E with respect to the Hermitian metric g_M .

Proposition 2.4 (Dinh-Nguyễn-Sibony). *Let $\mathcal{F} = (M, \mathcal{L}, E)$ be a Brody hyperbolic compact singular holomorphic foliation. Suppose that all the singularities of \mathcal{F} are non-degenerate. Then, there exists a constant $c > 1$ such that the Poincaré metric of \mathcal{F} satisfies*

$$c^{-1}d_M(x, E) \log^* d_M(x, E) \leq \eta(x) \leq cd_M(x, E) \log^* d_M(x, E), \quad x \in M \setminus E.$$

See [2, Proposition 4.1] for a proof, which is inspired by [7, Proposition 3.3].

3. LEAFWISE HEAT DIFFUSION OPERATORS

We will now define one of the two heat diffusion operators we are interested in. Let $\mathcal{F} = (M, \mathcal{L}, E)$ be a hyperbolic singular holomorphic foliation. The leafwise Poincaré metric g_P gives rise to its associated Laplacian Δ_P on leaves. More precisely, for $f \in \mathcal{C}^2(\mathbb{D})$ or for $f \in \mathcal{C}^2(L_x)$, Δ_P is defined by the following formula.

$$(\Delta_P f) g_P = \pi dd^c f = i\partial\bar{\partial}f \quad \text{on } \mathbb{D} \text{ or } L_x.$$

For any fixed $x \in M \setminus E$, we study the *heat equation* on L_x

$$(3.1) \quad \frac{\partial p(x, y, t)}{\partial t} = \Delta_{P, y} p(x, y, t), \quad \lim_{t \rightarrow 0^+} p(x, y, t) = \delta_x(y), \quad y \in L_x, t \in \mathbb{R}_+.$$

Here, we have denoted by δ_x the Dirac mass in x and by $\Delta_{P, y}$ the Laplacian Δ_P with respect to the variable y . The limit is taken in the sense of distribution.

The *heat kernel on L_x* , denoted by $p(x, y, t)$, is defined as the smallest positive solution of the heat equation (3.1) (see Chavel [4]). By diffusing $p(x, y, t)$, we get a one-parameter family of operators $\{D_t \mid t \in \mathbb{R}_+\}$ defined by

$$(3.2) \quad D_t f(x) = \int_{L_x} p(x, y, t) f(y) g_P(y), \quad x \in M \setminus E, \quad f \in L^\infty(M \setminus E).$$

This family is a semi-group of positive contractions of $L^\infty(M \setminus E)$ (see (3.3)). Its elements are called the *leafwise heat diffusion operators*.

$$(3.3) \quad D_0 = \text{id}; \quad D_t \mathbf{1} = \mathbf{1} \quad \text{and} \quad D_{t+s} = D_t \circ D_s, \quad \text{for } t, s \in \mathbb{R}_+,$$

where $\mathbf{1}$ is the function identically equal to 1.

The same results hold on \mathbb{D} itself. We denote by $p_{\mathbb{D}}(\theta, \zeta, t)$ the heat kernel on \mathbb{D} . The heat diffusions on a leaf L_x are then related to the one on \mathbb{D} by the formula

$$(3.4) \quad D_t(f \circ \phi_x) = (D_t f) \circ \phi_x, \quad \text{on } \mathbb{D}, \quad \text{for } t \in \mathbb{R}_+, \quad f \in L^\infty(L_x).$$

See [18, Proposition 2.7] for a proof.

4. DIRECTED CURRENTS AND HARMONIC MEASURES

We suppose that the reader is already familiar with the notion of currents, positive currents, harmonic currents. For a more detailed exposition, see [20]. For a new example, see [1]. Let $\mathcal{F} = (M, \mathcal{L}, E)$ be a hyperbolic singular holomorphic foliation. A directed harmonic current on \mathcal{F} can be decomposed leafwise and transversally using the following lemma. Its proof was done in [6, Proposition 2.3] (see also [20, Proposition 2.5]). We will need it to define the abstract heat diffusions.

Lemma 4.1. *Let T be a directed harmonic current on \mathcal{F} . Consider $\mathbb{U} \simeq \mathbb{D} \times \mathbb{T}$ a flow box the coordinates of which can be extended to a neighbourhood of its closure in M . Then, there is a positive Radon measure ν on \mathbb{T} and for ν -almost every $t \in \mathbb{T}$ a harmonic function h_t on \mathbb{D} such that*

- *The mass $\int_{\mathbb{T}} \|h_t\|_{L^1(\mathbb{D})} d\nu(t)$ is finite;*
- *For $\alpha \in \mathcal{D}^{1,1}(\mathcal{F})$ compactly supported in \mathbb{U} ,*

$$T(\alpha) = \int_{\mathbb{T}} \left(\int_{\mathbb{D}} h_t(y) \alpha(y, t) \right) d\nu(t).$$

If moreover T is positive, then for ν -almost every $t \in \mathbb{T}$, h_t is positive on \mathbb{D} .

We have a notion of harmonicity for measures, characterizing their behaviour with respect to the Laplacians on the leaves.

Definition 4.2. Let Δ_P be the aggregate of the leafwise Laplacians on the leaves of \mathcal{F} . A finite positive Borel measure μ on $M \setminus E$ is called *harmonic* if

$$\int_M \Delta_P f d\mu = 0, \quad f \in \mathcal{D}(\mathcal{F}),$$

where $\mathcal{D}(\mathcal{F}) = \mathcal{D}^{0,0}(\mathcal{F})$ denotes the space of test functions.

Harmonic measures and directed harmonic currents are strongly linked. More precisely, if T is a directed harmonic current and if the measure

$$(4.1) \quad \mu = T \wedge g_P \quad \text{on} \quad M \setminus E,$$

is finite, then μ is harmonic. Conversely, if μ is a harmonic measure, then there exists a directed harmonic current T such that (4.1). See [6] for an implicit proof. In any case, we call the total mass of μ defined by (4.1) the *Poincaré mass* of T . It may be infinite near singularities. We will need the following results. The first one is a weak version of a theorem by Skoda in [23]. The second is due to Dinh, Nguyễn and Sibony in [6].

Proposition 4.3. *Let T be a positive dd^c -closed $(0,0)$ -current on a ball $r_0\mathbb{B}$ and $\beta = dd^c \|z\|^2$ be the standard Kähler form. Then, the quantity $r^{-2} \|T \wedge \beta\|_{r\mathbb{B}}$ is bounded for $r \in (0, r_1]$, $r_1 < r_0$.*

Here, $\|T \wedge \beta\|_A$ denotes the mass of $T \wedge \beta$ as a positive measure on a measurable set A .

Proposition 4.4. *Let $\mathcal{F} = (M, \mathcal{L}, E)$ be a singular holomorphic foliation with isolated singularities and T be a directed positive harmonic current on $M \setminus E$. Then, T has locally finite mass near any singularity $a \in E$ and can be extended through a into a positive dd^c -closed current.*

The following criterion for the Poincaré mass of a dd^c -closed current to be finite will be very important to prove Theorem 1.2.

Proposition 4.5. *Let $\mathcal{F} = (M, \mathcal{L}, E)$ be a singular holomorphic foliation such that E is composed of isolated singularities. Suppose that in a neighbourhood of $a \in E$, the Poincaré metric of \mathcal{F} satisfies $\eta(x) \geq cd_M(x, a) \log^* d_M(x, a)$ for a constant $c > 0$. Then, any directed positive harmonic current on \mathcal{F} has locally finite Poincaré mass near a .*

Proof. The proof was implicitly done in [6, Proposition 4.2]. Basically, we work in coordinates and with the standard Kähler form $\beta = dd^c \|z\|^2$ of \mathbb{C}^k . Propositions 4.4 and 4.3 ensure that $\|T \wedge \beta\|_{r\mathbb{B}} \leq Cr^2$. Then, we use (2.2) and the proof ends with the following integration by parts.

$$\begin{aligned} \int_{r\mathbb{B}} T \wedge g_P &= 4 \int_{r\mathbb{B}} \frac{1}{\eta^2(z)} T \wedge \beta \leq \frac{4}{c^2} \int_{r\mathbb{B}} \frac{1}{\|z\|^2 \ln^2 \|z\|} T \wedge \beta \\ \int_{r\mathbb{B}} T \wedge g_P &\leq \frac{4}{c^2} \left[\frac{\|T \wedge \beta\|_{r\mathbb{B}}}{r^2 \ln^2(r)} + \int_0^r \|T \wedge \beta\|_{\rho\mathbb{B}} \left(\frac{2}{\rho^3 \ln^2 \rho} + \frac{2}{\rho^3 \ln^3 \rho} \right) d\rho \right] \\ &\leq \frac{8C}{c^2 |\ln r|} < +\infty. \end{aligned}$$

□

5. ABSTRACT HEAT DIFFUSION

Let $\mathcal{F} = (M, \mathcal{L}, E)$ be a hyperbolic singular holomorphic foliation and μ be a harmonic measure on \mathcal{F} . We want to have a solution for an abstract heat equation. It will be given by Hille-Yosida's theorem. First recall some notions and facts of functional analysis. The reader can find proofs and exposition in Brezis [3].

Let L be a Hilbert space and $\langle \cdot, \cdot \rangle$ be its scalar product. A linear operator $A: \text{Dom}(A) \rightarrow L$, for $\text{Dom}(A) \subset L$, is called *monotone* if $\langle Au, u \rangle \geq 0$, $u \in \text{Dom}(A)$. It is called *maximal monotone* if moreover for $f \in L$, there exists $u \in \text{Dom}(A)$ such that $Au + u = f$. If A is maximal monotone, then $\text{Dom}(A)$ is dense in L and the graph of A is closed. A *semi-group of contractions* is a family $S(t): L \rightarrow L$, $t \in \mathbb{R}_+$ satisfying

- $S(t + t') = S(t) \circ S(t')$, $t, t' \in \mathbb{R}_+$,
- $\|S(t)\| \leq 1$, $t \in \mathbb{R}_+$.

A maximal monotone operator A gives rise to a semi-group of contractions having A as infinitesimal generator by the following result.

Theorem 5.1 (Hille-Yosida). *Let A be a maximal monotone operator on a Hilbert space L . The equation*

$$(5.1) \quad \frac{\partial u(t, \cdot)}{\partial t} + Au(t, \cdot) = 0, \quad \text{and} \quad u(0, \cdot) = u_0,$$

for

$$(5.2) \quad u \in \mathcal{C}^1(\mathbb{R}_+, L) \cap \mathcal{C}^0(\mathbb{R}_+, \text{Dom}(A))$$

and $u_0 \in \text{Dom}(A)$, has a unique solution u . Moreover, u is given by a semi-group of contractions $S(t): L \rightarrow L$, $t \in \mathbb{R}_+$ acting on u_0 , i.e. $u(t, \cdot) = S(t)u_0$.

So, one will be able to solve the heat equation if one can show that the operator $-\Delta_P$ is maximal monotone. This work was done in [6, Proposition 5.6] (see also [20, Proposition 5.8]) and has led to the following result.

Proposition 5.2. *Let $\mathcal{F} = (M, \mathcal{L}, E)$ be a singular holomorphic foliation endowed with its Poincaré metric g_P . If μ is a harmonic measure on \mathcal{F} , then $-\Delta_P$ is maximal monotone on $L^2(\mu)$.*

In particular, there exists a semi-group of contractions $S(t)$, $t \in \mathbb{R}_+$ such that for $u_0 \in \text{Dom}(-\Delta_P)$, $u(t, \cdot) = S(t)u_0$ satisfies (5.2) and (5.1). The $S(t)$ are called the abstract heat diffusion operators.

The proof of [6] relies on the study of bilinear forms defined through the operator Δ_P and another Laplace type operator $\tilde{\Delta}_P$. We recall the definition of these bilinear forms and some of their properties we will need. Let T be a directed harmonic current associated to μ by (4.1). By Lemma 4.1, we can write in a given flow box $\mathbb{U} \simeq \mathbb{D} \times \mathbb{T}$

$$T(\alpha) = \int_{\mathbb{T}} \left(\int_{\mathbb{D}} h_t(y) \alpha(y, t) \right) d\nu(t),$$

where the h_t , for $t \in \mathbb{T}$, are positive harmonic functions on \mathbb{D} and ν is a Radon measure. Define

$$\tilde{\Delta}_P u = \Delta_P u + \langle h_t^{-1} \nabla h_t, \nabla u \rangle_{g_P}.$$

Here, the gradient is defined with respect to the Poincaré metric so that $\langle \nabla u, \xi \rangle_{g_P} = du(\xi)$, for $u \in \mathcal{D}(\mathcal{F})$ and ξ tangent to the leaf. Note that $|\nabla u| = |du|_P$. The gradient can be extended to an operator of domain $H^1(\mu)$ onto $L^2(\mu)$. The space $H^1(\mu)$ is the completion of $\mathcal{D}(\mathcal{F})$ for the norm

$$\|u\|_{H^1(\mu)} = \left(\|u\|_{L^2(\mu)}^2 + \|\nabla u\|_{L^2(\mu)}^2 \right)^{\frac{1}{2}}.$$

We have $\text{Dom}(-\Delta_P) \subset H^1(\mu)$. *A priori*, the definition of $\tilde{\Delta}_P$ depends on the choice of flow boxes, but the uniqueness of the functions h_t and the current T ensure that it is a global operator. Define the bilinear forms

$$q(u, v) = - \int (\Delta_P u) v d\mu, \quad \tilde{q}(u, v) = - \int (\tilde{\Delta}_P u) v d\mu,$$

for $u, v \in \mathcal{D}(\mathcal{F})$. We summarize the properties of q and \tilde{q} we will need in Section 6 in the following statement. The reader can find proofs in [6, Lemmas 5.4 and 5.5] (see also [20, Lemmas 5.4 and 5.5, Remark 5.7]).

Proposition 5.3. *Let μ be a harmonic measure. The bilinear forms q and \tilde{q} can be extended continuously to $H^1(\mu) \times H^1(\mu)$. Moreover,*

- (1) For $u \in H^1(\mu)$, $q(u, u) = \tilde{q}(u, u)$.
- (2) For $u \in \text{Dom}(-\Delta_P)$ and $v \in H^1(\mu)$, we have

$$\tilde{q}(u, v) = \int \langle \nabla u, \nabla v \rangle_{g_P} d\mu = \int i \partial u \wedge \bar{\partial} v \wedge T.$$

6. COINCIDENCE OF THE HEAT DIFFUSIONS : PROOF OF THEOREM 1.2

The aim of this section is to compare the diffusion operators $S(t)$ defined in Section 5 and D_t defined in Section 3, for any $t \in \mathbb{R}_+$. We begin by stating a very abstract criterion for both heat diffusions to be equal.

Theorem 6.1. *Let $\mathcal{F} = (M, \mathcal{L}, E)$ be a singular holomorphic foliation with isolated singularities. Consider on M the leafwise Poincaré metric g_P and a Hermitian metric g_M . Denote by η the ratio function defined by (2.1). Let μ be a harmonic measure on \mathcal{F} . Assume that*

(H1) η is locally bounded from above on $M \setminus E$ and in the neighbourhood of any $a \in E$, $\eta(x) \leq F_a(d_M(x, a))$ for some function $F_a: \mathbb{R}_+^* \rightarrow \mathbb{R}_+^*$ such that F_a is continuous and $\frac{1}{F_a}$ is not integrable near 0;

(H2) Any measurable function u on $M \setminus E$ satisfying

- (a) $\|u\|_{L^\infty} < \infty$,
 - (b) $\|du|_P\|_{L^\infty} < \infty$,
 - (c) $\|\Delta_P u\|_{L^\infty} < \infty$,
 - (d) for every $a \in E$, $\lim_{\varepsilon \rightarrow 0} \|u|_{\{d_M(x, a) < \varepsilon\}}\|_{L^\infty} = 0$,
- belongs to $H^1(\mu)$.

Then, the abstract heat diffusions operators and the leafwise heat diffusions operators coincide.

Remark 6.2. This theorem is analogous to a result by Nguyễn [20, Theorem 5.17]. In comparison, our hypothesis (H1) is stronger than his, and our hypothesis (H2) is weaker than his. Actually, he uses the first one in a context where he has far better. Therefore, he is forced to suppose quite specific singularities and currents in order to satisfy the second one. Namely, he has to work with hyperbolic singularities and a current that does not give mass to any leaf to have a vanishing Lelong number. Here, we have relaxed the second hypothesis to apply the theorem to more singular measures, and strengthened the first one in consequence. In particular, our assumption (H1) allows us to obtain the result of Lemma 6.3 below.

Before proving Theorem 6.1, we will need two lemmas. The first one is new and will be a crucial ingredient of our proof. The second one can be found in [20].

Lemma 6.3. *We keep the notations of Theorem 6.1 and suppose hypothesis (H1). Let $u_0 \in \mathcal{D}(\mathcal{F})$ and $u(t, \cdot) = D_t u_0$. Then, for $t > 0$ and $a \in E$, $\lim_{\varepsilon \rightarrow 0} \|u|_{\{d_M(x, a) < \varepsilon\}}(t, \cdot)\|_{L^\infty} = 0$.*

Proof. Let $a \in E$ and δ be such that there exists a Hermitian chart \mathbb{U} around a of radius δ . Without loss of generality, we can assume that g_M is the standard Kähler metric on $\mathbb{U} \simeq \delta\mathbb{B}$. Shrinking the chart if necessary, we also suppose that $\delta < d_M(\text{supp } u_0, a)$. Fix $t > 0$, take $\varepsilon < \delta$ and $x \in \varepsilon\mathbb{B}$. For $y \in L_x \cap \text{supp } u_0$, we have by (2.2),

$$d_P(x, y) \geq 2 \int_\varepsilon^\delta \frac{dt}{F_a(t)} = R_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} +\infty,$$

since F_a is not integrable near 0. It follows that $\phi_x^{-1}(\text{supp } u_0) \subset \mathbb{D} \setminus \mathbb{D}_{R_\varepsilon}$. In other words, in the uniformization of L_x centered at x , the support of u_0 is sent uniformly to the boundary when x goes to the singularity. By (3.4) and (3.2), we get

$$\begin{aligned} D_t u_0(x) &= u(t, x) = \int_{\mathbb{D}} p_{\mathbb{D}}(0, y, t) u_0(\phi_x(y)) g_P(y), \\ |u(t, x)| &\leq \|u_0\|_{L^\infty} \int_{\mathbb{D}} \mathbf{1}_{\mathbb{D} \setminus \mathbb{D}_{R_\varepsilon}}(y) p_{\mathbb{D}}(0, y, t) g_P(y), \end{aligned}$$

where $\mathbf{1}_{\mathbb{D} \setminus \mathbb{D}_{R_\varepsilon}}$ denotes the characteristic function of $\mathbb{D} \setminus \mathbb{D}_{R_\varepsilon}$. This bound is independent on x such that $d_M(x, a) < \varepsilon$. By Lebesgue's dominated convergence theorem and the fact that $R_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} +\infty$, the right hand side tends to 0 when ε goes to 0. This implies the desired conclusion. \square

Lemma 6.4. *We keep the notations and hypothesis of Lemma 6.3. Suppose moreover that u_0 is positive. Then u is measurable and satisfies*

- (1) (a) $\|u(\cdot, \cdot)\|_{L^\infty} < \infty$,
- (b) For $t > 0$, $\|du(t, \cdot)|_P\|_{L^\infty} < \infty$,
- (c) For $t > 0$, $\|\Delta_P u(t, \cdot)\|_{L^\infty} < \infty$;
- (2) $u \in \mathcal{C}^1(\mathbb{R}_+^*, L^2(\mu))$ and for $t \in \mathbb{R}_+^*$, $\frac{\partial u(t, \cdot)}{\partial t} - \Delta_P u(t, \cdot) = 0$;
- (3) $\lim_{t \rightarrow 0} u(t, \cdot) = u_0$ in $L^2(\mu)$.

See [20, Facts (5.15), (5.16) and (5.17)] for a proof relying on a precise estimate of the heat kernel.

Proof of Theorem 6.1. We follow the proof of [20, Theorem 5.17]. Let $u_0 \in \text{Dom}(-\Delta_P)$ and $t \in \mathbb{R}_+$. We have to show that

$$(6.1) \quad D_t u_0 = S(t)u_0.$$

Since the operators D_t and $S(t)$ are positive contractions and $\mathcal{D}(\mathcal{F})$ is dense in $\text{Dom}(-\Delta_P)$, it is enough to show (6.1) for a non-negative $u_0 \in \mathcal{D}(\mathcal{F})$. Take such a u_0 . Denote by $u(t, \cdot) = D_t u_0$ and $U(t, \cdot) = S(t)u_0$. By Lemmas 6.3 and 6.4 and hypothesis (H2), $u(t, \cdot)$, $U(t, \cdot)$ and $\tilde{u}(t, \cdot) = U(t, \cdot) - u(t, \cdot)$ belong to $H^1(\mu)$ for a fixed t . Moreover, they are $\mathcal{C}^1(\mathbb{R}_+^*, L^2(\mu))$. The heat equations (5.1) and (2) in Lemma 6.4 satisfied by $S(t)$ and D_t imply that

$$\frac{1}{2} \frac{\partial}{\partial t} \left(\|\tilde{u}(t, \cdot)\|_{L^2(\mu)}^2 \right) = \langle \Delta_P \tilde{u}(t, \cdot), \tilde{u}(t, \cdot) \rangle = -q(\tilde{u}(t, \cdot), \tilde{u}(t, \cdot)).$$

By an approximation of $\tilde{u}(t, \cdot)$ by elements of $\mathcal{D}(\mathcal{F})$ and Proposition 5.3, we get $q(\tilde{u}(t, \cdot), \tilde{u}(t, \cdot)) = \tilde{q}(\tilde{u}(t, \cdot), \tilde{u}(t, \cdot)) \geq 0$ (see [20, p. 50-51]). Hence, $\|\tilde{u}(t, \cdot)\|_{L^2(\mu)}^2$ is decreasing and goes to 0 for $t \rightarrow 0$ by the boundary conditions. This implies that it is identically zero and $U(t, \cdot) = u(t, \cdot)$. \square

Proof of Theorem 1.2. We are inspired by the proof of [20, Proposition 5.22]. Let T be a directed positive harmonic current on \mathcal{F} . By Proposition 2.4, we have

$$(6.2) \quad c^{-1} d_M(x, E) \log^* d_M(x, E) \leq \eta(x) \leq c d_M(x, E) \log^* d_M(x, E), \quad x \in M \setminus E.$$

Then, Proposition 4.5 ensures that the measure μ associated to T by (4.1) is finite. Hence, it is a harmonic measure. We want to apply Theorem 6.1. Hypothesis (H1) is clear by (6.2). It remains to check hypothesis (H2). Let u be a measurable function on $M \setminus E$ such that

$$(6.3) \quad \|u\|_{L^\infty} < \infty, \quad \|du|_P\|_{L^\infty} < \infty, \quad \|\Delta_P u\|_{L^\infty} < \infty, \quad \lim_{\varepsilon \rightarrow 0} \|u|_{\{d_M(x, a) < \varepsilon\}}\|_{L^\infty} = 0, \quad a \in E.$$

We need to show that u belongs to $H^1(\mu)$. Using the compactness of M , we can find a finite open covering $\mathcal{U} = (U_p)_{p \in I}$ by

- (1) flow boxes $U_p \simeq \mathbb{D} \times \mathbb{T}$ such that $d_M(U_p, E) > 0$,
- (2) Hermitian charts $U_a \simeq \mathbb{B}$ for $a \in E$ with $U_a \cap E = \{a\}$.

Since μ is finite, using a partition of the unity, we can suppose that u is compactly supported in a single U_p or U_a .

Case (1). We need to find $u_\varepsilon \in \mathcal{D}(\mathcal{F})$ such that $\|u - u_\varepsilon\|_{H^1(\mu)} < \varepsilon$. We will follow the three following steps.

- (i) Let $\chi^{(1)}: \mathbb{C} \rightarrow [0, 1]$ be a smooth function supported in $\overline{\mathbb{D}}$ such that $\int_{\mathbb{C}} \chi^{(1)} d\text{Leb} = 1$. Define $\chi_{\delta}^{(1)}(z) = \delta^{-2} \chi(\delta z)$ for $\delta \in (0, 1)$. Let u_1 be the leafwise convolution of u with $\chi_{\delta}^{(1)}$.
- (ii) Let v be a bounded continuous function such that $\|du_1 - v\|_{L^2(m)} < \varepsilon$ and w be a bounded continuous function such that $\|u_1(0, t) - w(t)\|_{L^2(\nu')} < \varepsilon$. Here, m and ν' are measures respectively on $\mathbb{D} \times \mathbb{T}$ and \mathbb{T} that our computation will specify. Recall that we can find such v and w if m and ν' are Radon measures by Luzin's Theorem. Define

$$u_2(z, t) = w(t) + \int_0^1 v(\tau z, t) \cdot z d\tau.$$

- (iii) Let $u_{\varepsilon} = u_3$ be the leafwise convolution of u_2 with $\chi_{\delta}^{(1)}$.

Note first that by (6.2) and (2.2), there is a constant $C > 1$ such that $C^{-1} \|dz\|^2 \leq g_P \leq C \|dz\|^2$ on U_P . It follows that the first and third steps will give $\|u_1 - u\|_{H^1(\mu)} < \frac{\varepsilon}{3}$ and $\|u_3 - u_2\|_{H^1(\mu)} < \frac{\varepsilon}{3}$, for sufficiently small δ . Moreover, it is clear that u_2 is continuous. Thus, $u_3 \in \mathcal{D}(\mathcal{F})$. So, it remains to prove that $\|u_1 - u_2\|_{H^1(\mu)} < \frac{\varepsilon}{3}$. Since u_1 is leafwise smooth, we have

$$u_1(z, t) = u_1(0, t) + \int_0^1 du_1(\tau z, t) \cdot z d\tau$$

everywhere on $(z, t) \in \mathbb{D} \times \mathbb{T}$. Recall that μ is given by $T \wedge g_P$ and T by Lemma 4.1. Decompose $u_1 - u_2$ into the transversal to 0 part and the integral part, and apply the Cauchy-Schwarz inequality to the second part. We get

$$\|u_1 - u_2\|_{L^2(\mu)} \leq \|w - u_1(0, \cdot)\|_{L^2(\nu')} + \|v - du_1\|_{L^2(m_1)},$$

where $\nu'(t) = \|h_t\|_{L^1(g_P)} \nu(t)$ and $m_1 = \lambda_*(\text{Leb} \times \mu)$ with $\lambda: (\tau, z, t) \mapsto (\tau z, t)$. Moreover, $du_2 = v$ and since $C^{-1} \|dz\|^2 \leq g_P \leq C \|dz\|^2$, we have

$$\|\nabla u_1 - \nabla u_2\|_{L^2(\mu)} \leq C \|v - du_1\|_{L^2(\mu)}.$$

It is clear that $m = m_1 + \mu$ is finite, hence is a Radon measure. So is ν' by Lemma 4.1. Choose v a bounded continuous function such that $\|du_1 - v\|_{L^2(m)} < \alpha\varepsilon$ and w a bounded continuous function such that $\|u_1(0, \cdot) - w\|_{L^2(\nu')} < \alpha\varepsilon$, for $\alpha \leq \frac{1}{3}(4 + C^2)^{-1/2}$. Then $\|u_2 - u_1\|_{H^1(\mu)} < \frac{\varepsilon}{3}$ and therefore, $\|u - u_{\varepsilon}\|_{H^1(\mu)} < \varepsilon$. This concludes Case (1).

Case (2). We work in coordinates $U_a \simeq \mathbb{B} \subset \mathbb{C}^k$ centered at a . Let $\chi: \mathbb{C}^k \rightarrow [0, 1]$ be a smooth function such that $\chi(z) = 1$ on $\frac{1}{2}\mathbb{B}$ and $\chi(z) = 0$ on $\mathbb{C}^k \setminus \mathbb{B}$. For $\varepsilon \in (0, 1)$, consider the function

$$v_{\varepsilon}(z) = (1 - \chi(\varepsilon^{-1}z)) u(z), \quad z \in \mathbb{B}.$$

By construction, we have $v_{\varepsilon} = 0$ on $\frac{\varepsilon}{2}\mathbb{B}$ and v_{ε} satisfies (6.3). By the same arguments as in Case (1), we get that $v_{\varepsilon} \in H^1(\mu)$. So, it will be sufficient to prove that $\lim_{\varepsilon \rightarrow 0} \|u - v_{\varepsilon}\|_{H^1(\mu)} = 0$. Moreover, $u - v_{\varepsilon}$ is supported in $\varepsilon\mathbb{B}$ and $|u - v_{\varepsilon}| \leq |u|$ everywhere. Since $\mu(\{a\}) = 0$,

$$\int_M |u - v_{\varepsilon}|^2 d\mu \leq \|u\|_{L^{\infty}}^2 \int_{\varepsilon\mathbb{B}} d\mu \xrightarrow{\varepsilon \rightarrow 0} 0.$$

Next, we address the gradient-part of the $H^1(\mu)$ -norm. We have

$$\int_M |\nabla(u - v_\varepsilon)|^2 d\mu = \int_{\varepsilon\mathbb{B}} i\partial(u(z)\chi(\varepsilon^{-1}z)) \wedge \bar{\partial}(u(z)\chi(\varepsilon^{-1}z)) \wedge T(z).$$

Up to multiplying by constants, the right hand side is lower than the sum of these three terms

$$I_1 = \int_{\varepsilon\mathbb{B}} i\partial u \wedge \bar{\partial} u \wedge T, \quad I_2 = \varepsilon^{-1} \int_{\varepsilon\mathbb{B} \setminus \frac{\varepsilon}{2}\mathbb{B}} |u| |du| T \wedge \beta, \quad I_3 = \varepsilon^{-2} \int_{\varepsilon\mathbb{B} \setminus \frac{\varepsilon}{2}\mathbb{B}} |u|^2 T \wedge \beta,$$

where $\beta = dd^c \|z\|^2$ is the standard Kähler metric. Since $\mu(\{a\}) = 0$, we obtain

$$I_1 = \int_{\varepsilon\mathbb{B}} |\nabla u|^2 d\mu \leq \| |du|_P \|^2 \mu(\varepsilon\mathbb{B}) \xrightarrow{\varepsilon \rightarrow 0} 0.$$

Recall that $|du| = \frac{|du|_P}{\eta} = \frac{|\nabla u|}{\eta}$. By (6.2) and the equivalence of g_M and the standard norm, we get

$$I_2 \leq \frac{C \|u\|_{L^\infty}}{\log^*(c\varepsilon)} \| |du|_P \|_{L^\infty} \varepsilon^{-2} \|T \wedge \beta\|_{\varepsilon\mathbb{B}}.$$

Then, Proposition 4.3 implies that $I_2 \rightarrow 0$ when $\varepsilon \rightarrow 0$. Finally,

$$I_3 \leq \|u|_{\{d_M(x,a) < \varepsilon\}}\|_{L^\infty}^2 \varepsilon^{-2} \|T \wedge \beta\|_{\varepsilon\mathbb{B}}.$$

Thus, Proposition 4.3 and (6.3) ensure that $I_3 \rightarrow 0$ when $\varepsilon \rightarrow 0$. It follows that $\|u - v_\varepsilon\|_{H^1(\mu)}$ goes to 0 when $\varepsilon \rightarrow 0$. Hence, $u \in H^1(\mu)$ and Theorem 1.2 is a consequence of Theorem 6.1. \square

Remark 6.5. In fact, I_1 is estimated exactly the same way as [20, p. 56-57] and I_2 similarly. The new point here is the estimate on I_3 which is based on Lemma 6.3.

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