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WEAK AND STRONG LAW OF LARGE NUMBERS FOR STRICTLY STATIONARY BANACH-VALUED RANDOM FIELDS

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ABSTRACT. In this paper, we investigate the law of large numbers for strictly stationary random fields, that is, we provide sufficient conditions on the moments and the dependence of the random field in order to guarantee the almost sure convergence to 0 and the convergence in \mathbb{L}^p of partials sums over squares or rectangles of \mathbb{Z}^d . Approximation by multi-indexed martingales as well as by m -dependent random fields are investigated. Applications to functions of d -independent Bernoulli shifts and to functionals of i.i.d. random fields are also provided.

1. INTRODUCTION

The law of large numbers is one of the most fundamental theorems in probability theory and statistics. It states that if $(X_i)_{i \geq 1}$ is an i.i.d. sequence such that $\mathbb{E}[|X_1|] < \infty$, then $n^{-1} \sum_{i=1}^n X_i$ converges to $\mathbb{E}[X_1]$ in the almost sure sense. A more general version, called Marcinkiewicz strong law of large numbers, states that if $1 \leq p < 2$ and $\mathbb{E}[|X_1|^p] < \infty$, then $n^{-1/p} \sum_{i=1}^n (X_i - \mathbb{E}[X_i])$ converges to 0 almost surely. Several authors investigated then the case of dependent sequences, see for instance [29, 6, 22, 25, 5].

We are interested in analogous results for collections of random variables indexed by \mathbb{Z}^d . Throughout the paper, we will denote for a positive integer d the set $\{1, \dots, d\}$ by $\llbracket 1, d \rrbracket$, for $\mathbf{i} = (i_\ell)_{\ell \in \llbracket 1, d \rrbracket}$ and $\mathbf{j} = (j_\ell)_{\ell \in \llbracket 1, d \rrbracket}$, $\mathbf{i} \preceq \mathbf{j}$ means that for each $\ell \in \llbracket 1, d \rrbracket$, $i_\ell \leq j_\ell$. We will write $\mathbf{1}$ for the element of \mathbb{Z}^d whose coordinates are all equal to one and for $\mathbf{n} = (n_\ell)_{\ell \in \llbracket 1, d \rrbracket}$, $|\mathbf{n}| = \prod_{\ell=1}^d n_\ell$. We are looking for conditions on the dependence and the moments of the centered random field $(X_i)_{i \in \mathbb{Z}^d}$ taking values in a Banach space $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$ in order to guarantee the convergence of

$$(1.1) \quad \frac{1}{|\mathbf{n}|} \left\| \sum_{\mathbf{1} \preceq \mathbf{i} \preceq \mathbf{n}} X_i \right\|_{\mathbb{B}}$$

to 0 in the almost sure sense or in \mathbb{L}^p for some p in a range that depends on the Banach space \mathbb{B} , as one of the coordinates of \mathbf{n} goes to infinity. We will also consider the almost sure convergence to 0 of the sum of squares, namely

$$(1.2) \quad \frac{1}{n^{d/p}} \left\| \sum_{\mathbf{1} \preceq \mathbf{i} \preceq n\mathbf{1}} X_i \right\|_{\mathbb{B}} \rightarrow 0$$

as n goes to infinity.

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In the i.i.d. and $\mathbb{B} = \mathbb{R}$ case, [14] provided a necessary and sufficient condition for the convergence the term involved in (1.1) as all the coordinates of \mathbf{n} go to infinity when $\mathbb{B} = \mathbb{R}$, namely, that

$$(1.3) \quad \mathbb{E} \left[|X_{\mathbf{0}}|^p (\log(1 + |X_{\mathbf{0}}|))^{d-1} \right] < \infty.$$

Still for independent random variables, a necessary and sufficient condition has been found in [24], Theorem 2.2, when the summation is done over sets that are not necessarily rectangles.

Under the assumption that the random field $(X_i)_{i \in \mathbb{Z}^d}$ is i.i.d. and satisfies a condition which is stronger than $\mathbb{E}[|X_1|] < \infty$ but weaker than $\mathbb{E} \left[|X_1| (\log(1 + |X_1|))^{d-1} \right]$, a strong law of large numbers for weighted sums over rectangle have been established in [19].

Under the independence but without the identical distribution assumption, a law of large number and convergence rates have been obtained in [15]. A result of the same spirit for pairwise independent random fields has been obtained in [4].

Assymmetric law of large numbers, that is, with the normalization $\prod_{\ell \in \llbracket 1, d \rrbracket} n_\ell^{\alpha_\ell}$ with potentially different exponents α_ℓ , for negatively associated random fields has been investigated in [17]. In Theorem 3.2 of [8], a strong law of large numbers for martingale random fields has been established. On one hand, there condition on the martingale property is more restrictive than orthomartingales (see Definition 2.3). On the other hand, we put restriction on the filtration, namely commutativity, while Dung and Duy Tien do not. When applied to identically distributed random fields, convergence (2.11) holds under a slightly stronger moment assumption since they need $\mathbb{E} \left[\|D_{\mathbf{0}}\|_{\mathbb{B}}^{p+\delta} \right]$ to be finite for some positive δ . This result was improved in [31], Theorem 3.3, where a similar result under the same martingale assumption as in [8], but with the optimal moment assumption $\mathbb{E} \left[\|D_{\mathbf{0}}\|_{\mathbb{B}}^p (\log(1 + \|D_{\mathbf{0}}\|_{\mathbb{B}}))^{d-1} \right] < \infty$. An other type of martingale difference random fields were investigated in [16], which does not seem to be directly comparable with orthomartingales. The case of real-valued orthomartingales have been treated in [18], but it is not easy to compare with our conditions because of the condition (4.2) in the aforementioned paper where some convergence rates on a conditional maximum are required. An other result concerning orthomartingale difference random fields which are not necessarily identically distributed or stochastically dominated has been studied in [30]. When $p = 1$, the convergence (2.11) has been shown in [23] in the context of pairwise independent and identically distributed random fields.

In this paper, we are interested in the strong law of large for random fields, that is, collections of random variables indexed by d -uples of integers and the summation over $\{1, \dots, n\}$ is replaced by rectangles of \mathbb{Z}^d . We will assume that the involved random variables take their values in a separable Banach space $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$.

We will use essentially two approaches: a first one by approximating via multi-indexed martingales and a second one by approximation by m -dependent random fields, giving different ranges of application.

The paper is organized as follows. In Section 2, we establish results on the Marcinkiewicz law of large numbers for stationary random fields using an approximation by multi-indexed martingales. We first complement the already obtained statement for the strong law of large numbers on squares and the convergence in \mathbb{L}^p . Then we provide results for stationary random fields by approximating by such martingale differences random fields. This method was performed in order to derive weak invariance principles [9, 34, 33, 20, 21, 27], quenched invariance principles [26, 37, 28] and the bounded law of the iterated logarithms [11].

In Section 3, the results obtained by orthomartingale approximation are used in order to derive a Marcinkiewicz strong law of large numbers for random fields that can be expressed as a Hölder continuous function of d mutually independent sequences expressable as functions of i.i.d. sequences.

The condition involves the exponent of Hölder regularity, the dimension d and the dependence coefficient of each involved sequence.

In Section 4, we investigate the aforementioned laws of large numbers for random fields expressible as a functional of an i.i.d. random field. We propose an application to a Hölder continuous functional of a Banach valued linear random field. The condition is written in terms of the exponent of Hölder regularity, the moments of the innovations and the operator norm of the coefficients.

Section 5 is devoted to the proofs of the results of Sections 2, 3 and 4. The needed auxiliary results are grouped in the Appendix.

2. WEAK AND STRONG LAW OF LARGE NUMBERS VIA ORTHOMARTINGALE APPROXIMATION

2.1. The orthomartingale case. The notion of multi-indexed martingale requires the notion of multi-indexed filtration. We will also require the filtration to be commuting in the following sense.

Definition 2.1. *We say that a collection of σ -algebras $(\mathcal{F}_i)_{i \in \mathbb{Z}^d}$ is a completely commuting filtration if*

- (1) *for each $\mathbf{i}, \mathbf{j} \in \mathbb{Z}^d$ such that $\mathbf{i} \preceq \mathbf{j}$, $\mathcal{F}_i \subset \mathcal{F}_j$ and*
- (2) *for each $Y \in \mathbb{L}^1$ and each $\mathbf{i}, \mathbf{j} \in \mathbb{Z}^d$,*

$$(2.1) \quad \mathbb{E}[\mathbb{E}[Y \mid \mathcal{F}_i] \mid \mathcal{F}_j] = \mathbb{E}[Y \mid \mathcal{F}_{\min\{\mathbf{i}, \mathbf{j}\}}],$$

where $\min\{\mathbf{i}, \mathbf{j}\}$ is the element of \mathbb{Z}^d defined as the coordinatewise minimum of \mathbf{i} and \mathbf{j} , that is, $\min\{\mathbf{i}, \mathbf{j}\} = (\min\{i_\ell, j_\ell\})_{\ell=1}^d$.

Let us give two examples of commuting filtrations.

Proposition 2.2. (1) *If $(\varepsilon_{\mathbf{u}})_{\mathbf{u} \in \mathbb{Z}^d}$ is i.i.d., then defining $\mathcal{F}_i = \sigma(\varepsilon_{\mathbf{u}}, \mathbf{u} \in \mathbb{Z}^d, \mathbf{u} \preceq \mathbf{i})$, the filtration $(\mathcal{F}_i)_{i \in \mathbb{Z}^d}$ is completely commuting.*

- (2) *Suppose that $(\mathcal{F}_{i_\ell}^{(\ell)})_{i_\ell \in \mathbb{Z}}$, $1 \leq \ell \leq d$, are filtrations on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Suppose that for each $i_1, \dots, i_d \in \mathbb{Z}$, the σ -algebras $\mathcal{F}_{i_\ell}^{(\ell)}$, $1 \leq \ell \leq d$, are independent. Let $\mathcal{F}_i = \bigvee_{\ell=1}^d \mathcal{F}_{i_\ell}^{(\ell)}$. Then $(\mathcal{F}_i)_{i \in \mathbb{Z}^d}$ is completely commuting.*

Both examples were introduced in Section 1 of [2], but without proof. The first item is a direct consequence of Proposition 2 p. 1693 of [34].

We are now in position to define orthomartingale martingale difference random field, which allows to exploit the martingale property in every direction. To formalize this, we need to denote by e_ℓ , $\ell \in \llbracket 1, d \rrbracket$, the element of \mathbb{Z}^d whose ℓ -th coordinate is 1 and all the others are zero.

Definition 2.3. *Let $(X_i)_{i \in \mathbb{Z}^d}$ be a random field taking values in a separable Banach space $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$. We say that $(X_i)_{i \in \mathbb{Z}^d}$ is an orthomartingale martingale difference random field with respect to the completely commuting filtration $(\mathcal{F}_i)_{i \in \mathbb{Z}^d}$ if for each $\mathbf{i} \in \mathbb{Z}^d$, $\|X_i\|_{\mathbb{B}}$ is integrable, X_i is \mathcal{F}_i -measurable and for each $\ell \in \llbracket 1, d \rrbracket$, $\mathbb{E}[X_i \mid \mathcal{F}_{i-e_\ell}] = 0$.*

Such a definition is very convenient because summation on a rectangular region of \mathbb{Z}^d can be treated with martingale properties when summing on a fixed coordinate.

The proof of the law of large number usually rest on satisfactory moment inequalities for martingales. Therefore, we will work with smooth Banach spaces in the following sense.

Definition 2.4. *Let $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$ be a separable Banach space. We say that \mathbb{B} is r -smooth for $1 < r \leq 2$ if there exists an equivalent norm $\|\cdot\|'_{\mathbb{B}}$ on \mathbb{B} such that*

$$(2.2) \quad \sup_{t>0} \sup_{x, y \in \mathbb{B}, \|x\|'_{\mathbb{B}} = \|y\|'_{\mathbb{B}} = 1} \frac{\|x + ty\|'_{\mathbb{B}} + \|x - ty\|'_{\mathbb{B}} - 2}{t^r} < \infty.$$

For example, if μ is σ -finite on the Borel σ -algebra of \mathbb{R} , then $\mathbb{L}^p(\mathbb{R}, \mu)$ is $\min\{p, 2\}$ -smooth. Moreover, a separable Hilbert space is 2-smooth.

Given a random variable X taking values in a separable Banach space $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$ and $p \geq 1$, $q \geq 0$, we define

$$(2.3) \quad \|X\|_{\mathbb{B},p} := (\mathbb{E} [\|X\|_{\mathbb{B}}^p])^{1/p} \text{ and}$$

$$(2.4) \quad \|X\|_{\mathbb{B},p,q} := \|\|X\|_{\mathbb{B}}\|_{p,q},$$

where, for a real valued random variable Y ,

$$(2.5) \quad \|Y\|_{p,q} = \inf \left\{ \lambda > 0, \varphi_{p,q} \left(\frac{|Y|}{\lambda} \right) \leq 1 \right\}, \varphi_{p,q}(t) = t^p (1 + \mathbf{1}_{t \geq 1} \ln t)^q.$$

Note that $\|X\|_{\mathbb{B},p,0} = \|X\|_{\mathbb{B},p}$. Denote by $\mathbb{L}_{p,q}$ the space of random variables Y such that $\mathbb{E} [\varphi_{p,q}(\|Y\|_{\mathbb{B}})] < \infty$. Conditions of the form

$$(2.6) \quad \mathbb{E} [\|D_{\mathbf{0}}\|_{\mathbb{B}}^p (\log(1 + \|D_{\mathbf{0}}\|_{\mathbb{B}}))^q] < \infty$$

for some p and q are usual in the context of random fields. For instance, the quenched functional central limit theorem on rectangles for a strictly stationary orthomartingale difference random field requires (2.6) for $p = 2$ and $q = d - 1$ (see [26]) and for the bounded law of the iterated logarithms, (2.6) for $p = 2$ and $q = 2d - 2$ (see [11]). Therefore, the condition on the moments can be expressed with the help of the norm $\|\cdot\|_{p,q}$.

We also define

$$(2.7) \quad \|X\|_{\mathbb{B},p,w} := \sup_{A: \mathbb{P}(A) > 0} \mathbb{P}(A)^{-1+1/p} \mathbb{E} [\|X\|_{\mathbb{B}} \mathbf{1}_A].$$

When $\mathbb{B} = \mathbb{R}$, we shall simply write $\|X\|_p$, $\|X\|_{p,q}$ and $\|X\|_{p,w}$ respectively. Note that there exists constants c_p and c'_p such that for each random variable X ,

$$(2.8) \quad c_p \left(\sup_{t>0} t^p \mathbb{P}(\|X\|_{\mathbb{B}} > t) \right)^{1/p} \leq \|X\|_{\mathbb{B},p,w} \leq c'_p \left(\sup_{t>0} t^p \mathbb{P}(\|X\|_{\mathbb{B}} > t) \right)^{1/p}.$$

For $\mathbf{n} = (n_\ell)_{\ell=1}^d \in \mathbb{N}^d$, we define $2^{\mathbf{n}} = (2^{n_\ell})_{\ell=1}^d$ and $\max \mathbf{n} = \max_{1 \leq \ell \leq d} n_\ell$ and we recall that $|\mathbf{n}| = \prod_{\ell=1}^d n_\ell$.

The following result has been obtained for the convergence of normalized partial sums on rectangles of an orthomartingale difference random field.

Theorem 2.5 (Theorem 2.2 in [13], law of large numbers on rectangles). *Let $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$ be a separable r -smooth Banach space for some $r \in (1, 2]$, $1 < p < r$ and $d \in \mathbb{N}$. There exists a constant $K_{p,d,\mathbb{B}}$ such that the following holds: if $(D_{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d}$ is an identically distributed orthomartingale difference random field such that $\|D_{\mathbf{0}}\|_{\mathbb{B},p,d-1} < \infty$, then for all positive t , the following inequality holds*

$$(2.9) \quad \sum_{\mathbf{N} \in \mathbb{N}^d} \mathbb{P} \left(|\mathbf{2}^{\mathbf{N}}|^{-1/p} \max_{\mathbf{1} \leq \mathbf{n} \leq \mathbf{2}^{\mathbf{N}}} \|S_{\mathbf{n}}\|_{\mathbb{B}} > t \right) \leq K_{p,d,\mathbb{B}} \mathbb{E} \left[\varphi_{p,d-1} \left(\frac{\|D_{\mathbf{0}}\|_{\mathbb{B}}}{t} \right) \right],$$

where $S_{\mathbf{n}} = \sum_{\mathbf{1} \leq \mathbf{i} \leq \mathbf{n}} D_{\mathbf{i}}$. In particular, for some constant $C_{\mathbb{B},d,p}$ depending only on \mathbb{B} , d and p ,

$$(2.10) \quad \left\| \sup_{\mathbf{n} \geq \mathbf{1}} \frac{\|S_{\mathbf{n}}\|_{\mathbb{B}}}{|\mathbf{n}|^{1/p}} \right\|_{\mathbb{B},p,w} \leq C_{\mathbb{B},d,p} \|D_{\mathbf{0}}\|_{\mathbb{B},p,d-1}$$

and the following convergence holds:

$$(2.11) \quad \lim_{N \rightarrow \infty} \sup_{n \geq 1, \max n \geq N} \frac{\|S_n\|_{\mathbb{B}}}{|n|^{1/p}} = 0 \text{ almost surely.}$$

Such a result was known for $d = 1$ (see [35] and Proposition 2.1 in [3]).

When we consider the summation over squares, that is, sets of the form $\llbracket 1, n \rrbracket^d$, where n is an integer bigger than one, it turns out that we can embed the orthomartingale into an one-dimensional martingale. As a consequence, only finite moments of order p are required.

Theorem 2.6 (Law of large numbers on squares). *Let $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$ be a separable r -smooth Banach space for some $r \in (1, 2]$, $1 < p < r$ and $d \in \mathbb{N}$. There exists a constant $C_{\mathbb{B}, d, p}$ such that the following holds: if $(D_i)_{i \in \mathbb{Z}^d}$ is an identically distributed orthomartingale difference random field such that $\|D_0\|_{\mathbb{B}} \in \mathbb{L}_p$, then the following inequality holds:*

$$(2.12) \quad \left\| \sup_{n \geq 1} \frac{\|S_{n\mathbf{1}}\|_{\mathbb{B}}}{n^{d/p}} \right\|_{p, w} \leq C_{\mathbb{B}, d, p} \|D_0\|_{\mathbb{B}, p}.$$

Moreover, the following convergence holds:

$$(2.13) \quad \lim_{n \rightarrow \infty} \frac{\|S_{n\mathbf{1}}\|_{\mathbb{B}}}{n^{d/p}} = 0 \text{ almost surely.}$$

For convergence in \mathbb{L}^p , we have the following result.

Theorem 2.7 (Convergence in \mathbb{L}^p). *Let $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$ be a separable r -smooth Banach space for some $r \in (1, 2]$, $1 < p < r$ and $d \in \mathbb{N}$. If $(D_i)_{i \in \mathbb{Z}^d}$ is an identically distributed orthomartingale difference random field such that $\|D_0\|_{\mathbb{B}} \in \mathbb{L}^p$, then*

$$(2.14) \quad \lim_{\max N \rightarrow \infty} \frac{1}{|N|^{1/p}} \left\| \max_{\mathbf{1} \leq n \leq N} \|S_n\|_{\mathbb{B}} \right\|_p = 0.$$

We point out that the convergence in (2.14) is as $\max N \rightarrow \infty$, in other words, we require that only one of the coordinates of N goes to infinity. Moreover, unlike in the case of almost sure convergence, the consideration of squares instead of rectangles would not give a less restrictive condition.

2.2. Orthomartingale approximation. In this section, we assume that the random field $(X_i)_{i \in \mathbb{Z}^d}$ is of the form

$$(2.15) \quad X_i = X_0 \circ T^i, \quad \mathcal{F}_i = T^{-i} \mathcal{F}_0$$

where $T^i: \Omega \rightarrow \Omega$ is such that $T^i \circ T^j = T^{i+j}$ for each $i, j \in \mathbb{Z}^d$.

For example, one can consider the case where $\Omega = \mathbb{R}^{\mathbb{Z}^d}$ endowed with the product measure of a probability measure μ and T^i is the shift operator given by $T^i((x_k)_{k \in \mathbb{Z}^d}) = (x_{k+i})_{k \in \mathbb{Z}^d}$.

An other example is the following: take probability spaces $(\Omega_\ell, \mathcal{A}_\ell, \mu_\ell)$ and let $\Omega = \prod_{\ell=1}^d \Omega_\ell$ endowed with the product σ -algebra and the product measure μ . Let $T_\ell: \Omega_\ell \rightarrow \Omega_\ell$ be a bijective bi-measure preserving map and let $\mathcal{F}_0^{(\ell)}$ be a sub- σ -algebra of \mathcal{A}_ℓ such that $T_\ell \mathcal{F}_0^{(\ell)} \subset \mathcal{F}_0^{(\ell)}$. Then define $\mathcal{F}_i := \bigvee_{\ell=1}^d T_\ell^{i_\ell} \mathcal{F}_0^{(\ell)}$.

For $i \in \mathbb{Z}^d$, we define the map U^i by $U^i(f)(\omega) = f(T^i \omega)$. In order to extend the results of Subsection 2.1 to a larger class of stationary random fields, we define the projection operator

$$(2.16) \quad P_{\mathbf{k}}(Y) := \sum_{I \subset \llbracket 1, d \rrbracket} (-1)^{|I|} \mathbb{E}[Y \mid \mathcal{F}_{\mathbf{k}-\mathbf{1}_I}],$$

where $|I|$ denotes the cardinality of I and $\mathbf{k} - \mathbf{1}_I = (k_\ell - \mathbf{1}_{\ell \in I})_{\ell=1}^d$. When $d = 1$, $P_{\mathbf{k}}(Y) = \mathbb{E}[Y | \mathcal{F}_{\mathbf{k}}] - \mathbb{E}[Y | \mathcal{F}_{\mathbf{k}-1}]$ and when $d = 2$,

$$(2.17) \quad P_{k_1, k_2}(Y) = \mathbb{E}[Y | \mathcal{F}_{k_1, k_2}] - \mathbb{E}[Y | \mathcal{F}_{k_1-1, k_2}] - \mathbb{E}[Y | \mathcal{F}_{k_1, k_2-1}] + \mathbb{E}[Y | \mathcal{F}_{k_1-1, k_2-1}].$$

The norm of such projectors is used in order to measure how far a random field from an orthomartingale difference random field is. Indeed, if $(D_{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d}$ is an orthomartingale difference random field, then $P_{\mathbf{k}}(D_{\mathbf{0}}) = 0$ if $\mathbf{k} \neq \mathbf{0}$. Moreover, the fact that the filtration is defined with the help of the action T gives, for $\mathbf{i}, \mathbf{k} \in \mathbb{Z}^d$,

$$(2.18) \quad \|P_{\mathbf{i}+\mathbf{k}}(X_{\mathbf{i}})\|_{p,q} = \|P_{\mathbf{k}}(X_{\mathbf{0}})\|_{p,q}.$$

It is tempting to express $X_{\mathbf{i}}$ as a sum of projectors, namely, $X_{\mathbf{i}} = \sum_{\mathbf{k} \in \mathbb{Z}^d} P_{\mathbf{k}}(X_{\mathbf{i}})$, where the sum is understood as $\lim_{m \rightarrow \infty} \sum_{\mathbf{k} \in \mathbb{Z}^d, \|\mathbf{k}\|_{\infty} \leq m} P_{\mathbf{k}}(X_{\mathbf{i}})$ and the limit in the sense of the $\|\cdot\|_{\mathbb{B}, p, q}$ for some p and q . Since for fixed m , the sum $\sum_{\mathbf{k} \in \mathbb{Z}^d, \|\mathbf{k}\|_{\infty} \leq m} P_{\mathbf{k}}(X_{\mathbf{i}})$ is telescopic, only 2^m terms are remaining: one of them is $\mathbb{E}[X_{\mathbf{i}} | \mathcal{F}_{m\mathbf{1}}]$ and the other one of the form $\mathbb{E}[X_{\mathbf{i}} | \mathcal{F}_{m\mathbf{1}_I - (m+1)\mathbf{1}_{[d] \setminus I}}]$ for some proper subset I of $[1, d]$, where $\mathbf{1}_I = \sum_{i \in I} \mathbf{e}_i$. In order to make their contribution negligible, we need the following assumptions:

$$(2.19) \quad \lim_{m \rightarrow \infty} \|X_{\mathbf{0}} - \mathbb{E}[X_{\mathbf{0}} | \mathcal{F}_{m\mathbf{1}}]\|_{\mathbb{B}, p, q} = 0,$$

$$(2.20) \quad \forall \ell_0 \in [1, d], \lim_{m \rightarrow \infty} \left\| \mathbb{E} \left[X_{\mathbf{0}} | \mathcal{F}_{m \sum_{\ell \in [1, d] \setminus \{\ell_0\}} \mathbf{e}_\ell - m \mathbf{e}_{\ell_0}} \right] \right\|_{\mathbb{B}, p, q} = 0.$$

The following results give a strong law and convergence in \mathbb{L}^p for stationary random fields.

Theorem 2.8 (Law of large numbers on rectangles). *Let $(X_{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d}$ be a strictly stationary random field taking values in a separable r -smooth Banach space $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$. and let $(\mathcal{F}_{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d}$ be a commuting filtration such that (2.15) is satisfied. Suppose that (2.19) and (2.20) hold with $q = d - 1$ and that*

$$(2.21) \quad \sum_{\mathbf{k} \in \mathbb{Z}^d} \|P_{\mathbf{k}}(X_{\mathbf{0}})\|_{\mathbb{B}, p, d-1} < \infty.$$

Then

$$(2.22) \quad \lim_{N \rightarrow \infty} \sup_{n \geq 1, \max \mathbf{n} \geq N} \frac{1}{|\mathbf{n}|^{1/p}} \left\| \sum_{\mathbf{1} \preceq \mathbf{i} \preceq \mathbf{n}} X_{\mathbf{i}} \right\|_{\mathbb{B}} = 0 \text{ almost surely.}$$

The corresponding result for sums over squares reads as follows.

Theorem 2.9 (Law of large numbers on squares). *Let $(X_{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d}$ be a strictly stationary random field taking values in a separable r -smooth Banach space $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$. and let $(\mathcal{F}_{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d}$ be a commuting filtration such that (2.15) is satisfied. Suppose that (2.19) and (2.20) hold with $q = 0$ and that*

$$(2.23) \quad \sum_{\mathbf{k} \in \mathbb{Z}^d} \|P_{\mathbf{k}}(X_{\mathbf{0}})\|_{\mathbb{B}, p} < \infty.$$

Then

$$(2.24) \quad \lim_{n \rightarrow \infty} \frac{1}{n^{d/p}} \left\| \sum_{\mathbf{1} \preceq \mathbf{i} \preceq n\mathbf{1}} X_{\mathbf{i}} \right\|_{\mathbb{B}} = 0 \text{ almost surely.}$$

A similar result can be formulated for the convergence in \mathbb{L}^p .

Theorem 2.10 (Convergence in \mathbb{L}^p). *Let $(X_{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d}$ be a strictly stationary random field taking values in a separable r -smooth Banach space $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$, $1 \leq p < r$, and let $(\mathcal{F}_{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d}$ be a commuting filtration such that (2.15) is satisfied. Suppose that (2.19) and (2.20) hold with $q = 0$ and that*

$$(2.25) \quad \sum_{\mathbf{k} \in \mathbb{Z}^d} \|P_{\mathbf{k}}(X_{\mathbf{0}})\|_{\mathbb{B}, p} < \infty.$$

Then

$$(2.26) \quad \lim_{N \rightarrow \infty} \sup_{n \geq 1, \max n \geq N} \frac{1}{|n|^{1/p}} \left\| \sum_{1 \leq i \leq n} X_{\mathbf{i}} \right\|_{\mathbb{B}, p} = 0.$$

3. WEAK AND STRONG LAW OF LARGE NUMBERS FOR FUNCTIONS OF INDEPENDENT BERNOULLI SHIFTS

In this section, we will provide a Marcinkiewicz strong law of large numbers and convergence in $\mathbb{L}^p(\mathbb{B})$ of normalized partials sums of random fields having the form

$$(3.1) \quad X_{\mathbf{i}} = g \left(f_1 \left(\left(\varepsilon_{i_1 - u_1}^{(1)} \right)_{u_1 \in \mathbb{Z}} \right), \dots, f_d \left(\left(\varepsilon_{i_d - u_d}^{(d)} \right)_{u_d \in \mathbb{Z}} \right) \right),$$

where $g: \mathbb{R}^d \rightarrow \mathbb{B}$ is Hölder continuous with exponent α , that is, there exists a constant C such that for each $x_1, \dots, x_d, y_1, \dots, y_d \in \mathbb{R}$,

$$(3.2) \quad \|g(x_1, \dots, x_d) - g(y_1, \dots, y_d)\|_{\mathbb{B}} \leq C \sum_{\ell=1}^d |x_{\ell} - y_{\ell}|^{\alpha},$$

$\left(\varepsilon_{u_{\ell}}^{(\ell)} \right)_{u_{\ell} \in \mathbb{Z}}$ are mutually independent i.i.d. sequences and f_1, \dots, f_d are measurable functions defined on $\mathbb{R}^{\mathbb{Z}^d}$ and taking values in \mathbb{R} .

The random fields defined via (3.1) are strictly stationary and the filtration $(\mathcal{F}_{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d}$ given by

$$(3.3) \quad \mathcal{F}_{\mathbf{i}} = \sigma \left(\varepsilon_{u_1}^{(1)}, u_1 \leq i_1, \dots, \varepsilon_{u_d}^{(d)}, u_d \leq i_d \right)$$

is commuting. We would like to apply the results of Subsection 2.2. However, the assumption that for each $\ell \in \llbracket 1, d \rrbracket$ and $j_k, k \in \llbracket 1, d \rrbracket \setminus \{\ell\}$, $\mathbb{E} \left[X_{\mathbf{i}} \mid \bigcap_{j_{\ell} \in \mathbb{Z}} \mathcal{F}_{j_{\ell}} \right] = 0$ may not be satisfied in most cases. Indeed, let $\mathbb{B} = \mathbb{R}$, $g(x_1, x_2) = x_1 + x_2$, the maps f_1, f_2 are projections on the coordinate of index zero and $\left(\varepsilon_{u_1}^{(1)} \right)_{u_1 \in \mathbb{Z}}, \left(\varepsilon_{u_2}^{(2)} \right)_{u_2 \in \mathbb{Z}}$ are centered independent i.i.d. sequences, then $\mathbb{E} \left[X_{i_1, i_2} \mid \bigcap_{j_2 \in \mathbb{Z}} \mathcal{F}_{i_1, j_2} \right] = \varepsilon_{i_1}^{(1)}$.

To overcome this problem, we define for a non-empty subset J of $\llbracket 1, d \rrbracket$ the σ -algebra

$$(3.4) \quad \mathcal{G}_J := \sigma \left(\varepsilon_{u_j}^{(j)}, u_j \in \mathbb{Z}, j \in J \right),$$

and \mathcal{G}_{\emptyset} is the trivial σ -algebra. We then define

$$(3.5) \quad X_{\mathbf{i}}^I := \sum_{J \subset I} (-1)^{|I|+|J|} \mathbb{E} [X_{\mathbf{i}} \mid \mathcal{G}_J].$$

Notice that for each $J \subset I$, $\mathbb{E} [X_{\mathbf{i}} \mid \mathcal{G}_J] = \mathbb{E} \left[X_{\sum_{\ell \in I} i_{\ell} \mathbf{e}_{\ell}} \mid \mathcal{G}_J \right]$ hence the coordinates of \mathbf{i} which do not belong to I do not play any role. For this reason, we write

$$(3.6) \quad X_{\mathbf{i}_I}^I = \sum_{J \subset I} (-1)^{|I|+|J|} \mathbb{E} [X_{\mathbf{i}} \mid \mathcal{G}_J], \quad \mathbf{i}_I = \sum_{\ell \in I} i_{\ell} \mathbf{e}_{\ell}.$$

Moreover, if $X_{\mathbf{0}}$ is such that $\|X_{\mathbf{0}}\|_{\mathbb{B}, p, q} < \infty$, then the random field $\left(X_{\mathbf{i}_I}^I \right)_{\mathbf{i}_I \in \mathbb{Z}^I}$ satisfies (2.19) and (2.20) with the filtration $(\mathcal{F}_{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d}$ replaced by $(\mathcal{F}_{\mathbf{i}_I})_{\mathbf{i}_I \in \mathbb{Z}^I}$. Finally, notice that $X_{\mathbf{i}} = \sum_{I \subset \llbracket 1, d \rrbracket} X_{\mathbf{i}_I}^I$. Therefore,

it is possible to apply Theorems 2.8, 2.9 and 2.10 to each random field $(X_{i_I}^I)_{i_I \in \mathbb{Z}^I}$ in order to derive a law of large numbers.

The appropriated normalization depends on an integer d_0 defined as follows: for some subset I_0 of cardinality d_0 , $\|X_{\mathbf{0}}^{I_0}\|_{\mathbb{B},1} \neq 0$ and for each set I of cardinality strictly smaller than d_0 , one has $\|X_{\mathbf{0}}^I\|_{\mathbb{B},1} = 0$. We then define for $\mathbf{n} \in (\mathbb{N} \setminus \{0\})^d$,

$$(3.7) \quad \pi_{d_0,p}(\mathbf{n}) = \max_{I: I \subset \llbracket 1, d \rrbracket, \text{Card}(I)=d_0} \prod_{\ell \in I} n_\ell^{1/p} \prod_{\ell' \in \llbracket 1, d \rrbracket \setminus I} n_{\ell'}.$$

When all the coordinates of \mathbf{n} are equal, say to N , one has $\pi_{d_0,p}(\mathbf{n}) = \pi_{d_0,p}(N\mathbf{1}) = N^{d_0/p+d-d_0}$.

In the previous example, if $\|\varepsilon_0^{(1)}\|_{\mathbb{R},1} \neq 0$, one has $d_0 = 1$ and if $X_{i_1, i_2} = \varepsilon_{i_1}^{(1)} \varepsilon_{i_2}^{(2)}$ then $d_0 = 2$. In higher dimension, for a prescribed D , one can construct similar examples with products over D coordinates in order to get $d_0 = D$.

It turns out that we can formulate a condition in terms of the measure of physical dependence introduced in [36]. We define for $i \in \mathbb{Z}$,

$$(3.8) \quad \delta_{\mathbb{B},p,q}^{(\ell)}(i) := \left\| f_\ell \left(\left(\varepsilon_{i-u_\ell}^{(\ell),*} \right)_{u_\ell \in \mathbb{Z}} \right) - f_\ell \left(\left(\varepsilon_{i-u_\ell}^{(\ell)} \right)_{u_\ell \in \mathbb{Z}} \right) \right\|_{\mathbb{B},p,q},$$

where $\varepsilon_{i-u_\ell}^{(\ell),*} = \varepsilon_{i-u_\ell}^{(\ell)}$ if $u_\ell \neq i$ and $\varepsilon_0^{(\ell),*}$ is a random variable independent of the sequence $(\varepsilon_u^{(\ell)})_{u \in \mathbb{Z}}$ and has the same distribution as $\varepsilon_0^{(\ell)}$.

We are now in position to state a strong law of large numbers for random fields of the form (3.1).

Theorem 3.1 (Law of large numbers on rectangles). *Let $(X_i)_{i \in \mathbb{Z}^d}$ be a strictly stationary random field having the form (3.1) and let $\alpha \in (0, 1]$ satisfying (3.2). Let $1/\alpha < p < r$. Suppose that*

$$(3.9) \quad \sum_{i \in \mathbb{Z}} |i|^{d-1} \left(\delta_{\mathbb{B},p\alpha,d-1}^{(\ell)}(i) \right)^\alpha < \infty.$$

Then

$$(3.10) \quad \lim_{N \rightarrow \infty} \sup_{\mathbf{n} \succcurlyeq \mathbf{1}, \max \mathbf{n} \geq N} \frac{1}{\pi_{d_0,p}(\mathbf{n})} \left\| \sum_{\mathbf{1} \preccurlyeq i \preccurlyeq \mathbf{n}} X_i \right\|_{\mathbb{B}} = 0 \text{ almost surely.}$$

Theorem 3.2 (Law of large numbers on squares). *Let $(X_i)_{i \in \mathbb{Z}^d}$ be a strictly stationary random field having the form (3.1) and let $\alpha \in (0, 1]$ satisfying (3.2). Let $1/\alpha < p < r$. Suppose that for each $\ell \in \llbracket 1, d \rrbracket$,*

$$(3.11) \quad \sum_{i \in \mathbb{Z}} |i|^{d-1} \left(\delta_{\mathbb{B},p\alpha}^{(\ell)}(i) \right)^\alpha < \infty.$$

Then

$$(3.12) \quad \lim_{n \rightarrow \infty} \frac{1}{n^{d_0/p+d-d_0}} \left\| \sum_{\mathbf{1} \preccurlyeq i \preccurlyeq n\mathbf{1}} X_i \right\|_{\mathbb{B}} = 0 \text{ almost surely.}$$

Theorem 3.3 (Convergence in \mathbb{L}^p). *Let $(X_i)_{i \in \mathbb{Z}^d}$ be a strictly stationary random field having the form (3.1) and let $\alpha \in (0, 1]$ satisfying (3.2). Let $1/\alpha < p < r$. Suppose that for each $\ell \in \llbracket 1, d \rrbracket$,*

$$(3.13) \quad \sum_{i \in \mathbb{Z}} |i|^{d-1} \left(\delta_{\mathbb{B},p\alpha}^{(\ell)}(i) \right)^\alpha < \infty.$$

Then

$$(3.14) \quad \lim_{N \rightarrow \infty} \sup_{n \succ 1, \max n \geq N} \frac{1}{\pi_{d_0, p}(n)} \left\| \sum_{1 \leq i \leq n} X_i \right\|_{\mathbb{B}, p} = 0.$$

4. WEAK AND STRONG LAW OF LARGE NUMBERS FOR FUNCTIONS OF INDEPENDENT RANDOM FIELDS

In this Section, we study the case where $(X_i)_{i \in \mathbb{Z}^d}$ has the form

$$(4.1) \quad X_i = f((\varepsilon_{i-k})_{k \in \mathbb{Z}^d}),$$

where $f: \mathbb{R}^{\mathbb{Z}^d} \rightarrow \mathbb{B}$ is measurable, $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$ is a separable r -smooth Banach space and $(\varepsilon_k)_{k \in \mathbb{Z}^d}$ is an i.i.d. random field. An approach via approximation by m -dependent random fields can be done. In order to quantify this dependence, we will use the natural extension of the physical dependence measure to random fields, which is defined as

$$(4.2) \quad \delta_{\mathbb{B}, p, q}(\mathbf{i}) := \left\| f((\varepsilon_{\mathbf{i}-\mathbf{k}}^*)_{\mathbf{k} \in \mathbb{Z}^d}) - f((\varepsilon_{\mathbf{i}-\mathbf{k}})_{\mathbf{k} \in \mathbb{Z}^d}) \right\|_{\mathbb{B}, p, q}, \quad \mathbf{i} \in \mathbb{Z}^d,$$

where $\varepsilon_{\mathbf{u}}^* = \varepsilon_{\mathbf{u}}$ if $\mathbf{u} \neq \mathbf{0}$ and $\varepsilon_{\mathbf{0}}^* = \varepsilon'_{\mathbf{0}}$, where $\varepsilon'_{\mathbf{0}}$ is independent of $(\varepsilon_{\mathbf{u}})_{\mathbf{u} \in \mathbb{Z}^d}$ and has the same distribution as $\varepsilon_{\mathbf{0}}$.

For $\mathbf{i} \in \mathbb{Z}^d$, we denote by $\|\mathbf{i}\|_{\infty}$ the quantity $\max_{\ell \in [1, d]} |i_{\ell}|$. We are now in position to state a strong law of large numbers for random fields of the form (4.1).

Theorem 4.1 (Law of large numbers on rectangles). *Let $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$ be a separable r -smooth Banach space and let $1 < p < r$. Let $(X_i)_{i \in \mathbb{Z}^d}$ be a centered random field admitting the representation (4.1). Suppose that*

$$(4.3) \quad \sum_{k=1}^{\infty} k^{d(1-1/p)} \left(\sum_{\mathbf{i}: \|\mathbf{i}\|_{\infty}=k} (\delta_{\mathbb{B}, p, d-1}(\mathbf{i}))^p \right)^{1/p} < \infty.$$

Then

$$(4.4) \quad \lim_{N \rightarrow \infty} \sup_{n \succ 1, \max n \geq N} \frac{1}{|n|^{1/p}} \left\| \sum_{1 \leq i \leq n} X_i \right\|_{\mathbb{B}} = 0 \text{ a.s..}$$

Theorem 4.2 (Law of large numbers on squares). *Let $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$ be a separable r -smooth Banach space and let $1 < p < r$. Let $(X_i)_{i \in \mathbb{Z}^d}$ be a centered random field admitting the representation (4.1). Suppose that*

$$(4.5) \quad \sum_{k=1}^{\infty} k^{d(1-1/p)} \left(\sum_{\mathbf{i}: \|\mathbf{i}\|_{\infty}=k} (\delta_{\mathbb{B}, p}(\mathbf{i}))^p \right)^{1/p} < \infty.$$

Then

$$(4.6) \quad \lim_{n \rightarrow \infty} \frac{1}{n^{d/p}} \left\| \sum_{1 \leq i \leq n \mathbf{1}} X_i \right\|_{\mathbb{B}} = 0 \text{ a.s..}$$

Theorem 4.3 (Convergence in \mathbb{L}^p). *Let $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$ be a separable r -smooth Banach space and let $1 < p < r$. Let $(X_i)_{i \in \mathbb{Z}^d}$ be a centered random field admitting the representation (4.1). Suppose that*

$$(4.7) \quad \sum_{k=1}^{\infty} k^{d(1-1/p)} \left(\sum_{\mathbf{i}: \|\mathbf{i}\|_{\infty}=k} (\delta_{\mathbb{B}, p}(\mathbf{i}))^p \right)^{1/p} < \infty.$$

Then

$$(4.8) \quad \lim_{N \rightarrow \infty} \sup_{n \succ 1, \max n \geq N} \frac{1}{|n|^{1/p}} \left\| \sum_{1 \leq i \leq n} X_i \right\|_{\mathbb{B}, p} = 0.$$

We provide an application of the previous result to Hölderian functions of a linear random field, which are represented as

$$(4.9) \quad X_i = g \left(\sum_{\mathbf{k} \in \mathbb{Z}^d} A_{\mathbf{k}} (\varepsilon_{i-\mathbf{k}}) \right),$$

where $g: \mathbb{B} \rightarrow \mathbb{B}$ is α -Hölder continuous for some $\alpha \in (0, 1]$, that is, there exists a constant $K(g)$ such that for each $x, x' \in \mathbb{B}$,

$$(4.10) \quad \|g(x) - g(x')\|_{\mathbb{B}} \leq K(g) \|x - x'\|_{\mathbb{B}}^{\alpha},$$

$(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$ is a separable r -smooth Banach space, $(\varepsilon_{\mathbf{u}})_{\mathbf{u} \in \mathbb{Z}^d}$ is i.i.d. and \mathbb{B} -valued, $\varepsilon_{\mathbf{0}}$ belongs to \mathbb{L}^p for some $1 < p < r$, $A_{\mathbf{k}}: \mathbb{B} \rightarrow \mathbb{B}$ is a linear bounded operator, $\sum_{\mathbf{k} \in \mathbb{Z}^d} \|A_{\mathbf{k}}\|_{\mathcal{B}(\mathbb{B})}^{p\alpha} < \infty$, where $\|A_{\mathbf{k}}\|_{\mathcal{B}(\mathbb{B})} = \sup_{u \in \mathbb{B}, u \neq 0} \|A_{\mathbf{k}}(u)\|_{\mathbb{B}} / \|u\|_{\mathbb{B}}$.

Corollary 4.4. *Let \mathbb{B} be a separable r -smooth Banach space and $1 < p < r$. Let $(X_i)_{i \in \mathbb{Z}^d}$ be a random field defined as in (4.9). Suppose that $p\alpha \geq 1$ and that*

$$(4.11) \quad \sum_{k=1}^{\infty} k^{d(1-1/p)} \left(\sum_{\mathbf{i}: \|\mathbf{i}\|_{\infty} = k} \|a_{\mathbf{i}}\|_{\mathcal{B}(\mathbb{B})}^{p\alpha} \right)^{1/p} < \infty.$$

- If $\|\varepsilon_{\mathbf{0}}\|_{\mathbb{B}, p\alpha} < \infty$, then

$$(4.12) \quad \lim_{N \rightarrow \infty} \sup_{n \succ 1, \max n \geq N} \frac{1}{|n|^{1/p}} \left\| \sum_{1 \leq i \leq n} (X_i - \mathbb{E}[X_i]) \right\|_{\mathbb{B}, p} = 0,$$

and

$$(4.13) \quad \lim_{N \rightarrow \infty} \sup_{n \geq N} \frac{1}{n^{d/p}} \left\| \sum_{1 \leq i \leq n\mathbf{1}} (X_i - \mathbb{E}[X_i]) \right\|_{\mathbb{B}} = 0 \text{ a.s.}$$

- If $\|\varepsilon_{\mathbf{0}}\|_{\mathbb{B}, p\alpha, d-1} < \infty$, then

$$(4.14) \quad \lim_{N \rightarrow \infty} \sup_{n \succ 1, \max n \geq N} \frac{1}{|n|^{1/p}} \left\| \sum_{1 \leq i \leq n} (X_i - \mathbb{E}[X_i]) \right\|_{\mathbb{B}} = 0 \text{ a.s.}$$

5. PROOFS

5.1. Proof of Theorem 2.6. It suffices to prove that there exists a constant $C(\mathbb{B}, d, p)$ such that for each identically distributed orthomartingale difference random field $(D_i)_{i \in \mathbb{Z}^d}$ and each positive t ,

$$(5.1) \quad \sum_{N \geq 1} \mathbb{P} \left(\max_{1 \leq n \leq 2^{N+1}} \|S_{n\mathbf{1}}\|_{\mathbb{B}} > 2^{dN/p} t \right) \leq C(\mathbb{B}, d, p) t^{-p} \|D_{\mathbf{0}}\|_{\mathbb{B}, p}^p.$$

Replacing D_i by tD_i , it suffices to prove (5.1) for $t = 1$. Define for $\ell \geq 1$ the set

$$(5.2) \quad I_{\ell} := \left\{ \mathbf{i} \in \mathbb{Z}^d, \mathbf{i} \succ \mathbf{1}, \max \mathbf{i} = \ell \right\}.$$

and

$$(5.3) \quad d_\ell = \sum_{i \in I_\ell} D_i.$$

For $N \geq 1$, let

$$(5.4) \quad d'_{N,\ell} := \sum_{i \in I_\ell} \sum_{J \subset [1,d]} (-1)^{|J|} \mathbb{E} \left[D_i \mathbf{1}_{\|D_i\|_{\mathbb{B}} \leq 2^{Nd/p}} \mid \mathcal{F}_{i-1_J} \right],$$

$$(5.5) \quad d''_{N,\ell} := \sum_{i \in I_\ell} \sum_{J \subset [1,d]} (-1)^{|J|} \mathbb{E} \left[D_i \mathbf{1}_{\|D_i\|_{\mathbb{B}} > 2^{Nd/p}} \mid \mathcal{F}_{i-1_J} \right].$$

The orthomartingale property of $(D_i)_{i \in \mathbb{Z}^d}$ guarantees that $d_\ell = d'_{N,\ell} + d''_{N,\ell}$. Therefore, it suffices to prove that

$$(5.6) \quad \sum_{N \geq 1} \mathbb{P} \left(\max_{1 \leq n \leq 2^N} \left\| \sum_{\ell=1}^n d'_{N,\ell} \right\|_{\mathbb{B}} > 2^{dN/p} \right) \leq C(\mathbb{B}, d, p) \|D_{\mathbf{0}}\|_{\mathbb{B},p}^p \text{ and}$$

$$(5.7) \quad \sum_{N \geq 1} \mathbb{P} \left(\max_{1 \leq n \leq 2^N} \left\| \sum_{\ell=1}^n d''_{N,\ell} \right\|_{\mathbb{B}} > 2^{dN/p} \right) \leq C(\mathbb{B}, d, p) \|D_{\mathbf{0}}\|_{\mathbb{B},p}^p.$$

Observe that for a fixed $N \geq 1$, the sequence $(d'_{N,\ell})_{\ell=1}^{2^N}$ is a martingale difference sequence, Doob's inequality combined with Proposition A.1 gives that

$$(5.8) \quad \mathbb{P} \left(\max_{1 \leq n \leq 2^N} \left\| \sum_{\ell=1}^n d'_{N,\ell} \right\|_{\mathbb{B}} > 2^{dN/p} \right) \leq 2^{-rdN/p} \left(\frac{r}{r-1} \right)^r C(\mathbb{B}, 1) \sum_{\ell=1}^{2^N} \mathbb{E} \left[\|d'_{N,\ell}\|_{\mathbb{B}}^r \right].$$

Moreover, using the orthomartingale property of $(\sum_{J \subset [1,d]} (-1)^{|J|} \mathbb{E} [D_i \mathbf{1}_{\|D_i\|_{\mathbb{B}} \leq 2^{Nd/p}} \mid \mathcal{F}_{i-1_J}])_{i \in \mathbb{Z}^d}$ followed by the triangle inequality and the fact that $(D_i)_{i \in \mathbb{Z}^d}$ is identically distributed, one gets

$$(5.9) \quad \begin{aligned} & \mathbb{P} \left(\max_{1 \leq n \leq 2^N} \left\| \sum_{\ell=1}^n d'_{N,\ell} \right\|_{\mathbb{B}} > 2^{dN/p} \right) \\ & \leq 2^{-rdN/p} \left(\frac{r}{r-1} \right)^r C(\mathbb{B}, 1)^r C(\mathbb{B}, d)^r \sum_{\ell=1}^{2^N} \sum_{i \in I_\ell} \mathbb{E} \left[\left\| \sum_{J \subset [1,d]} (-1)^{|J|} \mathbb{E} [D_i \mathbf{1}_{\|D_i\|_{\mathbb{B}} \leq 2^{Nd/p}} \mid \mathcal{F}_{i-1_J}] \right\|_{\mathbb{B}}^r \right] \\ & \leq 2^{Nd(1-r/p)} \left(\frac{r2^d}{r-1} \right)^r C(\mathbb{B}, 1)^r C(\mathbb{B}, d)^r \mathbb{E} \left[\left\| D_{\mathbf{0}} \mathbf{1}_{\|D_{\mathbf{0}}\|_{\mathbb{B}} \leq 2^{Nd/p}} \right\|_{\mathbb{B}}^r \right]. \end{aligned}$$

Then (5.6) follows from inequality (B.1).

Let us show (5.7). Using $\max_{1 \leq n \leq 2^N} \left\| \sum_{\ell=1}^n d''_{N,\ell} \right\|_{\mathbb{B}} \leq \sum_{\ell=1}^{2^N} \|d''_{N,\ell}\|_{\mathbb{B}}$ and Markov's inequality gives

$$\mathbb{P} \left(\max_{1 \leq n \leq 2^N} \left\| \sum_{\ell=1}^n d''_{N,\ell} \right\|_{\mathbb{B}} > 2^{dN/p} \right) \leq 2^{-dN/p} \sum_{\ell=1}^{2^N} \mathbb{E} \left[\left\| \sum_{i \in I_\ell} \sum_{J \subset [1,d]} (-1)^{|J|} \mathbb{E} [D_i \mathbf{1}_{\|D_i\|_{\mathbb{B}} > 2^{Nd/p}} \mid \mathcal{F}_{i-1_J}] \right\|_{\mathbb{B}}^r \right].$$

Then using the triangle inequality and the fact that $(D_i)_{i \in \mathbb{Z}^d}$ is identically distributed gives

$$(5.10) \quad \mathbb{P} \left(\max_{1 \leq n \leq 2^N} \left\| \sum_{\ell=1}^n d''_{N,\ell} \right\|_{\mathbb{B}} > 2^{dN/p} \right) \leq 2^{d-dN(1-1/p)} 2^N \mathbb{E} \left[\|D_{\mathbf{0}}\|_{\mathbb{B}} \mathbf{1}_{\|D_{\mathbf{0}}\|_{\mathbb{B}} > 2^{dN/p}} \right].$$

and (5.7) follows from inequality (B.2). This ends the proof of Theorem 2.6.

5.2. Proof of Theorem 2.7. The proof will be done via a truncation argument. Let $C(\mathbb{B}, d, p)$ be like in Corollary A.2. Fix a positive ε and choose R such that

$$(5.11) \quad \left\| D_0 \mathbf{1}_{\|D_0\|_{\mathbb{B}} > R} \right\|_{\mathbb{B}, p} \leq 2^{-d-1} \varepsilon / C(\mathbb{B}, d, p).$$

Define

$$(5.12) \quad D'_i := \sum_{I \subset [1, d]} (-1)^{|I|} \mathbb{E} \left[D_i \mathbf{1}_{\|D_i\|_{\mathbb{B}} \leq R} \mid \mathcal{F}_{i-1_I} \right], D''_i := \sum_{I \subset [1, d]} (-1)^{|I|} \mathbb{E} \left[D_i \mathbf{1}_{\|D_i\|_{\mathbb{B}} > R} \mid \mathcal{F}_{i-1_I} \right].$$

By the orthomartingale property of D_i , one has $D'_i + D''_i = D_i$. Consequently, for each $\mathbf{n} \succcurlyeq \mathbf{1}$,

$$(5.13) \quad |\mathbf{N}|^{-1/p} \max_{\mathbf{1} \preccurlyeq \mathbf{n} \preccurlyeq \mathbf{N}} \|S_{\mathbf{n}}\|_{\mathbb{B}} \leq |\mathbf{N}|^{-1/p} \max_{\mathbf{1} \preccurlyeq \mathbf{n} \preccurlyeq \mathbf{N}} \left\| \sum_{\mathbf{1} \preccurlyeq \mathbf{i} \preccurlyeq \mathbf{n}} D'_i \right\|_{\mathbb{B}} + |\mathbf{N}|^{-1/p} \max_{\mathbf{1} \preccurlyeq \mathbf{n} \preccurlyeq \mathbf{N}} \left\| \sum_{\mathbf{1} \preccurlyeq \mathbf{i} \preccurlyeq \mathbf{n}} D''_i \right\|_{\mathbb{B}}.$$

For the first term of the right hand side of (5.13), we use $\|\cdot\|_{\mathbb{B}, p} \leq \|\cdot\|_{\mathbb{B}, r}$ and Proposition A.1 in order to derive that

$$(5.14) \quad |\mathbf{N}|^{-1/p} \left\| \max_{\mathbf{1} \preccurlyeq \mathbf{n} \preccurlyeq \mathbf{N}} \left\| \sum_{\mathbf{1} \preccurlyeq \mathbf{i} \preccurlyeq \mathbf{n}} D'_i \right\|_{\mathbb{B}} \right\|_p \leq C(\mathbb{B}, d, p) |\mathbf{N}|^{-1/p} \left(\sum_{\mathbf{1} \preccurlyeq \mathbf{i} \preccurlyeq \mathbf{N}} \|D'_i\|_{\mathbb{B}, r}^r \right)^{1/r}.$$

Moreover, by definition of D'_i , one has

$$(5.15) \quad \|D'_i\|_{\mathbb{B}, r} \leq 2^d \left\| D_i \mathbf{1}_{\|D_i\|_{\mathbb{B}} \leq R} \right\|_{\mathbb{B}, r} = 2^d \left\| D_0 \mathbf{1}_{\|D_0\|_{\mathbb{B}} \leq R} \right\|_{\mathbb{B}, r}$$

where we used the fact that the random field $(D_i)_{i \in \mathbb{Z}^d}$ is identically distributed. We thus infer the bound

$$(5.16) \quad |\mathbf{N}|^{-1/p} \left\| \max_{\mathbf{1} \preccurlyeq \mathbf{n} \preccurlyeq \mathbf{N}} \left\| \sum_{\mathbf{1} \preccurlyeq \mathbf{i} \preccurlyeq \mathbf{n}} D'_i \right\|_{\mathbb{B}} \right\|_p \leq C(\mathbb{B}, d, p) 2^d |\mathbf{N}|^{1/r-1/p} \left\| D_0 \mathbf{1}_{\|D_0\|_{\mathbb{B}} \leq R} \right\|_{\mathbb{B}, r}.$$

Consequently, we can find N_0 such that if $\max \mathbf{N} \geq N_0$, then

$$(5.17) \quad |\mathbf{N}|^{-1/p} \left\| \max_{\mathbf{1} \preccurlyeq \mathbf{n} \preccurlyeq \mathbf{N}} \left\| \sum_{\mathbf{1} \preccurlyeq \mathbf{i} \preccurlyeq \mathbf{n}} D'_i \right\|_{\mathbb{B}} \right\|_p < \varepsilon/2.$$

For the second term of the right hand side of (5.13), we apply Corollary A.2 and get that

$$(5.18) \quad |\mathbf{N}|^{-1/p} \left\| \max_{\mathbf{1} \preccurlyeq \mathbf{n} \preccurlyeq \mathbf{N}} \left\| \sum_{\mathbf{1} \preccurlyeq \mathbf{i} \preccurlyeq \mathbf{n}} D''_i \right\|_{\mathbb{B}} \right\|_p \leq C(\mathbb{B}, d, p) |\mathbf{N}|^{-1/p} \left(\sum_{\mathbf{1} \preccurlyeq \mathbf{i} \preccurlyeq \mathbf{N}} \|D''_i\|_{\mathbb{B}, p}^p \right)^{1/p}.$$

The triangle inequality combined with the identical distribution of $(D_i \mathbf{1}_{\|D_i\|_{\mathbb{B}} > R})_{i \in \mathbb{Z}^d}$ and (5.11) shows that

$$(5.19) \quad |\mathbf{N}|^{-1/p} \left\| \max_{\mathbf{1} \preccurlyeq \mathbf{n} \preccurlyeq \mathbf{N}} \left\| \sum_{\mathbf{1} \preccurlyeq \mathbf{i} \preccurlyeq \mathbf{n}} D''_i \right\|_{\mathbb{B}} \right\|_p < \varepsilon/2.$$

The combination of (5.13), (5.17) and (5.19) concludes the proof of Theorem 2.7.

5.3. **Proof of Theorem 2.8.** It suffices to prove that for each positive ε ,

$$(5.20) \quad \lim_{N \rightarrow \infty} \mathbb{P} \left(\sup_{n \geq 1, \max n \geq N} \frac{1}{|\mathbf{n}|^{1/p}} \left\| \sum_{1 \leq i \leq n} X_i \right\|_{\mathbb{B}} > \varepsilon \right) = 0.$$

Let us define

$$(5.21) \quad X_i^{(K)} := \sum_{\mathbf{k}: |\mathbf{k}-i|_{\infty} \leq K} P_{i+\mathbf{k}}(X_i).$$

By assumption, the following convergence holds;

$$(5.22) \quad \lim_{K \rightarrow \infty} \left\| X_i - \sum_{\mathbf{k} \in \mathbb{Z}^d: \|\mathbf{k}\|_{\infty} \leq K} P_{i+\mathbf{k}}(X_i) \right\|_{\mathbb{B}, p, d-1} = 0,$$

where $\|\cdot\|_{p,q}$ is defined as in (2.4). By (2.8), one obtains

$$(5.23) \quad \mathbb{P} \left(\sup_{n \geq 1, \max n \geq N} \frac{1}{|\mathbf{n}|^{1/p}} \left\| \sum_{1 \leq i \leq n} X_i \right\|_{\mathbb{B}} > \varepsilon \right) \leq \mathbb{P} \left(\sup_{n \geq 1, \max n \geq N} \frac{1}{|\mathbf{n}|^{1/p}} \left\| \sum_{1 \leq i \leq n} X_i^{(K)} \right\|_{\mathbb{B}} > \varepsilon/2 \right) \\ + (c_p \varepsilon/2)^{-p} \left\| \sup_{n \geq 1, \max n \geq N} \frac{1}{|\mathbf{n}|^{1/p}} \left\| \sum_{\mathbf{k} \in \mathbb{Z}^d: \|\mathbf{k}\|_{\infty} > K} \sum_{1 \leq i \leq n} P_{i+\mathbf{k}}(X_i) \right\|_{\mathbb{B}, \mathbb{R}, p, w} \right\|_{\mathbb{B}, \mathbb{R}, p, w}^p$$

and the triangle inequality for the norm $\|\cdot\|_{\mathbb{B}, p, w}$ implies

$$(5.24) \quad \mathbb{P} \left(\sup_{n \geq 1, \max n \geq N} \frac{1}{|\mathbf{n}|^{1/p}} \left\| \sum_{1 \leq i \leq n} X_i \right\|_{\mathbb{B}} > \varepsilon \right) \leq \mathbb{P} \left(\sup_{n \geq 1, \max n \geq N} \frac{1}{|\mathbf{n}|^{1/p}} \left\| \sum_{1 \leq i \leq n} X_i^{(K)} \right\|_{\mathbb{B}} > \varepsilon/2 \right) \\ + (c_p \varepsilon/2)^{-p} \left(\sum_{\mathbf{k} \in \mathbb{Z}^d: \|\mathbf{k}\|_{\infty} > K} \left\| \sup_{n \geq 1, \max n \geq N} \frac{1}{|\mathbf{n}|^{1/p}} \sum_{1 \leq i \leq n} P_{i+\mathbf{k}}(X_i) \right\|_{\mathbb{B}, p, w} \right)^p.$$

Since $(P_{\mathbf{k}+i}(X_i))_{i \in \mathbb{Z}^d}$ is an identically distributed orthomartingale difference random field, Theorem 2.5 gives

$$(5.25) \quad \mathbb{P} \left(\sup_{n \geq 1, \max n \geq N} \frac{1}{|\mathbf{n}|^{1/p}} \left\| \sum_{1 \leq i \leq n} X_i \right\|_{\mathbb{B}} > \varepsilon \right) \leq \mathbb{P} \left(\sup_{n \geq 1, \max n \geq N} \frac{1}{|\mathbf{n}|^{1/p}} \left\| \sum_{1 \leq i \leq n} X_i^{(K)} \right\|_{\mathbb{B}} > \varepsilon/2 \right) \\ + (c_p \varepsilon/2)^{-p} \left(\sum_{\mathbf{k} \in \mathbb{Z}^d: \|\mathbf{k}\|_{\infty} > K} \|P_{\mathbf{k}}(X_0)\|_{\mathbb{B}, p, d-1} \right)^p$$

and since we can choose K such that the last term can be made as small as we wish, it suffices to show that for each K ,

$$(5.26) \quad \lim_{N \rightarrow \infty} \mathbb{P} \left(\sup_{n \geq 1, \max n \geq N} \frac{1}{|\mathbf{n}|^{1/p}} \left\| \sum_{1 \leq i \leq n} X_i^{(K)} \right\|_{\mathbb{B}} > \varepsilon \right) = 0,$$

which reduces, in view of the Borel-Cantelli lemma, to prove that

$$(5.27) \quad \sum_{N \in \mathbb{N}^d} \mathbb{P} \left(\frac{1}{|2N|^{1/p}} \max_{1 \leq n \leq 2^N} \left\| \sum_{1 \leq i \leq n} X_i^{(K)} \right\|_{\mathbb{B}} > \varepsilon \right) < \infty.$$

By the results of Section 4 in [32], we can express $X_i^{(K)}$ as

$$(5.28) \quad X_i^{(K)} = \sum_{I \subset \llbracket 1, d \rrbracket} \prod_{\ell \in \llbracket 1, d \rrbracket \setminus I} (\text{Id} - U^{e\ell}) D_i^{(I)},$$

where for $I \subsetneq \llbracket 1, d \rrbracket$, $(D_i^{(I)})_{i \in \mathbb{Z}, \ell \in I}$ is a strictly stationary orthomartingale difference random field, $D_0^{(I)} \in \mathbb{L}_{p, d-1}$. In view of (5.27) and (5.28), it suffices to prove that for each $I \subset \llbracket 1, d \rrbracket$,

$$(5.29) \quad \sum_{\mathbf{N} \in \mathbb{N}^d} \mathbb{P} \left(\frac{1}{|2^{\mathbf{N}}|^{1/p}} \max_{1 \preccurlyeq \mathbf{n} \preccurlyeq 2^{\mathbf{N}}} \left\| \sum_{1 \preccurlyeq \mathbf{i} \preccurlyeq \mathbf{n}} U^{\mathbf{i}} \prod_{\ell \in \llbracket 1, d \rrbracket \setminus I} (\text{Id} - U^{e\ell}) D_0^{(I)} \right\|_{\mathbb{B}} > \varepsilon \right) < \infty.$$

For $I = \llbracket 1, d \rrbracket$, this follows from Theorem 2.5 and for $I = \emptyset$, one derives from

$$(5.30) \quad \max_{1 \preccurlyeq \mathbf{n} \preccurlyeq 2^{\mathbf{N}}} \left\| \sum_{1 \preccurlyeq \mathbf{i} \preccurlyeq \mathbf{n}} U^{\mathbf{i}} \prod_{\ell \in \llbracket 1, d \rrbracket} (\text{Id} - U^{e\ell}) D_0^{(\emptyset)} \right\|_{\mathbb{B}} \leq \max_{1 \preccurlyeq \mathbf{n} \preccurlyeq 2^{\mathbf{N}+1}} U^{\mathbf{n}} \left(\|D_0^{(\emptyset)}\|_{\mathbb{B}} \right)$$

that

$$(5.31) \quad \sum_{\mathbf{N} \in \mathbb{N}^d} \mathbb{P} \left(\frac{1}{|2^{\mathbf{N}}|^{1/p}} \max_{1 \preccurlyeq \mathbf{n} \preccurlyeq 2^{\mathbf{N}}} \left\| \sum_{1 \preccurlyeq \mathbf{i} \preccurlyeq \mathbf{n}} U^{\mathbf{i}} \prod_{\ell \in \llbracket 1, d \rrbracket} (\text{Id} - U^{e\ell}) D_0^{(\emptyset)} \right\|_{\mathbb{B}} > \varepsilon \right) \\ \leq 2^d \sum_{\mathbf{N} \in \mathbb{N}^d} |2^{\mathbf{N}}| \mathbb{P} \left(\|D_0^{(\emptyset)}\|_{\mathbb{B}} > \varepsilon |2^{\mathbf{N}}|^{1/p} \right)$$

and using

$$(5.32) \quad \text{Card} \left(\left\{ \mathbf{N} \in \mathbb{N}^d : \sum_{\ell=1}^d N_{\ell} = k \right\} \right) \leq c_d k^{d-1},$$

one finds that

$$(5.33) \quad \sum_{\mathbf{N} \in \mathbb{N}^d} \mathbb{P} \left(\frac{1}{|2^{\mathbf{N}}|^{1/p}} \max_{1 \preccurlyeq \mathbf{n} \preccurlyeq 2^{\mathbf{N}}} \left\| \sum_{1 \preccurlyeq \mathbf{i} \preccurlyeq \mathbf{n}} U^{\mathbf{i}} \prod_{\ell \in \llbracket 1, d \rrbracket} (\text{Id} - U^{e\ell}) D_0^{(\emptyset)} \right\|_{\mathbb{B}} > \varepsilon \right) \\ \leq c_d \sum_{k=1}^{\infty} 2^k k^{d-1} \mathbb{P} \left(\|D_0^{(\emptyset)}\|_{\mathbb{B}} > \varepsilon 2^{k/p} \right),$$

which is finite using $D_0^{(\emptyset)} \in \mathbb{L}_{p, d-1}$ and (B.3).

Since the role played by the measure preserving maps is symmetric, it suffices to show that for each $\ell_0 \in \llbracket 1, d-1 \rrbracket$,

$$(5.34) \quad \sum_{\mathbf{N} \in \mathbb{N}^d} \mathbb{P} \left(\frac{1}{|2^{\mathbf{N}}|^{1/p}} \max_{1 \preccurlyeq \mathbf{n} \preccurlyeq 2^{\mathbf{N}}} \left\| \sum_{1 \preccurlyeq \mathbf{i} \preccurlyeq \mathbf{n}} U^{\mathbf{i}} \prod_{\ell \in \llbracket \ell_0+1, d \rrbracket} (\text{Id} - U^{e\ell}) D_0^{(\llbracket 1, \ell_0 \rrbracket)} \right\|_{\mathbb{B}} > \varepsilon \right) < \infty.$$

In order to ease the notations, we write for $\mathbf{n} = (n_{\ell})_{\ell \in \llbracket 1, d \rrbracket}$, $\mathbf{n}' \in \mathbb{N}^{\ell_0} := (n_{\ell})_{\ell \in \llbracket 1, \ell_0 \rrbracket}$ and $\mathbf{n}'' := (n_{\ell})_{\ell \in \llbracket \ell_0+1, d \rrbracket}$, and similar notations for $\mathbf{i}', \mathbf{i}''$. We start from

$$(5.35) \quad \max_{1 \preccurlyeq \mathbf{n} \preccurlyeq 2^{\mathbf{N}}} \left\| \sum_{1 \preccurlyeq \mathbf{i} \preccurlyeq \mathbf{n}} U^{\mathbf{i}} \prod_{\ell \in \llbracket \ell_0+1, d \rrbracket} (\text{Id} - U^{e\ell}) D_0^{(\llbracket 1, \ell_0 \rrbracket)} \right\|_{\mathbb{B}}$$

$$\leq \max_{\mathbf{1}' \preceq \mathbf{n}'' \preceq 2^{\mathbf{N}''} + \mathbf{1}''} U^{\mathbf{n}''} \max_{\mathbf{1}' \preceq \mathbf{n}' \preceq 2^{\mathbf{N}'}} \left\| \sum_{\mathbf{1}' \preceq \mathbf{i}' \preceq \mathbf{n}'} U^{\mathbf{i}'} D_{\mathbf{0}}^{(\llbracket 1, \ell_0 \rrbracket)} \right\|_{\mathbb{B}}$$

and we are therefore reduced to show that

$$(5.36) \quad \sum_{N \in \mathbb{N}^d} |\mathbf{2}^{\mathbf{N}''}| \mathbb{P} \left(\max_{\mathbf{1}' \preceq \mathbf{n}' \preceq 2^{\mathbf{N}'}} \left\| \sum_{\mathbf{1}' \preceq \mathbf{i}' \preceq \mathbf{n}'} U^{\mathbf{i}'} D_{\mathbf{0}}^{(\llbracket 1, \ell_0 \rrbracket)} \right\|_{\mathbb{B}} > \varepsilon |\mathbf{2}^{\mathbf{N}}|^{1/p} \right) < \infty$$

for each positive ε . To do so, define for each $N \in \mathbb{N}^d$ the orthomartingale difference random fields $(D_{N, \mathbf{0}} \circ T^{\mathbf{i}'})_{\mathbf{i}' \in \mathbb{Z}^{\ell_0}}$ and $(D'_{N, \mathbf{0}} \circ T^{\mathbf{i}'})_{\mathbf{i}' \in \mathbb{Z}^{\ell_0}}$ by

$$(5.37) \quad D_{N, \mathbf{0}} := \sum_{J \subset \llbracket 1, \ell_0 \rrbracket} (-1)^{\text{Card}(J)} \mathbb{E} \left[D_{\mathbf{0}}^{(\llbracket 1, \ell_0 \rrbracket)} \mathbf{1}_{\|D_{\mathbf{0}}^{(\llbracket 1, \ell_0 \rrbracket)}\|_{\mathbb{B}} \leq |\mathbf{2}^{\mathbf{N}}|^{1/p}} \mid \mathcal{F}_{\mathbf{0}-1, J} \right],$$

$$(5.38) \quad D'_{N, \mathbf{0}} := \sum_{J \subset \llbracket 1, \ell_0 \rrbracket} (-1)^{\text{Card}(J)} \mathbb{E} \left[D_{\mathbf{0}}^{(\llbracket 1, \ell_0 \rrbracket)} \mathbf{1}_{\|D_{\mathbf{0}}^{(\llbracket 1, \ell_0 \rrbracket)}\|_{\mathbb{B}} > |\mathbf{2}^{\mathbf{N}}|^{1/p}} \mid \mathcal{F}_{\mathbf{0}-1, J} \right].$$

The equality $D_{\mathbf{0}}^{(\llbracket 1, \ell_0 \rrbracket)} = D_{N, \mathbf{0}} + D'_{N, \mathbf{0}}$ reduces the proof of (5.36) to

$$(5.39) \quad \sum_{N \in \mathbb{N}^d} |\mathbf{2}^{\mathbf{N}''}| \mathbb{P} \left(\max_{\mathbf{1}' \preceq \mathbf{n}' \preceq 2^{\mathbf{N}'}} \left\| \sum_{\mathbf{1}' \preceq \mathbf{i}' \preceq \mathbf{n}'} U^{\mathbf{i}'} D_{N, \mathbf{0}} \right\|_{\mathbb{B}} > \varepsilon |\mathbf{2}^{\mathbf{N}}|^{1/p} \right) < \infty \text{ and}$$

$$(5.40) \quad \sum_{N \in \mathbb{N}^d} |\mathbf{2}^{\mathbf{N}''}| \mathbb{P} \left(\max_{\mathbf{1}' \preceq \mathbf{n}' \preceq 2^{\mathbf{N}'}} \left\| \sum_{\mathbf{1}' \preceq \mathbf{i}' \preceq \mathbf{n}'} U^{\mathbf{i}'} D'_{N, \mathbf{0}} \right\|_{\mathbb{B}} > \varepsilon |\mathbf{2}^{\mathbf{N}}|^{1/p} \right) < \infty.$$

In order to show (5.39), we use Corollary A.2 with $p = r$ combined with the identical distribution of $(U^{\mathbf{i}'} D_{N, \mathbf{0}})_{\mathbf{i}' \in \mathbb{Z}^{\ell_0}}$ and derive that

$$(5.41) \quad \begin{aligned} |\mathbf{2}^{\mathbf{N}''}| \mathbb{P} \left(\max_{\mathbf{1}' \preceq \mathbf{n}' \preceq 2^{\mathbf{N}'}} \left\| \sum_{\mathbf{1}' \preceq \mathbf{i}' \preceq \mathbf{n}'} U^{\mathbf{i}'} D_{N, \mathbf{0}} \right\|_{\mathbb{B}} > \varepsilon |\mathbf{2}^{\mathbf{N}}|^{1/p} \right) \\ \leq \varepsilon^{-r} C(\mathbb{B}, d, r) |\mathbf{2}^{\mathbf{N}}|^{-r/p} |\mathbf{2}^{\mathbf{N}}| \|D_{N, \mathbf{0}}\|_{\mathbb{B}, r}^r \\ \leq \varepsilon^{-r} C(\mathbb{B}, d, r) 2^{\ell_0} |\mathbf{2}^{\mathbf{N}}|^{(1-r/p)} \left\| D_{\mathbf{0}}^{(\llbracket 1, \ell_0 \rrbracket)} \mathbf{1}_{\|D_{\mathbf{0}}^{(\llbracket 1, \ell_0 \rrbracket)}\|_{\mathbb{B}} \leq |\mathbf{2}^{\mathbf{N}}|^{1/p}} \right\|_{\mathbb{B}, r}^r. \end{aligned}$$

Then by (5.32), we reduce the proof of (5.39) to

$$(5.42) \quad \sum_{k=1}^{\infty} 2^{k(1-r/p)} k^{d-1} \left\| D_{\mathbf{0}}^{(\llbracket 1, \ell_0 \rrbracket)} \mathbf{1}_{\|D_{\mathbf{0}}^{(\llbracket 1, \ell_0 \rrbracket)}\|_{\mathbb{B}} \leq 2^{k/p}} \right\|_{\mathbb{B}, r}^r < \infty,$$

which follows from (B.4). In order to show (5.40), we start by Markov's inequality, which gives

$$|\mathbf{2}^{\mathbf{N}''}| \mathbb{P} \left(\max_{\mathbf{1}' \preceq \mathbf{n}' \preceq 2^{\mathbf{N}'}} \left\| \sum_{\mathbf{1}' \preceq \mathbf{i}' \preceq \mathbf{n}'} U^{\mathbf{i}'} D'_{N, \mathbf{0}} \right\|_{\mathbb{B}} > \varepsilon |\mathbf{2}^{\mathbf{N}}| \right) \leq 2^d \varepsilon^{-1} |\mathbf{2}^{\mathbf{N}}| \mathbb{E} \left[\left\| D_{\mathbf{0}}^{(\llbracket 1, \ell_0 \rrbracket)} \right\|_{\mathbb{B}} \mathbf{1}_{\|D_{\mathbf{0}}^{(\llbracket 1, \ell_0 \rrbracket)}\|_{\mathbb{B}} > |\mathbf{2}^{\mathbf{N}}|^{1/p}} \right],$$

then we sum over $N \in \mathbb{N}^d$, use (5.32) in order to reduce the proof of (5.40) to that of

$$(5.43) \quad \sum_{k=0}^{\infty} 2^k k^{d-1} \mathbb{E} \left[\left\| D_{\mathbf{0}}^{(\llbracket 1, \ell_0 \rrbracket)} \right\|_{\mathbb{B}} \mathbf{1}_{\|D_{\mathbf{0}}^{(\llbracket 1, \ell_0 \rrbracket)}\|_{\mathbb{B}} > 2^{k/p}} \right] < \infty.$$

Then we use (B.5) to get (5.40) and conclude the proof of Theorem 2.8.

5.4. Proof of Theorem 2.9. The proof follows essentially the lines of that of Theorem 2.8 hence we will mention only the crucial steps. The reduction to the case where the random field $(X_i)_{i \in \mathbb{Z}^d}$ admits the representation (5.28) works in an analogous way, replacing each occurrence of $\sup_{n \geq 1, \max n \geq N}$ by a supremum over the n having identical coordinates. The analogue of (5.29) reads as

$$(5.44) \quad \sum_{N=1}^{\infty} \mathbb{P} \left(\frac{1}{2^{Nd/p}} \max_{1 \leq n \leq 2^N} \left\| \sum_{1 \leq i \leq n} U^i \prod_{\ell \in \llbracket 1, d \rrbracket \setminus I} (\text{Id} - U^{e_\ell}) D_0^{(I)} \right\|_{\mathbb{B}} > \varepsilon \right) < \infty.$$

For $I = \llbracket 1, d \rrbracket$, this follows from Theorem 2.6 and for $I = \emptyset$, we use the bound

$$(5.45) \quad \left\| \sum_{1 \leq i \leq n} U^i \prod_{\ell \in \llbracket 1, d \rrbracket} (\text{Id} - U^{e_\ell}) D_0^{(\emptyset)} \right\|_{\mathbb{B}} \leq U^1 \max_{J \subset \llbracket 1, d \rrbracket} U^{n \mathbf{1}_J} \left(\left\| D_0^{(\emptyset)} \right\|_{\mathbb{B}} \right)$$

from which it follows that

$$(5.46) \quad \mathbb{P} \left(\frac{1}{2^{Nd/p}} \max_{1 \leq n \leq 2^N} \left\| \sum_{1 \leq i \leq n} U^i \prod_{\ell \in \llbracket 1, d \rrbracket \setminus I} (\text{Id} - U^{e_\ell}) D_0^{(I)} \right\|_{\mathbb{B}} > \varepsilon \right) \leq 2^{Nd} \mathbb{P} \left(\left\| D_0^{(\emptyset)} \right\|_{\mathbb{B}} > \varepsilon 2^{Nd/p} \right)$$

and the convergence of the series in (5.44) is a consequence of the fact that $\left\| D_0^{(\emptyset)} \right\|_{\mathbb{B}}$ belongs to \mathbb{L}^p . For $\emptyset \subsetneq I \subsetneq \llbracket 1, d \rrbracket$, we use the same symmetry argument to deal with the case $I = \llbracket 1, \ell_0 \rrbracket$ for some $\ell_0 \in \llbracket 1, d-1 \rrbracket$. Denoting as before $i' = (i_\ell)_{\ell \in \llbracket 1, \ell_0 \rrbracket}$ for $i = (i_\ell)_{\ell \in \llbracket 1, d \rrbracket}$, we have

$$(5.47) \quad \max_{1 \leq n \leq 2^N} \left\| \sum_{1 \leq i \leq n} U^i \prod_{\ell \in \llbracket \ell_0+1, d \rrbracket} (\text{Id} - U^{e_\ell}) D_0^{(I)} \right\|_{\mathbb{B}} \leq U^{1''} \max_{J \subset \llbracket \ell_0+1, d \rrbracket} \max_{1 \leq n \leq 2^N} U^{n \mathbf{1}_J} \left\| \sum_{1' \leq i' \leq n'} D_0^{(\llbracket 1, \ell_0 \rrbracket)} \right\|_{\mathbb{B}}$$

hence

$$(5.48) \quad \mathbb{P} \left(\max_{1 \leq n \leq 2^N} \left\| \sum_{1 \leq i \leq n} U^i \prod_{\ell \in \llbracket 1, d \rrbracket \setminus I} (\text{Id} - U^{e_\ell}) D_0^{(I)} \right\|_{\mathbb{B}} > \varepsilon 2^{Nd/p} \right) \leq 2^{d+\ell_0 N} \mathbb{P} \left(\max_{1 \leq n \leq 2^N} U^{n \mathbf{1}_J} \left\| \sum_{1' \leq i' \leq n'} D_0^{(\llbracket 1, \ell_0 \rrbracket)} \right\|_{\mathbb{B}} > \varepsilon 2^{Nd/p} \right).$$

The control of the tail is done in a similar way, this time with $2^{Nd/p}$ as a truncation level, which leads to show the convergence of analogous series as in (5.42) and (5.43) but without the term k^{d-1} and where k is a multiple of d , which follows from the fact that $\left\| D_0^{(\llbracket 1, \ell_0 \rrbracket)} \right\|_{\mathbb{B}}$ belongs to \mathbb{L}^p . This ends the proof Theorem 2.9.

5.5. Proof of Theorem 2.10. Define $X_i^{(K)}$ as in (5.21). Observe that $X_i = X_i^{(K)} + \sum_{k: |k-i|_\infty > K} P_k(X_i)$, hence

$$(5.49) \quad \frac{1}{|N|^{1/p}} \left\| \max_{1 \leq n \leq N} \left\| \sum_{1 \leq i \leq n} X_i \right\|_{\mathbb{B}} \right\|_p \leq \frac{1}{|N|^{1/p}} \left\| \max_{1 \leq n \leq N} \left\| \sum_{1 \leq i \leq n} X_i^{(K)} \right\|_{\mathbb{B}} \right\|_p + \sum_{k: |k-i|_\infty > K} \frac{1}{|N|^{1/p}} \left\| \max_{1 \leq n \leq N} \left\| \sum_{1 \leq i \leq n} P_{i+k}(X_i) \right\|_{\mathbb{B}} \right\|_p.$$

Moreover, using Corollary A.2 then (2.18), one has

$$(5.50) \quad \frac{1}{|\mathbf{N}|^{1/p}} \left\| \left\| \max_{1 \leq n \leq N} \left\| \sum_{1 \leq i \leq n} P_{i+k}(X_i) \right\|_{\mathbb{B}} \right\|_p \leq \frac{C(\mathbb{B}, d, p)}{|\mathbf{N}|^{1/p}} \left(\sum_{1 \leq i \leq N} \|P_{i+k}(X_i)\|_{\mathbb{B}, p}^p \right)^{1/p}$$

$$(5.51) \quad \leq C(\mathbb{B}, d, p) \|P_k(X_0)\|_{\mathbb{B}, p}$$

hence in view of assumption (2.25), it suffices to prove that for each K ,

$$(5.52) \quad \lim_{R \rightarrow \infty} \sup_{N \geq 1, \max N \geq R} \frac{1}{|\mathbf{N}|^{1/p}} \left\| \left\| \max_{1 \leq n \leq N} \left\| \sum_{1 \leq i \leq n} U^i X_0^{(K)} \right\|_{\mathbb{B}} \right\|_p = 0.$$

In view of the decomposition (5.28), this reduces to check that for each $I \subset \llbracket 1, d \rrbracket$,

$$(5.53) \quad \lim_{R \rightarrow \infty} \sup_{N \geq 1, \max N \geq R} \frac{1}{|\mathbf{N}|^{1/p}} \left\| \left\| \max_{1 \leq n \leq N} \left\| \sum_{1 \leq i \leq n} \prod_{\ell \in \llbracket 1, d \rrbracket \setminus I} (\text{Id} - U^{e_\ell}) U^i D_0^{(I)} \right\|_{\mathbb{B}} \right\|_p = 0.$$

The case $I = \llbracket 1, d \rrbracket$ corresponds to Theorem 2.7. Moreover, for $I = \emptyset$, using $\|\max_{j \in J} \|Y_j\|_{\mathbb{B}}\|_p \leq \text{Card}(J)^{1/p} \max_{j \in J} \|Y_j\|_{\mathbb{B}, p}$, one has

$$(5.54) \quad \left\| \left\| \max_{1 \leq i \leq N} \left\| \sum_{1 \leq i \leq n} \prod_{\ell \in \llbracket 1, d \rrbracket} (\text{Id} - U^{e_\ell}) D_i^{(\emptyset)} \right\|_{\mathbb{B}} \right\|_p \leq 2^d \left\| \left\| \max_{1 \leq i \leq N+1} \|U^i D_0^{(\emptyset)}\|_{\mathbb{B}} \right\|_p$$

$$(5.55) \quad \leq 2^d \tau + |\mathbf{N} + 1|^{1/p} \left\| D_0^{(\emptyset)} \mathbf{1}_{\|D_0^{(\emptyset)}\|_{\mathbb{B}} > \tau} \right\|_p$$

hence (5.53) holds for $I = \emptyset$. Since the role played by the coordinates is symmetric, it suffices to treat the case $I = \llbracket 1, \ell_0 \rrbracket$ for each $\ell_0 \in \llbracket 1, d-1 \rrbracket$. For such an ℓ_0 , denote for $\mathbf{i} = (i_\ell)_{\ell \in \llbracket 1, d \rrbracket} \in \mathbb{N}^d$, $\mathbf{i}' := (i_\ell)_{\ell \in \llbracket 1, \ell_0 \rrbracket}$ and $\mathbf{i}'' := (i_\ell)_{\ell \in \llbracket \ell_0+1, d \rrbracket}$ and similarly for $\mathbf{n}', \mathbf{n}'', \mathbf{1}'$ and $\mathbf{1}''$. One has

$$(5.56) \quad \max_{1 \leq n \leq N} \left\| \left\| \sum_{1 \leq i \leq n} \prod_{\ell \in \llbracket 1, d \rrbracket \setminus I} (\text{Id} - U^{e_\ell}) U^i D_0^{(I)} \right\|_{\mathbb{B}} \right\|_p \leq \max_{\mathbf{1}'' \leq \mathbf{i}'' \leq \mathbf{N}''+1} U^{\mathbf{i}''} \max_{\mathbf{1}' \leq \mathbf{n}' \leq \mathbf{N}'} \left\| \left\| \sum_{\mathbf{1}' \leq \mathbf{i}' \leq \mathbf{n}'} U^{\mathbf{i}'} D_0^{(I)} \right\|_{\mathbb{B}} \right\|_p.$$

and by Theorem 2.7, the family

$$\left\{ M_{\mathbf{N}'} := \frac{1}{|\mathbf{N}'|} \max_{\mathbf{1}' \leq \mathbf{n}' \leq \mathbf{N}'} \left\| \left\| \sum_{\mathbf{1}' \leq \mathbf{i}' \leq \mathbf{n}'} U^{\mathbf{i}'} D_0^{(I)} \right\|_{\mathbb{B}} \right\|_p^p, \mathbf{N}' \in \mathbb{N}^{\ell_0} \right\},$$

is uniformly integrable and $\lim_{R \rightarrow \infty} \max_{\mathbf{N}': \max \mathbf{N}' \geq R} \|M_{\mathbf{N}'}\|_{\mathbb{B}, p} = 0$. With the observation that

$$(5.57) \quad A_N := \frac{1}{|\mathbf{N}|^{1/p}} \left\| \left\| \max_{1 \leq n \leq N} \left\| \sum_{1 \leq i \leq n} \prod_{\ell \in \llbracket 1, d \rrbracket \setminus I} (\text{Id} - U^{e_\ell}) U^i D_0^{(I)} \right\|_{\mathbb{B}} \right\|_p$$

$$(5.58) \quad \leq \frac{1}{|\mathbf{N}''|^{1/p}} \max_{\mathbf{1}'' \leq \mathbf{i}'' \leq \mathbf{N}''+1} U^{\mathbf{i}''} M_{\mathbf{N}'}^{1/p},$$

one get

$$(5.59) \quad A_N \leq 2^p \min \left\{ \left\| M_{\mathbf{N}'}^{1/p} \right\|_{\mathbb{B}, p}, \frac{\tau}{|\mathbf{N}''|^{1/p}} + \left\| M_{\mathbf{N}'} \mathbf{1}_{\|M_{\mathbf{N}'}\|_{\mathbb{B}} > \tau} \right\|_{\mathbb{B}, p} \right\}.$$

This ends the proof of Theorem 2.7.

5.6. Proof of the results of Section 3. As pointed out before the statements of theorems, we can decompose the random field $(X_i)_{i \in \mathbb{Z}^d}$ as a sum indexed by subsets I of $\llbracket 1, d \rrbracket$ of random fields whose coordinates are the restriction of those of \mathbf{i} to the set I , namely,

$$(5.60) \quad X_{\mathbf{i}} = \sum_{I \subset \llbracket 1, d \rrbracket, \text{Card}(I) \geq d_0} X_{\mathbf{i}_I}^I,$$

where $X_{\mathbf{i}_I}^I$ is defined as in (3.6). Consequently, for each $\mathbf{n} \succcurlyeq \mathbf{1}$,

$$(5.61) \quad \frac{1}{\pi_{d_0, p}(\mathbf{n})} \left\| \sum_{\mathbf{1} \preccurlyeq \mathbf{i} \preccurlyeq \mathbf{n}} X_{\mathbf{i}} \right\|_{\mathbb{B}} \leq \sum_{I \subset \llbracket 1, d \rrbracket, \text{Card}(I) \geq d_0} \frac{1}{\prod_{\ell \in I} n_{i_\ell}^{1/p}} \left\| \sum_{\mathbf{1}_I \preccurlyeq \mathbf{i}_I \preccurlyeq \mathbf{n}_I} X_{\mathbf{i}_I}^I \right\|_{\mathbb{B}}.$$

We are thus reduced to show that conditions (3.13) and (3.9) imply that for each $I \subset \llbracket 1, d \rrbracket$ having d_0 or more elements, the random field $(X_{\mathbf{i}_I}^I)_{\mathbf{i}_I \in \mathbb{Z}^I}$ satisfies the assumptions of Theorems 2.8, 2.9 and 2.10. By the symmetric role played by the coordinates, it suffices to show that (3.13) (respectively (3.9)) implies that for each $\ell_0 \in \llbracket d_0, d \rrbracket$,

$$(5.62) \quad \sum_{\mathbf{k} \in \mathbb{Z}^{\ell_0}} \left\| P_{\mathbf{k}} \left(X_{\mathbf{0}}^{\llbracket 1, \ell_0 \rrbracket} \right) \right\|_{\mathbb{B}, p} < \infty$$

(respectively

$$(5.63) \quad \sum_{\mathbf{k} \in \mathbb{Z}^{\ell_0}} \left\| P_{\mathbf{k}} \left(X_{\mathbf{0}}^{\llbracket 1, \ell_0 \rrbracket} \right) \right\|_{\mathbb{B}, p, d-1} < \infty).$$

To do so, we shall first prove that for $q \geq 0$,

$$(5.64) \quad \forall \ell \in \llbracket 1, d \rrbracket, \sum_{i \in \mathbb{Z}} |i|^{d-1} \left(\delta_{\mathbb{B}, p, q}^{(\ell)}(i) \right)^\alpha < \infty$$

implies that

$$(5.65) \quad \sum_{\mathbf{k} \in \mathbb{Z}^d} \|P_{\mathbf{k}}(X_{\mathbf{0}})\|_{\mathbb{B}, p, q} < \infty,$$

then that (5.65) implies that for each $\ell_0 \in \llbracket d_0, d \rrbracket$,

$$(5.66) \quad \sum_{\mathbf{k} \in \mathbb{Z}^{\ell_0}} \left\| P_{\mathbf{k}} \left(X_{\mathbf{0}}^{\llbracket 1, \ell_0 \rrbracket} \right) \right\|_{\mathbb{B}, p, q} < \infty.$$

- Proof that (5.64) implies (5.65).

First, observe that by commutativity of the operators $P_{k_{\ell} e_{\ell}}$ and the fact that $\|P_{\mathbf{k} - k_{\ell} e_{\ell}}(X_{\mathbf{0}})\|_{\mathbb{B}, p, q} \leq 2^{d-1} \|X_{\mathbf{0}}\|_{\mathbb{B}, p, q}$, we get that for each $\mathbf{k} \in \mathbb{Z}^d$,

$$(5.67) \quad \|P_{\mathbf{k}}(X_{\mathbf{0}})\|_{\mathbb{B}, p, q} \leq \min_{1 \leq \ell \leq d} \|P_{k_{\ell} e_{\ell}}(X_{\mathbf{0}})\|_{\mathbb{B}, p, q}.$$

Define the random variable $X_{\mathbf{0}}^{(\mathbf{k}, \ell)}$ by

$$(5.68) \quad X_{\mathbf{0}}^{(\mathbf{k}, \ell)} = g \left(f_1 \left(\left(\varepsilon_{i_1 - u_1}^{*, (1)} \right)_{u_1 \in \mathbb{Z}} \right), \dots, f_d \left(\left(\varepsilon_{i_d - u_d}^{*, (d)} \right)_{u_d \in \mathbb{Z}} \right) \right),$$

where $\varepsilon_u^{*, (\ell')} = \varepsilon^{(\ell')}$ if $\ell \neq \ell'$, $\varepsilon_u^{*, (\ell)} = \varepsilon_u$ for $u \neq k_{\ell}$ and $\varepsilon_{k_{\ell}}^{*, (\ell)} = \varepsilon_{k_{\ell}}'^{(\ell)}$. In other words, only the random variable $\varepsilon_{k_{\ell}}^{(\ell)}$ is replaced by a copy independent of the sequences $\left(\varepsilon_{u_q}^{(q)} \right)_{u_q \in \mathbb{Z}^d}$. Since

$P_{k_{\ell} e_{\ell}} \left(X_{\mathbf{0}}^{(\mathbf{k}, \ell)} \right) = 0$, we get that

$$(5.69) \quad \|P_{\mathbf{k}}(X_{\mathbf{0}})\|_{\mathbb{B}, p, q} \leq 2 \min_{1 \leq \ell \leq d} \left\| X_{\mathbf{0}} - X_{\mathbf{0}}^{(\mathbf{k}, \ell)} \right\|_{\mathbb{B}, p, q}.$$

. Using Hölder regularity of g and inequality (A.8), we derive that

$$(5.70) \quad \|P_{\mathbf{k}}(X_0)\|_{\mathbb{B},p,q} \leq 2C \min_{1 \leq \ell \leq d} \left\| \left\| f_{\ell} \left(\left(\varepsilon_{k_{\ell}-u_{\ell}}^{(\ell),*} \right)_{u_{\ell} \in \mathbb{Z}} \right) - f_{\ell} \left(\left(\varepsilon_{k_{\ell}-u_{\ell}}^{(\ell)} \right)_{u_{\ell} \in \mathbb{Z}} \right) \right\|_{\mathbb{B},p,q}^{\alpha} \right\|_{\mathbb{B},p,q}$$

$$(5.71) \quad \leq 2C \kappa_{\alpha,p,q} \min_{1 \leq \ell \leq d} \left\| \left\| f_{\ell} \left(\left(\varepsilon_{k_{\ell}-u_{\ell}}^{(\ell),*} \right)_{u_{\ell} \in \mathbb{Z}} \right) - f_{\ell} \left(\left(\varepsilon_{k_{\ell}-u_{\ell}}^{(\ell)} \right)_{u_{\ell} \in \mathbb{Z}} \right) \right\|_{\mathbb{B},p\alpha,q}^{\alpha} \right\|_{\mathbb{B},p\alpha,q}$$

$$(5.72) \quad = 2C \kappa_{\alpha,p,q} \min_{1 \leq \ell \leq d} \left(\delta_{\mathbb{B},p\alpha,q}^{(\ell)}(k_{\ell}) \right)^{\alpha}.$$

Defining $a_k := \max_{1 \leq \ell \leq d} \left(\delta_{\mathbb{B},p\alpha,q}^{(\ell)}(k) \right)^{\alpha}$, we are thus reduced to prove that $\sum_{\mathbf{k} \in \mathbb{Z}^d} \min_{1 \leq \ell \leq d} a_{k_{\ell}} < \infty$. To do so, take a bijective map $\tau: \mathbb{N} \rightarrow \mathbb{Z}$ such that the sequence $(b_i)_{i \in \mathbb{N}} = (a_{\tau(i)})_{i \in \mathbb{N}}$ is non-increasing. We have to show that $\sum_{i_1, \dots, i_d \in \mathbb{N}} c_{i_1, \dots, i_d} < \infty$, where $c_{i_1, \dots, i_d} = \min_{1 \leq \ell \leq d} b_{i_{\ell}}$. By invariance of c_{i_1, \dots, i_d} under permutation of the indexes i_1, \dots, i_d , it suffices to prove that $\sum_{i_1, \dots, i_d, 0 \leq i_1, \dots, i_{d-1} \leq i_d} c_{i_1, \dots, i_d} < \infty$. Since $(b_i)_{i \geq 1}$ is non-increasing, we derive that for each $i_d \geq 0$, $\sum_{0 \leq i_1, \dots, i_{d-1} \leq i_d} c_{i_1, \dots, i_d} \leq b_{i_d} i_d^{d-1}$. Summing over i_d and using (5.64) allows us to derive (5.65).

- Proof that (5.65) implies (5.66)

From the definition of $X_0^{[1, \ell_0]}$ given in (3.6), the following equality holds for each $\mathbf{k} \in \mathbb{Z}^{\ell_0}$:

$$(5.73) \quad P_{\mathbf{k}} \left(X_0^{[1, \ell_0]} \right) = \sum_{K \subset [1, \ell_0]} (-1)^{\text{Card}(K)} \sum_{J \subset [1, \ell_0]} (-1)^{\text{Card}(J)} \mathbb{E} \left[\mathbb{E} [X_0 \mid \mathcal{G}_J] \mid \mathcal{F}_{\mathbf{k}-1_K} \right].$$

Observe that if J is such $[1, \ell_0] \setminus J$ contains some j_0 , then for each $K \subset [1, \ell_0]$ such that $j_0 \notin K$,

$$(5.74) \quad \mathbb{E} \left[\mathbb{E} [X_0 \mid \mathcal{G}_J] \mid \mathcal{F}_{\mathbf{k}-1_K} \right] = \mathbb{E} \left[\mathbb{E} [X_0 \mid \mathcal{G}_J] \mid \mathcal{F}_{\mathbf{k}-1_{K \cup \{j_0\}}} \right].$$

As a consequence, only the term for $J = [1, \ell_0]$ in the right hand side of (5.73) remains and using commutativity of $(\mathcal{F}_i)_{i \in \mathbb{Z}^d}$ gives

$$(5.75) \quad P_{\mathbf{k}} \left(X_0^{[1, \ell_0]} \right) = (-1)^{\ell_0} \sum_{K \subset [1, \ell_0]} (-1)^{\text{Card}(K)} \mathbb{E} \left[\mathbb{E} [X_0 \mid \mathcal{F}_{\mathbf{k}-1_K}] \mid \mathcal{G}_{[1, \ell_0]} \right]$$

and it follows that

$$(5.76) \quad \left\| P_{\mathbf{k}} \left(X_0^{[1, \ell_0]} \right) \right\|_{\mathbb{B},p,q} \leq \left\| P_{k_1, \dots, k_{\ell_0}, 0, \dots, 0} (X_0) \right\|_{\mathbb{B},p,q},$$

from which (5.66) can be easily derived.

This ends the proof of the results of Section 3.

5.7. Proof of the results of Section 4. The proof of the results of Section 4 will first require some preliminary notations and intermediate results. Define for $m \geq 0$ and $\mathbf{i} \in \mathbb{Z}^d$ the random variables

$$(5.77) \quad Y_{\mathbf{i},m} = \mathbb{E} [X_{\mathbf{i}} \mid \sigma(\varepsilon_{\mathbf{k}} : \|\mathbf{k} - \mathbf{i}\|_{\infty} \leq m)], \quad X_{\mathbf{i},m} = Y_{\mathbf{i},m} - Y_{\mathbf{i},m-1}$$

and $Y_{\mathbf{i},-1} = 0$. Then $X_{\mathbf{i}} = \sum_{m=0}^{\infty} X_{\mathbf{i},m}$ and the convergence holds in the almost sure sense. For a fixed m , define $m' := 2m + 1$ and for $\mathbf{a} \in \mathbb{Z}^d$ such that $\mathbf{1} \preceq \mathbf{a} \preceq m' \mathbf{1}$, let $I_{\mathbf{a}}$ be set of elements $\mathbf{i} \in \mathbb{Z}^d$ such that there exists $\mathbf{j} \in \mathbb{Z}^d$ for which $\mathbf{i} = m' \mathbf{j} + \mathbf{a}$. Notice that $(X_{\mathbf{i},m})_{\mathbf{i} \in I_{\mathbf{a}}}$ is an i.i.d. random field, or equivalently, that $(X_{m' \mathbf{j} + \mathbf{a}, m})_{\mathbf{j} \in \mathbb{Z}^d}$ is an i.i.d. random field.

Also, observe that for each integer $m \geq 0$ and each $\mathbf{n} \succeq \mathbf{1}$, it is possible to bound the partial sums of $(X_{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d}$ over rectangles via partial sums of the i.i.d. random fields $(X_{m' \mathbf{j} + \mathbf{a}, m})_{\mathbf{j} \in \mathbb{Z}^d}$, namely,

$$(5.78) \quad \left\| \sum_{\mathbf{1} \preceq \mathbf{i} \preceq \mathbf{n}} X_{\mathbf{i}} \right\|_{\mathbb{B}} \leq \sum_{\mathbf{1} \preceq \mathbf{a} \preceq m' \mathbf{1}} \sum_{\delta \in \{0,1\}^d} \left\| \sum_{\mathbf{0} \preceq \mathbf{j} \preceq \lfloor \frac{\mathbf{1}}{m'} \mathbf{n} \rfloor - \delta} X_{m' \mathbf{j} + \mathbf{a}, m} \right\|_{\mathbb{B}},$$

where for $\mathbf{x} = (x_\ell)_{\ell \in [1,d]} \in \mathbb{R}^d$, $[\mathbf{x}] = ([x_\ell])_{\ell \in [1,d]}$ and for a real number t , $[t]$ is the unique integer satisfying $[t] \leq t < [t] + 1$. Indeed, since the sets $I_{\mathbf{a}}$, $\mathbf{1} \preceq \mathbf{a} \preceq m'\mathbf{1}$ are pairwise disjoint, the following inequality takes place

$$(5.79) \quad \left\| \sum_{\mathbf{1} \preceq \mathbf{i} \preceq \mathbf{n}} X_{\mathbf{i}} \right\|_{\mathbb{B}} \leq \sum_{\mathbf{1} \preceq \mathbf{a} \preceq m'\mathbf{1}} \left\| \sum_{\mathbf{1} \preceq \mathbf{i} \preceq \mathbf{n}, \mathbf{i} \in I_{\mathbf{a}}} X_{\mathbf{i}} \right\|_{\mathbb{B}}.$$

Then we express $\mathbf{i} \in I_{\mathbf{a}}$ as $\mathbf{i} = m'\mathbf{j} + \mathbf{a}$ and translate the inequalities $\mathbf{1} \preceq \mathbf{i} \preceq \mathbf{n}$ as $\mathbf{0} \preceq \mathbf{j} \preceq [m^{-1}(\mathbf{n} - \mathbf{a})]$. The sum over $\delta \in \{0,1\}^d$ comes from the fact that for each ℓ , $[m^{-1}(n_\ell - a_\ell)] \in \{[m^{-1}n_\ell], [m^{-1}n_\ell] - 1\}$.

We will express bounds on norm of maximal functions in terms of some norm of $X_{\mathbf{0},m}$. For this reason, we need the following intermediate result.

Lemma 5.1. *Let $1 < p \leq r$ and $q \geq 0$. There exists a constant C depending only on \mathbb{B} , p and q such that*

$$(5.80) \quad \|X_{\mathbf{0},m}\|_{\mathbb{B},p,q} \leq C \left(\sum_{\mathbf{i}: \|\mathbf{i}\|_\infty = m} (\delta_{\mathbb{B},p,q}(\mathbf{i}))^p \right)^{1/p}.$$

Proof. This follows the idea of proof of Corollary 1 in [10] and Corollary 1.5 in [12], which is to express $X_{\mathbf{0},m}$ as a sum of a martingale difference sequence and apply a Burkholder type inequality, namely, Proposition A.3 in the case $d = 1$. \square

Proof of Theorem 4.1. We have to show that (4.8) implies that for each positive ε ,

$$(5.81) \quad \lim_{N \rightarrow \infty} \mathbb{P} \left(\sup_{\mathbf{n} \succeq \mathbf{1}, \max \mathbf{n} \geq N} \frac{1}{|\mathbf{n}|^{1/p}} \left\| \sum_{\mathbf{1} \preceq \mathbf{i} \preceq \mathbf{n}} X_{\mathbf{i}} \right\|_{\mathbb{B}} > \varepsilon \right) = 0.$$

Using the decomposition $X_{\mathbf{i}} = Y_{\mathbf{i},M-1} + \sum_{m=M}^{\infty} X_{\mathbf{i},m}$, we derive that

$$(5.82) \quad \begin{aligned} & \mathbb{P} \left(\sup_{\mathbf{n} \succeq \mathbf{1}, \max \mathbf{n} \geq N} \frac{1}{|\mathbf{n}|^{1/p}} \left\| \sum_{\mathbf{1} \preceq \mathbf{i} \preceq \mathbf{n}} X_{\mathbf{i}} \right\|_{\mathbb{B}} > \varepsilon \right) \\ & \leq \mathbb{P} \left(\sup_{\mathbf{n} \succeq \mathbf{1}, \max \mathbf{n} \geq N} \frac{1}{|\mathbf{n}|^{1/p}} \left\| \sum_{\mathbf{1} \preceq \mathbf{i} \preceq \mathbf{n}} Y_{\mathbf{i},M-1} \right\|_{\mathbb{B}} > \frac{\varepsilon}{2} \right) + \mathbb{P} \left(\sup_{\mathbf{n} \succeq \mathbf{1}, \max \mathbf{n} \geq N} \frac{1}{|\mathbf{n}|^{1/p}} \left\| \sum_{m=M}^{\infty} \sum_{\mathbf{1} \preceq \mathbf{i} \preceq \mathbf{n}} X_{\mathbf{i},m} \right\|_{\mathbb{B}} > \frac{\varepsilon}{2} \right) \\ & \leq \mathbb{P} \left(\sup_{\mathbf{n} \succeq \mathbf{1}, \max \mathbf{n} \geq N} \frac{1}{|\mathbf{n}|^{1/p}} \left\| \sum_{\mathbf{1} \preceq \mathbf{i} \preceq \mathbf{n}} Y_{\mathbf{i},M-1} \right\|_{\mathbb{B}} > \frac{\varepsilon}{2} \right) + \mathbb{P} \left(\sum_{m=M}^{\infty} \sup_{\mathbf{n} \succeq \mathbf{1}} \frac{1}{|\mathbf{n}|^{1/p}} \left\| \sum_{\mathbf{1} \preceq \mathbf{i} \preceq \mathbf{n}} X_{\mathbf{i},m} \right\|_{\mathbb{B}} > \frac{\varepsilon}{2} \right). \end{aligned}$$

Observe that if $\|\mathbf{k}\|_\infty \geq M$, then $P_{\mathbf{k}}(Y_{\mathbf{0},M-1}) = 0$ hence by Theorem 2.8, one has

$$(5.83) \quad \lim_{N \rightarrow \infty} \mathbb{P} \left(\sup_{\mathbf{n} \succeq \mathbf{1}, \max \mathbf{n} \geq N} \frac{1}{|\mathbf{n}|^{1/p}} \left\| \sum_{\mathbf{1} \preceq \mathbf{i} \preceq \mathbf{n}} Y_{\mathbf{i},M-1} \right\|_{\mathbb{B}} > \frac{\varepsilon}{2} \right) = 0.$$

Using (2.8) and the triangle inequality for the norm $\|\cdot\|_{\mathbb{B},p,w}$, it suffices to show that

$$(5.84) \quad \sum_{m=1}^{\infty} \left\| \sup_{\mathbf{n} \succeq \mathbf{1}} \frac{1}{|\mathbf{n}|^{1/p}} \left\| \sum_{\mathbf{1} \preceq \mathbf{i} \preceq \mathbf{n}} X_{\mathbf{i},m} \right\|_{\mathbb{B}} \right\|_{p,w} < \infty.$$

To do so, we use (5.78) and replacing \mathbf{n} by $m'\mathbf{n}' + \mathbf{b}$, where $\mathbf{0} \preceq \mathbf{b} \preceq (m' - 1)\mathbf{1}$, we derive that for each fixed m ,

$$\left\| \sup_{\mathbf{n} \succeq \mathbf{1}} \frac{1}{|\mathbf{n}|^{1/p}} \left\| \sum_{\mathbf{1} \preceq \mathbf{i} \preceq \mathbf{n}} X_{i,m} \right\|_{\mathbb{B}} \right\|_{p,w} \leq \sum_{\mathbf{1} \preceq \mathbf{a} \preceq m'\mathbf{1}} \sum_{\delta \in \{0,1\}^d} \left\| \sup_{\mathbf{n}' \succeq \mathbf{0}} \sup_{\mathbf{0} \preceq \mathbf{b} \preceq (m'-1)\mathbf{1}} \frac{1}{|m\mathbf{n}' + \mathbf{b}|^{1/p}} \left\| \sum_{\mathbf{0} \preceq \mathbf{j} \preceq \mathbf{n}' - \delta} X_{m'\mathbf{j} + \mathbf{a}, m} \right\|_{\mathbb{B}} \right\|_{p,w}$$

and by Theorem 2.5, that

$$(5.85) \quad \left\| \sup_{\mathbf{n} \succeq \mathbf{1}} \frac{1}{|\mathbf{n}|^{1/p}} \left\| \sum_{\mathbf{1} \preceq \mathbf{i} \preceq \mathbf{n}} X_{i,m} \right\|_{\mathbb{B}} \right\|_{p,w} \leq C m^{d(1-1/p)} \|X_{\mathbf{0},m}\|_{\mathbb{B},p,d-1}.$$

We conclude by Lemma 5.1. \square

Proof of Theorem 4.2. We have to show that (4.5) implies that for each positive ε ,

$$(5.86) \quad \lim_{N \rightarrow \infty} \mathbb{P} \left(\sup_{n \geq N} \frac{1}{n^{d/p}} \left\| \sum_{\mathbf{1} \preceq \mathbf{i} \preceq n\mathbf{1}} X_{\mathbf{i}} \right\|_{\mathbb{B}} > \varepsilon \right) = 0.$$

By replacing in the beginning of proof of Theorem 4.1 the suprema $\sup_{\mathbf{n} \succeq \mathbf{1}, \max \mathbf{n} \geq N}$ by $\sup_{n \geq N}$ and \mathbf{n} by $n\mathbf{1}$ we reduce the proof to

$$(5.87) \quad \sum_{m=1}^{\infty} \left\| \sup_{n \geq 1} \frac{1}{n^{d/p}} \left\| \sum_{\mathbf{1} \preceq \mathbf{i} \preceq n\mathbf{1}} X_{i,m} \right\|_{\mathbb{B}} \right\|_{p,w} < \infty.$$

By (5.78) applied with $\mathbf{n} = n\mathbf{1}$, and by decomposing the supremum over n according to the remainder of n for the Euclidean division by m' , we derive that

$$(5.88) \quad \left\| \sup_{n \geq 1} \frac{1}{n^{d/p}} \left\| \sum_{\mathbf{1} \preceq \mathbf{i} \preceq n\mathbf{1}} X_{i,m} \right\|_{\mathbb{B}} \right\|_{p,w} \leq \sum_{\mathbf{1} \preceq \mathbf{a} \preceq m'\mathbf{1}} \sum_{\delta \in \{0,1\}^d} \left\| \sup_{N \geq 1} \left(\frac{1}{Nm'} \right)^{d/p} \left\| \sum_{\mathbf{0} \preceq \mathbf{j} \preceq N\mathbf{1} + \delta} X_{m'\mathbf{j} + \mathbf{a}, m} \right\|_{\mathbb{B}} \right\|_{p,w}.$$

A minor modification of the proof of Theorem 2.6 shows that (2.12) holds with $S_{n\mathbf{1}}$ replaced by $\sum_{\mathbf{0} \preceq \mathbf{j} \preceq N\mathbf{1} + \delta} X_{m'\mathbf{j} + \mathbf{a}, m}$. Therefore, by (5.88),

$$(5.89) \quad \left\| \sup_{n \geq 1} \frac{1}{n^{d/p}} \left\| \sum_{\mathbf{1} \preceq \mathbf{i} \preceq n\mathbf{1}} X_{i,m} \right\|_{\mathbb{B}} \right\|_{p,w} \leq C m^{d(1-1/p)} \|X_{\mathbf{0},m}\|_{\mathbb{B},p}$$

and (5.87) follows from Lemma 5.1. \square

Proof of Theorem 4.3. Using similar arguments as at the beginning of the proof of Theorem 4.1, it suffices to prove that

$$(5.90) \quad \sum_{m=0}^{\infty} \sup_{n \geq 1} \frac{1}{|\mathbf{n}|^{1/p}} \left\| \sum_{\mathbf{1} \preceq \mathbf{i} \preceq \mathbf{n}} X_{i,m} \right\|_{\mathbb{B},p} < \infty.$$

By (5.78), it is possible to bound $\left\| \sum_{\mathbf{1} \preceq \mathbf{i} \preceq \mathbf{n}} X_{i,m} \right\|_{\mathbb{B},p}$ by the sum of m^d terms, each of them being the $\|\cdot\|_{\mathbb{B},p}$ -norm of an independent random field on a rectangle having at most $\prod_{\ell=1}^d (n_{\ell}/m' + 1)$ elements hence by Corollary A.2, we infer that

$$(5.91) \quad \sup_{\mathbf{n} \succeq \mathbf{1}} \frac{1}{|\mathbf{n}|^{1/p}} \left\| \sum_{\mathbf{1} \preceq \mathbf{i} \preceq \mathbf{n}} X_{i,m} \right\|_{\mathbb{B},p} \leq C m^{d(1-1/p)} \|X_{\mathbf{0},m}\|_{\mathbb{B},p}.$$

We conclude by Lemma 5.1. \square

Proof of Corollary 4.4. In view of Theorems 4.1, 4.2 and 4.3, it suffices to prove that there exists a constant C such that for each $\mathbf{i} \in \mathbb{Z}^d$,

$$(5.92) \quad \delta_{\mathbb{B},p,d-1}(\mathbf{i}) \leq C \|A_{\mathbf{i}}\|_{\mathcal{B}(\mathbb{B})}^{\alpha}.$$

To do so, observe that by (4.10) and linearity of $A_{\mathbf{i}}$, the following inequalities hold almost surely

$$(5.93) \quad \|X_{\mathbf{i}} - X_{\mathbf{i}}^*\|_{\mathbb{B}} \leq K(g) \left\| \sum_{\mathbf{k} \in \mathbb{Z}^d} A_{\mathbf{k}}(\varepsilon_{\mathbf{i}-\mathbf{k}}) - \sum_{\mathbf{k} \in \mathbb{Z}^d} A_{\mathbf{k}}(\varepsilon_{\mathbf{i}-\mathbf{k}}^*) \right\|_{\mathbb{B}}^{\alpha} = K(g) \|A_{\mathbf{i}}(\varepsilon_{\mathbf{0}}) - A_{\mathbf{i}}(\varepsilon'_{\mathbf{0}})\|_{\mathbb{B}}^{\alpha}.$$

Since $A_{\mathbf{i}}$ is bounded, we infer that

$$(5.94) \quad \|X_{\mathbf{i}} - X_{\mathbf{i}}^*\|_{\mathbb{B}} \leq K(g) \|A_{\mathbf{i}}\|_{\mathcal{B}(\mathbb{B})}^{\alpha} (\|\varepsilon_{\mathbf{0}}\|_{\mathbb{B}}^{\alpha} + \|\varepsilon'_{\mathbf{0}}\|_{\mathbb{B}}^{\alpha})$$

and we conclude by (A.8). \square

A. MOMENT INEQUALITIES FOR ORTHOMARTINGALES

First recall the following, rewritten in terms of norm $\|\cdot\|_{\mathbb{B},r}$.

Proposition A.1 (Proposition A.1 in [13]). *Let $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$ be a separable r -smooth Banach space. For each $d \geq 1$, there exists a constant $C(\mathbb{B})$ such that for each orthonormal martingale difference random field $(D_{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d}$,*

$$(A.1) \quad \left\| \sum_{1 \leq \mathbf{i} \leq \mathbf{n}} D_{\mathbf{i}} \right\|_{\mathbb{B},r} \leq C(\mathbb{B}, d) \left(\sum_{1 \leq \mathbf{i} \leq \mathbf{n}} \|D_{\mathbf{i}}\|_{\mathbb{B},r}^r \right)^{1/r}.$$

Keeping in mind that an r -smooth Banach space is also p -smooth for $1 < p \leq r$, we derive the following consequence of Proposition A.1 and iterations of Doob's inequality.

Corollary A.2. *Let $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$ be a separable r -smooth Banach space and let $1 < p \leq r$. For each $d \geq 1$, there exists a constant $C(\mathbb{B}, d, p)$ such that for each orthonormal martingale difference random field $(D_{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d}$,*

$$(A.2) \quad \left\| \max_{1 \leq \mathbf{n} \leq N} \left\| \sum_{1 \leq \mathbf{i} \leq \mathbf{n}} D_{\mathbf{i}} \right\|_{\mathbb{B}} \right\|_{\mathbb{R},p} \leq C(\mathbb{B}, d, p) \left(\sum_{1 \leq \mathbf{i} \leq N} \|D_{\mathbf{i}}\|_{\mathbb{B},p}^p \right)^{1/p}.$$

We will also need to control the Orlicz-norm associated to the function $\varphi_{p,q}$ defined in (2.5) of sums of an martingale difference sequence.

Proposition A.3. *Let $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$ be a separable r -smooth Banach space. For each $1 < p \leq r$ and $q > 0$, there exists a constant $C(\mathbb{B}, p, q)$ such that for each martingale difference sequence $(D_{\mathbf{i}})_{\mathbf{i} \geq 1}$,*

$$(A.3) \quad \left\| \sum_{i=1}^n D_{\mathbf{i}} \right\|_{\mathbb{B},p,q} \leq C \left(\sum_{i=1}^n \|D_{\mathbf{i}}\|_{\mathbb{B},p,q} \right)^{1/p}.$$

Proof. By Lemma 2.2 in [7], we can find a constant K depending only on \mathbb{B} and p such that for each $x > 0$, $\beta > 1$ and $\delta \in (0, \beta - 1)$,

$$(A.4) \quad \mathbb{P} \left(\max_{1 \leq n \leq N} \left\| \sum_{i=1}^n D_{\mathbf{i}} \right\|_{\mathbb{B}} > \beta x \right) \leq \left(\frac{K(\mathbb{B}) \delta}{\beta - \delta - 1} \right)^r \mathbb{P} \left(\max_{1 \leq n \leq N} \left\| \sum_{i=1}^n D_{\mathbf{i}} \right\|_{\mathbb{B}} > x \right)$$

$$+ \mathbb{P} \left(\max \left\{ \max_{1 \leq i \leq N} \|D_i\|_{\mathbb{B}}, \left(\sum_{i=1}^N \mathbb{E} [\|D_i\|_{\mathbb{B}}^p | \mathcal{F}_{i-1}] \right)^{1/p} \right\} > \delta x \right).$$

Proceeding as in the proof of Theorem 21.1 in [1], we find that there exists a constant $K(\mathbb{B}, p, q)$ such that for each λ ,

$$(A.5) \quad \mathbb{E} \left[\varphi_{p,q} \left(\frac{\|\sum_{i=1}^n D_i\|_{\mathbb{B}}}{\lambda} \right) \right] \leq K(\mathbb{B}, p, q) \mathbb{E} \left[\varphi_{p,q} \left(\frac{\max \left\{ \max_{1 \leq i \leq n} \|D_i\|_{\mathbb{B}}, \left(\sum_{i=1}^n \mathbb{E} [\|D_i\|_{\mathbb{B}}^p | \mathcal{F}_{i-1}] \right)^{1/p} \right\}}{\lambda} \right) \right].$$

Using the fact that there exists a constant $\kappa_{p,q}$ such that for each $x, y \geq 0$, $\varphi_{p,q}(xy) \leq \kappa_{p,q} \varphi_{p,q}(x) \varphi_{p,q}(y)$, we get that for each positive λ, R ,

$$(A.6) \quad \mathbb{E} \left[\varphi_{p,q} \left(\frac{\|\sum_{i=1}^n D_i\|_{\mathbb{B}}}{R\lambda} \right) \right] \leq K(\mathbb{B}, p, q) \kappa_{p,q} \varphi_{p,q} \left(\frac{1}{R} \right) \mathbb{E} \left[\varphi_{p,q} \left(\frac{\max \left\{ \max_{1 \leq i \leq n} \|D_i\|_{\mathbb{B}}, \left(\sum_{i=1}^n \mathbb{E} [\|D_i\|_{\mathbb{B}}^p | \mathcal{F}_{i-1}] \right)^{1/p} \right\}}{\lambda} \right) \right].$$

Take R_0 such that $K(\mathbb{B}, p, q) \kappa_{p,q} \varphi_{p,q} \left(\frac{1}{R_0} \right) \leq 1$ in order to get that

$$(A.7) \quad \left\| \sum_{i=1}^n D_i \right\|_{\mathbb{B}, p, q} \leq R_0 \left\| \max \left\{ \max_{1 \leq i \leq n} \|D_i\|_{\mathbb{B}}, \left(\sum_{i=1}^n \mathbb{E} [\|D_i\|_{\mathbb{B}}^p | \mathcal{F}_{i-1}] \right)^{1/p} \right\} \right\|_{\mathbb{B}, p, q}.$$

We derive (A.3) using

$$\max \left\{ \max_{1 \leq i \leq n} \|D_i\|_{\mathbb{B}}, \left(\sum_{i=1}^n \mathbb{E} [\|D_i\|_{\mathbb{B}}^p | \mathcal{F}_{i-1}] \right)^{1/p} \right\} \leq \left(\sum_{i=1}^n \|D_i\|_{\mathbb{B}}^p \right)^{1/p} + \left(\sum_{i=1}^n \mathbb{E} [\|D_i\|_{\mathbb{B}}^p | \mathcal{F}_{i-1}] \right)^{1/p},$$

the triangle inequality and the fact that there exists a constant $\kappa'_{\alpha, p, q}$ such that for each non-negative random variable Y ,

$$(A.8) \quad \|Y^\alpha\|_{p, q} \leq \kappa'_{\alpha, p, q} \|Y\|_{p\alpha, q}^\alpha; \quad \|Y^p\|_{1, q} \leq \kappa'_{p, q} \|Y\|_{\alpha, p, q}.$$

□

B. BOUNDS ON SERIES OF TRUNCATED RANDOM VARIABLES

The truncation arguments we will use throughout the proofs lead to consideration of bounds of series having the form $\sum_{k=1}^\infty a_k \mathbb{P}(Y > b_k)$, $\sum_{k=1}^\infty a_k \mathbb{E}[Y \mathbf{1}_{Y \leq b_k}]$ or $\sum_{k=1}^\infty a_k \mathbb{E}[Y \mathbf{1}_{Y > b_k}]$ for some non-negative random variable Y and some sequences $(a_k)_{k \geq 1}$ and $(b_k)_{k \geq 1}$.

Proposition B.1. *For a non-negative random variable Y , $d \geq 1$, $1 < p < r$, the following inequalities take place:*

$$(B.1) \quad \sum_{N \geq 1} 2^{Nd(1-r/p)} \mathbf{1}_{Y \leq 2^{Nd/p}} \leq \kappa_{d, p, r} (\mathbf{1}_{Y \leq 1} + Y^{p-r} \mathbf{1}_{Y > 1}),$$

$$(B.2) \quad \sum_{N \geq 1} 2^{-dN(1-1/p)} \mathbf{1}_{Y > 2^{dN/p}} \leq \kappa_{d, p} Y^{p-1},$$

$$(B.3) \quad \sum_{k=1}^{\infty} 2^k k^{d-1} \mathbb{P} \left(Y > \varepsilon 2^{k/p} \right) \leq c_d \mathbb{E} [\varphi_{p,d-1}(Y)],$$

$$(B.4) \quad \sum_{k=1}^{\infty} 2^{k(1-r/p)} k^{d-1} \mathbb{E} [Y^r \mathbf{1}_{Y \leq 2^{k/p}}] \leq \kappa_{p,r,d} \mathbb{E} [\varphi_{p,d-1}(Y)],$$

$$(B.5) \quad \sum_{k=1}^{\infty} 2^{k(1-1/p)} k^{d-1} \mathbb{E} [Y \mathbf{1}_{Y > 2^{k/p}}] \leq c_{p,d} \mathbb{E} [\varphi_{p,d-1}(Y)].$$

Proof. All these inequalities follow from the decompositions

$$(B.6) \quad \mathbf{1}_{Y > 2^{ak}} = \sum_{j=k}^{\infty} \mathbf{1}_{2^{aj} < Y \leq 2^{a(j+1)}} \text{ and}$$

$$(B.7) \quad \mathbf{1}_{Y \leq 2^{ak}} = \mathbf{1}_{Y \leq 1} + \sum_{j=1}^k \mathbf{1}_{2^{a(j-1)} < Y \leq 2^{aj}},$$

and the fact that

$$(B.8) \quad \sum_{k=1}^j 2^{ka} k^{d-1} \leq \kappa_{a,d} 2^j j^{d-1}.$$

□

REFERENCES

- [1] D. L. Burkholder, *Distribution function inequalities for martingales*, Ann. Probability **1** (1973), 19–42. MR 365692
- [2] R. Cairoli and J. B. Walsh, *Stochastic integrals in the plane*, Acta Math. **134** (1975), 111–183. MR 420845
- [3] C. Cuny, *A compact LIL for martingales in 2-smooth Banach spaces with applications*, Bernoulli **21** (2015), no. 1, 374–400. MR 3322323
- [4] E. B. Czerebak-Mrozowicz, O. I. Klesov, and Z. Rychlik, *Marcinkiewicz-type strong law of large numbers for pairwise independent random fields*, Probab. Math. Statist. **22** (2002), no. 1, 127–139. MR 1944146
- [5] J. Dedecker and F. Merlevède, *Convergence rates in the law of large numbers for Banach-valued dependent variables*, Teor. Veroyatn. Primen. **52** (2007), no. 3, 562–587. MR 2743029
- [6] J. Dedecker and F. Merlevède, *Vitesses de convergence dans la loi forte des grands nombres pour des variables dépendantes*, C. R. Math. Acad. Sci. Paris **339** (2004), no. 12, 883–886. MR 2111727
- [7] J. Ding, J. R. Lee, and Y. Peres, *Markov type and threshold embeddings*, Geom. Funct. Anal. **23** (2013), no. 4, 1207–1229. MR 3077911
- [8] L. V. Dung and N. D. Tien, *Strong laws of large numbers for random fields in martingale type p Banach spaces*, Statist. Probab. Lett. **80** (2010), no. 9-10, 756–763. MR 2608813
- [9] D. Giraud, *Invariance principle via orthomartingale approximation*, Stoch. Dyn. **18** (2018), no. 6, 1850043, 29. MR 3869881
- [10] D. Giraud, *Convergence rates in the central limit theorem for weighted sums of Bernoulli random fields*, Mod. Stoch. Theory Appl. **6** (2019), no. 2, 251–267. MR 3963881
- [11] D. Giraud, *Maximal function associated to the bounded law of the iterated logarithms via orthomartingale approximation*, J. Math. Anal. Appl. **496** (2021), no. 1, Paper No. 124792, 25. MR 4186670
- [12] D. Giraud, *Bound on the maximal function associated to the law of the iterated logarithms for Bernoulli random fields*, Stochastics **94** (2022), no. 2, 248–276. MR 4372657
- [13] D. Giraud, *Deviation inequality for Banach-valued orthomartingales*, 2022.
- [14] A. Gut, *Marcinkiewicz laws and convergence rates in the law of large numbers for random variables with multidimensional indices*, Ann. Probability **6** (1978), no. 3, 469–482. MR 494431
- [15] N. V. Huan, *Complete convergence and complete moment convergence for independent random fields in Banach spaces*, Publ. Math. Debrecen **101** (2022), no. 3-4, 509–522. MR 4515464

- [16] O. Klesov and I. Molchanov, *Moment conditions in strong laws of large numbers for multiple sums and random measures*, *Statist. Probab. Lett.* **131** (2017), 56–63. MR 3706696
- [17] M.-H. Ko, *On the asymmetric Marcinkiewicz-Zygmund strong law of large numbers for linear random fields*, *Comm. Statist. Theory Methods* **42** (2013), no. 8, 1553–1562. MR 3211166
- [18] A. Kuczmaszewska and Z. A. Lagodowski, *Convergence rates in the SLLN for some classes of dependent random fields*, *J. Math. Anal. Appl.* **380** (2011), no. 2, 571–584. MR 2794415
- [19] P. Kurasiński, P. Matuła, and André Adler, *Exact strong laws of large numbers for independent random fields*, *Bull. Pol. Acad. Sci. Math.* **66** (2018), no. 2, 179–188. MR 3892761
- [20] H.-M. Lin, *On the weak invariance principle for non-adapted stationary random fields under projective criteria*, *Stoch. Dyn.* **22** (2022), no. 5, Paper No. 2250013, 33. MR 4486233
- [21] H.-M. Lin and F. Merlevède, *On the weak invariance principle for ortho-martingale in Banach spaces. Application to stationary random fields*, *Stochastic Process. Appl.* **153** (2022), 198–220. MR 4477977
- [22] S. Louhichi, *Convergence rates in the strong law for associated random variables*, *Probab. Math. Statist.* **20** (2000), no. 1, 203–214. MR 1785247
- [23] P. Matuła and M. Seweryn, *On Etemadi’s subsequences and the strong law of large numbers for random fields*, *Rocky Mountain J. Math.* **45** (2015), no. 5, 1555–1563. MR 3452228
- [24] T. Mikosch and R. Norvaiša, *Strong laws of large numbers for fields of Banach space valued random variables*, *Probab. Theory Related Fields* **74** (1987), no. 2, 241–253. MR 871253
- [25] M. Peligrad, *Convergence rates of the strong law for stationary mixing sequences*, *Z. Wahrsch. Verw. Gebiete* **70** (1985), no. 2, 307–314. MR 799152
- [26] M. Peligrad and D. Volný, *Quenched invariance principles for orthomartingale-like sequences*, *J. Theoret. Probab.* **33** (2020), no. 3, 1238–1265. MR 4125956
- [27] M. Peligrad and N. Zhang, *Martingale approximations for random fields*, *Electron. Commun. Probab.* **23** (2018), Paper No. 28, 9. MR 3798239
- [28] L. Reding and N. Zhang, *On the Quenched Functional Central Limit Theorem for Stationary Random Fields under Projective Criteria*, working paper or preprint, February 2022.
- [29] E. Rio, *A maximal inequality and dependent Marcinkiewicz-Zygmund strong laws*, *Ann. Probab.* **23** (1995), no. 2, 918–937. MR 1334177
- [30] T. C. Son and D. H. Thang, *The Brunk-Prokhorov strong law of large numbers for fields of martingale differences taking values in a Banach space*, *Statist. Probab. Lett.* **83** (2013), no. 8, 1901–1910. MR 3069895
- [31] T. C. Son, D. H. Thang, and L. Van Dung, *Rate of complete convergence for maximums of moving average sums of martingale difference fields in Banach spaces*, *Statist. Probab. Lett.* **82** (2012), no. 11, 1978–1985. MR 2970301
- [32] D. Volný and Y. Wang, *An invariance principle for stationary random fields under Hannan’s condition*, *Stochastic Process. Appl.* **124** (2014), no. 12, 4012–4029. MR 3264437
- [33] D. Volný and Y. Wang, *Erratum to “An invariance principle for stationary random fields under Hannan’s condition” [Stochastic Process. Appl. 124 (12) (2014) 4012–4029] [MR3264437]*, *Stochastic Process. Appl.* **127** (2017), no. 6, 2088–2091. MR 3646441
- [34] Y. Wang and M. Woodroffe, *A new condition for the invariance principle for stationary random fields*, *Statist. Sinica* **23** (2013), no. 4, 1673–1696. MR 3222815
- [35] W. A. Woyczyński, *Asymptotic behavior of martingales in Banach spaces. II*, *Martingale theory in harmonic analysis and Banach spaces* (Cleveland, Ohio, 1981), *Lecture Notes in Math.*, vol. 939, Springer, Berlin-New York, 1982, pp. 216–225. MR 668549
- [36] W. B. Wu, *Nonlinear system theory: another look at dependence*, *Proc. Natl. Acad. Sci. USA* **102** (2005), no. 40, 14150–14154. MR 2172215
- [37] N. Zhang, L. Reding, and M. Peligrad, *On the quenched central limit theorem for stationary random fields under projective criteria*, *J. Theoret. Probab.* **33** (2020), no. 4, 2351–2379. MR 4166203