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On Manipulation in Merging Epistemic States*

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Abstract

In this work we study the problem of manipulation in the framework of merging complex epistemic states. We adopt the techniques concerning representation and impossibility in merging complex epistemic states proposed by Mata Díaz and Pino Pérez in 2017. We introduce here the notion of belief lifting, aiming to capture the preferences of an agent over formulas. This allows us to define a general qualitative notion of manipulability. We prove that, given some rational properties, a merging operator is either manipulable, with respect to any well-behaved belief lifting, or it admits a powerful agent (a dictator or a nominator) who, in some way, imposes his will. We also study the behaviour of some concrete epistemic state merging operators, showing that most of them are manipulable. By means of this study, we prove that strategy-proofness cannot be characterised in terms of dictators or nominators in this qualitative framework of manipulability.

Keywords: Merging, Epistemic states, Manipulation, Strategy-proof, Dictator, Nominators.

2000 MSC: 68T37, 68T30, 68T27, 03B42, 91B14

1. Introduction

When several sources (agents/devices) give information and we want to extract a coherent and relevant information from this group of sources, we face a typical problem of fusion. This problem is important in many areas (decision making, medical diagnosis, policy planning, geographical information systems, automatic aggregation of data, etc.) and it has been studied extensively in recent years. The nature of the data appearing in these problems can be very diverse. Numerical data and qualitative data are the two big families of data in which different methods for merging information are developed. Our interest is in the second family. More precisely, in understanding fusion of data having a representation close to a logical representation. Our starting point is the logical based model of belief merging of Konieczny and Pino Pérez [43].

In that work, they introduced belief merging with integrity constraints, a logical framework that states the rational properties that an aggregating method of information must satisfy in order to obtain a coherent piece of information from several sources, which might be mutually contradictory. That framework (the IC merging framework) has been conceived as an extension of the AGM belief revision theory [1]. Indeed, it extends to a multi-source setting, the finite version of belief revision proposed by Katsuno and Mendelzon [39], where the pieces of information are encoded by a propositional formula (a belief base). In this setting, a belief merging process (called an IC merging operator) aims to merge the belief bases of the agents, which are collected in a “bag”, under an integrity constraint (also codified by a propositional formula), which has to be satisfied by the outcome of the merging process. In order to regulate this class of processes, the authors presented a set of rational syntactic postulates that merging operators should satisfy and stated a semantic representation result, which is indeed an essential tool to analyse this kind of processes.

*This work is an extended version of the work presented in ECSQARU-2021 [60].

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In the present work we consider a generalization of the classical belief merging framework of Konieczny and Pino Pérez [43]. In this generalization, proposed by Mata Díaz and Pino Pérez [56], instead of propositional formulas, abstract objects called epistemic states, are considered. Thus, an epistemic states merging operator receives as input a profile of epistemic states and a constraint, which is also an epistemic state and, as output, it also produces an epistemic state. What is important is that the epistemic states have some logical information attached which allows one to state, in logical terms, some rationality properties. Of course, this abstraction needs to be made concrete in order to define tangible operators. A very useful concrete realization of epistemic states will be a total preorder over interpretations. As a matter of fact, the reader who desires to avoid abstractions, may think of epistemic states as total preorders. Even if that is not entirely true, this is a paradigmatic and very useful example to understand the main ideas in this work.

Two aspects which deserve attention when considering methods for merging, are:

1. The methods in which an agent can impose his point of view (powerful agents).
2. The methods in which an agent can misrepresent his information in order to obtain a more advantageous outcome at the end of the process (manipulator agents).

Merging methods should obey a very basic principle: the output should be in at least partial agreement with all the different sources of information. Thus, methods in which an agent can impose his point of view are to be avoided. In the same spirit, the misrepresentation of information in order to obtain a favorable output is not desirable. Thus, it is important to understand which processes can avoid this.

As a matter of fact, these aspects have been studied for quite some time in social choice theory [69]. Indeed, the belief merging processes are related to the methods of preference aggregation studied in social choice theory, as was pointed out by Konieczny and Pino Pérez [44]. Among the first works showing these relationships we find Everaere et al. [24, 25], Eckert and Pigozzi [22], Chopra et al. [16], Gabbay et al. [31]. Thus, it is natural considering “social” features (group properties) coming from social choice theory in merging when considering these aspects. For this reason, we have also studied some of these relations, in particular focusing on the two aspects mentioned above. Concerning the first one, in [56, 58] and concerning the second one, a few very preliminary steps were done in [57, 59]. In this work we deepen the study of the second question and obtain more general and interesting results.

Let us recall that the central questions in social choice theory are the following: given a set of alternatives and a set of voters with their preferences over the alternatives, how to select the best alternatives for the group and in what measure a method for selecting the alternatives is good. The most general models of election are based on social choice functions. The method for defining good (rational) social choice functions evokes the methods for defining belief merging operators given by the representation theorem [44].

One measure of the goodness of a social choice function is the fact that it satisfies some reasonable democracy criteria presented by Arrow [3]: *non-imposition, the Pareto condition, independence of irrelevant alternatives and absence of dictators*. In his work, Arrow showed that, if a social choice function satisfies the first three criteria, then it admits a dictator. This is his famous impossibility theorem.

Exploiting the similarities between social choice functions and belief merging operators, we translated the Arrovian criteria to the logical framework of belief merging in [56]. This gave an answer to the first aspect mentioned above. Additionally, in that work we showed that this type of operators meets the majority of those criteria, stating also an impossibility result which generalises Arrow’s Theorem (cf. Theorem 3 below).

As mentioned earlier, in order to be able to carry out this study, we have adopted a more complex representation of the information, namely, we use the notion of *epistemic states*. Roughly, an epistemic state is an object with some attached logical information, called the “entrenched beliefs”.

As a matter of fact, this complex representation of information is necessary in many situations in the study of belief dynamics. This was revealed by Darwiche and Pearl [17] in the case of revision operators with a good iterative behaviour. In their setting, it is a key point to have the possibility that two different objects (epistemic states) have the same logical information attached. The necessity of considering a complex manner of representing information in belief merging was showed by us in [56], when we extended the IC merging framework to the setting of Epistemic State Fusion (the ESF framework for short). In the ESF framework, the agents’ information and integrity constraints are encoded in complex epistemic states (a

complex model of representation of information stated by Benferhat et al. [11]), the IC merging postulates are adapted to this setting and a semantic representation result was also stated.

The abstract setting of representation of information stated by Benferhat et al. captures the main idea that an epistemic state must contain the entrenched belief of an agent as well as additional information which is necessary for an adequate and precise evolutionary treatment of information. This setting is indeed a formalization of the concept stated by Darwiche and Pearl [17]. The most common concrete realization of this notion are total preorders over interpretations but it is not the only way to represent epistemic states (see for instance [2, 65]). The notion of epistemic states can also be instantiated in other concrete models: ranking functions [68]; rational consequence relations [50] and, of course, the classical representations of belief bases: logical theories [1] and simple formulas [39]. However, these last two instantiations are not adequate for the representation of our operators.

Considering (complex) epistemic states for modelling merging processes, instead of simple belief bases, has considerable advantages. First, it is an abstract setting which allows a uniform treatment for many structures representing pieces of information. Second, this representation allows satisfaction of some rational properties such as the standard domain condition (which says intuitively that all the outputs are possible, cf. Subsection 2.5), a key property for the proof of important results supporting this work. Note that in the context of the representation of epistemic states as simple formulas, this property is not satisfied (see [56] for details). Third, this setting allows a treatment of information richer than formulas as we can see in the examples. Moreover, it is possible to encode in our setting problems of voting theory (see [56]). Fourth, the setting of epistemic states allows for our merging operators to be, indeed, a generalization of revision operators with a good iterative behaviour¹ (see [55] for details).

The following example illustrates a scenario of a merging process in which the complex representation of information plays a decisive role. This is an adaptation of Example 1 presented in [56] and it will be formally addressed in Subsection 2.3 (cf. Example 1 revisited).

Example 1. *Anne, Bob and Charles are good friends. They are discussing the divorce of a couple of friends of them, Tom and Mary. They try to determine who's to blame. Anne expresses that she does not believe that Mary is to blame. To a lesser degree she believes in any other possibility. Bob, who had witnessed some terrible fights between Tom and Mary, believes that both of them are to blame. To a lesser degree he believes that only one is to blame and finally he does not believe at all that none is to blame for the divorce. Charles believes, on the contrary, that none is to blame for the divorce. To a lesser degree he believes in any other possibility. The three friends learn that, in fact, there is at least one member of the couple to blame (the integrity constraint). Under this constraint, a reasonable fusion process will say that the more consensual belief is that Tom is to blame, regardless of whether Mary is to blame or not. In a lesser degree, in this process comes the belief that Mary is to blame but not Tom and, finally, that none is to blame is the less believed fact in this process.*

In order to see how a complex constraint can play an interesting role, we consider the following example which will be formally addressed in Subsection 2.3 (cf. Example 2 revisited).

Example 2. *Anne and Bob are now discussing about the replacement of two electronic devices, x and y , necessary for the correct operation of their coffee machine. They both believe that the most advantageous option is to replace both devices. They also believe that perhaps the replacement of the two devices can wait a few weeks more. Finally, they don't believe that replacing only one device is a good option. Now, an expert of the coffee machine's manufacturer informs them that it is not necessary to replace device y . But if it is replaced, then device x has to be replaced too. The worst option is to replace y and not to replace x . After that, a reasonable way to merge the beliefs of Anne and Bob under the advice of the expert (taken as a constraint), the most consensual belief is to wait for replacing the devices. But in case of replacing, to replace device x and not device y is better than replacing the two devices. The worst option being to replace y and not x .*

¹Restricted to single-profiles, ESF operators are iterated belief revision operators in the framework of Benferhat et al. [11].

Taking into account all the information from the expert leads one to consider that to replace device x and not device y is better than replacing the two devices, a conclusion that can hardly be deduced from only the fact that it is not necessary to replace device y .

We must note that it is not necessary to consider epistemic states at the level of integrity constraints in order to state our results. However, as in the previous example, the use of complex integrity constraints is useful to model a problem in which the constraints can be seen as an infallible agent and in this manner to obtain a finer representation of the outputs. Thus, in this paper we adopt this point of view: all the basic pieces of information are epistemic states.

Regarding aspect 2, mentioned before, it is indeed another interesting topic addressed in social choice theory: the manipulability of electoral processes. Manipulation occurs when a voter, misrepresenting his preferences obtains a result which is more advantageous for him than the result obtained when he votes according to his true preferences. Gibbard [35] and Satterthwaite [63] are the pioneers in this topic. In their works, they independently stated the necessary and sufficient conditions for a *non-imposed* (all the alternatives have a chance of being chosen) and *resolute scheme of vote* (a special social choice function for which all the alternatives are always available and only one alternative is chosen) to be nonmanipulable: this is the case when it admits a dictator. Therefore, any natural scheme of vote is either manipulable or dictatorial. This is the well-known Gibbard-Satterthwaite theorem [71].

Manipulability situations are also present in belief merging: an agent could “lie” in order to obtain a result that fits better with his beliefs. The following example illustrates a manipulability situation. This is an adaptation of Example 2 in [56], and it will be addressed in Subsection 3.3 (cf. Example 3 revisited).

Example 3. *Alan and Ben have to travel from point X to point Y . There are four possible paths, which do not meet each other: w_1 , w_2 , w_3 and w_4 . Alan believes that the best path is w_1 . Actually, he believes that w_i is better than w_{i+1} , for $i = 1, 2, 3$. Ben believes that w_1 and w_4 are the best paths, while he believes the other two paths are indifferently less good. In this case, by consensus, they would agree in believing that the best path to travel is w_1 . This agreement would fit very well with each other’s beliefs.*

However, the latest news inform that path w_1 is obstructed, whereas on paths w_2 , w_3 and w_4 there is light traffic. In the light of this information, Ben estimates that they would agree in believing that w_2 is the best path to travel, if he expresses his real beliefs. His reasoning is that since w_1 has to be dismissed, the choice has to be done among w_2 , w_3 and w_4 . Since Alan prefers w_2 to w_3 and for Ben these two paths are indifferent, the reasonable choice between w_2 and w_3 , is w_2 . And since Alan prefers w_2 ‘much more’ than w_4 while Ben prefers w_4 over w_2 with ‘less strength’, the reasonable choice between w_2 and w_4 , is w_2 . This result would fit Alan’s beliefs, but it would clash with Ben’s more entrenched beliefs: it is better to travel along path w_1 or path w_4 .

Hence, searching for a result that fits better with his beliefs, Ben expresses that the best way to travel is path w_4 , and that in a decreasing preference come path w_1 , path w_3 and path w_2 . In this case, taking into account the relative positions of paths, Alan and Ben should agree in believing that w_4 is the best path (the reasoning for this agreement is analogous to the case where Ben expresses his true beliefs, see Example 3 revisited in Section 3.3 for a formal treatment). This result, due to a manipulation of the merging process, fits better with Ben’s beliefs.

An inherent issue in the study of manipulation is to measure how suitable the outcome of a merging process is for an agent. Everaere et al. [25] stated a quantitative manner to do so, introducing the notion of *indexes of satisfaction*. In their work, the manipulability of some particular classes of IC merging operators (with respect to some specific indexes) was characterised, but no general manipulability result was established. We introduce a qualitative method to measure such suitability using *belief liftings*.

A *belief lifting* is a mapping that establishes preferences over information (formulas) from preferences over interpretations of propositional formulas, in a similar way in which a lifting over sets extends preferences over objects to preferences over sets of objects. The idea of using liftings goes back to the work of de Finetti [28], which provides a characterisation of qualitative probabilities. Liftings have been considered in the study of some logical frameworks in order to state preferences over formulas: van Benthem et al. [12] stated a global preference relation between propositions, which has an essential *ceteris paribus* rider; Halpern [36] extended

preferences over worlds to a ‘likelihood’ ordering on sets of worlds, and examines the resulting logic of these processes. Liftings have also been considered to address manipulation issues in both social choice theory and belief merging: Barberà [4] and Kelly [40] independently showed that, if a natural social choice function is strategy-proof with respect to a “well-behaved” lifting, then it admits a powerful voter (a *nominator*), who imposes his will by always including at least one of his preferred alternatives in the outcome of the election process.

In this work we study strategy-proofness in the extended context of the ESF framework, using belief liftings as a tool to establish preferences over information in a qualitative manner. We would like to highlight two important features of our work:

Firstly, the view of epistemic states considered here is more general than the propositional view, although it preserves the main logical aspects. This abstract representation of information allows for the results stated here to hold for any concrete instantiation of epistemic states rich enough in order to have operators satisfying the postulates. We have to note that this view of epistemic states is even more general than total preorders or rankings over worlds. Moreover, the voting problems of social choice theory, in particular preferences, can be encoded faithfully in our framework [56].

Secondly, we enrich the set of rational postulates by introducing some new postulates: *strong standard domain*, *non-imposition*, *stability* and *absence of nominators*. Although they are inspired on the classical Arrow’s criteria, they are natural and some classical instances of ES merging operators satisfy them (cf. Table 3).

Based on these properties, we state some general manipulability results, which show a dichotomy between manipulability (with respect to any “well-behaved” belief lifting) and the existence of a powerful agent. Thus, we can summarize the main contributions of this work in the following ones:

- A general definition of manipulability in the framework of ESF using the concept of belief liftings.
- A general theorem on manipulability of merging operators.
- A hierarchical classification of powerful agents.

This work is organised as follows: Section 2 contains the fundamental notions and tools that will be used along this paper. More specifically, in that section we present some special orders (over objects and sets of objects) that will be used throughout the paper. We give the formal concept of epistemic spaces, epistemic states and illustrate these with some examples. We also present the concept of merging operators of epistemic states as well as the social postulates and the main impossibility theorem stated in [56]. This impossibility theorem is a basic tool for obtaining the new results of this work. We also introduce some concrete examples of operators and establish their properties. These operators are used to illustrate our concepts and results throughout the paper. Section 3 is the core of this work. Therein, we present the notion of belief lifting, other new social postulates and we establish the main results of this paper about nonmanipulation. Section 4 contains some results involving the powerful agents mentioned before. To do so, we present there a taxonomic survey of the powerful agents. Section 5 contains some links with related works. Finally, we make some concluding remarks in Section 6. Appendix A is devoted to the proofs of our results.

2. Preliminaries

In this section we introduce succinctly the fundamental concepts and notation which is necessary for developing the main ideas of this work. First, we present some basic notions concerning *preorders* and *liftings*. These notions will allow the establishment of individual preferences between several pieces of information, in a qualitative way. Later on, we present the concept of epistemic spaces, epistemic states and then we introduce the notion of *epistemic state merging operators* (*ES merging operators* for short) and recall some basic representation theorems. We define some concrete operators and analyse their properties. We give some properties related to their Arrovian behaviour. Based on these properties, we present an Arrovian impossibility result which will be very useful to establish our results about nonmanipulability.

2.1. Orders and liftings

A *preorder* over a set A is a binary relation \succeq over A which is reflexive and transitive. Its associated strict relation, \succ , and indifference relation, \simeq , are given respectively as follows: $x \succ y$ iff $x \succeq y$ and $y \not\succeq x$; $x \simeq y$ iff $x \succeq y$ and $y \succeq x$.

A *total preorder* over a set A is a preorder which is total. Given a subset X of A , we say that x in X is a maximal element of X (resp. minimal element of X), with respect to a total preorder \succeq , if $x \succeq y$ (resp. $y \succeq x$), for all y in X . The set of maximal elements (resp. minimal elements) of X , with respect to \succeq , will be denoted by $\max(X, \succeq)$ (resp. $\min(X, \succeq)$). We will write $\max(\succeq)$ and $\min(\succeq)$, instead of $\max(A, \succeq)$ and $\min(A, \succeq)$, to respectively denote the set of maximal and minimal elements of the whole set A , with respect to the total preorder \succeq . From now on, $\succeq|_X$ denotes the restriction of \succeq to the set X and $P(A)$ denotes the set of all total preorders over a set A .

Two simple instances of total preorders are linear orders and the *indifference order* (also called the *flat order* and denoted \simeq by abuse of notation), for which all the elements are indifferent. Another instance of a total preorder, which is often used in this paper, is the *lexicographical combination* of two total preorders, \succeq_1 and \succeq_2 , denoted $\succeq^{\text{lex}(\succeq_1, \succeq_2)}$:

$$x \succeq^{\text{lex}(\succeq_1, \succeq_2)} y \text{ iff } \begin{cases} x \succ_1 y, & \text{or} \\ x \simeq_1 y & \text{and } x \succeq_2 y \end{cases}$$

Observation 1. *It is easy to see that, for any pair of total preorders \succeq_1, \succeq_2 , over a set A , the relation $\succeq^{\text{lex}(\succeq_1, \succeq_2)}$ is also a total preorder over A , for which the following holds:*

$$\max(\succeq^{\text{lex}(\succeq_1, \succeq_2)}) = \max(\max(\succeq_1), \succeq_2) \quad (1)$$

Moreover, if either \succeq_1 or \succeq_2 is a linear order over A , then $\succeq^{\text{lex}(\succeq_1, \succeq_2)}$ is also a linear order over A . In addition, if \succeq_2 (resp. \succeq_1) is the flat order, or $\succeq_1 = \succeq_2$, then $\succeq^{\text{lex}(\succeq_1, \succeq_2)} = \succeq_1$ (resp. $\succeq^{\text{lex}(\succeq_1, \succeq_2)} = \succeq_2$).

A more complex example of a total preorder is the *precise-leximax order* introduced by Leal and Pino Pérez [49]. It is defined over finite ordered tuples of elements of a finite set A . More precisely, given a nonempty set A with n elements, and a total preorder \succeq over A , we consider the set $(A)_\succeq$ formed by all the tuples of size less or equal to n , whose inputs are not repeated elements of A , ordered in decreasing manner by \succeq . Thus, given a total preorder \succeq over A , we define the *precise-leximax order* $\succeq_\succeq^{\text{plm}}$ as follows, for every pair of tuples $\vec{x} = (x_1, x_2, \dots, x_k)$ and $\vec{y} = (y_1, y_2, \dots, y_m)$ in $(A)_\succeq$:

$$\vec{x} \succeq_\succeq^{\text{plm}} \vec{y} \text{ iff } \begin{cases} k \leq m \text{ and } x_i \simeq y_i, \text{ for all } i \leq k, \text{ or} \\ \text{there is } j \leq \min\{k, m\} \text{ s.t. } x_i \simeq y_i, \text{ for all } i < j, \text{ and } x_j \succ y_j \end{cases}$$

A precise-leximax order is actually a total preorder that discriminates the ordered chains by considering the lexicographical order and privileging proper initial segments.

As usual, we think of a preorder as a *preference* relation over the elements of a set: $x \succeq y$ expresses that x is *at least as preferred as* y , while $x \succ y$ tell us that x is *more preferred than* y , and $x \simeq y$ says that x and y are *indifferent*.

It is possible to extend preferences over single elements of a set A to preferences over its subsets in different rational manners. This might be done either quantitatively or qualitatively. A quantitative manner to perform such extensions is related to probability: if p is a probability measure over a finite nonempty set A , we can think of p as representing the preferences over A as follows: $x \succeq y$ iff $p(\{x\}) \geq p(\{y\})$. As p extends additively the preferences of points in A to subsets (events) of A , we can define the probabilistic relation \sqsupseteq^p over $\mathcal{P}(A)$ (the powerset of A) as follows: $X \sqsupseteq^p Y$ iff $p(X) \geq p(Y)$. As far as we have checked, this is historically the first method used to achieve such extensions.

The idea of using qualitative methods for extending preferences over points to preferences over sets of points goes back to de Finetti [28]. This idea has also been considered in logical frameworks by Halpern [36] and van Benthem et al. [12]. In this paper, these methods are called *liftings over sets*.

More precisely, a *lifting over sets* is an application $\succeq \mapsto \sqsupseteq_{\succeq}$ that maps any total preorder \succeq over a set A into a preorder \sqsupseteq_{\succeq} over $\mathcal{P}(A)$, which satisfies the following:

$$\{x\} \sqsupseteq_{\succeq} \{y\} \text{ iff } x \succeq y \quad (2)$$

From now on, given a total preorder \succeq , the preorder \sqsupseteq_{\succeq} is also called *lifting* of \succeq , by abuse of notation.

In the literature, several specific well known liftings have been stated. The *possibilistic lifting* and the *pessimistic lifting* are two very standard instances of liftings. The *possibilistic lifting*, $\succeq \mapsto \sqsupseteq_{\succeq}^{\Pi}$, was introduced by Shackle [67] in a preliminary form, and proposed in different forms by Lewis [51], Spohn [68], Dubois et al. [21] and Friedman and Halpern [30]. In this paper, it is defined as follows:

$$X \sqsupseteq_{\succeq}^{\Pi} Y \text{ iff } Y = \emptyset, \text{ or there exists } x \text{ in } X \text{ such that, for all } y \text{ in } Y; x \succeq y$$

For a preference relation \succeq , the preorder $\sqsupseteq_{\succeq}^{\Pi}$ can be seen as a “comparative preference” relation associated with \succeq , as Dubois et al. [21] highlighted. Actually, $X \sqsupseteq_{\succeq}^{\Pi} Y$ expresses that the “best” elements in X are at least as good as the “best” elements in Y , with respect to \succeq .

The *pessimistic lifting*, $\succeq \mapsto \sqsupseteq_{\succeq}^{\text{W}}$, has a similar behavior to the possibilistic lifting, but in pessimistic terms. This lifting has been addressed in social choice by Barberà et al. [5] and Taylor [70, 71], and in logical frameworks by Lang and van der Torre [48]. It is defined as follows:

$$X \sqsupseteq_{\succeq}^{\text{W}} Y \text{ iff } Y = \emptyset, \text{ otherwise } X \neq \emptyset \text{ and there exists } y \text{ in } Y \text{ such that, for all } x \text{ in } X; x \succeq y$$

$X \sqsupseteq_{\succeq}^{\text{W}} Y$ expresses that the “worst” elements in X are at least as good as the “worst” elements in Y .

Another classical lifting is the so-called *Kelly lifting*, $\succeq \mapsto \sqsupseteq_{\succeq}^{\text{K}}$, which was introduced by Kelly [40]. It says that the “worst” elements in a set X are at least as good as the “best” elements in a set Y . More precisely:

$$X \sqsupseteq_{\succeq}^{\text{K}} Y \text{ iff } X = Y \text{ or, for all } x \text{ in } X \text{ and for all } y \text{ in } Y, x \succeq y$$

The following is the so-called *precise-leximax lifting*, denoted by $\sqsupseteq_{\succeq}^{\text{plm}}$. This lifting was proposed by Leal and Pino Pérez [49] as a variant of the leximax lifting [5, 13, 14, 15]. To define it, given a total preorder over a set A , we use the precise-leximax order over $(A)_{\succeq}$ and, for every subset Z of A , we consider the subset $[Z]_{\succeq}$ of $(A)_{\succeq}$, formed by all the vectors of length $|Z|$ whose inputs are in Z .

$$X \sqsupseteq_{\succeq}^{\text{plm}} Y \text{ iff for all } \vec{y} \text{ in } [Y]_{\succeq} \text{ there exists } \vec{x} \text{ in } [X]_{\succeq} \text{ such that } \vec{x} \succeq_{\succeq}^{\text{plm}} \vec{y}$$

This lifting tries to capture the idea that one has to prefer a group of alternatives which is smaller than another group of alternatives if the alternatives of the first group are at least as preferred as the alternatives of the second group. In order to give a real example of this kind of preference, we can imagine that alternatives are workers and for economical reasons (e.g. economy of salaries) a corporation prefers a team of workers with the same degree of skills than another one if the former has fewer persons.

Barberà et al. [5] characterized many natural liftings through their properties. Among them, we find a pair of very basic properties, which were firstly stated by Gärdenfors [32]. These properties are the so-called *dominance conditions* or *Gärdenfors properties*.

G1: $\{x, y\} \sqsupseteq_{\succeq} \{y\}$, whenever $x \succ y$.

G2: $\{x\} \sqsupseteq_{\succeq} \{x, y\}$, whenever $x \succ y$.

The Gärdenfors properties have a very natural interpretation: **G1** expresses that “good company” improves the group; while **G2** says “bad company” worsens the group. These properties have been also considered in the computational social choice framework by Geist and Endriss [34], whose work concerns the automatized search of impossibility theorems.

It is not hard to see that the Kelly lifting and the precise-leximax lifting satisfy both instances of the Gärdenfors properties, unlike the possibilistic lifting, which satisfies **G1**, but **G2** does not hold, and the pessimistic lifting, for which **G2** holds but **G1** does not. Other two liftings that also satisfy both instances of Gärdenfors properties are *Fishburn lifting* [29] and *Gärdenfors lifting* [32] (see [5] for more examples).

2.2. Epistemic States

In knowledge dynamics (revision, contraction, update, merging, etc.) there are many representations of the state of information of the agents. For instance, in the AGM framework for revision [1, 33], the epistemic states are logical theories (*i.e.*, logically closed sets of formulas), also known as belief sets. In the Katsuno and Mendelzon’s framework for revision [43], the epistemic states are simple propositional formulas. This is also the case in the belief merging framework of Konieczny and Pino Pérez [43]. In Hansson’s framework for revision [37], the epistemic states are sets of propositional formulas, not necessarily logically closed, also known as belief bases. In the Darwiche and Pearl [17] framework for revision, the epistemic states can be mainly thought as total preorders over interpretations, even if their framework is really much more general (see for instance [2, 65]). Another important representation of epistemic states in revision is through the ordinal conditional functions introduced by Spohn [68]. Conditionals are also a very useful representation of epistemic states (see for instance Beierle and Kern-Isberner [8]). Possibilistic bases are also a representation of epistemic states which are very useful in the treatment of conditionals as shown by Benferhat et al. [9] and in fusion by Benferhat and Kaci [10].

The previous list, which is far from being exhaustive, gives an idea of the diversity of representations of epistemic states. However, there are some features which these different representations have in common: they all have some distinguished logical information attached (explicit or implicit) and an ordering relation over the worlds (which can be also explicit or implicit). In what follows we will try to capture in an abstract concept (*epistemic state*) the most representative features of these structures.

One of the ingredients of our abstract representation is a finite propositional language. More precisely, let us consider $\mathcal{L}_{\mathcal{P}}$, a set of non contradictory formulas built from a given finite set of atomic propositions \mathcal{P} , and assume that $\mathcal{W}_{\mathcal{P}}$ is its associated set of interpretations (models). If φ is a formula in $\mathcal{L}_{\mathcal{P}}$, we denote by $\llbracket \varphi \rrbracket$ the set of its models, that is, $\llbracket \varphi \rrbracket = \{w \in \mathcal{W}_{\mathcal{P}} : w \models \varphi\}$. If M is a nonempty subset of $\mathcal{W}_{\mathcal{P}}$, φ_M denotes a formula whose set of models is exactly M . For this case, we write φ_w and $\varphi_{w,w'}$ instead of $\varphi_{\{w\}}$ and $\varphi_{\{w,w'\}}$, respectively. We say that a formula φ in $\mathcal{L}_{\mathcal{P}}$ is complete if it has a unique model.

We are now going to introduce the notions of epistemic spaces and epistemic states. But before doing that, let us say that one of the main interests of considering these abstract concepts is similar to that of considering an abstract notion like that of vector space (in linear algebra): proving general properties about vector spaces (the general concept) keeps us from proving the same properties in particular cases, such as the space \mathbb{R}^n , the space of polynomials, the space of functions, etc. We have also to say that in a similar way in which the space \mathbb{R}^n is considered a paradigm of vector spaces, we will have a paradigmatic structure for the concept of epistemic space. Actually, we have already mentioned this structure: a total preorder over interpretations.

An *epistemic space* is a triple $(\mathcal{E}, B, \mathcal{L}_{\mathcal{P}})$, where \mathcal{E} is a nonempty set, whose elements are called *epistemic states*, $\mathcal{L}_{\mathcal{P}}$ is a set of non contradictory formulas and B is a surjection from \mathcal{E} into $\mathcal{L}_{\mathcal{P}}$. For every E in \mathcal{E} , $B(E)$ is interpreted as the *belief base* or *the most entrenched beliefs* in E .² This abstract concept of epistemic space was stated by Benferhat et al. [11] and, indeed, most of the concrete models proposed for representing epistemic states are instantiations of it: belief bases [39, 43], logical theory [1] and, of course, total preorders over $\mathcal{W}_{\mathcal{P}}$ [17]. Later, in Section 2.4 and in Examples 1 and 3 revisited (cf. Section 3), we adopt this last concrete and well known model as our paradigm to define some specific ES merging operators and to illustrate our general results. Moreover, this concrete model is rich enough to represent real preferences between alternatives (see [56]).

Let us now illustrate the previous concepts.

Example 4 (Some epistemic spaces).

1. The tpo epistemic space \mathfrak{S}_{tpo} defined by $\mathfrak{S}_{tpo} = (\mathcal{E}_{tpo}, B_{tpo}, \mathcal{L}_{\mathcal{P}})$ where:
 - \mathcal{E}_{tpo} is the set of all total preorders over the set of all worlds over \mathcal{P} ;

²A *belief base* is a finite nonempty set of formulas in $\mathcal{L}_{\mathcal{P}}$, which encodes the set of propositions believed by an agent at some point in time [39, 43]. Since our work is in the context of finite propositional formulas, any belief base can be identified by an equivalent formula, if syntactic representations are not relevant.

- B_{tpo} is the mapping associating each total preorder \succeq from \mathcal{E}_{tpo} with a formula $\psi \in \mathcal{L}_{\mathcal{P}}$ such that $\llbracket \psi \rrbracket = \max(\succeq)$.

This will be our paradigmatic example. In this case the epistemic states are total preorders over interpretations.

2. The epistemic space of formulas $\mathfrak{S}_{\mathcal{L}}$, defined by $\mathfrak{S}_{\mathcal{L}} = (\mathcal{E}_{\mathcal{L}}, B_{\mathcal{L}}, \mathcal{L}_{\mathcal{P}})$, with $\mathcal{E}_{\mathcal{L}} = \mathcal{L}_{\mathcal{P}}$ and $B_{\mathcal{L}}$ the identity function.
3. The ocf epistemic space \mathfrak{S}_{ocf} defined by $\mathfrak{S}_{ocf} = (\mathcal{E}_{ocf}, B_{ocf}, \mathcal{L}_{\mathcal{P}})$ where:
 - An ocf κ is a function associating each world with a non-negative integer³ such that there is a world ω such that $\kappa(\omega) = 0$.
 - \mathcal{E}_{ocf} is the set of all ocf's over $\mathcal{W}_{\mathcal{P}}$;
 - B_{ocf} is the mapping associating each ocf κ from \mathcal{E}_{ocf} with a formula ψ such that $\llbracket \psi \rrbracket = \{w : \kappa(w) = 0\}$.

2.3. Merging Complex Epistemic States

Having introduced the notion of epistemic states, we give the concept of *Epistemic State merging operators*. To do so, we first state the notions of *agents*, *epistemic profiles* and the notation that will be used throughout this work.

Let us now consider a well ordered set $(\mathcal{S}, <)$, whose elements are called *agents*.⁴ A *finite society* (or simply a *society*) is a finite and nonempty subset $N = \{i_1, i_2, \dots, i_n\}$ of \mathcal{S} , whose elements, we assume are ordered in an increasing fashion, that is, $i_j < i_k$ if $j < k$. A *partition* of a finite society N is a finite family $\{N_1, \dots, N_k\}$ of pairwise disjoint societies whose union is N . $\mathcal{F}^*(\mathcal{S})$ denotes the set of all the finite societies.

In order to introduce the concept of *epistemic profiles*, we fix an epistemic space $(\mathcal{E}, B, \mathcal{L}_{\mathcal{P}})$ and a set of agents $(\mathcal{S}, <)$. Given a society N of agents in \mathcal{S} , an *N-profile of epistemic states* (also called *epistemic profile* or *N-profile* for short) is a tuple $\Phi = (E_{i_1}, \dots, E_{i_n})$ ordered increasingly by the elements in N . Sometimes we will write $\Phi(i)$ in order to represent the epistemic state E_i in Φ , and we think of it as the epistemic state of the agent i , for every i in N . Thus, an *N-profile* Φ collects in ordered way the information expressed by those agents in N . $E_i \in \Phi$ will denote that E_i is an epistemic state in Φ and we will write $\bigwedge_{E_i \in \Phi} B(E_i)$ to denote the conjunction of all the entrenched beliefs $B(E_i)$ of the epistemic states in Φ . If N is a singleton, suppose $N = \{i\}$, by abuse of notation, E_i will denote the *N-profile* (E_i) , and we will call it *single-profile*. The set of all the epistemic profiles is denoted $P(\mathcal{S}, \mathcal{E})$.

In order to present some special notations for epistemic profiles, consider the societies $N = \{i_1, \dots, i_n\}$ and $M = \{j_1, \dots, j_m\}$, the *N-profile* $\Phi = (E_{i_1}, \dots, E_{i_n})$ and the *M-profile* $\Phi' = (E'_{j_1}, \dots, E'_{j_m})$.

We say that Φ and Φ' are equivalent, denoted $\Phi \equiv \Phi'$, if $n = m$ and $E_{i_k} = E'_{j_k}$, for $k = 1, \dots, n$. We must note that, although the notion of equivalence of epistemic profiles suggests that, as tuples, Φ and Φ' must be equals, as profiles they could be different. This is due to the fact that every epistemic profile is indeed a function whose domain is a society of agents. Thus, every epistemic profile depends on the society of agents for which it is defined (cf. [56] for more details). Note that, given two single-profiles E_i and E_j , we have $E_i = E_j$ (respectively $E_i \neq E_j$) if and only if $E_i \equiv E_j$ (respectively $E_i \not\equiv E_j$).

If N and M are disjoint, we define a new $(N \cup M)$ -profile, $\Phi \sqcup \Phi'$, as follows: $(\Phi \sqcup \Phi')(i)$ is $\Phi(i)$ if $i \in N$, otherwise it is $\Phi'(i)$. If $M \subseteq N$, Φ_M denotes the *M-profile* obtained by the restriction of Φ to M .

From now on, Φ and Φ' will denote the profiles $(E_{i_1}, \dots, E_{i_n})$ and $(E'_{i_1}, \dots, E'_{i_n})$, respectively. In addition, $\Phi[E^*_i]$ will denote the profile obtained from Φ by replacing E_i with E^* . More precisely, $\Phi[E^*_i](j)$ is $\Phi(j)$ if $j \neq i$, otherwise it is E^* .

Now we are able to present the notion of belief merging in the complex context of epistemic states. An *Epistemic States combination operator* (*ES combination operator* for short) is a function of the form

³Note that in the original definition [68] *ocfs* are defined on ordinals. But here, as in most cases, the integer restriction is enough.

⁴A tuple $(\mathcal{S}, <)$ is a *well ordered set* if \mathcal{S} is a nonempty set, $<$ is a linear order over \mathcal{S} and every nonempty subset of \mathcal{S} has a minimal element. The set of natural numbers, \mathbb{N} , with its usual order, is an example of a well ordered set.

$\nabla : \mathcal{P}(\mathcal{S}, \mathcal{E}) \times \mathcal{E} \longrightarrow \mathcal{E}$, where $\nabla(\Phi, E)$ is the result of combining the epistemic states in Φ under an *integrity constraint* E . There is a set of postulates, which is listed below, that rules the rationality of ES combination operators. Such postulates were proposed and widely studied in [55, 56] in the context of epistemic states. Most of them are adapted from the IC merging postulates, proposed by Konieczny and Pino Pérez [43]. In order to present such postulates, let N and M be any pair of finite societies of agents in \mathcal{S} ; j, k be any pair of agents in \mathcal{S} ; $\{N_1, N_2\}$ be any partition of N ; Φ be any N -profile; Φ' be any M -profile; E_j be any j -profile; E_k be any k -profile; and E, E', E'' be any triple of epistemic states in \mathcal{E} .

(ESF1) $B(\nabla(\Phi, E)) \vdash B(E)$.

(ESF2) If $\Phi \equiv \Phi'$ and $B(E) \equiv B(E')$, then $B(\nabla(\Phi, E)) \equiv B(\nabla(\Phi', E'))$.

(ESF3) If $B(E) \equiv B(E') \wedge B(E'')$, then $B(\nabla(\Phi, E')) \wedge B(E'') \vdash B(\nabla(\Phi, E))$.

(ESF4) If $B(E) \equiv B(E') \wedge B(E'')$ and $B(\nabla(\Phi, E')) \wedge B(E'') \not\vdash \perp$, then $B(\nabla(\Phi, E)) \vdash B(\nabla(\Phi, E')) \wedge B(E'')$.

(ESF5) If $E_j \neq E_k$, then there exists E^* in \mathcal{E} such that $B(\nabla(E_j, E^*)) \not\equiv B(\nabla(E_k, E^*))$.

(ESF6) If $\bigwedge_{E_i \in \Phi} B(E_i) \wedge B(E) \not\vdash \perp$, then $B(\nabla(\Phi, E)) \equiv \bigwedge_{E_i \in \Phi} B(E_i) \wedge B(E)$.

(ESF7) $B(\nabla(\Phi_{N_1}, E)) \wedge B(\nabla(\Phi_{N_2}, E)) \vdash B(\nabla(\Phi, E))$.

(ESF8) If $B(\nabla(\Phi_{N_1}, E)) \wedge B(\nabla(\Phi_{N_2}, E)) \not\vdash \perp$, then $B(\nabla(\Phi, E)) \vdash B(\nabla(\Phi_{N_1}, E)) \wedge B(\nabla(\Phi_{N_2}, E))$.

(ESF8W) If $B(\nabla(\Phi_{N_1}, E)) \wedge B(\nabla(\Phi_{N_2}, E)) \not\vdash \perp$, then $B(\nabla(\Phi, E)) \vdash B(\nabla(\Phi_{N_1}, E)) \vee B(\nabla(\Phi_{N_2}, E))$.

We must note that this enumeration, which differs from the classical organisation of IC merging postulates [43] and AGM postulates [1, 33], was previously proposed in [55, 56]. This is due to the fact that this organisation provides a simpler classification of the ES combination operator and an easy handling of the merging postulates via the semantics, as we can see later in Theorems 1 and 2, and Proposition 1.

The postulates **(ESF1)**-**(ESF4)** are jointly called the *basic ESF postulates*. **(ESF1)** assures that, at the level of the most entrenched beliefs, the output of the merging process must satisfy the restriction of the system. **(ESF2)** is a weak form of the anonymity at the profile level and a syntax irrelevance property at the level of beliefs for the integrity restrictions. **(ESF3)** and **(ESF4)** together determine the manner in which the beliefs are chosen. These are the minimal rationality requirements that any ES combination operator must satisfy in order to admit a preferences-based semantic representation (cf. Theorem 1), characterizing thus an important class of ES combination operators: the *basic ESF operators*.

Definition 1. An ES combination operator is said to be an *epistemic states basic merging operator* (ES basic merging operator for short) if it satisfies **(ESF1)**-**(ESF4)**.

The postulates **(ESF5)**-**(ESF8)** and **(ESF8W)** describe mainly the relationships between the results of merging as a whole society and the results of merging in its subsocieties: **(ESF5)** says that given two different epistemic states, there is a restriction E that leads to different results. **(ESF6)** expresses that, at the level of the most entrenched beliefs, if all the agents involved in a merging process have a consensus and it is consistent with the restriction of the system, such consensus coincides with the output of the merging process. **(ESF7)** assures that, for any partition of a group into two subgroups, the conjunction of the beliefs resulting from applying the operator to each subgroup are logically stronger than the beliefs resulting from applying the operator to the whole group. **(ESF8)** expresses that if we can divide a group into two subgroups such that the application of the operator to each subgroup leads to beliefs which are mutually consistent, then the conjunction of these beliefs will be the beliefs resulting from applying the operator to the whole group. **(ESF8W)** is an important weakened variant of **(ESF8)**. It says that, under the same assumptions of **(ESF8)**, the beliefs resulting from the merging process applied to the whole group must satisfy the disjunction of the beliefs obtained when applying the merging operator to each subgroup. These postulates allow introducing other two subclasses of ES combination operators:

Definition 2 (ES merging operators). Let ∇ be an ES combination operator.

- (i) ∇ is said to be an *ES merging operator* if it satisfies **(ESF1)**-**(ESF8)**.
- (ii) ∇ is said to be an *ES quasi-merging operator* if it satisfies **(ESF1)**-**(ESF7)** and **(ESF8W)**.

Thus, any ES merging operator is indeed an ES quasi-merging operator and any ES (quasi-)merging operator is actually an ES basic merging operator.

It is possible to study the semantic behaviour of an ES basic merging operator through an *assignment*. An *assignment* is a function $\Phi \mapsto \succeq_\Phi$ which maps each epistemic profile Φ into a total preorder \succeq_Φ over interpretations in $\mathcal{W}_\mathcal{P}$, satisfying that $\succeq_\Phi = \succeq_\Psi$, for every pair of equivalent epistemic profiles Φ and Ψ .

The intended meaning of an assignment is to encode in terms of preferences over interpretations the result of a merging process. This will become particularly clear when we see the statements of Theorems 1 and 2. The next list of properties aims to systematize the rational behaviour of these assignments. In order to present it, let N be any finite society of agents in \mathcal{S} ; j, k be any pair of agents in \mathcal{S} ; $\{N_1, N_2\}$ be any partition of N , Φ be any N -profile, E_j be any j -profile, E_k be any k -profile, and w, w' be any pair of interpretations in $\mathcal{W}_\mathcal{P}$.

1. If $E_j \neq E_k$, then $\succeq_{E_j} \neq \succeq_{E_k}$.
2. If $\bigwedge_{E_i \in \Phi} B(E_i) \not\vdash \perp$, then $\llbracket \bigwedge_{E_i \in \Phi} B(E_i) \rrbracket = \max(\succeq_\Phi)$.
3. If $w \succeq_{\Phi_{N_1}} w'$ and $w \succeq_{\Phi_{N_2}} w'$ then $w \succeq_\Phi w'$.
4. If $w \succeq_{\Phi_{N_1}} w'$ and $w \succ_{\Phi_{N_2}} w'$, then $w \succ_\Phi w'$.
- 4'. If $w \succ_{\Phi_{N_1}} w'$ and $w \succ_{\Phi_{N_2}} w'$, then $w \succ_\Phi w'$.

Property 1 expresses that different epistemic states lead to different total preorders (injectivity of the assignment restricted to single-profiles). Property 2 assures that, at the level of the most entrenched beliefs, the models of the conjunction of the epistemic states in the profile Φ (if they exist) are the maximal models of the total preorder associated to this profile. Property 3 states that, if one model w is at least as preferred as a model w' for one group, and the same occurs for a second group, then for the group resulting of the union of these groups, w will be at least as preferred as w' . Property 4 is similar to the previous one, except that if there is one strict preference for one of the subgroups, this will be the case for the whole group. Property 4' is weaker than 4. It states that, given two alternatives, w, w' , if for two subgroups, separately w is more preferred than w' , then, putting the groups together, w still is more preferred than w' .

Two important classes of well behaved assignments are the *faithful assignments* and the *quasi-faithful assignments*. A *faithful assignment* is an assignment for which 1-4 hold, while a *quasi-faithful assignment* is an assignment that satisfies 1-3 and 4'.

Observation 2. *If E_i is the epistemic state of an agent i , then \succeq_{E_i} is a representation in terms of preferences over interpretations of the information conveyed by the agent.⁵ Indeed, Property 2 entails that the most preferred models represent the most entrenched beliefs of any agent, that is, the next equality holds, for every single-profile E : $\llbracket B(E) \rrbracket = \max(\succeq_E)$. We call this equality the maximality condition (the max condition for short).*

The converse of this result does not hold, that is, there are some instances of assignments satisfying the max condition for which property 2 does not hold. However, in presence of properties 3 and 4, we get that property 2 and the max condition are equivalent (see [56] for details).

Now we are able to present some results that describe the semantic behaviour of the ES basic merging operators at the level of the belief bases. These preference-based semantic results were proved in [56].

⁵For some concrete instantiations of epistemic states, for instance in terms of total preorders, \succeq_{E_i} will coincide with the agent's preferences over the worlds (see the *structure preserving* condition in Section 2.4).

Theorem 1 (Weak representation of ES basic merging operators). *An ES combination operator ∇ is an ES basic merging operator iff there exists a unique assignment, $\Phi \mapsto \succeq_\Phi$, such that:*⁶

$$\llbracket B(\nabla(\Phi, E)) \rrbracket = \max(\llbracket B(E) \rrbracket, \succeq_\Phi) \quad (\text{B-Rep})$$

Theorem 1 may be applied to different instantiations of epistemic states. Moreover, although this result does not explicitly mention the additional information of epistemic states, it allows the possibility of having postulates which capture such information. This is the case of the *standard domain postulate*, which is presented in Subsection 2.5, and its new strong version, introduced in Subsection 3.2. Using this result it is also possible to show the tight relationship between the (syntactic) postulates for ES merging operators and the (semantic) properties for assignments. This can be seen in the next result.

Proposition 1. *The following assertions hold, for every ES basic merging operator ∇ and its associated assignment $\Phi \mapsto \succeq_\Phi$:*

- (i) ∇ satisfies **(ESF5)** iff $\Phi \mapsto \succeq_\Phi$ satisfies property **1**.
- (ii) ∇ satisfies **(ESF6)** iff $\Phi \mapsto \succeq_\Phi$ satisfies property **2**.
- (iii) ∇ satisfies **(ESF7)** iff $\Phi \mapsto \succeq_\Phi$ satisfies property **3**.
- (iv) ∇ satisfies **(ESF8)** iff $\Phi \mapsto \succeq_\Phi$ satisfies property **4**.
- (v) ∇ satisfies **(ESF8W)** iff $\Phi \mapsto \succeq_\Phi$ satisfies property **4'**.

Due to the one-to-one relationship between ES basic merging operators and assignments, shown in Theorem 1 (at least at the level of the entrenched beliefs), and the duality between their properties (exposed in Proposition 1) we can associate the max condition to ES basic merging operators. Thus, from now on we say that an ES basic merging operator satisfies the max condition if its associated assignment satisfies it.

As a straightforward consequence of Theorem 1 and Proposition 1, we get the next representation result:

Theorem 2. *The following assertions hold for every ES basic merging operator ∇ , and its associated assignment, $\Phi \mapsto \succeq_\Phi$:*

- (i) ∇ is an ES merging operator iff $\Phi \mapsto \succeq_\Phi$ is a faithful assignment
- (ii) ∇ is an ES quasi-merging operator iff $\Phi \mapsto \succeq_\Phi$ is a quasi-faithful assignment

2.4. Some concrete merging operators

In this section we present some concrete examples of ES basic merging operators that will be used throughout this paper in order to illustrate the properties and results presented here. In order to introduce them, from now on we assume in this section that the epistemic space is $\mathfrak{S}_{tpo} = (\mathcal{E}_{tpo}, B_{tpo}, \mathcal{L}_{\mathcal{P}})$ given in Example 4 which for simplicity of notation we call $(\mathcal{E}, B, \mathcal{L}_{\mathcal{P}})$.

One important feature of this particular information representation framework is the fact that the whole structure of an agent's epistemic state can be captured by means of assignments. More precisely, we say that an assignment $\Phi \mapsto \succeq_\Phi$ is *structure preserving* if, for every single-profile \succeq^* , we have $\succeq_{\succeq^*} = \succeq^*$.

Note that the structure preserving assignments are really able to encode, in a faithful manner, the preferences of the agent. This supports in a strong way the claim made in Observation 2: in such a case \succeq_{E_i} is (not only encode) the preferences of the agent i .

Another remarkable feature of this framework is the fact that, given an assignment $\Phi \mapsto \succeq_\Phi$, we can easily build an ES combination operator by means of the lexicographical combination of two total preorders:

$$\nabla(\Phi, \succeq) = \succeq^{\text{lex}(\succeq, \succeq_\Phi)} \quad (3)$$

From the equalities (1) and (B-Rep), and the definition of the belief function, we straightforwardly get that an operator built in this way is indeed a concrete instance of a ES basic merging operator.

In what follows, we will make use of this method in order to build our concrete examples of ES merging operators. For simplicity, those operators and their associated assignment will be called in the same manner.

⁶From now on, such an assignment, $\Phi \mapsto \succeq_\Phi$, is so-called the assignment associated to ∇ .

Observation 3. *It is worth noting that any structure preserving assignment satisfies the max condition and Property 1. Therefore, its associated ES basic merging operators satisfy the max condition and (ESF5).*

The first operator we introduce is the simplest ES combination operator that can be defined: the *indifference operator*. This operator maps each epistemic profile and each integrity constraint into the integrity constraint itself. More precisely:

Indifference operator: $\nabla^{\text{Ind}}(\Phi, \succeq) = \succeq$

The *indifference operator* is actually an ES basic merging operator. Indeed, if we consider the flat order over $\mathcal{W}_{\mathcal{P}}$, \simeq , from Observation 1 we obtain $\succeq^{\text{lex}(\succeq, \simeq)} = \succeq$. Hence, the *indifference assignment*, $\Phi \mapsto \succeq_{\Phi}^{\text{Ind}}$, is such that $\succeq_{\Phi}^{\text{Ind}} = \simeq$, for every epistemic profile Φ . Therefore, this assignment is not structure preserving.

Concerning the remaining merging postulates, we have the following result:

Proposition 2. *The indifference operator, ∇^{Ind} , is an ES basic merging operator for which (ESF7), (ESF8), (ESF8W) hold, but the max condition, (ESF5) and (ESF6) do not hold.*

The following is a variant of the indifference operator that we call *quasi-indifference operator*. For this operator, its associated assignment has the same behaviour as the indifference assignment, except for when it is restricted to single-profiles. In this case, it preserves the structure of the epistemic states.

Quasi-indifference operator: $\nabla^{\text{Ind}^*}(\Phi, \succeq) = \succeq^{\text{lex}(\succeq, \succeq_{\Phi}^{\text{Ind}^*})}$; where $\succeq_{\Phi}^{\text{Ind}^*} = \succeq_i$, if Φ has a unique input, namely, \succeq_i ; otherwise $\succeq_{\Phi}^{\text{Ind}^*} = \simeq$.

Since the quasi-indifference assignment is structure preserving, from Observation 3 it follows that the quasi-indifference assignment satisfies the max condition. The next result shows which properties the quasi-indifference operator satisfies.

Proposition 3. *The quasi-indifference operator, ∇^{Ind^*} , is an ES basic merging operator that satisfies the max condition, (ESF5) and (ESF7), but it does not satisfy neither (ESF6), (ESF8) nor (ESF8W).*

In what follows, we will consider a class of basic merging operators called *aggregation-based ES basic merging operators*. They are built from aggregation functions and they are classical in the study of belief merging, being widely studied in [42, 43, 44, 45, 46] in the context of belief bases, and then adapted to the ESF framework in [55, 56]. In order to introduce them, let us recall what an *aggregation function* is.

A (symmetric) *aggregation function* F is a total function which associates a nonnegative value to every finite tuple of nonnegative numbers, and for which the following holds for any tuple x_1, \dots, x_n, x, y :

- $F(x_1, \dots, x, \dots, x_n) \geq F(x_1, \dots, y, \dots, x_n)$, whenever $x \geq y$ [Monotony]
- $F(x_1, \dots, x_n) = 0$ if, and only if, $x_1 = \dots = x_n = 0$ [Minimality]
- $F(x) = x$ [Identity]
- For any permutation σ , $F(x_1, \dots, x_n) = F(\sigma(x_1, \dots, x_n))$ [Symmetry]

Well known examples of aggregation functions are the *sum* function, $\sum(x_1, \dots, x_n) = \sum x_i$; the *max* function, $\max(x_1, \dots, x_n) = \max\{x_1, \dots, x_n\}$; and the *Gmax* function, $Gmax(x_1, \dots, x_n) = (y_1, \dots, y_n)$, where (y_1, \dots, y_n) is the reordering of (x_1, \dots, x_n) decreasingly.⁷

Any aggregation function F induces a total preorder, \succeq_{Φ}^F over $\mathcal{W}_{\mathcal{P}}$, for every epistemic profile Φ . More precisely, given an epistemic profile $\Phi = (\succeq_{i_1}, \dots, \succeq_{i_n})$, we define the total preorder \succeq_{Φ}^F as follows:

$$w \succeq_{\Phi}^F w' \text{ iff } F(r_{\succeq_{i_1}}(w), \dots, r_{\succeq_{i_n}}(w)) \geq F(r_{\succeq_{i_1}}(w'), \dots, r_{\succeq_{i_n}}(w')) \quad (4)$$

⁷Despite the *Gmax* function is defined using lexicographic sequences, it can be represented by reals to fit the above definition (see [41] for details).

where, r_{\succeq_i} is the natural ranking function of \succeq_i .⁸

Thus, given an aggregation function F , from (4) we build the assignment $\Phi \mapsto \succeq_{\Phi}^F$ and, from (3), we build the ES basic merging operator ∇^F . This method allows to define three examples of aggregation-based ES basic merging operators: *sum*, *max* and *Gmax*, respectively denoted ∇^{Σ} , ∇^{\max} and ∇^{Gmax} .

From the identity property it straightforwardly follows that, given an aggregation function F , the assignment $\Phi \mapsto \succeq_{\Phi}^F$ is structure preserving. Therefore, these assignments also satisfy the max condition.

In addition to the basic properties of aggregation (monotony, minimality, identity and symmetry), there are other properties that hold for some natural aggregation functions. We list three of them below:

- If $F(x_1, \dots, x_n) \geq F(y_1, \dots, y_n)$, then $F(x_1, \dots, x_n, z) \geq F(y_1, \dots, y_n, z)$. [Composition]
- If $F(x_1, \dots, x_n, z) \geq F(y_1, \dots, y_n, z)$, then $F(x_1, \dots, x_n) \geq F(y_1, \dots, y_n)$ [Decomposition]
- For any triple of nonnegative integers, $x > y > z$; $F(x, z) \geq F(y, z + 1)$. [Discreteness]

The sum and Gmax functions are classic examples of aggregation functions which satisfy each of these properties, as can be easily shown. The max function is another classical instance of an aggregation function, satisfying composition and discreteness, but not decomposition, as it is easy to show.

The following is an example of an aggregation function satisfying composition and decomposition, for which discreteness does not hold: For any tuple $\vec{x} = (x_1, \dots, x_n)$ of nonnegative real values, $F(\vec{x}) = x$, if \vec{x} has a sole input, namely x ; otherwise $F(\vec{x}) = \sum_{i=1}^n \sqrt{x_i}$. It is quite clear that F is indeed an aggregation function for which composition and decomposition hold. To see that discreteness fails, consider $x = 2$, $y = 1$ and $z = 0$, and note that $F(x, z) = \sqrt{2}$ and $F(y, z + 1) = 2$. Thus, $x > y > z$, but $F(y, z + 1) > F(x, z)$.

Composition and decomposition have been previously addressed in the study of belief merging [41, 46] and, to our knowledge, discreteness is considered for the first time in this work. These properties are going to be useful for our study. Indeed, through composition and decomposition we state the next result, which gives a characterisation of those ES merging operators that are built from an aggregation function.

Proposition 4. *Given an aggregation F , the operator ∇^F is an ES merging operator iff, restricted to the tuples (x_1, \dots, x_n) of nonnegative integers with $|\mathcal{W}_{\mathcal{P}}| \geq x_i$, F satisfies composition and decomposition.*

A first result of this type was stated by Konieczny et al. [41]. In their work, they used composition and decomposition in order to characterise those IC merging operators that are built from a distance over interpretations and an aggregation function. We do the same, but in a more general setting: that in which a total preorder is not structured from a distance. Later, we are going to study another set of rational properties that this kind of operators satisfies.

From Proposition 4 it straightforwardly follows that the sum and Gmax operators are two instances of ES merging operators. For its part, the max operator is an instance of a quasi-merging operator for which **(ESF8)** does not hold, as it was proved in [56].

The last class of operators to present concerns those built from a projection: the *projection-based ES basic merging operators*. To do so, we denote the maximal element of any finite society N by d .

The *projective operator* is the first operator of this type to be addressed. For this operator, its associated assignment is defined by the projection of any epistemic profile over its last input. More precisely:

Projective operator: $\nabla^{\pi}(\Phi, \succeq) = \succeq^{\text{lex}(\succeq, \succeq_{\Phi}^{\pi})}$; where $\succeq_{\Phi}^{\pi} = \succeq_d$, for every profile Φ .

Hence, the output of a merging process, using the *projective operator*, totally depends on the epistemic states of the agent d and the integrity constraints.

Observation 4. *From its definition, it straightforwardly follows that the projective assignment is structure preserving and, therefore, the projective operator satisfies the max condition. Concerning the remaining merging postulates, this operator is an ES basic merging operator that satisfies the max condition, **(ESF5)**, **(ESF7)** and **(ESF8W)**, but it does not satisfy **(ESF6)** nor **(ESF8)** as it was shown in [56].*

⁸The natural ranking function of a preorder \succeq is given by $r_{\succeq}(x) = \max\{n \in \mathbb{N} : \exists x_0, \dots, x_n \text{ s.t. } x_{k+1} \succ_i x_k \text{ and } x_n = x\}$.

The last operator to be defined is the one we call Σ -projective operator.⁹ Its associated assignment is defined by the lexicographical combination of the projective assignment and the sum assignment.

Σ -projective operator: $\nabla^{\Sigma\pi}(\Phi, \succeq) = \succeq^{\text{lex}(\succeq, \succeq_{\Phi}^{\Sigma\pi})}$; where $\succeq_{\Phi}^{\Sigma\pi} = \succeq^{\text{lex}(\succeq_d, \succeq_{\Phi}^{\Sigma})}$.

The Σ -projective operator allows for weak participation of all the agents involved in a merging process: the most preferred models are determined by the agent d , and the remaining agents participate in the merging process if the agent d expresses indifference between two interpretations, breaking the tie by a majority.

Observation 5. *The Σ -projective assignment, $\Phi \mapsto \succeq_{\Phi}^{\Sigma\pi}$, is structure preserving. This is due to Observation 1 and the fact that the sum assignment is structure preserving. From this and Observation 3, we get that the Σ -projective operator satisfies the max condition. Concerning the remaining merging postulates, we have that this operator is an ES basic merging operator that satisfies (ESF5) and (ESF6), but it does not satisfy (ESF7), (ESF8) and (ESF8W), as it was also shown in [56].*

We now return to Example 1 in order to show how a rational merging operator, like the sum operator, works.

Example 5 (Example 1 revisited). *Anne, Bob and Charles try to determine who to blame in the case of divorce of their couple of friends Tom and Mary under the constraint that there is at least one of the members of the couple to blame. We will use a language built from the two propositional variables, m , t (considered in that order) to model this situation: m symbolises the sentence ‘Mary is to blame’ and t represents the sentence ‘Tom is to blame’. Thus, the interpretations are given by $w_1 = 00$, $w_2 = 01$, $w_3 = 10$, $w_4 = 11$.*

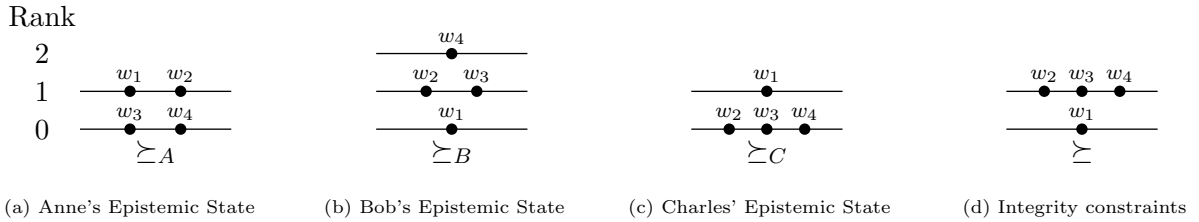


Figure 1: Representation of the individual epistemic states and the integrity constraints.

Anne believes that Mary is not to blame, regardless of whether Tom is to blame or not, and, to a lesser degree, she believes that the remaining options are possible. Thus, her epistemic state is encoded by \succeq_A , where $w_1 \simeq_A w_2 \succ_A w_3 \simeq_A w_4$ (cf. Figure 1a). Bob believes that both are to blame; to a lesser degree, he believes that only one is to blame and finally he doesn't believe at all that none is to blame. Thus, his epistemic state can be represented by \succeq_B , where $w_4 \succ_B w_2 \simeq_B w_3 \succ_B w_1$ (cf. Figure 1b). Charles believes that none is to blame and, to a lesser degree, he believes in any other option. So, Charles' epistemic state can be represented by \succeq_C : $w_1 \succ_C w_2 \simeq_C w_3 \simeq_C w_4$ (cf. Figure 1c). The integrity constraint, there is at least one to blame, can be represented by \succeq given in Figure 1d.

In this situation, the epistemic profile is given by $\Phi = (\succeq_A, \succeq_B, \succeq_C)$. Its aggregation, using the sum assignment, is calculated in Table 1, and the resulting epistemic state is represented in Figure 2. Recall that the sum assignment is defined as follows: $w \succeq_{\Phi}^{\Sigma} w'$ iff $\sum r_{\succeq_i}(w) \geq \sum r_{\succeq_i}(w')$.

⁹This operator is called Σ -pseudoprojective operator in [56], but we change its name and notation for simplicity.

| $\mathcal{W}_{\mathcal{P}}$ | $\text{rank}(\succeq_A)$ | $\text{rank}(\succeq_B)$ | $\text{rank}(\succeq_C)$ | $\text{rank}(\succeq_{\Phi}^{\Sigma})$ |
|-----------------------------|--------------------------|--------------------------|--------------------------|--|
| w_1 | 1 | 0 | 1 | 2 |
| w_2 | 1 | 1 | 0 | 2 |
| w_3 | 0 | 1 | 0 | 1 |
| w_4 | 0 | 2 | 0 | 2 |

Table 1: Calculation of the ranks with sum

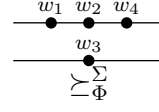


Figure 2: Preorder resulting from the aggregation process

The epistemic state $\nabla^{\Sigma}(\Phi, \succeq)$, resulting from the merging process with the sum operator, is represented in Figure 3. This is obtained from the lexicographical combination of the integrity constraints, \succeq , with the total preorder \succeq_{Φ}^{Σ} , that is: $\nabla^{\Sigma}(\Phi, \succeq) = \succeq^{\text{lex}}(\succeq, \succeq_{\Phi}^{\Sigma})$. The models of the entrenched beliefs in this epistemic state are those pointed out on the top of the obtained preorder.

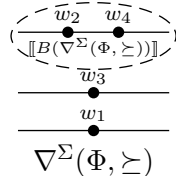


Figure 3: Epistemic states resulting from the merging processes with the sum operator.

Since w_2 and w_4 are the models of $B(\nabla^{\Sigma}(\Phi, \succeq))$, Anne, Bob and Charles should agree that the most consensual belief is that Tom is to blame, regardless of whether Mary is to blame or not. This is due to the fact that, among the available options, blaming Tom is the closest belief to what they really believe as a group. Moreover, to a lesser degree, they should agree in believing that Mary is to blame but not Tom, and finally the less believed fact is that none is to blame.

Let us now return to Example 2 in order to see more formally the role of complex constraints.

Example 6 (Example 2 revisited). Remember that Anne and Bob were discussing about replacing two electronic devices, x and y , of their coffee machine. For modelling this problem, we will use a language built from two variables, denoted x, y , by abuse of notation, considered in that order. The proposition x represents the sentence ‘ x should be replaced’ and proposition y symbolises ‘ y should be replaced’. The interpretations are given by $w_1 = 00$, $w_2 = 01$, $w_3 = 10$, $w_4 = 11$.

Anne believes that the most advantageous option is to replace both devices. To a lesser degree, she believes that the replacement of the two devices can wait a few weeks more, but she does not believe that replacing only one device is a good option. Thus, her epistemic state is encoded by \succeq_A , represented in Figure 4a. As Bob fully agrees with Anne, his epistemic state, encoded by \succeq_B , is the same as hers, i.e. $\succeq_B = \succeq_A$ (cf. Figure 4b).

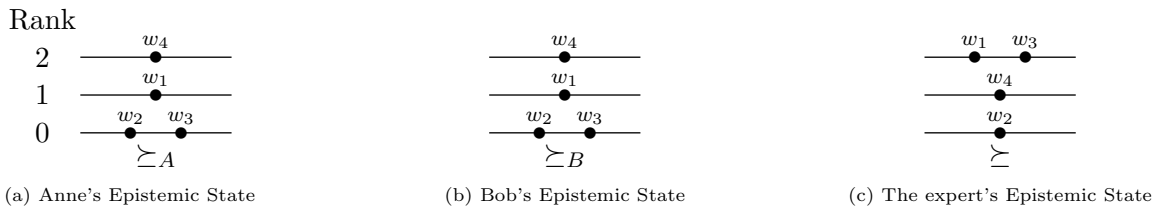


Figure 4: Representation of the individual epistemic states and the integrity constraints.

In this situation, the epistemic profile is given by $\Phi = (\succeq_A, \succeq_B)$. Its aggregation, using any rational assignment $\Phi \mapsto \succeq_{\Phi}$ (like sum, max and Gmax), will result in the epistemic state of Anne (the same as Bob), that is, $\succeq_{\Phi} = \succeq_A$. Thus, naturally, they should still agree on what they initially expressed.

However, an expert from the coffee machine's manufacturer advised them that is not necessary to replace device y , but if it is replaced then device x has to be replaced too. The worst option is to replace y and not to replace x . Thus, the expert's epistemic state is encoded by \succeq , given in Figure 4c. Due to the expert's knowledge, Anne and Bob should consider his advice in order to agree on what to do. In this way, the expert's epistemic state plays the role of integrity constraints in the merging process.

The epistemic state $\nabla(\Phi, \succeq)$, resulting from the merging process, is represented in Figure 5. This is obtained from the lexicographical combination of the integrity constraints, \succeq , with the total preorder \succeq_Φ , that is: $\nabla(\Phi, \succeq) = \succeq^{\text{lex}(\succeq, \succeq_\Phi)}$.

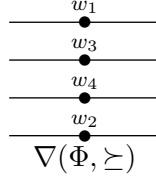


Figure 5: Epistemic state resulting from the merging processes.

Since w_1 is the model of $B(\nabla(\Phi, \succeq))$, Anne and Bob should agree to replace neither device x nor device y . Moreover, since $w_3 \succ_{\nabla(\Phi, \succeq)} w_4 \succ_{\nabla(\Phi, \succeq)} w_2$, then they should also agree on believing the following: if they decide to replace device x , then it is better not to replace device y than to replace it. In any case, they should furthermore agree on believing that to replace y but not x is the worst option.

2.5. Arrovian behaviour of ES merging operators

Now, we present some properties that capture the social choice principles appearing in the seminal work of Arrow [3]: *Standard domain*, *Pareto condition*, *independence of irrelevant alternatives* and *existence of a dictator*. These properties were earlier proposed in [56] in this setting of belief merging. In order to give a formulation of them in logical terms, from now on we suppose that \mathcal{P} has at least two propositional variables. Thus, there are at least four interpretations available in $\mathcal{W}_{\mathcal{P}}$.

(ESF-SD) For every agent i in \mathcal{S} and every triple w, w', w'' in $\mathcal{W}_{\mathcal{P}}$, the following conditions hold:

- (i) There exists E_i in \mathcal{E} such that $B(\nabla(E_i, E_{w,w'})) \equiv \varphi_{w,w'}$ and $B(\nabla(E_i, E_{w',w''})) \equiv \varphi_{w'}$,
- (ii) There exists E_i in \mathcal{E} such that $B(\nabla(E_i, E_{w,w'})) \equiv \varphi_w$ and $B(\nabla(E_i, E_{w',w''})) \equiv \varphi_{w',w''}$,
- (iii) There exists E_i in \mathcal{E} such that $B(\nabla(E_i, E_{w,w'})) \equiv \varphi_w$ and $B(\nabla(E_i, E_{w',w''})) \equiv \varphi_{w'}$,

where $E_{w,w'}$ and $E_{w',w''}$ in \mathcal{E} are such that $\llbracket B(E_{w,w'}) \rrbracket = \{w, w'\}$ and $\llbracket B(E_{w',w''}) \rrbracket = \{w', w''\}$.

(ESF-P) For all N in $\mathcal{F}^*(\mathcal{S})$, every N -profile Φ in $\mathcal{P}(\mathcal{S}, \mathcal{E})$ and every pair of epistemic states E, E' in \mathcal{E} , if $\bigwedge_{E_i \in \Phi} B(\nabla(E_i, E)) \not\vdash \perp$ and $B(\nabla(E_i, E)) \wedge B(E') \vdash \perp$, for all i in N , then $B(\nabla(\Phi, E)) \wedge B(E') \vdash \perp$.

(ESF-I) For every N in $\mathcal{F}^*(\mathcal{S})$, each pair of N -profiles Φ, Φ' and every E in \mathcal{E} , $B(\nabla(\Phi, E)) \equiv B(\nabla(\Phi', E))$, whenever $B(\nabla(E_i, E')) \equiv B(\nabla(E'_i, E'))$, for every i in N , and every E' in \mathcal{E} , with $B(E') \vdash B(E)$.

(ESF-D) For every N in $\mathcal{F}^*(\mathcal{S})$ there exists an agent d in N such that $B(\nabla(\Phi, E)) \vdash B(\nabla(E_d, E))$, for all N -profile Φ and every E in \mathcal{E} .

(ESF-SD) is the *standard domain* condition. It expresses that, given an integrity constraint whose beliefs have at most two models, an agent may express an epistemic state in order to obtain a resulting epistemic state such that the models of its beliefs could be any subset of the set of models, stating thus some “richness” in the results of a merging process. **(ESF-P)** is referred to as the *Pareto* condition. This property expresses that, at the level of the most entrenched beliefs, if all the agents reject a piece of information, and all the agents have a consensus, such information will be rejected by the result of merging. **(ESF-I)** is the *independence* condition. It captures the following principle: the merging process depends only on how the

restrictions in the individual epistemic states are related. This property essentially expresses that, given an integrity constraint, if each agent in a merging process has two possible choices of epistemic states, and if revising such epistemic states by integrity constraints having beliefs stronger than the original integrity constraint, the beliefs of the resulting epistemic states coincide, then the result of the merging process is the same, at the level of entrenched beliefs, regardless of the choice of the epistemic state made by each agent. It is worth mentioning that the independence condition was also addressed by Schwind et al. [64], characterizing the distance-based merging operators that satisfy it.¹⁰ **(ESF-D)** evokes the existence of a *dictator*, that is, an agent (an *Arrovian dictator*) that always imposes his will. This is the property that good operators should avoid. Operators satisfying this property are called *dictatorial operators*, while those for which it does not hold are called *non-dictatorial operators*.

Observation 6. *Any ES (quasi)merging operator satisfies the Pareto Condition. This is because **(ESF7)** and **(ESF8W)** entail **(ESF-P)**. Moreover, any dictatorial operator also satisfies the Pareto Condition, since **(ESF-D)** straightforwardly entails **(ESF-P)** (see [56] for details).*

Each aforementioned Arrovian property has a semantic counterpart, which is listed below. These properties will be useful in order to study the operators presented in Section 2.4.

(SD) For every total preorder \succeq over $\mathcal{W}_{\mathcal{P}}$ (except perhaps the flat order) and for every triple of interpretations w, w', w'' in $\mathcal{W}_{\mathcal{P}}$, there exists a single-profile E_i in $\mathbf{P}(\mathcal{S}, \mathcal{E})$ s.t. $\succeq_{E_i} \upharpoonright_{\{w, w', w''\}} = \succeq \upharpoonright_{\{w, w', w''\}}$

(P) If $w \succ_{E_i} w'$, for all i in N , then $w \succ_{\Phi} w'$

(I) If $\succeq_{E_i} \upharpoonright_{\{w, w'\}} = \succeq_{E'_i} \upharpoonright_{\{w, w'\}}$, for all i in N , then $\succeq_{\Phi} \upharpoonright_{\{w, w'\}} = \succeq_{\Phi'} \upharpoonright_{\{w, w'\}}$

(D) For every N in $\mathcal{F}^*(\mathcal{S})$ there exists d in N s.t. for every N -profile Φ , if $w \succ_{E_d} w'$ then $w \succ_{\Phi} w'$

The next result collects certain results in [56], which show that, modulo the basic merging postulates, the Arrovian properties and their semantic counterparts are indeed in a one-to-one relationship.

Proposition 5. *The following assertions hold for every ES basic merging operator ∇ and its associated assignment $\Phi \mapsto \succeq_{\Phi}$:*

- (i) ∇ satisfies **(ESF-SD)** iff $\Phi \mapsto \succeq_{\Phi}$ satisfies **(SD)**
- (ii) ∇ satisfies **(ESF-P)** iff $\Phi \mapsto \succeq_{\Phi}$ satisfies **(P)**
- (iii) ∇ satisfies **(ESF-I)** iff $\Phi \mapsto \succeq_{\Phi}$ satisfies **(I)**
- (iv) ∇ satisfies **(ESF-D)** iff $\Phi \mapsto \succeq_{\Phi}$ satisfies **(D)**

The next result shows which Arrovian properties are satisfied by each of those merging operators that were presented in Subsection 2.4. We should mention that the Arrovian behaviour of the sum, the max, the projective and the Σ -projective operators, which is pointed out in this result, was previously studied in [56]. Concerning the indifference operator, the quasi-indifference operator, and every ES aggregation based-merging operator in general, their Arrovian behaviour is addressed for the first time in this work.

Proposition 6. *In the context of the tpo epistemic space, \mathfrak{S}_{tpo} , the following assertions hold:*

- (i) *An ES basic merging operator satisfies **(ESF-SD)**, if its associated assignment is structure preserving. Therefore, **(ESF-SD)** holds for the quasi-indifference operator, ∇^{Ind^*} , the projective operator, ∇^{π} , the Σ -projective operator, $\nabla^{\Sigma\pi}$, and every aggregation-based ES basic merging operator; in particular, for the sum operator, ∇^{Σ} , the Gmax operator, ∇^{Gmax} , and the max operator, ∇^{max} . In contrast, **(ESF-SD)** does not hold for the indifference operator, ∇^{Ind} .*

¹⁰A distance-based merging operator is a merging operator defined using a distance over interpretations and an aggregation function (cf. [41, 43, 64]).

- (ii) **(ESF-P)** holds for the indifference operator, ∇^{Ind} , the max operator, ∇^{max} , the projective operator, ∇^π , the Σ -projective operator, $\nabla^{\Sigma\pi}$, and every aggregation-based ES basic merging operator, ∇^F , defined by means of an aggregation function, F , satisfying composition and decomposition; in particular, for the sum operator, ∇^Σ , and the Gmax operator, ∇^{Gmax} . In contrast, **(ESF-P)** does not hold for the quasi-indifference operator, ∇^{Ind^*} .
- (iii) **(ESF-I)** holds for the indifference operator, ∇^{Ind} , the quasi-indifference operator, ∇^{Ind^*} , and the projective operator, ∇^π . In contrast, **(ESF-I)** does not hold for the max operator, ∇^{max} , the Σ -projective operator, $\nabla^{\Sigma\pi}$, and any aggregation-based ES basic merging operator, ∇^F , defined by means of an aggregation function F satisfying composition, decomposition and discreteness; in particular, for the sum operator, ∇^Σ , and the Gmax operator, ∇^{Gmax} .
- (iv) **(ESF-D)** holds for the indifference operator, ∇^{Ind} , the projective operator, ∇^π , and the Σ -projective operator, $\nabla^{\Sigma\pi}$. In addition, every agent in a finite society N is a dictator for the indifference operator. In contrast, **(ESF-D)** does not hold for the max operator, ∇^{max} , the quasi-indifference operator, ∇^{Ind^*} , and any aggregation-based ES basic merging operator, ∇^F , defined by means of an aggregation function F satisfying composition, decomposition and discreteness; in particular, for the sum operator, ∇^Σ , and the Gmax operator, ∇^{Gmax} .

Now we present an Arrovian impossibility theorem, which is similar to that stated by Sen [66] for social choice functions. This result expresses that standard domain, the Pareto condition, independence and absence of dictators cannot be satisfied jointly. This theorem was proved in [56], and it will be very useful to state our manipulability results.

Theorem 3 (Main impossibility theorem). *Every ES basic merging operator that satisfies **(ESF-SD)**, **(ESF-P)** and **(ESF-I)** also satisfies **(ESF-D)**.*

(ESF-SD), **(ESF-P)** and **(ESF-I)** are only sufficient conditions for an ES basic merging operator to admit a dictator, as was pointed out in [56]. Indeed, as we saw in Proposition 6, there exist two instances of dictatorial ES basic merging operators for which either the standard domain condition or the independence condition does not hold: The first example is the Σ -projective operator that satisfies **(ESF-SD)**, **(ESF-P)** and **(ESF-D)**, but does not satisfy **(ESF-I)**. The other example is the indifference operator, which satisfies **(ESF-P)**, **(ESF-I)** and **(ESF-D)**, but does not satisfy **(ESF-SD)**. As a matter of fact, for the indifference operator, any agent involved in the merging process is an Arrovian dictator, as can be easily shown from its definition. However, as it is shown in the proof of Theorem 3 in [56], modulo the basic ESF postulates and the Arrovian properties (standard domain, the Pareto condition and independence), the Arrovian dictator is unique. We wonder if **(ESF-SD)** is the reason for such uniqueness.

3. On manipulability of ES merging operators

In this section we deal with another interesting issue which is present in the logical framework of belief merging, namely, *manipulability of merging information processes*. In general, surveys on manipulability address situations under which an agent (the *manipulator*) obtains a result which is more advantageous for him when he expresses an information that he does not truly believe. Some preliminary studies on manipulability have been carried out in the framework of logic-based belief merging by Everaere et al. [24, 25] and more recently by us [57, 59]. Continuing with these studies, in this section we state the necessary conditions to ensure that a merging process of epistemic states cannot be manipulated. To do so, firstly, we are going to present a qualitative method for establishing preferences over information (formulas), namely, *belief liftings*. Then, we will carry out a study on other rational properties that any rational belief merging operator must satisfy: the *strong standard domain condition*, *non-imposition*, *stability* and *absence of nominator*.

3.1. Belief liftings

One issue that arises from the study of the manipulation of a merging operator lies in determining when the outcome of a merging process is better for an agent than other. Everaere et al. [25] proposed a

quantitative method to do so, via the *satisfaction indexes*. In this work, we suggest a qualitative method to solve this issue, namely, *belief liftings*.

A *belief lifting* is a mapping $\succeq \mapsto \sqsupseteq_{\succeq}$ that associates a preorder \sqsupseteq_{\succeq} over $\mathcal{L}_{\mathcal{P}}$ to each total preorder \succeq over $\mathcal{W}_{\mathcal{P}}$, which satisfies the following: $\varphi_w \sqsupseteq_{\succeq} \varphi_{w'}$ iff $w \succeq w'$; and $\varphi \simeq_{\succeq} \psi$ if $\varphi \equiv \psi$.

Belief liftings extend individual preferences over interpretations to individual preferences over information as follows: as we mentioned earlier in Subsection 2.3, a total preorder \succeq_{E_i} semantically encodes the preferences of an agent i whose epistemic state is E_i , via an assignment. Then, the preorder $\sqsupseteq_{\succeq_{E_i}}$ transfers the preferences encoded in \succeq_{E_i} to a preference over formulas, establishing thus a preference over information: $\varphi \sqsupseteq_{\succeq_{E_i}} \psi$ expresses that, for the agent i , the information encoded in φ is at least as preferred as the one encoded in ψ , while $\varphi \sqsupset_{\succeq_{E_i}} \psi$ says that, for i , φ is more preferred than ψ . Thus, this method for establishing preferences over formulas depends partly on the way in which the beliefs of the agent i are semantically encoded in \succeq_{E_i} . Actually, the only constraint on this preference is that it has to coincide with the relation of the agent on the complete information (a coherence condition), that is $\varphi_w \sqsupseteq_{\succeq} \varphi_{w'}$ iff $w \succeq w'$. In any other aspect the preference relation $\sqsupseteq_{\succeq_{E_i}}$ has a lot of freedom. The best way to think about it, is as a qualitative way that the agent i has to determine when a formula is more advantageous for him than another. Above all, it should not be thought of as a relation coding the beliefs of the agent i .

Through a lifting over subsets of $\mathcal{W}_{\mathcal{P}}$, $\succeq \mapsto \sqsupseteq_{\succeq}^{\mathcal{W}_{\mathcal{P}}}$, it is possible to define a belief lifting $\succeq \mapsto \sqsupseteq_{\succeq}^{\mathcal{L}_{\mathcal{P}}}$, as follows:

$$\varphi \sqsupseteq_{\succeq}^{\mathcal{L}_{\mathcal{P}}} \psi \text{ iff } \llbracket \varphi \rrbracket \sqsupseteq_{\succeq}^{\mathcal{W}_{\mathcal{P}}} \llbracket \psi \rrbracket \quad (5)$$

Indeed, the relationship established in (5) between lifting over subsets of $\mathcal{W}_{\mathcal{P}}$ and belief liftings is one-to-one. Thus, we can translate the Gärdenfors properties (cf. Section 2.1) into the logical setting as follows:

G1: $\varphi_{w,w'} \sqsupseteq_{\succeq} \varphi_{w'}$, whenever $w \succ w'$.

G2: $\varphi_w \sqsupseteq_{\succeq} \varphi_{w,w'}$, whenever $w \succ w'$.

We think that these are the most basic and natural properties that any well-behaved belief lifting must satisfy. From now on, we say that a belief lifting is a *G-belief lifting* if it satisfies both instances.

The relationship established in (5) also allows us to define four concrete instances of belief lifting from the liftings presented in Section 2.1: the *possibilistic belief lifting*, $\succeq \mapsto \sqsupseteq_{\succeq}^{\Pi}$; the *pessimistic belief lifting*, $\succeq \mapsto \sqsupseteq_{\succeq}^{\mathbb{W}}$; the *Kelly belief lifting*, $\succeq \mapsto \sqsupseteq_{\succeq}^{\mathbb{K}}$; the *precise-leximax belief lifting*, $\succeq \mapsto \sqsupseteq_{\succeq}^{\text{plm}}$. We must note that Kelly belief lifting and precise-leximax belief lifting are examples of G-belief liftings, in contrast to the possibilistic and the pessimistic belief liftings (cf. Subsection 2.1).

3.2. Other rational properties of merging

Now we introduce other properties that will be very useful in order to state our general manipulability results: *strong standard domain condition*, *non-imposition*, *stability* and *absence of nominator*.

The first property to be introduced is the *strong standard domain condition*:

(ESF-SSD) For every agent i in \mathcal{S} and every total preorder \succeq over $\mathcal{W}_{\mathcal{P}}$, there exists an i -profile E_i such that $\llbracket B(\nabla(E_i, E)) \rrbracket = \max(\llbracket B(E) \rrbracket, \succeq)$, for all E in \mathcal{E} .

Semantically, **(ESF-SSD)** captures the fact that, restricted to single-profiles, the assignment associated to a merging operator ∇ , which satisfies such property, is a surjective function, as we see in the next result. Thus, any total preorder is a semantic encoding of an epistemic state.

Proposition 7. *An ES basic merging operator ∇ satisfies **(ESF-SSD)** iff $\Phi \mapsto \succeq_{\Phi}$, the assignment associated to ∇ by Theorem 1, satisfies the following:*

(SSD) *For every agent i in \mathcal{S} and every total preorder \succeq over $\mathcal{W}_{\mathcal{P}}$, there is an i -profile E_i such that $\succeq_{E_i} = \succeq$.*

Observation 7. Note that **(ESF-SSD)** is stronger than **(ESF-SD)**, modulo the basic ESF postulates. This is due to Propositions 5 and 7 and the fact that, as it is quite clear, **(SSD)** is stronger than **(SD)**. As a matter of fact, there are some instances of ES basic merging operators satisfying **(ESF-SD)** for which **(ESF-SSD)** does not hold. To show this, consider $\mathbb{T}(\mathcal{W}_{\mathcal{P}})$, the set of all the total preorders \succeq over $\mathcal{W}_{\mathcal{P}}$ with at most three levels, and note that, if the set of epistemic states \mathcal{E} is such that $|\mathcal{E}| \geq |\mathbb{T}(\mathcal{W}_{\mathcal{P}})|$,¹¹ then we can consider a surjection $E \mapsto \succeq_E^*$ that associates a total preorder \succeq_E^* in $\mathbb{T}(\mathcal{W}_{\mathcal{P}})$ to each epistemic state E in \mathcal{E} .

Thus, consider the assignment $\Phi \mapsto \succeq_{\Phi}^*$ that maps every epistemic profile $\Phi = (E_{i_1}, E_{i_2}, \dots, E_{i_d})$ into the total preorder $\succeq_{\Phi}^* = \succeq_{E_d}^*$. It is easy to see from its definition that this assignment satisfies **(SD)**, but **(SSD)** does not hold. Now, if for every epistemic profile Φ and every epistemic state E we assume $E_{(\Phi, E)}$ in \mathcal{E} such that $\llbracket B(E_{(\Phi, E)}) \rrbracket = \max(\llbracket B(E) \rrbracket, \succeq_{\Phi}^*)$, then the operator ∇^* , defined by putting $\nabla^*(\Phi, E) = E_{(\Phi, E)}$, is an ES basic merging operator that satisfies **(ESF-SD)**, but it does not satisfy **(ESF-SSD)**.

The next result states which of those operators in Subsection 2.4 satisfy the strong standard domain condition. This result straightforwardly follows from structure preserving and Proposition 7.

Proposition 8. In the context of the tpo epistemic space, \mathfrak{S}_{tpo} , if the associated assignment of an ES basic merging operator is structure preserving, then the ES basic merging operator satisfies **(ESF-SSD)**. In particular, the quasi-indifference operator, ∇^{Ind^*} , the projective operator, ∇^{π} , the Σ -projective operator, $\nabla^{\Sigma\pi}$, and every aggregation-based ES basic merging operator, ∇^F , satisfy this property. In contrast, the indifference operator, ∇^{Ind} , does not satisfy **(ESF-SSD)**.

Another quite natural property is *non-imposition*. This property expresses the fact that, if a complete information entails the entrenched belief of the integrity constraint, then it can be the entrenched belief of one possible output of the merging process. More precisely, we have the following:

(ESF-NI) For every N in $\mathcal{F}^*(\mathcal{S})$, every E in \mathcal{E} and every w in $\mathcal{W}_{\mathcal{P}}$ there exists an N -profile Φ in $\mathbb{P}(\mathcal{S}, \mathcal{E})$ such that $B(\nabla(\Phi, E)) \equiv \varphi_w$, whenever $w \models B(E)$.

We say that an ES combination operator is *non-imposed* if **(ESF-NI)** holds, otherwise it is *imposed*. The next result establishes a semantic characterisation of non-imposition.

Proposition 9. An ES basic merging operator ∇ satisfies **(ESF-NI)** iff its associated assignment, $\Phi \mapsto \succeq_{\Phi}$, satisfies the following property:

(NI) For every N in $\mathcal{F}^*(\mathcal{S})$ and every w in $\mathcal{W}_{\mathcal{P}}$, there exists an N -profile Φ in $\mathbb{P}(\mathcal{S}, \mathcal{E})$ s.t. $\max(\succeq_{\Phi}) = \{w\}$.

It is not hard to see that every ES (quasi)merging operator is non-imposed, as the next result reveals.

Proposition 10. If **(ESF6)** holds, then **(ESF-NI)** also holds.

Another instance of a non-imposed operator is the projective operator, as we show in the next result. Indeed, this is an example of a non-imposed operator for which **(ESF6)** does not hold (cf. Observation 4). This shows that the converse of Proposition 10 does not hold and, therefore, **(ESF6)** is stronger than **(ESF-NI)**. In this result we also show that many of the merging operators defined in Subsection 2.4 are non-imposed, especially the most rational.

Proposition 11. In the context of the tpo epistemic space, \mathfrak{S}_{tpo} , **(ESF-NI)** holds for the max operator, ∇^{max} , the projective operator, ∇^{π} , the Σ -projective operator, $\nabla^{\Sigma\pi}$, and every aggregation-based ES basic merging operator, ∇^F , defined by means of an aggregation function F satisfying composition and decomposition; in particular, for the sum operator, ∇^{Σ} , and the Gmax operator, ∇^{Gmax} . In contrast, **(ESF-NI)** does not hold for the indifference operator, ∇^{Ind} , and the quasi-indifference operator, ∇^{Ind^*} .

¹¹This is the case of the tpo epistemic space, \mathfrak{S}_{tpo} , which was presented in Subsection 2.2.

The next property to be presented is *stability*.¹² This property captures the fact that, at the level of entrenched beliefs, if the revised beliefs of an agent by the restriction of a merging process coincide with the belief of such restriction, then the agent can change his mind expressing a new epistemic state, and thus any information accepted by both the outcome of the original merging process (when the agent expresses his true beliefs) and the revision of the new epistemic state with the same restriction, will remain accepted by the outcome of the merging process obtained when the agent expresses the new epistemic state. Formally:

(ESF-S) For all N in $\mathcal{F}^*(\mathcal{S})$, every N -profile Φ in $\mathsf{P}(\mathcal{S}, \mathcal{E})$ and every pair E, E^* in \mathcal{E} , if $B(\nabla(E_i, E)) \equiv B(E)$, then $B(\nabla(\Phi, E)) \wedge B(\nabla(E^*, E)) \vdash B(\nabla(\Phi[E^*_i], E))$.

We say that an ES combination operator is *stable* if **(ESF-S)** holds, otherwise we say that it is *unstable*.

In order to illustrate the rationality of the stability criteria let us revisit Example 1, using again the *sum operator* as a rational instance of an ES merging operator.

Example 7 (Example 1 revisited). *As we said previously, Anne, Bob and Charles have considered determining who to blame in the case of divorce of their pair of friends, Tom and Mary, under the constraint that there is at least one of the members of the couple to blame. This situation have been modeled by means of the propositional language $\mathcal{L}_{\mathcal{P}}$, which is built from the variables m (Mary is to blame) and t (Tom is to blame), considered in that order and whose interpretations are given by $w_1 = 00$, $w_2 = 01$, $w_3 = 10$, $w_4 = 11$.*

In Example 5, Anne's epistemic state is encoded by \succeq_A , where $w_1 \simeq_A w_2 \succ_A w_3 \simeq_A w_4$, while Bob's and Charles' epistemic states are respectively encoded by \succeq_B and \succeq_C , where $w_4 \succ_B w_2 \simeq_B w_3 \succ_B w_1$ and $w_1 \succ_C w_2 \simeq_C w_3 \simeq_C w_4$. The integrity constraint is encoded by the total preorder \succeq , which is defined by $w_2 \simeq w_3 \simeq w_4 \succ w_1$, and the epistemic profile is given by $\Phi = (\succeq_A, \succeq_B, \succeq_C)$ (cf. Figure 1).

In Example 5, we also saw that, under this situation, Anne, Bob and Charles should agree on the fact that the most consensual belief is that Tom is to blame, regardless of whether Mary is to blame or not. This is due to the fact that w_2 and w_4 are the models of $B(\nabla^\Sigma(\Phi, \succeq))$, the entrenched belief of the epistemic state resulting from the merging process.

Now, suppose that Anne and Bob are still expressing their beliefs, but Charles changes his mind and expresses that he fervently believes that Tom is to blame but not Mary – that is, his epistemic state \succeq^ is such that $w_2 \succ^* w_i$, for $i = 3, 4$, and therefore $w_2 \models B(\nabla^\Sigma(\Phi, \succeq)) \wedge B(\nabla^\Sigma(\succeq^*, \succeq))$ – then Charles' new entrenched belief, modeled by w_2 , also satisfies the entrenched belief of the new epistemic state resulting from the merging processes when Charles expresses his new belief, that is, $w_2 \models B(\nabla^\Sigma(\Phi[\succeq^*/_C], \succeq))$. This is due to the following: (1) Anne and Bob are still expressing their true epistemic states; (2) w_2 is on the top of \succeq_Φ^Σ (cf. Figure 2); (3) in Charles' original epistemic state, \succeq_C , w_2 is indifferent to all the available models (those in $\llbracket B(\succeq) \rrbracket = \max(\succeq)$; cf. Figure 1); and (4) in Charles' new epistemic state, \succeq^* , w_2 is more preferred than the remaining available models. Thus, the rank value of w_2 cannot be downgraded in $\succeq_{\Phi[\succeq^*/_C]}^\Sigma$, that is, $w_2 \succeq_{\Phi[\succeq^*/_C]}^\Sigma w$, for all $w \models B(\succeq)$. This can be seen in the next result, which involves a semantic characterisation of our stability criteria.*

Proposition 12. *An ES basic merging operator ∇ satisfies **(ESF-S)** iff $\Phi \mapsto \succeq_\Phi$, the assignment associated to ∇ by Theorem 1, satisfies the following property:*

(S) *For all N in $\mathcal{F}^*(\mathcal{S})$, every N -profile Φ in $\mathsf{P}(\mathcal{S}, \mathcal{E})$, every E^* in \mathcal{E} and every pair w, w' in $\mathcal{W}_{\mathcal{P}}$, if $w \succeq_\Phi w'$, $w \simeq_{E_i} w'$ and $w \succeq_{E^*} w'$, then $w \succeq_{\Phi[E^*_i]} w'$.*

Another hint that lead us to consider stability as a rational property, is the fact that some natural operators, like sum and Gmax, satisfy it. Indeed, every ES merging operator is stable, as the following result shows:

¹²Preliminary versions of *stability* were previously proposed in [57] and [60]. The version of stability proposed in [57] seems not to be so rational, since some classic and natural operators do not satisfy it. This is the case with the sum operator, as it was pointed out in that work. The version of stability proposed in [60] is less compact than the version proposed here, but they are equivalent, as can be easily verified.

Proposition 13. *If (ESF1), (ESF7) and (ESF8) hold, then (ESF-S) holds.*

There are some important remarkable facts concerning Proposition 13, which are addressed in what follows:

Observation 8. *On one hand, one might be tempted to substitute (ESF8) for its weaker version, (ESF8W), in Proposition 13. However, as we will see in the next result, the max operator is an operator for which (ESF1), (ESF7) and (ESF8W) hold, but it does not satisfy (ESF-S). This shows that Proposition 13 does not hold for ES quasi-merging operators, but for its strong counterpart: the ES merging operators.*

On the other hand, the converse of Proposition 13 does not hold, that is, there exist some instances of ES combination operators for which the converse of this result does not hold. Indeed, as we see in the next result, the Σ -projective operator is a stable ES basic merging operator for which neither (ESF7) nor (ESF8) hold (cf. Observation 5). This shows that modulo the basic ESF postulates, (ESF7) and (ESF8) are jointly stronger than (ESF-S).

The following result shows which of those operators presented in Subsection 2.4 are stable.

Proposition 14. *In the context of the tpo epistemic space, \mathfrak{S}_{tpo} , (ESF-S) holds for the indifference operator, ∇^{Ind} , the quasi-indifference operator, ∇^{Ind^*} , the projective operator, ∇^π , the Σ -projective operator, $\nabla^{\Sigma\pi}$, and every aggregation-based ES basic merging operator, ∇^F , which is defined by means of an aggregation function F satisfying composition and decomposition; in particular, for the sum operator, ∇^Σ , and the Gmax operator, ∇^{Gmax} . In contrast, (ESF-S) does not hold for the max operator, ∇^{max} .*

The last property to be presented aims at capturing the notion of *nominator*. The “good” operators are those that do not satisfy the following:

(ESF-N) For all N in $\mathcal{F}^*(\mathcal{S})$, there exists d in N such that, $B(\nabla(\Phi, E)) \wedge B(\nabla(E_d, E)) \not\vdash \perp$, for every N -profile Φ in $\text{P}(\mathcal{S}, \mathcal{E})$ and every E in \mathcal{E} .

An operator satisfying (ESF-N), admits an agent (called a *nominator*) in any society, which has a similar behaviour to that of dictators by imposing their will, but in a weaker manner: at the level of the entrenched beliefs, the revision of the nominator’s epistemic state with the integrity constraint is consistent with the outcome of the whole merging process under such restriction, regardless of the epistemic states expressed by the remaining agents. We say that an ES combination operator *admits a nominator* if it satisfies (ESF-N), otherwise we say that it does not admit any.

The next proposition establishes a semantic characterisation of (ESF-N).

Proposition 15. *An ES basic merging operator ∇ satisfies (ESF-N) iff its associated assignment, $\Phi \mapsto \succeq_\Phi$, satisfies the following property:*

(N) *For every society of agents N in $\mathcal{F}^*(\mathcal{S})$ there exists an agent d in N such that, for every N -profile Φ in $\text{P}(\mathcal{S}, \mathcal{E})$ and every couple of interpretations w, w' in \mathcal{W}_P , if $w \succ_{E_d} w'$ then $w \succeq_\Phi w'$.*

Observation 9. *From the consistency of the merging processes at the level of entrenched beliefs it straightforwardly follows that (ESF-D) entails (ESF-N). As a matter of fact, if d is a dictator for a merging operator, then d is also a nominator for it. However, the converse of the last assertion is not true. Indeed, as we will see in the next result, the quasi-indifference operator is a non-dictatorial operator (cf. Proposition 6) for which any agent involved in the merging process is a nominator.*

The following result shows which of those operators in Subsection 2.4 admit a nominator.

Proposition 16. *In the context of the tpo epistemic space, \mathfrak{S}_{tpo} , (ESF-N) holds for the indifference operator, ∇^{Ind} , the quasi-indifference operator, ∇^{Ind^*} , the projective operator, ∇^π and the Σ -projective operator, $\nabla^{\Sigma\pi}$. Furthermore, every agent in a finite society N is a nominator for the indifference operator and the quasi-indifference operator. In contrast, (ESF-N) does not hold for the max operator, ∇^{max} , and any aggregation-based ES basic merging operator, ∇^F , which is defined by means of an aggregation function F satisfying composition, decomposition and discreteness; in particular, neither the sum operator, ∇^Σ , nor the Gmax operator, ∇^{Gmax} , satisfies it.*

3.3. Some general results on manipulability

In what follows, we present some general results of manipulability through belief liftings. To do so, we first present the notion of *manipulability* in the ESF framework, based on belief lifting. An ES basic merging operator is manipulable by an agent if such an agent, knowing the restriction of the merging process and the information that will be expressed by the other agents, changes his mind in order to obtain a result that “fits” better his true beliefs. More precisely:

Definition 3. An ES basic merging operator ∇ is *manipulable* if there exist a belief lifting $\succeq \mapsto \sqsupseteq_{\succeq}$, a finite society N in $\mathcal{F}^*(\mathcal{S})$, an N -profile Φ , an agent i in N and a pair of epistemic states E, E^* in \mathcal{E} , such that:

$$B(\nabla(\Phi^{[E^*/i]}, E)) \sqsupseteq_{\succeq_{E_i}} B(\nabla(\Phi, E))$$

In this case, ∇ is said to be manipulable with respect to the belief lifting $\succeq \mapsto \sqsupseteq_{\succeq}$, while the tuple formed by N, Φ, i, E, E^* is called a *manipulability situation* for ∇ with respect to $\succeq \mapsto \sqsupseteq_{\succeq}$.

We say that ∇ is *absolutely manipulable* (resp. *absolutely G-manipulable*) if it is manipulable with respect to any belief lifting (resp. to any G-belief lifting). We also say that ∇ is *strategy-proof* or *nonmanipulable* if the statement in Definition 3 does not hold. If ∇ does not admit a manipulability situation with respect to a prefixed belief lifting, we say that ∇ is *strategy-proof* with respect to that belief lifting.

Let us make a comment on our definition of manipulability. Note that in the setting of \mathfrak{S}_{tpo} , that is, the setting in which the epistemic states are preferences over worlds, \succeq_{E_i} is actually the preferences of agent i when the assignment is structure preserving. Thus, Definition 3 is natural and really very close to the definition of manipulation in social choice theory. In the fully abstract case, we rely on the representation theorem (Theorem 1) in order to have a way for ‘materializing’ the preferences of agent i . This is done via the total preorder \succeq_{E_i} given by the assignment.

Later, we present some results that state the condition in order for an ES basic merging operator to be absolutely G-manipulable (cf. Theorems 4-6). For the moment, we illustrate the notion of manipulability by revisiting Example 3. To do so, we will use the sum operator.

Example 8 (Example 3 revisited.) *Alan and Ben have to travel from point X to point Y. There are four paths to accomplish the trip: w_1, w_2, w_3 and w_4 . We will use a language built from four propositional variables $\{p_1, \dots, p_4\}$. Since no path meets with another, by abuse of notation, we assume that w_i is the model of the sentence “to travel only path w_i ”, that is, $w_1 = 1000, w_2 = 0100, w_3 = 0010, w_4 = 0001$.*

On one hand, Alan believes that w_i is better than w_{i+1} , for $i = 1, 2, 3$. Thus, Alan’s epistemic state can be represented by \succeq_A , given in Figure 6a. On the other hand, as Ben believes that the best paths to travel are w_1 and w_4 , while he believes that the other two paths are really equally bad, then his epistemic state is encoded by \succeq_B , represented in Figure 6b. In addition, since the last traffic news inform that path w_1 is obstructed, whereas on paths w_2, w_3 and w_4 there is light traffic, the integrity constraint is encoded by \succeq , given in Figure 6c. The thick line at the bottom of each ordering in Figure 1 represents the interpretations different from w_1, w_2, w_3 and w_4 .

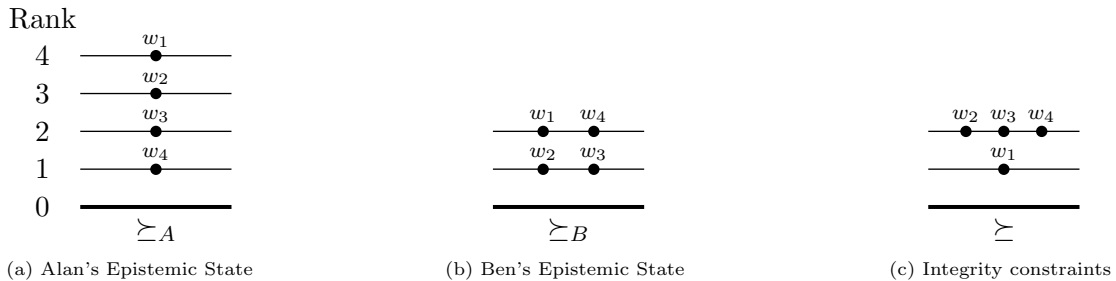


Figure 6: Representation of the individual epistemic states and the integrity constraints.

The preorder obtained from the aggregation of $\Phi = (\succeq_A, \succeq_B)$ using sum assignment is graphically represented in Figure 7, and its calculations are given in Table 2.

| $\mathcal{W}_{\mathcal{P}}$ | $\text{rank}(\succeq_A)$ | $\text{rank}(\succeq_B)$ | $\text{rank}(\succeq_{\Phi}^{\Sigma})$ |
|-----------------------------|--------------------------|--------------------------|--|
| w_1 | 4 | 2 | 6 |
| w_2 | 3 | 1 | 4 |
| w_3 | 2 | 1 | 3 |
| w_4 | 1 | 2 | 3 |

Table 2: Calculation of the ranks

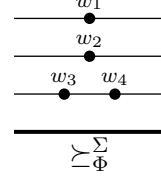


Figure 7: Preorder obtained from the aggregation process

The epistemic state resulting from the merging process using the sum operator can graphically be represented by Figure 8. We must recall that the sum operator is defined by putting $\nabla^{\Sigma}(\Phi, \succeq) = \succeq^{\text{lex}}(\succeq, \succeq_{\Phi}^{\Sigma})$.

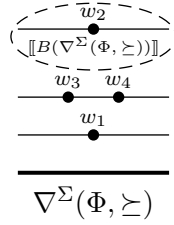
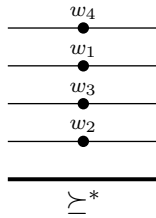


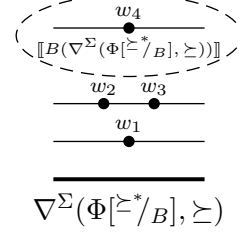
Figure 8: Epistemic state resulting from the merging process.

Based on the fact that $B(\nabla^{\Sigma}(\Phi, \succeq)) \equiv \varphi_{w_2}$, Ben estimates that they should agree that path w_2 is the best option for travelling. This is not Ben's most entrenched belief.

However, as Ben really believes that w_4 is one of the best paths to travel, he lies and says that he really believes that path w_4 is the best one, and that the others paths are in decreasing order of goodness w_1 , w_3 and w_2 (cf. Figure 9a).



(a) Ben's lie



(b) Outcome of the merging process when Ben lies

Figure 9: Ben's lie and manipulated outcome of the merging process.

In this case, based on the outcome of the merging process with sum, which is given in Figure 9b, Alan and Ben should agree that w_4 is the best path to travel from X to Y , that is, $B(\nabla^{\Sigma}(\Phi[\succeq^*/_B], \succeq)) \equiv \varphi_{w_4}$. This result fits very well with Ben's beliefs. As a matter of fact, given a belief lifting $\succeq \mapsto \sqsupset_{\succeq_B}$, since $w_4 \succ_B w_2$, we have $B(\nabla^{\Sigma}(\Phi[\succeq^*/_B], \succeq)) \sqsupset_{\succeq_B} B(\nabla^{\Sigma}(\Phi, \succeq))$. Then, Ben has manipulated the merging process.

The situation given in Example 8 is indeed an absolute manipulability situation for the sum operator. This is also a manipulability situation for the Gmax and max operators, as we can see after some simple calculations. Concerning the manipulation of the remaining operators presented in Subsection 2.4, we have the following result:

Proposition 17. *In the context of the tpo epistemic space, \mathfrak{S}_{tpo} , the following holds:*

- (i) The indifference operator, ∇^{Ind} , is strategy-proof.
- (ii) The quasi-indifference operator, ∇^{Ind^*} , and the projective operator, ∇^π , are both manipulable with respect to the precise-leximax belief lifting, but they are strategy-proof with respect to the possibilistic, the pessimistic and the Kelly belief liftings.
- (iii) Given an aggregation function F , the aggregation-based ES basic merging operator ∇^F is absolutely G -manipulable, if F satisfies composition, decomposition and discreteness.
- (iv) The sum operator, ∇^Σ , the Gmax operator, ∇^{Gmax} , the max operator, ∇^{max} and the Σ -projective operator, $\nabla^{\Sigma\pi}$, are absolutely manipulable.

Thus, the quasi-indifference operator and the projective operator are two instances of merging operators that are manipulable but not absolutely manipulable. The Σ -projective operator is an interesting instance of a merging operator, since it is indeed a dictatorial operator which is absolutely manipulable. This leads us to avoid thinking that a gauge for the goodness of a belief lifting is the strategy behaviour of a merging operator, which we expect to be nonmanipulable. This is the case with dictatorial operators.

Table 3 summarises all the results presented along this work about the behaviour of the concrete operators introduced in Subsection 2.4, thus offering a landscape of those rational properties satisfied by each of them and their manipulation with respect to the belief lifting presented in Subsection 3.1. In this table, for the operators pointed in the first row, each \checkmark denotes the satisfaction of the property indicated in the first column, while \times denotes that such property does not hold.

| Operators | ∇^{Ind} | ∇^{Ind^*} | ∇^Σ | ∇^{Gmax} | ∇^{max} | ∇^π | $\nabla^{\Sigma\pi}$ |
|---|-----------------------|-------------------------|-----------------|------------------------|-----------------------|--------------|----------------------|
| Properties | | | | | | | |
| Max condition | \times | \checkmark | \checkmark | \checkmark | \checkmark | \checkmark | \checkmark |
| (ESF5) | \times | \checkmark | \checkmark | \checkmark | \checkmark | \checkmark | \checkmark |
| (ESF6) | \times | \times | \checkmark | \checkmark | \checkmark | \times | \checkmark |
| (ESF7) | \checkmark | \checkmark | \checkmark | \checkmark | \checkmark | \checkmark | \times |
| (ESF8) | \checkmark | \times | \checkmark | \checkmark | \times | \times | \times |
| (ESF8W) | \checkmark | \times | \checkmark | \checkmark | \checkmark | \checkmark | \times |
| (ESF-SD) | \times | \checkmark | \checkmark | \checkmark | \checkmark | \checkmark | \checkmark |
| (ESF-P) | \checkmark | \times | \checkmark | \checkmark | \checkmark | \checkmark | \checkmark |
| (ESF-I) | \checkmark | \checkmark | \times | \times | \times | \checkmark | \times |
| (ESF-D) | \checkmark | \times | \times | \times | \times | \checkmark | \checkmark |
| (ESF-SSD) | \times | \checkmark | \checkmark | \checkmark | \checkmark | \checkmark | \checkmark |
| (ESF-NI) | \times | \times | \checkmark | \checkmark | \checkmark | \checkmark | \checkmark |
| (ESF-S) | \checkmark | \checkmark | \checkmark | \checkmark | \times | \checkmark | \checkmark |
| (ESF-N) | \checkmark | \checkmark | \times | \times | \times | \checkmark | \checkmark |
| Strategy-Proofness | | | | | | | |
| $\succ \mapsto \sqsupset_{\Sigma}^{\Pi}$ | \checkmark | \checkmark | \times | \times | \times | \checkmark | \times |
| $\succ \mapsto \sqsupset_{\Sigma}^{\text{W}}$ | \checkmark | \checkmark | \times | \times | \times | \checkmark | \times |
| $\succ \mapsto \sqsupset_{\Sigma}^{\text{K}}$ | \checkmark | \checkmark | \times | \times | \times | \checkmark | \times |
| $\succ \mapsto \sqsupset_{\Sigma}^{\text{plm}}$ | \checkmark | \times | \times | \times | \times | \times | \times |

Table 3: Behaviour of the concrete basic merging operators that were introduced in Subsection 2.4.

Strategy-proofness entails some rational properties when other good properties are involved. This can be seen through the following result.

Proposition 18. *The following assertions hold for every ES basic merging operator ∇ , which is strategy-proof with respect to a G-belief lifting:*

- (i) *If ∇ satisfies **(ESF-NI)**, then it also satisfies **(ESF-P)**.*
- (ii) *∇ satisfies **(ESF-S)** iff it also satisfies **(ESF-I)**.*

We must note that the converse of Proposition 18, Part (i), fails: the *indifference operator* is an instance of a strategy-proof operator satisfying **(ESF-P)** for which **(ESF-NI)** does not hold, as can be verified in Table 3. Another example is the *sum operator*, which satisfies **(ESF-P)** and **(ESF-NI)**, but is absolutely manipulable. The converse of Proposition 18, Part (ii), also fails: the max operator is an operator for which neither **(ESF-S)** nor **(ESF-I)** hold, but it is absolutely manipulable, as we can also verify in Table 3.

We are now able to state our main strategy-proofness results. The first of them is a straightforward corollary of Theorem 3 and Proposition 18, establishing a dichotomy between absolute G-manipulability and the existence of an Arrovian dictator.

Theorem 4. *Any ES basic merging operator satisfying **(ESF-SD)**, **(ESF-NI)** and **(ESF-S)** is either absolutely G-manipulable, or it satisfies **(ESF-D)**.*

In particular, if a non-imposed and stable ES basic merging operator that satisfies the standard domain condition is strategy-proof (with respect to a G-belief lifting), then it admits an Arrovian dictator. The converse of this assertion fails. Indeed, as it is exposed in Table 3, the Σ -projective operator is an example of a non-imposed, stable ES basic merging operator satisfying the standard domain condition, which is dictatorial, but it is absolutely manipulable. Moreover, in Table 3 we can also see that the *projective operator* is a non-imposed, stable and dictatorial basic merging operator that satisfies the standard domain condition, which is strategy-proof with respect to a G-belief lifting, namely, the Kelly lifting, but it is manipulable with respect to the precise-leximax lifting. Thus, Theorem 4 heavily relies on the lifting chosen.

The next result is a corollary of Propositions 10 and 13 and Theorem 4. It shows that any non-dictatorial ES merging operator satisfying standard domain is absolutely G-manipulable.

Theorem 5. *An ES merging operator satisfying **(ESF-SD)** is either absolutely G-manipulable, or it satisfies **(ESF-D)**.*

Proposition 13 is widely used in the proof of Theorem 5 in order to obtain **(ESF-S)** from **(ESF7)** and **(ESF8)**. However, as we mentioned in Observation 8, **(ESF8)** cannot be replaced by **(ESF8W)** in Proposition 13 in order to obtain stability. Therefore, Theorem 5 cannot be applied in general to ES quasi-merging operators. In order to fix this issue, we state the following strategy-proofness result, which can be applied to a wider range of merging operators: those for which stability does not hold.

Theorem 6. *If an ES basic merging operator satisfies **(ESF-SSD)** and **(ESF-NI)**, then it is either absolutely G-manipulable, or it satisfies **(ESF-N)**.*

Theorem 6 also leads us to state that any non-imposed and strategy-proof (with respect to a G-belief lifting) ES basic merging operator satisfying the strong standard domain condition, admits a nominator. The converse of this fact does not hold, as can easily be shown through the Σ -projective operator: this is a non-imposed ES basic merging operator satisfying the strong standard domain condition, which admits a nominator but is absolutely manipulable (cf. Table 3).

Our last strategy-proofness result is a straightforward corollary of Proposition 10 and Theorem 6. Similar to Theorem 5, this result shows that, under the strong standard domain condition, any ES quasi-merging operator, which does not admit a nominator, is absolutely G-manipulable.

Theorem 7. *An ES quasi-merging operator satisfying **(ESF-SSD)** is either absolutely G-manipulable, or it satisfies **(ESF-N)**.*

Theorems 5 and 7 provide two powerful tools to determine if an ES (quasi)merging operator is manipulable with respect to any G-belief lifting. This is the case when the operator satisfies the (strong) standard domain condition but it does not admit a nominator. We wonder if there is an ES (quasi)merging operator satisfying both properties. If the answer to this question is “No” (as we think), then any ES (quasi)merging operator satisfying (strong) standard domain is absolutely G-manipulable.

4. More on powerful agents

In [57], we introduced a notion of powerful agents, who have a behaviour which is similar to that of Arrovian dictators and nominators who impose their beliefs, but differ in the way they do so. Such agents, called *weak-dictators*, can express an epistemic state that allows a prefixed interpretation to always satisfy the entrenched beliefs resulting from a merging process, regardless of the epistemic states expressed by the remaining agents and the integrity constraint, which the interpretation must satisfy. More precisely:

(ESF-WD) For every society N , there exists an agent d in N (called weak-dictator) for which the following assertion holds: for all interpretation w in $\mathcal{W}_{\mathcal{P}}$ there exists an epistemic state E^w in \mathcal{E} such that $w \models B(\nabla(\Phi^{[E^w/d]}, E))$, for every N -profile Φ and every epistemic state E in \mathcal{E} , with $w \models B(E)$.

An ES combination operator is said to be *weak-dictatorial* if it satisfies **(ESF-WD)**, otherwise we say that it is not *weak-dictatorial*.

Although the notion of weak-dictator seems to be unnatural, it is closely related with the notions of Arrovian dictator and nominator in the presence of some rational properties. In order to show such relationships, in the following we present a taxonomic survey on these three types of powerful agents.

The next result states a strict hierarchy between Arrovian dictators, nominators and weak-dictators. In order to establish such a hierarchy, we group the ES basic merging operators that satisfy the max condition and admit this type of powerful agents, in the following three classes: \mathfrak{D} , the class of the ES basic merging operators that satisfy the max condition and admit Arrovian dictators; \mathfrak{N} , the class formed by all of the ES basic merging operators for which the max condition holds and admit a nominator; and \mathfrak{W} , the class of all the weak-dictatorial ES basic merging operators satisfying the max condition.

Theorem 8. *The following inclusions hold: $\mathfrak{D} \subsetneq \mathfrak{N} \subsetneq \mathfrak{W}$. Moreover, if ∇ is an ES basic merging operator satisfying the max condition, then every Arrovian dictator for ∇ is also a nominator for ∇ , and every nominator for ∇ is also a weak-dictator for ∇ .*

If we remove the max condition from this classification, then the inclusions stated in Theorem 8 do not hold. Now we present an example of a dictatorial ES basic merging operator, which is not a weak-dictatorial operator. To do so, consider again the tpo epistemic space, \mathfrak{S}_{tpo} , given in Example 4, and prefix an interpretation w^* in $\mathcal{W}_{\mathcal{P}}$. In this example, d denotes again the maximal element of any society N .

$$\nabla^{w^*}(\Phi, \succeq) = \succeq^{\text{lex}(\succeq, \succeq_d^{w^*})}, \text{ where } \succeq_d^{w^*} \text{ is such that } \min(\succeq_d^{w^*}) = \{w^*\} \text{ and } \succeq_d^{w^*} \upharpoonright_{\mathcal{W}_{\mathcal{P}} \setminus \{w^*\}} = \succeq \upharpoonright_{\mathcal{W}_{\mathcal{P}} \setminus \{w^*\}}$$

It is quite clear that this is an ES basic merging operator for which the max condition does not hold. Moreover, from its definition it straightforwardly follows that ∇^{w^*} also satisfies **(ESF-D)**, and then **(ESF-N)** also holds (cf. Observation 9). In order to show that **(ESF-WD)** does not hold, we only have to note that, if \succeq is a total preorder over $\mathcal{W}_{\mathcal{P}}$ with at least two models at its top level, then for every society N , every agent i in N , every N -profile Φ and every epistemic state \succeq^* , we obtain $w^* \not\models B(\nabla^{w^*}(\Phi[\succeq^*/i], \succeq))$.

Although the inclusions of the classes given in Theorem 8 are strict, under certain hypotheses the classes \mathfrak{N} and \mathfrak{W} collapse. The next result establishes precisely the hypothesis under which such a collapse occurs. In order to state this result, let us consider the following classes of ES basic merging operators: \mathfrak{D}^* , formed by those operators in \mathfrak{D} that are strategy-proof with respect to a belief lifting satisfying **G1**; \mathfrak{N}^* , the subclass of \mathfrak{N} of all those operators which are strategy-proof with respect to a belief lifting satisfying **G1**; and \mathfrak{W}^* , the class of all the operators in \mathfrak{W} which are strategy-proof with respect to a belief lifting satisfying **G1**.

Theorem 9. *The following inclusions hold: $\mathfrak{D}^* \subsetneq \mathfrak{N}^* = \mathfrak{W}^*$. Moreover, if ∇ is an ES basic merging operator satisfying the max condition which is strategy-proof with respect to a belief lifting satisfying **G1**, then an agent d is a nominator for ∇ if, and only if, d is a weak-dictator for ∇ .*

Figure 10 summarises Theorems 8 and 9 jointly, providing a graphical representation of the taxonomy of powerful agents presented in this paper. This shows that the rational ES basic merging operators that are nonmanipulable with respect to a well-behaved belief lifting are concentrated in \mathfrak{N} , the class of those operators that admit a nominator.

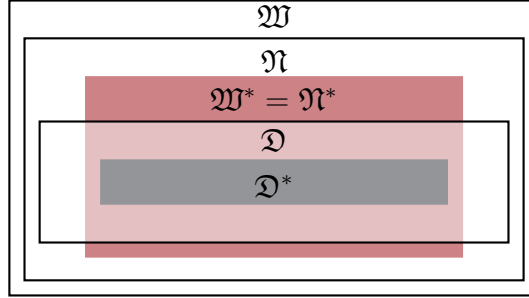


Figure 10: Diagram summarising the taxonomy of powerful agents in belief merging

In [57], there was also presented a preliminary result on manipulability via belief liftings, similar to the ones presented here (Theorems 4 and 6), but involving weak-dictators instead of Arrovian dictators or nominators. That result is as follows:

*Let ∇ be an ES basic merging operator satisfying the max condition (restricted to single-profiles with complete beliefs), **(ESF-SD)** and **(ESF-NI)**. If ∇ is strategy-proof with respect to a G -belief lifting, then ∇ is a weak dictatorial operator.*

However, we have found some problems when we look closely at the sketch of the proof proposed in that work. In that proof, a complete ES basic merging operator ∇' is built from a prefixed ES basic merging operator ∇ (which is strategy-proof with respect to a G -belief lifting) by forcing an alignment of the assignment associated to ∇ .¹³ That deformation of the original assignment makes it impossible to deduce the strategy-proofness of ∇' from the strategy-proofness of ∇ due to the following three reasons: (1) The alignment of the assignment breaks the indifferences in the original preferences, leading to define a totally new merging operator; (2) the preference over the outcomes of a merging operator depend on the way in which the individual epistemic states are semantically encoded, as we mentioned in Subsection 3.1. Therefore, the two liftings that encode the preferences over information of an agent (each of which respectively corresponds to the total preorders resulting from the original assignment and its deformation) are different in general; and (3) the Gärdenfors properties do not provide any information about the preference between two non-equivalent formulas φ and ψ , with $\varphi \vdash \psi$.

To solve these problems, we replace the standard domain condition by the strong standard domain condition. More specifically, we present the following result which, by Theorem 9, is equivalent to Theorem 6.

Theorem 10. *If an ES basic merging operator satisfies **(ESF-SSD)**, **(ESF-NI)** and the max condition, then it is either absolutely G -manipulable, or it satisfies **(ESF-WD)**.*

5. Related Works

Manipulability and the existence of a powerful agent are two issues that have been historically considered as inherent to electoral processes and, as such, have been widely studied in social choice theory. However, as we mentioned earlier, these problems might also arise when a group of agents, in search of a consensus, “aggregate” their individual considerations (preference, belief or acceptation) about certain information. In this way, both of these issues, manipulability and the existence of a powerful agent, have also been addressed in other aggregation settings. Namely in belief merging the problem of manipulation is studied and also in judgment aggregation. These two different approaches have in common the fact of being logic-based frameworks of aggregating information.

¹³We say that ∇' is a complete ES basic merging operator if $B(\nabla'(\Phi, E))$ is a complete formula, for every epistemic profile Φ and every epistemic state E .

Everaere, Konieczny and Marquis [24, 25] and, later, Haret and Wallner [38], addressed the manipulation issue in belief merging from a quantitative view, using *satisfaction indexes* in order to measure how suitable the outcome of a merging process is for an agent. The nature of epistemic states they consider, simply formulas, make this choice very natural. In their framework the notion of preferences of an agent is absent. In our framework, this can be represented by means of tight articulations between the operators and assignments. In some cases, as in the paradigmatic instantiation of epistemic states in the space \mathfrak{S}_{tpo} , these representations coincide totally with the preferences of the agents. We use these representations together with the notion of lifting to define our ordinal notion of satisfaction.

Another difference concerns the obtained results. In contrast to ours, which search for general properties that any epistemic state merging operator must satisfy in order to be nonmanipulable (or strategy-proof), Everaere et al. [24, 25] and Haret and Wallner [38] characterize the nonmanipulation of the sum, max and Gmax operators, with respect to some specific indexes, by means of the number of agents involved in the merging process, the existence of an integrity constraint and the completeness of the real belief base of the manipulator agent. In the case of Everaere et al., they considered the *weak drastic*, the *strong drastic* and the *probabilistic* indexes, which are closely related to the notion of satisfaction of information, while Haret and Wallner consider other two indexes which are closely related to the notions of skepticism and credulity. We must note that, in [59], we showed that, in the setting of distance-based IC merging operators, manipulation via the possibilistic belief lifting and a refined variant of this (the *refined possibilistic* belief lifting) generalize the manipulation via weak drastic and strong drastic indexes, respectively.

Another important aspect in our work which is not considered by Everaere et al. [24, 25] nor Haret and Wallner [38] in their respective works, is the study of powerful agents and their connection with manipulability.

In *Judgment aggregation*, another logical-based framework in which the task of aggregating information coming from several sources is present, problems of impossibility and manipulability have been studied. This framework was introduced by List and Pettit [52, 53] and then addressed by Eckert and Pigozzi [22], Dietrich and List [19, 20], Nehring and Puppe [61], Lang, Pigozzi, Slavkovik and van der Torre [47], Everaere, Konieczny and Marquis [26, 27] and others. In its generalised form, proposed by Dietrich [18], it consists of associating a set of collective judgment with a profile of n individual judgments, similar to a voting processes. In this setting, the judgment A_i of an agent i is a subset of a set of logical formulas X (in a fixed logic). The fact that $p \in A_i$ captures the support (acceptance) of the formula p by the agent i .

Beyond the differences in the nature of their inputs and outputs, judgment aggregation functions and belief merging operators (specially the ESF operators) differ in the nature of their properties, as Everaere, Konieczny and Marquis pointed out in [27]. Indeed, as highlighted earlier, in belief merging some sets of postulates characterize the behaviour of rational operators, and semantic representations have been also established. In judgment aggregation, some properties have been identified as well, most of them inspired by voting theory (none related to the existence of a powerful agent) and introduced in the seminal work of List and Pettit [52, 53] and also by Dietrich and List [20] and Everaere et al. [26, 27].

In spite of these differences, some links between these two settings have been established, specifically with the IC merging framework. For instance, Eckert and Pigozzi [22], and then Pigozzi [62], used the distance-based sum IC merging operator to build a judgment aggregation function and dissolved the *doctrinal paradox*.¹⁴ Based on this idea, we might think that there is a duality between these two frameworks of aggregating information. However, Everaere et al. [26] proved that there is not such a duality, showing also the conditions under which this duality could arise.

Some impossibility results have been stated in judgment aggregation. For instance, List and Pettit [53] showed that there is no judgment aggregation function satisfying all the properties that they consider rational for this type of function. In contrast to our impossibility result (Theorem 3), which involves the existence of a powerful agent and generalizes Arrow's Theorem [56], their impossibility result is not related to powerful agents.

¹⁴The *doctrinal paradox* is a problem that arises when majority voting is applied to propositions. The result obtained by this method may be in contradiction with the doctrine [52, 53, 22, 62].

Manipulation has also been addressed in judgment aggregation. Dietrich and List [20] were the first in addressing this issue in this framework. In that work, they established two notions of manipulation for judgment aggregation functions. The first one concerns the acceptance of a piece of information; the second one is similar to our notion of manipulation, using a specific lifting which is defined by means of their acceptance-based notion of manipulability. Dietrich and List show that these two notions of manipulability are equivalent. Furthermore, they also show that these notions are equivalent to other properties of judgment aggregation functions; none of them related with the existence of powerful agents unlike our results. Thus, our manipulation results and those established by Dietrich and List are quite different.

As a matter of fact, our results on manipulability and impossibility are more related with the properties and techniques studied in social choice theory in which there are tight links between powerful agents, impossibility theorems and manipulability.

Baumeister, Rothe and Selker [7] study computational aspects of manipulation in judgment aggregation processes. They presented a survey on complexity of manipulation of judgment aggregation processes based on the work of Dietrich and List [20]. An interesting work in perspective is a study on the complexity of manipulation in the ESF framework.

6. Final remarks

We have stated some general results on manipulability for ES basic merging operators that show a dichotomy between absolute G-manipulation and the existence of a powerful agent (a dictator, a nominator or even a weak-dictator) when some rational properties are involved. One very interesting feature of our approach is that it allows instantiating the strategy-proofness theorems to different representations of epistemic states: ordinal conditional functions, rational relations, and of course total preorders. However, with the representation of epistemic states as formulas, our results do not hold because it is impossible to have the standard domain property in the presence of a good representation of beliefs, namely, the max condition [56, 58]. This fact highlights the necessity of using complex epistemic states if we want properties like standard domain to hold.

Exploiting the dichotomy obtained from the manipulability results, it is possible to determine when an ES merging operator is absolutely G-manipulable – exactly when it satisfies the standard domain condition and it does not admit Arrovian dictators (cf. Theorem 5) – and more generally, when an ES quasi-merging operator is absolutely G-manipulable – exactly when it satisfies the strong standard domain condition and it does not admit a nominator. This technique leads us to determine that ES merging operators with good behaviour, like sum, Gmax and even max, are absolutely G-manipulable.

Although our results also reveal that, under rational properties, strategy-proofness entails the existence of a powerful agent (an Arrovian dictator or even a nominator), it is surprising to find a dictatorial operator which is absolutely manipulable, namely the Σ -projective operator. This fact shows that strategy-proofness cannot be characterised in terms of either Arrovian dictators or nominators.

In this work, we adopt the generalized logic based merging approach of epistemic states. In this framework we have considered, together with the reformulations of classical IC merging postulates, social postulates which are indeed natural postulates given that most of them are entailed by the usual merging postulates. This framework has allowed to establish in the past a general impossibility result which is “polymorph” in the sense that it applies to different types of structures in which some kinds of epistemic states can be realized. Now, in this work, we have established a general result of manipulation which is also “polymorph”. A by-product is a manipulability result in social choice which can be obtained when epistemic states are instantiated in the epistemic space of total preorders.

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Appendix A. Proofs

PROOF OF PROPOSITION 2. Let us show that the indifference operator, ∇^{Ind} , satisfies **(ESF7)**, **(ESF8)** and **(ESF8W)**, but it does not satisfy neither the max condition, **(ESF5)** nor **(ESF6)**.

From the definition of ∇^{Ind} , we straightforwardly get that **(ESF7)**, **(ESF8)** and **(ESF8W)** hold. From this we also get $B(\nabla^{\text{Ind}}(\succeq_i, \succeq)) \equiv B(\nabla^{\text{Ind}}(\succeq_j, \succeq))$, for $\succeq_i \neq \succeq_j$. This shows that **(ESF5)** does not hold.

In order to show that the max condition does not hold, note that, given a total preorder \succeq over $\mathcal{W}_{\mathcal{P}}$, $\max(\succeq^{\text{Ind}}) = \mathcal{W}_{\mathcal{P}}$. This is due to the fact that \succeq^{Ind} is indeed the flat order, for every total preorder \succeq . Then, if we consider a total preorder \succeq such that $B(\succeq) \neq \mathcal{W}_{\mathcal{P}}$, we obtain $B(\succeq) \neq \max(\succeq^{\text{Ind}})$, as desired.

Now, since the indifference operator is an ES basic merging operator for which **(ESF7)** and **(ESF8)** hold, from Proposition 1 it follows that the indifference assignment satisfies properties **3** and **4**. Hence, since this assignment does not satisfy the max condition, from Observation 2 we obtain that property **2** does not hold. From this, by Proposition 1, we get that **(ESF6)** does not hold either. \square

PROOF OF PROPOSITION 3. Let us show that the quasi-indifference operator, ∇^{Ind^*} , satisfies the max condition, **(ESF5)** and **(ESF7)**, but it does not satisfy **(ESF6)**, **(ESF8)** and **(ESF8W)**.

Since the quasi-indifference assignment, $\Phi \mapsto \succeq_{\Phi}^{\text{Ind}^*}$, is structure preserving, from Observation 3 it follows that the quasi-indifference operator satisfies the max condition and **(ESF5)**.

In order to show that **(ESF7)** holds, by Proposition 1, we only have to see that the quasi-indifference assignment satisfies property **3**. To do so, let us consider a society of agents N , a partition $\{N_1, N_2\}$ of N , an N -profile Φ and a pair of alternatives w, w' , and suppose that $w \succeq_{\Phi_{N_1}}^{\text{Ind}^*} w'$ and $w \succeq_{\Phi_{N_2}}^{\text{Ind}^*} w'$. Since N_1 and N_2 are disjoint nonempty sets, then N has at least two agents. From this it follows that $\succeq_{\Phi}^{\text{Ind}^*}$ is the flat order over $\mathcal{W}_{\mathcal{P}}$. Therefore, $w \succeq_{\Phi}^{\text{Ind}^*} w'$, as desired.

To show that neither **(ESF6)**, **(ESF8)** nor **(ESF8W)** hold, by Proposition 1, we only have to see that the quasi-indifference assignment does not satisfy the properties **2**, **4** and **4'**.

Firstly, we are going to show that property **2** does not hold. To do so, we only have to consider the finite society $N = \{1, 2\}$ and a N -profile $\Phi = (\succeq_1, \succeq_2)$ such that $\succeq_1 = \succeq_2$ and $[[B(\succeq_i)]] \neq \mathcal{W}_{\mathcal{P}}$, for $i = 1, 2$. Thus, $[[\bigwedge_{\succeq_i \in \Phi} B(\succeq_i)]] \neq \mathcal{W}_{\mathcal{P}}$, but $\succeq_{\Phi}^{\text{Ind}^*}$ is the flat order. Therefore, $[[\bigwedge_{\succeq_i \in \Phi} B(\succeq_i)]] \neq \max(\succeq_{\Phi}^{\text{Ind}^*})$, as desired.

To show that neither property **4** nor property **4'** hold, consider again the society $N = \{1, 2\}$, and also consider a N -profile $\Phi = (\succeq_1, \succeq_2)$ and a couple of interpretations w, w' in $\mathcal{W}_{\mathcal{P}}$ such that $w \succ_1 w'$ and $w \succ_2 w'$. Since $\Phi \mapsto \succeq_{\Phi}^{\text{Ind}^*}$ is structure preserving, we get that $w \succ_{\succeq_1}^{\text{Ind}^*} w'$ and $w \succ_{\succeq_2}^{\text{Ind}^*} w'$, but as $\succeq_{\Phi}^{\text{Ind}^*}$ is the flat order, we get $w \simeq_{\Phi}^{\text{Ind}^*} w'$. Therefore, $w \not\succeq_{\Phi}^{\text{Ind}^*} w'$, showing that neither **4** nor **4'** hold. \square

PROOF OF PROPOSITION 4. Let F be an aggregation function. Let us first show that ∇^F is an ES merging operator (that is, ∇^F satisfies **(ESF1)**-**(ESF8)**) iff F satisfies composition and decomposition, at least when it is restricted to the set of all the finite tuples (x_1, \dots, x_n) for which $|\mathcal{W}_{\mathcal{P}}| \geq x_i$.

In order to show this, by Theorem 2, we only have to prove that $\Phi \mapsto \succeq_{\Phi}^F$ is a faithful assignment iff F satisfies composition and decomposition, at least when it is restricted to the set of all the finite tuples (x_1, \dots, x_n) for which $|\mathcal{W}_{\mathcal{P}}| \geq x_i$. To do so, consider the following two remarks:

Remark 1. $F(x_1, \dots, x_n, w_1, \dots, w_m) \geq F(y_1, \dots, y_n, z_1, \dots, z_m)$, whenever $F(x_1, \dots, x_n) \geq F(y_1, \dots, y_n)$ and $F(w_1, \dots, w_m) \geq F(z_1, \dots, z_m)$.

Remark 2. $F(x_1, \dots, x_n, w_1, \dots, w_m) > F(y_1, \dots, y_n, z_1, \dots, z_m)$, whenever $F(x_1, \dots, x_n) \geq F(y_1, \dots, y_n)$ and $F(w_1, \dots, w_m) > F(z_1, \dots, z_m)$.

Remark 1 straightforwardly follows from composition and symmetry. To show Remark 2, assume $F(x_1, \dots, x_n) \geq F(y_1, \dots, y_n)$ and $F(w_1, \dots, w_m) > F(z_1, \dots, z_m)$ and, toward a contradiction, suppose $F(y_1, \dots, y_n, z_1, \dots, z_m) \geq F(x_1, \dots, x_n, w_1, \dots, w_m)$. On one hand, from the hypothesis and Remark 1, we have $F(x_1, \dots, x_n, w_1, \dots, w_m) \geq F(y_1, \dots, y_n, z_1, \dots, z_m)$. From this and our assumption we obtain $F(y_1, \dots, y_n, z_1, \dots, z_m) = F(x_1, \dots, x_n, w_1, \dots, w_m)$. On the other hand, from composition and the fact that $F(x_1, \dots, x_n) \geq F(y_1, \dots, y_n)$, we obtain $F(x_1, \dots, x_n, w_1, \dots, w_m) \geq F(y_1, \dots, y_n, w_1, \dots, w_m)$. From

this and the equality above we get $F(y_1, \dots, y_n, z_1, \dots, z_m) \geq F(y_1, \dots, y_n, w_1, \dots, w_m)$. Thus, by symmetry and decomposition, $F(z_1, \dots, z_m) \geq F(w_1, \dots, w_m)$, a contradiction.

Continuing with the main proof, let us show the *if* part of this result. Suppose that F satisfies composition and decomposition, at least when it is restricted to tuples whose inputs are not greater than $|\mathcal{W}_{\mathcal{P}}|$. Note that, for every epistemic profile $\Phi = (\succeq_{i_1}, \dots, \succeq_{i_n})$ and every interpretation w in $\mathcal{W}_{\mathcal{P}}$, the tuple $(r_{\succeq_{i_1}}(w), \dots, r_{\succeq_{i_n}}(w))$ is such that $|\mathcal{W}_{\mathcal{P}}| \geq r_{\succeq_i}(w)$. Thus, Remarks 1 and 2 lead us to the fact that $\Phi \mapsto \succeq_{\Phi}^F$ satisfies properties **3** and **4**. Additionally, since $\Phi \mapsto \succeq_{\Phi}^F$ is structure preserving, by Observation 3, property **1** holds. Moreover, by the same reason, this assignment satisfies the max condition. From this and Observation 2 it follows that property **2** also holds. This shows that $\Phi \mapsto \succeq_{\Phi}^F$ is a faithful assignment.

To prove the *only if* part, we suppose that $\Phi \mapsto \succeq_{\Phi}^F$ is a faithful assignment and will show that the aggregation function F satisfies composition and decomposition, if we restrict to tuples whose inputs are not greater than $|\mathcal{W}_{\mathcal{P}}|$. Let $x_1, \dots, x_n, y_1, \dots, y_n, z$ be a list of non-negative integers of this type. Note that, given a finite society $N = \{i_1, \dots, i_{n+1}\}$ and a pair of interpretations w, w' in $\mathcal{W}_{\mathcal{P}}$, we can build an N -profile $\Phi = (\succeq_{i_1}, \dots, \succeq_{i_{n+1}})$ such that $r_{\succeq_{i_k}}(w) = x_k$ and $r_{\succeq_{i_k}}(w') = y_k$, for $k = 1, \dots, n$, and $r_{\succeq_{i_{n+1}}}(w) = r_{\succeq_{i_{n+1}}}(w') = z$. Consider $\{N_1, N_2\}$, a partition of N , with $N_1 = \{i_1, \dots, i_n\}$ and $N_2 = \{i_{n+1}\}$.

Thus, if $F(x_1, \dots, x_n) \geq F(y_1, \dots, y_n)$, from (4) and identity we get $w \succeq_{\Phi_{N_1}}^F w'$ and $w \succeq_{\Phi_{N_2}}^F w'$, respectively. Therefore, by property **3**, $w \succeq_{\Phi}^F w'$, that is, $F(x_1, \dots, x_n, z) \geq F(y_1, \dots, y_n, z)$. This shows that composition holds. To show that decomposition also holds, assume that $F(x_1, \dots, x_n, z) \geq F(y_1, \dots, y_n, z)$ and, toward a contradiction, suppose that $F(y_1, \dots, y_n) > F(x_1, \dots, x_n)$. From (4) and identity it follows that $w' \succ_{\Phi_{N_1}}^F w$ and $w' \succeq_{\Phi_{N_2}}^F w$. Then, by property **4**, $w' \succ_{\Phi}^F w$, that is, $F(y_1, \dots, y_n, z) > F(x_1, \dots, x_n, z)$, a contradiction. Therefore, decomposition also holds. \square

PROOF OF PROPOSITION 6. Suppose that the epistemic space is given by the two epistemic space, \mathfrak{S}_{tpo} . A survey on the Arrovian behaviour of the sum operator, ∇^{Σ} , the Gmax operator, ∇^{Gmax} , the max operator, ∇^{max} , the projective operator, ∇^{π} , the Σ -projective operator, $\nabla^{\Sigma\pi}$, was presented in [56]. Thus we are going to study the Arrovian behaviour of the indifference operator, ∇^{Ind} , the quasi-indifference operator $\nabla^{\text{Ind*}}$, and the aggregation-based ES basic merging operators, ∇^F :

- (i) Let us study first the satisfaction of **(ESF-SD)**. It is quite clear that any structure preserving assignment also satisfies **(SD)**. From this, by Proposition 5, we obtain that the quasi-indifference operator and any aggregation-based ES basic merging operators satisfy **(ESF-SD)**. It is also clear that the indifference assignment does not satisfy **(SD)**. Thus, from Proposition 5 it also follows that the indifference operator does not satisfy **(ESF-SD)**.
- (ii) Now we are going to study the satisfaction of **(ESF-P)**. By vacuity, the indifference assignment satisfies **(P)**. This is due to the fact that $\succeq_{\succeq}^{\text{Ind}}$ is the flat order, for every epistemic state (total preorder) \succeq . Thus, from Proposition 5 we obtain that the indifference operator satisfies **(ESF-P)**. Now, if an aggregation function F satisfies composition and decomposition, from Proposition 4 and Observation 6 it straightforwardly follows that the aggregation-based operator ∇^F satisfies **(ESF-P)**. To show that the quasi-indifference operator does not satisfy **(ESF-P)**, we proceed in a similar manner to that in the Proof of Proposition 3, where it is proved that neither **(ESF8)** nor **(ESF8W)** hold for this operator.
- (iii) Let us study now the satisfaction of **(ESF-I)**. It is quite clear that the indifference assignment and the quasi-indifference assignment satisfy **(I)**. From this, by Proposition 5, we get that both the indifference operator and the quasi-indifference operator satisfy **(ESF-I)**. Now suppose that an aggregation function F satisfies composition, decomposition and discreteness. As we will see below in the proof of part (iv) of this result, the operator ∇^F does not satisfy **(ESF-D)**. Therefore, since ∇^F satisfies **(ESF-SD)** and **(ESF-P)**, but **(ESF-D)** does not hold, from Theorem 3 it follows that ∇^F does not satisfy **(ESF-I)**.
- (iv) Finally, we study the satisfaction of **(ESF-D)**. From its definition it straightforwardly follows that the indifference operator satisfies **(ESF-D)**. Moreover, given a finite society N , it is easy to see that every agent in N is an Arrovian dictator for ∇^{Ind} .

To show that the quasi-indifference operator does not satisfy **(ESF-D)**, by Observation 6, we only have to note that **(ESF-P)** does not hold, as we have shown in the part (ii) of this result.

Let us show now that, given an aggregation function F satisfying composition, decomposition and discreteness, the aggregation-based operator ∇^F does not satisfy **(ESF-D)**. This is a straightforward consequence of the next remark:

Remark 3. If F is an aggregation function satisfying composition, decomposition and discreteness, then there is no finite society N , with at least two agents, that admits an Arrovian dictator for ∇^F .

To show this, suppose that $\mathcal{L}_{\mathcal{P}}$ is a language built from two variables, and assume $\mathcal{W}_{\mathcal{P}} = \{w_1, w_2, w_3, w_4\}$. Based on an inductive argument on the size of the finite societies, firstly, we assume $N = \{1, 2\}$, and consider the N -profile $\Phi = (\succeq_1, \succeq_2)$ and the epistemic state \succeq , which are represented in Figure A.11.

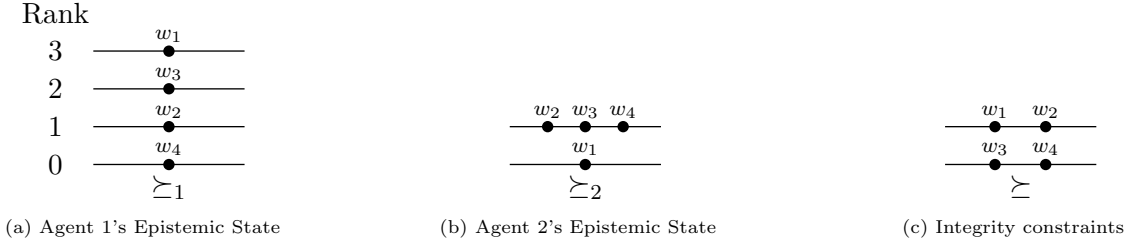


Figure A.11: Representation of the individual epistemic states and the integrity constraints.

On the one hand, since $\Phi \mapsto \succeq_{\Phi}^F$ is structure preserving, from **(B-Rep)** we get $B(\nabla^F(\succeq_2, \succeq)) \equiv \varphi_{w_2}$. On the other hand, note that $r_{\succeq_1}(w_1) > r_{\succeq_1}(w_3) > r_{\succeq_1}(w_2)$ and $r_{\succeq_1}(w_2) = r_{\succeq_2}(w_2) = r_{\succeq_2}(w_1) + 1$. From this, by Remark 2 and the identity property, we have:

$$F(r_{\succeq_1}(w_1), r_{\succeq_2}(w_1)) > F(r_{\succeq_1}(w_3), r_{\succeq_2}(w_1)) \quad (\text{A.1})$$

Since $r_{\succeq_1}(w_3) > r_{\succeq_1}(w_2) > r_{\succeq_2}(w_1)$, from the discreteness property it straightforwardly follows that $F(r_{\succeq_1}(w_3), r_{\succeq_2}(w_1)) \geq F(r_{\succeq_1}(w_2), r_{\succeq_2}(w_1) + 1)$, that is, $F(r_{\succeq_1}(w_3), r_{\succeq_2}(w_1)) \geq F(r_{\succeq_1}(w_2), r_{\succeq_2}(w_2))$. From this and the inequality (A.1) we obtain $F(r_{\succeq_1}(w_1), r_{\succeq_2}(w_1)) > F(r_{\succeq_1}(w_2), r_{\succeq_2}(w_2))$. Thus, $w_1 \succ_{\Phi}^F w_2$ and, by structure preserving, $w_2 \succ_{\succeq_2}^F w_1$. From this, by **(B-Rep)**, we get $B(\nabla^F(\Phi, \succeq)) \equiv \varphi_{w_1}$ and $B(\nabla^F(\succeq_2, \succeq)) \equiv \varphi_{w_2}$. Therefore, $B(\nabla^F(\Phi, \succeq)) \not\equiv B(\nabla^F(\succeq_2, \succeq))$, showing that agent 2 is not an Arrovian dictator for ∇^F . From this and the symmetry of F we also get that agent 1 is not an Arrovian dictator for ∇^F , by switching the epistemic states of the two agents.

Now, assume $N = \{1, 2, \dots, n\}$, with $n > 2$, and consider the N -partition $\{N_1, N_2\}$, where $N_1 = \{1, 2\}$ and $N_2 = N \setminus N_1$. Also consider the N -Profile $\Phi = (\succeq_1, \succeq_2, \dots, \succeq_n)$ and the epistemic state \succeq , where \succeq_i is given in Figure A.11a, for all $i \neq 2$, and \succeq_2 and \succeq are given in Figures A.11b and A.11c, respectively.

On the one hand, as we saw in the previous case, $w_1 \succ_{\Phi_{N_1}}^F w_2$. On the other hand, as **(ESF-P)** holds, by Proposition 5, $\Phi \mapsto \succeq_{\Phi}^F$ satisfies **(P)**. Hence, from structure preserving and since $w_1 \succ_i w_2$, for all i in N_2 , we have $w_1 \succ_{\Phi_{N_2}}^F w_2$. From this and **(B-Rep)** we get $B(\nabla^F(\Phi_{N_1}, \succeq)) \equiv B(\nabla^F(\Phi_{N_2}, \succeq)) \equiv \varphi_{w_1}$, following from **(ESF7)** and **(ESF8)** that $B(\nabla^F(\Phi, \succeq)) \equiv \varphi_{w_1}$ and, as we show in the previous case, $B(\nabla^F(\succeq_2, \succeq)) \equiv \varphi_{w_2}$. Therefore, $B(\nabla^F(\Phi, \succeq)) \not\equiv B(\nabla^F(\succeq_2, \succeq))$, showing that agent 2 is not a dictator for ∇^F . To show that any other agent in N is not a dictator for ∇^F either, by the symmetry of F , we only have to switch the epistemic state of any agent i in N with that of agent 2. \square

PROOF OF PROPOSITION 7. To show the *only if* part, suppose that ∇ is an ES basic merging operator satisfying **(ESF-SSD)** and let us see that its associated assignment, $\Phi \mapsto \succeq_{\Phi}$, satisfies **(SSD)**. To do so, consider an agent i in \mathcal{S} , a total preorder \succeq over $\mathcal{W}_{\mathcal{P}}$ and, toward a contradiction, assume that $\succeq_{E_i^*} \neq \succeq$, for every i -profile E_i^* .

By **(ESF-SSD)**, let us consider an i -profile E_i such that $\llbracket B(\nabla(E_i, E)) \rrbracket = \max(\llbracket B(E) \rrbracket, \succeq)$, for every epistemic state E in \mathcal{E} . Now, since $\succeq_{E_i} \neq \succeq$, there exist a couple of interpretations w, w' in $\mathcal{W}_{\mathcal{P}}$ such that $w \succeq_{E_i} w'$ and $w' \succ w$, or $w \succ_{E_i} w'$ and $w' \succeq w$. In any case, if we consider an epistemic state E in \mathcal{E} such that $\llbracket B(E) \rrbracket = \{w, w'\}$, from (B-Rep) we get $\llbracket B(\nabla(E_i, E)) \rrbracket \neq \max(\llbracket B(E) \rrbracket, \succeq)$, a contradiction.

The *if* part straightforwardly follows from (B-Rep) and the assumption of **(SSD)**. \square

PROOF OF PROPOSITION 8. Suppose that the epistemic space is given by the tpo epistemic space, \mathfrak{S}_{tpo} . Let us show that if the assignment associated to an ES basic merging operator is structure preserving, then such operator satisfies **(ESF-SSD)**. In particular, the quasi-indifference operator, ∇^{Ind^*} , the projective operator, ∇^π , the Σ -projective operator, $\nabla^{\Sigma\pi}$, and every aggregation-based ES basic merging operator satisfy **(ESF-SSD)**. In addition, we also show that the indifference operator, ∇^{Ind} , does not satisfy it.

It is quite clear that, in this context of representation of epistemic states, if an assignment is structure preserving, then it is a surjection when it is restricted to the set of single-profiles. Therefore, such assignment satisfies **(SSD)** and, by Proposition 7, **(ESF-SSD)** holds for its associated ES basic merging operator. From this we straightforwardly get that the quasi-indifference operator, the projective operator, the Σ -projective operator and every aggregation-based ES basic merging operator satisfy **(ESF-SSD)**.

To show that the indifference operator does not satisfy **(ESF-SSD)**, by Proposition 7, we only have to show that the indifference assignment does not satisfy **(SSD)**, that is, the indifference assignment is not a surjection when it is restricted to single-profiles. But this follows from the fact that the indifference assignment maps every epistemic state into the flat order. \square

PROOF OF PROPOSITION 9. Let ∇ be an ES basic merging operator, and suppose that $\Phi \mapsto \succeq_\Phi$ is its associated assignment. We show that ∇ satisfies **(ESF-NI)** iff $\Phi \mapsto \succeq_\Phi$ satisfies **(NI)**. To do so, let us fix a society N in $\mathcal{F}^*(\mathcal{S})$.

To show the *only if* part, let us consider an interpretation w in $\mathcal{W}_{\mathcal{P}}$ and an epistemic state E in \mathcal{E} such that $B(E) \equiv \top$. Thus, by **(ESF-NI)**, if we consider an N -profile Φ such that $B(\nabla(\Phi, E)) \equiv \varphi_w$, from (B-Rep) we get $\max(\succeq_\Phi) = \{w\}$, as desired.

To show the *if* part, let us consider an epistemic state E in \mathcal{E} and a model w of $B(E)$. Thus, by **(NI)**, if Φ is an N -profile such that $\max(\succeq_\Phi) = \{w\}$, from (B-Rep) we also get $B(\nabla(\Phi, E)) \equiv \varphi_w$, as desired. \square

PROOF OF PROPOSITION 10. Let ∇ be an ES combination operator satisfying **(ESF6)**. To show that ∇ satisfies **(ESF-NI)**, given a society N in $\mathcal{F}^*(\mathcal{S})$, an epistemic state E in \mathcal{E} and a model w of $B(E)$, it is enough to consider an N -profile Φ such that $\llbracket B(E_i) \rrbracket = \{w\}$, for all i in N . Then, apply **(ESF6)**. \square

PROOF OF PROPOSITION 11. Let us show that the max operator, ∇^{max} , the projective operator, ∇^π , the Σ -projective operator, $\nabla^{\Sigma\pi}$, and every aggregation-based ES basic merging operator ∇^F (defined from an assignment F satisfying composition and decomposition) satisfy **(ESF-NI)**, but neither the indifference operator, ∇^{Ind} , nor the quasi-indifference operator, ∇^{Ind^*} , satisfy it.

Note that **(ESF6)** holds for the max operator, the Σ -projective operator and every aggregation-based ES basic merging operator ∇^F , that has been defined from an assignment F satisfying composition and decomposition. From this, by Proposition 10, we get that **(ESF-NI)** holds for these operators.

To show that the projective operator satisfies **(ESF-NI)**, we only have to see that the projective assignment satisfies **(NI)**, that is, it satisfies the following: *given a finite society N and an interpretation w in $\mathcal{W}_{\mathcal{P}}$, there exists an N -profile Φ such that $\max(\succeq_\Phi) = \{w\}$* . To do so, given a finite society N , we only have to consider an N -profile Φ such that $\max(\succeq_d) = \{w\}$, where \succeq_d is its last input. The result follows from the fact that $\succeq_\Phi = \succeq_d$.

To prove that neither the indifference operator nor the quasi-indifference operator satisfy **(ESF-NI)**, by Proposition 9, we only have to show that there exist a finite society N and an interpretation w in $\mathcal{W}_{\mathcal{P}}$ such that, for every N -profile Φ , $\max(\succeq_\Phi^{\text{Ind}}) \neq \{w\}$ (resp. $\max(\succeq_\Phi^{\text{Ind}^*}) \neq \{w\}$). To do so, we only have to consider a finite society N with at least two agents and any interpretation w in $\mathcal{W}_{\mathcal{P}}$, and note that, for any N -profile Φ , both $\succeq_\Phi^{\text{Ind}}$ and $\succeq_\Phi^{\text{Ind}^*}$ are the flat order. \square

PROOF OF PROPOSITION 12. Let ∇ be an ES basic merging operator, and suppose that $\Phi \mapsto \succeq_\Phi$ is its associated assignment. We will show that ∇ satisfies **(ESF-S)** iff $\Phi \mapsto \succeq_\Phi$ satisfies **(S)**. In order to do this, let N be a society in $\mathcal{F}^*(\mathcal{S})$, i be an agent in N , Φ be an N -profile and E^* be an epistemic state in \mathcal{E} .

To show the *only if part*, consider a couple of interpretations w, w' in \mathcal{W}_P such that $w \succeq_\Phi w'$, $w \simeq_{E_i} w'$ and $w \succeq_{E^*} w'$, and let us show that $w \succeq_{\Phi[E^*/i]} w'$. Let us consider an epistemic state E in \mathcal{E} such that $\llbracket B(E) \rrbracket = \{w, w'\}$ and note that, by (B-Rep), $B(\nabla(E_i, E)) \equiv B(E)$ and $w \models B(\nabla(\Phi, E)) \wedge B(\nabla(E^*, E))$. From this and **(ESF-S)**, we obtain $w \models B(\nabla(\Phi[E^*/i], E))$. The result follows from (B-Rep).

To show the *if part*, we assume that E is an epistemic state in \mathcal{E} such that $B(\nabla(E_i, E)) \equiv B(E)$. We must show that $B(\nabla(\Phi, E)) \wedge B(\nabla(E^*, E)) \vdash B(\nabla(\Phi[E^*/i], E))$. If $B(\nabla(\Phi, E)) \wedge B(\nabla(E^*, E))$ is inconsistent, then the result is straightforward. Then, assume that $B(\nabla(\Phi, E)) \wedge B(\nabla(E^*, E))$ is consistent, and consider w , a model of $B(\nabla(\Phi, E)) \wedge B(\nabla(E^*, E))$. Thus, by **(ESF1)**, w is a model of $B(E)$. Now, if w' is a model of $B(E)$, from (B-Rep) we get that $w \succeq_\Phi w'$, $w \simeq_{E_i} w'$ and $w \succeq_{E^*} w'$. From this and **(S)**, we obtain $w \succeq_{\Phi[E^*/i]} w'$. Again, by (B-Rep), $w \models B(\nabla(\Phi[E^*/i], E))$. Therefore $B(\nabla(\Phi, E)) \wedge B(\nabla(E^*, E)) \vdash B(\nabla(\Phi[E^*/i], E))$. \square

PROOF OF PROPOSITION 13. Let ∇ be an ES combination operator satisfying **(ESF1)**, **(ESF7)** and **(ESF8)**. Let us show that ∇ satisfies **(ESF-S)**. To do so, let us show that, given a society N in $\mathcal{F}^*(\mathcal{S})$, an N -profile Φ and a couple of epistemic states E, E^* in \mathcal{E} , we get $B(\nabla(\Phi, E)) \wedge B(\nabla(E^*, E)) \vdash B(\nabla(\Phi[E^*/i], E))$, whenever $B(\nabla(E_i, E)) \equiv B(E)$.

Suppose then $B(\nabla(E_i, E)) \equiv B(E)$. If Φ is a single-profile, the result straightforwardly follows from the assumption and **(ESF1)**. Thus, suppose that Φ has at least two inputs and consider $M = N \setminus \{i\}$. Since $B(\nabla(E_i, E)) \equiv B(E)$, by **(ESF1)** we get $B(\nabla(\Phi_M, E)) \wedge B(\nabla(E_i, E)) \equiv B(\nabla(\Phi_M, E))$. Moreover, $B(\nabla(\Phi_M, E)) \wedge B(\nabla(E_i, E)) \not\vdash \perp$, **(ESF7)** and **(ESF8)** entail $B(\nabla(\Phi_M, E)) \wedge B(\nabla(E_i, E)) \equiv B(\nabla(\Phi, E))$. Thus, $B(\nabla(\Phi_M, E)) \equiv B(\nabla(\Phi, E))$. Now, by **(ESF7)**, $B(\nabla(\Phi_M, E)) \wedge B(\nabla(E^*, E)) \vdash B(\nabla(\Phi[E^*/i], E))$. From this, we obtain that $B(\nabla(\Phi, E)) \wedge B(\nabla(E^*, E)) \vdash B(\nabla(\Phi[E^*/i], E))$, as desired. \square

PROOF OF PROPOSITION 14. Let us show that the indifference operator, ∇^{Ind} , the quasi-indifference operator, ∇^{Ind^*} , the projective operator, ∇^π , the Σ -projective operator, $\nabla^{\Sigma\pi}$, and every aggregation-based ES basic merging operator ∇^F (defined from an aggregation function F satisfying composition and decomposition) satisfy **(ESF-S)**, but the max operator, ∇^{max} , does not satisfy it.

As we see in Propositions 2 and 4, **(ESF7)** and **(ESF8)** hold for the indifference operator and for any aggregation-based ES basic merging operator ∇^F , defined by means of an aggregation function F satisfying composition and decomposition. Thus, from Proposition 13 it follows that they satisfy **(ESF-S)**.

Let us show now that the quasi-indifference operator and the Σ -projective operator satisfy **(ESF-S)**. To do so, it suffices to show that their respective associated assignments satisfy **(S)**. Let us consider thus a finite society N , an agent i in N , a N -profile Φ , an epistemic state (total preorder) \succeq^* , and a pair of interpretations w, w' in \mathcal{W}_P .

To show that the quasi-indifference assignment, $\Phi \mapsto \succeq_\Phi^{\text{Ind}^*}$, satisfies **(S)**, suppose that $w \succeq_\Phi^{\text{Ind}^*} w'$ and $w \succeq_{\succeq^*} w'$. We have to prove that $w \succeq_{\Phi[\succeq^*/i]}^{\text{Ind}^*} w'$. To do so let us consider the following two cases about the number of agents in N : (1) N has a sole agent, and (2) N has at least two agents.

Case 1 [N has a sole agent]: In this case $\Phi[\succeq^*/i] = \succeq^*$. Thus, since the quasi-indifference assignment is structure preserving, we have $\succeq_{\Phi[\succeq^*/i]}^{\text{Ind}^*} = \succeq^*$ and $\succeq_{\succeq^*}^{\text{Ind}^*} = \succeq^*$. From this and the fact that $w \succeq_{\succeq^*}^{\text{Ind}^*} w'$, we obtain $w \succeq_{\Phi[\succeq^*/i]}^{\text{Ind}^*} w'$.

Case 2 [N has at least two agents]: In this case, by the definition of the quasi-indifference assignment we obtain that $\succeq_{\Phi[\succeq^*/i]}^{\text{Ind}^*}$ is the flat order. From this we straightforwardly get that $w \succeq_{\Phi[\succeq^*/i]}^{\text{Ind}^*} w'$.

In any case we have got $w \succeq_{\Phi[\succeq^*/i]}^{\text{Ind}^*} w'$. This shows that the quasi-indifference operator satisfies **(S)**.

To show that the Σ -projective assignment, $\Phi \mapsto \succeq_\Phi^{\Sigma\pi}$, also satisfies **(S)**, assume $w \succeq_\Phi^{\Sigma\pi} w'$ and $w \succeq_{\succeq^*}^{\Sigma\pi} w'$. We have to see that $w \succeq_{\Phi[\succeq^*/i]}^{\Sigma\pi} w'$.

To do so, assume first that $i \neq d$. Thus, from the definition of the lexicographical preorder, since $w \succeq_\Phi^{\Sigma\pi} w'$ and $\succeq_\Phi^{\Sigma\pi} = \succeq^{\text{lex}(\succeq_d, \succeq_\Phi^{\Sigma\pi})}$, we get the following two cases: (3) $w \succ_d w'$, and (4) $w \simeq_d w'$ and $w \succeq_\Phi^{\Sigma\pi} w'$.

Case 3 [$w \succ_d w'$]: In this case, since $i \neq d$, we get $w \succeq^{\text{lex}(\succeq_d, \succeq_{\Phi[\succeq^*/i]})} w'$, that is, $w \succeq_{\Phi[\succeq^*/i]}^{\Sigma\pi} w'$.

Case 4 [$w \simeq_d w'$ and $w \succeq_{\Phi}^{\Sigma} w'$]: In this case, since $w \simeq_{\Phi}^{\Sigma\pi} w'$, $w \succeq_{\Phi}^{\Sigma\pi} w'$, by structure preserving, we obtain $w \simeq_i w'$ and $w \succeq^* w'$. Thus, $r_{\succeq_i}(w) = r_{\succeq_i}(w')$, $r_{\succeq^*}(w) \geq r_{\succeq^*}(w')$ and from the definition of the sum assignment we get $\sum r_{\succeq_j}(w) \geq \sum r_{\succeq_j}(w')$. From this, by addition properties, we obtain $r_{\succeq^*}(w) + \sum_{j \neq i} r_{\succeq_j}(w) \geq r_{\succeq^*}(w') + \sum_{j \neq i} r_{\succeq_j}(w')$, that is, $w \succeq_{\Phi[\succeq^*/i]}^{\Sigma} w'$. Therefore, $w \simeq_d w'$ and $w \succeq_{\Phi[\succeq^*/i]}^{\Sigma} w'$. This leads us to $w \succeq_{\Phi[\succeq^*/i]}^{\Sigma\pi} w'$.

Now, suppose $i = d$ and note that, by our assumption, we have $w \simeq_{\succeq_d}^{\Sigma\pi} w'$, $w \succeq_{\succeq^*}^{\Sigma\pi} w'$ and $w \succeq_{\Phi}^{\Sigma\pi} w'$. From this, by structure preserving, we obtain $w \simeq_d w'$, $w \succeq^* w'$. Moreover, since $\succeq_{\Phi}^{\Sigma\pi} = \succeq^{\text{lex}(\succeq_d, \succeq_{\Phi}^{\Sigma})}$, we get $w \succeq_{\Phi}^{\Sigma} w'$. Thus, since $w \succeq^* w'$, we have the following two cases: (5) $w \succ^* w'$, and (6) $w \simeq^* w'$.

Case 5 [$w \succ^* w'$]: In this case, we straightforwardly get $w \succeq^{\text{lex}(\succeq^*, \succeq_{\Phi[\succeq^*/d]})} w'$, that is, $w \succeq_{\Phi[\succeq^*/d]}^{\Sigma\pi} w'$.

Case 6 [$w \simeq^* w'$]: In this case, we proceed in a similar way to that in Case 4 to obtain $w \succeq_{\Phi[\succeq^*/d]}^{\Sigma} w'$. From this and the fact that $w \simeq^* w'$, we get $w \succeq_{\Phi[\succeq^*/d]}^{\Sigma\pi} w'$.

In any case, we have got $w \succeq_{\Phi[\succeq^*/i]}^{\Sigma\pi} w'$. This shows that the Σ -projective assignment satisfies **(S)**.

Finally, let us prove that the max operator does not satisfy **(ESF-S)**. To do so, by Proposition 12, we only have to show that the max assignment, $\Phi \mapsto \succeq_{\Phi}^{\text{max}}$, does not satisfy **(S)**. Consider thus the set of interpretations $\mathcal{W}_{\mathcal{P}} = \{w_1, w_2, w_3, w_4\}$ of a finite propositional language $\mathcal{L}_{\mathcal{P}}$ built from two propositional variables. Let $N = \{1, 2\}$ be a society of agents and consider the N -profile Φ given in Figure A.12a, and the epistemic state \succeq^* represented in Figure A.12b.



Figure A.12: Instability situation for the *max* operator

From structure preserving we get $w_1 \simeq_{\succeq_2}^{\text{max}} w_2$ and $w_1 \simeq_{\succeq_2}^{\text{max}*} w_2$, and some simple calculations lead us to $w_1 \simeq_{\Phi}^{\text{max}} w_2$ and $w_2 \succ_{\Phi[\succeq^*/2]}^{\text{max}} w_1$. From this we obtain $w_1 \succeq_{\Phi}^{\text{max}} w_2$, $w_1 \simeq_{\succeq_2}^{\text{max}} w_2$ and $w_1 \succeq_{\succeq^*}^{\text{max}} w_2$, but $w_2 \succ_{\Phi[\succeq^*/2]}^{\text{max}} w_1$. This shows that **(S)** does not hold. \square

PROOF OF PROPOSITION 15. Let ∇ be an ES basic merging operator, and suppose that $\Phi \mapsto \succeq_{\Phi}$ is its associated assignment. We will show that ∇ satisfies **(ESF-N)** iff $\Phi \mapsto \succeq_{\Phi}$ satisfies **(N)**. To do so, let us consider a society N in $\mathcal{F}^*(\mathcal{S})$.

In order to show the *only if* part, let us suppose that ∇ satisfies **(ESF-N)** and assume that d in N is a nominator for ∇ . Let us consider w, w' in $\mathcal{W}_{\mathcal{P}}$ such that $w \succ_{E_d} w'$ and, toward a contradiction, suppose that $w' \succ_{\Phi} w$. Thus, if E in \mathcal{E} is an epistemic state such that $\llbracket B(E) \rrbracket = \{w, w'\}$, from (B-Rep) we get that $B(\nabla(\Phi, E))$ and $B(\nabla(E_d, E))$ are inconsistent, a contradiction.

To show the *if* part, assume that $\Phi \mapsto \succeq_{\Phi}$ satisfies **(N)**. Thus, there exists an agent d in N such that, if $w \succ_{E_d} w'$, then $w \succeq_{\Phi^*} w'$, for every N -profile Φ^* and every pair of interpretations w, w' in $\mathcal{W}_{\mathcal{P}}$. Toward a contradiction, suppose that d is not a nominator for ∇ . Then, there exist an N -profile Φ in $\text{P}(\mathcal{S}, \mathcal{E})$ and an epistemic state E in \mathcal{E} such that $B(\nabla(E_d, E)) \wedge B(\nabla(\Phi, E)) \vdash \perp$. Let us consider thus w, w' in $\mathcal{W}_{\mathcal{P}}$ such that $w \models B(\nabla(E_d, E))$ and $w' \models B(\nabla(\Phi, E))$. From (B-Rep) we get $w \succ_{E_d} w'$ and $w' \succ_{\Phi} w$. But, since $w \succ_{E_d} w'$, from **(N)** we get $w \succeq_{\Phi} w'$, a contradiction. \square

PROOF OF PROPOSITION 16. Let us show that **(ESF-N)** holds for the indifference operator, ∇^{Ind} , the quasi-indifference operator, ∇^{Ind^*} , the projective operator, ∇^π and the Σ -projective operator, $\nabla^{\Sigma\pi}$, but it does not hold for the max operator, ∇^{max} or for any aggregation-based ES basic merging operators, ∇^F , which is defined through an aggregation function F satisfying composition, decomposition and discreteness; in particular, nor the sum operator, ∇^Σ , nor the Gmax operator, ∇^{Gmax} , satisfy it.

As the indifference operator, the projective operator and the Σ -projective operator are dictatorial ES basic merging operators (cf. Proposition 6), from Observation 9 we get that **(ESF-N)** holds for them. In order to prove that the quasi-indifference operator satisfies **(ESF-N)**, consider a finite society N and let us prove that there exists an agent d in N such that $B(\nabla^{\text{Ind}^*}(\Phi, \succeq)) \wedge B(\nabla^{\text{Ind}^*}(\succeq_d, \succeq)) \not\vdash \perp$, for every N -profile Φ and every epistemic state \succeq . If N has a unique agent, then the result is straightforward. Suppose, then, that N has at least two agents. Thus, $\succeq_\Phi^{\text{Ind}^*}$ is the flat order over \mathcal{W}_P . From this, by (B-Rep), we get $B(\nabla^{\text{Ind}^*}(\Phi, \succeq)) \equiv B(\succeq)$. Thus, by **(ESF1)**, we obtain $B(\nabla^{\text{Ind}^*}(\Phi, \succeq)) \wedge B(\nabla^{\text{Ind}^*}(\succeq_d, \succeq)) \not\vdash \perp$, for every agent d in N . This also shows that every agent in N is a nominator for ∇^{Ind^*} .

The next remark shows that **(ESF-N)** does not hold for any aggregation-based ES basic merging operator ∇^F , built from the max function or an aggregation function F satisfying composition, decomposition and discreteness.

Remark 4. If F is the max function or an aggregation function satisfying composition, decomposition and discreteness, then there is no finite society N , with at least two agents, that admits a nominator for ∇^F .

To show this remark, we proceed in a similar way to that of the proof of Remark 3 (cf. Proof of Proposition 6), by noting that, in any of the cases presented there, $B(\nabla^F(\Phi, \succeq)) \wedge B(\nabla^F(\succeq_d, \succeq)) \vdash \perp$. \square

PROOF OF PROPOSITION 17. Let us study the manipulability of the indifference operator, ∇^{Ind} , the quasi-indifference operator, ∇^{Ind^*} , the max operator, ∇^{max} , the projective operator, ∇^π , the Σ -projective operator, $\nabla^{\Sigma\pi}$, and every aggregation-based ES basic merging operator ∇^F , defined from an aggregation function F for which composition, decomposition and discreteness hold; especially the sum operator, ∇^Σ , and the Gmax operator, ∇^{Gmax} .

- (i) Let us show that the indifference operator, ∇^{Ind} , is strategy-proof. To do so note that, given a finite society N , an N -profile Φ , an agent i in N and a couple of epistemic states \succeq, \succeq^* , we have $B(\nabla^{\text{Ind}}(\Phi, \succeq)) \equiv B(\nabla^{\text{Ind}}(\Phi[\succeq^*/i], \succeq))$. Therefore, $B(\nabla^{\text{Ind}}(\Phi, \succeq)) \sqsubseteq_{\succeq_{\succeq_i}^{\text{Ind}}} B(\nabla^{\text{Ind}}(\Phi[\succeq^*/i], \succeq))$, for every belief lifting $\succeq \mapsto \sqsupset_{\succeq}$, every finite society N , every N -profile Φ , every agent i in N and every couple of epistemic states \succeq, \succeq^* . This shows the strategy-proofness of the indifference operator.
- (ii) Let us show now that the quasi-indifference operator, ∇^{Ind^*} , and the projective operator, ∇^π , are both strategy-proof with respect to the possibilistic, the pessimistic and Kelly belief liftings, but they are manipulable with respect to the precise-leximax belief lifting.

To show the strategy-proofness of the quasi-indifference operator with respect to the possibilistic belief lifting, $\succeq \mapsto \sqsupset_{\succeq}^{\Pi}$, consider a finite society N , an N -profile Φ , an agent i in N , a couple of epistemic states \succeq, \succeq^* in \mathcal{E} , and consider the following two cases concerning the size of N : (1) N has a sole agent, and (2) N has at least two agents.

Case 1 [N has a sole agent]: In this case, N is formed by the agent i , $\Phi = \succeq_i$ and $\Phi[\succeq^*/i] = \succeq^*$. By (B-Rep), because $\Phi \mapsto \succeq_\Phi^{\text{Ind}^*}$ is structure preserving, we get $w \succeq_{\succeq_i}^{\text{Ind}^*} w'$, for very couple of interpretations w, w' in \mathcal{W}_P , with $w \models B(\nabla^{\text{Ind}^*}(\Phi, \succeq))$ and $w' \models B(\nabla^{\text{Ind}^*}(\Phi[\succeq^*/i], \succeq))$. Therefore, from the definition of the possibilistic lifting, we have $B(\nabla^{\text{Ind}^*}(\Phi, \succeq)) \sqsubseteq_{\succeq_{\succeq_i}^{\text{Ind}^*}} B(\nabla^{\text{Ind}^*}(\Phi[\succeq^*/i], \succeq))$.

Case 2 [N has at least two agents]: In this case, from the definition of the quasi-indifference operator we obtain $B(\nabla^{\text{Ind}^*}(\Phi[\succeq^*/i], \succeq)) \equiv B(\nabla^{\text{Ind}^*}(\Phi, \succeq))$. From this it straightforwardly follows that $B(\nabla^{\text{Ind}^*}(\Phi, \succeq)) \sqsubseteq_{\succeq_{\succeq_i}^{\text{Ind}^*}} B(\nabla^{\text{Ind}^*}(\Phi[\succeq^*/i], \succeq))$.

In any case, we have got $B(\nabla^{\text{Ind}^*}(\Phi, \succeq)) \sqsubseteq_{\succeq_{\succeq_i}^{\text{Ind}^*}} B(\nabla^{\text{Ind}^*}(\Phi[\succeq^*/i], \succeq))$, showing the strategy-proofness of the quasi-indifference operator with respect to the possibilistic belief lifting.

Similarly, one can show the strategy-proofness of the quasi-indifference operator with respect to the pessimistic belief lifting and Kelly belief lifting.

To show that the quasi-indifference operator is manipulable with respect to the precise-leximax belief lifting, $\succeq \mapsto \sqsupset_{\succeq}^{\text{plm}}$, consider a pair of interpretations w, w' in $\mathcal{W}_{\mathcal{P}}$, a single-profile $\Phi = \succeq_i$ and a couple of epistemic states \succeq, \succeq^* such that $w \simeq_i w', w \succ^* w'$ and $\llbracket B(\succeq) \rrbracket = \{w, w'\}$. Since the quasi-indifference assignment is structure preserving, from (B-Rep) we get $B(\nabla^{\text{Ind}^*}(\Phi, \succeq)) \equiv \varphi_{w, w'}$ and $B(\nabla^{\text{Ind}^*}(\Phi[\succeq^*/i], \succeq)) \equiv \varphi_w$. Moreover, because $w \simeq_i w'$, we have $w \simeq_{\succeq_i}^{\text{Ind}^*} w'$. From this and the definition of the precise-leximax order, we obtain $(w) \succ_{\succeq_{\succeq_i}^{\text{plm}}}^{\text{plm}}(w, w')$ and $(w) \succ_{\succeq_{\succeq_i}^{\text{plm}}}^{\text{plm}}(w', w)$. Thus, by the definition of the precise-leximax belief lifting, $B(\nabla^{\text{Ind}^*}(\Phi[\succeq^*/i], \succeq)) \sqsupset_{\succeq_{\succeq_i}^{\text{plm}}}^{\text{plm}} B(\nabla^{\text{Ind}^*}(\Phi, \succeq))$.

Now, let us show that the projective operator, ∇^{π} , is strategy-proof with respect to the possibilistic, the pessimistic and Kelly belief liftings and its manipulation with respect to the precise-leximax lifting. To show that ∇^{π} is strategy-proof with respect to the possibilistic lifting, $\succeq \mapsto \sqsupset_{\succeq}^{\Pi}$, consider a finite society N , an agent i in N , an N -profile Φ , a couple of epistemic states \succeq, \succeq^* , and consider the following two cases for i : (3) $i \neq d$, and (4) $i = d$; where $d = \max(N)$.

Case 3 [$i \neq d$]: In this case, by definition of the projective operator, $B(\nabla^{\pi}(\Phi, \succeq)) \equiv B(\nabla^{\pi}(\Phi[\succeq^*/i], \succeq))$. From this we straightforwardly get $B(\nabla^{\pi}(\Phi, \succeq)) \sqsupset_{\succeq_{\succeq_i}^{\Pi}}^{\Pi} B(\nabla^{\pi}(\Phi[\succeq^*/i], \succeq))$.

Case 4 [$i = d$]: In this case, from structure preserving and the definition of the projective assignment, we obtain $\succeq_{\succeq_d}^{\pi} = \succeq_{\succeq_{\Phi}}^{\pi}$. Thus, if w, w' in $\mathcal{W}_{\mathcal{P}}$ are interpretations such that $w \models B(\nabla^{\pi}(\Phi, \succeq))$ and $w' \models B(\nabla^{\pi}(\Phi[\succeq^*/i], \succeq))$, by (B-Rep), $w \succeq_{\succeq_{\Phi}}^{\pi} w'$, that is, $w \succeq_{\succeq_d}^{\pi} w'$. Then, from the definition of the possibilistic belief lifting we obtain $B(\nabla^{\pi}(\Phi, \succeq)) \sqsupset_{\succeq_{\succeq_i}^{\Pi}}^{\Pi} B(\nabla^{\pi}(\Phi[\succeq^*/i], \succeq))$.

Therefore, in any case we have got $B(\nabla^{\pi}(\Phi, \succeq)) \sqsupset_{\succeq_{\succeq_i}^{\Pi}}^{\Pi} B(\nabla^{\pi}(\Phi[\succeq^*/i], \succeq))$. This shows the strategy-proofness of the projective operator with respect to the possibilistic lifting.

Using a similar argument to the previous one, we can show that ∇^{π} is strategy-proof with respect to the pessimistic and Kelly belief liftings. Furthermore, similarly to how we have proved the manipulation of the quasi-indifference operator with respect to the precise-leximax belief lifting, we can show the manipulation of the projective operator with respect to that belief lifting.

- (iii) Let us show that, for any aggregation function F satisfying composition, decomposition and discreteness, the aggregation-based ES basic merging operator ∇^F is absolutely G-manipulable. To do so, note that, by Propositions 4 and 6, ∇^F is an ES basic merging operator for which **(ESF-D)** does not hold. From this, by Theorem 4, we get that ∇^F is absolutely G-manipulable.
- (iv) Finally, let us see that the sum operator, ∇^{Σ} , the Gmax operator, ∇^{Gmax} , the max operator, ∇^{max} and the Σ -projective operator, $\nabla^{\Sigma\pi}$, are absolutely manipulable.

As we highlighted earlier, the situation given in Example 8 is an absolute manipulability situation for the sum operator, as well as for the Gmax and max operators, as it might be seen after some simple calculations. Thus, we only have to show that the Σ -projective operator is absolutely manipulable.

Assume that $\mathcal{W}_{\mathcal{P}} = \{w_1, w_2, w_3, w_4\}$ is the set of interpretations of a language $\mathcal{L}_{\mathcal{P}}$, and consider the following manipulability situation: the society $N = \{1, 2, 3\}$, the N -profile $\Phi = (\succeq_1, \succeq_2, \succeq_3)$ represented in Figure A.13a, the epistemic state \succeq^* given in Figure A.13b, and a restriction \succeq such that $\llbracket B(\succeq) \rrbracket = \{w, w'\}$.¹⁵

Some simple calculations lead us to $w_2 \succ_{\Phi}^{\Sigma} w_1$ and $w_1 \succ_{\Phi[\succeq^*/1]}^{\Sigma} w_2$. Thus, since $w_1 \simeq_3 w_2$ and $\succeq_{\Phi}^{\Sigma\pi} = \succeq^{\text{lex}}(\succeq_3, \succeq_{\Phi}^{\Sigma})$, we obtain $w_2 \succ_{\Phi}^{\Sigma\pi} w_1$ and $w_1 \succ_{\Phi[\succeq^*/1]}^{\Sigma\pi} w_2$. From this, by (B-Rep), we have $B(\nabla^{\Sigma\pi}(\Phi, \succeq)) \equiv \varphi_{w_2}$ and $B(\nabla^{\Sigma\pi}(\Phi[\succeq^*/1], \succeq)) \equiv \varphi_{w_1}$. Then, since $w_1 \succ_1 w_2$, from structure preserving we get $w_1 \succ_{\succeq_1}^{\Sigma\pi} w_2$. Thus, for any belief lifting $\succeq \mapsto \sqsupset_{\succeq}$, $B(\nabla^{\Sigma\pi}(\Phi[\succeq^*/1], \succeq)) \sqsupset_{\succeq_{\succeq_1}^{\Sigma\pi}}^{\Sigma\pi} B(\nabla^{\Sigma\pi}(\Phi, \succeq))$. \square

PROOF OF PROPOSITION 18. Let ∇ be an ES basic merging operator, assume that $\Phi \mapsto \succeq_{\Phi}$ is its associated assignment and suppose that ∇ is nonmanipulable with respect to a G-belief lifting $\succeq \mapsto \sqsupset_{\succeq}$.

¹⁵We may also use this situation to show the absolute manipulation of the sum, the Gmax and the max operators.

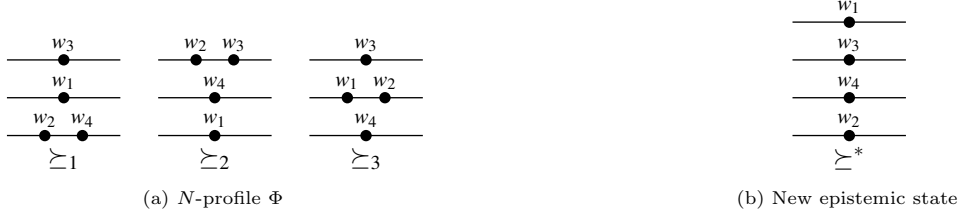


Figure A.13: Manipulability situation for the Σ -projective operator

- (i) In order to show that **(ESF-NI)** entails **(ESF-P)**, assume that ∇ satisfies **(ESF-NI)** and, towards a contradiction, suppose that **(ESF-P)** does not hold. Thus, there exist a finite society N , an N -profile Φ and a pair of epistemic states E, E' in \mathcal{E} such that, for every agent i , $B(\nabla(E_i, E)) \wedge B(E') \vdash \perp$, $\bigwedge_{E_i \in \Phi} B(\nabla(E_i, E)) \not\vdash \perp$, but $B(\nabla(\Phi, E)) \wedge B(E') \not\vdash \perp$.

Consider then a couple of models w, w' such that $w \models \bigwedge_{E_i \in \Phi} B(\nabla(E_i, E))$ and $w' \models B(\nabla(\Phi, E)) \wedge B(E')$ and an epistemic state E'' in \mathcal{E} such that $\llbracket B(E'') \rrbracket = \{w, w'\}$. Note that, by **(ESF1)**, $B(E'') \vdash B(E)$ and therefore $B(E'') \equiv B(E) \wedge B(E'')$. Then, given an agent i in N , since $w \models B(\nabla(E_i, E))$, $B(\nabla(E_i, E)) \wedge B(E') \vdash \perp$ and $w' \models B(E')$, then $B(\nabla(E_i, E)) \wedge B(E'')$ has a sole model, namely w . Furthermore, from **(ESF3)** and **(ESF4)** we get $B(\nabla(E_i, E'')) \equiv B(\nabla(E_i, E)) \wedge B(E'')$. Thus, w is the unique model of $B(\nabla(E_i, E''))$, for all i in N . Moreover, since $w' \models B(\nabla(\Phi, E)) \wedge B(E'')$, from **(ESF3)** and **(ESF4)** we have $B(\nabla(\Phi, E'')) \equiv B(\nabla(\Phi, E)) \wedge B(E'')$. Therefore, $w' \models B(\nabla(\Phi, E''))$, that is, $B(\nabla(\Phi, E'')) \not\equiv \varphi_w$.

Now, by **(ESF-NI)**, we can consider an N -profile Φ^* such that $B(\nabla(\Phi^*, E'')) \equiv \varphi_w$. Let us define then the sequence $\Phi_0, \Phi_1, \dots, \Phi_n$ as follows: $\Phi_0 = \Phi$ and $\Phi_t = \Phi_{t-1}^{[E_{i_t}^*/i_t]}$, for $t > 0$. Note that $\Phi_n = \Phi^*$. Then $B(\nabla(\Phi_0, E'')) \not\equiv \varphi_w$ and $B(\nabla(\Phi_n, E'')) \equiv \varphi_w$.

Let Φ_k be the first in the sequence such that $B(\nabla(\Phi_k, E)) \equiv \varphi_w$, and note that $k > 0$. Let us assume $r = k - 1$ and $i = i_k$. Then, $\Phi_k = \Phi_r^{[E_i^*/i]}$ and, $B(\nabla(\Phi_r, E)) \equiv \varphi_{w'}$ or $B(\nabla(\Phi_r, E)) \equiv \varphi_{w, w'}$. Since w is the unique model of $B(\nabla(E_i, E''))$, from (B-Rep) we get $w \succ_{E_i} w'$. Thus, $\varphi_w \sqsupset_{\succeq_{E_i}} \varphi_{w'}$ and from **G2** we get $\varphi_w \sqsupset_{\succeq_{E_i}} \varphi_{w, w'}$. Therefore, for the two possible outcomes of $B(\nabla(\Phi_r, E))$, we obtain $B(\nabla(\Phi_r^{[E_i^*/i]}, E)) \sqsupset_{\succeq_{E_i}} B(\nabla(\Phi_r, E))$. This shows the manipulation of ∇ , a contradiction.

- (ii) Let us show now that **(ESF-S)** holds iff **(ESF-I)** also holds.

In order to show the *only if* part, assume that **(ESF-S)** holds and, towards a contradiction, suppose that **(ESF-I)** does not hold. Consider thus a pair of epistemic profiles $\Phi = (E_{i_1}, \dots, E_{i_n})$ and $\Phi^* = (E_{i_1}^*, \dots, E_{i_n}^*)$, and an epistemic state E in \mathcal{E} such that $B(\nabla(E_i, E')) \equiv B(\nabla(E_i^*, E'))$, for every E' in \mathcal{E} , with $B(E') \vdash B(E)$, but $B(\nabla(\Phi, E)) \not\equiv B(\nabla(\Phi^*, E))$.

Suppose $B(\nabla(\Phi, E)) \not\vdash B(\nabla(\Phi^*, E))$ (the other case is analogous) and consider a pair w, w' in $\mathcal{W}_{\mathcal{P}}$ such that $w \models B(\nabla(\Phi, E)) \wedge \neg B(\nabla(\Phi^*, E))$ and $w' \models B(\nabla(\Phi^*, E))$. Let E'' in \mathcal{E} be an epistemic state such that $\llbracket B(E'') \rrbracket = \{w, w'\}$. Thus, from **(ESF1)**, we get $B(E'') \equiv B(E) \wedge B(E'')$.

Since $w \models B(\nabla(\Phi, E)) \wedge B(E'')$, by **(ESF3)** and **(ESF4)**, $B(\nabla(\Phi, E'')) \equiv B(\nabla(\Phi, E)) \wedge B(E'')$. Thus, $w \models B(\nabla(\Phi, E''))$. Similarly, because $w' \models B(\nabla(\Phi^*, E)) \wedge B(E'')$, again, from **(ESF3)** and **(ESF4)**, we get $B(\nabla(\Phi^*, E'')) \equiv B(\nabla(\Phi^*, E)) \wedge B(E'')$. Indeed, w' is the unique model of $B(\nabla(\Phi^*, E''))$. Thus, we have $B(\nabla(\Phi, E'')) \not\equiv \varphi_{w'}$ and $B(\nabla(\Phi^*, E'')) \equiv \varphi_{w'}$.

Consider thus the sequence $\Phi_0, \Phi_1, \dots, \Phi_n$ built as in part (i): $\Phi_0 = \Phi$ and $\Phi_t = \Phi_{t-1}^{[E_{i_t}^*/i_t]}$, for $t > 0$. Then, $\Phi_n = \Phi^*$, $B(\nabla(\Phi_0, E'')) \not\equiv \varphi_{w'}$ and $B(\nabla(\Phi_n, E'')) \equiv \varphi_{w'}$.

Let Φ_k be the first term of the sequence such that $B(\nabla(\Phi_k, E'')) \equiv \varphi_{w'}$, and note that $k > 0$. Then, if $r = k - 1$ and $i = i_k$, we obtain $B(\nabla(\Phi_r^{[E_i^*/i]}, E'')) \equiv \varphi_{w'}$ and $B(\nabla(\Phi_r, E'')) \not\equiv \varphi_{w'}$, that is, $B(\nabla(\Phi_r, E'')) \equiv \varphi_w$ or $B(\nabla(\Phi_r, E'')) \equiv \varphi_{w, w'}$.

Consider now the following cases for $B(\nabla(E_i^*, E''))$: (1) $w \not\models B(\nabla(E_i^*, E''))$, and (2) $w \models B(\nabla(E_i^*, E''))$.

Case 1 [$w \not\models B(\nabla(E_i^*, E''))$]: Since $B(\nabla(E_i^*, E'')) \equiv B(\nabla(E_i, E''))$, from (B-Rep) we get $w' \succ_{E_i} w$.

Thus, $\varphi_{w'} \sqsupset_{\succeq_{E_i}} \varphi_w$ and, by **G2**, $\varphi_{w'} \sqsupset_{\succeq_{E_i}} \varphi_{w, w'}$. Therefore, for any of the two possible outcomes of $B(\nabla(\Phi_r, E''))$, we have $B(\nabla(\Phi_r^{[E_i^*/i]}, E'')) \sqsupset_{\succeq_{E_i}} B(\nabla(\Phi_r, E''))$, contradicting the non-manipulation of ∇ .

Case 2 [$w \models B(\nabla(E_i^*, E''))$]: In this case we get $w \models B(\nabla(\Phi_r, E'')) \wedge B(\nabla(E_i^*, E''))$. From this and since $B(\nabla(\Phi_r[E_i^*/i], E'')) \equiv \varphi_{w'}$, we have $B(\nabla(\Phi_r, E'')) \wedge B(\nabla(E_i^*, E'')) \not\models B(\nabla(\Phi_r[E_i^*/i], E''))$. Thus, from **(ESF-S)** we obtain $B(\nabla(E_i, E'')) \not\equiv B(E'')$ and, since $B(\nabla(E_i, E'')) \equiv B(\nabla(E_i^*, E''))$, we also have $w' \not\models B(\nabla(E_i^*, E''))$. Then, by (B-Rep), $w \succ_{E_i^*} w'$. From this we get $\varphi_w \sqsupset_{\succeq_{E_i^*}} \varphi_{w'}$ and, by **G1**, $\varphi_{w, w'} \sqsupset_{\succeq_{E_i^*}} \varphi_{w'}$. Now, since $\Phi_r = \Phi_k[E_i^*/i]$, we get $B(\nabla(\Phi_k[E_i^*/i], E'')) \equiv \varphi_w$ or $B(\nabla(\Phi_k[E_i^*/i], E'')) \equiv \varphi_{w, w'}$. Thus, for any of the two possible outcomes of $B(\nabla(\Phi_k[E_i^*/i], E''))$ we obtain $B(\nabla(\Phi_k[E_i^*/i], E'')) \sqsupset_{\succeq_{E_i^*}} B(\nabla(\Phi_k, E''))$, contradicting the non-manipulation of ∇ .

In any case, we have obtained a contradiction. Therefore **(ESF-S)** entails **(ESF-I)**.

To show the *if* part, assume that **(ESF-I)** holds and, towards a contradiction, suppose that **(ESF-S)** does not hold. Consider then a society N , an agent i in N , an N -profile Φ and a pair of epistemic states E, E^* in \mathcal{E} such that $B(\nabla(E_i, E)) \equiv B(E)$ and $B(\nabla(\Phi, E)) \wedge B(\nabla(E^*, E)) \not\models B(\nabla(\Phi[E^*/i], E))$. Let w, w' be a pair of models in $\mathcal{W}_{\mathcal{P}}$ such that $w \models B(\nabla(\Phi, E)) \wedge B(\nabla(E^*, E)) \wedge \neg B(\nabla(\Phi[E^*/i], E))$ and $w' \models B(\nabla(\Phi[E^*/i], E))$, and let E'' be an epistemic state in \mathcal{E} such that $\llbracket B(E'') \rrbracket = \{w, w'\}$. Let us note that, by **(ESF1)**, $B(E'') \equiv B(E) \wedge B(E'')$. Moreover, since $B(\nabla(E_i, E)) \equiv B(E)$, $w' \models B(\nabla(\Phi[E^*/i], E)) \wedge B(E'')$ and $w \models B(\nabla(E^*, E)) \wedge B(\nabla(\Phi, E)) \wedge B(E'')$, from **(ESF3)** and **(ESF4)** we have the following equivalences:

- $B(\nabla(E_i, E'')) \equiv B(E'')$,
- $B(\nabla(E^*, E'')) \equiv B(\nabla(E^*, E)) \wedge B(E'')$, and
- $B(\nabla(\Phi, E'')) \equiv B(\nabla(\Phi, E)) \wedge B(E'')$,
- $B(\nabla(\Phi[E^*/i], E'')) \equiv B(\nabla(\Phi[E^*/i], E)) \wedge B(E'')$

From this we get $w \models B(\nabla(\Phi, E'')) \wedge B(\nabla(E^*, E''))$ and w' is the sole model of $B(\nabla(\Phi[E^*/i], E''))$. Furthermore, $B(\nabla(\Phi, E'')) \not\equiv B(\nabla(\Phi[E^*/i], E''))$.

Since $w \models B(\nabla(E^*, E''))$, from (B-Rep) we get the following cases: (3) $w \simeq_{E^*} w'$, and (4) $w \succ_{E^*} w'$.

Case 3 [$w \simeq_{E^*} w'$]: In this case note that, by (B-Rep), $B(\nabla(E_i, E')) \equiv B(\nabla(E^*, E'))$, for every E' in \mathcal{E} , with $B(E') \vdash B(E'')$. As a matter of fact, if $B(E')$ has a unique model, by **(ESF1)**, both $B(\nabla(E_i, E'))$ and $B(\nabla(E^*, E'))$ are equivalent to $B(E')$. If $B(E')$ has two models, from (B-Rep) we obtain that both $B(\nabla(E_i, E'))$ and $B(\nabla(E^*, E'))$ are equivalent to $B(E'')$. Therefore, from **(ESF-I)** we have $B(\nabla(\Phi, E'')) \equiv B(\nabla(\Phi[E^*/i], E''))$, a contradiction.

Case 4 [$w \succ_{E^*} w'$]: In this case, assume $\Phi^* = \Phi[E^*/i]$ and note that $\Phi^*[E_i^*/i] = \Phi$. Thus, we have $B(\nabla(\Phi^*, E'')) \equiv \varphi_{w'}$ and, $B(\nabla(\Phi^*[E_i^*/i], E'')) \equiv \varphi_w$ or $B(\nabla(\Phi^*[E_i^*/i], E'')) \equiv \varphi_{w, w'}$. Now, since $w \succ_{E^*} w'$, we have $\varphi_w \sqsupset_{\succeq_{E^*}} \varphi_{w'}$ and, by **G1**, $\varphi_{w, w'} \sqsupset_{\succeq_{E^*}} \varphi_{w'}$. Therefore, for any of the two possible outcomes of $B(\nabla(\Phi^*[E_i^*/i], E''))$, we get $B(\nabla(\Phi^*[E_i^*/i], E'')) \sqsupset_{\succeq_{E^*}} B(\nabla(\Phi^*, E''))$, contradicting the non-manipulation of ∇ .

In any case, we have obtained a contradiction. Then **(ESF-I)** entails **(ESF-S)**, as desired. \square

PROOF OF THEOREM 6. To show this result, let us first state the following remark which will be very useful for this purpose.

Remark 5. If an ES basic merging operator ∇ is strategy-proof with respect to a belief lifting satisfying the **(G2)** property, then ∇ also satisfies the following:

If $B(\nabla(\Phi, E)) \equiv \varphi_w$ and E^* in \mathcal{E} is such that $\max(\succeq_{E^*}) = \{w\}$, then $B(\nabla(\Phi[E^*/i], E)) \equiv \varphi_w$.

In order to prove Remark 5, suppose that ∇ is strategy-proof with respect to a belief lifting $\succeq \mapsto \sqsupset_{\succeq}$ satisfying the **(G2)** property. Assume that $B(\nabla(\Phi, E)) \equiv \varphi_w$, E^* in \mathcal{E} is such that $\max(\succeq_{E^*}) = \{w\}$ and, toward a contradiction, suppose $B(\nabla(\Phi[E^*/i], E)) \not\equiv \varphi_w$.

Let w' be a model of $B(\nabla(\Phi[E^*/i], E))$, with $w' \neq w$, and consider an epistemic state E' in \mathcal{E} such that $\llbracket B(E') \rrbracket = \{w, w'\}$. By **(ESF1)**, we get $B(E') \equiv B(E) \wedge B(E')$, $B(\nabla(\Phi, E)) \wedge B(E') \not\models \perp$ and $B(\nabla(\Phi[E^*/i], E)) \wedge B(E') \not\models \perp$. From this, by **(ESF3)** and **(ESF4)**, we obtain $B(\nabla(\Phi, E')) \equiv \varphi_w$ and $B(\nabla(\Phi[E^*/i], E')) \not\equiv \varphi_w$, that is, $B(\nabla(\Phi[E^*/i], E')) \equiv \varphi_{w'}$ or $B(\nabla(\Phi[E^*/i], E')) \equiv \varphi_{w, w'}$.

Put $\Phi^* = \Phi^{[E^*/i]}$, and note that $\Phi^*[E^*/i] = \Phi$. Thus, $B(\nabla(\Phi^*[E^*/i], E')) \equiv \varphi_w$, and $B(\nabla(\Phi^*, E')) \equiv \varphi_{w'}$ or $B(\nabla(\Phi^*, E')) \equiv \varphi_{w, w'}$.

Now, since $w \succ_{E^*} w'$, we have $\varphi_w \sqsupset_{\succeq_{E^*}} \varphi_{w'}$ and, by the **(G2)** property, $\varphi_w \sqsupset_{\succeq_{E^*}} \varphi_{w, w'}$. Therefore, $B(\nabla(\Phi^*[E^*/i], E')) \sqsupset_{\succeq_{E^*}} B(\nabla(\Phi^*, E'))$, for any of the two possible outcomes of $B(\nabla(\Phi^*, E'))$. This shows the manipulation of ∇ with respect to $\succeq \mapsto \sqsupset_{\succeq}$, a contradiction.

Continuing with the proof of Theorem 6, suppose that ∇ is a strategy-proof ES basic merging with respect to a G-belief lifting $\succeq \mapsto \sqsupset_{\succeq}$, and assume that $\Phi \mapsto \succeq_{\Phi}$ is its associated assignment. Consider also the new set of epistemic states \mathcal{E}^* formed by all the epistemic states E in \mathcal{E} such that \succeq_E is a linear order over $\mathcal{W}_{\mathcal{P}}$. Note that, since ∇ satisfies **(ESF-SSD)**, the set of epistemic states \mathcal{E}^* is not empty.

Aiming to show that ∇ satisfies **(ESF-N)**, let us consider the operator $\nabla^* : P(\mathcal{S}, \mathcal{E}^*) \times \mathcal{E} \rightarrow \mathcal{E}$ defined by $\nabla^*(\Phi, E) = \nabla(\Phi, E)$, and note that $\Phi \mapsto \succeq_{\Phi}$ also is the associated assignment to ∇^* , when the assignment is restricted to $P(\mathcal{S}, \mathcal{E}^*)$. Also note that, since ∇ is strategy-proof with respect to $\succeq \mapsto \sqsupset_{\succeq}$, ∇^* also is. Now, let us see that ∇^* satisfies part (iii) of **(ESF-SD)**, as well as **(ESF-P)** and **(ESF-I)**.

Since ∇ satisfies **(ESF-SSD)**, we straightforwardly get that ∇^* satisfies the part (iii) of **(ESF-SD)**. To show that ∇^* satisfies **(ESF-P)**, it suffices to prove that ∇ satisfies it; but this fact straightforwardly follows from Proposition 18, since ∇ satisfies **(ESF-NI)** and it is nonmanipulable with respect to a G-belief lifting, namely, $\succeq \mapsto \sqsupset_{\succeq}$.

In order to show that ∇^* satisfies **(ESF-I)**, by Proposition 18, we only have to show that ∇^* satisfies **(ESF-S)**, since ∇^* is strategy-proof with respect to the G-belief lifting $\succeq \mapsto \sqsupset_{\succeq}$. Before that, we must note that Proposition 18 also holds when we restrict the whole set $P(\mathcal{S}, \mathcal{E})$ to the set $P(\mathcal{S}, \mathcal{E}^*)$. The reader may replicate the proof of Proposition 18 for this case.

Let us then show that **(ESF-S)** is fulfilled for ∇^* . To do so, consider a society N , an N -profile Φ in $P(\mathcal{S}, \mathcal{E}^*)$, and a couple of epistemic states, E in \mathcal{E} and E' in \mathcal{E}^* , such that $B(\nabla^*(E_i, E)) \equiv B(E)$. Since Φ is in $P(\mathcal{S}, \mathcal{E}^*)$, then \succeq_{E_i} is a linear order over $\mathcal{W}_{\mathcal{P}}$. From this and (B-Rep), we get that both $B(\nabla^*(E_i, E))$ and $B(E)$ are complete. Thus, from **(ESF1)** we get $B(\nabla^*(\Phi, E)) \wedge B(\nabla^*(E^*, E)) \equiv B(E)$ and $B(\nabla^*(\Phi^{[E^*/i]}, E)) \equiv B(E)$. Therefore, $B(\nabla^*(\Phi, E)) \wedge B(\nabla^*(E^*, E)) \vdash B(\nabla^*(\Phi^{[E^*/i]}, E))$, as desired.

Now, since ∇^* satisfies part (iii) of **(ESF-SD)**, **(ESF-P)** and **(ESF-I)**, from Theorem 3, we have that ∇^* is a dictatorial operator – the reader can also replicate the proof of Theorem 3 in [56] to show that this result also holds if one considers $P(\mathcal{S}, \mathcal{E}^*)$ instead of the whole set $P(\mathcal{S}, \mathcal{E})$.

Finally, let us show that ∇ satisfies **(ESF-N)**. To do so we only have to show that every dictator for ∇^* , is also a nominator for ∇ . Consider then a society N , an N -dictator d for ∇^* and, toward a contradiction, suppose that d is not a nominator for ∇ . Thus, there exist an N -profile Φ in $P(\mathcal{S}, \mathcal{E})$ and an epistemic state E in \mathcal{E} such that $B(\nabla(\Phi, E)) \wedge B(\nabla(E_d, E)) \vdash \perp$.

Let w be a model of $B(\nabla(E_d, E))$ and w' be a model of $B(\nabla(\Phi, E))$, and consider an epistemic state E' in \mathcal{E} such that $\llbracket B(E') \rrbracket = \{w, w'\}$. Thus, $B(E') \equiv B(E) \wedge B(E')$, $B(\nabla(E_d, E)) \wedge B(E') \not\vdash \perp$ and $B(\nabla(\Phi, E)) \wedge B(E') \not\vdash \perp$. From this, by **(ESF3)** and **(ESF4)**, we have $B(\nabla(E_d, E')) \equiv B(\nabla(E_d, E)) \wedge B(E')$ and $B(\nabla(\Phi, E')) \equiv B(\nabla(\Phi, E)) \wedge B(E')$. Therefore, $B(\nabla(\Phi, E')) \equiv \varphi_{w'}$ and, by (B-Rep), $w \succ_{E_d} w'$.

Let Φ^* be an N -profile in $P(\mathcal{S}, \mathcal{E})$ such that for all i in $N \setminus \{d\}$, E_i^* is an epistemic state in \mathcal{E}^* , $\max(\succeq_{E_i^*}) = \{w'\}$, and $E_d^* = E_d$. Thus, from Remark 5, we obtain $B(\nabla(\Phi^*, E')) \equiv \varphi_{w'}$.

If E^* in \mathcal{E}^* is such that $\max(\succeq_{E^*}) = \{w\}$, then both E^* and $\Phi^*[E^*/d]$ are epistemic profiles in $P(\mathcal{S}, \mathcal{E}^*)$ and, by (B-Rep), $B(\nabla^*(E^*, E')) \equiv \varphi_w$. Moreover, because d is a dictator for ∇^* , from **(ESF-SD)** we get $B(\nabla^*(\Phi^*[E^*/d], E')) \vdash B(\nabla^*(E^*, E'))$. From this, we have $B(\nabla(\Phi^*[E^*/d], E')) \equiv \varphi_w$. Now, since $w \succ_{E_d} w'$, we get $\varphi_w \sqsupset_{\succeq_{E_d}} \varphi_{w'}$, that is, $B(\nabla(\Phi^*[E^*/d], E')) \sqsupset_{\succeq_{E_d}} B(\nabla(\Phi^*, E'))$. This shows the manipulation of ∇ with respect to $\succeq \mapsto \sqsupset_{\succeq}$, a contradiction. \square

PROOF OF THEOREM 8. Consider \mathfrak{D} , the set of all dictatorial ES basic merging operators that satisfy the max condition; \mathfrak{N} the set of all the ES basic merging operators that admit nominators and for which the max condition holds; and \mathfrak{W} the set of all the weak-dictatorial ES basic merging operators satisfying the max condition. Let us show that $\mathfrak{D} \subsetneq \mathfrak{N} \subsetneq \mathfrak{W}$.

From what we highlighted in Observation 9, it straightforwardly follows that $\mathfrak{D} \subset \mathfrak{N}$. To show that $\mathfrak{N} \not\subset \mathfrak{D}$, we only have to consider the quasi-indifference operator. As we saw in Propositions 3, 6 and 16, this

operator satisfies the max condition and **(ESF-N)**, but **(ESF-D)** does not hold. Thus the quasi-indifference operator is in \mathfrak{N} , but it is not in \mathfrak{D} . This shows $\mathfrak{D} \subsetneq \mathfrak{N}$.

Now, let us show $\mathfrak{N} \subsetneq \mathfrak{W}$. In order to show $\mathfrak{N} \subset \mathfrak{W}$, assume that ∇ is an ES basic merging operator in \mathfrak{N} , and suppose that $\Phi \mapsto \succeq_\Phi$ is its associated assignment. Also suppose that d , in a society N , is a nominator for ∇ . Let us prove that d is also a weak-dictator for ∇ . To do so, let us consider an interpretation w in \mathcal{W}_P . We have to show that there exists an epistemic state E^* in \mathcal{E} such that $w \models B(\nabla(\Phi[E^*/d], E))$, for every N -profile Φ and every epistemic state E in \mathcal{E} , for which $w \models B(E)$.

Consider an epistemic state E^* in \mathcal{E} such that $\llbracket B(E^*) \rrbracket = \{w\}$. Since ∇ satisfies the max condition, we have $\max(\succeq_{E^*}) = \{w\}$. Thus, from **(B-Rep)** we obtain $\llbracket B(\nabla(E^*, E')) \rrbracket = \{w\}$, for every E' in \mathcal{E} , with $w \models B(E')$. We claim that $w \models B(\nabla(\Phi[E^*/d], E))$, for every N -profile Φ and every E in \mathcal{E} , with $w \models B(E)$. In effect, let Φ be any N -profile and E in \mathcal{E} be any epistemic state such that $w \models B(E)$. Since d in N is a nominator for ∇ , then $B(\nabla(\Phi', E')) \wedge B(\nabla(E'_d, E')) \not\vdash \perp$, for every N -profile $\Phi' = (E_{i_1}, \dots, E'_d, \dots, E_{i_n})$ and every epistemic state E' in \mathcal{E} ; especially for $\Phi' = \Phi[E^*/d]$ and $E' = E$. Therefore, since $B(\nabla(\Phi[E^*/d], E)) \wedge B(\nabla(E^*, E')) \not\vdash \perp$ and w is the unique model of $B(\nabla(E^*, E))$, we get $w \models B(\nabla(\Phi[E^*/d], E))$, as desired.

To show $\mathfrak{W} \not\subset \mathfrak{N}$, consider the epistemic space given in Section 2.4 and, for every finite society N , assume $d = \max(N)$. Based on this, we define the following ES basic merging operator:

$$\nabla(\Phi, \succeq) = \succeq^{\text{lex}(\succeq, \succeq_\Phi)}, \text{ where } \succeq_\Phi = \succeq_d, \text{ if } \Phi \text{ is a singleton, namely, } \succeq_d, \text{ otherwise } w \succeq_\Phi w' \text{ iff } w' \succeq_d w.$$

Since the assignment $\Phi \mapsto \succeq_\Phi$ is structure preserving, then ∇ satisfies the max condition, as we highlighted in Observation 3. In order to show that **(ESF-WD)** holds, given a finite society N and an interpretation w in \mathcal{W}_P , we only have to consider an epistemic state \succeq^* in \mathcal{E} such that $\max(\succeq^*) = \{w\}$, if N has a sole agent, namely d ; otherwise $\min(\succeq^*) = \{w\}$. Hence, by **(B-Rep)**, in any case we have $w \models B(\nabla(\Phi[\succeq^*/d], \succeq))$, for every N -profile Φ and every epistemic state \succeq in \mathcal{E} , with $w \models B(\succeq)$.

To show that **(ESF-N)** does not hold, consider the finite society $N = \{1, 2\}$, two interpretations w, w' in \mathcal{W}_P , an N -profile $\Phi = (\succeq_1, \succeq_2)$, such that $w \succ_i w'$, for $i = 1, 2$, and an epistemic state \succeq in \mathcal{E} such that $\llbracket B(\succeq) \rrbracket = \{w, w'\}$. Let us note that $w' \succ_\Phi w$. Thus, from **(B-Rep)** it follows that, w is the sole model of $B(\nabla(\succeq_i, \succeq))$, for $i = 1, 2$; while w' is the unique model of $B(\nabla(\Phi, \succeq))$. Therefore, for every agent i in N we have $B(\nabla(\succeq_i, \succeq))$ is inconsistent with $B(\nabla(\Phi, \succeq))$. \square

PROOF OF THEOREM 9. Consider the following sets: \mathfrak{D}^* , formed by those dictatorial operators satisfying the max condition that are strategy-proof with respect to a belief lifting satisfying **G1**; \mathfrak{N}^* , the class of all those operators having nominators, satisfying the max condition and are strategy-proof with respect to a belief lifting satisfying **G1**; and \mathfrak{W}^* , the class of all those operators in \mathfrak{W} , which admit a weak dictator, satisfy the max condition and are strategy-proof with respect to a belief lifting satisfying **G1**. Let us show that $\mathfrak{D}^* \subsetneq \mathfrak{N}^* = \mathfrak{W}^*$.

From Theorem 8 follows that $\mathfrak{D}^* \subset \mathfrak{N}^* \subset \mathfrak{W}^*$. To show that $\mathfrak{N}^* \not\subset \mathfrak{D}^*$, consider again the quasi-indifference operator, and note that this is an operator that satisfies the max condition, **(ESF-N)** (cf. Proposition 16) and it is strategy-proof with respect to a belief lifting satisfying **G1**, (namely the possibilistic belief lifting and the Kelly belief lifting; cf. Proposition 17), for which **(ESF-D)** does not hold (cf. Proposition 6).

To prove $\mathfrak{W}^* \subset \mathfrak{N}^*$, by Theorem 8, it is sufficient to show that, if d is a weak-dictator for an ES basic merging operator ∇ in \mathfrak{W}^* , then d is also a nominator for ∇ .

Assume that an agent d in a society N is a weak-dictator for an operator ∇ in \mathfrak{W} , and suppose that ∇ is strategy-proof with respect to a belief lifting $\succeq \mapsto \sqsupseteq_\succeq$ satisfying **G1**. Toward a contradiction, also suppose that d is not a nominator for ∇ . Thus, there exists an N -profile Φ and an epistemic state E' in \mathcal{E} such that $B(\nabla(\Phi, E')) \wedge B(\nabla(E_d, E')) \vdash \perp$. Let us consider two interpretations w, w' in \mathcal{W}_P such that $w \models B(\nabla(E_d, E'))$ and $w' \models B(\nabla(\Phi, E'))$ and suppose that $\Phi \mapsto \succeq_\Phi$ is the assignment associated to ∇ . Thus, from **(B-Rep)** we get $w \succ_{E_d} w'$ and $w' \succ_\Phi w$.

Since d is a weak-dictator for ∇ , there is an epistemic state E^w in \mathcal{E} such that $w \models B(\nabla(\Phi^*[E^w/d], E))$, for every N -profile Φ^* and every epistemic state E in \mathcal{E} , with $w \models B(E)$. Let us thus consider an epistemic state E in \mathcal{E} such that $\llbracket B(E) \rrbracket = \{w, w'\}$, and note that, by **(ESF1)**, either $\llbracket B(\nabla(\Phi^*[E^w/d], E)) \rrbracket = \{w\}$ or

$\llbracket B(\nabla(\Phi^{E_w/d}), E) \rrbracket = \{w, w'\}$. Also note that, by (B-Rep), $\llbracket B(\nabla(\Phi, E)) \rrbracket = \{w'\}$. Now, since $w \succ_{E_d} w'$, we get $\varphi_w \sqsupset_{E_d} \varphi_{w'}$ and, by **(G1)**, $\varphi_{w, w'} \sqsupset_{E_d} \varphi_{w'}$. Then, for the two possible outcomes of $B(\nabla(\Phi^{E_w/d}), E)$, we have $B(\nabla(\Phi^{E_w/d}), E) \sqsupset_{E_d} B(\nabla(\Phi, E))$. This shows the manipulation of ∇ with respect to $\sqsupset \mapsto \sqsupset_{\geq}$, a contradiction. \square

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