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Morpho-Logic from a Topos Perspective – Application to Symbolic AI

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Abstract

Modal logics have proved useful for many reasoning tasks in symbolic artificial intelligence (AI), such as belief revision, spatial reasoning, among others. On the other hand, mathematical morphology (MM) is a theory for non-linear analysis of structures, that was widely developed and applied in image analysis. Its mathematical bases rely on algebra, complete lattices, topology. Strong links have been established between MM and mathematical logics, mostly modal logics. In this paper, we propose to further develop and generalize this link between mathematical morphology and modal logic from a topos perspective, i.e. categorial structures generalizing space, and connecting logics, sets and topology. Furthermore, we rely on the internal language and logic of a topos. We define structuring elements, dilations and erosions as morphisms. Then we introduce the notion of structuring neighborhoods, and show that the dilations and erosions based on them lead to a constructive modal logic, for which a sound and complete proof system is proposed. We then show that the modal logic thus defined (called morpho-logic here), is well adapted to define concrete and efficient operators for revision, merging, and abduction of new knowledge, or even spatial reasoning.

Key words: Mathematical morphology, Topos, Constructive modal logic, Neighborhood semantics, Symbolic reasoning.

1 Introduction

Modal logics have proved useful for many reasoning tasks in symbolic artificial intelligence (AI), such as belief revision, spatial reasoning, among others.

On the other hand, mathematical morphology (MM) is a theory for non-linear analysis of structures, that was widely developed and applied in image analysis. In its deterministic setting, its mathematical bases rely on algebra, complete lattices, topology. Strong links have been established between MM and mathematical logics, mostly modal logics. Necessity \Box and possibility \Diamond modalities are then interpreted by the two basic MM operators, namely erosion and dilation [19,47,37]. This interpretation allows for easy formulations of non-classical reasoning, including revision, merging, abduction [3,4,20,33], and spatial reasoning [2,5,13].

Erosion ε and dilation δ are the two basic operations of MM often defined from a structuring element B used to probe spatial structures to either dilate or erode them. More formally, in the set theoretical case, let E be a Euclidean space (often \mathbb{R}^d or \mathbb{Z}^d where d is the space dimension). Let B (the structuring element) be a subset of E , and let $B_x = \{x + b \mid b \in B\}$ be its translation at a point $x \in E$. The dilation of a set X by a structuring element B is then defined as $\delta[B](X) = \{x \in E \mid \check{B}_x \cap X \neq \emptyset\}$, where \check{B} is the symmetric of B with respect to the origin of space, and the erosion of X is defined as $\varepsilon[B](X) = \{x \in E \mid B_x \subseteq X\}$. The structuring element B can be equivalently defined as a binary relation on E , i.e. $B(x, y)$ iff $y \in B_x$. In this setting, the following properties hold¹:

- erosion commutes with intersection and preserves E ,
- dilation commutes with union and preserves the empty set, and
- erosion and dilation defined from the same structuring element are dual operators with respect to complementation.

All this means that erosion and dilation are modal operators, that is the tuple $(\mathcal{P}(E), \cap, \cup, \text{ }^c, \emptyset, E, \varepsilon[B], \delta[B])$, where $\mathcal{P}(E)$ is the powerset of E , is a modal algebra. Hence, under this interpretation, modal logic is a tool for talking about spatial transformation, and in this setting, modal logic has been applied efficiently to symbolic artificial intelligence [5,13,16].

Until now, this link between MM and logic has been studied in the set framework (with extensions to fuzzy sets). Since then MM has been extended to a large family of algebraic structures such as graphs [27,28,45,61], hypergraphs [17,18], simplicial complexes [29], various logics, etc. All these extensions proved useful for knowledge representation and reasoning, taking into account low level information (points or neighborhood of points), structural information (e.g. based on spatial relations between regions or objects), prior

¹ Note that, more generally in MM, algebraic erosions and dilations on lattices are actually defined as operations that commute with the infimum and supremum, respectively (i.e. intersection and union in $(\mathcal{P}(E), \subseteq)$). These more general forms of operators do not necessarily involve structuring elements. The two first properties are then rather definitions in a more general setting.

knowledge, semantics, etc.

To take into account all of these extensions abstractly, we now propose to deepen this link between binary MM and modal logic from a topos perspective². Hence, paraphrasing a remark by O. Caramello in [25], every topos embodies a certain domain of reality, susceptible of becoming an object of knowledge (i.e. the idealized instantiations of this reality are the points of that topos). Toposes constitute a categorical structure defined by A. Grothendieck in the early sixties [35], which generalizes the notions of space, mathematical universe, and for what concerns us here, knowledge representation. The reason why the choice of toposes is natural is as follows. It remains in the spirit of toposes, as developed in [60], which establish a correspondence between logics, sets and topology³. Hence, the classical logic can be handled by considering sorts as objects, functional symbols and terms as morphisms, predicates as subobjects. The setting we propose in this paper is based on this view of toposes, by considering objects X as collections of states and morphisms $X \rightarrow PX$ as transitions, and by interpreting modal formulas as subobjects (and then as collections of states which satisfy them).

The proposed definitions and operations will then enhance the reasoning ability of MM, extending previous work on morphological modal logic [5,13]. They will allow, among others, giving morphological semantics to modalities with a topological flavor [59] conventionally used for spatial reasoning [1], and which could not be obtained directly from erosions and dilations; actually, the properties of these modalities are closer to the morphological operators of opening and closing (in their particular form of composition of erosion and dilation) due to a double quantification \forall/\exists in their definition. In [34], MM has been extended to structuring elements based on a notion of neighborhood close to a similar topological notion. We then propose to extend this first work to the framework of toposes. To obtain the usual expected properties of erosion and dilation in this new framework, we will have to impose a supplementary condition on these new structuring neighborhoods which will be an adaptation of the notion of filter, standard in topology and logic, to the framework of toposes. From all this, we will then have an internal CS4-modal algebra according to the meaning given to this notion in [7] (i.e. an internal interior algebra for erosions and a weaker version of internal closure algebra for dilations because dilations will not distribute over upper bounds (therefore,

² All the previous mentioned extensions can be represented by presheaves whose categories are toposes.

³ In this correspondence, we can find [60]:

space	~	logical theory
point	~	model of the theory
open set	~	propositional formula
sheaf	~	predicate formula
continuous map	~	transformation of models that is definable within geometric logic

there are not dilations in strict sense)⁴). This will then allow us to give a neighborhood semantics to constructive modal logic from a topos perspective.

Related extensions can be found, on the logical side, in the context of institutions [5] and satisfaction systems [3,4], thus encompassing many different logics in a federative framework. Applications to typical reasoning problems (revision, abduction, spatial reasoning) were instantiated in this framework.

The interpretation of modal logic in toposes has already been approached by others. We can cite [8,56]. To obtain modal operators possessing the right properties (commutativity with the upper and lower bounds, and preservation of the maximum and minimum elements), these works consider adjoints between internal Heyting algebras which therefore have the good properties of preservation and commutation [44]. Here, we propose a less general but more constructive definition of modal operators by defining them from erosion and dilation.

Some preliminaries on toposes are given in Section 2. We review some concepts, notations and terminology about toposes, more specifically about elementary toposes of Lawvere and Tierney [41]. One important contribution of this paper is to rely on the internal language of toposes, based on their logical account, which allows reasoning on them in a way close to reasoning on sets and functions. This is even more relevant in the scope of this paper where the algebraic setting of MM is considered. In Section 3, we formulate the notion of structuring element and two basic operators, dilation and erosion, in the framework of elementary toposes. In Section 4, we further extend the notion of structuring element to the notion of structuring neighborhood system. This notion was first introduced in [34], and is now generalized in toposes, with the aim of applying MM to logic for reasoning. In Section 5, we propose a new way of considering modalities in propositional modal logic, inspired by the interpretation of these modalities as dilation and erosion with a structuring neighborhood modeling an accessibility relation (useful for instance for spatial reasoning, among others). The proposed extension formalizes constructive modal logic via MM in toposes. This also extends the neighborhood semantics, usually considered on sets [49], to toposes. Syntax and semantics are defined, as well as a sound and complete proof system. Finally, in Section 6, we illustrate the proposed approach on typical examples in symbolic AI and knowledge representation, namely belief revision, merging, abduction, and spatial reasoning. Useful notations are summarized in Appendix.

⁴ In [7], the authors have shown that in the intuitionistic setting, this last property could be rejected, motivated by computer applications [32,51,62].

2 Preliminaries: elementary toposes

This paper relies on many terms and notations from the categorical theory of elementary toposes. The notions introduced here use basic notions of category theory (category, functor, natural transformation, limits, colimits, Cartesian closed) which are not recalled here, but interested readers may refer to textbooks such as [12,42].

2.1 Notations

In the whole paper, \mathcal{C} denotes a generic category, X, Y , and Z denote objects of \mathcal{C} . When \mathcal{C} is Cartesian closed, we denote by X^Y the exponential object of X and Y . The symbols f, g , and h denote morphisms, and given a morphism $f : X \rightarrow Y$, we denote by $\text{dom}(f) = X$ the domain of f and by $\text{cod}(f) = Y$ the co-domain of f ; $F, G, H : \mathcal{C} \rightarrow \mathcal{C}$ denote functors, and $\alpha, \beta : F \Rightarrow G$ denote natural transformations. Identity morphisms are denoted by Id , and initial and terminal objects by \emptyset and $\mathbb{1}$, respectively. Finally, monomorphisms are denoted by \rightarrow , that is if m is a monomorphism from X into Y , then we denote it by $m : X \rightarrow Y$.

2.2 Basic definitions

A topos \mathcal{C} is a finitely complete Cartesian closed category with a subobject classifier Ω . Having a subobject classifier means that there is a morphism out of the terminal object $true : \mathbb{1} \rightarrow \Omega$ such that for every monomorphism $m : Y \rightarrow X$ there is a unique morphism $\chi_m : X \rightarrow \Omega$ (called the characteristic morphism of m) such that the following diagram is a pullback:

$$\begin{array}{ccc} Y & \xrightarrow{\quad ! \quad} & \mathbb{1} \\ m \downarrow & & \downarrow true \\ X & \xrightarrow{\quad \chi_m \quad} & \Omega \end{array}$$

Let $X \in |\mathcal{C}|$ be an object, its set of subobjects is defined as:

$$\text{Sub}(X) = \{[m] \mid \text{cod}(m) = X \text{ and } m \text{ is a monomorphism}\}$$

where $[m]$ is the equivalence class of m according to the equivalence relation

$$m \simeq m' \text{ iff } \text{cod}(m) = \text{cod}(m') \text{ and } \text{dom}(m) \text{ is isomorphic to } \text{dom}(m')$$

that is, we have:

$$\begin{array}{ccc} \text{dom}(m) & & \\ \downarrow \simeq & \searrow m & \\ \text{dom}(m') & \xrightarrow{m'} & X \end{array}$$

If there is no ambiguity, we may write simply m instead of $[m]$ in the sequel.

Obviously, we have the bijection $\text{Sub}(X) \simeq \text{Hom}(X, \Omega)$ by the mapping $m \mapsto \chi_m$.

Let us define the partial order \leq_X on $\text{Sub}(X)$ as follows: for all $f : Y \twoheadrightarrow X$ and $g : Z \twoheadrightarrow X$

$$[f] \leq_X [g] \iff \exists h : Y \twoheadrightarrow Z, f = g \circ h$$

As usual, \geq_X denotes the reverse order. It is known that $\text{Sub}(X)$ is a Heyting algebra [38], that is $(\text{Sub}(X), \leq_X)$ is a distributive bounded lattice with $[Id_X]$ and $[\emptyset \twoheadrightarrow X]$ as the largest and the smallest elements, respectively, and which admits an implication \rightarrow right-adjoint to the meet operation \wedge (see Appendix A for a presentation of these results).

As \mathcal{C} is finitely complete, subobjects give rise to the contravariant functor $\text{Sub} : \mathcal{C}^{op} \rightarrow \text{Pos}$, where Pos is the category of posets, which to every $X \in |\mathcal{C}|$ (object of \mathcal{C}) associates $\text{Sub}(X)$ and to every morphism $f : X \rightarrow X'$ associates the mapping $\text{Sub}(f) : \text{Sub}(X') \rightarrow \text{Sub}(X)$ which to every $[Y' \twoheadrightarrow X']$ associates $[Y \twoheadrightarrow X]$ making the diagram

$$\begin{array}{ccc} Y & \longrightarrow & Y' \\ \downarrow & & \downarrow \\ X & \longrightarrow & X' \end{array}$$

a pullback.

Every topos has further the following properties [11,38]:

- It has also finite colimits, and then it has an initial object \emptyset and a terminal object $\mathbb{1}$ which are respectively the colimit and the limit of the empty diagram.
- Every morphism f can be factorized uniquely as $m_f \circ e_f$ where e_f is an epimorphism and m_f is a monomorphism. The codomain of e_f is often denoted by $\text{Im}(f)$ and is called the *image of f* , and then $(A \xrightarrow{f} B) = (A \xrightarrow{e_f} \text{Im}(f) \xrightarrow{m_f} B)$.
- Every object $X \in \mathcal{C}$ has a *power object* defined by Ω^X and denoted by PX . As a power object, it satisfies the following adjunction property:

$$\text{Hom}_{\mathcal{C}}(X \times Y, \Omega) \simeq \text{Hom}_{\mathcal{C}}(X, PY)$$

Given a morphism $f \in \text{Hom}_{\mathcal{C}}(X \times Y, \Omega)$ (respectively $f \in \text{Hom}_{\mathcal{C}}(X, PY)$)

we denote by $f^\#$ its equivalent by the above bijection. The morphism $f^\#$ is called the *transpose* of f . Note that by construction, we have $(f^\#)^\# = f$.

In particular, the transpose of the identity $Id_{PX} : PX \rightarrow PX$ is the characteristic morphism of a subobject $\epsilon_X \twoheadrightarrow X \times PX$ such that for every object $Y \in \mathcal{C}$ and every monomorphism $R \twoheadrightarrow X \times Y$, there exists a unique morphism $R \rightarrow \epsilon_X$ making the following diagram a pullback:

$$\begin{array}{ccc} R & \longrightarrow & \epsilon_X \\ \downarrow & & \downarrow \\ X \times Y & \xrightarrow{Id_X \times \chi_R^\#} & X \times PX \end{array}$$

Similarly to the category Set , the power object function which maps every object $X \in |\mathcal{C}|$ to its power object PX can be extended both into a covariant functor $\exists : \mathcal{C} \rightarrow \mathcal{C}$ and a contravariant functor $\mathcal{P} : \mathcal{C}^{op} \rightarrow \mathcal{C}$. The covariant functor \exists associates to every morphism $f : X \rightarrow Y \in \mathcal{C}$ the morphism $\exists f : PX \rightarrow PY$ whose transpose classifies the image of the morphism $g : \exists_X \twoheadrightarrow PX \times X \xrightarrow{Id_X \times f} PX \times Y$, i.e. $\exists f = \chi_{\text{Im}(g)}^\#$.

Likewise, the contravariant functor \mathcal{P} associates to every morphism $f : X \rightarrow Y$, the morphism $\mathcal{P}f : PY \rightarrow PX$ whose transpose classifies the morphism $R \twoheadrightarrow PY \times X$ where R is the pullback of the diagram

$$\begin{array}{ccc} R & \longrightarrow & \exists_Y \\ \downarrow & & \downarrow \\ PY \times X & \xrightarrow{Id_X \times f} & PY \times Y \end{array}$$

Toposes are sufficiently set-behaved to internalize a logic in which one may reason as if they were picking elements in sets, and accomodate internally constructive proofs, i.e. using neither the law of excluded middle nor the axiom of choice. We will use this internal language of toposes extensively in the paper. Now, in order not to overburden the presentation of the paper, we refer the reader to the details of this internal language in Appendix B.

2.3 Example: the category of presheaves $\text{Set}^{\mathcal{C}^{op}}$

An interesting feature of this family of categories is that it subsumes most of algebraic structures (e.g. sets, graphs, hypergraphs, etc.). Let \mathcal{C} be a small category, i.e. both collections of objects and arrows are sets. Let us denote by $\text{Set}^{\mathcal{C}^{op}}$ the category of contravariant functors $F : \mathcal{C}^{op} \rightarrow \text{Set}$ (presheaves) where Set is the category of sets. When the category \mathcal{C} is a small category, it is known that the category $\text{Set}^{\mathcal{C}^{op}}$ is complete and co-complete (i.e. it has all limits and colimits). Let us recall why it is also a topos.

First of all, observe that the functor $\text{Sub} : \text{Set}^{\mathcal{C}^{op}} \rightarrow \text{Set}$, which maps every presheaf F to its set of subobjects $\text{Sub}(F)$, is naturally isomorphic to the functor which maps each presheaf F to the set of its sub-presheaves. Therefore, we can assume that $G(C) \subseteq F(C)$ for all $G \in \text{Sub}(F)$ and $C \in |\mathcal{C}|$, and then such a subobject will be denoted by $G \subseteq F$.

$\text{Set}^{\mathcal{C}^{op}}$ is Cartesian closed. The product of two functors $F, G : \mathcal{C}^{op} \rightarrow \text{Set}$ is the functor $H : \mathcal{C}^{op} \rightarrow \text{Set}$ defined for every $C \in |\mathcal{C}|$ by $H(C) = F(C) \times G(C)$, and for every $f : A \rightarrow B \in \mathcal{C}$ by the mapping $H(f) : H(B) \rightarrow H(A)$ defined by $(a, b) \mapsto (F(f)(a), G(f)(b))$.

By the Yoneda Lemma, the exponential of functors $F, G : \mathcal{C}^{op} \rightarrow \text{Set}$ to the object $C \in |\mathcal{C}|$ should give an isomorphism $G^F(C) \simeq \text{Nat}(\text{Hom}(_, C), G^F)$. But, the definition of ‘‘Cartesian closed’’ requires that $\text{Nat}(\text{Hom}(_, C), G^F) \simeq \text{Nat}(\text{Hom}(_, C) \times F, G)$. This leads naturally to define the exponential of F and G by the functor G^F which associates to any object $C \in |\mathcal{C}|$ the set of natural transformations from $\text{Hom}(_, C) \times F$ to G . For every $f : A \rightarrow B \in \mathcal{C}$, $G^F(f) : G^F(B) \rightarrow G^F(A)$ is the mapping which associates to any natural transformation $\alpha : \text{Hom}(_, B) \times F \Rightarrow G$ the natural transformation $\beta : \text{Hom}(_, A) \times F \Rightarrow G$ defined for every object $C \in |\mathcal{C}|$ by $\beta_C(g : C \rightarrow A, c \in F(C)) = \alpha_C(f \circ g, c)$.

$\text{Set}^{\mathcal{C}^{op}}$ has a subobject classifier. For every $A \in |\mathcal{C}|$, a set S of arrows f in \mathcal{C} is said to be a **sieve on A** if:

- (1) For all arrows $f \in S$ we have $\text{cod}(f) = A$, and
- (2) For all arrows $f \in S$ and $g \in \text{Hom}(C)$ such that $\text{cod}(g) = \text{dom}(f)$, we have $f \circ g \in S$.

We denote by $\text{Sieve}(A)$ the set of sieves on A . Moreover, the map $\text{Sieve} : \mathcal{C} \rightarrow \text{Set}$ is naturally extended to a *contravariant* functor $\Omega : \mathcal{C} \rightarrow \text{Set}$, *i.e.* a presheaf $\Omega \in \text{Set}^{\mathcal{C}^{op}}$, as follows:

$$\Omega : \left\{ \begin{array}{ll} \mathcal{C} & \longrightarrow \text{Set} \\ A & \longmapsto \text{Sieve}(A) \\ f : A \rightarrow B & \longmapsto \Omega(f) : \left\{ \begin{array}{ll} \text{Sieve}(B) & \longrightarrow \text{Sieve}(A) \\ S & \longmapsto \{g : C \rightarrow A \mid f \circ g \in S\} \end{array} \right. \end{array} \right.$$

In fact, Ω is the subobject classifier; indeed, let us consider:

- the natural transformation $true : \mathbb{1} \Rightarrow \Omega$ which⁵ for every $A \in \mathcal{C}$ associates to the unique element in $\mathbb{1}(A)$ the maximal sieve on A (i.e. the unique sieve which contains Id_A);
- for every presheaf $F \in |\mathbf{Set}^{\mathcal{C}^{op}}|$, the bijection:

$$\chi : \begin{cases} \text{Sub}(F) \longrightarrow & \text{Hom}(F, \Omega) \\ G \subseteq F \longmapsto \chi(G)_A : \begin{cases} F(A) \longrightarrow & \text{Sieve}(A) \\ x \longmapsto \{f : B \rightarrow A \mid F(f)(x) \in G(B)\} \end{cases} \end{cases}$$

whose inverse is:

$$\chi^{-1} : \begin{cases} \text{Hom}(F, \Omega) \longrightarrow & \text{Sub}(F) \\ \xi \longmapsto A \mapsto \{x \in F(A) \mid Id_A \in \xi(A)(x)\} \end{cases}$$

Then we clearly have a correspondence between subobjects of $F \in |\mathbf{Set}^{\mathcal{C}^{op}}|$ and morphisms $F \rightarrow \Omega$, via the following pullback:

$$\begin{array}{ccc} G & \xrightarrow{!} & \mathbb{1} \\ \downarrow i & & \downarrow true \\ F & \xrightarrow{\chi(G)} & \Omega \end{array}$$

This makes Ω a subobject classifier in $\mathbf{Set}^{\mathcal{C}^{op}}$.

Hence, given a presheaf $X : \mathcal{C}^{op} \rightarrow \mathbf{Set}$, the power object $PX : \mathcal{C}^{op} \rightarrow \mathbf{Set}$ is the presheaf which, given an object $C \in |\mathcal{C}|$, gives the set

$$PX(C) = \text{Nat}(\text{Hom}_{\mathcal{C}}(-, C) \times X, \Omega) \simeq \text{Sub}(\text{Hom}_{\mathcal{C}}(-, C) \times X)$$

Examples of categories \mathcal{C} . Sets, undirected graphs, directed graphs, rooted trees and hypergraphs can be defined as presheaves. Indeed, for each of them, the (base) category \mathcal{C} can be graphically defined as:

$$\begin{array}{c} \bullet \\ \downarrow \end{array} \quad V \begin{array}{c} \xrightarrow{s} \\ \xleftarrow{t} \end{array} E \begin{array}{c} \text{inv} \\ \curvearrowright \end{array} \quad V \begin{array}{c} \xrightarrow{s} \\ \xleftarrow{t} \end{array} E \quad (\mathbb{N}, \leq) \quad V \xrightarrow{a} R \xleftarrow{h} E$$

The category \mathcal{C} for undirected graphs satisfies in addition the two equations: $s \circ \text{inv} = t$ and $t \circ \text{inv} = s$.

⁵ $\mathbb{1} : \mathcal{C}^{op} \rightarrow \mathbf{Set}$ is the presheaf which associates to any $A \in |\mathcal{C}|$ the terminal object $\mathbb{1}$ in \mathbf{Set} .

As an example, let us develop more precisely the case of directed graphs in its three aspects:

- (1) object classifier,
- (2) power object, and
- (3) object PX where X is a directed graph.

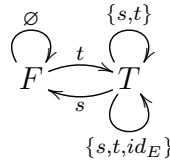
Object classifier. Let $G : \mathcal{C}^{op} \rightarrow \text{Set}$ be a directed graph. We have seen that $\Omega(V)$ and $\Omega(E)$ are respectively $\text{Sieve}(V)$ and $\text{Sieve}(E)$, i.e. $\Omega(V) = \{\emptyset, \{id_V\}\}$ and $\Omega(E) = \{\emptyset, \{s\}, \{t\}, \{s, t\}, \{s, t, id_E\}\}$. For any presheaf category $\text{Set}^{\mathcal{C}^{op}}$, given a morphism $f : X \rightarrow Y$ in \mathcal{C} , there is a straightforward way to compute $\Omega(f) : \Omega(Y) \rightarrow \Omega(X)$: it is defined for every sieve S on Y by:

$$\Omega(f)(S) = \{g : Z \rightarrow X \mid Z \in \mathcal{C} \text{ and } f \circ g \in S\}$$

Hence, we have that:

$$\begin{array}{ll} \Omega(s)(\emptyset) = \emptyset & \Omega(t)(\emptyset) = \emptyset \\ \Omega(s)(\{s\}) = \{Id_V\} & \Omega(t)(\{s\}) = \emptyset \\ \Omega(s)(\{t\}) = \emptyset & \Omega(t)(\{t\}) = \{Id_V\} \\ \Omega(s)(\{s, t\}) = \{Id_V\} & \Omega(t)(\{s, t\}) = \{Id_V\} \\ \Omega(s)(\{s, t, Id_E\}) = \{Id_V\} & \Omega(t)(\{s, t, Id_E\}) = \{Id_V\} \end{array}$$

Therefore, the subobject classifier is the directed graph:



where we denote by F the vertex \emptyset and by T the vertex $\{Id_V\}$.

The true arrow $true : \mathbb{1} \rightarrow \Omega$ maps the unique vertex to T and the unique arrow to $\{s, t, id_E\}$.

Hence, given a subgraph $m : G' \hookrightarrow G$, the characteristic mapping χ_m works as follows:

- all vertices which are not in G' are mapped to F ;
- all vertices which are in G' are mapped to T ;
- if an edge is not in G' , we have 4 possibilities:
 - (1) edges whose source and target are not in G' are mapped to \emptyset ;

- (2) edges whose source is in G' but the target is not are mapped to s ;
- (3) edges whose target is in G' but the source is not are mapped to t ;
- (4) edges whose source and target are in G' are mapped to $\{s, t\}$;
- edges in G' are mapped to $\{s, t, Id_E\}$.

Power object. Let us consider two graphs $X, Y \in |\text{Set}^{C^{op}}|$. Then X^Y is a graph defined as follows:

- *Vertex set:* the vertices of X^Y are in correspondence with the maps $f_V : Y(V) \rightarrow X(V)$.

$$\begin{aligned} X^Y(V) &= \text{Nat}(\text{Hom}_C(-, V) \times Y, X) \\ &\cong X(V)^{Y(V)} \end{aligned}$$

- *Edge set:* The edges of X^Y are in correspondence with the triples of maps $(f_E : Y(E) \rightarrow X(E), f_E^s : Y(V) \rightarrow X(V), f_E^t : Y(V) \rightarrow X(V))$ that make the following diagrams commutative:

$$\begin{array}{ccc} Y(E) & \xrightarrow{f_E} & X(E) \\ Y(s) \downarrow & & \downarrow X(s) \\ Y(V) & \xrightarrow{f_E^s} & X(V) \end{array} \quad \begin{array}{ccc} Y(E) & \xrightarrow{f_E} & X(E) \\ Y(t) \downarrow & & \downarrow X(t) \\ Y(V) & \xrightarrow{f_E^t} & X(V) \end{array}$$

$$\begin{aligned} X^Y(E) &= \text{Nat}(\text{Hom}_C(-, E) \times Y, X) \\ &\cong \{(f_E, f_E^s, f_E^t) \in X(E)^{Y(E)} \times X(V)^{Y(V)} \times X(V)^{Y(V)} \mid \\ &\quad X(s) \circ f_E = f_E^s \circ Y(s) \text{ and } X(t) \circ f_E = f_E^t \circ Y(t)\} \end{aligned}$$

- *Source and target maps:*

$$\begin{aligned} X^Y(s) &: \begin{cases} X^Y(E) & \longrightarrow X^Y(V) \\ (f_E, f_E^s, f_E^t) & \longmapsto f_E^s \end{cases} \\ X^Y(t) &: \begin{cases} X^Y(E) & \longrightarrow X^Y(V) \\ (f_E, f_E^s, f_E^t) & \longmapsto f_E^t \end{cases} \end{aligned}$$

Note that the graph morphisms $\text{Hom}_{\text{Set}^{C^{op}}}(Y, X)$ correspond to the morphisms $\mathbb{1} \rightarrow X^Y$, i.e. the loops in the graph X^Y .

Object PX . In particular, the graph $PX = \Omega^X$ is characterized by :

- *Vertex set:* the vertices of PX are in correspondence with the subsets of $X(V)$.

$$\begin{aligned} PX(V) &\cong \text{Sub}(\text{Hom}_{\mathcal{C}}(-, V) \times X) \\ &\cong \wp(X(V)) \end{aligned}$$

$(\wp(X(V)))$ denotes here the powerset of $X(V)$

- *Edge set:* The edges of PX correspond to the triples $(\chi_E \subseteq X(E), S_E \subseteq X(V), T_E \subseteq X(V))$ such that $X(s)(\chi_E) \subseteq S_E$ and $X(t)(\chi_E) \subseteq T_E$.
- *Source and target maps:*

$$\begin{aligned} PX(s) &: \begin{cases} PX(E) &\longrightarrow PX(V) \\ (\chi_E, S_E, T_E) &\longmapsto S_E \end{cases} \\ PX(t) &: \begin{cases} PX(E) &\longrightarrow PX(V) \\ (\chi_E, S_E, T_E) &\longmapsto T_E \end{cases} \end{aligned}$$

Intuitively, PX is the graph whose vertices are the subsets of $X(V)$, and edges between two vertices $\sigma, \tau \subseteq X(V)$ are the subsets $\chi \subseteq X(E)$ such that $s(\chi_E) \subseteq \sigma$ and $t(\chi_E) \subseteq \tau$.

3 Mathematical Morphology in Topos

3.1 Definitions

MM based on structuring elements extends fairly simply to toposes. Let \mathcal{C} be a topos.

Definition 1 (Structuring element). *A structuring element is a morphism $b: X \rightarrow PX$ for $X \in \mathcal{C}$.*

By the bijections $\text{Hom}_{\mathcal{C}}(X, PX) \simeq \text{Sub}(X \times X) \simeq \text{Hom}_{\mathcal{C}}(X \times X, \Omega)$, given a structuring element $b: X \rightarrow PX$, there exists a unique subobject $r_b: R_b \rightarrow X \times X \in \text{Sub}(X \times X)$ such that $b = \chi_{r_b}^\#$, i.e.

$$\begin{array}{ccc} R_b & \longrightarrow & \mathbb{1} \\ \downarrow & & \downarrow \\ X \times X & \xrightarrow{\chi_{r_b}} & \Omega \end{array}$$

is a pullback diagram.

Let us denote by $\check{b}: X \rightarrow PX$ the transpose of the morphism which classifies the image of the morphism $R_b \rightarrow X \times X \xrightarrow{\Delta_{X \times X}} (X \times X) \times (X \times X) \xrightarrow{p_2 \times p_1} X \times X$,

where Δ denotes the diagonal morphism, and p_i the projection on the i th space. Its description in the internal language of the topos \mathcal{C} is the following:

$$\check{b}(y) = \{x : X \mid y \in_X b(x)\}$$

Definition 2 (Erosion). *Let $b : X \rightarrow PX$ be a structuring element. The **erosion** by b is the morphism $\varepsilon[b] : PX \rightarrow PX$ whose transpose classifies the morphism $r : R \rightarrow PX \times X$ (i.e. $\varepsilon[b] = \chi_r^\#$) where R is the pullback of the diagram:*

$$\begin{array}{ccc} R & \longrightarrow & \succeq_X \\ \downarrow & & \downarrow \\ PX \times X & \xrightarrow{Id \times b} & PX \times PX \end{array}$$

In the internal logic of the topos \mathcal{C} , this is expressed as follows:

$$\varepsilon[b](Y) = \{x : X \mid b(x) \leq_X Y\}$$

Definition 3 (Dilation). *Let $b : X \rightarrow PX$ be a structuring element. The **dilation** by b is the morphism $\delta[b] : PX \rightarrow PX$ which classifies the image of the morphism $R \rightarrow X \times X \times PX \rightarrow PX \times X$ where the second morphism is the projection in the last and the first arguments, and R is the pullback of the diagram:*

$$\begin{array}{ccc} R & \longrightarrow & R_b \times \epsilon_X \\ \downarrow & & \downarrow \\ X \times X \times PX & \xrightarrow{Id \times \Delta_X \times Id} & X \times X \times X \times PX \end{array}$$

In the internal logic of the topos \mathcal{C} , this leads to:

$$\delta[b](Y) = \{x : X \mid \exists y. y \in_X \check{b}(x) \wedge y \in_X Y\}$$

Remark 1. *We could easily have extended erosions and dilations to any structuring element of the form $b : X \rightarrow PY$, and then defining morphisms $\varepsilon[b], \delta[b] : PY \rightarrow PX$. All the results given in Section 3.2 are easily adaptable to such structuring elements.*

Given a structuring object $b : X \rightarrow PX$, we have the two mappings $\overline{\varepsilon[b]}_X, \overline{\delta[b]}_X : \text{Sub}(X) \rightarrow \text{Sub}(X)$ given by the two commuting diagrams:

$$\begin{array}{ccc} \text{Hom}(\mathbb{1}, PX) \dashrightarrow \text{Sub}(X) & & \text{Hom}(\mathbb{1}, PX) \dashrightarrow \text{Sub}(X) \\ \text{Hom}(Id_{\mathbb{1}}, \varepsilon[b]) \downarrow & & \downarrow \overline{\varepsilon[b]}_X \\ \text{Hom}(\mathbb{1}, PX) \dashrightarrow \text{Sub}(X) & & \text{Hom}(\mathbb{1}, PX) \dashrightarrow \text{Sub}(X) \end{array} \quad \begin{array}{ccc} \text{Hom}(\mathbb{1}, PX) \dashrightarrow \text{Sub}(X) & & \text{Hom}(\mathbb{1}, PX) \dashrightarrow \text{Sub}(X) \\ \text{Hom}(Id_{\mathbb{1}}, \delta[b]) \downarrow & & \downarrow \overline{\delta[b]}_X \\ \text{Hom}(\mathbb{1}, PX) \dashrightarrow \text{Sub}(X) & & \text{Hom}(\mathbb{1}, PX) \dashrightarrow \text{Sub}(X) \end{array}$$

We will see in the next section that $\varepsilon[b]$ and $\delta[b]$ are monotonous. Hence,

$\overline{\varepsilon[b]}_X, \overline{\delta[b]}_X : \text{Sub}(X) \rightarrow \text{Sub}(X)$ are functorial when $\text{Sub}(X)$ is considered as a category.

We will also see in the next section that when the structuring element b satisfies the formula:

$$\forall x. x \in_X b(x)$$

then, $\overline{\delta[b]}_X$ is a Čech closure operator.

3.2 Results

We find all the results of MM in the set framework. The proofs of these different results mimic in the internal logic of the topos \mathcal{C} the classical proofs that we can find in the setting of MM on sets. The reason is that these classical proofs are constructive intuitionistic proofs, i.e. they use neither the axiom of choice nor the law of excluded middle. The proofs being quite simple but very formal, we refer the reader to the details of the proofs given in Appendix C, so as not to overload the presentation.

Proposition 1 (Adjunction). *The following formula in the internal logic is valid:*

$$\forall Y. \forall Z. \delta[b](Y) \leq_X Z \iff Y \leq_X \varepsilon[b](Z)$$

By Proposition 1, $(PX, \wedge, \vee, \Rightarrow, \varepsilon[b], \delta[b], \perp, \top)$ is an internal HGC-algebra, i.e. a Heyting algebra equipped with an order-preserving Galois connection [30], and then can be used to give a semantic to the intuitionistic propositional logic with Galois connections (IntGC) introduced in [30] from a topos perspective.

Proposition 2. *Erosion and dilation are monotonous for \leq_X and preserve least upper bound and greater lower bound, respectively. Moreover, we have that:*

- $\varepsilon[b](X) = X$, and
- $\delta[b](\emptyset) = \emptyset$.

Finally, $\varepsilon[b]$ and $\delta[b]$ are, respectively, anti-extensive and extensive for \leq_X iff the formula $\forall x. x \in_X b(x)$ is valid. More formally, these last two properties mean that both statements are valid:

- (1) $\vdash (\forall x. x \in_X b(x)) \Leftrightarrow (\forall Y. \varepsilon[b](Y) \leq_X Y)$
- (2) $\vdash (\forall x. x \in_X b(x)) \Leftrightarrow (\forall Y. Y \leq_X \delta[b](Y))$

When \mathcal{C} is complete (e.g. topos of presheaves), if we update our syntax to include set-indexed limits and colimits on any power object PX , then the preservation properties can be extended as follows:

$$\varepsilon[b](\bigwedge_{i \in I} Y_i) = \bigwedge_{i \in I} (\varepsilon[b](Y_i)) \quad ; \quad \delta[b](\bigvee_{i \in I} Y_i) = \bigvee_{i \in I} (\delta[b](Y_i))$$

for any index set I . This implies the monotonicity of $\varepsilon[b], \delta[b]$.

Proposition 3. $\varepsilon[b](\neg_X Y) = \neg_X \delta[b](Y)$ and $\delta[b](\neg_X Y) \leq_X \neg_X \varepsilon[b](Y)$ where \neg_X is the pseudo-complement of the internal Heyting algebra PX .

Proof. This directly results from the fact that the underlying internal logic for \mathcal{C} is intuitionistic. \square

If \mathcal{C} is a Boolean topos, we further have that $\neg_X \varepsilon[b](Y) \leq_X \delta[b](\neg_X Y)$, and hence the equality.

As usual, the composition of erosion and dilation is not equal to the identity, but produces two other operators, called opening (defined as $\delta[b] \circ \varepsilon[b]$) and closing (defined as $\varepsilon[b] \circ \delta[b]$). Opening and closing have the following properties.

Proposition 4. $\varepsilon[b] \circ \delta[b]$ (closing) and $\delta[b] \circ \varepsilon[b]$ (opening) satisfy the following properties:

- $\varepsilon[b] \circ \delta[b]$ and $\delta[b] \circ \varepsilon[b]$ are monotonous;
- $\varepsilon[b] \circ \delta[b]$ is extensive;
- $\delta[b] \circ \varepsilon[b]$ is anti-extensive;
- $\varepsilon[b] \circ \delta[b] \circ \varepsilon[b] = \varepsilon[b]$;
- $\delta[b] \circ \varepsilon[b] \circ \delta[b] = \delta[b]$;
- $\varepsilon[b] \circ \delta[b]$ and $\delta[b] \circ \varepsilon[b]$ are idempotent.

3.3 Mathematical morphology in presheaves

In toposes $\text{Set}^{\mathcal{C}^{op}}$, given a presheaf $F : \mathcal{C}^{op} \rightarrow \text{Set}$, a structuring element is a natural transformation $b : F \Rightarrow PF$, and then for every $C \in |\mathcal{C}|$ and every $c \in F(C)$, $b_C(c)$ is a sub-presheaf of $\text{Hom}(_, C) \times F$.

Then, given a structuring element $b : F \Rightarrow PF$, $\varepsilon[b] : PF \Rightarrow PF$ is defined as follows: for every $C \in |\mathcal{C}|$

$$\varepsilon[b]_C : \begin{cases} PF(C) \rightarrow PF(C) \\ X \mapsto \varepsilon[b]_C(X) \end{cases}$$

where X and $\varepsilon[b]_C(X)$ are sub-presheaves of $\text{Hom}(_, C) \times F$, and $\varepsilon[b]_C(X) : \mathcal{C}^{op} \rightarrow \text{Set}$ is defined as follows:

- For every $D \in |\mathcal{C}|$, we have:

$$\varepsilon[b]_C(X)(D) = \{(f: D \rightarrow C, d) \mid b_D(d) \subseteq PF(f)(X)\}$$

- For every $f : D \rightarrow E$, we have:

$$\varepsilon[b]_{\mathcal{C}(X)}(f) : \begin{cases} \varepsilon[b]_{\mathcal{C}(X)}(E) \rightarrow \varepsilon[b]_{\mathcal{C}(X)}(D) \\ (g : E \rightarrow C, e) \mapsto (g \circ f, F(f)(e)) \end{cases}$$

Note that $(g \circ f, F(f)(e))$ belongs indeed to $\varepsilon[b]_{\mathcal{C}(X)}(D)$ as we can see in the following commutative diagram:

$$\begin{array}{ccccc} D & & F(D) & \xrightarrow{b_D} & PF(D) \\ \downarrow f & & \uparrow F(f) & & \uparrow PF(f) \\ E & e \in & F(E) & \xrightarrow{b_E} & PF(E) \ni PF(g)(X) \end{array}$$

More precisely, knowing that $b_E(e) \subseteq PF(g)(X)$ and $PF(f)$ being isotone, we also have $b_D(F(f)(e)) = PF(f)(b_E(e)) \subseteq PF(f)(PF(g)(X)) = PF(g \circ f)(X)$.

Likewise, $\delta[b] : PF \Rightarrow PF$ is defined as follows: for every $C \in |\mathcal{C}|$

$$\delta[b]_{\mathcal{C}} : \begin{cases} PF(C) \rightarrow PF(C) \\ X \mapsto \delta[b]_{\mathcal{C}}(X) \end{cases}$$

where X and $\delta[b]_{\mathcal{C}}(X)$ are sub-presheaves of $\text{Hom}(_, C) \times F$, and $\delta[b]_{\mathcal{C}}(X) : \mathcal{C}^{op} \rightarrow \text{Set}$ is defined at all $D \in \mathcal{C}$ by:

$$\delta[b]_{\mathcal{C}}(X)(D) = \{(f : D \rightarrow C, d) \mid \exists d' \in X(D), (Id_D, d') \in \check{b}_D(d)(D) \cap PF(f)(X)(D)\}$$

where $\check{b} : F \Rightarrow PF$ is defined for all $C \in \mathcal{C}$ by:

$$\check{b}_C : \begin{cases} F(C) \rightarrow PF(C) \\ c \mapsto (D \mapsto \{(f : D \rightarrow C, d) \mid (Id_D, F(f)(c)) \in b_D(d)(D)\}) \end{cases}$$

Example 1 (Set). *Due to the structure of the base category, presheaves can be directly represented by sets, and power objects by the power set operator $X \mapsto 2^X$. Then, let S be a set. A structuring element is a map $b : S \rightarrow 2^S$, corresponding by transposition to the subset $R_b = \{(x, y) \mid y \in b(x)\} \subseteq S \times S$.*

When MM is applied in image processing, S is equipped with an additive law + (i.e. $(S, +)$ is an Abelian group), and then given a point $x \in S$, $b(x) = \{x + b \mid b \in B\}$ where $B \subseteq S$. So, given a set $X \subseteq S$, $\varepsilon[b](X) = \{x \in S \mid b(x) \subseteq X\}$, and $\delta[b](X) = \{x \in S \mid \check{b}(x) \cap X \neq \emptyset\}$.

The proposed definitions indeed include the classical definitions on sets.

Example 2 (Directed graph). Let $G : \mathcal{C}^{op} \rightarrow \text{Set}$ be a graph (\mathcal{C} is the base category given in Section 2.3). Many examples of structuring elements can be defined. Here, we provide three examples:

- (1) Star of a vertex. Given a vertex $v \in G(V)$, we consider the subgraph composed of vertices with an edge with v . This defines the following structuring element $b : G \Rightarrow PG$:

$$b_V : \begin{cases} G(V) \rightarrow & PG(V) \\ v \mapsto & \{v' \mid \exists e \in G(E), G(s)(e) = v, G(t)(e) = v'\} \end{cases}$$

$$b_E : \begin{cases} G(E) \rightarrow & PG(E) \\ e \mapsto & \begin{pmatrix} \{e' \mid G(t)(e) = G(s)(e')\} \\ b_V(G(s)(e)) \\ b_V(G(t)(e)) \end{pmatrix} \end{cases}$$

- (2) Path from a vertex. Given a vertex $v \in G(V)$, we extend the previous structuring element to paths from v of length $n \in \mathbb{N}^*$. This defines the following structuring element $b : G \Rightarrow PG$:

$$b_V : \begin{cases} G(V) \rightarrow PG(V) \\ v \mapsto Y \end{cases}$$

where $Y \subseteq G(V)$ is the set defined by:

$$v' \in Y \Leftrightarrow \begin{cases} \exists e_1, \dots, e_k \in G(E), k \leq n \\ \exists v_0, \dots, v_k \in G(V), \\ \forall i, 1 \leq i \leq k, G(s)(e_i) = v_{i-1} \text{ and } G(t)(e_i) = v_i \\ v = v_0 \text{ and } v' = v_k \end{cases}$$

$$b_E : \begin{cases} G(E) \rightarrow & PG(E) \\ e \mapsto & \begin{pmatrix} Y \\ b_V(G(s)(e)) \\ b_V(G(t)(e)) \end{pmatrix} \end{cases}$$

where $Y \subseteq G(E)$ is the set defined by:

$$e' \in Y \Leftrightarrow \begin{cases} \exists e_1, \dots, e_k \in G(E), k \leq n+1 \\ \exists v_0, \dots, v_k \in G(V), \\ \forall i, 1 \leq i \leq k, G(s)(e_i) = v_{i-1} \text{ and } G(t)(e_i) = v_i \\ e = e_1 \text{ and } e' = e_k \end{cases}$$

(3) Star of an edge. Given an edge $e \in G(E)$, we consider the subgraph composed of edges sharing a vertex with e . Here, the structuring element $b : G \Rightarrow PG$ is defined by: given an edge $e \in G(E)$, we denote $S_e = \{G(s)(e), G(t)(e)\}$

$$b_V : \begin{cases} G(V) \rightarrow & PG(V) \\ v \mapsto & \{v' \mid \exists (e, e') \in G(E), v \in S_e, G(s)(e') = v \text{ and } G(t)(e') = v'\} \end{cases}$$

$$b_E : \begin{cases} G(E) \rightarrow & PG(E) \\ e \mapsto & \left(\begin{array}{l} \{e' \mid G(s)(e) = G(s)(e') \text{ or } G(t)(e) = G(s)(e')\} \\ b_V(G(s)(e)) \\ b_V(G(t)(e)) \end{array} \right) \end{cases}$$

Then, $\varepsilon[b] : PG \Rightarrow PG$ is composed of the mappings $\varepsilon[b]_V$ and $\varepsilon[b]_E$ defined respectively by:

$$\varepsilon[b]_V : \begin{cases} PG(V) \longrightarrow & PG(V) \\ X \longmapsto & \{v \in G(V) \mid b_V(v) \subseteq X\} \end{cases}$$

$$\varepsilon[b]_E : \begin{cases} PG(E) \longrightarrow & PG(E) \\ (Y, S, T) \longmapsto & \left(\begin{array}{l} \{e \in G(E) \mid b_E(e) \subseteq Y\} \\ \{v \in G(V) \mid b_V(v) \subseteq S\} \\ \{v \in G(V) \mid b_V(v) \subseteq T\} \end{array} \right) \end{cases}$$

Likewise, $\delta[b] : PG \Rightarrow PG$ is composed of the mappings $\delta[b]_V$ and $\delta[b]_E$ defined respectively by:

$$\delta[b]_V : \begin{cases} PG(V) \longrightarrow & PG(V) \\ X \longmapsto & \{v \in G(V) \mid \exists w \in X, w \in b_V(v)\} \end{cases}$$

$$\delta[b]_E : \begin{cases} PG(E) \rightarrow & PG(E) \\ (Y, S, T) \mapsto & \left(\begin{array}{l} \{e \in G(E) \mid \check{b}_E(e) \cap Y \neq \emptyset\} \\ \{v \in G(V) \mid \exists w \in S, w \in \check{b}_V(v)\} \\ \{v \in G(V) \mid \exists w \in T, w \in \check{b}_V(v)\} \end{array} \right) \end{cases}$$

Example 3 (Hypergraphs). Here also, many examples of structuring elements can be provided. For example, we can adapt the structuring element defined as the “star of a vertex” to hypergraphs.

Let us consider a hypergraph $H : C^{op} \rightarrow \text{Set}$, and define more precisely the structure of PH . According to the same construction process as that for directed graphs, we can set:

- Vertex set: $PH(V) = \mathcal{P}(H(V))$
- Edge set: $PH(E) = \mathcal{P}(H(E))$
- Relation set: $PH(R) = \{(\chi_R, A_R, H_R) \mid \chi_R \subseteq H(R), A_R \subseteq H(V), H_R \subseteq H(E)\}$ such that $H(a)(\chi_R) \subseteq A_R$ and $H(h)(R) \subseteq H_R$.
- Maps $PH(a)$ and $PH(h)$:

$$PH(a) : \begin{cases} PH(R) \rightarrow PH(V) \\ (\chi_R, A_R, H_R) \mapsto A_R \end{cases}$$

$$PH(h) : \begin{cases} PH(R) \rightarrow PH(V) \\ (\chi_R, A_R, H_R) \mapsto A_H \end{cases}$$

Then, we may define the structuring element $b : H \Rightarrow PH$ as:

$$b_V : \begin{cases} H(V) \rightarrow & PH(V) \\ v \mapsto & \left\{ \begin{array}{l} \left. \begin{array}{l} \mid \exists r_1, r_2 \in H(R), H(a)(r_1) = v, \\ \mid H(a)(r_2) = v', \\ \mid H(h)(r_1) = H(h)(r_2) \end{array} \right\} \end{array} \right\} \end{cases}$$

$$b_E : \begin{cases} H(E) \rightarrow & PH(E) \\ e \mapsto & \left\{ \begin{array}{l} \left. \begin{array}{l} \mid \exists r_1, r_2 \in H(R), H(h)(r_1) = e, \\ \mid H(h)(r_2) = e', \\ \mid H(a)(r_1) = H(a)(r_2) \end{array} \right\} \end{array} \right\} \end{cases}$$

$$b_R : \begin{cases} H(R) \rightarrow & PH(R) \\ r \mapsto & (\{r' \mid H(a)(r) = H(a)(r')\}, b_V(H(a)(e)), b_E(H(h)(e))) \end{cases}$$

All examples of Example 2 can be extended to hypergraphs similarly.

Likewise, the definitions of erosion and dilation are easily adaptable from those for directed graphs.

4 Structuring Neighborhoods: Internal Topology

In [34], MM based on sets has been generalized to a new setting called structuring neighborhood systems. Here, we propose to extend this generalization to the topos framework. Hence, we propose to generalize all the notions of Section 3 using a lax notion of structuring element, called *structuring neighborhood*. The motivation of this extension is mainly logical. Indeed, the motivations of the paper is to apply MM to logic, so-called morpho-logic (see Section 5), which has been proved useful to model knowledge, beliefs or preferences, and to model classical reasoning methods such as revision, fusion, abduction or spatial reasoning [3–5,13,21,24].

4.1 Structuring Neighborhood: Definitions and Results

In the proposed framework of structuring neighborhoods, many properties of classical erosion and dilation can be recovered, but at the price of supplementary conditions on structuring neighborhoods. These supplementary conditions led to a mathematical construction which is very important in topology and logic, namely filters.

Let us consider an object X in a topos \mathcal{C} .

Definition 4 (Filter). *We formalize the notion of **filter** in X by the following axioms on the variable $F : PPX$:*

- *Closed under finite intersections:*

$$\forall A. \forall B. A \in_{PX} F \wedge B \in_{PX} F \Rightarrow A \wedge B \in_{PX} F$$

- *Upward closed:*

$$\forall A. \forall B. A \in_{PX} F \wedge A \leq_X B \Rightarrow B \in_{PX} F$$

- *Non-empty:*

$$X \in_X F$$

- *Strict:*

$$\forall A. A \in_{PX} F \Rightarrow (\exists x. x \in_X A)$$

The conjunction of the formulas above is again a formula, hence a morphism $\mathcal{F} : PPX \rightarrow \Omega$.

Filters will allow us to internally define a topology over a given topos, based on the notion of topological neighborhood (see Definition 7).

Definition 5 (Structuring neighborhood). A **structuring neighborhood** is a morphism $N : X \rightarrow PPX$ which validates both formulas:

- (1) $\forall x. \mathcal{F}(N(x))$, where \mathcal{F} is the morphism introduced in Definition 4
- (2) $\forall x. \forall A. A \in_{PX} N(x) \Rightarrow x \in_X A$

Intuitively, Definition 5 means that neighborhoods of x have a structure of filter, and therefore makes the notion of structuring neighborhood similar to the notion of neighborhood in topology (the set of neighborhoods of a point x in a topology is a filter), and all neighborhoods of x contain x . This last point will allow us to have erosions and dilations verifying the property of anti-extensivity and extensivity, respectively.

Definition 6 (Erosion and dilation). Let us consider a structuring neighborhood $N : X \rightarrow PPX$. Let us define the morphism $\varepsilon[N] : PX \rightarrow PX$ by the formula:

$$\forall Y. \varepsilon[N](Y) = \{x : X \mid Y \in_{PX} N(x)\}$$

and the morphism $\delta[N] : PX \rightarrow PX$ by the formula:

$$\begin{aligned} \forall Y. \delta[N](Y) &= \{x : X \mid \exists F. \mathcal{F}(F) \wedge Y \in_{PX} F \wedge N(x) \leq_{PX} F\} \\ &= \{x : X \mid \forall A. A \in_{PX} N(x) \Rightarrow \exists y. y \in_X A \wedge Y\} \end{aligned}$$

Hence, the erosion of Y contains all the elements x which have Y as neighborhood, and the dilation of Y contains all elements x whose all neighborhoods intersect with Y .

Given a structuring neighborhood $N : X \rightarrow PPX$, we also have two mappings $\overline{\varepsilon[N]}_X, \overline{\delta[N]}_X : \text{Sub}(X) \rightarrow \text{Sub}(X)$ given by the two commuting diagrams:

$$\begin{array}{ccc} \text{Hom}(\mathbb{1}, PX) \dashrightarrow \text{Sub}(X) & & \text{Hom}(\mathbb{1}, PX) \dashrightarrow \text{Sub}(X) \\ \text{Hom}(Id_{\mathbb{1}}, \varepsilon[b]) \downarrow & & \text{Hom}(Id_{\mathbb{1}}, \delta[b]) \downarrow \\ \text{Hom}(\mathbb{1}, PX) \dashrightarrow \text{Sub}(X) & & \text{Hom}(\mathbb{1}, PX) \dashrightarrow \text{Sub}(X) \end{array} \quad \begin{array}{ccc} & & \\ & \downarrow \varepsilon[N]_X & \\ & & \\ & \downarrow \delta[N]_X & \\ & & \end{array}$$

By Proposition 5 just below, $\varepsilon[N]$ and $\delta[N]$ are monotonous, and then $\overline{\varepsilon[N]}_X$ and $\overline{\delta[N]}_X$ are functorial when $\text{Sub}(X)$ is considered as a category.

Proposition 5. $\varepsilon[N]$ and $\delta[N]$ are monotonous. Moreover:

- $\varepsilon[N]$ verifies:
 - $\forall A. \forall B. \varepsilon[N](A \wedge B) = \varepsilon[N](A) \wedge \varepsilon[N](B)$.
 - $\varepsilon[N](X) = X$.

- $\forall Y. \varepsilon[N](Y) \leq_X Y$.
- $\delta[N]$ verifies:
 - $\forall A. \forall B. \delta[N](A \vee B) \geq_X \delta[N](A) \vee \delta[N](B)$.
 - $\delta[N](\emptyset) = \emptyset$.
 - $\forall Y. Y \leq_X \delta[N](Y)$.
- $\forall Y. \varepsilon[N](\neg_X Y) \leq_X \neg_X \delta[N](Y)$

Proof. • *Monotonicity:* Considering variables $x : X, Y : PX, Z : PX$, let $Y \leq_X Z$. Since $N(x)$ is a filter, $Y \in_{PX} N(x)$ implies $Z \in_{PX} N(x)$ (filters are upward closed by definition). It follows that

$$Y \leq_X Z \vdash \varepsilon[N](Y) \leq_X \varepsilon[N](Z)$$

- *Distributivity of $\varepsilon[N]$ with \wedge :* Let us consider variables $A : PX$ and $B : PX$. By monotonicity (increasingness) we already have $\varepsilon[N](A \wedge B) \leq_X \varepsilon[N](A) \wedge \varepsilon[N](B)$. Let us prove the other inclusion.
For a variable $x : X$, we have:

$$\begin{aligned} x \in_X (\varepsilon[N](A) \wedge \varepsilon[N](B)) &\Rightarrow A \in_{PX} N(x) \wedge B \in_{PX} N(x) \\ &\Rightarrow (A \wedge B) \in_{PX} N(x) \\ &\Rightarrow x \in_X \varepsilon[N](A \wedge B) \end{aligned}$$

- *Identity element:* For a variable $x : X$, we have:

$$\begin{aligned} x \in_X \varepsilon[N](X) &\Leftrightarrow X \in_{PX} N(x) \\ &\Leftrightarrow x \in_X X \end{aligned}$$

- *Anti-extensivity:* For a variable $x : X$, we have:

$$\begin{aligned} x \in_X \varepsilon[N](Y) &\Leftrightarrow Y \in_{PX} N(x) \\ &\Rightarrow x \in_X Y \end{aligned}$$

- For dilation $\delta[N]$, the proofs of monotonicity, commutativity and preservation of lower bound are similar (monotonicity argument and definition of \emptyset). Let us prove the extensivity property: Let $x : X$ be a variable. Let $x \in_X Y$. Let us assume that $\forall A. A \in_{PX} N(x)$. Then $x \in_X A$. And since $x \in_X Y$, we have $x \in_X A \wedge Y$. Therefore $x \in_X \delta[N](Y)$ (by taking $y = x$ in the definition of the dilation).

- *Duality*: For variables $Y : PX$ and $x : X$, we have $\neg_X Y \wedge Y = \emptyset$ and then:

$$\begin{aligned}
x \in_X \varepsilon[N](\neg_X Y) &\Rightarrow \neg_X Y \in_{PX} N(x) \\
&\Rightarrow \neg_X Y \in_{PX} N(x) \wedge (\forall y. \neg(y \in_X \neg_X Y \wedge Y)) \\
&\Rightarrow \exists A. A \in_{PX} N(x) \wedge (\forall y. \neg(y \in_X A \wedge Y)) \\
&\Rightarrow \exists A. A \in_{PX} N(x) \wedge \neg(\exists y. y \in_X A \wedge Y) \\
&\Rightarrow \exists A. \neg(A \in_{PX} N(x) \Rightarrow \exists y. y \in_X A \wedge Y) \\
&\Rightarrow \neg(\forall A. A \in_{PX} N(x) \Rightarrow \exists y. y \in_X A \wedge Y) \\
&\Rightarrow \neg(x \in \delta[N](Y)) \\
&\Rightarrow x \in \neg_X \delta[N](Y)
\end{aligned}$$

which concludes the proof. \square

The definitions of $\varepsilon[N]$ and $\delta[N]$ have a topological flavor. However, they do not satisfy some properties required for interior and closure operators, respectively (they are not idempotent). In Section 4.2, we will restrict the set of structuring neighborhoods to topological neighborhoods (see Definition 7). Whereas structuring neighborhoods are sufficient to give semantics to the modal logic IT, this new family of structuring neighborhoods will allow us to define erosions and dilations so that they will form an internal CS4-modal algebra over PX .

Example 4 (From structuring element to neighborhood). *Each structuring element $b : X \rightarrow PX$ provides a natural structuring neighborhood $N_b : X \rightarrow PPX$ defined by:*

$$\forall x. N_b(x) = \{Y : PX \mid Y \succeq_X b(x)\}$$

Proposition 6. $\varepsilon[N_b] = \varepsilon[b]$ and $\delta[N_b] = \delta[\check{b}]$.

Proof.

$$\begin{aligned}
x \in_X \varepsilon[N_b](Y) &\Leftrightarrow Y \in_{PX} N_b(x) \\
&\Leftrightarrow Y \succeq_X b(x) \\
&\Leftrightarrow x \in_X \varepsilon[b](Y)
\end{aligned}$$

$$\begin{aligned}
x \in_X \delta[N_b](Y) &\Rightarrow \exists F. \mathcal{F}(F) \wedge Y \in_{PX} F \wedge N_b(x) \leq_{PX} F \\
&\Rightarrow \exists F. \mathcal{F}(F) \wedge Y \in_{PX} F \wedge b(x) \in_{PX} F \\
&\Rightarrow \exists F. \mathcal{F}(F) \wedge (Y \wedge b(x)) \in_{PX} F \\
&\Rightarrow \exists y. y \in_X (Y \wedge b(x)) \\
&\Rightarrow x \in \delta[\check{b}](Y)
\end{aligned}$$

$$\begin{aligned}
x \in \delta[\check{b}](Y) &\Rightarrow \exists y. y \in_X (Y \wedge b(x)) \\
&\Rightarrow \exists F. F = \{Z : PX \mid Z \succeq_X (Y \wedge b(x))\} \\
&\quad \wedge \mathcal{F}(F) \wedge Y \in_{PX} F \wedge N_b(x) \preceq_{PX} F \\
&\Rightarrow x \in_X \delta[N_b](Y)
\end{aligned}$$

In the second implication, we use the fact that if $Z \succeq_X Y \wedge b(x)$ and $\exists y. y \in_X Y \wedge b(x)$ then $\exists y. y \in_X Z$, which gives us the last axiom of \mathcal{F} . The other axioms are obviously satisfied. \square

Hence, we proved that dilation and erosion based on structuring neighborhoods actually generalize those based on structuring elements, through the identification of each structuring element $b : X \rightarrow PX$ such that $x \in_X b(x)$ with its corresponding structuring neighborhood $N_b : X \rightarrow PPX$.

Proposition 7. *For all structuring neighborhoods $M, N : X \rightarrow PPX$, the following assertions are equivalent:*

- (1) *Adjunction: $\forall Y. \forall Z. \delta[N](Y) \preceq_X Z \Leftrightarrow Y \preceq_X \varepsilon[M](Z)$.*
- (2) *Dual structuring elements: there exists a structuring element $b : X \rightarrow PX$ such that $M = N_b$ and $\delta[N] = \delta[N_{\check{b}}]$.*

Proof. We already proved $2 \Rightarrow 1$. Let us prove $1 \Rightarrow 2$.

If we have the adjunction property between $\delta[N]$ and $\varepsilon[M]$, then by considering $Y = \{x\}$ for any variable $x : X$, we have in particular:

$$\forall x. \forall Z. \delta[N](\{x\}) \preceq_X Z \Leftrightarrow \{x\} \preceq_X \varepsilon[M](Z) \Leftrightarrow Z \in M(x)$$

Hence, we have necessarily:

$$\forall x. M(x) = \{Z : PX \mid \delta[N](\{x\}) \preceq_X Z\} = N_b(x)$$

where $b : X \rightarrow PX$ is the structuring element defined by:

$$\forall x. b(x) = \delta[N](\{x\})$$

We have therefore constructed a structuring element b satisfying $M = N_b$. Let us now prove that we have necessarily $\delta[N] = \delta[N_{\check{b}}]$.

By Proposition 6, we have $\delta[N_{\check{b}}] = \delta[b]$ and $\varepsilon[M] = \varepsilon[N_b] = \varepsilon[b]$. Therefore,

Theorem 1, combined with our adjunction hypothesis on N and M , results in:

$$\begin{aligned}
\forall Y. \forall Z. \quad \delta[N_{\check{b}}](Y) \leq_X Z &\Leftrightarrow \delta[b](Y) \leq_X Z \\
&\Leftrightarrow Y \leq_X \varepsilon[b](Z) \\
&\Leftrightarrow Y \leq_X \varepsilon[M](Z) \\
&\Leftrightarrow \delta[N](Y) \leq_X Z
\end{aligned}$$

In particular, by considering respectively $Z = \delta[N_{\check{b}}](Y)$ and $Z = \delta[N](Y)$:

$$\forall Y. \delta[N](Y) \leq_X \delta[N_{\check{b}}](Y) \quad ; \quad \forall Y. \delta[N_{\check{b}}](Y) \leq_X \delta[N](Y)$$

which concludes the proof. \square

Remark 2. When \mathcal{C} is a Boolean topos, then the assumption $\neg\neg\varphi = \varphi$ holds, we have moreover $N = N_{\check{b}}$. Indeed, we have $\varepsilon[N](Y) = \neg_X \delta[N](\neg_X Y) = \neg_X \delta[b](\neg_X Y) = \varepsilon[\check{b}](Y)$, i.e. $Y \in N(x) \Leftrightarrow \check{b}(x) \leq_X Y$.

4.2 Topological Neighborhood

Definition 7 (Topological neighborhood). A **topological neighborhood** is a structuring neighborhood $N : X \rightarrow PPX$ such that:

$$\forall x. \forall A. A \in_{PX} N(x) \Rightarrow \left(\begin{array}{l} \exists B. B \in_{PX} N(x) \wedge \\ (\forall y. y \in_X B \Rightarrow A \in_{PX} N(y)) \end{array} \right)$$

Proposition 8 (Interior operator). If $N : X \rightarrow PPX$ is a topological neighborhood, then $\varepsilon[N]$ is an internal interior operator in PX , i.e. it satisfies the following properties:

- (1) $\forall Y. \varepsilon[N](Y) \leq_X Y$.
- (2) $\varepsilon[N](X) = X$.
- (3) $\forall Y. \forall Z. \varepsilon[N](Y \wedge Z) = \varepsilon[N](Y) \wedge \varepsilon[N](Z)$.
- (4) $\forall Y. \varepsilon[N](\varepsilon[N](Y)) = \varepsilon[N](Y)$.

Proof. The first three properties have been proved in Proposition 5. Let us prove the last property. Let us consider a variable $Y : PX$. We already have

$\varepsilon[N](\varepsilon[N](Y)) \leq_X \varepsilon[N](Y)$. Conversely, for a variable $x : X$, we have:

$$\begin{aligned}
x \in_X \varepsilon[N](Y) &\Rightarrow Y \in_{PX} N(x) \\
&\Rightarrow \exists B. B \in_{PX} N(x) \wedge (\forall y. y \in_X B \Rightarrow Y \in_{PX} N(y)) \\
&\Rightarrow \exists B. B \in_{PX} N(x) \wedge (\forall y. y \in_X B \Rightarrow y \in_X \varepsilon[N](Y)) \\
&\Rightarrow \exists B. B \in_{PX} N(x) \wedge B \leq_X \varepsilon[N](Y) \\
&\Rightarrow \varepsilon[N](Y) \in_{PX} N(x) \\
&\Rightarrow x \in_X \varepsilon[N](\varepsilon[N](Y))
\end{aligned}$$

□

Proposition 9. *If $N : X \rightarrow PPX$ is a topological neighborhood, then $\delta[N]$ satisfies the following properties:*

- (1) $\forall Y. Y \leq_X \delta[N](Y)$.
- (2) $\delta[N](\emptyset) = \emptyset$.
- (3) $\forall Y. \delta[N](\delta[N](Y)) = \delta[N](Y)$.

Proof. The first two properties have already been proved in Proposition 5. Let us prove the last one. Let us consider a variable $x : X$. Then, we have the following implications :

$$x \in_X \delta[N](\delta[N](Y)) \Rightarrow (\forall A. A \in_{PX} N(x) \Rightarrow \exists y. y \in_X \delta[N](Y) \wedge A) \quad (1)$$

But

$$A \in_{PX} N(x) \Rightarrow \left(\begin{array}{l} \exists B. B \in_{PX} N(x) \wedge \\ (\forall y \in_X B \Rightarrow A \in_{PX} N(y)) \end{array} \right) \quad (2)$$

Moreover, as $B \in_{PX} N(x)$, we have that

$$\exists y. y \in_X \delta[N](Y) \wedge B \quad (3)$$

Now,

$$y \in_X \delta[N](Y) \Rightarrow \forall C. C \in_{PX} N(y) \Rightarrow \exists z. z \in_X Y \wedge C \quad (4)$$

From Implications 2 and 4, we have that $A \in_{PX} N(y)$, and then

$$\exists z. z \in_X Y \wedge A$$

from which we can conclude that $x \in \delta[N](Y)$, and hence $\delta[N](\delta[N](Y)) \leq_X \delta[N](Y)$.

Finally, $\delta[N](Y) \leq_X \delta[N](\delta[N](Y))$ is a direct consequence of the first property, and then we have $\forall Y. \delta[N](\delta[N](Y)) = \delta[N](Y)$. □

Hence, for any topological neighborhood N , $\overline{\delta[N]}_X$ is a closure operator to which we can associate the two natural transformations $\eta: Id_{Sub(X)} \Rightarrow \overline{\delta[N]}_X$ and $\mu: \overline{\delta[N]}_X \circ \overline{\delta[N]}_X \Rightarrow \overline{\delta[N]}_X$, and then $(\overline{\delta[N]}_X, \eta, \mu)$ is a monad⁶.

By Propositions 8 and 9, as $\delta[N]$ does not distribute over \vee , we have that the tuple $(PX, \leq_X, \wedge, \vee, \neg_X, \emptyset, \varepsilon[N], \delta[N])$ is an internal $CS4$ -modal algebra [7], and then is well adapted to give a semantic to the constructive modal logic $CS4$ from a topos perspective (which we will do in Section 5). Actually, we can easily show that $\delta[N]$ distributes over \wedge . Hence, $\delta[N]$ is a Lawvere-Tierney topology [43]. Now, we are not far from obtaining a closure algebra. For this purpose, we must impose the supplementary condition:

$$\forall A. \forall B. \delta[N](A \vee B) \leq_X \delta[N](A) \vee \delta[N](B)$$

Note that there are actually some $\delta[N]$ for which this inequality holds (and then this condition combined with extensivity makes $\delta[N]$ commute with \vee). An example of N for which $\delta[N]$ distributes over \vee is N_b for a structuring element b satisfying: $\forall x. x \in_X b(x)$. When this holds, we have defined a notion of internal topology over $X \in |\mathcal{C}|$, and $\varepsilon[N]$ and $\delta[N]$ are interpreted as topological notions of interior and closure, respectively.

In this context, we can define the two morphisms $open, closed: PX \rightarrow \Omega$ as the equalizers of $\varepsilon[N]$ and Id_{PX} , and $\delta[N]$ and Id_{PX} , respectively, which gives in the internal language of the topos \mathcal{C} the following characterizations:

- $open(Y) \Leftrightarrow \varepsilon[N](Y) = Y$
- $closed(Y) \Leftrightarrow \delta[N](Y) = Y$

We then recover the following result, which is standard for topological spaces (and also proved in a previous paper [5]).

Proposition 10. *Adjunction holds if and only if opening is equivalent to closing, that is the following statement is valid: for every $N: X \rightarrow PPX$*

$$Adjunction(\varepsilon[N], \delta[N]) \Leftrightarrow (\forall W, open(W) \Leftrightarrow closed(W))$$

where $Adjunction(\varepsilon[N], \delta[N])$ is defined as:

$$Adjunction(\varepsilon[N], \delta[N]) \equiv (\forall Y. \forall Z. \delta[N](Y) \leq_X Z \Leftrightarrow Y \leq_X \varepsilon[N](Z))$$

Proof. (\Rightarrow) Let us assume adjunction. Let us suppose that Y is closed, that is $\delta[N](Y) = Y$, and then $\delta[N](Y) \leq_X Y$. By adjunction, we then have that $Y \leq_X \varepsilon[N](Y)$. But, we also know that $\varepsilon[N](Y) \leq_X Y$, and then $\varepsilon[N](Y) = Y$. Therefore Y is also open. Applying this to the complement allows us to conclude that all opens are closed.

⁶ Here, $Sub(X)$ is seen as a category.

(\Leftarrow) Let us suppose that $\delta[N](Y) \leq_X Z$. By Theorem 5, we have that

$$\varepsilon[N](\delta[N](Y)) \leq_X \varepsilon[N](Z)$$

We know that $\delta[N](Y)$ is closed, and then it is open by hypothesis. We can then conclude that $\varepsilon[N](\delta[N](Y)) = \delta[N](Y)$. But, $\delta[N]$ is extensive, and then $Y \leq_X \delta[N](Y)$, whence we can conclude that $Y \leq_X \varepsilon[N](Z)$.

The other direction of the implication of adjunction is proved similarly. \square

5 Morpho-Logic: Interpretation of Modalities in Topos

Taking advantage of the fact that the tuple $(PX, \wedge, \vee, \neg_X, \emptyset, \varepsilon[N], \delta[N])$ is an internal modal algebra for the family of structuring neighborhoods N defined in Definition 7, we will give in this section a neighborhood semantics of the constructive modal logic from a topos perspective. As is customary in categorical logic, we propose an entailment system formulated as a sequent calculus for which we prove a completeness result.

Syntax. Let PV be a countable set whose elements are called **propositional variables** and denoted by letters p, q, \dots . The set Φ of formulas is defined by the following grammar:

$$\varphi, \psi ::= \top \mid \perp \mid p \mid \neg\varphi \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \varphi \Rightarrow \psi \mid \Box\varphi \mid \Diamond\varphi$$

where p runs through PV .

Semantics. As structuring neighborhoods subsume structuring elements $b : X \rightarrow PX$ such that $\forall x. x \in_X b(x)$ (cf. Example 4), we interpret the modal operators \Box and \Diamond respectively by the morphological operators $\overline{\varepsilon[N]}_X$ and $\overline{\delta[N]}_X$ for $N : X \rightarrow PPX$ with $X \in |\mathcal{C}|$ where \mathcal{C} is a topos.

Given a set of propositional variables PV , a **PV -model**, or simply a model, \mathcal{M} is a triple (X, N, ν) where:

- X is an object of a topos \mathcal{C} ,
- $N : X \rightarrow PPX$ is a structuring neighborhood, and
- $\nu : PV \rightarrow \text{Sub}(X)$ is a mapping called **valuation**.

If we want to give semantics to CS4-modal logic, we restrict ourselves to models (X, N, ν) where N is a topological neighborhood.

We use Mod to denote the class of PV -models.

The semantics of formulas in a model \mathcal{M} is a mapping $\llbracket \mathcal{M} \rrbracket (-) : \Phi \rightarrow \text{Sub}(X)$ defined by structural induction on formulas as follows:

- $\llbracket \mathcal{M} \rrbracket (\top) = Id_X$,
- $\llbracket \mathcal{M} \rrbracket (\perp) = \emptyset \twoheadrightarrow X$ ($\emptyset \twoheadrightarrow X$ is the least subobject of $\text{Sub}(X)$),
- $\llbracket \mathcal{M} \rrbracket (p) = \nu(p)$,
- $\llbracket \mathcal{M} \rrbracket (\neg\varphi) = \llbracket \mathcal{M} \rrbracket (\varphi) \rightarrow \llbracket \mathcal{M} \rrbracket (\perp)$
- $\llbracket \mathcal{M} \rrbracket (\varphi \wedge \psi) = \llbracket \mathcal{M} \rrbracket (\varphi) \wedge \llbracket \mathcal{M} \rrbracket (\psi)$ (the infimum of $\llbracket \mathcal{M} \rrbracket (\varphi)$ and $\llbracket \mathcal{M} \rrbracket (\psi)$),
- $\llbracket \mathcal{M} \rrbracket (\varphi \vee \psi) = \llbracket \mathcal{M} \rrbracket (\varphi) \vee \llbracket \mathcal{M} \rrbracket (\psi)$ (the supremum of $\llbracket \mathcal{M} \rrbracket (\varphi)$ and $\llbracket \mathcal{M} \rrbracket (\psi)$),
- $\llbracket \mathcal{M} \rrbracket (\varphi \Rightarrow \psi) = \llbracket \mathcal{M} \rrbracket (\varphi) \rightarrow \llbracket \mathcal{M} \rrbracket (\psi)$
- $\llbracket \mathcal{M} \rrbracket (\Box\varphi) = \varepsilon[N]_X(\llbracket \mathcal{M} \rrbracket (\varphi))$
- $\llbracket \mathcal{M} \rrbracket (\Diamond\varphi) = \delta[N]_X(\llbracket \mathcal{M} \rrbracket (\varphi))$

We write $\mathcal{M} \models \varphi$ if and only if $\llbracket \mathcal{M} \rrbracket (\varphi) = Id_X$, and then for every $\iota \in \text{Sub}(X)$, we write $\mathcal{M} \models_\iota \varphi$ to mean that $\iota \leq_X \llbracket \mathcal{M} \rrbracket (\varphi)$ (\leq_X is the ordering on $\text{Sub}(X)$). Let us denote by $Mod(\varphi)$ the class of models \mathcal{M} such that $\mathcal{M} \models \varphi$. Finally, given a set of formulas Γ and a formula φ , we write $\Gamma \models \varphi$ to mean that for every model \mathcal{M} which verifies $\mathcal{M} \models \psi$ for every formula $\psi \in \Gamma$, we have that $\mathcal{M} \models \varphi$.

By the properties of implication, the Kripke schema holds. Indeed, let $\mathcal{M} = (X, N, \nu)$ be a model. By definition of implication in Heyting algebra, we have:

$$\llbracket \mathcal{M} \rrbracket (\varphi) \wedge \llbracket \mathcal{M} \rrbracket (\varphi \Rightarrow \psi) \leq_X \llbracket \mathcal{M} \rrbracket (\psi)$$

Then, we have:

$$\llbracket \mathcal{M} \rrbracket (\Box\varphi) \wedge \llbracket \mathcal{M} \rrbracket (\Box(\varphi \Rightarrow \psi)) \leq_X \llbracket \mathcal{M} \rrbracket (\Box\psi)$$

Now, we have that $(_ \wedge \llbracket \mathcal{M} \rrbracket (\Box\varphi)) \dashv (\llbracket \mathcal{M} \rrbracket (\Box\varphi) \rightarrow _)$, from which we can conclude that $\llbracket \mathcal{M} \rrbracket (\Box(\varphi \Rightarrow \psi)) \leq_X \llbracket \mathcal{M} \rrbracket (\Box\varphi \Rightarrow \Box\psi)$.

Lemma 1. *For any model $\mathcal{M} = (X, N, \nu)$, and any formulas φ, ψ , we have:*

$$\mathcal{M} \models \varphi \Rightarrow \psi \text{ iff } \llbracket \mathcal{M} \rrbracket (\varphi) \leq_X \llbracket \mathcal{M} \rrbracket (\psi)$$

Proof. $\text{Sub}(X)$ is a Heyting algebra, and then it satisfies the following property:

$$\llbracket \mathcal{M} \rrbracket (\varphi) \leq_X \llbracket \mathcal{M} \rrbracket (\psi) \iff (\llbracket \mathcal{M} \rrbracket (\varphi) \rightarrow \llbracket \mathcal{M} \rrbracket (\psi) = Id_X)$$

□

A sound and complete entailment system. As is customary in categorical logic, we propose an entailment system formulated as a sequent calculus.

Definition 8 (Sequent). Given two formulas $\varphi, \psi \in \Phi$, a **sequent** is an expression of the form $\varphi \vdash \psi$. It is **valid** for a model $\mathcal{M} = (X, N, \nu)$, denoted $\varphi \vDash_{\mathcal{M}} \psi$, if we have that: $[[\mathcal{M}]](\varphi) \leq_X [[\mathcal{M}]](\psi)$. It is **valid**, denoted $\varphi \vDash \psi$, if for every model \mathcal{M} , $\varphi \vDash_{\mathcal{M}} \psi$.

The sequent calculus will consist of inference rules enabling us to derive a sequent (or a collection of sequents) from a collection of others, which will be written:

$$\frac{\Gamma}{\sigma}$$

to mean that the sequent σ can be inferred from a collection of sequents Γ . Note that the axioms will be rules in which Γ , the premises, is the empty set. In such a case we write simply σ .

Consider the following rules, where $\varphi \dashv\vdash \psi$ is a shortened notation for $\varphi \vdash \psi$ and $\psi \vdash \varphi$:

- *Identity rule:*

$$\varphi \vdash \varphi$$

- *Axioms:*

- **Preservation.** $\Box \top \dashv\vdash \top$, and $\Diamond \perp \dashv\vdash \perp$
- **Duality.** $\Box \neg \varphi \vdash \neg \Diamond \varphi$
- **Distributivity.** $\Box(\varphi \wedge \psi) \dashv\vdash \Box \varphi \wedge \Box \psi$ and $\Diamond \varphi \vee \Diamond \psi \vdash \Diamond(\varphi \vee \psi)$
- **Axiome K.** $\Box(\varphi \Rightarrow \psi) \vdash \Box \varphi \Rightarrow \Box \psi$
- **Axiom T.** $\Box \varphi \vdash \varphi$, and $\varphi \vdash \Diamond \varphi$
- **Axiom S4.** $\Box \varphi \vdash \Box \Box \varphi$, and $\Diamond \Diamond \varphi \vdash \Diamond \varphi$ (when models are restricted to topological neighborhoods)
- **Classical.** $\neg \neg \varphi \vdash \varphi$ (When \mathcal{C} is a Boolean topos)

- *Inconsistency:*

$$\perp \vdash \psi$$

- *Tautology:*

$$\varphi \vdash \top$$

- *Cut rule:*

$$\frac{\varphi \vdash \psi \quad \psi \vdash \chi}{\varphi \vdash \chi}$$

- *Rules for conjunction:*

$$\varphi \wedge \psi \vdash \varphi \quad \varphi \wedge \psi \vdash \psi \quad \varphi \wedge \varphi \dashv\vdash \varphi \quad \varphi \wedge \psi \dashv\vdash \psi \wedge \varphi$$

$$\frac{\varphi \vdash \psi \quad \varphi \vdash \chi}{\varphi \vdash \psi \wedge \chi}$$

- *Rules for disjunction:*

$$\varphi \vdash \varphi \vee \psi \quad \psi \vdash \varphi \vee \psi \quad \varphi \vee \psi \dashv\vdash \psi \vee \varphi$$

$$\frac{\varphi \vdash \chi \quad \psi \vdash \chi}{\varphi \vee \psi \vdash \chi} \quad \frac{\varphi \vee \psi \vdash \chi}{\varphi \vdash \chi} \quad \frac{\varphi \vee \psi \vdash \chi}{\psi \vdash \chi}$$

- *Distributivity:*

$$\varphi \wedge (\psi \vee \chi) \dashv\vdash (\varphi \wedge \psi) \vee (\varphi \wedge \chi)$$

- *Rules for implication:*

$$\frac{\varphi \wedge \psi \vdash \chi}{\varphi \vdash \psi \Rightarrow \chi} \quad \frac{\varphi \vdash \psi \Rightarrow \chi}{\varphi \wedge \psi \vdash \chi}$$

- *Rule for negation:*

$$\neg\varphi \dashv\vdash \varphi \Rightarrow \perp$$

- *Rules for modalities:*

$$\frac{\varphi \vdash \psi}{\Box\varphi \vdash \Box\psi} \quad \frac{\varphi \vdash \psi}{\Diamond\varphi \vdash \Diamond\psi}$$

A sequent $\varphi \vdash \psi$ is **provable** if there exists a finite sequence of sequents $(\varphi_1 \vdash \psi_1, \dots, \varphi_n \vdash \psi_n)$ such that $\varphi_n \vdash \psi_n = \varphi \vdash \psi$, and each sequent of the sequence is either an axiom or is obtained by applying inference rules from sequents that precede it in the sequence.

Theorem 1 (Soundness). *If $\varphi \vdash \psi$ is a sequent which is provable, then $\varphi \vDash \psi$.*

Proof. This is proved by structural induction on derivation. This is trivial in almost every case from the way in which semantics has been defined. For instance, the rules for modalities are a direct consequence of the fact that for every model $\mathcal{M} = (X, N, \nu)$, $\varepsilon[N]_X$ and $\delta[N]_X$ are monotonic. \square

In the following, we call **interior modal algebra** any tuple $(A, \wedge_A, \vee_A, \rightarrow_A, 1, 0, \Box_A, \Diamond_A)$ where $(A, \wedge_A, \vee_A, \rightarrow_A, 1, 0)$ is a Heyting algebra, and \Box_A and \Diamond_A are respectively, interior and closure operators. Let us define the binary relation \Vdash on $\Phi_{\dashv\vdash}$ (where $\Phi_{\dashv\vdash}$ is the quotient of Φ by the equivalence relation $\dashv\vdash$, and given a formula φ , $[\varphi]_{\dashv\vdash}$ is its equivalence class) by:

$$[\varphi]_{\dashv\vdash} \Vdash [\psi]_{\dashv\vdash} \text{ iff } \varphi \vdash \psi \text{ is provable}$$

Proposition 11. *$(\Phi_{\dashv\vdash}, \Vdash)$ is a Heyting algebra.*

Proof. By the identity rule and the definition of $\dashv\vdash$, \Vdash is reflexive and anti-symmetric. Now, by the cut rule, it is further transitive, and then \Vdash is a partial ordering.

Let us now show that $(\Phi_{/\multimap}, \Vdash)$ is a Heyting algebra. It is a bounded lattice where the least and the greatest elements in $\Phi_{/\multimap}$ are, respectively, $[\perp]_{\multimap}$ and $[\top]_{\multimap}$.

Let $[\varphi]_{\multimap}$ and $[\psi]_{\multimap}$ be two elements of $\Phi_{/\multimap}$. Then, we have:

$$\begin{aligned} [\varphi]_{\multimap} \wedge [\psi]_{\multimap} &= [\varphi \wedge \psi]_{\multimap} \\ [\varphi]_{\multimap} \vee [\psi]_{\multimap} &= [\varphi \vee \psi]_{\multimap} \\ [\varphi]_{\multimap} \rightarrow [\psi]_{\multimap} &= [\varphi \Rightarrow \psi]_{\multimap} \end{aligned}$$

By the rules of conjunction and disjunction, \wedge and \vee clearly define the lower and upper bound operators.

It remains to prove the following equivalence:

$$[\varphi]_{\multimap} \wedge [\psi]_{\multimap} \Vdash [\chi]_{\multimap} \iff [\varphi]_{\multimap} \Vdash [\psi]_{\multimap} \rightarrow [\chi]_{\multimap}$$

This amounts to show

$$\varphi \wedge \psi \vdash \chi \iff \varphi \vdash \psi \Rightarrow \chi$$

which is the rule for implication. □

Hence, if we define the two operators $\Box, \Diamond : \Phi_{/\multimap} \rightarrow \Phi_{/\multimap}$ by:

$$\Box : [\varphi]_{\multimap} \mapsto [\Box \varphi]_{\multimap}$$

and

$$\Diamond : [\varphi]_{\multimap} \mapsto [\Diamond \varphi]_{\multimap}$$

then we have that the tuple

$$(\Phi_{/\multimap}, \wedge, \vee, \rightarrow, [\top]_{\multimap}, [\perp]_{\multimap}, \Box, \Diamond)$$

is an interior modal algebra.

Given an interior modal algebra $\mathcal{A} = (A, \wedge_A, \vee_A, \rightarrow_A, 0, 1, \Box_A, \Diamond_A)$ and a map-

ping $\nu_A : PV \rightarrow A$, we can define the mapping:

$$[[\mathcal{A}]]_{\nu_A} : \begin{cases} \Phi & \rightarrow \mathcal{A} \\ \top & \mapsto 1 \\ \perp & \mapsto 0 \\ p & \mapsto \nu_A(p) \\ \varphi @ \psi & \mapsto [[\mathcal{A}]]_{\nu_A}(\varphi) @_A [[\mathcal{A}]]_{\nu_A}(\psi) \text{ where } @ \in \{\wedge, \vee, \rightarrow\} \\ \Box \varphi & \mapsto \Box_A [[\mathcal{A}]]_{\nu_A}(\varphi) \\ \Diamond \varphi & \mapsto \Diamond_A [[\mathcal{A}]]_{\nu_A}(\varphi) \end{cases}$$

Let us set:

$$\varphi \models \psi \text{ iff } \begin{cases} \text{for every interior modal algebra } \mathcal{A}, \text{ and every } \nu_A : PV \rightarrow A, \\ [[\mathcal{A}]]_{\nu_A}(\varphi) \leq_A [[\mathcal{A}]]_{\nu_A}(\psi) \end{cases}$$

Let us denote $IntAlg$ the category whose objects are couples (\mathcal{A}, ν_A) where \mathcal{A} is an interior modal algebra and $\nu_A : PV \rightarrow A$ is a valuation, and morphisms $\mu : (\mathcal{A}, \nu_A) \rightarrow (\mathcal{B}, \nu_B)$ are interior modal algebra morphisms (i.e. μ is a Heyting algebra morphism which preserves modal operators) such that for every $p \in PV$, $\nu_B(p) = \mu(\nu_A(p))$.

Proposition 12. $(\Phi_{\rightarrow+}, \nu : p \mapsto [p]_{\rightarrow+})$ is initial in $IntAlg$.

Proof. Let $\mu_A : (\Phi_{\rightarrow+}, \nu) \rightarrow (\mathcal{A}, \nu_A)$ be the mapping defined by:

$$\mu_A : \begin{cases} \Phi_{\rightarrow+} & \rightarrow \mathcal{A} \\ [\top]_{\rightarrow+} & \mapsto 1 \\ [\perp]_{\rightarrow+} & \mapsto 0 \\ [p]_{\rightarrow+} & \mapsto \nu_A(p) \\ [\varphi @ \psi]_{\rightarrow+} & \mapsto \mu_A([\varphi]_{\rightarrow+}) @_A \mu_A([\psi]_{\rightarrow+}) \text{ where } @ \in \{\wedge, \vee, \rightarrow\} \\ [\Box \varphi]_{\rightarrow+} & \mapsto \Box_A \mu_A([\varphi]_{\rightarrow+}) \\ [\Diamond \varphi]_{\rightarrow+} & \mapsto \Diamond_A \mu_A([\varphi]_{\rightarrow+}) \end{cases}$$

By construction, μ_A defines an interior modal algebra morphism which is unique. \square

Corollary 1. If $\varphi \vdash \psi$ is provable, then for every $(\mathcal{A}, \nu_A) \in |IntAlg|$, $[[\mathcal{A}]]_{\nu_A}(\varphi) \leq_A [[\mathcal{A}]]_{\nu_A}(\psi)$.

Proof. From the hypothesis, we have that $[\varphi]_{\dashv\vdash} \Vdash [\psi]_{\dashv\vdash}$. As μ_A is a Heyting morphism, we have that $\mu_A([\varphi]_{\dashv\vdash}) \leq_A \mu_A([\psi]_{\dashv\vdash})$. It is quite direct to show that $\mu_A(\varphi) = \llbracket \mathcal{A} \rrbracket_{\nu_A}(\varphi)$. \square

Theorem 2. $\varphi \vdash \psi$ is provable iff $\varphi \Vdash \psi$

Proof. The *Only-if part* is a direct consequence of Corollary 1. The *If part* is a direct consequence of the fact that $(\Phi_{/\dashv\vdash}, \nu : p \mapsto [p]_{\dashv\vdash})$ is in *IntAlg*. \square

Theorem 3 (Completeness). $\varphi \Vdash \psi$ iff $\varphi \models \psi$.

Proof. The *Only-if part* is obvious because for every model $\mathcal{M} = (X, N, \nu)$, $(Sub(X), \wedge, \vee, \rightarrow, [Id_X], [\emptyset \mapsto X], \varepsilon[N]_X, \delta[N]_X)$ is an interior modal algebra, and then $(Sub(X), \nu) \in |IntAlg|$.

To prove the *If part*, let us start by introducing some notions. Let $\mathcal{A} = (A, \wedge_A, \vee_A, \rightarrow_A, 0, 1, \Box_A, \Diamond_A)$ be an interior modal algebra. Let $\Gamma \subseteq A$ be a subset of A . Γ is said *consistent* if for all finite subsets $\Delta \subseteq \Gamma$, $\wedge_A \Delta \neq 0$. Likewise, Γ is said *maximally consistent* if Γ is consistent and there does not exist a consistent subset $\Theta \subseteq A$ such that $\Gamma \subseteq \Theta$. By Zorn's lemma, given a subset $\Gamma \subseteq A$, there exists a maximally consistent set Δ that contains Γ .⁷ Now, add to \mathcal{A} a mapping $\nu_A : PV \rightarrow A$, and define the model $\mathcal{M}_A = (X, N, \nu)$ in the topos *Set* as follows:

- X is the set of all maximal consistent sets of A ;
- $N_b : X \rightarrow \wp(\wp(X))$ is the structuring neighborhood obtained from the structuring element $b : X \rightarrow \wp(X)$ (see Example 4) defined by:

$$\Gamma' \in b(\Gamma) \text{ iff } \begin{cases} \Box_A a \in \Gamma \Rightarrow a \in \Gamma' \\ a \in \Gamma \Rightarrow \Diamond_A a \in \Gamma' \end{cases}$$

- $\nu : \{\Gamma \in X \mid \nu_A(p) \in \Gamma\}$

Lemma 2.

$$\Box_A a \in \Gamma \text{ iff } \forall \Gamma' \in b(\Gamma), a \in \Gamma'$$

⁷ The inductive set to which Zorn's lemma is applied is $S = \{\Gamma' \subseteq A \mid \Gamma' \text{ is consistent and } \Gamma \subseteq \Gamma'\}$. And then, by Zorn's lemma, Δ is the maximal element of S . It is quite direct to show that it is maximally consistent.

Proof. The *Only-if part* is a direct consequence of Definition of b . To show the *If-part*, let us suppose that for every $\Gamma' \in b(\Gamma)$, we have that $a \in \Gamma'$. Hence, the set $\{b \in A \mid \Box_A b \in \Gamma\} \cup \{a\}$ is consistent. Hence, for all finite subsets $\Delta \subseteq \{b \in A \mid \Box_A b \in \Gamma\}$, $0 \leq_A \bigwedge_A \Delta \wedge_A a$, and then $0 \leq_A \bigwedge_A \Box_A \Delta \wedge_A \Box_A a$, from which we can conclude that $\Box_A a \in \Gamma$. \square

Lemma 3. *For every formula φ , and every $\Gamma \in X$, we have:*

$$\Gamma \in [[\mathcal{M}_A]](\varphi) \text{ iff } [[\mathcal{A}]]_{\nu_A}(\varphi) \in \Gamma$$

Proof. By structural induction on φ . The basic case is obvious by definition. The cases of propositional operators are not difficult. Let us prove the property for modalities.

- φ is $\Box\psi$. We have the following equivalences:

$$\begin{aligned} \Gamma \in [[\mathcal{M}_A]](\Box\psi) &\Leftrightarrow [[\mathcal{M}_A]](\psi) \subseteq b(\Gamma) \\ &\Leftrightarrow \forall \Gamma' \in [[\mathcal{M}_A]](\psi), \Gamma' \in b(\Gamma) \\ &\Leftrightarrow \forall \Gamma' \in b(\Gamma), [[\mathcal{A}]]_{\nu_A}(\psi) \in \Gamma' \text{ (Induction hypothesis)} \\ &\Leftrightarrow [[\mathcal{A}]]_{\nu_A}(\Box\psi) \in \Gamma \text{ (Lemma 2)} \end{aligned}$$

- φ is $\Diamond\psi$. We have the following equivalences:

$$\begin{aligned} \Gamma \in [[\mathcal{M}_A]](\Diamond\psi) &\Leftrightarrow \exists \Gamma' \in X, \Gamma \in b(\Gamma') \wedge \Gamma' \in [[\mathcal{M}_A]](\psi) \\ &\Leftrightarrow \exists \Gamma' \in X, \Gamma \in b(\Gamma') \wedge [[\mathcal{A}]]_{\nu_A}(\psi) \in \Gamma' \text{ (Induction hypothesis)} \\ &\Leftrightarrow [[\mathcal{A}]]_{\nu_A}(\Diamond\psi) \in \Gamma \text{ (definition of } b) \end{aligned}$$

\square

Let us suppose that for every model \mathcal{M} , $\varphi \vDash_{\mathcal{M}} \psi$. This is equivalent to $\mathcal{M} \vDash \varphi \Rightarrow \psi$. Hence, for every interior modal algebra \mathcal{A} and every $\nu_A : PV \rightarrow A$, we have that $[[\mathcal{M}_A]](\varphi \Rightarrow \psi) = X$, and then by Lemma 3, for every $\Gamma \in X$, $[[\mathcal{A}]]_{\nu_A}(\varphi \Rightarrow \psi) \in \Gamma$, from which we can conclude that $[[\mathcal{A}]]_{\nu_A}(\varphi \Rightarrow \psi) = 1$. \square

In the previous proof, we showed completeness independently of a given topos, hence the definition of the set model \mathcal{M}_A for an interior modal algebra \mathcal{A} is similar to the canonical model. Let us now propose a proof of completeness for a model class reduced to all models defined on a same topos \mathcal{C} . To show this result, we assume a topos \mathcal{C} which is non-degenerate (i.e. $\emptyset \neq \mathbb{1}$, and then *true* \neq *false*), and therefore Mod is restricted to models $\mathcal{M} = (X, N, \nu)$ where $X \in |\mathcal{C}|$. Hence, in the remaining of this section, a sequent $\varphi \vdash \psi$ is valid if it is valid for all models \mathcal{M} in Mod .

Let $\Gamma \subseteq \Phi$ be a set of formulas. For every $\varphi \in \Phi$, we write $\Gamma \vdash \varphi$ to mean that the sequent $\varphi \dashv\vdash \top$ can be obtained by using the inference rules from all sequents $\psi \dashv\vdash \top$ where $\psi \in \Gamma$. Γ is said consistent if $\Gamma \not\vdash \perp$ (i.e. $\perp \dashv\vdash \top$ cannot be obtained from Γ).

By Zorn's lemma, there exists a maximally consistent set of formulas $\bar{\Gamma}$ which contains Γ when Γ is consistent.⁸

Proposition 13. *Let Δ be a maximally consistent set of formulas. There exists a model \mathcal{M}_Δ such that, for every $\varphi \in \Phi$:*

$$[[\mathcal{M}_\Delta]](\varphi) = Id_X \text{ iff } \varphi \in \Delta$$

Proof. $\mathcal{M}_\Delta = (X, N, \nu)$ is any model such that

- $X \in |\mathcal{C}|$ is an object different from the initial one (this is always possible since we have assumed that \mathcal{C} is non degenerated).
- $\nu : p \in PV \mapsto \begin{cases} Id_X & \text{if } p \in \Delta \\ \emptyset \twoheadrightarrow X & \text{otherwise} \end{cases}$

Note that no constraint is imposed on N .

The equivalence in the proposition is proved by structural induction on φ . The cases of basic formulas and propositional connectives are easily provable. Then, let us prove the property for the modality \Box (the proof for the modality \Diamond is substantially identical). So, let φ be of the form $\Box\psi$.

(\implies) Let us suppose that $[[\mathcal{M}_\Delta]](\varphi) = Id_X$. By anti-extensivity, we necessarily have that $[[\mathcal{M}_\Delta]](\psi) = Id_X$. Hence, by the induction hypothesis, we have that $\psi \in \Delta$, i.e. $\psi \dashv\vdash \top$ is obtained from Δ , and then so is $\Box\psi \dashv\vdash \top$ by the rule for modalities, whence we can then conclude that

$$\Box\psi \in \Delta$$

(\impliedby) Let us suppose that $\Box\psi \in \Delta$. We then have that $\psi \in \Delta$. Otherwise, this means that the sequent $\psi \dashv\vdash \perp$ can be obtained from Δ from which we can conclude by the axioms associated to modalities that $\varphi \dashv\vdash \perp$ which contradicts the hypothesis. So, by the induction hypothesis, we have that $[[\mathcal{M}_\Delta]](\psi) = Id_X$, from which we can conclude that $[[\mathcal{M}_\Delta]](\varphi) = Id_X$. \square

⁸ As previously, the inductive set to which Zorn's lemma is applied is $S = \{\Gamma' \subseteq Sen(\Sigma) \mid \Gamma' \text{ is consistent and } \Gamma \subseteq \Gamma'\}$. The poset (S, \subseteq) is inductive. Therefore, by Zorn's lemma, $\bar{\Gamma}$ is the maximal element of S , and quite directly it is maximally consistent.

Theorem 4 (Completeness). *If $\varphi \vdash \psi$ is a sequent which is valid, then it is provable.*

Proof. If $\varphi \vdash \psi$ is not provable, then the set $\Gamma = \{\neg(\varphi \Rightarrow \psi)\}$ is consistent. Let $\bar{\Gamma}$ be a maximally consistent set of formulas which contains Γ . By Proposition 13, there exists a model $\mathcal{M}_{\bar{\Gamma}} = (X, N, \nu)$ such that $[[\mathcal{M}_{\bar{\Gamma}}]](\neg(\varphi \Rightarrow \psi)) = Id_X$, and then $\varphi \not\vdash \psi$. \square

6 Application to Symbolic AI

6.1 Belief Revision

Belief revision is the process that makes an agent's beliefs evolve with newly acquired knowledge. In a logical framework, agent's beliefs and knowledge are formally defined by formulas. In practice, in this setting, the problem is then characterized by the resolution of the inconsistency of a theory after the addition of a new formula. To facilitate the presentation, we will assume that agent's beliefs and knowledge are in a finite number, and therefore can be represented by a simple formula. In the general case of an infinite set of knowledge and therefore defined by a theory, we refer the reader to our article [3] whose revision operators are easily adaptable to the formalism developed in this paper.

Modal logics are classically used to formalize beliefs. The reason is that they allow expressing beliefs about the beliefs of the other agents. Hence, we suppose here that agent's beliefs are formalized by modal formulas, and belief revision will be defined by an operator \circ between two formulas φ and ψ which will express how to transform φ into a formula φ' such that $\varphi' \wedge \psi$ is consistent [39].⁹

An axiomatization has imposed itself, the AGM theory [6], to describe the proper functioning of the revision operator \circ . AGM postulates are the following:

- (G1) If ψ is a consistent formula, then so is $\varphi \circ \psi$
- (G2) $Mod(\varphi \circ \psi) \subseteq Mod(\psi)$
- (G3) If $\varphi \wedge \psi$ is consistent, then $\varphi \circ \psi = \varphi \wedge \psi$
- (G4) If $\varphi \equiv \varphi'$ and $\psi \equiv \psi'$, then $Mod(\varphi \circ \psi) = Mod(\varphi' \circ \psi')$ (\equiv means that formulas are logically equivalent)

⁹ φ will often be the conjunction of a finite set of beliefs themselves defined by formulas [39].

- (G4') If $\psi \equiv \psi'$, then $Mod(\varphi \circ \psi) = Mod(\varphi \circ \psi')$
- (G5) $Mod((\varphi \circ \psi) \wedge \chi) = Mod(\varphi \circ (\psi \wedge \chi))$ if $(\varphi \circ \psi) \wedge \chi$ is consistent

Postulate (G4) means a complete independence of the syntax. But, we will see next that when the revision operator applies a syntactic transformation to knowledge bases (here specified by the formula φ) Postulate (G4) cannot be ensured anymore. This will be the case for our revision operator dedicated to the logic CS4 (see below), hence the introduction of Postulate (G4').

Revision operators based on dilations. By the way we interpreted modalities, we have for every $\varphi \in \Phi$ that

$$Mod(\varphi) \subseteq Mod(\diamond\varphi)$$

(This is just a consequence of the fact that dilations are extensive).

Following the approach developed in [16] for the propositional logic, the idea is then to dilate the class of models of φ until meeting the class of models of ψ . The problem is that the set $S = \{k \in \mathbb{N} \mid \diamond^k\varphi \wedge \psi \text{ is consistent}\}$ can be empty. In this case, we force $\varphi \circ \psi$ to be ψ . This leads to:

$$\varphi \circ \psi = \begin{cases} \diamond^n\varphi \wedge \psi & \text{if } S \neq \emptyset \text{ and } n = \min S \\ \psi & \text{otherwise} \end{cases}$$

with $\diamond^n\varphi = \underbrace{\diamond \dots \diamond}_{n \text{ times}}\varphi$.

Proposition 14. *The operator \circ satisfies the AGM postulates (G1)-(G5).*

Proof. \circ obviously satisfies Postulates (G1), (G2) and (G3).

Let us prove (G4). Let $\varphi \equiv \varphi'$ and $\psi \equiv \psi'$. This means that $Mod(\varphi) = Mod(\varphi')$ and $Mod(\psi) = Mod(\psi')$. Obviously, we have for every $k \in \mathbb{N}$ that $Mod(\diamond^k\varphi) = Mod(\diamond^k\varphi')$. This entails that $Mod(\varphi \circ \psi) = Mod(\varphi' \circ \psi')$.

Let us now prove (G5). Here, two cases have to be considered:

- (1) there does not exist $n \in \mathbb{N}$ such that $\diamond^n\varphi \wedge \psi$ is consistent. In this case, $Mod(\varphi \circ \psi) = Mod(\psi)$, and then $Mod((\varphi \circ \psi) \wedge \chi) = Mod(\psi \wedge \chi)$. Now, there does not exist $m \in \mathbb{N}$ such that $\diamond^m\varphi \wedge (\psi \wedge \chi)$ is consistent, otherwise $\diamond^m\varphi \circ \psi$ would be consistent too, which would be a contradiction. Hence, we have that $Mod(\varphi \circ (\psi \wedge \chi)) = Mod(\psi \wedge \chi)$.
Likewise, if there does not exist $n \in \mathbb{N}$ such that $\diamond^n\varphi \wedge (\psi \wedge \chi)$ is consistent, then as $(\varphi \circ \psi) \wedge \chi$ is consistent, this means that there does not exist

$m \in \mathbb{N}$ such that $\diamond^m \varphi \circ \psi$ is consistent. Hence, in both cases, we have that $Mod((\varphi \circ \psi) \wedge \chi) = Mod(\psi \wedge \chi)$ and $Mod(\varphi \circ (\psi \wedge \chi)) = Mod(\psi \wedge \chi)$.

- (2) $\varphi \circ \psi = \diamond^n \varphi \wedge \psi$. Therefore, as $(\varphi \circ \psi) \wedge \chi$ is consistent, we have that $Mod((\varphi \circ \psi) \wedge \chi) = Mod(\diamond^n \varphi \wedge \psi \wedge \chi) \neq \emptyset$. Hence, $S = \{k \mid \diamond^k \varphi \wedge \psi \wedge \chi \text{ consistent}\} \neq \emptyset$ ($n \in S$). Then $\varphi \circ (\psi \wedge \chi) = \diamond^m \varphi \wedge \psi \wedge \chi$ where $m = \min S$. This means that $\diamond^m \varphi \circ \psi$ is consistent, and then $n \leq m$. Moreover, since $n \in S$ and $m = \min S$, then $m \leq n$, from which we can conclude that $n = m$.

Likewise, if we suppose that $\varphi \circ (\psi \wedge \chi) = \diamond^n \varphi \wedge \psi \wedge \chi$, then this means that $\varphi \circ \psi = \diamond^n \varphi \wedge \psi$. Indeed, if there were $m \in \mathbb{N}$ such that $\varphi \circ \psi = \diamond^m \varphi \wedge \psi$, as $(\varphi \circ \psi) \wedge \chi$ is consistent, we would have that $n \leq m$.

□

The above approach is constraining because it imposes to check that such an index n exists. And this is not algorithmically definable. Moreover, it does not work if we consider Axiom S4 of modal logics to be valid. Indeed, in this case, the sequent $\diamond \diamond \varphi \dashv\vdash \diamond \varphi$ is valid, and then $Mod(\diamond \diamond \varphi) = Mod(\diamond \varphi)$. Another way to define a revision operator algorithmically definable because syntactic, is then to change necessity modalities to possibility ones. Indeed, it is quite intuitive that if the revision cannot be consistent for all states, it can be for some of them. A similar approach has been adopted in [3] for defining revision operators in first-order logic and modal logic in a set framework. The definition of these revision operators took advantage of Boolean reasoning. Thus, the revision operators were defined on formulas in normal form. In an intuitionistic framework, such normal forms for formulas do not exist. So, when the topos \mathcal{C} is not Boolean, we propose the following definition for the revision operator: let us define first two mappings $\rho, \kappa : \Phi \rightarrow \Phi$ on formulas as

- $\rho(\top) = \top$ and $\kappa(\top) = \top$
- $\rho(\perp) = \perp$ and $\kappa(\perp) = \perp$
- $\rho(p) = p$ and $\kappa(p) = p$ with $p \in PV$.
- $\rho(\varphi \Rightarrow \psi) = (\kappa(\varphi) \Rightarrow \psi) \vee (\varphi \Rightarrow \rho(\psi))$ and $\kappa(\varphi \Rightarrow \psi) = (\varphi \Rightarrow \kappa(\psi)) \vee (\rho(\varphi) \Rightarrow \psi)$
- $\rho(\varphi @ \psi) = (\rho(\varphi) @ \psi) \vee (\varphi @ \rho(\psi))$ and $\kappa(\varphi @ \psi) = (\kappa(\varphi) @ \psi) \vee (\varphi @ \kappa(\psi))$ with $@ \in \{\wedge, \vee\}$
- $\rho(\neg \varphi) = \neg \kappa(\varphi)$ and $\kappa(\neg \varphi) = \neg \rho(\varphi)$
- $\rho(\Box \varphi) = \diamond \varphi$ and $\kappa(\Box \varphi) = \Box \kappa(\varphi)$
- $\rho(\diamond \varphi) = \diamond \rho(\varphi)$ and $\kappa(\diamond \varphi) = \Box \varphi$

Intuitively, the mapping ρ transforms a formula φ by changing at each application a modal operator \Box to \diamond when the latter is in the positive part of φ (i.e. the operator is not in the scope of a negation or in the premise of an implication), and a \diamond into \Box when the latter is in the negative part (i.e. it is in the scope of a negation or in the premise of an implication). The mapping

κ is the dual of ρ , that is it transforms a formula φ by changing at each application a modal operator \diamond into \square when the latter is in the positive part of the formula, and a \square in \diamond when the latter is in the negative part.

Proposition 15. *For every formula φ and every model $\mathcal{M} = (X, N, \nu)$, we have $\llbracket \mathcal{M} \rrbracket(\varphi) \leq_X \llbracket \mathcal{M} \rrbracket(\rho(\varphi))$ and $\llbracket \mathcal{M} \rrbracket(\kappa(\varphi)) \leq_X \llbracket \mathcal{M} \rrbracket(\varphi)$.*

Proof. The proof is done by structural induction on φ . The basic cases are obvious. For the general case, several forms of φ must be considered. Here, we prove the result for implication and negation. The other cases are quite direct.

- φ is of the form $\psi \Rightarrow \chi$.
 - For the mapping ρ . By the definition of implication, we have that

$$\llbracket \mathcal{M} \rrbracket(\psi \Rightarrow \chi) = \llbracket \mathcal{M} \rrbracket(\psi) \rightarrow \llbracket \mathcal{M} \rrbracket(\chi)$$

By definition of \rightarrow , $\llbracket \mathcal{M} \rrbracket(\psi) \rightarrow \llbracket \mathcal{M} \rrbracket(\chi)$ is the greatest element ι in $\text{Sub}(X)$ such that

$$\llbracket \mathcal{M} \rrbracket(\psi) \wedge \iota \leq_X \llbracket \mathcal{M} \rrbracket(\chi)$$

By the induction hypothesis, we have both $\llbracket \mathcal{M} \rrbracket(\kappa(\psi)) \leq_X \llbracket \mathcal{M} \rrbracket(\psi)$ and $\llbracket \mathcal{M} \rrbracket(\chi) \leq_X \llbracket \mathcal{M} \rrbracket(\rho(\chi))$. Thus, we have that

$$\llbracket \mathcal{M} \rrbracket(\kappa(\psi)) \wedge \llbracket \mathcal{M} \rrbracket(\psi) \rightarrow \llbracket \mathcal{M} \rrbracket(\chi) \leq_X \llbracket \mathcal{M} \rrbracket(\chi)$$

and

$$\llbracket \mathcal{M} \rrbracket(\psi) \wedge \llbracket \mathcal{M} \rrbracket(\psi) \rightarrow \llbracket \mathcal{M} \rrbracket(\chi) \leq_X \llbracket \mathcal{M} \rrbracket(\rho(\chi))$$

and then by adjunction (i.e. the fact that $(_ \wedge \llbracket \mathcal{M} \rrbracket(\kappa(\psi))) \dashv \dashv (\llbracket \mathcal{M} \rrbracket(\kappa(\psi)) \rightarrow _)$ and $(_ \wedge \llbracket \mathcal{M} \rrbracket(\psi)) \dashv \dashv (\llbracket \mathcal{M} \rrbracket(\psi) \rightarrow _)$), we have that

$$\llbracket \mathcal{M} \rrbracket(\psi \Rightarrow \chi) \leq_X \llbracket \mathcal{M} \rrbracket(\kappa(\psi) \Rightarrow \chi)$$

and

$$\llbracket \mathcal{M} \rrbracket(\psi \Rightarrow \chi) \leq_X \llbracket \mathcal{M} \rrbracket(\psi \Rightarrow \rho(\chi))$$

that is

$$\llbracket \mathcal{M} \rrbracket(\psi \Rightarrow \chi) \leq_X \llbracket \mathcal{M} \rrbracket(\rho(\psi \Rightarrow \chi))$$

- For the mapping κ . By definition, we have that

$$\llbracket \mathcal{M} \rrbracket(\psi \Rightarrow \kappa(\chi)) = \llbracket \mathcal{M} \rrbracket(\psi) \rightarrow \llbracket \mathcal{M} \rrbracket(\kappa(\chi))$$

and

$$\llbracket \mathcal{M} \rrbracket(\rho(\psi) \Rightarrow \chi) = \llbracket \mathcal{M} \rrbracket(\rho(\psi)) \rightarrow \llbracket \mathcal{M} \rrbracket(\chi)$$

By definition of \rightarrow , $\llbracket \mathcal{M} \rrbracket(\psi) \rightarrow \llbracket \mathcal{M} \rrbracket(\kappa(\chi))$ and $\llbracket \mathcal{M} \rrbracket(\rho(\psi)) \rightarrow \llbracket \mathcal{M} \rrbracket(\chi)$ are the greatest elements ι and θ of $\text{Sub}(X)$ such that

$$\llbracket \mathcal{M} \rrbracket(\psi) \wedge \iota \leq_X \llbracket \mathcal{M} \rrbracket(\kappa(\chi))$$

and

$$[[\mathcal{M}]](\rho(\psi)) \wedge \theta \leq_X [[\mathcal{M}]](\chi)$$

By the induction hypothesis, we have that $[[\mathcal{M}]](\kappa(\chi)) \leq_X [[\mathcal{M}]](\chi)$ and $[[\mathcal{M}]](\psi) \leq_X [[\mathcal{M}]](\rho(\psi))$, and then again by the adjunction property, we have that

$$[[\mathcal{M}]](\psi \Rightarrow \kappa(\chi)) \leq_X [[\mathcal{M}]](\psi \Rightarrow \chi)$$

and

$$[[\mathcal{M}]](\rho(\psi) \Rightarrow \chi) \leq_X [[\mathcal{M}]](\psi \Rightarrow \chi)$$

- φ is of the form $\neg\psi$.
- For the mapping ρ . By definition, we have that

$$[[\mathcal{M}]](\neg\psi) = [[\mathcal{M}]](\psi) \rightarrow [[\mathcal{M}]](\perp)$$

By definition of \rightarrow , we have

$$[[\mathcal{M}]](\psi) \wedge [[\mathcal{M}]](\neg\psi) \leq_X [[\mathcal{M}]](\perp)$$

By the induction hypothesis, we have that $[[\mathcal{M}]](\kappa(\psi)) \leq_X [[\mathcal{M}]](\psi)$, and then by the adjunction property, we can conclude that

$$[[\mathcal{M}]](\neg\psi) \leq_X [[\mathcal{M}]](\neg\kappa(\psi))$$

and therefore $[[\mathcal{M}]](\varphi) \leq_X [[\mathcal{M}]](\rho(\varphi))$.

- For the mapping κ . By definition, we have that

$$[[\mathcal{M}]](\kappa(\neg\psi)) \leq_X [[\mathcal{M}]](\rho(\psi)) \rightarrow [[\mathcal{M}]](\perp)$$

By adjunction property, we then have that

$$[[\mathcal{M}]](\rho(\psi)) \wedge [[\mathcal{M}]](\kappa(\neg\psi)) \leq_X [[\mathcal{M}]](\perp)$$

By the induction hypothesis, and the adjunction property, we can conclude that

$$[[\mathcal{M}]](\kappa(\neg\psi)) \leq_X [[\mathcal{M}]](\neg\psi)$$

□

Let us define the mapping $\tau : \Phi \rightarrow \Phi$ as follows:

$$\tau : \begin{cases} \Phi \rightarrow \Phi \\ \varphi \mapsto \begin{cases} \chi & \text{if } \rho(\varphi) = \varphi \\ \rho(\varphi) & \text{otherwise} \end{cases} \end{cases}$$

where χ is a tautology.

Proposition 16. *For every formula φ , we have:*

- **Extensivity.** $Mod(\varphi) \subseteq Mod(\tau(\varphi))$
- **Exhaustivity.** *There exists $k \in \mathbb{N}$ such that $Mod(\tau^k(\varphi)) = Mod$*

The operator τ is then an extensive operator that is not a dilation.

Proof. Obviously, exhaustivity holds. Let $\mathcal{M} \in Mod(\varphi)$. By definition of semantics, this means that $\llbracket \mathcal{M} \rrbracket(\varphi) = Id_X$. If φ is a propositional formula, we know that $Mod(\varphi) \subseteq Mod(\diamond\varphi)$. Otherwise, by Proposition 15, we have that $\llbracket \mathcal{M} \rrbracket(\varphi) \leq_X \llbracket \mathcal{M} \rrbracket(\rho(\varphi))$, and then $\llbracket \mathcal{M} \rrbracket(\rho(\varphi)) = Id_X$. The case where $\tau(\varphi) = \chi$ is obvious. This proves the extensivity of τ . \square

Let us define the following revision operator \circ :

$$\varphi \circ \psi = \tau^n(\varphi) \wedge \psi$$

with $n = \min\{k \in \mathbb{N} \mid \tau^k(\varphi) \wedge \psi \text{ is consistent}\}$ and $\tau^n(\varphi) = \underbrace{\tau(\dots\tau(\varphi)\dots)}_{n \text{ times}}$.

Proposition 17. *The operator \circ satisfies the AGM postulates (G1)-(G3), (G4'), and (G5).*

Proof. The proof is substantially similar to that of Proposition 14. \square

Minimality result. In a previous paper [3], we showed independently of any logic a minimization result demonstrated first by H. Katsuno and A.-O. Mendelzon [39] in the framework of propositional logic. This result is a representation theorem which means that revision operators satisfying AGM postulates induce minimal changes, that is the models of $\varphi \circ \psi$ are the models of ψ that are the closest to models of φ according to some distance for measuring how close models are. This result is based on the notion of faithful assignment [39] whose definition we recall below.

Definition 9 (Faithful assignment). *An **assignment** is a mapping that assigns to each formula φ a binary relation \leq_φ on models. We say that this assignment is **faithful (FA)** if the following two conditions are satisfied:*

- (1) *If $\mathcal{M}, \mathcal{M}' \in Mod(\varphi)$, then $\mathcal{M} \not\prec_\varphi \mathcal{M}'$*
- (2) *For every $\mathcal{M} \in Mod(\varphi)$ and for every model $\mathcal{M}' \notin Mod(\varphi)$, $\mathcal{M} \prec_\varphi \mathcal{M}'$.*

($\mathcal{M} \prec_\varphi \mathcal{M}'$ means that $\mathcal{M} \leq_\varphi \mathcal{M}'$ and $\mathcal{M}' \not\leq_\varphi \mathcal{M}$).

Proposition 18. *A revision operator \circ satisfies the AGM postulates if and only if there exists a FA that maps each formula φ to a binary relation \leq_φ such that for every formula ψ*

$$Mod(\varphi \circ \psi) = Min(Mod(\psi), \leq_\varphi)$$

where $Min(Mod(\psi), \leq_{\varphi}) = \{\mathcal{M} \in Mod(\psi) \mid \forall \mathcal{M}' \in Mod(\psi), \mathcal{M}' \not\prec_{\varphi} \mathcal{M}\}$.

The proof of Proposition 18 is given in Appendix. It is substantially similar to that of the general result given in [3] (see Theorem 1) except that the result here relates to formulas and not to theories. This then required redefining a particular FA in the proof. The FA used in [3] to prove this representation result was an adaptation of the FA proposed in the original paper [39], but the latter was not adaptable within the framework of modal logic¹⁰.

6.2 Contraction

Other important operators considered in the domain of belief change are contraction operators [6]. Unlike revision operators, the goal of contraction operators is to remove the new piece of information.

Using the results of the previous subsection in which revision operators have been defined, we can now define, via the Harper identity [36], contraction operators in the following way:

$$\varphi \dot{-} \psi = (\varphi \circ \neg\psi) \vee \varphi \quad \text{(Harper identity)}$$

The idea behind $\varphi \dot{-} \psi$ (φ contracted by ψ) is that the information provided by ψ is “removed” from φ , thus guaranteeing that the result of the contraction does not prove ψ . This is why the Harper identity links contraction of φ by ψ to revision of φ by $\neg\psi$.

The well behaved contraction operators should satisfy the following postulates:¹¹

- (C1) $\varphi \vdash \varphi \dot{-} \psi$
- (C2) If $\varphi \not\vdash \psi$, then $\varphi \dot{-} \psi \vdash \varphi$
- (C3) If $\varphi \dot{-} \psi \vdash \psi$, then $\vdash \psi$
- (C4) If $\varphi \vdash \psi$, then $(\varphi \dot{-} \psi) \wedge \psi \vdash \varphi$
- (C5) If $\psi_1 \equiv \psi_2$ then $\varphi \dot{-} \psi_1 \equiv \varphi \dot{-} \psi_2$
- (C6) $\varphi \dot{-} (\psi \wedge \beta) \vdash (\varphi \dot{-} \psi) \vee (\varphi \dot{-} \beta)$
- (C7) If $\varphi \dot{-} (\psi \wedge \beta) \not\vdash \psi$, then $\varphi \dot{-} \psi \vdash \varphi \dot{-} (\psi \wedge \beta)$

¹⁰ The reason is that, unlike in propositional logic, the set of valid modal formulas for a model is not necessarily definable by a single modal formula.

¹¹ We present the formulation of postulates as in [26].

Now we can establish the following result, the proof of which is a straightforward consequence of Proposition 17 and standard facts about the duality between revision and contraction [6]:

Proposition 19. *Let $\dot{-}$ be the operator defined by the Harper identity where \circ is the revision operator of Proposition 17. Then, $\dot{-}$ satisfies (C1-C7).*

6.3 Merging

When the beliefs play symmetrical roles, another widely addressed problem is the one of belief merging, or fusion. Let $\varphi_1, \dots, \varphi_m$ be m formulas modeling agents' beliefs. As for revision, their fusion can be defined from a dilation δ or better by a τ operator (defined as in Section 6.1) as:

$$\text{Merging}(\varphi_1, \dots, \varphi_m) = \tau^n(\varphi_1) \wedge \dots \wedge \tau^n(\varphi_m)$$

where τ is defined as for the revision, and

$$n = \min\{k \in \mathbb{N} \mid \bigwedge_{i=1}^m \tau^k(\varphi_i) \text{ is consistent}\}$$

It has been shown in [21,24] that this dilation-based merging is equivalent to a merging operator defined from a specific distance and aggregation function, and satisfies the set of rationality postulates introduced in [40], except the independence of the syntax, since syntactic operations are involved in τ .¹²

6.4 Abduction

Abduction is the process of finding, given a theory T and an observation φ , the best explanation ψ such that $T \cup \{\psi\} \models \varphi$. Candidate explanations of φ with respect to T are many. In a logical framework, following our previous work in [4] where abduction has been studied independently of a given logic, we will also study abduction as a form of inference. Intuitively, finding candidate explanations for φ with respect to T consists in cutting/eroding the class of models of T while remaining consistent with φ . Hence, abduction can be seen as the dual of revision, and then it will consist in cutting in $\text{Mod}(T \cup \{\varphi\})$ as

¹² Of course, if we define the fusion by

$$\text{Merging}(\varphi_1, \dots, \varphi_m) = \diamond^n(\varphi_1) \wedge \dots \wedge \diamond^n(\varphi_m)$$

under the condition that n exists, then all the rationality postulates in [40] are satisfied because the complete independence of the syntax holds.

much as possible while preserving minimality properties. For this, we propose to instantiate the abstract approach developed in [4] to our framework of categorical morpho-logic¹³. In an intuitionistic framework, to work around the problem of missing normal forms of formulas, we propose the mapping $\zeta : \Phi \rightarrow \Phi$ defined as follows: let χ be an antilogy

$$\zeta : \begin{cases} \Phi \rightarrow \Phi \\ \varphi \mapsto \begin{cases} \chi & \text{if } \kappa(\varphi) = \varphi \\ \kappa(\varphi) & \text{otherwise} \end{cases} \end{cases}$$

where κ is the mapping defined in Section 6.1.

Thus, the mapping ζ will transform $\wedge T$ at each step by changing an operator \diamond into \square in order to reduce the class of models, and thus hope to only encounter the models of φ . If this is not possible (there are none left) and therefore we no longer have a way to sufficiently reduce the models of $\wedge T$ to keep only those in common with φ , we force the transformation into an antilogy that trivially ensures the inclusion of models.

Proposition 20. *The mapping ζ verifies for every consistent formula $\varphi \in \Phi$:*

- **Anti-extensivity.** $Mod(\zeta(\varphi)) \subseteq Mod(\varphi)$
- **Vacuum.** *There exists $k \in \mathbb{N}$ such that $Mod(\zeta^k(\varphi)) = \emptyset$.*

In [4], mappings satisfying such conditions are called retractions.

Proof. ζ is anti-extensive because by Proposition 15 κ is. The vacuum property holds because in a finite number of steps, we always reach the antilogy χ . \square

Following [4], from ζ we can define two families of model classes \mathcal{C}_{lcr} and \mathcal{C}_{lnr} as follows¹⁴: Let T be a finite set of modal formulas and let φ be a formula such that $T \cup \{\varphi\}$ is consistent

$$\mathcal{C}_{lcr}^\varphi = \{Mod(\zeta^k(\wedge T) \wedge \varphi) \mid k \in \mathbb{N}, Mod(\zeta^k(\wedge T) \wedge \varphi) \neq \emptyset\}$$

$$\mathcal{C}_{lnr}^\varphi = \{Mod(\zeta^k(\wedge T \wedge \varphi)) \mid k \in \mathbb{N}, Mod(\zeta^k(\wedge T \wedge \varphi)) \neq \emptyset\}$$

where $\wedge T = \varphi_1 \wedge \dots \wedge \varphi_n$ if $T = \{\varphi_1, \dots, \varphi_n\}$.

Let us observe that in both $\mathcal{C}_{lcr}^\varphi$ and $\mathcal{C}_{lnr}^\varphi$ we have a unique maximal chain of finite size due to the vacuum property. Hence, both $\mathcal{C}_{lcr}^\varphi$ and $\mathcal{C}_{lnr}^\varphi$ are closed

¹³The abstract approach in [4] has been applied to the case of modal logic but in the set-theoretic framework. Here, as before, we must adapt this instantiation to the intuitionistic framework.

¹⁴*lcr* for *last consistent retraction* and *lnr* for *last non-trivial retraction*

under set-theoretical inclusion and are well-founded. In [4], any sub-family satisfying such conditions is called cutting.

These two cuttings give rise to the following two explanatory relations:

$$\varphi \triangleright_{\mathcal{C}_{lcr}} \psi \iff \begin{cases} Mod(T \cup \{\psi\}) \neq \emptyset, \text{ and} \\ Mod(T \cup \{\psi\}) \subseteq Mod(\zeta^n(\wedge T) \wedge \varphi) \end{cases}$$

where $n = \sup\{k \in \mathbb{N} \mid Mod(\zeta^k(\wedge T) \wedge \varphi) \neq \emptyset\}$;

$$\varphi \triangleright_{\mathcal{C}_{lnr}} \psi \iff \begin{cases} Mod(T \cup \{\psi\}) \neq \emptyset, \text{ and} \\ Mod(T \cup \{\psi\}) \subseteq Mod(\zeta^m(\wedge T \wedge \varphi)) \end{cases}$$

where $m = \sup\{k \in \mathbb{N} \mid Mod(\zeta^k(\wedge T \wedge \varphi)) \neq \emptyset\}$.

Directly from Theorems 2, 3 and 4 in [4], we derive that these two explanatory relations satisfy all or part of the (rationality) postulates defined in [52] which we summarize in Table 1.

Rationality postulates	$\triangleright_{\mathcal{C}_{lcr}}$	$\triangleright_{\mathcal{C}_{lnr}}$
LLE and RLE	✓	✓
RS	✓	✓
E-Con	✓	✓
ROR	✓	✓
E-Reflexivity	✓	✓
E-CM	✓	
E-C-Cut	✓	

Table 1

Links between rationality postulates in [52] and properties satisfied by $\triangleright_{\mathcal{C}_{lcr}}$ and $\triangleright_{\mathcal{C}_{lnr}}$.

The original postulates of [52] are recalled in Table 2.

6.5 Spatial Reasoning: RCC-8

A part of the domain of qualitative spatial reasoning deals with topological relations between spatial entities. Here we propose to apply the defined morpho-logic to mereotopology, and more precisely to the RCC-8 model [55]. This theory allows defining several topological relations from a connection

LLE:	If $\vdash_T \alpha \leftrightarrow \alpha'$ and $\alpha \triangleright \gamma$ then $\alpha' \triangleright \gamma$.
RLE:	If $\vdash_T \gamma \leftrightarrow \gamma'$ and $\alpha \triangleright \gamma$ then $\alpha \triangleright \gamma'$.
E-CM:	If $\alpha \triangleright \gamma$ and $\gamma \vdash_T \beta$ then $(\alpha \wedge \beta) \triangleright \gamma$.
E-C-Cut:	If $(\alpha \wedge \beta) \triangleright \gamma$ and $\forall \delta [\alpha \triangleright \delta \Rightarrow \delta \vdash_T \beta]$ then $\alpha \triangleright \gamma$.
RS:	If $\alpha \triangleright \gamma$, $\gamma' \vdash_T \gamma$ and $\gamma' \not\vdash_T \perp$ then $\alpha \triangleright \gamma'$.
ROR:	If $\alpha \triangleright \gamma$ and $\alpha \triangleright \delta$ then $\alpha \triangleright (\gamma \vee \delta)$.
E-Reflexivity:	If $\alpha \triangleright \gamma$ then $\gamma \triangleright \gamma$.
E-Con:	$\not\vdash_T \neg\alpha$ iff there is γ such that $\alpha \triangleright \gamma$.

Table 2

Rationality postulates of [52], expressed according to a theory T .

predicate C , and has been often formalized in first order logic in the literature. Let us recall the eight relations of RCC-8 theory (without their formalization):

- **Disconnection** DC . $DC(XY)$ means that region X is disconnected from region Y ;
- **External Connection** EC . $EC(X, Y)$ means that X is externally connected to Y (close to the notion of adjacency);
- **Partial Overlap** PO . $PO(X, Y)$ means that X and Y intersect each other but are not equal;
- **Tangential Proper Part (resp. inverse)** TPP (resp. TPP_i). $TPP(X, Y)$ (resp. $TPP_i(X, Y)$) means that X (resp. Y) is a tangential proper part of Y (resp. of X);
- **Non-Tangential Proper Part (resp. inverse)** $NTPP$ (resp. $NTPP_i$). $NTPP(X, Y)$ (resp. $NTPP_i(X, Y)$) means that X (resp. Y) is a non-tangential proper part of Y (resp. of X);
- **Equality** EQ . $EQ(X, Y)$ means that X and Y are identical regions.

As shown in [5,13,16], morpho-logic, as defined in this paper, provides a simple axiomatization of some of these relations.

Let $\mathcal{M} = (X, N, \nu)$ be a model. The sub-objects of X are spatial entities (i.e. regions), and formulas are combinations of such entities. The RCC-8 relations can then be formulated as follows:

- $C(X, Y) : \varphi \wedge \psi$;
- $DC(XY) : \neg(\varphi \wedge \psi)$;
- $EC(X, Y) : \neg(\varphi \wedge \psi)$ and $\diamond\varphi \wedge \psi$ and $\varphi \wedge \diamond\psi$;
- $PO(X, Y) : \varphi \wedge \psi$ and $\varphi \wedge \neg\psi$ and $\neg\varphi \wedge \psi$;
- $TPP(X, Y) : \varphi \Rightarrow \psi$ and $\diamond\varphi \wedge \neg\psi$;
- $NTPP(X, Y) : \varphi \Rightarrow \psi$ and $\varphi \Rightarrow \square\psi$;
- $EQ(X, Y) : \varphi \Leftrightarrow \psi$.

where φ and ψ are formulas defining regions X and Y , respectively. Let us detail the meaning of some of these formulas:

- For *EC*: the two regions X and Y do not intersect, but as soon as one of them is dilated (using \diamond) then they do intersect.
- For *TPP*: X is included in Y but the dilation of X (represented by $\diamond\varphi$) is not.
- For *NTPP*: X is included in Y and also in the erosion of Y .

The meaning of the other relations is very natural.

7 Conclusion

In this article, we deepened in the topos framework the strong link between MM and modal logic initiated twenty years ago in [13]. The interest of toposes is that they generalize the notion of space and subspace, and then they include a large family of algebraic structures which have proved useful for knowledge representation and reasoning.

We have then generalized MM from a topos perspective that we extended by moving from the power object PX to PPX , for X an object of a topos \mathcal{C} , to define structuring neighborhoods. We then showed that the morphological operations of erosion and dilation have all the properties to make PX an internal modal algebra, and then can be used to give a neighborhood semantics to constructive modal logic in the topos setting.

To our knowledge, MM has never been studied independently of a given structure. In the complete lattice setting, basic MM operators are defined without referring to the notion of structuring element in the algebraic definitions. Our work is then the first proposition of a generalization of MM independently of a particular structure (set, graph...). Taking into account the notion of structuring element and structuring neighborhood allowed us to generalize the notion of Kripke model and neighborhood model in the topos framework.

The proposed extension of basic operators relies on structuring elements defined as morphisms $X \rightarrow PX$. Dilations and erosions using a structuring element are then defined as morphisms $PX \rightarrow PX$ with specific properties. Dilations, erosions, as well as their compositions are shown to have similar properties as in classical morphology (adjunction, monotonicity...). Interestingly enough, the proofs of the main results in this paper highly benefit from the internal logic of topos. The underlying logic being intuitionistic, as future work, it might be interesting to certify these proofs using a formal proof

management system, such as Coq¹⁵.

Finally, we introduced structuring neighborhoods and topological neighborhoods, giving rise to modal algebras, for which a sound and complete entailment system is proposed.

This led us to suggest applications to typical reasoning problems in AI, such as revision, merging, abduction, spatial reasoning based on RCC-8 relations.

Let us now mention some other directions for future work.

Towards a unification of morphological operators via the notion of predicate lifting.

By the bijection correspondence $\text{Hom}(X, PY) \simeq \text{Sub}(X \times Y)$, structuring elements and structuring neighborhoods can be seen as mappings which match each element x with elements in relation with it. A generalization of that can be obtained via the notion of co-algebras [57]. In computing science, co-algebras have been studied in Set (the category of sets), and in this framework the notion of predicate liftings [50] has been identified as the concept underlying modal operator semantics. This notion of predicate lifting can be extended to arbitrary topos. Indeed, suppose a topos \mathcal{C} and a functor $T : \mathcal{C} \rightarrow \mathcal{C}$. A *predicate lifting* is then a natural transformation $\lambda : \mathcal{P} \Rightarrow \mathcal{P} \circ T$ (recall that \mathcal{P} is the contravariant functor presented in Section 2). A structuring element b is then defined as a morphism (co-algebra) $b : X \rightarrow T(X)$. If we define the morphism $b^{-1} : PT(X) \rightarrow PX$ as follows:

$$b^{-1}(Y) = \{x : X \mid \exists y. y \in_{T(X)} Y \wedge b(x) = y\}$$

a *morphological operator* $op[b] : PX \rightarrow PX$ is then defined by: $op[b] = b^{-1} \circ \lambda_X$. From there, a morphological operator $op[b]$ is an *erosion* (respectively a *dilation*) if λ_X commutes with \wedge (respectively \vee).

Studying this unifying framework in more depth should allow us to generalize the theory of co-algebras, to other toposes than the category Set , but also to go further in the generalization of MM and its application to modal logic. Among other things, this will allow us to study concepts such as bisimulation and associated preservation results which have been extensively studied in the theory of co-algebras, at a more abstract level.

Extension to the fuzzy case. It is usual to introduce uncertainty in qualitative spatial reasoning. Then, as future work, we propose to extend our approach to the fuzzy case. This will first require us to introduce fuzziness in a topos. There are some works showing that fuzzy sets do not form toposes in

¹⁵ <https://coq.inria.fr/>

general [10,31,53], but there are very few which extend toposes to the fuzzy case. Fuzzy sets may form a topos when different maps between them are considered [63]. An extension of toposes to fuzzy sets has been proposed in [54]. We can then rely on this work to introduce fuzziness in toposes. This will allow us to extend erosion and dilation based on structuring elements to the fuzzy case in this framework. In the set theory framework, several definitions of MM on fuzzy sets with structuring elements have been proposed in the literature, since the early work in [9,22] (see e.g [14,23,46] for reviews). We will be able to take inspiration from the approach developed in [14,15,58] using conjunctions and implications from a topos perspective. More generally, we could replace the traditional $[0, 1]$ interval by any lattice L , thus opening the way towards bipolar information (positive as preferences, arguments, observations, and negative as constraints, attacks, etc.). This could find applications in preference-based reasoning, argumentation, spatial reasoning, among others.

Pretopos structures. There are spatial structures which are not toposes but only pretoposes. The most emblematic example is the category of compact Hausdorff spaces. Now the properties of pretoposes (exactness and extensivity) make them a very set-like category. Among others, they have an internal logic. The difficulty for extending our work to such structures will be to see how to get around the absence of the power object, a notion on which many of the concepts presented in the paper are built.

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A Algebraization of subobjects

Let \mathcal{C} be a topos. Let X be an object of \mathcal{C} . Let us define the partial ordering \leq on $Sub(X)$ as follows: for all $f : A \rightarrow X$ and $g : B \rightarrow X$

$$[f] \leq [g] \iff \exists h : A \rightarrow B, f = g \circ h$$

Proposition 21. *($Sub(X), \leq$) is a bounded lattice. It is a complete lattice if \mathcal{C} is a complete topos (i.e. \mathcal{C} has all limits and colimits).*

Proof. Obviously, \leq is reflexive and transitive. To show anti-symmetry, let us suppose that $[f] \leq [g]$ and $[g] \leq [f]$. This means that there exist h and h' such that $f = g \circ h$ and $g = f \circ h'$ whence we have both that $f = f \circ h' \circ h$ and $g = g \circ h \circ h'$. We can deduce that $h' \circ h = Id_A$ and $h \circ h' = Id_B$, and then $A \simeq B$ that is $[f] = [g]$. Given two elements $[f]$ and $[g]$ of $Sub(X)$ with $f : A \twoheadrightarrow X$ and $g : B \twoheadrightarrow X$, the infimum of $[f]$ and $[g]$ is $[A \cap B \twoheadrightarrow A \xrightarrow{f} X]$ which is equivalent to $[A \cap B \twoheadrightarrow B \xrightarrow{g} X]$ (we recall that every topos has pullbacks, and pullbacks of monics are monics), and the supremum of $[f]$ and $[g]$ is $[A \cup B \twoheadrightarrow X]$ where $A \cup B \twoheadrightarrow X$ is the unique morphism, consequence of the fact that $A \cup B$ is the pushout of $A \cap B$. This pushout exists because pushouts of monics are monics in toposes. Hence, $(Sub(X), \leq)$ is a lattice. Finally, it is bounded because \mathcal{C} is a topos, and then $[\emptyset \twoheadrightarrow X]$ (in any topos, the unique morphism from the initial object \emptyset to any other object is always a monic) and $[id_X]$ belongs to $Sub(X)$. These definitions of supremum and infimum extent to any family of elements of $Sub(X)$ when \mathcal{C} is complete, and then, if \mathcal{C} is complete, then so is the lattice $(Sub(X), \leq)$. \square

Actually, $(Sub(X), \leq)$ satisfies a stronger result, it is a Heyting algebra. This has generated many works (initiated by Lawvere and Tierney [41]) which establish strong connections between elementary toposes and intuitionistic logics.

Proposition 22. *($Sub(X), \leq$) is a Heyting algebra.*

Proof. Let us show first that $Sub(\mathbb{1})$ where $\mathbb{1}$ is the terminal object of the topos \mathcal{C} is a Heyting algebra, i.e. categorically $Sub(\mathbb{1})$ is finitely complete and finitely co-complete, and it is Cartesian closed. We have seen in Proposition 21 that $Sub(\mathbb{1})$ is both finitely complete and finitely co-complete. Finite product is defined by the infimum of equivalence classes of monics introduced in Proposition 21. It remains to show that $Sub(\mathbb{1})$ has exponentials. Let $[U \twoheadrightarrow \mathbb{1}]$ and $[V \twoheadrightarrow \mathbb{1}]$ be two subobjects of $\mathbb{1}$. As \mathcal{C} is a topos, U and V have an exponential V^U . As $\mathbb{1}$ is terminal in \mathcal{C} , there is a unique morphism $v^u : V^U \rightarrow \mathbb{1}$. It remains to show that the morphism v^u is a monic. Let $f, g : S \rightarrow V^U$ be two morphisms. Under the (exponential) bijection, we have that $Hom_{\mathcal{C}}(S, V^U) \simeq Hom_{\mathcal{C}}(S \times U, V)$, and then we have for f and g corresponding morphisms $\bar{f}, \bar{g} : S \times U \rightarrow V$. Let us suppose that $v^u \circ f = v^u \circ g$.

This means that $\bar{f} \circ u = \bar{g} \circ u$. But u is a monic and then $\bar{f} = \bar{g}$ which implies by bijection that $f = g$. We then deduce that v^u is a monic, and we conclude that $Sub(\mathbb{1})$ is a Heyting algebra.

It is not difficult to show that $Sub(X) \simeq Sub_{\mathcal{C}/X}(Id_X)$ where \mathcal{C}/X is the slice category over X and $Sub_{\mathcal{C}/X}(Id_X)$ is the set of subobjects of the object Id_X in the slice category \mathcal{C}/X . By the fundamental theorem of topos theory¹⁶ \mathcal{C}/X is also a topos, and then $Sub_{\mathcal{C}/X}(Id_X)$ is a Heyting algebra (Id_X is terminal in \mathcal{C}/X), whence we conclude under the isomorphism $Sub(X) \simeq Sub_{\mathcal{C}/X}(Id_X)$ that $Sub(X)$ is a Heyting algebra. \square

B Logic and Internal Language in Topos

An interesting feature of toposes is that we can reason on objects and morphisms of a topos “as if they were sets and functions” [38,41]. The reason is that we can do logic in toposes. Indeed, we can define logical connectives in toposes. Here, we recall the definition of propositional connectives $\{\wedge, \vee, \neg, \Rightarrow\}$ and of constants *true*, *false*.

- By definition of subobject classifiers, we have a monomorphism $true : \mathbb{1} \rightarrow \Omega$, and then we also have a morphism $(true, true) : \mathbb{1} \rightarrow \Omega \times \Omega$ which is also a monomorphism. So, by the subobject classifier definition, $\wedge : \Omega \times \Omega \rightarrow \Omega$ is its characteristic morphism.
- $\vee : \Omega \times \Omega \rightarrow \Omega$ classifies the image of the morphism $[(true, Id_\Omega), (Id_\Omega, true)] : \Omega + \Omega \rightarrow \Omega \times \Omega$, where $+$ denotes the co-product.
- the morphism $\Rightarrow : \Omega \times \Omega \rightarrow \Omega$ is the characteristic morphism of $\leq \Rightarrow \Omega \times \Omega$ where \leq is the equalizer of \wedge and the projection on the first argument $p_1 : \Omega \times \Omega \rightarrow \Omega$.
- Finally, the unique morphism $\emptyset \rightarrow \mathbb{1}$ is a monomorphism. Let us denote by $false : \mathbb{1} \rightarrow \Omega$ its characteristic morphism. Then, $\neg : \Omega \rightarrow \Omega$ is defined as the composite $\Rightarrow \circ (Id_\Omega \times false)$.

¹⁶This fundamental theorem states that if \mathcal{C} is a topos, then for every object $X \in |\mathcal{C}|$, \mathcal{C}/X is a topos (see [11] for a proof of this result).

Consequently the power object $\Omega = P\mathbb{1}$ is an internal Heyting algebra¹⁷ and then the logic is intuitionistic. Actually, through the bijection $\text{Sub}(X \times Y) \simeq \text{Hom}_{\mathcal{C}}(X, PY)$, for every object X in a topos \mathcal{C} , PX is an internal Heyting algebra. We can then define a partial order \leq_X as an object of \mathcal{C} such that \leq_X is the equalizer of $\wedge : PX \times PX \rightarrow PX$ and $p_1 : PX \times PX \rightarrow PX$ where p_1 is the projection on the first argument of couples.

For every topos \mathcal{C} , we can define an internal language $\mathcal{L}_{\mathcal{C}}$ composed of types defined by the objects of \mathcal{C} , from which we can define terms as follows:

- $true : X$;
- $x : X$ where x is a variable and X is a type;
- $f(t) : Y$ where $f : X \rightarrow Y$ is a morphism of \mathcal{C} and $t : X$ is a term;
- $\langle t_1, \dots, t_n \rangle : X_1 \times \dots \times X_n$ if for every i , $1 \leq i \leq n$, $t_i : X_i$ is a term;
- $(t)_i : X_i$ if $t : X_1 \times \dots \times X_n$ is a term;
- $\{x : X \mid \alpha\} : PX$ if $\alpha : \Omega$ is a term;
- $\sigma = \tau : \Omega$ if σ and τ are terms of the same type;
- $\sigma \in_X \tau : \Omega$ if $\sigma : X$ and $\tau : PX$ are terms;
- $\sigma \leq_X \tau : \Omega$ if $\sigma, \tau : PX$ are terms;
- $\varphi @ \psi : \Omega$ if $\varphi : \Omega$ and $\psi : \Omega$ are terms with $@ \in \{\wedge, \vee, \Rightarrow\}$;
- $\neg\varphi : \Omega$ if $\varphi : \Omega$ is a term;
- $Qx. \varphi : \Omega$ if $x : X$ and $\varphi : \Omega$ are terms and $Q \in \{\forall, \exists\}$.

Terms of type Ω are called *formulas*.

Semantics of terms will depend on their type. Hence, semantics of terms of type $X \neq \Omega$ will be defined by morphisms, and terms of type Ω will be interpreted as subobjects.

We say that a sequence of variables $\vec{x} = (x_1, \dots, x_n)$ is a *suitable context* for a term or a formula if each free variable of this term or this formula occurs in \vec{x} . Let us denote by $X_{\vec{x}}$ the product $X_1 \times \dots \times X_n$ when $\vec{x} = (x_1, \dots, x_n)$ and each $x_i : X_i$. Then the *semantics* of $t : X$ in the context \vec{x} , denoted by $[[t]]_{\vec{x}}$, is a morphism from $X_{\vec{x}}$ to X . It is defined recursively on the structure of t as

¹⁷ An *internal Heyting algebra* in a topos is an internal lattice L , that is equipped with two morphisms $\wedge, \vee : L \times L \rightarrow L$ such that the diagrams expressing the standard laws for \wedge and \vee commute, and with top and bottom which are morphisms $\perp, \top : \mathbb{1} \rightarrow L$ such that $\wedge \circ (Id_L \times \top) = Id_L$ and $\vee \circ (Id_L \times \perp) = Id_L$, together with an additional morphism $\Rightarrow : L \times L \rightarrow L$ which satisfies the diagrams given by the identities:

- $x \Rightarrow x = \top$
- $x \wedge (x \Rightarrow y) = x \wedge y$ and $y \wedge (x \Rightarrow y) = y$
- $x \Rightarrow (y \wedge z) = (x \Rightarrow y) \wedge (x \Rightarrow z)$

follows:

- $\llbracket x_i : X_i \rrbracket_{\bar{x}} = p_i$ where $p_i : X_{\bar{x}} \rightarrow X_i$ is the obvious projection on the i^{th} argument;
- $\llbracket f(t) \rrbracket_{\bar{x}} = f \circ \llbracket t \rrbracket_{\bar{x}}$;
- $\llbracket \langle t_1, \dots, t_n \rangle \rrbracket_{\bar{x}} = (\llbracket t_1 \rrbracket_{\bar{x}}, \dots, \llbracket t_n \rrbracket_{\bar{x}})$;
- $\llbracket (t)_i \rrbracket_{\bar{x}} = p_i \circ \llbracket t \rrbracket_{\bar{x}}$ where p_i is the projection on the i^{th} argument of the tuple;
- $\llbracket \{x : X \mid \alpha\} \rrbracket_{\bar{x}}$ is the unique morphism $r : X_{\bar{x}} \rightarrow PX$ making the diagram below a pullback square

$$\begin{array}{ccc} R & \longrightarrow & \epsilon_X \\ \llbracket \alpha \rrbracket_{(x, \bar{x})} \downarrow & & \downarrow \\ X \times X_{\bar{x}} & \xrightarrow{Id_X \times r} & X \times PX \end{array}$$

The semantics of a formula $\varphi : \Omega$ in the context \bar{x} , denoted by $\llbracket \varphi \rrbracket_{\bar{x}}$, is interpreted as a subobject of $\text{Sub}(X_{\bar{x}})$ and is recursively defined as follows:

- $\llbracket \text{true} \rrbracket_{\bar{x}} = Id_{X_{\bar{x}}}$;
- when $\varphi = \sigma = \tau$, then $\llbracket \varphi \rrbracket_{\bar{x}}$ equalizes $\llbracket \sigma \rrbracket_{\bar{x}}$ and $\llbracket \tau \rrbracket_{\bar{x}}$;
- when $\varphi = \sigma \in_X \tau$, then $\llbracket \varphi \rrbracket_{\bar{x}} : R \rightarrow X_{\bar{x}}$ where R is the pullback of the diagram

$$\begin{array}{ccc} R & \longrightarrow & \epsilon_X \\ \llbracket \varphi \rrbracket_{\bar{x}} \downarrow & & \downarrow \\ X_{\bar{x}} & \xrightarrow{\llbracket \sigma \rrbracket_{\bar{x}} \times \llbracket \tau \rrbracket_{\bar{x}}} & X \times PX \end{array}$$

- if $\varphi = \sigma \leq_X \tau$, then $\llbracket \varphi \rrbracket_{\bar{x}} : R \rightarrow X_{\bar{x}}$ where R is the pullback of the diagram

$$\begin{array}{ccc} R & \longrightarrow & \leq_X \\ \llbracket \varphi \rrbracket_{\bar{x}} \downarrow & & \downarrow \\ X_{\bar{x}} & \xrightarrow{\llbracket \sigma \rrbracket_{\bar{x}} \times \llbracket \tau \rrbracket_{\bar{x}}} & PX \times PX \end{array}$$

- if $\varphi = \varphi_1 @ \varphi_2$, then $\llbracket \varphi \rrbracket_{\bar{x}} = \llbracket \varphi_1 \rrbracket_{\bar{x}} @ \llbracket \varphi_2 \rrbracket_{x_t}$ where $@$ is the operator in $\{\wedge, \vee, \Rightarrow\}$ in the Heyting algebra $\text{Sub}(X_{\bar{x}})$;
- $\llbracket \neg \varphi \rrbracket_{\bar{x}} = \neg_{X_{\bar{x}}}(\llbracket \varphi \rrbracket_{\bar{x}})$ where $\neg_{X_{\bar{x}}}(\llbracket \varphi \rrbracket_{\bar{x}})$ is the pseudo-complement of $\llbracket \varphi \rrbracket_{\bar{x}}$ in $\text{Sub}(X_{\bar{x}})$;
- $\llbracket \forall x. \varphi \rrbracket_{\bar{x}} = \forall_p(\llbracket \varphi \rrbracket_{(\bar{x}, x)})$ where $p : X_{\bar{x}} \times X \rightarrow X_{\bar{x}}$ is the projection, and \forall_p is the right adjoint to the pullback functor $p^* : \text{Sub}(X_{\bar{x}}) \rightarrow \text{Sub}(X_{\bar{x}} \times X)$ when the Heyting algebras $\text{Sub}(X_{\bar{x}})$ and $\text{Sub}(X_{\bar{x}} \times X)$ are regarded as categories.
- $\llbracket \exists x. \varphi \rrbracket_{\bar{x}}$ is the image of $p \circ \llbracket \varphi \rrbracket_{(\bar{x}, x)}$ where p is the same projection as above.

Equivalently, semantics of any formula $\varphi : \Omega$ could be defined by a morphism from $X_{\bar{x}}$ to Ω , by interpreting φ as the classifying morphism of $\llbracket \varphi \rrbracket_{\bar{x}}$.

We write $\mathcal{C} \models_{\bar{x}} \varphi$ if $\llbracket \varphi \rrbracket_{\bar{x}} = Id_{X_{\bar{x}}}$ ($Id_{X_{\bar{x}}}$ is the top element in $\text{Sub}(X_{\bar{x}})$). How-

ever, to actually prove properties on toposes, the best way is to use the internal logic. This internal logic is defined as for any logic by a set of inference rules which allow us to derive true statements from other true statements. In order to describe this better we need to introduce the notion of a sequent.

Definition 10 (Sequent). *Given two formulas φ and ψ , a **sequent** is an expression of the form $\varphi \vdash_{\vec{x}} \psi$ and means that ψ is a logical consequence of φ in the context \vec{x} .*

Note that if the sequence \vec{x} is empty, then we will note simply \vdash and \vDash .

The deduction system is then defined as a sequent calculus, i.e. a set of inference rules which will allow us to infer a sequent from other sequents (see [38,43,48] for a presentation of this deduction system).

C Proofs of Section 3.2

C.1 Proof of Proposition 1 (adjunction)

Let us first recall a useful lemma.

Lemma 4.

$$\varphi \Rightarrow \psi \dashv\vdash \varphi \wedge \psi \Leftrightarrow \varphi$$

where $A \dashv\vdash B$ is a shortened notation for $A \vdash B$ and $B \vdash A$.

The result in Proposition 1 then rests on the two following propositions.

Proposition 23.

$$\vdash_{\vec{x}} Y \leq_X Z \Leftrightarrow (\forall x. x \in_X Y \Rightarrow x \in_X Z)$$

Proof.

$$\begin{aligned} Y \leq_X Z &\dashv\vdash_{Y,Z} Y \wedge Z = Y \\ &\dashv\vdash_{Y,Z,x} x \in_X Y \wedge x \in_X Z \Leftrightarrow x \in_X Y \\ &\dashv\vdash_{Y,Z,x} x \in_X Y \Rightarrow x \in_X Z \\ &\dashv\vdash_{Y,Z} \forall x. x \in_X Y \Rightarrow x \in_X Z \end{aligned}$$

□

Proposition 24. *The following statement is valid:*

$$\vdash \forall y. y \in_X Y \Rightarrow (\forall x. x \in_X b(y) \Rightarrow x \in_X \delta[b](Y))$$

Proof. By using the internal logic, by both universal quantification and implication rules, this amounts to prove that

$$y \in_X Y \wedge y \in_X \check{b}(x) \vdash_{Y,y,x} x \in_X \delta[b](Y)$$

(We use the fact that $x \in_X b(y) \Leftrightarrow y \in_X \check{b}(x)$). The existential quantification rule combined with the definition of dilation concludes. Indeed, by the existential quantification rule, we have

$$\exists y. y \in_X Y \wedge y \in_X \check{b}(x) \vdash_{Y,x} \text{true}$$

and then, by introduction of hypothesis, we can write

$$\exists y. y \in_X Y \wedge y \in_X \check{b}(x) \vdash_{Y,x} \exists y. y \in_X Y \wedge y \in_X \check{b}(x)$$

Then, by the definition of dilation, we have

$$\exists y. y \in_X Y \wedge y \in_X \check{b}(x) \vdash_{Y,x} x \in_X \delta[b](Y)$$

By using the existential quantification rule, this is equivalent to

$$y \in_X Y \wedge y \in_X \check{b}(x) \vdash_{Y,y,x} x \in_X \delta[b](Y)$$

□

Note that the opposite implication \Leftarrow in Proposition 24 is also valid.

It is now easy to prove the adjunction property. Its proof is substantially similar to that above.

Proof. of Proposition 1: By definition, both dilation and erosion of any object $Y : PX$ define subobjects $\varepsilon[b](Y) \rightrightarrows X$ and $\delta[b](Y) \rightrightarrows X$. Dilation and erosion both result in applications $\text{Sub}(X) \rightarrow \text{Sub}(X)$. Moreover, by proposition 24 we have that

$$\vdash_Y \forall y. y \in_X Y \Rightarrow (\forall x. x \in_X b(y) \Rightarrow x \in_X \delta[b](Y))$$

From Proposition 23, we deduce that

$$\vdash_Y \forall y. y \in_X Y \Rightarrow b(y) \leq_X \delta[b](Y)$$

Then, we have the following equivalences:

$$\begin{aligned} \vdash_{Y,Z} \delta[b](Y) \leq_X Z &\Leftrightarrow \forall y. y \in_X Y \Rightarrow b(y) \leq_X Z \\ &\Leftrightarrow \forall y. y \in_X Y \Rightarrow y \in_X \varepsilon[b](Z) \quad (\text{definition of erosion}) \\ &\Leftrightarrow Y \leq_X \varepsilon[b](Z) \quad (\text{Proposition 23}) \end{aligned}$$

□

C.2 Proof of Proposition 2

Let us show the properties for erosion. Let us show first the monotonicity property:

$$\vdash_{Y,Z} Y \leq_X Z \implies \varepsilon[b](Y) \leq_X \varepsilon[b](Z)$$

We have the following implications:

$$\begin{aligned} Y \leq_X Z \vdash_{Y,Z} \forall x. x \in_X \varepsilon[b](Y) &\Rightarrow b(x) \leq Y && \text{(definition of erosion)} \\ &\Rightarrow b(x) \leq Z && \text{(hypothesis and transitivity of } \leq_X \text{)} \\ &\Rightarrow x \in_X \varepsilon[b](Z) && \text{(definition of erosion)} \\ &\Rightarrow \varepsilon[b](Y) \leq_X \varepsilon[b](Z) && \text{(Proposition 23)} \end{aligned}$$

Let us show the commutativity of erosion with conjunction:

$$\vdash \forall Y. \forall Z. \varepsilon[b](Y \wedge Z) = \varepsilon[b](Y) \wedge \varepsilon[b](Z)$$

By definition of erosion, we have that

$$\vdash_{Y,Z} \forall x. x \in_X \varepsilon[b](Y \wedge Z) \Leftrightarrow b(x) \leq_X Y \wedge Z$$

and then we have that

$$\vdash_{Y,Z} \forall x. x \in_X \varepsilon[b](Y \wedge Z) \Leftrightarrow b(x) \leq_X Y \wedge b(x) \leq_X Z$$

By definition of erosion, we can then derive that

$$\vdash_{Y,Z} \forall x. x \in_X \varepsilon[b](Y \wedge Z) \Leftrightarrow x \in_X \varepsilon[b](Y) \wedge x \in_X \varepsilon[b](Z)$$

that is:

$$\vdash_{Y,Z} \forall x. x \in_X \varepsilon[b](Y \wedge Z) \Leftrightarrow x \in_X \varepsilon[b](Y) \wedge \varepsilon[b](Z)$$

This extends to any conjunction.

Now, let us show that $\vdash \varepsilon[b](X) = X$. By definition of erosion, we have that

$$\vdash \forall x. x \in_X \varepsilon[b](X) \Leftrightarrow b(x) \leq_X X$$

Now, by the topos properties, we have that

$$\vdash \forall x. b(x) \leq_X X$$

and then, by the cut rule, we can write

$$\vdash \forall x. x \in_X \varepsilon[b](X)$$

from which we derive

$$\vdash \varepsilon[b](X) = X$$

To finish, let us prove the necessary and sufficient condition for anti-extensivity of erosion, and first show that:

$$\vdash (\forall x. x \in_X b(x)) \Rightarrow (\forall Y. \varepsilon[b](Y) \leq_X Y)$$

By the rules for the implication connective and universal quantifier, this is equivalent to prove the following statement:

$$\forall x. x \in_X b(x) \vdash_Y \varepsilon[b](Y) \leq_X Y$$

By introduction of hypothesis, we have

$$\forall x. x \in_X b(x) \vdash_Y \forall x. x \in_X b(x)$$

and then, by the rule for the universal quantifier, this leads to

$$\forall x. x \in_X b(x) \vdash_{Y,x} x \in_X b(x)$$

By the thinning rule, we have

$$\forall x. x \in_X b(x), x \in_X \varepsilon[b](Y) \vdash_{Y,x} x \in_X b(x)$$

Now, by the introduction of hypothesis and the definition of $\varepsilon[b]$, we have

$$\forall x. x \in_X b(x), x \in_X \varepsilon[b](Y) \vdash_{Y,x} b(x) \leq_X Y$$

and then by Proposition 23, we deduce

$$\forall x. x \in_X b(x), x \in_X \varepsilon[b](Y) \vdash_{Y,x} x \in_X Y$$

By the rule for the implication, we then have

$$\forall x. x \in_X b(x) \vdash_{Y,x} x \in_X \varepsilon[b](Y) \Rightarrow x \in_X Y$$

which gives, by the rule for the universal quantifier:

$$\forall x. x \in_X b(x) \vdash_Y \forall x. x \in_X \varepsilon[b](Y) \Rightarrow x \in_X Y$$

which, by Proposition 23, allows us to conclude

$$\forall x. x \in_X b(x) \vdash_Y \varepsilon[b](Y) \leq_X Y$$

Let us now prove the opposite sequent: we have

$$\forall Y. \varepsilon[b](Y) \leq_X Y \vdash_Y \varepsilon[b](Y) \leq_X Y$$

and then by substitution:

$$\forall Y. \varepsilon[b](Y) \leq_X Y \vdash_x \varepsilon[b](b(x)) \leq_X b(x)$$

We have by definition $\vdash_x x \in_X \varepsilon[b](b(x))$, so by thinning and finite conjunction rules, we have:

$$\begin{aligned} \forall Y. \varepsilon[b](Y) \leq_X Y \vdash_x x \in_X \varepsilon[b](b(x)) \wedge \varepsilon[b](b(x)) \leq_X b(x) \\ \vdash_x x \in_X b(x) \end{aligned}$$

which concludes by the universal quantification rule:

$$\forall Y. \varepsilon[b](Y) \leq_X Y \vdash_Y \forall x. x \in_X b(x)$$

The proofs for dilation are substantially similar.

Let us only prove the necessary and sufficient condition for extensivity of dilation. We have:

$$\begin{aligned} (\forall x. x \in_X b(x)) \wedge x \in_X Y \vdash_{Y,x} x \in_X b(x) \wedge x \in_X Y \\ \vdash_{Y,x,y} y = x \wedge y \in_X \check{b}(x) \wedge y \in_X Y \\ \vdash_{Y,x} \exists y. y = x \wedge y \in_X \check{b}(x) \wedge y \in_X Y \\ \vdash_{Y,x} \exists y. y \in_X \check{b}(x) \wedge y \in_X Y \\ \vdash_{Y,x} x \in \delta[b](Y) \end{aligned}$$

and by implication and universal quantification rules, we derive:

$$\forall x. x \in_X b(x) \vdash \forall Y. Y \leq_X \delta[b](Y)$$

Conversely:

$$\begin{aligned} \forall Y. Y \leq_X \delta[b](Y) \vdash_x \{x\} \leq_X \delta[b](\{x\}) \\ \vdash_x x \in_X \delta[b](\{x\}) \\ \vdash_x \exists y. x \in_X b(y) \wedge y \in_X \{x\} \\ \vdash_x \exists y. x \in_X b(x) \\ \vdash_x x \in_X b(x) \end{aligned}$$

and by universal quantification:

$$\forall Y. Y \leq_X \delta[b](Y) \vdash \forall x. x \in_X b(x)$$

C.3 Proof of Proposition 4

- *Monotonicity* is derived directly from the monotonicity of $\varepsilon[b]$ and $\delta[b]$.

- *Extensivity of closing.* Obviously, we have that

$$\vdash_Y \delta[b](Y) \leq_X \delta[b](Y)$$

and then by the adjunction property, we can conclude that

$$\vdash_Y Y \leq_X \varepsilon[b](\delta[b](Y))$$

- *Anti-extensivity of opening.* Obviously, we have that

$$\vdash_Y \varepsilon[b](Y) \leq_X \varepsilon[b](Y)$$

and then by the adjunction property, we can conclude that

$$\vdash_Y \delta[b](\varepsilon[b](Y)) \leq_X Y$$

- *Preservation.* By anti-extensivity of opening we have that

$$\vdash_Y \delta[b](\varepsilon[b](Y)) \leq_X Y$$

and then by monotonicity of erosion, we can conclude that

$$\vdash_Y \varepsilon[b](\delta[b](\varepsilon[b](Y))) \leq_X \varepsilon[b](Y)$$

To show the opposite direction, we obviously have that

$$\vdash_Y \delta[b](\varepsilon[b](Y)) \leq_X \delta[b](\varepsilon[b](Y))$$

and then by the adjunction property, we can conclude that

$$\vdash_Y \varepsilon[b](Y) \leq_X \varepsilon[b](\delta[b](\varepsilon[b](Y)))$$

The equation $\delta[b] \circ \varepsilon[b] \circ \delta[b] = \delta[b]$ can be proved similarly.

- *Idempotence.* Direct consequence of the preservation properties.

C.4 Proof of Proposition 18

(\implies) Let us suppose that the revision operator \circ satisfies AGM Postulates. Let us define the binary relation \leq_ψ^φ on $Mod(\psi)$ as follows:

$$\mathcal{M} \leq_\psi^\varphi \mathcal{M}' \text{ iff } \mathcal{M} \in Mod(\varphi \circ \psi) \text{ and } \mathcal{M}' \notin Mod(\varphi \circ \psi)$$

Let us set $f(t) =_{\leq_\varphi} \bigcup_\psi \leq_\psi^\varphi$. Let us first show that \leq_φ is a FA. Then, let us show that the mapping f is a FA.

- Let $\mathcal{M}, \mathcal{M}' \in Mod(\varphi)$, and let us suppose that $\mathcal{M} \leq_\varphi \mathcal{M}'$. This means that there exists ψ such that $\mathcal{M} \leq_\psi^\varphi \mathcal{M}'$, i.e. $\mathcal{M}, \mathcal{M}' \in Mod(\psi)$, $\mathcal{M} \in Mod(\varphi \circ \psi)$

and $\mathcal{M}' \notin Mod(\varphi \circ \psi)$. Hence, we have that $\varphi \wedge \psi$ is consistent, and then by Postulate (G3), $\varphi \circ \psi = \varphi \wedge \psi$. We then have that $\mathcal{M}' \in Mod(\varphi \circ \psi)$ which is a contradiction.

- Let $\mathcal{M} \in Mod(\varphi)$ and let $\mathcal{M}' \notin Mod(\varphi)$. We have that $\mathcal{M} \leq_{\varphi}^{\varphi} \mathcal{M}'$ by definition of \leq_{φ} . Now, let us suppose that $\mathcal{M}' \leq_{\varphi} \mathcal{M}$. This means that there exists ψ such that $\mathcal{M}' \leq_{\psi}^{\varphi} \mathcal{M}$. Hence, we have that $\mathcal{M}, \mathcal{M}' \in Mod(\psi)$, $\mathcal{M}' \in Mod(\varphi \circ \psi)$ and $\mathcal{M} \notin Mod(\varphi \circ \psi)$, and then as $\mathcal{M} \in Mod(\varphi)$ we have that $\varphi \wedge \psi$ is consistent. By Postulate (G3), we then have that $\varphi \circ \psi = \varphi \wedge \psi$, and then $\mathcal{M} \in Mod(\varphi \circ \psi)$ which is a contradiction.

Let us show now that $Mod(\varphi \circ \psi) = Min(Mod(\psi), \leq_{\varphi})$.

- Let $\mathcal{M} \in Mod(\varphi \circ \psi)$, and let us suppose that $\mathcal{M} \notin Min(Mod(\psi), \leq_{\varphi})$. By Postulate (G2), $\mathcal{M} \in Mod(\psi)$. By hypothesis, there exists $\mathcal{M}' \in Mod(\psi)$ such that $\mathcal{M}' <_{\varphi} \mathcal{M}$. This means that there exists a formula δ such that $\mathcal{M}' \in Mod(\varphi \circ \delta)$ and $\mathcal{M} \notin Mod(\varphi \circ \delta)$. By definition of the relation \leq_{δ}^{φ} , we have that $\psi \wedge \delta$ is consistent, and then by Postulate (G5), we can write

$$Mod((\varphi \circ \psi) \wedge \delta) = Mod((\varphi \circ \delta) \wedge \psi) = Mod(\varphi \circ (\psi \wedge \delta))$$

Hence, we have that $\mathcal{M} \in Mod(\varphi \circ \delta)$ which is a contradiction.

- Let $\mathcal{M} \in Min(Mod(\psi), \leq_{\varphi})$. Let us assume that $\mathcal{M} \notin Mod(\varphi \circ \psi)$. Let us suppose $\mathcal{M}' \in Mod(\varphi \circ \psi)$. By definition of \leq_{ψ}^{φ} , we have that $\mathcal{M}' \leq_{\psi}^{\varphi} \mathcal{M}$, and then $\mathcal{M}' \leq_{\varphi} \mathcal{M}$. Now, let us suppose that $\mathcal{M} \leq_{\varphi} \mathcal{M}'$. This means that there exists δ such that $\mathcal{M} \in Mod(\varphi \circ \delta)$ and $\mathcal{M}' \notin Mod(\varphi \circ \delta)$. But, $\mathcal{M}' \in Mod(\psi \wedge \delta)$, and by Postulate (G5) we have that

$$Mod((\varphi \circ \psi) \wedge \delta) = Mod((\varphi \circ \delta) \wedge \psi) = Mod(\varphi \circ (\psi \wedge \delta))$$

Therefore, we have that $\mathcal{M} \in Mod(\varphi \circ \delta)$ which is a contradiction. Hence, we just showed that $\mathcal{M}' <_{\varphi} \mathcal{M}$ which is a contradiction.

(\Leftarrow) Let us now suppose that for a revision operation \circ there exists a FA which maps any formula φ to a binary relation \leq_{φ} satisfying the conditions of Proposition 18. Let us prove that \circ satisfies the AGM Postulates.

- (G1) As ψ is consistent, we have that $Min(Mod(\psi), \leq_{\varphi}) \neq \emptyset$, and then by the hypothesis $\varphi \circ \psi$ is consistent.
- (G2) Let $\mathcal{M} \in Mod(\varphi \circ \psi)$. By hypothesis, we have that

$$\mathcal{M} \in Min(Mod(\psi), \leq_{\varphi})$$

and then $\mathcal{M} \in Mod(\psi)$.

- (G3) Let us suppose that $\varphi \circ \psi$ is consistent.
 - Let us prove that $Mod(\varphi \circ \psi) \subseteq Mod(\varphi \wedge \psi)$. Let $\mathcal{M} \in Mod(\varphi \circ \psi)$. Hence, we have that $\mathcal{M} \in Mod(\psi)$. Let us suppose that $\mathcal{M} \notin Mod(\varphi)$.

Let $\mathcal{M}' \in \text{Mod}(\varphi)$. By the definition of FA, we have that $\mathcal{M}' <_{\varphi} \mathcal{M}$ which contradicts the fact that $\mathcal{M} \in \text{Min}(\text{Mod}(\psi), \leq_{\varphi})$.

- Let us prove that $\text{Mod}(\varphi \wedge \psi) \subseteq \text{Mod}(\varphi \circ \psi)$. Let $\mathcal{M} \in \text{Mod}(\varphi \wedge \psi)$ be a model such that $\mathcal{M} \notin \text{Mod}(\varphi \circ \psi)$. We have that $\mathcal{M} \in \text{Mod}(\psi)$. By hypothesis, there exists a model $\mathcal{M}' \in \text{Mod}(\psi)$ such that $\mathcal{M}' <_{\varphi} \mathcal{M}$, and then $\mathcal{M} \notin \text{Mod}(\varphi)$ which is a contradiction.
- (G4) Let us suppose that $\psi \equiv \psi'$. This means that $\text{Mod}(\psi) = \text{Mod}(\psi')$, and then $\text{Min}(\text{Mod}(\psi), \leq_{\varphi}) = \text{Min}(\text{Mod}(\psi'), \leq_{\varphi})$. We can then conclude that $\text{Mod}(\varphi \circ \psi) = \text{Mod}(\varphi \circ \psi')$.
- (G5) Let us suppose that $(\varphi \circ \psi) \wedge \chi$ is consistent.
 - Let us suppose that $\mathcal{M} \in \text{Mod}((\varphi \circ \psi) \wedge \chi)$. Let us suppose that $\mathcal{M} \notin \text{Min}(\varphi \circ (\psi \wedge \chi))$. This means that there exists $\mathcal{M}' \in \text{Mod}(\psi \wedge \chi)$ such that $\mathcal{M}' <_{\varphi} \mathcal{M}$. We have then that $\mathcal{M}' \in \text{Mod}(\psi)$, and then $\mathcal{M} <_{\varphi} \mathcal{M}'$ which is a contradiction.
 - Let us suppose that $\mathcal{M} \in \text{Mod}(\varphi \circ (\psi \wedge \chi))$. We then have that $\mathcal{M} \in \text{Mod}(\psi \wedge \chi)$. Now, we have that

$$\text{Min}(\text{Mod}(\psi \wedge \chi), \leq_{\varphi}) = \text{Min}(\text{Mod}(\psi), \leq_{\varphi}) \cap \text{Mod}(\chi)$$

and then $\mathcal{M} \in \text{Mod}((\varphi \circ \psi) \wedge \chi)$.