



**HAL**  
open science

# The gradient's limit of a definable family of functions is a conservative set-valued field

Sholom Schechtman

► **To cite this version:**

Sholom Schechtman. The gradient's limit of a definable family of functions is a conservative set-valued field. 2024. hal-04452981

**HAL Id: hal-04452981**

**<https://hal.science/hal-04452981>**

Preprint submitted on 12 Feb 2024

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# The gradient's limit of a definable family of functions is a conservative set-valued field

Sholom Schechtman\*

February 12, 2024

## Abstract

It is well-known that the convergence of a family of smooth functions does not imply the convergence of its gradients. In this work, we show that if the family is definable in an o-minimal structure (for instance semialgebraic, subanalytic, or any composition of the previous with  $\exp$ ,  $\log$ ), then the gradient's limit is a conservative set-valued field in the sense introduced by Bolte and Pauwels. Immediate implications of this result on convergence guarantees of smoothing methods are discussed. Finally, a more general result is established, where the functions in the original family might be merely Lipschitz continuous, vector-valued and the gradients are replaced by their Clarke's Jacobians or an arbitrary definable conservative mapping.

**Keywords.** conservative mappings, Clarke subgradient, o-minimal structures, semialgebraic, nonsmooth optimization, smoothing methods

## 1 Introduction

In this work, given a family of real-valued functions  $(f_a)_{a>0}$  that converges, when  $a \rightarrow 0$ , to some function  $F : \mathbb{R}^d \rightarrow \mathbb{R}$ , we are interested in the behavior of the limit of the corresponding family of (sub)-gradients  $(\partial f_a)_{a>0}$ . Formally, we are interested in properties that satisfies

$$D_F(x) := \{v \in \mathbb{R}^d : \text{there is } (x_n, v_n, a_n) \rightarrow (x, v, 0) \text{ with } v_n \in \partial f_{a_n}(x_n)\}. \quad (1)$$

While characterizing such limits is interesting from a purely theoretical perspective,  $D_F$  naturally appears in the analysis of smoothing methods. In these, given a nonsmooth function  $F$ , the general construction, dating back to at least Mayne and Polak ([31]), goes as follows (see e.g. [13]). *i*) First, construct  $(f_a)_{a>0}$  a family of approximations of  $F$  such that every  $f_a$  is smooth. *ii*) Second, for fixed  $a_k, \varepsilon_k > 0$ , find  $x_k \in \mathbb{R}^d$  an approximate stationary point:  $\|\nabla f_{a_k}(x_k)\| \leq \varepsilon_k$ . *iii*) Finally, decrease  $a_k, \varepsilon_k$  and go back to step *ii*). The interest of this procedure lies in the fact that we replace the original nonsmooth problem with a sequence of smooth optimization problems, for which there exists an abundance of algorithms with theoretical convergence guarantees ([33]).

In the case of a smoothing method, and assuming that  $(x_k)$  has an accumulation point  $x^*$ , we immediately obtain that  $x^*$  is  $D_F$ -critical:  $0 \in D_F(x^*)$ . Thus, ideally, one would wish  $D_F$

---

\*SAMOVAR, Télécom SudParis, Institut Polytechnique de Paris, 91120 Palaiseau, France

to be reduced to one of the common first-order operators: to the gradient if  $F$  is differentiable, to the convex subgradient if it is convex or to the Frechet’s or Clarke’s subgradient if  $F$  is merely locally Lipschitz continuous. Famously, Attouch in [2] has shown that when every function in  $(f_a)_{a>0}$  is convex,  $D_F$  is indeed the convex subgradient. This result was extended in numerous ways, example given: for Banach spaces ([4]), weakly-convex functions ([29, 36]), and equi-lower semidifferentiable functions ([16, 47, 48]).

Regrettably, the following simple example shows that there is no hope to state an equivalent result in full generality. Let  $f_a := a \sin(x/a)$ , then  $f'_a(x) = \sin(x/a)$  and for any  $x \neq 0$ ,  $D_F(x) = [-1, 1]$ , which is obviously larger than any (sub)-gradient of  $F \equiv 0$ . While unfortunate, this counter-example shows that the convergence’s failure is due to a highly oscillatory behavior of  $f_a$ . Thus, naturally, one might wonder what could happen if we restrict ourselves to a class of functions where such an oscillation phenomena does not occur.

Fortunately, the nonsmooth optimization literature ([6, 18, 26]) have already established a setting, where precisely this pathological behavior is ruled out: the class of functions *definable in an o-minimal structure* ([15, 40]). The class of such functions is large. It includes any semialgebraic function, any analytic function restricted to a semialgebraic compact, the exponential and the logarithm. Moreover, definability is stable by most of common operators such as  $\{+, -, \times, \circ, \circ^{-1}, \sup, \inf\}$ , explaining their ubiquity in optimization.

Definable functions may be nonsmooth, nevertheless, their differentiability properties are relatively well-understood. For instance, it is well-known that the domain of a definable function can be partitioned (or more precisely *stratified*) into manifolds (or *strata*) such that restricted to each element of the partition the function is smooth (see [40]). Furthermore, in the seminal work [6], it was established that the Clarke’s subgradient of such functions admits the so-called *variational stratification*: the partition can be chosen in a way ensuring that the projection of the Clarke’s subgradient onto the tangent plane of the corresponding manifold (on which the objective is smooth) is simply the Riemannian gradient of the smooth restriction.

Closely related to this geometric description is the notion of a *conservative set-valued field*. Recently introduced by Bolte and Pauwels ([7]), conservative set-valued fields are a “differential-akin” object, which, roughly speaking, act as a usual directional derivative along any smooth curve. For definable functions, examples of conservative fields are the Clarke’s subgradient, but also the output of automatic differentiation provided in popular API’s such as TensorFlow or PyTorch ([1, 34]). In fact, as shown in [17, 30, 35], in the definable setting, being a conservative set-valued field is equivalent to admitting a variational stratification analogous to the one satisfied by the Clarke’s subgradient. In particular, while a given conservative set-valued field might be different from the Clarke’s subgradient, the two objects are equal almost everywhere. Furthermore, even on the set of points where the equality does not hold, the field possesses a transparent geometric description.

All necessary objects being introduced, we are ready to state the main result of this work.

**Theorem 1.1** (Informal). *Assume that the family  $(f_a)_{a>0}$  be definable in an o-minimal structure. Then,  $D_F$  is a conservative set-valued field of  $F$ .*

Let us comment on some immediate implications of this result. First, as previously mentioned,  $D_F$  might be strictly larger than the set of (sub)-gradients of  $F$ . Nevertheless, from the variational stratification property of  $D_F$  we immediately obtain that  $D_F(x) = \partial F(x) = \{\nabla F(x)\}$  on an open dense set. In the vocabulary of smoothing methods this means that the *gradient consistency* property holds almost everywhere (see [13]). Second, even for

points where  $D_F(x) \neq \partial F(x)$ , the variational stratification gives a pleasant geometric description of  $D_F$ . Finally, due to the remarkable stability of definable functions, most of the smoothing techniques produce a family  $(f_a)_{a>0}$  which is indeed definable in the same o-minimal structure. Thus, for such methods, the guarantee that  $0 \in D_F(x^*)$  (or more precisely  $0 \in \text{conv } D_F(x^*)$ ) is a meaningful and necessary condition of optimality.

Let us also mention that our result is established in a slightly more general setting. In fact, in the construction of  $D_F$  in (1) the Clarke's subgradients  $\partial f_a$  can be replaced by an arbitrary (definable) conservative field  $D_a$  of  $f_a$ . Furthermore, similar conclusions hold when the functions are vector-valued, with  $\partial f_a$  being replaced by an arbitrary definable conservative mapping.

**Paper organization.** In Section 2 we introduce the necessary definitions of o-minimal structures, subgradients and conservative mappings. In Section 3 we present our main theorem. Implications of the latter for smoothing methods are discussed in Section 4. Finally, Section 5 is devoted to the proof of our main result.

## 2 Preliminaries

**Notations.** We say that  $D : \mathbb{R}^d \rightrightarrows \mathbb{R}^m$  is a set-valued map if for all  $x \in \mathbb{R}^d$ ,  $D(x) \subset \mathbb{R}^m$ . For such map,  $\text{Graph } D := \{(x, y) : y \in D(x)\}$ , and we say that  $D$  is closed if  $\text{Graph } D$  is a closed subset of  $\mathbb{R}^{d+m}$ . It is said to have nonempty (respectively compact, convex) values if for all  $x \in \mathbb{R}^d$ ,  $D(x)$  is nonempty (respectively compact, convex). Finally, it is locally bounded if every  $x \in \mathbb{R}^d$  admits a neighborhood  $\mathcal{U} \subset \mathbb{R}^d$  and  $C > 0$  such that  $\sup_{x \in \mathcal{U}, v \in D(x)} \|v\| \leq C$ . For  $A \subset \mathbb{R}^d$ , we denote  $\text{conv } A$  its convex hull,  $\bar{A}$  its closure. Similarly,  $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ .

### 2.1 Functions definable in an o-minimal structure

We collect here few elementary facts about functions and sets definable in an o-minimal structure. For more details, we refer to Appendix A and the monographs [15, 40, 42]. A nice review of their importance in optimization is [26].

The definition of an o-minimal structure is inspired by properties that are satisfied by semialgebraic sets.

**Definition 2.1.** *We say that  $\mathcal{O} := (\mathcal{O}_n)$ , where for each  $n \in \mathbb{N}$ ,  $\mathcal{O}_n$  is a collection of sets in  $\mathbb{R}^n$ , is an o-minimal structure if the following holds.*

1. *If  $Q : \mathbb{R}^n \rightarrow \mathbb{R}$  is a polynomial, then  $\{x \in \mathbb{R}^n : Q(x) = 0\} \in \mathcal{O}_n$ .*
2. *For each  $n \in \mathbb{N}$ ,  $\mathcal{O}_n$  is a boolean algebra: if  $A, B \in \mathcal{O}_n$ , then  $A \cup B, A \cap B$  and  $A^c$  are in  $\mathcal{O}_n$ .*
3. *If  $A \in \mathcal{O}_n$  and  $B \in \mathcal{O}_m$ , then  $A \times B \in \mathcal{O}_{n+m}$ .*
4. *If  $A \in \mathcal{O}_{n+1}$ , then the projection of  $A$  onto its first  $n$  coordinates is in  $\mathcal{O}_n$ .*
5. *Every element of  $\mathcal{O}_1$  is exactly a finite union of intervals and points of  $\mathbb{R}$ .*

Sets contained in  $\mathcal{O}$  are called *definable*. We call a map  $f : \mathbb{R}^d \rightarrow \mathbb{R}^m$  definable if its graph is definable.

Definable sets and maps have remarkable stability properties. For instance, if  $f$  and  $A$  are definable, then  $f(A)$  and  $f^{-1}(A)$  are definable and definability is stable by most of the common operators such as  $\{+, -, \times, \circ, \circ^{-1}\}$ .

Let us look at some examples of o-minimal structures.

**Semialgebraic.** Semialgebraic sets form an o-minimal structure. This follows from the celebrated result of Tarski [39]. A set  $A \subset \mathbb{R}^n$  is semialgebraic if it is a finite union of intersections of sets of the form  $\{Q(x) \leq 0\}$ , where  $Q : \mathbb{R}^n \rightarrow \mathbb{R}$  is some polynomial. A function is semialgebraic if its graph is a semialgebraic set. Examples of such functions include any piecewise polynomial and rational functions but also functions such as  $x \mapsto x^q$ , where  $q$  is any rational number. In fact, *any o-minimal structure* contains every semialgebraic set.

**Globally subanalytic.** There is an o-minimal structure that contains, for every  $n \in \mathbb{N}$ , sets of the form  $\{(x, t) : t = f(x)\}$ , where  $f : [-1, 1]^n \rightarrow \mathbb{R}$  is an analytic function. This comes from the fact that subanalytic sets are stable by projection, which was established by Gabrielov [21, 22]. The sets belonging to this structure are called globally subanalytic (see [5] for more details).

**Log-exp.** There is an o-minimal structure that contains, semialgebraic sets, globally subanalytic sets as well as the graph of the exponential and the logarithm (see [41, 43]).

With these examples in mind it is usually easy to verify that a function is definable. This will be the case as soon as the function is constructed by a finite number of definable operations on definable functions. From this, we see that *most of the functions* used in optimization are definable in the structure *Log-exp*.

## 2.2 Conservative set-valued fields

Conservative set-valued fields were introduced by Bolte and Pauwels in [7] as an elegant description of the output of automatic differentiation provided by numerical libraries such as TensorFlow or PyTorch ([1, 34]). Since then, several works have worked out some geometrical properties of conservative set-valued mappings of definable functions ([17, 30, 35]). They constitute an important tool for establishing the convergence of first-order methods in nonsmooth optimization ([7–9, 18, 28, 44]).

**Definition 2.2** ([7]). *We say that a locally bounded, closed set-valued map  $D : \mathbb{R}^d \rightrightarrows \mathbb{R}^d$  with nonempty values is a conservative field for a potential function  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  if for any absolutely continuous curve  $x : [0, 1] \rightarrow \mathbb{R}^d$  and any measurable function  $v : [0, 1] \rightarrow \mathbb{R}^d$ , such that for all  $t \in [0, 1]$ ,  $v(t) \in D(x(t))$ , it holds that*

$$f(x(1)) = f(x(0)) + \int_0^1 \langle v(t), \dot{x}(t) \rangle dt. \quad (2)$$

*Functions that are potentials of some conservative field are called path differentiable.*

**Remark 2.3.** *Fix an o-minimal structure  $\mathcal{O}$ . If  $f$  and  $D$  are definable in  $\mathcal{O}$ , then, as shown in [35, Theorem 2] and [17, Theorem 3.2], it is sufficient to verify Equation (2) only for  $\mathcal{C}^1$  definable curves  $x$  and definable selections  $v$  (in  $\mathcal{O}$ ).*

Definable functions always admit a conservative field. The most important example of one is the Clarke’s subgradient (this was proven in [19] but see also [18]). Recall that for a set  $A$ ,  $\text{conv } A$  denotes its convex hull.

**Definition 2.4** (Clarke's subgradient [14]). *Let  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  be a locally Lipschitz function. The Clarke's subgradient of  $f$  at  $x$  is defined as*

$$\partial f(x) := \text{conv}\{v \in \mathbb{R}^d : \text{there is } x_n \rightarrow x, \text{ with } f \text{ differentiable at } x_n \text{ and } \nabla f(x_n) \rightarrow v\}.$$

From the optimization perspective, note that the Clarke's subgradient provides a necessary condition of optimality: if  $x$  is a local minimum of  $f$ , then  $0 \in \partial f(x)$  ([14]).

As shown in [7], the Clarke's subgradient is the smallest convex-valued conservative field.

**Proposition 2.5** ([7, Corollary 1]). *Let  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  be a definable locally Lipschitz continuous function. Then  $\partial f : \mathbb{R}^d \rightrightarrows \mathbb{R}^d$  is a conservative set-valued field for  $f$ . Moreover, if  $D$  is any other conservative field of  $f$ , then so is  $x \rightrightarrows \text{conv } D(x)$  and*

$$\partial f(x) \subset \text{conv } D(x).$$

In particular, a point  $x \in \mathbb{R}^d$  is a local minimum only if  $0 \in \text{conv } D(x)$ .

From a geometric perspective, it turns out that in the definable setting conservative fields are exactly those maps that admit the so-called *variational stratification* (see [7, Section 4.2]). We refer to [10] for the necessary definitions of manifolds and Riemannian gradients.

**Proposition 2.6** ([30, Theorem 2.2] and [17, 35]). *Let  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  be definable and let  $D : \mathbb{R}^d \rightrightarrows \mathbb{R}^d$  be a definable conservative field of  $f$ . For any  $p \geq 1$ , there is  $(\mathcal{M}_i)$ , a finite partition of  $\mathbb{R}^d$  into  $\mathcal{C}^p$  manifolds such that  $f$  restricted to any  $\mathcal{M}_i$  is  $\mathcal{C}^p$  and for any  $x \in \mathcal{M}_i$ , it holds that*

$$D(x) \subset \nabla_{\mathcal{M}_i} f(x) + \mathcal{N}_x \mathcal{M}_i, \quad (3)$$

where  $\nabla_{\mathcal{M}_i} f(x)$  is the Riemannian gradient of  $f|_{\mathcal{M}_i}$  at  $x$  and  $\mathcal{N}_x \mathcal{M}_i$  is the normal plane of  $\mathcal{M}_i$  at  $x$ .

Conversely, if there exists a closed, locally bounded mapping  $D : \mathbb{R}^d \rightrightarrows \mathbb{R}^d$  and a finite partition of  $\mathbb{R}^d$  into  $\mathcal{C}^p$  manifolds  $(\mathcal{M}_i)$  such that (3) holds, then  $D$  is a conservative set-valued field of  $f$ .

Let  $\mathcal{U}$  be the union of full-dimensional manifolds from the partition given by Proposition 2.6. Then,  $f$  is differentiable on  $\mathcal{U}$  and for all  $x \in \mathcal{U}$ ,  $D(x) = \{\nabla f(x)\}$ . Note also that  $\mathcal{U}$  is an open and dense subset of  $\mathbb{R}^d$ . Therefore, in the definable setting, any conservative field is equal to the gradient on an open and dense set. Actually, as shown in [7, Theorem 1], even if  $f$  is not definable it is true that its conservative field is equal to the gradient *almost everywhere*.

**Remark 2.7.** *One could convince himself on why Equation (3) implies Definition 2.2 in the following way. Consider  $\mathbf{x} : [0, 1] \rightarrow \mathbb{R}^d$  an absolutely continuous curve. Then, informally, we can partition  $[0, 1]$  into intervals on each of which  $\mathbf{x}$  stays on the same manifold  $\mathcal{M}$ . Since  $f|_{\mathcal{M}}$  is smooth, we immediately obtain that  $(f \circ \mathbf{x})'(t) = \langle \nabla_{\mathcal{M}} f(\mathbf{x}(t)), \dot{\mathbf{x}}(t) \rangle$ , which under Equation (3) implies Equation (2). This is almost a complete proof of the implication, since we can not always find such partition for an arbitrary absolutely continuous curve (for a full proof see [18, Theorem 5.8] or [7, Section 4]).*

## 2.3 Conservative mappings

Definitions of the previous section readily extend to the case where the potential function is vector-valued.

**Definition 2.8** ([7]). *Let  $f : \mathbb{R}^d \rightarrow \mathbb{R}^m$  be a locally Lipschitz continuous function. We say that  $D : \mathbb{R}^d \rightrightarrows \mathbb{R}^{m \times d}$  is a conservative mapping for  $f$ , if for any absolutely continuous curve  $x : [0, 1] \rightarrow \mathbb{R}^d$ ,*

$$f(x(1)) = f(x(0)) + \int_0^1 D(x(t))\dot{x}(t) dt.$$

Of course, when  $m = 1$ , this definition is exactly the one of a conservative set-valued field. Not surprisingly, rows of a conservative mapping are actually conservative fields for the corresponding coordinate of  $f$ .

**Proposition 2.9.** [7, Section 3.3] *Let  $f : \mathbb{R}^d \rightarrow \mathbb{R}^m$  be locally Lipschitz continuous and  $D : \mathbb{R}^d \rightrightarrows \mathbb{R}^{m \times d}$  be a conservative mapping for  $f$ . For  $1 \leq i \leq m$ , denote*

$$D_i(x) := \{v_i \in \mathbb{R}^d : \exists v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_m \in \mathbb{R}^d, [v_1, \dots, v_m]^\top \in D(x)\}.$$

*Then,  $x \rightrightarrows D_i(x)$  is a conservative field of  $f_i$ .*

*Conversely, if  $D_i$  is a conservative field for  $f_i$ , then*

$$x \rightrightarrows [v_1, \dots, v_m]^\top \quad v_i \in D_i(x), i = 1, \dots, m$$

*is a conservative mapping of  $f$ .*

## 3 Definable limits of conservative fields

### 3.1 Main result

In the following, we fix an o-minimal structure  $\mathcal{O}$ . Definable will always mean definable in  $\mathcal{O}$ .

Consider a function  $f : \mathbb{R}^d \times \mathbb{R}_+^* \rightarrow \mathbb{R}$ . Denoting, for  $a > 0$ ,  $f_a(\cdot) := f(\cdot, a)$ , we think of  $(f_a)_{a>0}$  as a parametrized family of functions. Consider a set-valued map  $D : \mathbb{R}^d \times \mathbb{R}_+^* \rightrightarrows \mathbb{R}^d$  such that for each  $a > 0$ ,  $D_a := D(\cdot, a)$  is a conservative field for  $f_a$ . Note that by definability there is  $F : \mathbb{R}^d \rightarrow \overline{\mathbb{R}}$  such that  $f_a \xrightarrow{a \rightarrow 0} F$  pointwise (see Lemma A.2).

**Assumption 1.**

1. *The function  $f$  and the set-valued map  $D$  are definable.*
2. *For any compact set  $K \subset \mathbb{R}^d$ ,*

$$\limsup_{a_1 \rightarrow 0} \sup\{\|v\| : v \in D_a(x), a \leq a_1, x \in K\} < +\infty.$$

Let us emphasize that if for every  $a$ ,  $D_a \equiv \partial f_a$ , then the set-valued map  $D$  is definable as soon as  $f$  is. Therefore, Assumption 1 encompasses the setting presented in the introduction. Note, moreover, that Assumption 1 implies that  $F$  is locally Lipschitz continuous.

Define the set-valued map  $D_F : \mathbb{R}^d \rightrightarrows \mathbb{R}^d$  as

$$D_F(x) := \{v \in \mathbb{R}^d : \text{there is } (x_n, v_n, a_n) \rightarrow (x, v, 0), \text{ with } v_n \in D(x_n, a_n)\}.$$

Finally, denote  $\text{conv } D_F : \mathbb{R}^d \rightrightarrows \mathbb{R}^d$  the set-valued map  $x \rightrightarrows \text{conv } D_F(x)$ .

**Theorem 3.1.** *Let Assumption 1 hold. The maps  $D_F$  and  $\text{conv } D_F$  are definable and are conservative fields for  $F$ .*

**Remark 3.2.** *Conservative set-valued fields being a local object, conclusions of Theorem 3.1 continue to hold when the functions  $f_a, F$  (as well as  $D_a, D_F$ ) are defined only on some definable open set  $\mathcal{U} \subset \mathbb{R}^d$ .*

**Remark 3.3.** *Observe that it is not sufficient to have  $f_a$  definable for every  $a > 0$ . Indeed, recall the example from the introduction:  $f_a(x) = a \sin(x/a)$ ,  $f'_a(x) = \sin(x/a)$ ,  $F \equiv 0$  and  $D_F \equiv [-1, 1]$ . If we restrict the functions to  $[-1, 1]$ , then for all  $a$ ,  $f_a$  is definable in the structure of subanalytic sets. Nevertheless, the assumption of the theorem do not hold since  $\text{Graph } f = \{(x, a, y) : f_a(x) = y\}$  is not definable.*

If for each  $a$ ,  $D_a = \partial f_a$ , one might wonder if  $\text{conv } D_F$  is equal to  $\partial F$ ? Unfortunately, the following semialgebraic example (given to the author by Edouard Pauwels) shows that this is generally not true. Nonetheless, note that by Theorem 3.1 we always have  $\partial F \subset \text{conv } D_F$ . Furthermore, since  $D_F$  is a conservative field it holds that  $D_F = \{\nabla F\}$  almost everywhere.

**Example 3.4.** *Consider  $f_a(x) = a - |x|$  if  $|x| < a$  and  $f_a(x) = 0$  otherwise. A direct computation shows that  $f'_a(a/2) = -1$  and as a consequence  $-1 \in D_F(0)$ . Thus,  $D_F$  has elements that are different from the Clarke's subgradient of  $F \equiv 0$ . Note that by smoothing the corners it is easy to construct a similar example where each  $f_a$  is smooth.*

Theorem 3.1 has an easy generalization to the case where the functions are vector-valued.

**Assumption 2.**

1. *There is  $f : \mathbb{R}^d \times \mathbb{R}_+^* \rightarrow \mathbb{R}^m$  and  $D : \mathbb{R}^d \times \mathbb{R}_+^* \rightrightarrows \mathbb{R}^{m \times d}$  such that for each  $a \in \mathbb{R}_+^*$ ,  $D(\cdot, a)$  is a conservative mapping of  $f(\cdot, a)$ .*
2. *The function  $f$  and the set-valued map  $D$  are definable in the same o-minimal structure.*
3. *For any compact set  $K \subset \mathbb{R}^d$ ,*

$$\limsup_{a_1 \rightarrow 0} \{ \|v\| : v \in D_a(x), a \leq a_1, x \in K \} < +\infty.$$

As previously, let  $D_F$  be defined as

$$D_F(x) := \{B \in \mathbb{R}^{m \times d} : \text{there is } (x_n, B_n, a_n) \rightarrow (x, B, 0), \text{ with } B_n \in D(x_n, a_n)\}.$$

**Theorem 3.5.** *Let Assumption 2 hold. The map  $D_F$  and  $\text{conv } D_F$  are conservative mappings for  $F$ .*

*Proof.* It is sufficient to prove the claim on  $D_F$ . For each  $1 \leq i \leq m$ , the map  $x \rightrightarrows D_i(x, a)$  is a conservative field of  $f_i$  by Proposition 2.9. Denote  $D_{F_i}$  the corresponding limit. By Theorem 3.1,  $D_{F_i}$  is a conservative field for  $F_i$ . Thus, by Proposition 2.9 the mapping  $\tilde{D}_F : x \rightrightarrows [D_{F_1}(x), \dots, D_{F_m}(x)]^\top$  is a conservative mapping for  $F$ . The proof is completed by noticing that  $D_F \subset \tilde{D}_F$ .  $\square$



To finish this section, let us mention that, from a technical point of view, the proof of Theorems 3.1 and 3.5 uses very little of Lipschitz properties of  $f_a, F$ . Mainly, this assumption is used to ensure the fact that  $D_F$  is nonempty. In particular, we conjecture that a similar statement should hold, when we no longer assume local Lipschitz continuity of  $(f_a)$  and  $F$ .

However, the notion of conservative set-valued fields (or mappings) being for now defined only for Lipschitz functions, this would require a proper definition of conservativity for non Lipschitz mappings. Whence, we leave such considerations for future works.

## 4 Applications to smoothing methods

Assume that we are interested in the optimization problem

$$\min_{x \in \mathbb{R}^d} F(x),$$

where  $F : \mathbb{R}^d \rightarrow \mathbb{R}$  is continuous but is neither convex nor smooth. A large body of work (see for example [10, 11, 11–13, 20, 23, 25, 31, 32, 37, 45, 46]) suggests tackling this problem by designing a smooth function  $f : \mathbb{R}^d \times \mathbb{R}_+^* \rightarrow \mathbb{R}$  such that for each  $a > 0$ ,  $f_a$  is continuously differentiable and  $f_a \xrightarrow{a \rightarrow 0} F$ . The general optimization procedure then find, for decreasing values of  $a_k, \varepsilon_k > 0$ , an approximately stationary point  $x_k : \|\nabla f_{a_k}(x_k)\| \leq \varepsilon_k$ .

Notice that when  $a_k, \varepsilon_k \rightarrow 0$ , any accumulation point  $x^*$  of  $(x_k)$  satisfies  $0 \in D_F(x^*)$ . Therefore, as soon as  $f$  is definable, Theorem 3.1 establishes that  $D_F$  is a conservative set-valued field of  $F$ . Since  $0 \in \text{conv } D_F(x^*)$  is a necessary condition for  $x^*$  to be a local minimum (see Section 2.2), Theorem 3.1 gives theoretical guarantees of convergence for a large class of smoothing methods.

Let us also notice, that here the non-intrinsic properties of conservative fields are becoming apparent. Indeed,  $D_F$  is implicitly defined by the design of the smoothing function  $f$ . Depending on the latter,  $D_F$  might be just the Clarke’s subgradient or might be larger (or smaller). Thus, the guarantees provided by Theorem 3.1 will be dependent on the design of the smoothing function by the practitioner.

Let us look at some common examples of smoothing functions. In the first two,  $\text{conv } D_F$  turns out to be simply the Clarke’s subgradient. In the third-one,  $\text{conv } D_F$  might contain other elements.

**Example 4.1** (Lasry-Lions regularization ([27])). *Let  $F : \mathbb{R}^d \rightarrow \mathbb{R}$  be a continuous definable function such that  $\frac{F(x)}{1+\|x\|^2}$  is bounded from below. Then, for every  $a > 0$ ,*

$$f_a(x) = \sup_{z \in \mathbb{R}^d} \inf_{y \in \mathbb{R}^d} \left( F(y) + \frac{1}{2a} \|y - z\|^2 - \frac{1}{4a} \|z - x\|^2 \right)$$

*is the Lasry-Lions regularization (with parameters  $(a, a/2)$ ). For  $a$  small enough,  $f_a$  is continuously differentiable, with Lipschitz continuous gradients (see [3, 27]). As shown in [24], in this case  $\text{conv } D_F = \partial F$ , without any requirement on definability.*

*Note that an implementation of a method using the Lasry-Lions regularization would require to solve an optimization problem at every point of interest.*

**Example 4.2** (Mollifiers). *Perhaps the most common construction of smoothing approximations given by convolution of  $F$  with smooth kernels. Let  $(\psi_a)_{a>0}$  be a sequence of bounded,*

measurable functions  $\psi_a : \mathbb{R}^d \rightarrow \mathbb{R}_+$ , with  $\int_{\mathbb{R}^d} \psi_a(x) dx = 1$  such that for every  $a > 0$ ,  $\psi_a$  is equal to zero outside a ball of radius  $r(a)$ , with  $\lim_{a \rightarrow 0} r(a) = 0$ . Then, the family of function  $(f_a)$ , defined as

$$f_a(x) = \int_{\mathbb{R}^d} F(x - y) \psi_a(y) dy = \int_{B(x, r(a))} F(y) \psi_a(x - y) dy,$$

converges uniformly to  $F$  on every compact of  $\mathbb{R}^d$ , with moreover  $\text{conv } D_F = \partial F$  ([38, Example 7.19 and Theorem 9.67] and [20]). Note that no assumption on definability is needed here.

The typical choice here is  $\psi_a$  being the indicator of the open ball of radius  $a$ . From the computational perspective, the limitation of a method using a mollifying sequence would be in the evaluation of a  $d$ -dimensional integral.

**Example 4.3** (Max structure [12, 13]). Many interesting optimization problems include a nonsmooth function  $F$  that can be represented as a composition of smooth functions with the function  $(t)_+ := \max(0, t)$ . Examples include nonlinear complementarity problems, mixed complementarity problems or regularized minimization problems (see [13] for more details). Replacing  $(t)_+$  by a smoothing function  $\varphi(t, a)$  we obtain a smoothing function for  $F$ . In this case,  $\text{conv } D_F$  might be larger than  $\partial F$ . Nevertheless, Theorem 3.1 ensures the fact that  $D_F$  is a conservative set-valued field of  $F$ .

**Gradient consistency.** A desirable property of a smoothing method is the gradient consistency (see [13]):  $D_F(x) \subset \partial F(x)$ . Indeed, in this case  $D_F$  does not produce any additional critical points. As we have seen in Example 3.4, such a property does not hold in full generality. However, using the fact that  $D_F$  is a conservative field, by Proposition 2.6 we immediately obtain gradient consistency *almost everywhere*. Interestingly, we note in the following proposition that the failure of gradient consistency is intimately related to the failure of the chain rule of the Clarke's subgradient.

**Proposition 4.4.** Extend  $f : \mathbb{R}^d \times \mathbb{R}_+^*$  to  $a = 0$ , by putting  $f(x, 0) = F(x)$ . Assume that this extended function is locally Lipschitz continuous and that for every  $a > 0$ ,  $D_a \equiv \partial f_a$ . Then,

$$\text{conv } D_F(x) \subset P_d(\partial f(x, 0)),$$

where  $P_d : \mathbb{R}^{d+1} \rightarrow \mathbb{R}^d$  denotes the orthogonal projection onto the first  $d$  coordinates.

Note that  $F$  can be written as  $f \circ \varphi$ , with  $\varphi(x) = (x, 0)$ . Since  $\varphi$  is differentiable, this implies that  $\partial F \subset J_\varphi^\top \partial f(x, 0)$ , where the inclusion could be strict, depending on the regularity properties of  $f$  ([14, Chapter 2]). Noting that  $J_\varphi^\top = P_d$ , we see that the gradient consistency is determined by the strictness of this inclusion.

*Proof.* Consider  $x_n, v_n, a_n \rightarrow x, v, 0$ , with  $v_n \in \partial f_{a_n}(x_n)$ . Denoting  $\varphi_n(x) = (x, a_n)$ , it holds that  $f_{a_n} = f \circ \varphi_n$ . Thus, since the transpose of the Jacobian of  $\varphi_n$  is equal to  $P_d$ , from the chain rule for the Clarke subgradient ([14, Theorem 2.5]) we obtain that  $v_n \in \partial f_{a_n}(x_n) \subset P_d \partial f(x_n, a_n)$ . By the continuity of the Clarke subgradient we obtain that  $v \in P_d \partial f(x, 0)$ .  $\square$

**Remark 4.5.** The proof of Proposition 4.4 indicates a potentially alternative proof of Theorem 3.1. Indeed, if  $f$  extended to  $a = 0$  by  $f(x, 0) = F(x)$  is locally Lipschitz continuous and  $\tilde{D}$  is a conservative field of  $f$ , then by the chain rule for conservative fields ([7, Lemma

6]),  $x \rightrightarrows P_d \tilde{D}(x, a)$  is conservative for  $f_a$ . Thus, if we are able to construct  $\tilde{D}$  such that  $P_d \tilde{D}(x, a) = D_a(x)$ , then from the fact that  $\tilde{D}$  is closed the statement of Theorem 3.1 is immediate. We believe that, relying on stratification properties of definable sets and maps, such construction should be possible but is not immediate. Note, furthermore, that  $f$  is not even assumed to be continuous in Assumption 1. Hence, one should be careful with the meaning and definitions of conservativity for potentially discontinuous maps.

## 5 Proof of Theorem 3.1

The fact that  $D_F$  (and thus  $\text{conv } D_F$ ) is closed and locally bounded is immediate from its definition and Assumption 1. To show that  $D_F$  (and thus  $\text{conv } D_F$ ) is definable note that

$$\text{Graph } D_F = \{(x, v) : \forall \varepsilon > 0, \text{ there is } (x', v', a') \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+^*, \text{ such that } v' \in D_{a'}(x') \text{ and } \|x' - x\| + \|v' - v\| + a' \leq \varepsilon\}.$$

The right-hand side is defined through a definable first-order formula and thus is definable (see Proposition A.1).

Before going further, let us first describe the idea of the proof. Consider definable functions  $x, v : [0, 1] \rightarrow \mathbb{R}^d$  such that  $x$  is  $\mathcal{C}^1$  and for all  $t \in [0, 1]$ ,  $v(t) \in D_F(x(t))$ . By Remark 2.3 it is sufficient to show that

$$F(x(1)) - F(x(0)) - \int_0^1 \langle \dot{x}(t), v(t) \rangle dt = 0.$$

This will imply that  $D_F$  is conservative and will imply this property for  $\text{conv } D_F$  (see [7, Remark 3]).

The key idea is then to note that if one can construct a definable (and sufficiently regular) family of curves  $x_a, v_a \xrightarrow{a \rightarrow 0} x, v$  such that,  $v_a(t) \in D_a(x(t))$  and, furthermore,  $\dot{x}_a \xrightarrow{a \rightarrow 0} \dot{x}$ , then, up to some careful interchange of limit and expectation, one will obtain

$$F(x(1)) - F(x(0)) = \lim_{a \rightarrow 0} f_a(x_a(1)) - f_a(x_a(0)) = \lim_{a \rightarrow 0} \int_0^1 \langle \dot{x}_a(t), v_a(t) \rangle dt = \int_0^1 \langle \dot{x}(t), v(t) \rangle dt,$$

completing the proof. While such a family of curves does not necessarily exist, we prove that it always exists on a subset of  $[0, 1]$  of measure arbitrary close to 1. This is sufficient to complete the proof.

Let us describe the organization of this section.

- First, Lemmas 5.1–5.4 establish that if a family of curves  $(x_a)_{a>0}$  converges to a curve  $x$ , then almost surely  $\dot{x}_a \rightarrow \dot{x}$ . In some sense, it can be viewed as a one-dimensional version of the theorem.
- In Lemma 5.5 we show the existence of  $x_a, v_a \in D_a(x_a)$  such that  $x_a, v_a \xrightarrow{a \rightarrow 0} x, v$ .
- Combining the two previous results, we complete the proof of Theorem 3.1 by showing that for almost every  $t$ ,  $\langle \dot{x}_a, v_a \rangle \xrightarrow{a \rightarrow 0} \frac{d}{dt} F(x(t))$ .

To simplify the notations, until the end of the proof we will use the following convention.

Whenever  $t_0 < \dots < t_k$  is a partition of some open interval  $I = ]t_0, t_k[$ , we will denote  $I_i := ]t_i, t_{i+1}[$  and for  $\delta > 0$  small enough,  $I_{i,\delta} := ]t_i + \delta, t_{i+1} - \delta[$ .

**Lemma 5.1.** *Let  $I$  be an open interval,  $d > 0$  be an integer and  $x : I \times \mathbb{R}_+^* \rightarrow \mathbb{R}^d$  be a definable function. There is a finite collection of points  $t_0 < t_1 < \dots < t_k$  such that  $I = ]t_0, t_k[$  and for any  $\delta > 0$ , there is  $0 < a_\delta$  such that for  $0 \leq i \leq k-1$ , every coordinate  $1 \leq j \leq d$  and  $I_{i,\delta}$ , one of the following holds.*

- i) For all  $t \in I_{i,\delta}$ ,  $x(t, \cdot)_j$  increases on  $]0, a_\delta[$ .
- ii) For all  $t \in I_{i,\delta}$ ,  $x(t, \cdot)_j$  decreases on  $]0, a_\delta[$ .
- iii) For all  $t \in I_{i,\delta}$ ,  $x(t, \cdot)_j$  is constant on  $]0, a_\delta[$ .

*Proof.* It is sufficient to prove the lemma in the case where  $d = 1$ , since the general case will follow by finding a common refinement for all coordinates.

Assume that  $d = 1$ . By the monotonicity lemma and the definable choice (see Appendix A) we can find a definable function  $t \mapsto a_t$  such that  $x(t, \cdot)$  is decreasing (respectively increasing, constant) on  $]0, a_t[$ . Therefore, we can find  $t_0 < \dots < t_k$  such that  $t \mapsto a_t$  is continuous on each  $I_i = ]t_i, t_{i+1}[$  and on each  $I_i$ , the function  $x(t, \cdot)$  is either increasing, decreasing, or constant (in the same manner for all  $t \in I_i$ ) on  $]0, a_t[$ . Since  $t \mapsto a_t$  is continuous on  $I_i$  and since for any  $\delta > 0$ ,  $I_{i,\delta}$  is compact, it holds that  $a_{i,\delta} := \inf_{t \in I_{i,\delta}} a_t > 0$ . Denoting  $a_\delta = \min_i a_{i,\delta}$  completes the proof.  $\square$

**Lemma 5.2.** *Let  $I$  be an open interval,  $d > 0$  be an integer and  $x : I \times \mathbb{R}_+^* \rightarrow \mathbb{R}^d$  be a definable function. For any  $p > 0$ , there is a finite collection of points  $t_0 < t_1 < \dots < t_k$  such that  $I = ]t_0, t_k[$  and for any  $\delta > 0$ , there is  $0 < a_\delta$  such that for  $0 \leq i \leq k-1$ ,  $x$  is  $\mathcal{C}^p$  on  $I_{i,\delta} \times ]0, a_\delta[$ .*

*Proof.* By standard stratification properties of definable functions  $I \times \mathbb{R}_+^*$  can be partitioned into cells  $C_1, \dots, C_{k_0}$  such that  $x$  is  $\mathcal{C}^p$  on each of the cells. For every  $i \in \{1, \dots, k_0\}$ , consider the sets

$$A_i = \{(t, a) \in I \times \mathbb{R}_+^* : \forall a' \leq a, (t, a') \in C_i\} \quad \text{and} \quad I_i = \{t : \exists a \in \mathbb{R}_+^*, (t, a) \in A_i\}.$$

*Claim:*  $I = \bigcup I_i$ . Indeed, by definability, for every  $t \in I$ , the function  $a \mapsto t + a$ , for small enough  $a$ , will be contained in one of  $A_i$ . *End of the proof of the claim.*

Hence, we can find  $t_0 < t_1 < \dots < t_k$  such that  $]t_0, t_k[ = I$  and such that for each  $j \in \{0, \dots, k-1\}$ , there is  $i \in \{1, \dots, k_0\}$  such that  $]t_j, t_{j+1}[ \subset I_i$ . For such  $j$  and  $I_i$ , consider the definable function  $\mathbf{a}_j : ]t_j, t_{j+1}[ \rightarrow A_i$ :

$$\mathbf{a}_j(t) = \frac{1}{2} \sup\{a : (t, a) \in A_i\}.$$

Refining, if needed, the subdivision, we can assume that  $\mathbf{a}$  is continuous on  $]t_j, t_{j+1}[$ . Therefore, for every  $\delta > 0$ , there is  $0 < a_j < \inf\{\mathbf{a}_j(t) : t \in ]t_j + \delta, t_{j+1} - \delta[\}$ . Choosing  $a_\delta = \min_j a_j$ , completes the proof.  $\square$

In the statement of the following lemma, for a function  $y : I \rightarrow \overline{\mathbb{R}}$ , we say that the integral  $\int_I y$  is well-defined if either  $\int_I |y(t)| dt < +\infty$ , or  $y$  (respectively  $-y$ ) is lower bounded by an integrable function  $z$  and  $\int_I y(t) - z(t) dt = +\infty$  (respectively  $\int_I -y(t) - z(t) dt = +\infty$ ). In the latter case,  $\int_I y(t) dt = +\infty$  (respectively  $\int_I y(t) dt = -\infty$ ).

**Lemma 5.3** (Fuzzy integral convergence). *Consider an open interval  $I$  and a definable family of function  $x : I \times \mathbb{R}_+^* \rightarrow \mathbb{R}$ . Denote  $y : I \rightarrow \overline{\mathbb{R}}$  the definable function that satisfies, for all  $t \in I$ ,*

$$\lim_{a \rightarrow 0} |x(t, a) - y(t)| = 0.$$

*Then, there is  $t_0 < \dots < t_k$  such that  $I = ]t_0, t_k[$  and such that for any  $\delta > 0$  and any  $0 \leq i \leq k - 1$ ,  $\int_{I_{i,\delta}} y$  is well-defined. Moreover, there is  $a_\delta$  such that for  $a < a_\delta$ ,  $\int_{I_{i,\delta}} x(\cdot, a)$  is well-defined and*

$$\int_{I_{i,\delta}} x_a(t) dt \xrightarrow{a \rightarrow 0} \int_{I_{i,\delta}} y(t) dt.$$

*Proof.* Consider  $t_0 < \dots < t_k$  given by a common refinement of Lemmas 5.1 and 5.2 applied to  $x$ . Since  $y$  is definable, refining if needed the subdivision, we can also assume that, on each  $I_i$ ,  $y$  is either real-valued and continuous or is constant and is equal to  $\pm\infty$ . Hence,  $\int_{I_{i,\delta}} y$  is well-defined.

Moreover, for any  $\delta > 0$ , there is  $a_\delta$  such that the sequence  $(x(\cdot, a))_{a \leq a_\delta}$  is monotone and such that  $x$  is  $\mathcal{C}^p$  on  $\overline{I_{i,\delta}} \times ]0, a_\delta]$ . In particular, when  $a \leq a_\delta$  and  $a \rightarrow 0$ , either  $0 \leq x(\cdot, a) - x(\cdot, a_\delta) \nearrow y(\cdot) - x(\cdot, a_\delta)$  or  $0 \leq -x(\cdot, a) + x(\cdot, a_\delta) \nearrow -y(\cdot) + x(\cdot, a_\delta)$ . Since  $x(\cdot, a_\delta)$  is continuous, it is integrable on  $I_{i,\delta}$  and thus  $\int_{I_{i,\delta}} x(\cdot, a)$  is well-defined for  $a \leq a_\delta$ . Finally, applying the monotone convergence theorem, we obtain one of the following

$$\begin{aligned} \int_{I_{i,\delta}} x(t, a) - x(t, a_\delta) dt &\xrightarrow{a \rightarrow 0} \int_{I_{i,\delta}} y(t) - x(t, a_\delta) dt, \\ \int_{I_{i,\delta}} -x(t, a) + x(t, a_\delta) dt &\xrightarrow{a \rightarrow 0} \int_{I_{i,\delta}} -y(t) + x(t, a_\delta) dt. \end{aligned}$$

□

**Lemma 5.4.** *Consider  $d > 0$  an integer, an open interval  $I$  and a definable family of functions  $x : I \times \mathbb{R}_+^* \rightarrow \mathbb{R}^d$  such that for  $a > 0$ ,  $x_a := x(\cdot, a)$  is  $\mathcal{C}^1$ . Assume that there is a definable function  $y : I \rightarrow \mathbb{R}^d$  such that for every  $t \in I$ ,*

$$\lim_{a \rightarrow 0} \|y(t) - x(t, a)\| = 0.$$

*Then, there is  $t_0 < \dots < t_k$ , such that  $I = ]t_0, t_k[$  and for any  $t \in I \setminus \{t_0, \dots, t_k\}$ ,  $y$  is differentiable at  $t$  and  $\dot{x}(t, a) \xrightarrow{a \rightarrow 0} \dot{y}(t)$ .*

*Proof.* It is sufficient to prove the lemma in the case where  $d = 1$ , since the general case will follow by finding a common refinement for all the coordinates.

Assume that  $d = 1$ . Since  $y$  is definable we can find  $t_0 < \dots < t_k$  such that  $y$  is  $\mathcal{C}^1$  on each  $I_i$ . For every  $t \in I$ ,  $\lim_{a \rightarrow 0} \dot{x}(t, a) = g_t$  exists and is in  $\overline{\mathbb{R}}$ . By Lemma 5.3 we can refine the partition such that for any  $\delta > 0$ , and any  $t', t'' \in I_{i,\delta}$ ,

$$\int_{t'}^{t''} \dot{x}(t, a) dt \xrightarrow{a \rightarrow 0} \int_{t'}^{t''} g_t dt.$$

Since the left-hand side is equal to  $x(t'', a) - x(t', a)$ , we obtain

$$\int_{t'}^{t''} \dot{y}(t) dt = y(t'') - y(t') = \int_{t'}^{t''} g_t dt.$$

Since  $\delta$  and  $t', t'' \in I_{i,\delta}$  were arbitrary, this shows that  $g_t = \dot{y}$  almost everywhere. The latter, by definability, implies that  $g_t = \dot{y}$  everywhere except a finite number of points.  $\square$

We return to the proof of Theorem 3.1. Consider the definable functions  $x, v : [0, 1] \rightarrow \mathbb{R}^d$  such that  $x$  is  $\mathcal{C}^1$  and for all  $t \in [0, 1]$ ,  $v(t) \in D_F(x(t))$ . The purpose of the next lemma is to show that we can always partition  $I$  into subintervals such that, up to an arbitrary precision, there are curves  $x_a, v_a \xrightarrow{a \rightarrow 0} x, v$ , with  $v_a(t) \in D_a(x_a(t))$ .

**Lemma 5.5.** *There is  $t_0 < \dots < t_k$  such that for any  $\delta > 0$ , there is  $a_\delta > 0$  and two definable functions  $\tilde{x}, \tilde{v} : I_{i,\delta} \times ]0, a_\delta] \rightarrow \mathbb{R}^d$ , (defined for each  $0 \leq i \leq k-1$ ) such that for any  $(t, a) \in I_{i,\delta} \times ]0, a_\delta]$ ,*

$$v_a(t) \in D_a(x_a(t)) \quad \text{and} \quad x_a(t), v_a(t) \xrightarrow{a \rightarrow 0} x(t), v(t),$$

where we denoted  $x_a(t) = \tilde{x}(t, a)$  and  $v_a(t) = \tilde{v}(t, a)$ .

*Proof.* Denote  $S$  the definable set

$$S := \{(t, \varepsilon, \tilde{x}, \tilde{v}, a) : (a, t, \varepsilon) \in ]0, 1]^3, \|\tilde{v} - v(t)\| + \|\tilde{x} - x(t)\| + a \leq \varepsilon, \tilde{v} \in D_a(\tilde{x})\}.$$

Notice that for any  $(t, \varepsilon) \in ]0, 1]^2$ ,  $S_{t,\varepsilon} \neq \emptyset$ , where  $S_{t,\varepsilon} := \{(\tilde{x}, \tilde{v}, a) : (t, \varepsilon, \tilde{x}, \tilde{v}, a) \in S\}$ . Thus, there is a definable selection  $(\tilde{x}_1, \tilde{v}_1, a) : ]0, 1]^2 \rightarrow \mathbb{R}^{2d+1}$  such that for all  $(t, \varepsilon) \in ]0, 1]^2$ ,  $(\tilde{x}_1(t, \varepsilon), \tilde{v}_1(t, \varepsilon), a(t, \varepsilon)) \in S_{t,\varepsilon}$ . Notice that  $\tilde{v}_1(t, \varepsilon) \in D_{a(t,\varepsilon)}(\tilde{x}_1(t, \varepsilon))$  and that

$$\tilde{x}_1(t, \varepsilon), \tilde{v}_1(t, \varepsilon) \xrightarrow{\varepsilon \rightarrow 0} x(t), v(t).$$

We will complete the proof if we show that there is a definable function  $e(t, a)$  such that  $a(t, e(t, a)) = a$ .

For every  $t \in [0, 1]$ , there is  $\varepsilon_t$  such that  $a(t, \cdot)$  is continuous and increasing on  $]0, \varepsilon_t]$ , with  $\lim_{\varepsilon \rightarrow 0} a(t, \varepsilon) = 0$ . Denote  $a_t = a(t, \varepsilon_t) > 0$ . The function  $t \mapsto a_t$  being definable, there is  $0 = t_0 < t_1 < \dots < t_k = 1$  such that  $a_t$  is continuous on any interval  $I_i$ . Therefore, for any  $i, \delta$ ,  $a_{i,\delta} := \min_{t \in I_{i,\delta}} a_t > 0$ .

Denote  $a_\delta := \min_i a_{i,\delta}$  and note that the set

$$\{(t, a, \varepsilon) : t \in I_{i,\delta}, \varepsilon \in ]0, \varepsilon_t], a(t, \varepsilon) = a\}$$

is definable and that by the mean-value theorem, for any  $(t, a) \in I_{i,\delta} \times ]0, a_\delta]$ , the set

$$\{\varepsilon \in ]0, \varepsilon_t] : a(t, \varepsilon) = a\}$$

is non-empty. Hence, by Definable Choice, there is  $e : I_{i,\delta} \times ]0, a_\delta] \rightarrow ]0, 1]$  such that  $e(t, a) \leq \varepsilon_t$  and  $a(t, e(t, a)) = a$ .

*Claim:* for any  $t \in I_{i,\delta}$ ,  $\lim_{a \rightarrow 0} e(t, a) = 0$ . Indeed, by definability  $\lim_{a \rightarrow 0} e(t, a) = \varepsilon_a$  exists. If  $\varepsilon_a \neq 0$ , then from the continuity of  $a(t, \cdot)$  on  $]0, \varepsilon_t]$  we obtain  $a(t, e(t, a)) = a \rightarrow a(t, \varepsilon_a) = 0$ . Since  $a(t, \cdot)$  takes only positive values, this shows that necessarily  $\varepsilon_a = 0$ .

Thus, denoting  $\tilde{x}(t, a), \tilde{v}(t, a) = \tilde{x}_1(t, e(t, a)), \tilde{v}_1(t, e(t, a))$ , we obtain the statements of the lemma.  $\square$

*End of the proof of Theorem 3.1.* Consider  $t_0 < \dots < t_k$  given by Lemma 5.5, and for any  $\delta_1 > 0$  the associated functions  $\tilde{x}, \tilde{v} : I_{i, \delta_1} \times ]0, a_{\delta_1}] \rightarrow \mathbb{R}^d$ .

By Lemma 5.2, for any  $0 \leq i \leq k-1$ , we can partition  $I_{i, \delta_1}$  by  $t_i + \delta_1 = t_0^i < \dots < t_{k_i}^i = t_{i+1} - \delta_1$  such that, denoting, for  $0 \leq j \leq k_i - 1$  and  $\delta_2 > 0$ ,  $I_{i, j, \delta_2} := ]t_j^i + \delta_2, t_{j+1}^i - \delta_2[$ , there is  $a_{\delta_2} \leq a_{\delta_1}$  such that  $\tilde{x}$  and  $f \circ \tilde{x}$  is  $\mathcal{C}^1$  on  $I_{i, j, \delta_2} \times ]0, a_{\delta_2}]$ .

By Lemma 5.4, for all  $t \in I_{i, j, \delta_2}$  except a finite number of points, we obtain that

$$\dot{x}_a(t) \xrightarrow{a \rightarrow 0} \dot{x}(t) \quad \text{and} \quad \frac{d}{dt}(f_a \circ x_a)(t) \xrightarrow{a \rightarrow 0} \dot{y}(t),$$

where  $y(t)$  is the limit of  $(f_a \circ x_a)(t)$ . Note that by Assumption 1 and Arzelà-Ascoli theorem, there is  $a_n \rightarrow 0$ , such that  $f_{a_n} \rightarrow F$  uniformly on compact sets. Thus,  $y(t) = F(x(t))$  and

$$\dot{x}_a(t) \xrightarrow{a \rightarrow 0} \dot{x}(t) \quad \text{and} \quad \frac{d}{dt}(f_a \circ x_a)(t) \xrightarrow{a \rightarrow 0} \frac{d}{dt}(F \circ x)(t). \quad (4)$$

Now, consider any sequence  $(a_n)_{n \in \mathbb{N}}$ , upper bounded by  $a_{\delta_2}$ , such that  $a_n \searrow 0$ . For every  $n \in \mathbb{N}$ , by path differentiability of  $f_{a_n}$  there is a set  $T_n$  of full measure in  $I_{i, j, \delta_2}$  such that for all  $t \in T_n$ ,  $\frac{d}{dt}(f_{a_n} \circ x_{a_n})(t) = \langle \dot{x}_{a_n}(t), v_{a_n}(t) \rangle$ . In particular, for all  $t \in \bigcap T_n$ , it holds that

$$\frac{d}{dt}(f_{a_n} \circ x_{a_n})(t) = \langle \dot{x}_{a_n}(t), v_{a_n}(t) \rangle \xrightarrow{n \rightarrow \infty} \langle \dot{x}(t), v(t) \rangle. \quad (5)$$

Combining Equations (4) and (5), we obtain for almost all  $t \in I_{i, j, \delta_2}$ ,

$$\frac{d}{dt}(F \circ x)(t) = \langle \dot{x}(t), v(t) \rangle. \quad (6)$$

Since  $\delta_2$  was arbitrary, Equation (6) also holds for almost every  $t \in I_{i, j} = ]t_j^i, t_{j+1}^i[$  and thus for almost every  $t \in I_{i, \delta_1}$ . Similarly, since  $\delta_1$  was arbitrary, this implies that it holds for almost every  $t \in I_i$  and thus for almost every  $t \in [0, 1]$ . Finally,

$$F(x(1)) - F(x(0)) = \int_0^1 \frac{d}{dt}(F \circ x)(t) dt = \int_0^1 \langle \dot{x}(t), v(t) \rangle dt,$$

which completes the proof.  $\square$

**Acknowledgments.** The author would like to thank Jérôme Bolte and Edouard Pauwels for interesting discussions.

## References

- [1] Martín Abadi, Ashish Agarwal, Paul Barham, Eugene Brevdo, Zhifeng Chen, Craig Citro, Greg S. Corrado, Andy Davis, Jeffrey Dean, Matthieu Devin, Sanjay Ghemawat, Ian Goodfellow, Andrew Harp, Geoffrey Irving, Michael Isard, Yangqing Jia, Rafal Jozefowicz, Lukasz Kaiser, Manjunath Kudlur, Josh Levenberg, Dandelion Mané, Rajat Monga, Sherry Moore, Derek Murray, Chris Olah, Mike Schuster, Jonathon Shlens, Benoit Steiner, Ilya Sutskever, Kunal Talwar, Paul Tucker, Vincent Vanhoucke, Vijay Vasudevan, Fernanda Viégas, Oriol Vinyals, Pete Warden, Martin Wattenberg, Martin Wicke, Yuan Yu, and Xiaoqiang Zheng. TensorFlow: Large-scale machine learning on heterogeneous systems, 2015. Software available from tensorflow.org.

- [2] Hedy Attouch. Convergence de fonctionnelles convexes. In *Journées d'Analyse Non Linéaire: Proceedings, Besançon, France, June 1977*, pages 1–40. Springer, 2006.
- [3] Hedy Attouch and Dominique Aze. Approximation and regularization of arbitrary functions in hilbert spaces by the lasry-lions method. In *Annales de l'Institut Henri Poincaré C, Analyse non linéaire*, volume 10, pages 289–312. Elsevier, 1993.
- [4] Hedy Attouch and Gerald Beer. On the convergence of subdifferentials of convex functions. *Archiv der Mathematik*, 60(4):389–400, 1993.
- [5] Edward Bierstone and Pierre D Milman. Semianalytic and subanalytic sets. *Publications Mathématiques de l'IHÉS*, 67:5–42, 1988.
- [6] J. Bolte, A. Daniilidis, A. Lewis, and M. Shiota. Clarke subgradients of stratifiable functions. *SIAM Journal on Optimization*, 18(2):556–572, 2007.
- [7] Jérôme Bolte and Edouard Pauwels. Conservative set valued fields, automatic differentiation, stochastic gradient methods and deep learning. *Mathematical Programming*, 188:19–51, 2021.
- [8] Jérôme Bolte, Edouard Pauwels, and Antonio Silvetti-Falls. Differentiating nonsmooth solutions to parametric monotone inclusion problems. *SIAM Journal on Optimization*, 34(1):71–97, 2024.
- [9] Jérôme Bolte, Edouard Pauwels, and Samuel Vaiter. One-step differentiation of iterative algorithms. *arXiv preprint arXiv:2305.13768*, 2023.
- [10] James V Burke, Xiaojun Chen, and Hailin Sun. The subdifferential of measurable composite max integrands and smoothing approximation. *Mathematical Programming*, 181:229–264, 2020.
- [11] James V Burke and Tim Hoheisel. Epi-convergence properties of smoothing by infimal convolution. *Set-Valued and Variational Analysis*, 25:1–23, 2017.
- [12] Chunhui Chen and Olvi L Mangasarian. A class of smoothing functions for nonlinear and mixed complementarity problems. *Computational Optimization and Applications*, 5(2):97–138, 1996.
- [13] Xiaojun Chen. Smoothing methods for nonsmooth, nonconvex minimization. *Mathematical programming*, 134:71–99, 2012.
- [14] Francis H Clarke, Yuri S Ledyaev, Ronald J Stern, and Peter R Wolenski. *Nonsmooth analysis and control theory*, volume 178. Springer-Verlag, New York, 1998.
- [15] Michel Coste. *An introduction to o-minimal geometry*. Istituti editoriali e poligrafici internazionali Pisa, 2000.
- [16] Marc-Olivier Czarnecki and Ludovic Rifford. Approximation and regularization of lipschitz functions: convergence of the gradients. *Transactions of the American Mathematical Society*, 358(10):4467–4520, 2006.



- [17] Damek Davis and Dmitriy Drusvyatskiy. Conservative and semismooth derivatives are equivalent for semialgebraic maps. *Set-Valued and Variational Analysis*, 30(2):453–463, 2022.
- [18] Damek Davis, Dmitriy Drusvyatskiy, Sham Kakade, and Jason D Lee. Stochastic subgradient method converges on tame functions. *Foundations of computational mathematics*, 20(1):119–154, 2020.
- [19] Dmitriy Drusvyatskiy, Alexander D Ioffe, and Adrian S Lewis. Curves of descent. *SIAM Journal on Control and Optimization*, 53(1):114–138, 2015.
- [20] Yuri M Ermoliev, Vladimir I Norkin, and Roger JB Wets. The minimization of semi-continuous functions: mollifier subgradients. *Siam journal on control and optimization*, 33(1):149–167, 1995.
- [21] Andrei Gabrielov. Complements of subanalytic sets and existential formulas for analytic functions. *Inventiones mathematicae*, 125(1):1–12, 1996.
- [22] Andrei M Gabrielov. Projections of semi-analytic sets. *Functional Analysis and its applications*, 2(4):282–291, 1968.
- [23] R Garmanjani and LN Vicente. Smoothing and worst-case complexity for direct-search methods in nonsmooth optimization. *IMA Journal of Numerical Analysis*, 33(3):1008–1028, 2013.
- [24] Pando Gr Georgiev and Nadia P Zlateva. Reconstruction of the clarke subdifferential by the lasry–lions regularizations. *Journal of mathematical analysis and applications*, 248(2):415–428, 2000.
- [25] Ming Hu and Masao Fukushima. Smoothing approach to nash equilibrium formulations for a class of equilibrium problems with shared complementarity constraints. *Computational Optimization and Applications*, 52:415–437, 2012.
- [26] Alexander D Ioffe. An invitation to tame optimization. *SIAM Journal on Optimization*, 19(4):1894–1917, 2009.
- [27] Jean-Michel Lasry and Pierre-Louis Lions. A remark on regularization in hilbert spaces. *Israel Journal of Mathematics*, 55:257–266, 1986.
- [28] Tam Le. Nonsmooth nonconvex stochastic heavy ball. *arXiv preprint arXiv:2304.13328*, 2023.
- [29] Adam B Levy, R Poliquin, and Lionel Thibault. Partial extensions of attouch’s theorem with applications to proto-derivatives of subgradient mappings. *Transactions of the American Mathematical Society*, 347(4):1269–1294, 1995.
- [30] Adrian S Lewis and Tonghua Tian. The structure of conservative gradient fields. *SIAM Journal on Optimization*, 31(3):2080–2083, 2021.
- [31] David Q Mayne and Elijah Polak. Nondifferential optimization via adaptive smoothing. *Journal of Optimization Theory and Applications*, 43(4):601–613, 1984.

- [32] Yu Nesterov. Smooth minimization of non-smooth functions. *Mathematical programming*, 103:127–152, 2005.
- [33] Jorge Nocedal and Stephen J Wright. *Numerical optimization*. Springer, 1999.
- [34] Adam Paszke, Sam Gross, Soumith Chintala, Gregory Chanan, Edward Yang, Zachary DeVito, Zeming Lin, Alban Desmaison, Luca Antiga, and Adam Lerer. Automatic differentiation in pytorch, 2017.
- [35] Edouard Pauwels. Conservative parametric optimality and the ridge method for tame min-max problems. *Set-Valued and Variational Analysis*, 31(3):1–24, 2023.
- [36] René A Poliquin. An extension of attouch’s theorem and its application to second-order epi-differentiation of convexly composite functions. *Transactions of the American Mathematical Society*, 332(2):861–874, 1992.
- [37] Liqun Qi and Xiaojun Chen. A globally convergent successive approximation method for severely nonsmooth equations. *SIAM Journal on control and Optimization*, 33(2):402–418, 1995.
- [38] R Tyrrell Rockafellar and Roger J-B Wets. *Variational analysis*, volume 317. Springer Science & Business Media, 2009.
- [39] Alfred Tarski. A decision method for elementary algebra and geometry. In *Quantifier elimination and cylindrical algebraic decomposition*, pages 24–84. Springer, 1951.
- [40] Lou Van den Dries. *Tame topology and o-minimal structures*, volume 248. Cambridge university press, 1998.
- [41] Lou van den Dries, Angus Macintyre, and David Marker. The elementary theory of restricted analytic fields with exponentiation. *Annals of Mathematics*, 140(1):183–205, 1994.
- [42] Lou van den Dries and Chris Miller. Geometric categories and o-minimal structures. *Duke Math. J.*, 85(1):497–540, 1996.
- [43] Alex Wilkie. Model completeness results for expansions of the ordered field of real numbers by restricted pfaffian functions and the exponential function. *Journal of the American Mathematical Society*, 9(4):1051–1094, 1996.
- [44] Nachuan Xiao, Xiaoyin Hu, Xin Liu, and Kim-Chuan Toh. Adam-family methods for nonsmooth optimization with convergence guarantees. *arXiv preprint arXiv:2305.03938*, 2023.
- [45] Israel Zang. A smoothing-out technique for min—max optimization. *Mathematical Programming*, 19:61–77, 1980.
- [46] Israel Zang. Discontinuous optimization by smoothing. *Mathematics of operations research*, 6(1):140–152, 1981.
- [47] Tullio Zolezzi. Continuity of generalized gradients and multipliers under perturbations. *Mathematics of operations research*, 10(4):664–673, 1985.

[48] Tullio Zolezzi. Convergence of generalized gradients. *Set-Valued Analysis*, 2(1-2):381–393, 1994.

## A O-minimal structures

In the following we fix some o-minimal structure  $\mathcal{O}$ . Definable will always mean definable in  $\mathcal{O}$ .

An attractive property of definable sets is that they can be constructed by means of *first order formulas*. A first order formula is constructed according to the following rules.

- i) If  $Q : \mathbb{R}^n \rightarrow \mathbb{R}$  is a polynomial, then  $Q(x) = 0$  and  $Q(x) > 0$  are first order formulas.
- ii) If  $A \subset \mathbb{R}^n$  is definable, then  $x \in A$  is a first order formula.
- iii) If  $\Phi(x)$  and  $\Psi(x)$  are first order formulas, “ $\Psi(x)$  and  $\Phi(x)$ ”, “ $\Psi(x)$  or  $\Phi(x)$ ”, “not  $\Phi(x)$ ” and “ $\Psi(x) \implies \Phi(x)$ ” are first order formulas.
- iv) If  $\Phi(x, y)$  is a first order formula, where  $(x, y) \in \mathbb{R}^n \times \mathbb{R}^l$ , and  $A \subset \mathbb{R}^n$  is definable, then “ $\exists x \in A \ \Psi(x, y)$ ” and “ $\forall x \in A \ \Psi(x, y)$ ” are first order formulas.

**Proposition A.1** ([15, Theorem 1.13]). *If  $\Phi(x)$  is a first order formula, then the set of  $x$  that satisfies  $\Phi(x)$  is a definable set.*

The following two lemmas will be used throughout the proof of Theorem 3.1.

**Lemma A.2** (Monotonicity lemma [42, Theorem 4.1]). *Let  $f : ]a, b[ \rightarrow \mathbb{R}$ , with  $-\infty \leq a < b \leq +\infty$ , be a definable function and  $p \geq 0$ . There is a finite subdivision  $a = a_0 < \dots < a_k = b$  such that on each interval  $]a_i, a_{i+1}[$   $f$  is  $C^p$  and either constant or strictly monotone.*

In particular, that shows that any one-dimensional function admits right and left limits (that can be infinite) near every point.

The following lemma shows that we can construct definable selections from definable sets.

**Lemma A.3** (Definable choice [15, Theorem 3.1]). *Let  $A \subset \mathbb{R}^n \times \mathbb{R}^l$  be a definable set. Let  $\pi_n$  denote the projection on the first  $n$  coordinates. Then there is a definable function  $\rho : \pi_n(A) \rightarrow \mathbb{R}^l$  s.t. for any  $x \in \pi_n(A)$ ,  $(x, \rho(x)) \in A$ .*