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# Highly singular (frequently sparse) steady solutions for the 2D Navier–Stokes equations on the torus

Pierre Gilles Lemarié-Rieusset\*

## Abstract

We construct non-trivial steady solutions in  $H^{-1}$  for the 2D Navier–Stokes equations on the torus. In particular, the solutions are not square integrable, so that we have to redefine the notion of solutions.

**Keywords :** Navier–Stokes equations, steady solutions, lacunary Fourier series, nonuniqueness, Koch and Tataru theorem.

**AMS classification :** 35K55, 35Q30, 76D05.

## Introduction

In this paper, we are looking for steady solutions  $\vec{u}$  of the 2D Navier–Stokes equations on the torus  $\mathbb{T}^d = \mathbb{R}^2/2\pi\mathbb{Z}^2$ , i.e. for solutions of the equations

$$\begin{cases} \Delta \vec{u} - \mathbb{P}(\vec{u} \cdot \vec{\nabla} \vec{u}) = 0 \\ \operatorname{div} \vec{u} = 0 \end{cases} \quad (1)$$

where  $\vec{u}$  is a periodical distribution vector field, with mean value 0:

$$\int_{\mathbb{T}^2} \vec{u}(x) dx = 0.$$

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Such a periodical distribution vector field  $\vec{u}$  can be written as a Fourier series

$$\vec{u}(x) = \sum_{k \in \mathbb{Z}_+^2} \cos(k \cdot x) \vec{v}_k + \sin(k \cdot x) \vec{w}_k$$

where  $k \in \mathbb{Z}_+^2$  if and only if  $k = (k_1, k_2) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$  and  $\arg(k_1 + ik_2) \in (-\pi/2, \pi/2]$ . The convergence in  $\mathcal{D}'$  is given by a slow growth of the coefficients:

$$|\vec{v}_k| + |\vec{w}_k| \leq C|k|^N$$

for some constants  $C$  and  $N$ ; in particular,  $\vec{u} \in H^s(\mathbb{T}^2)$  for  $s < -N - 1$ . If  $k \in \mathbb{Z}^2$ ,  $k \neq (0, 0)$  and  $k \notin \mathbb{Z}_+^2$ , then we may replace  $k$  with  $-k$  (with  $-k \in \mathbb{Z}_+^2$ ) and write  $\cos(k \cdot x) \vec{v}_k + \sin(k \cdot x) \vec{w}_k = \cos((-k) \cdot x) \vec{v}_k + \sin((-k) \cdot x) (-\vec{w}_k)$ , hence the condition  $k \in \mathbb{Z}_+^2$  is not essential.

In our equations,  $\mathbb{P}$  is the Leray projection operator on solenoidal vector fields, defined by

$$\mathbb{P}\left(\sum_{k \in \mathbb{Z}_+^2} \cos(k \cdot x) \vec{v}_k + \sin(k \cdot x) \vec{w}_k\right) = \sum_{k \in \mathbb{Z}_+^2} \rho_k \cos(k \cdot x + \theta_k) k^\perp \quad (2)$$

with

$$\rho_k \cos(k \cdot x + \theta_k) = \cos(k \cdot x) \frac{\vec{v}_k \cdot k^\perp}{|k|^2} + \sin(k \cdot x) \frac{\vec{w}_k \cdot k^\perp}{|k|^2}$$

where

$$(k_1, k_2)^\perp = (-k_2, k_1), |(k_1, k_2)|^2 = k_1^2 + k_2^2 \text{ and } \rho_k = \frac{\sqrt{(\vec{v}_k \cdot k^\perp)^2 + (\vec{w}_k \cdot k^\perp)^2}}{|k|^2}.$$

It is easy to check that, if the solution  $\vec{u}$  satisfies  $\vec{u} \in H^1(\mathbb{T}^2)$ , then  $\vec{u} = 0$ . Thus, we shall not require (weak) differentiability for  $\vec{u}$ . Usually, it is customary to rewrite  $\vec{u} \cdot \vec{\nabla} \vec{u}$  in the equations as  $\partial_1(u_1 \vec{u}) + \partial_2(u_2 \vec{u})$  (since  $\text{div} \vec{u} = 0$ ), where the derivatives are taken in the sense of distributions. In order to define  $u_i \vec{u}$ , one then usually requires that  $\vec{u}$  be square integrable.

As we shall see it, it is easy to check that, if the solution  $\vec{u}$  satisfies  $\vec{u} \in L^p(\mathbb{T}^2)$  for some  $p > 2$ , then  $\vec{u} = 0$ . This is even true when  $\vec{u}$  belongs to the Lorentz space  $L^{2,1}(\mathbb{T}^2)$ . This is still probably the case for  $\vec{u} \in L^2(\mathbb{T}^2)$ , thus we will search for some solution  $\vec{u}$  which is not square integrable. We need however to be able to define  $\mathbb{P}(\vec{u} \cdot \vec{\nabla} \vec{u})$  when  $\vec{u}$  is no longer square integrable.

**Definition 1** (Admissible vector fields). *A divergence free periodical distribution vector field is admissible for the steady problem on  $\mathbb{T}^2$  if it is of the form*

$$\vec{u} = \sum_{k \in \mathbb{Z}_+^2} \rho_k \cos(k \cdot x + \theta_k) k^\perp \quad (3)$$

with

$$\sum_{k_1 \in \mathbb{Z}_+^2} \sum_{k_2 \in \mathbb{Z}_+^2} \rho_{k_1} \rho_{k_2} \|\mathbb{P}(\cos(k_1 \cdot x + \theta_{k_1}) \sin(k_2 \cdot x + \theta_{k_2}) (k_1^\perp \cdot k_2) k_2^\perp)\|_{H^{-N}} < +\infty \quad (4)$$

for some  $N$ .

If  $\vec{u} = \sum_{k \in \mathbb{Z}_+^2} \rho_k \cos(k \cdot x + \theta_k) k^\perp = \sum_{k \in \mathbb{Z}_+^2} \vec{u}_k$  is an admissible vector field, we will then define  $\mathbb{P}(\vec{u} \cdot \vec{\nabla} \vec{u})$  as

$$\mathbb{P}(\vec{u} \cdot \vec{\nabla} \vec{u}) = \sum_{k_1 \in \mathbb{Z}_+^2} \sum_{k_2 \in \mathbb{Z}_+^2} \mathbb{P}(\vec{u}_{k_1} \cdot \vec{\nabla} \vec{u}_{k_2}). \quad (5)$$

The main result in this paper is then the following one:

**Theorem 1.** *There exists non-trivial solutions to the equations*

$$\begin{cases} \Delta \vec{u} - \mathbb{P}(\vec{u} \cdot \vec{\nabla} \vec{u}) = 0 \\ \operatorname{div} \vec{u} = 0 \end{cases} \quad (6)$$

where  $\vec{u}$  is an admissible vector field (with mean value 0) with  $\vec{u} \in H^{-1}(\mathbb{T}^2) \cap BMO^{-1}$ .

The paper is organized in the following manner: in Section 1, we recall classical results on steady solutions on  $\mathbb{T}^d$ ; in section 2, we describe some examples of admissible vector fields; in section 3, we prove Theorem 1; in section 4, we comment on the Koch and Tataru theorem.

## 1 Steady solutions for the Navier-Stokes problem on $\mathbb{T}^d$ : known results.

In this section, we recall known results on steady solutions for the Navier-Stokes problem in  $L^2(\mathbb{T}^d)$ , for  $d \geq 2$ .

**Case**  $\vec{u} \in H^1(\mathbb{T}^d) \cap L^4(\mathbb{T}^d)$ .

If  $\vec{u} \in H^1(\mathbb{T}^d) \cap L^4(\mathbb{T}^d)$  (recall that, when  $d \leq 4$ ,  $H^1(\mathbb{T}^d) \subset L^4(\mathbb{T}^d)$ ), we can compute

$$- \int_{\mathbb{T}^d} |\vec{\nabla} \otimes \vec{u}|^2 dx = \int_{\mathbb{T}^d} \vec{u} \cdot \mathbb{P}(\vec{u} \cdot \nabla \vec{u}) dx = \int_{\mathbb{T}^d} \vec{u} \cdot (\vec{u} \cdot \nabla \vec{u}) dx = \int_{\mathbb{T}^d} \operatorname{div} \left( \frac{|\vec{u}|^2}{2} \vec{u} \right) dx = 0.$$

Thus  $\vec{u} = 0$  (we are interested in vector fields with null mean value).

**Case**  $\vec{u} \in L^p(\mathbb{T}^d)$ ,  $p > d$ .

As  $p$  is subcritical when  $p > d$ , the initial value problem

$$\begin{cases} \partial_t \vec{v} = \Delta \vec{v} - \mathbb{P}(\vec{v} \cdot \nabla \vec{v}) \\ \operatorname{div} \vec{v} = 0 \\ \vec{v}(0, x) = \vec{u}_0(x) \end{cases} \quad (7)$$

with  $\vec{u}_0 \in L^p$  has a unique solution in  $\mathcal{C}([0, T], L^p)$  for some time  $T$ . This solution  $\vec{v}$  is smooth on  $(0, T) \times \mathbb{T}^d$ .

If  $\vec{u}$  is a steady solution of (1), then  $\vec{v}(t, x) = \vec{u}(x)$  defines a solution of the Cauchy problem (7) with initial value  $\vec{u}_0 = \vec{u}$ . Hence, if  $\vec{u} \in L^p$ , we find that  $\vec{u} \in H^1 \cap L^\infty$  and finally  $\vec{u} = 0$ .

**Case**  $\vec{u} \in L^d(\mathbb{T}^d)$ ,  $d \geq 3$ .

The value  $p = d$  is critical for the Cauchy problem (7). When  $\vec{u}_0 \in L^d$ , the problem has a solution  $\vec{v}$  in  $\mathcal{C}([0, T], L^d)$  for some time  $T$  and this solution  $\vec{v}$  is smooth on  $(0, T) \times \mathbb{T}^d$ . But uniqueness of solutions in  $\mathcal{C}([0, T], L^d)$  is known only for  $d \geq 3$ . Thus, if  $\vec{u}$  is a steady solution of (1) with  $\vec{u} \in L^d(\mathbb{T}^d)$  and if  $d \geq 3$ , then  $\vec{u} = 0$ .

**Case**  $\vec{u} \in L^2(\mathbb{T}^d)$ ,  $d \geq 4$ .

Recently, Luo [7] constructed non trivial steady solutions in  $L^2(\mathbb{T}^d)$ ,  $d \geq 4$  (this solution belongs to  $L^p(\mathbb{T}^d)$  for some  $p \in (2, d)$ ). His proof was following the scheme of convex integration developed by De Lellis and Székelyhidi [2] in the case of non-steady solutions for the Euler equations and by Buckmaster and Vicol [1] in the case of non-steady solutions for the Navier–Stokes equations. However, his proof requires the spatial dimension  $d$  to be no lesser than 4.

**Case  $\vec{u} \in L^2(\mathbb{T}^2)$ .**

Uniqueness of solutions of the Cauchy problem (7) in  $\mathcal{C}([0, T], L^2(\mathbb{T}^2))$  is not known. Proofs of uniqueness in  $\mathcal{C}([0, T], L^3(\mathbb{T}^3))$  are based on maximal regularity properties which are no longer true in the 2D case [3, 6, 9, 10, 4, 8]. On the other hand, the proof of non-uniqueness in  $\mathcal{C}([0, T], L^2(\mathbb{T}^3))$  is based on convex integration methods which cannot be applied in the 2D case [1].

However, we have uniqueness in  $\mathcal{C}([0, T], L^{2,1}(\mathbb{T}^2))$ , where  $L^{2,1}$  is a Lorentz space:

**Proposition 1.**

*If  $\vec{v}_1$  and  $\vec{v}_2$  are two solutions of the Cauchy problem (7) with  $\vec{v}_1, \vec{v}_2 \in \mathcal{C}([0, T], L^{2,1}(\mathbb{T}^2))$  and  $\vec{v}_1(0, \cdot) = \vec{v}_2(0, \cdot) = \vec{u}_0$ , then  $\vec{v}_1 = \vec{v}_2$ .*

*Proof.* We follow the lines of [4] and [9]. If

$$T^* = \sup\{S \geq 0 / \vec{v}_1 = \vec{v}_2 \text{ on } [0, S)\}$$

and if  $T^* < T$ , then  $\vec{v}_1(T^*, \cdot) = \vec{v}_2(T^*, \cdot)$ : it is obvious if  $T^* = 0$ , and is a consequence of continuity if  $0 < T^* < T$ . Moreover, we can write the integral formulation of the Navier–Stokes equations with initial time  $T^*$ : for  $T^* \leq t < T$  and  $j = 1, 2$

$$\vec{v}_j(t, \cdot) = e^{(t-T^*)\Delta} \vec{v}_j(T^*, \cdot) + \int_{T^*}^t \int_{\mathbb{R}^2} K(t-s, \cdot - y) (\vec{v}_j(s, y) \otimes \vec{v}_j(s, y)) dy ds$$

with

$$|K(t, x)| \leq C \frac{1}{(\sqrt{t} + |x|)^3} \leq C(\mathbf{1}_{|x| < \pi} \frac{1}{t^{3/2}} + \mathbf{1}_{|x| \geq \pi} \frac{1}{|x|^3}).$$

Let

$$K_{\text{per}}(t, x) = \sum_{k \in \mathbb{Z}^2} K(t, x - 2\pi k).$$

Then  $K_{\text{per}}(t, \cdot) \in L^1(\mathbb{T}^2) \cap L^\infty(\mathbb{T}^2)$ , with

$$\|K_{\text{per}}(t, \cdot)\|_1 \leq C \frac{1}{\sqrt{t}} \text{ and } \|K_{\text{per}}(t, \cdot)\|_\infty \leq C(1 + \frac{1}{t^{3/2}}).$$

We write  $\vec{v}_j = \vec{w}_0 + \vec{w}_j$ , with  $\vec{w}_0 = e^{(t-T^*)\Delta} \vec{v}_j(T^*, \cdot)$ . By density of  $L^\infty(\mathbb{T}^2)$  in  $L^{2,1}(\mathbb{T}^2)$ , we have

$$\lim_{\delta \rightarrow 0^+} \sup_{T^* < t < T^* + \delta} \sqrt{t - T^*} \|\vec{w}_0(t, \cdot)\|_\infty = 0,$$

while, by continuity of  $\vec{v}_j$  and  $\vec{w}_0$ , we have, for  $j = 1, 2$ ,

$$\lim_{\delta \rightarrow 0^+} \sup_{T^* < t < T^* + \delta} \|\vec{w}_j(t, \cdot)\|_{L^{2,1}(\mathbb{T}^2)} = 0.$$

We now write, for  $\vec{w} = \vec{v}_1 - \vec{v}_2$  and  $T^* \leq t < T$ ,

$$\begin{aligned} \vec{w}(t, x) &= \int_{T^*}^t \int_{\mathbb{T}^2} K_{\text{per}}(t-s, x-y) (\vec{w}(s, y) \otimes \vec{w}_0(s, y)) dy ds \\ &\quad + \int_{T^*}^t \int_{\mathbb{T}^2} K_{\text{per}}(t-s, x-y) (\vec{w}(s, y) \otimes \vec{w}_1(s, y)) dy ds \\ &\quad + \int_{T^*}^t \int_{\mathbb{T}^2} K_{\text{per}}(t-s, x-y) (\vec{w}_0(s, y) \otimes \vec{w}(s, y)) dy ds \\ &\quad + \int_{T^*}^t \int_{\mathbb{T}^2} K_{\text{per}}(t-s, x-y) (\vec{w}_2(s, y) \otimes \vec{w}(s, y)) dy ds \\ &= \vec{z}_1(t, x) + \vec{z}_2(t, x) + \vec{z}_3(t, x) + \vec{z}_4(t, x) \end{aligned}$$

and we estimate  $\|\vec{w}(t, \cdot)\|_{L^{2,\infty}(\mathbb{T}^2)}$ .

We first write, for  $\vec{w} = \vec{v}_1 - \vec{v}_2$  and  $T^* \leq t < T$ ,

$$\begin{aligned} \|\vec{z}_1(t, \cdot)\|_{L^{2,\infty}} + \|\vec{z}_3(t, \cdot)\|_{L^{2,\infty}} &\leq C \int_{T^*}^t \|K_{\text{per}}(t-s, \cdot)\|_1 \|\vec{w}_0(s, \cdot)\|_{\infty} \|\vec{w}(s, \cdot)\|_{L^{2,\infty}} ds \\ &\leq C' \int_{T^*}^t \frac{1}{\sqrt{t-s}\sqrt{s-T^*}} ds \sup_{T^* < s < t} \sqrt{s-T^*} \|\vec{w}_0(s, \cdot)\|_{\infty} \sup_{T^* < s < t} \|\vec{w}(s, \cdot)\|_{L^{2,\infty}} \\ &= \pi C' \sup_{T^* < s < t} \sqrt{s-T^*} \|\vec{w}_0(s, \cdot)\|_{\infty} \sup_{T^* < s < t} \|\vec{w}(s, \cdot)\|_{L^{2,\infty}}. \end{aligned}$$

For  $A > 0$ , we write

$$\begin{aligned} \vec{z}_2(t, x) + \vec{z}_4(t, x) &= \int_{\sup(t-A, T^*)}^t \int_{\mathbb{T}^2} K_{\text{per}}(t-s, x-y) (\vec{w}(s, y) \otimes \vec{w}_1(s, y)) dy ds \\ &\quad + \int_{T^*}^{\sup(t-A, T^*)} \int_{\mathbb{T}^2} K_{\text{per}}(t-s, x-y) (\vec{w}(s, y) \otimes \vec{w}_1(s, y)) dy ds \\ &\quad + \int_{\sup(t-A, T^*)}^t \int_{\mathbb{T}^2} K_{\text{per}}(t-s, x-y) (\vec{w}_2(s, y) \otimes \vec{w}(s, y)) dy ds \\ &\quad + \int_{T^*}^{\sup(t-A, T^*)} \int_{\mathbb{T}^2} K_{\text{per}}(t-s, x-y) (\vec{w}_2(s, y) \otimes \vec{w}(s, y)) dy ds \\ &= \vec{z}_{5,A}(t, x) + \vec{z}_{6,A}(t, x) + \vec{z}_{7,A}(t, x) + \vec{z}_{8,A}(t, x). \end{aligned}$$

Since the pointwise product is bounded from  $L^{2,1} \times L^{2,\infty}$  to  $L^1$ , we have

$$\begin{aligned} \|\vec{z}_{5,A}(t, \cdot)\|_1 &\leq C \int_{\sup(t-A, T^*)}^t \|K_{\text{per}}(t-s, \cdot)\|_1 \|\vec{w}_1(s, \cdot)\|_{L^{2,1}} \|\vec{w}(s, \cdot)\|_{L^{2,\infty}} ds \\ &\leq C' \int_{\sup(t-A, T^*)}^t \frac{1}{\sqrt{t-s}} ds \sup_{T^* < s < t} \|\vec{w}_1(s, \cdot)\|_{L^{2,1}} \sup_{T^* < s < t} \|\vec{w}(s, \cdot)\|_{L^{2,\infty}} \\ &\leq 2C' \sqrt{A} \sup_{T^* < s < t} \|\vec{w}_1(s, \cdot)\|_{L^{2,1}} \sup_{T^* < s < t} \|\vec{w}(s, \cdot)\|_{L^{2,\infty}}. \end{aligned}$$

Similarly

$$\|\vec{z}_{7,A}(t, \cdot)\|_1 \leq C \sqrt{A} \sup_{T^* < s < t} \|\vec{w}_2(s, \cdot)\|_{L^{2,1}} \sup_{T^* < s < t} \|\vec{w}(s, \cdot)\|_{L^{2,\infty}}.$$

On the other hand, we have (for  $T^* \leq t < \min(T, T^* + 1)$ )

$$\begin{aligned} \|\vec{z}_{6,A}(t, \cdot)\|_\infty &\leq C \int_{T^*}^{\sup(t-A, T^*)} \|K_{\text{per}}(t-s, \cdot)\|_\infty \|\vec{w}_1(s, \cdot)\|_{L^{2,1}} \|\vec{w}(s, \cdot)\|_{L^{2,\infty}} ds \\ &\leq C' \int_{T^*}^{\sup(t-A, T^*)} \frac{1}{(t-s)^{3/2}} ds \sup_{T^* < s < t} \|\vec{w}_1(s, \cdot)\|_{L^{2,1}} \sup_{T^* < s < t} \|\vec{w}(s, \cdot)\|_{L^{2,\infty}} \\ &\leq 2C' \frac{1}{\sqrt{A}} \sup_{T^* < s < t} \|\vec{w}_1(s, \cdot)\|_{L^{2,1}} \sup_{T^* < s < t} \|\vec{w}(s, \cdot)\|_{L^{2,\infty}}. \end{aligned}$$

Similarly

$$\|\vec{z}_{8,A}(t, \cdot)\|_\infty \leq C \frac{1}{\sqrt{A}} \sup_{T^* < s < t} \|\vec{w}_2(s, \cdot)\|_{L^{2,1}} \sup_{T^* < s < t} \|\vec{w}(s, \cdot)\|_{L^{2,\infty}}.$$

As  $L^{2,\infty} = [L^1, L^\infty]_{\frac{1}{2}, \infty}$ , we find that, for  $T^* \leq t < \min(T, T^* + 1)$ ,

$$\|\vec{z}_2(t, \cdot)\|_{L^{2,\infty}} \leq C \sup_{T^* < s < t} \|\vec{w}_1(s, \cdot)\|_{L^{2,1}} \sup_{T^* < s < t} \|\vec{w}(s, \cdot)\|_{L^{2,\infty}}$$

and

$$\|\vec{z}_4(t, \cdot)\|_{L^{2,\infty}} \leq C \sup_{T^* < s < t} \|\vec{w}_2(s, \cdot)\|_{L^{2,1}} \sup_{T^* < s < t} \|\vec{w}(s, \cdot)\|_{L^{2,\infty}}.$$

Putting together those estimates, we get that, for  $0 < \delta < \min(1, T - T^*)$ ,

$$\sup_{T^* \leq t \leq T^* + \delta} \|\vec{w}(t, \cdot)\|_{L^{2,\infty}} \leq CA(\delta) \sup_{T^* \leq t \leq T^* + \delta} \|\vec{w}(t, \cdot)\|_{L^{2,\infty}}$$



with

$$A(\delta) = \sup_{T^* < t < T^* + \delta} \sqrt{t - T^*} \|\vec{w}_0(t, \cdot)\|_\infty + \|\vec{w}_1(t, \cdot)\|_{L^{2,1}(\mathbb{T}^2)} + \|\vec{w}_2(t, \cdot)\|_{L^{2,1}(\mathbb{T}^2)}.$$

As

$$\lim_{\delta \rightarrow 0^+} A(\delta) = 0,$$

we get that  $\vec{w} = \vec{v}_1 - \vec{v}_2$  is equal to 0 on  $[0, T^* + \delta]$  for  $\delta$  small enough, in contradiction with the definition of  $T^*$ . Thus  $T^* = T$ , and  $\vec{v}_1 = \vec{v}_2$ .  $\square$

**Corollary 1.**

If  $\vec{u}$  is a steady solution of (1) with  $\vec{u} \in L^{2,1}(\mathbb{T}^2)$ , then  $\vec{u} = 0$ .

*Proof.* We consider the Cauchy problem (7) where the initial value  $\vec{u}_0$  is equal to our steady solution  $\vec{u}$ . We can construct a mild solution  $\vec{v}_1$  on a small time interval  $[0, T]$  such that  $\vec{v}_1 \in \mathcal{C}([0, T], L^{2,1})$ ,  $\sup_{0 < t < T} \sqrt{t} \|\vec{v}_1(t, \cdot)\|_\infty < +\infty$  and  $\lim_{t \rightarrow 0^+} \sqrt{t} \|\vec{v}_1(t, \cdot)\|_\infty = 0$ . We have another solution in  $\mathcal{C}([0, T], L^{2,1})$ , namely  $\vec{v}_2(t, \cdot) = \vec{u}$ . By uniqueness, we find that  $\vec{u} = \vec{v}_1(\frac{T}{2}, \cdot) \in L^\infty$ , and thus  $\vec{u} = 0$ .  $\square$

## 2 Admissible vector fields.

In this section, we describe some examples of admissible divergence free periodical distribution vector fields

$$\vec{u} = \sum_{k \in \mathbb{Z}_+^2} \rho_k \cos(k \cdot x + \theta_k) k^\perp = \sum_{k \in \mathbb{Z}_+^2} \vec{u}_k. \quad (8)$$

### Square integrable vector fields

The most obvious example is the case  $\vec{u} \in L^2$ , i.e.  $\sum_{k \in \mathbb{Z}_+^2} \|\vec{u}_k\|_2^2 < +\infty$ . We have  $\vec{u}_{k_1} \otimes \vec{u}_{k_2} \in L^1$  while the frequencies appearing in  $\vec{u}_{k_1} \otimes \vec{u}_{k_2}$  are  $k_1 + k_2$  and  $k_1 - k_2$  (if  $k_1 \neq k_2$ , since  $\vec{u}_k \cdot \vec{\nabla} \vec{u}_k = 0$ ). Thus,

$$\|\mathbb{P}(\vec{u}_{k_1} \cdot \vec{\nabla} \vec{u}_{k_2})\|_{H^{-N}} \leq C_N \|\vec{u}_{k_1}\|_2 \|\vec{u}_{k_2}\|_2 (|k_1 + k_2|^{-N-1} + |k_1 - k_2|^{-N-1}).$$

If  $N > 1$ , we have

$$\sum_{j \in \mathbb{Z}^2 \setminus \{(0,0)\}} |j|^{-N-1} < +\infty,$$

hence

$$\sum_{j \in \mathbb{Z}^2 \setminus \{(0,0)\}} |j|^{-N-1} \|\vec{u}_{k_2+j}\|_2 \in l^2$$

and thus

$$\sum_{k_1 \in \mathbb{Z}_+^2} \sum_{k_2 \in \mathbb{Z}_+^2} \|\mathbb{P}(\vec{u}_{k_1} \cdot \vec{\nabla} \vec{u}_{k_2})\|_{H^{-N}} < +\infty.$$

## Lacunary Fourier series

Let us consider a lacunary Fourier series

$$\vec{u} = \sum_{j=0}^{+\infty} \rho_{k_j} \cos(k_j \cdot x + \theta_{k_j}) k_j^\perp = \sum_{j=0}^{+\infty} \vec{u}_{k_j}$$

with

$$|k_{j+1}| > 8|k_j|$$

and

$$\sum_{j=0}^{+\infty} \rho_{k_j}^2 |k_j|^{-2N} < +\infty$$

where  $N \geq 0$  (so that  $\vec{u} \in H^{-N-1}$ ). We have.

$$\begin{aligned} \|\mathbb{P}(\vec{u}_{k_j} \cdot \vec{\nabla} \vec{u}_{k_p})\|_{H^{-2N-3}} &\leq C \rho_{k_j} \rho_{k_p} |k_j| |k_p| (|k_j| + |k_p|)^{-2N-2} \\ &\leq C' |k_j|^{-N} \rho_{k_j} |k_p|^{-N} \rho_{k_p} \left( \frac{\min(|k_j|, |k_p|)}{\max(|k_j|, |k_p|)} \right)^{N+1}. \end{aligned}$$

Noticing that

$$\sum_{j=1}^{+\infty} \sum_{0 \leq p \leq j-1} \rho_{k_j} \rho_{k_p} \frac{|k_j| |k_p|}{(|k_j| + |k_p|)^{2N+2}} \leq C \sum_{j=1}^{+\infty} \sum_{0 \leq p \leq j-1} \frac{\rho_{k_j}}{|k_j|^N} \frac{\rho_{k_p}}{|k_p|^N} 8^{-(j-p)(N+1)},$$

we find that

$$\sum_{p=0}^{+\infty} \sum_{j=0}^{+\infty} \|\mathbb{P}(\vec{u}_{k_p} \cdot \vec{\nabla} \vec{u}_{k_j})\|_{H^{-2N-3}} < +\infty.$$

**Remark:** We could have proved that  $\mathbb{P}(\vec{u} \cdot \vec{\nabla} \vec{u}) \in H^{-2N-3}$  in another way: we have  $\vec{u} \in H^{-N-1} \cap B_{\infty, \infty}^{-N-1}$ ; using paradifferential calculus and decomposing the product  $\vec{u} \cdot \vec{\nabla} \vec{u}$  in two paraproducts and a remainder, we see that the paraproducts are controlled in  $H^{-2N-3}$  by  $\|\vec{u}\|_{H^{-N-1}} \|\vec{u}\|_{B_{\infty, \infty}^{-N-1}}$ , while the remainder is equal to 0.

## Lacunary resonant Fourier modes

$$\vec{u} = \sum_{j=0}^{+\infty} \rho_{k_j} (\cos(k_j \cdot x + \theta_{k_j}) k_j^\perp + \cos((k_j + \omega_j) \cdot x + \eta_{k_j}) (k_j + \omega_j)^\perp) = \sum_{j=0}^{+\infty} \vec{u}_{k_j}$$

with

$$|k_{j+1}| > 8|k_j|, \quad |k_j| > 8|\omega_j|, \quad \omega_j \cdot k_j = 0$$

and

$$\sum_{j=0}^{+\infty} \rho_{k_j}^2 \frac{|k_j|}{|\omega_j|} < +\infty.$$

We write

$$\vec{v}_{k_j} = \rho_{k_j} \cos(k_j \cdot x + \theta_{k_j}) k_j^\perp \quad \text{and} \quad \vec{w}_{k_j} = \rho_{k_j} \cos((k_j + \omega_j) \cdot x + \eta_{k_j}) (k_j + \omega_j)^\perp.$$

In particular, we have that  $\sum_{j=0}^{+\infty} \rho_{k_j}^2 < +\infty$ , so that  $\vec{u} \in H^{-1} \cap B_{\infty, \infty}^{-1}$ . Following the computations of the case of lacunary solutions, we find that

$$\begin{aligned} \sum_{p=0}^{+\infty} \sum_{0 \leq j, j \neq p} \|\mathbb{P}(\vec{v}_{k_p} \cdot \vec{\nabla} \vec{v}_{k_j})\|_{H^{-3}} &< +\infty, \quad \sum_{p=0}^{+\infty} \sum_{0 \leq j, j \neq p} \|\mathbb{P}(\vec{v}_{k_p} \cdot \vec{\nabla} \vec{w}_{k_j})\|_{H^{-3}} < +\infty, \\ \sum_{p=0}^{+\infty} \sum_{0 \leq j, j \neq p} \|\mathbb{P}(\vec{w}_{k_p} \cdot \vec{\nabla} \vec{v}_{k_j})\|_{H^{-3}} &< +\infty, \quad \sum_{p=0}^{+\infty} \sum_{0 \leq j, j \neq p} \|\mathbb{P}(\vec{w}_{k_p} \cdot \vec{\nabla} \vec{w}_{k_j})\|_{H^{-3}} < +\infty. \end{aligned}$$

We now estimate the diagonal terms  $\mathbb{P}(\vec{u}_{k_j} \cdot \vec{\nabla} \vec{u}_{k_j})$ . We have

$$\vec{v}_{k_j} \cdot \vec{\nabla} \vec{v}_{k_j} = \vec{w}_{k_j} \cdot \vec{\nabla} \vec{w}_{k_j} = 0,$$

while

$$\begin{aligned} \vec{v}_{k_j} \cdot \vec{\nabla} \vec{w}_{k_j} &= \rho_{k_j}^2 (k_j^\perp \cdot \omega_j) \cos(k_j \cdot x + \theta_{k_j}) \cos((k_j + \omega_j) \cdot x + \eta_{k_j} + \frac{\pi}{2}) (k_j + \omega_j)^\perp \\ &= \frac{1}{2} \rho_{k_j}^2 (k_j^\perp \cdot \omega_j) \cos((2k_j + \omega_j) \cdot x + \theta_{k_j} + \eta_{k_j} + \frac{\pi}{2}) (k_j + \omega_j)^\perp \\ &\quad + \frac{1}{2} \rho_{k_j}^2 (k_j^\perp \cdot \omega_j) \cos(\omega_j \cdot x - \theta_{k_j} + \eta_{k_j} - \frac{\pi}{2}) (k_j + \omega_j)^\perp \end{aligned}$$

and

$$\begin{aligned}\vec{w}_{k_j} \cdot \vec{\nabla} \vec{v}_{k_j} &= \rho_{k_j}^2 (k_j \cdot \omega_j^\perp) \cos(k_j \cdot x + \theta_{k_j} + \frac{\pi}{2}) \cos((k_j + \omega_j) \cdot x + \eta_{k_j}) k_j^\perp \\ &= \frac{1}{2} \rho_{k_j}^2 (k_j \cdot \omega_j^\perp) \cos((2k_j + \omega_j) \cdot x + \theta_{k_j} + \eta_{k_j} + \frac{\pi}{2}) k_j^\perp \\ &\quad + \frac{1}{2} \rho_{k_j}^2 (k_j \cdot \omega_j^\perp) \cos(\omega_j \cdot x - \theta_{k_j} + \eta_{k_j} - \frac{\pi}{2}) k_j^\perp.\end{aligned}$$

We have

$$\|\mathbb{P}(\rho_{k_j}^2 (k_j^\perp \cdot \omega_j) \cos((2k_j + \omega_j) \cdot x + \theta_{k_j} + \eta_{k_j} + \frac{\pi}{2}) (k_j + \omega_j)^\perp)\|_{H^{-3}} \leq C \rho_{k_j}^2 \frac{|\omega_j|}{|k_j|}$$

and

$$\|\mathbb{P}(\rho_{k_j}^2 (k_j \cdot \omega_j^\perp) \cos((2k_j + \omega_j) \cdot x + \theta_{k_j} + \eta_{k_j} - \frac{\pi}{2}) k_j^\perp)\|_{H^{-3}} \leq C \rho_{k_j}^2 \frac{|\omega_j|}{|k_j|}.$$

On the other hand, we have

$$\mathbb{P}(\rho_{k_j}^2 (k_j \cdot \omega_j^\perp) \cos(\omega_j \cdot x - \theta_{k_j} + \eta_{k_j} + \frac{\pi}{2}) k_j^\perp) = 0$$

and

$$\mathbb{P}(\rho_{k_j}^2 (k_j^\perp \cdot \omega_j) \cos(\omega_j \cdot x - \theta_{k_j} + \eta_{k_j} - \frac{\pi}{2}) (\omega_j + k_j)^\perp) = \rho_{k_j}^2 (k_j^\perp \cdot \omega_j) \cos(\omega_j \cdot x - \theta_{k_j} + \eta_{k_j} - \frac{\pi}{2}) \omega_j^\perp$$

so that

$$\|\mathbb{P}(\rho_{k_j}^2 (k_j^\perp \cdot \omega_j) \cos(\omega_j \cdot x - \theta_{k_j} + \eta_{k_j} - \frac{\pi}{2}) (\omega_j + k_j)^\perp)\|_{H^{-3}} \leq C \rho_{k_j}^2 \frac{|k_j|}{|\omega_j|}.$$

Thus, we get

$$\sum_{j=0}^{+\infty} \|\mathbb{P}(\vec{v}_{k_j} \cdot \vec{\nabla} \vec{w}_{k_j})\|_{H^{-3}} < +\infty, \quad \sum_{j=0}^{+\infty} \|\mathbb{P}(\vec{w}_{k_j} \cdot \vec{\nabla} \vec{v}_{k_j})\|_{H^{-3}} < +\infty,$$

and  $\mathbb{P}(\vec{u} \cdot \vec{\nabla} \vec{u})$  is well defined in  $H^{-3}$ .

### 3 2D steady solutions.

We are going to prove Theorem 1 following the lines of [1] and [7], i.e. applying the convex integration scheme by using intermittencies in the Fourier spectrum of the solution. In our case, however, computations will be much more simple than in the ones in [1] and [7], as we don't bother on convergence in  $L^2$ .

We shall look for a solution

$$\vec{u} = \sum_{j=0}^{+\infty} \vec{u}_j = \vec{u}_0 + \sum_{j=1}^{+\infty} \vec{v}_j + \vec{w}_j$$

where

- $\vec{u}_0 = \rho_0 \cos(k_0 \cdot x) k_0^\perp$  with  $0 < \rho_0 < 1$  and  $k_0 \in \mathbb{Z}^2 \setminus \{(0, 0)\}$ ,
- $\vec{v}_j = \rho_j \cos(k_j \cdot x) k_j^\perp$  and  $\vec{w}_j = \rho_j \cos((k_j + \omega_j) \cdot x + \eta_j) (k_j + \omega_j)^\perp$  with  $\rho_j > 0$  and  $k_j, \omega_j \in \mathbb{Z}^2 \setminus \{(0, 0)\}$ ,
- for  $j \geq 1$ ,  $|k_j| > 8|k_{j-1}|$ ,  $|k_j| > 8|\omega_j|$ ,  $\omega_j \cdot k_j = 0$ .

$k_j, \omega_j$  and  $\eta_j$  will be constructed by induction and we'll check that

$$\sum_{j=1}^{+\infty} \rho_j^2 \frac{|k_j|}{|\omega_j|} < +\infty,$$

so that  $\vec{u}$  is an admissible vector field such that  $\vec{u} \in H^{-1}$  (hence  $\Delta \vec{u} \in H^{-3}$ ) and  $\mathbb{P}(\vec{u} \cdot \vec{\nabla} \vec{u}) \in H^{-3}$ .

Defining  $\vec{U}_n = \sum_{j=0}^n \vec{u}_j$ , we have the convergence of  $\Delta \vec{U}_n - \mathbb{P}(\vec{U}_n \cdot \vec{\nabla} \vec{U}_n)$  to  $\Delta \vec{u} - \mathbb{P}(\vec{u} \cdot \vec{\nabla} \vec{u})$  in  $H^{-3}$ . We write, for  $n \geq 1$ ,

$$\Delta \vec{U}_n - \mathbb{P}(\vec{U}_n \cdot \vec{\nabla} \vec{U}_n) = \vec{V}_0 + \sum_{j=1}^n \vec{V}_j + \vec{W}_j$$

- $\vec{V}_0 = \Delta \vec{u}_0 (= \Delta \vec{u}_0 - \mathbb{P}(\vec{u}_0 \cdot \vec{\nabla} \vec{u}_0)) = -\rho_0 |k_0|^2 \cos(k_0 \cdot x) k_0^\perp$
- for  $n \geq 1$ ,

$$\begin{aligned} \vec{V}_n &= \Delta \vec{U}_n - \mathbb{P}(\vec{u}_n \cdot \vec{\nabla} \vec{U}_{n-1}) - \mathbb{P}(\vec{U}_{n-1} \cdot \vec{\nabla} \vec{u}_n) \\ &\quad - \frac{1}{2} (k_n^\perp \cdot \omega_n) \rho_n^2 \mathbb{P}(\cos((2k_n + \omega_n) \cdot x + \eta_n + \frac{\pi}{2}) (k_n + \omega_n)^\perp) \\ &\quad - \frac{1}{2} (k_n \cdot \omega_n^\perp) \rho_n^2 \mathbb{P}(\cos((2k_n + \omega_n) \cdot x + \eta_n - \frac{\pi}{2}) k_n^\perp) \end{aligned}$$

- for  $n \geq 1$ ,  $\vec{W}_n = -\frac{1}{2}\rho_n^2(k_n^\perp \cdot \omega_n) \cos(\omega_n \cdot x + \eta_n - \frac{\pi}{2})\omega_n^\perp$

Let us write  $A_n$  for the set of frequencies involved in the expansion of  $\vec{V}_n$ :

$$\vec{V}_n = \sum_{k \in A_n} \cos(k \cdot x + \alpha_{n,k}) \vec{v}_{n,k} = \sum_{k \in A_n} \lambda_{n,k} \cos(k \cdot x + \alpha_{n,k}) k^\perp,$$

with  $\lambda_{n,k} = \frac{\vec{v}_{n,k} \cdot k^\perp}{|k|^2}$ . Using the formula

$$\begin{aligned} & \mathbb{P}(\cos(\alpha \cdot x + \theta) \alpha^\perp \cdot \vec{\nabla}(\cos(\beta \cdot x + \eta) \beta^\perp) + \cos(\beta \cdot x + \eta) \beta^\perp \cdot \vec{\nabla}(\cos(\alpha \cdot x + \theta) \alpha^\perp)) \\ &= -\mathbb{P}((\alpha^\perp \cdot \beta) \cos(\alpha \cdot x + \theta) \sin(\beta \cdot x + \eta) \beta^\perp) \\ & \quad - \mathbb{P}((\beta^\perp \cdot \alpha) \sin(\alpha \cdot x + \theta) \cos(\beta \cdot x + \eta) \alpha^\perp) \\ &= -\frac{1}{2} \mathbb{P}((\cos((\alpha + \beta) \cdot x + \theta + \eta - \frac{\pi}{2}))((\alpha^\perp \cdot \beta) \beta^\perp + (\beta^\perp \cdot \alpha) \alpha^\perp)) \\ & \quad - \frac{1}{2} \mathbb{P}((\cos((\alpha - \beta) \cdot x + \theta - \eta - \frac{\pi}{2}))(-(\alpha^\perp \cdot \beta) \beta^\perp + (\beta^\perp \cdot \alpha) \alpha^\perp)) \\ &= -\frac{1}{2} \cos((\alpha + \beta) \cdot x + \theta + \eta + \frac{\pi}{2}) (\beta^\perp \cdot \alpha) \frac{|\beta|^2 - |\alpha|^2}{|\alpha + \beta|^2} (\alpha + \beta)^\perp \\ & \quad - \frac{1}{2} \cos((\alpha - \beta) \cdot x + \theta - \eta + \frac{\pi}{2}) (\alpha^\perp \cdot \beta) \frac{|\beta|^2 - |\alpha|^2}{|\alpha + \beta|^2} (\alpha - \beta)^\perp \end{aligned}$$

we see that we have more precisely  $8n - 1$  frequencies in  $A_n$  for  $n \geq 1$ :

- $k = k_n$  with  $\lambda_{n,k} = -\rho_n |k_n|^2$  and  $\eta_{n,k} = 0$
- $k = k_n + \omega_n$  with  $\lambda_{n,k} = -\rho_n |k_n + \omega_n|^2$  and  $\eta_{n,k} = \eta_n$
- $k = 2k_n + \omega_n$  with  $\lambda_{n,k} = -\frac{1}{2} \frac{(k_n^\perp \cdot \omega_n)^2}{|2k_n + \omega_n|^2} \rho_n^2$  and  $\eta_{n,k} = \eta_n - \frac{\pi}{2}$
- for  $j = 0, \dots, n-1$ ,  $k = k_n + k_j$  with  $\lambda_{n,k} = \frac{1}{2} \rho_n \rho_j (k_j^\perp \cdot k_n) \frac{|k_j|^2 - |k_n|^2}{|k_n + k_j|^2}$  and  $\eta_{n,k} = \frac{\pi}{2}$
- for  $j = 0, \dots, n-1$ ,  $k = k_n - k_j$  with  $\lambda_{n,k} = \frac{1}{2} \rho_n \rho_j (k_n^\perp \cdot k_j) \frac{|k_j|^2 - |k_n|^2}{|k_n - k_j|^2}$  and  $\eta_{n,k} = \frac{\pi}{2}$
- for  $j = 1, \dots, n-1$ ,  $k = k_n + k_j + \omega_j$  with  $\eta_{n,k} = \eta_j + \frac{\pi}{2}$  and  $\lambda_{n,k} = \frac{1}{2} \rho_n \rho_j ((k_j + \omega_j)^\perp \cdot k_n) \frac{|k_j + \omega_j|^2 - |k_n|^2}{|k_n + k_j + \omega_j|^2}$

- for  $j = 1, \dots, n-1$ ,  $k = k_n - k_j - \omega_j$  with  $\eta_{n,k} = -\eta_j + \frac{\pi}{2}$  and  $\lambda_{n,k} = \frac{1}{2}\rho_n\rho_j(k_n^\perp \cdot (k_j + \omega_j)) \frac{|k_j + \omega_j|^2 - |\vec{k}_n|^2}{|k_n - k_j - \omega_j|^2}$
- for  $j = 0, \dots, n-1$ ,  $k = k_n + \omega_n + k_j$  with  $\eta_{n,k} = \eta_n + \frac{\pi}{2}$  and  $\lambda_{n,k} = \frac{1}{2}\rho_n\rho_j(k_j^\perp \cdot (k_n + \omega_n)) \frac{|k_j|^2 - |k_n + \omega_n|^2}{|k_n + \omega_n + k_j|^2}$
- for  $j = 0, \dots, n-1$ ,  $k = k_n + \omega_n - k_j$  with  $\eta_{n,k} = \eta_n + \frac{\pi}{2}$  and  $\lambda_{n,k} = \frac{1}{2}\rho_n\rho_j((k_n + \omega_n)^\perp \cdot k_j) \frac{|k_j|^2 - |\vec{k}_n + \omega_n|^2}{|k_n + \omega_n - k_j|^2}$
- for  $j = 1, \dots, n-1$ ,  $k = k_n + \omega_n + k_j + \omega_j$  with  $\eta_{n,k} = \eta_n + \eta_j + \frac{\pi}{2}$  and  $\lambda_{n,k} = \frac{1}{2}\rho_n\rho_j((k_j + \omega_j)^\perp \cdot (k_n + \omega_n)) \frac{|k_j + \omega_j|^2 - |k_n + \omega_n|^2}{|k_n + \omega_n + k_j + \omega_j|^2}$
- for  $j = 1, \dots, n-1$ ,  $k = k_n + \omega_n - k_j - \omega_j$  with  $\eta_{n,k} = \eta_n + -\eta_j + \frac{\pi}{2}$  and  $\lambda_{n,k} = \frac{1}{2}\rho_n\rho_j((k_n + \omega_n)^\perp \cdot (k_j + \omega_j)) \frac{|k_j + \omega_j|^2 - |\vec{k}_n + \omega_n|^2}{|k_n + \omega_n - k_j - \omega_j|^2}$

For  $k \in A_n$ , we find that  $\frac{5}{8}|k_n| \leq |k| \leq \frac{11}{8}|k_n|$ , with

$$\frac{11}{8}|k_n| \leq \frac{11}{64}|k_{n+1}| \leq \frac{3}{8}\left(\frac{5}{8}|k_{n+1}|\right)$$

and the frequencies occuring in  $A_{n+1}$  are greater than those occuring in  $A_n$ .

We then write

$$A_0 = \{\gamma_1\}, A_1 = \{\gamma_2, \dots, \gamma_8\}, \dots, A_n = \{\gamma_{4n^2-5n+3}, \dots, \gamma_{4n^2+3n+1}\}, \dots$$

We write, for  $j \geq 0$ ,

$$\vec{V}_j = \sum_{\gamma_p \in A_j} \lambda_p \cos(\gamma_p \cdot x + \alpha_p) \gamma_p^\perp = \sum_{\gamma_p \in A_j} |\lambda_p| \cos(\gamma_p \cdot x + \alpha_p + \epsilon_p \pi) \gamma_p^\perp$$

with  $\epsilon_p \in \{0, 1\}$ . Thus we have

$$\begin{aligned} \Delta \vec{U}_n - \mathbb{P}(\vec{U}_n \cdot \vec{\nabla} \vec{U}_n) &= \sum_{j=0}^n \sum_{\gamma_p \in A_j} |\lambda_p| \cos(\gamma_p \cdot x + \alpha_p + \epsilon_p \pi) \gamma_p^\perp \\ &\quad - \sum_{j=1}^n \frac{1}{2} \rho_j^2 (k_j^\perp \cdot \omega_j) \cos(\omega_j \cdot x + \eta_j - \frac{\pi}{2}) \omega_j^\perp. \end{aligned}$$

We know the values of  $\rho_0, k_0$ , hence of  $\gamma_1 = k_0$ ,  $|\lambda_1| = \rho_0 |k_0|^2$  and  $\alpha_0 + \epsilon_0 \pi = \pi$ . We shall define by induction  $\omega_n, k_n, \rho_n$  and  $\eta_n$  for  $n \geq 1$ : we remark that

$\gamma_n \in A_{j(n)}$  for some  $j(n) < n$  (as  $n < 4n^2 - 5n + 3$ ). Thus, if we already know  $\omega_j$ ,  $k_j$ ,  $\rho_j$  and  $\eta_j$  for  $0 \leq j \leq n-1$ , we already know  $\gamma_n$ ,  $|\lambda_n|$  and  $\alpha_n + \epsilon_n\pi$ . The main idea is then to require that

$$|\lambda_n| \cos(\gamma_n \cdot x + \alpha_n + \epsilon_n\pi) \gamma_n^\perp = \frac{1}{2} \rho_n^2 (k_n^\perp \cdot \omega_n) \cos(\omega_n \cdot x + \eta_n - \frac{\pi}{2}) \omega_n^\perp.$$

We thus make the following choices:

- We take  $\omega_n = \gamma_n$ .
- We take  $k_n = N_n \omega_n^\perp$ , where the integer  $N_n$  will fulfill some requirements. Our first requirement will be that  $N_n \in \mathbb{N}$  is large enough to grant that  $N_n > 8$  and  $|k_n| > 8|k_{n-1}|$ .

- We then have

$$\frac{1}{2} \rho_n^2 (k_n^\perp \cdot \omega_n) \cos(\omega_n \cdot x + \eta_n - \frac{\pi}{2}) \omega_n^\perp = \frac{1}{2} \rho_n^2 N_n |\omega_n|^2 \cos(\omega_n \cdot x + \eta_n + \frac{\pi}{2}) \omega_n^\perp.$$

Thus, we take

$$\rho_n = \sqrt{\frac{2|\lambda_n|}{N_n |\omega_n|^2}} \text{ and } \eta_n = \alpha_n + \epsilon_n\pi - \frac{\pi}{2}.$$

- We shall add another requirement on  $N_n$  in order to grant that

$$\sum_{j=1}^{+\infty} \rho_j^2 \frac{|k_j|}{|\omega_j|} < +\infty,$$

Recall that  $0 < \rho_0 < 1$ . Take  $N_0 = 1$ . We first check by induction that  $\rho_n \leq \rho_0 N_n^{-1/4} (\leq 1)$ . Indeed, there is a constant  $C_0$  such that  $|\omega_n| \geq \frac{1}{C_0} |k_{j(n)}|$  and

$$|\lambda_n| \leq C_0 |k_{j(n)}|^2 \rho_{j(n)} \sup(1, \rho_0, \dots, \rho_{j(n)})$$

so that, by induction,  $|\lambda_n| \leq C_0 |k_{j(n)}|^2 \frac{\rho_0}{N_{j(n)}^{1/4}}$  and  $\rho_n \leq \sqrt{\frac{2C_0^3}{N_n}} \leq \rho_0 N_n^{-1/4}$  (if we take  $N_n \geq 4C_0^6 \rho_0^{-4}$ ). We have

$$\rho_n^2 \frac{|k_n|}{|\omega_n|} = N_n \rho_n^2 = \frac{2|\lambda_n|}{|\omega_n|^2} \leq 2C_0^3 \rho_0 N_{j(n)}^{-1/4}$$



Thus,

$$\sum_{j=1}^{+\infty} \rho_j^2 \frac{|k_j|}{|\omega_j|} \leq 2C_0^3 \rho_0 \sum_{n=0}^{+\infty} \sum_{j \in A_n} N_n^{-1/4} \leq 2C_0^3 \rho_0 (1 + \sum_{n=1}^{+\infty} (8n-1) N_n^{-1/4}).$$

Hence, our last requirement on  $N_n$  will be that  $N_n \geq (8n-1)^{12}$ .

Theorem 1 is proved.

## 4 A remark on the Koch–Tataru theorem.

In our construction, we have  $\vec{u} \in H^{-1} \cap BMO^{-1}$  with

$$\|\vec{u}\|_{H^{-1}} + \|\vec{u}\|_{BMO^{-1}} < C\rho_0.$$

Moreover,

$$\|\vec{u} - \vec{U}_n\|_{BMO^{-1}} \leq C\rho_0 j(n)^{-3} \rightarrow_{n \rightarrow +\infty} 0.$$

By the Koch–Tataru theorem [5], for  $\rho_0$  small enough, the evolutionary problem

$$\begin{cases} \partial_t \vec{v} = \Delta \vec{v} - \mathbb{P}(\vec{v} \cdot \nabla \vec{v}) \\ \operatorname{div} \vec{v} = 0 \\ \vec{v}(0, x) = \vec{u}(x) \end{cases} \quad (9)$$

will have a smooth solution on  $(0, +\infty) \times \mathbb{T}^2$  such that

- $\sup_{t>0} \sqrt{t} \|\vec{v}(t, \cdot)\|_\infty < +\infty$
- $\sup_{t>0} t \|\vec{\nabla} \otimes \vec{v}(t, \cdot)\|_\infty < +\infty$
- $\vec{v} \in \mathcal{C}([0, +\infty), BMO^{-1})$ .

The steady solution  $\vec{u}$  is another solution of the evolutionary problem (9), with  $\vec{u} \in \mathcal{C}([0, +\infty), BMO^{-1})$ . Of course,  $\vec{v} \neq \vec{u}$  as  $\lim_{t \rightarrow +\infty} \|\vec{v}(t, \cdot)\|_{H^{-1}} = 0$ .

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